A DC–PROGRAMMING APPROACH FOR SPARSE PDE OPTIMAL CONTROL PROBLEMS WITH NONCONVEX FRACTIONAL COSTS

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ABSTRACT. We propose a local regularization of elliptic optimal control problems which involves the nonconvex L^q fractional penalizations in the cost function. The proposed *Huber type* regularization allows us to formulate the PDE constrained optimization formulation as a DC programming problem (difference of convex functions) that is useful to obtain necessary optimality conditions and tackle its numerical solution by applying the well known DC algorithm used in nonconvex optimization problems. By this procedure we approximate the original problem in terms of a consistent family of parameterized problems for which there are efficient numerical methods available. Finally, we present numerical experiments to illustrate our theory with different configurations associated to the parameters of the problem.

1. INTRODUCTION

Several optimal control problems governed by PDE's with sparse solutions have been considered in recent years. One of the pioneer works on this subject c.f.[19] introduced optimal control problems with L^1–norm penalization in order to promote sparse optimal solutions. These solutions are characterized by having small supports, which are interpreted as a "localized" action of the optimal control. This particular feature of sparse optimal controls is relevant in applications because is rather difficult to implement optimal controls taking values on the whole domain in practice, which is the usual case of optimal control problems with the usual L^2–term in its cost functional. Another interesting class of optimal control problems involving sparsity where considered in [3] and [2] where the set of feasible controls is chosen in the space of regular Borel measures, therefore optimal controls can be supported in a zero Lebesgue measure set.

Another less explored approach that offers sparse solutions induced by a penalization term was considered in [15] which refers to penalizations consisting in nonconvex L^q–functionals with $q \in [0, 1)$. These kind of penalizations have many important applications, for instance: in inverse problems on the reconstruction of the sparsest solution in undetermined systems [17], image restoration [10], compressive sensing [9] and optimal control problems [15]. In particular, the L^0–functional is a difficult problem which corresponds to the selection of the most representative variables of the optimization process, extending the notion of cardinality of the control variable in finite dimensions, represented by the ℓ^0 norm, which is well known to be a NP–hard problem. L^q–functionals with $q \in (0, 1)$ on the other hand, are a natural approximation to L^0–functional. However, they are neither convex nor differentiable. In [15] a similar problem is considered involving a penalization term for the control variable involving the H_0^1–norm, this allows to get and explicit optimality system that can be solved directly by semi–smooth Newton methods. In our case, we consider a Tikhonov term in the L^2–norm. Although existence of optimal controls can

be argued in this case, uniqueness of the solution is not guaranteed as showed in a simple example below.

Due to the lack of convexity and differentiability these type of costs are difficult to tackle numerically. In this paper, we address the numerical solution of this type of problems by regularizing the fractional L^q–functionals; for this purpose, we introduce a Huber–like smoothing function which regularizes the nonconvex L^q term. In this way, we obtain a family of regularized problems whose objective functional can be expressed as a DC-function ("DC" stands for difference of convex functions), which reveals the underlying convexity of this class of problems.

Although the regularized problem remains nonconvex and nondifferentiable, we can take advantage of the DC structure of the functional by applying known tools from the convex analysis and DC programming theories in order to derive optimality conditions and prove that the regularization is consistent. Moreover, we propose a numerical method based on the *DC-Algorithm*. It follows that the proposed DC splitting leads to a primal–dual updating that only requires the numerical resolution of a convex L^1–norm penalized optimal control problem in each iteration, for which there are efficient numerical methods.

This paper is organized as follows. In section 1 we introduce the non convex optimal control problems endowed with L^q–functionals with $q = \frac{1}{p}$, and $p > 1$. In Section 2 we introduce a Huber–like smoothing function in order to regularize the nonconvex optimal control problems. We show that the regularized problems can be expressed as a difference of convex functions and derive optimality conditions in Section 3. The box–constrained case is discussed at the end of this section. In addition, we provide a proof that the solution of the regularized version of the optimal control problem approximates its solution when the regularized parameter tends to infinity. Section 4 is devoted to the numerical solution by proposing a DC–Algorithm based method. We finish this research by showing numerical examples and numerical evidence of the efficiency of the proposed method.

For $p > 1$, let us define the mapping Υ_p defined by

$$(1) \qquad\qquad u \mapsto \Upsilon_p(u) := \int_\Omega |u|^{\frac{1}{p}}.$$

Let Ω a bounded Lipschitz domain in $\mathbb{R}^n$ ($n = 2$ or $n = 3$) with boundary Γ. We are interested in the following optimal control problem with a $L^{1/p}$–fractional penalization term (6), formulated as follows:

$$(P) \qquad \begin{cases} \min_{(y,u)} \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\Upsilon_p(u) \\ \text{subject to} \\ \qquad\quad Ay = \ u + f, \quad \text{in } \Omega, \\ \qquad\quad\ y = \ 0, \qquad \text{on } \Gamma. \end{cases}$$

Where f is a given function in $L^2(\Omega)$ and A is a uniformly elliptic second order differential operator, of the form

$$(2) \qquad\qquad (Ay)(x) = -\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial y(x)}{\partial x_j}\right) + c_0 y(x),$$

where the coefficients $a_{ij} \in C^{0,1}(\bar{\Omega})$, and $c_0 \in L^\infty(\Omega)$. Moreover, the matrix $(a_{ij})_{i,j}$ is symmetric and fulfill the uniform ellipticity condition:

$$\exists \sigma > 0: \quad \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq \sigma|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \text{for almost all } x \in \Omega.$$

We will denote the adjoint of A by A^*. By defining the bilinear form $a(y,v) := \langle Ay, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}$ we consider the following linear elliptic problem

$$(3) \qquad\qquad a(y,v) = (w,v), \quad \forall v \in H_0^1(\Omega).$$

It is well known that (3) has a unique solution belonging to the space $H_0^1(\Omega)$. Let $S : L^2(\Omega) \to H_0^1(\Omega)$ be the linear and continuous operator which assigns to every $v \in L^2(\Omega)$ the corresponding solution $y = y(w) \in H_0^1(\Omega)$ satisfying (3). Thus, the state equation: $Ay = u$ in Ω with homogeneous Dirichlet boundary conditions considered in (P), is understood in the weak sense of equation (3). In this way, the state y associated to the control u has the representation $y = S(u + f)$, which in turn, allows us to formulate the usual reduced problem:

$$(P') \qquad\qquad \min_u J(u) := \frac{1}{2}\|Su + Sf - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\Upsilon_p(u).$$

Theorem 1. *There exists a solution $\bar{u} \in L^2(\Omega)$ for the regularized problem* (P').

Proof. Existence of a solution can be argued by standard techniques. Let $(u_n)_{n \in \mathbb{N}}$ be a minimizing sequence for problem (P'). By the definition of J in (P') it follows that $(u_n)_{n \in \mathbb{N}}$ is bounded in $L^2(\Omega)$. Therefore, we extract a weakly convergent subsequence $(u_n)_{n \in \mathbb{N}}$ with weak limit $\bar{u} \in L^2(\Omega)$. Moreover, by the usual compact embedding $H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, the corresponding state $y_n := y(u_n) = S(u_n)$ converges strongly to $\bar{y}_\gamma := y(\bar{u}_\gamma)$ in $L^2(\Omega)$. Consequently, it also converges strongly in $L^1(\Omega)$. By Lemma [15, Lemma 5.1] Υ_p is a continuous and quasiconvex functional, then it is weakly lower semi-continuous, see [5, pg. 26] or [16, Lemma 2.1], therefore we have

$$\inf J(u) = \lim_{n\to\infty} J(u_n) = \liminf_{n\to\infty} J(u_n)$$
$$= \liminf_{n\to\infty} f(u_n) + \Upsilon_p(u_n)$$
$$\geq \liminf_{n\to\infty} f(u_n) + \liminf_{n\to\infty} \Upsilon_p(u_n)$$
$$\geq f(\bar{u}) + \Upsilon_p(\bar{u}) = J(\bar{u}),$$

which implies that J attains its minimum value at $\bar{u}$ and thus $\bar{u}_\gamma$ is a solution for problem (7) and therefore it is an optimal control for (P'). ∎

Remark 1. *The question of uniqueness is more delicate. The following example of the minimization of a real function has two solutions. Let $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \frac{1}{2}(x - a)^2 + \beta|x|^{\frac{1}{2}}$. By choosing $a = 1 + \frac{1}{2}$ and $\beta = 1$, it is easy to verify that f has two minimum points at $x_1 = 0$ and $x_2 = 1$ with the minimum value $f(0) = f(1) = \frac{9}{8}$. Therefore, we can not expect uniqueness of the solution for problem* (P') *in view of nonconvexity of cost function.*

As in the work of Stadler [19] where L^1–norm penalization optimal control problems are considered, we expect that some analogous properties also hold for problem (P). For example, it is expected that a local solution for (P) vanishes if the parameter β is large enough. We address to this question in the following lemma.

Lemma 1. *Let S^* be the adjoint operator of S, and let $M > 0$. If $\beta \geq \beta_0$ with $\beta_0 = M^{\frac{p-1}{p}} \|S^*(Sf - y_d)\|_{L^\infty(\Omega)}$, then problem (P) has a local minimum at $\bar{u} = 0$ in $B_\infty(0, M)$ with associated state $\bar{y} = Sf$.*

Proof. Taking into account the reduced form (P'), we argue analogously to [19, Lemma 3.1]. Let us take $u \in B_\infty(0, M)$, then $|u(x)| < M$ for almost all x in Ω. Computing the difference of the cost values we have:

$$
\begin{aligned}
J(u) - J(0) =& \frac{1}{2}\|Su + Sf - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\Upsilon_p(u) \\
& - \frac{1}{2}\|Sf - y_d\|_{L^2(\Omega)}^2, \\
=& \frac{1}{2}\|Su\|_{L^2(\Omega)}^2 + (Su, Sf - y_d)_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\Upsilon_p(u), \\
\geq& \frac{1}{2}\|Su\|_{L^2(\Omega)}^2 - \|u\|_{L^1(\Omega)}\|S^*(Sf - y_d)\|_{L^\infty(\Omega)} + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\Upsilon_p(u), \\
& - \|u\|_{L^1(\Omega)}\|S^*(Sf - y_d)\|_{L^\infty(\Omega)} + \beta\Upsilon_p(u), \\
=& \int_\Omega \beta|u|^{\frac{1}{p}} - |u|\|S^*(Sf - y_d)\|_{L^\infty(\Omega)}\, dx, \\
\geq& \int_\Omega \beta_0|u|^{\frac{1}{p}} - |u|\|S^*(Sf - y_d)\|_{L^\infty(\Omega)}\, dx,
\end{aligned}
$$

by the definition of β_0 it follows that

$$
J(u) - J(0) \geq \int_\Omega \left(M^{\frac{p-1}{p}} - |u|^{\frac{p-1}{p}}\right)|u|^{\frac{1}{p}}\|S^*(Sf - y_d)\|_{L^\infty(\Omega)}\, dx > 0,
$$

where the non-negativity is obtained by our assumption $u \in B_\infty(0, M)$.   ∎

2. Huber–type Regularization of the Optimal Control Problem

In order to analyze this problem we formulate a family of regularized problems, by means of the following Huber–type regularization of the absolute value. Extending the classical Huber C^1 regularization of the absolute value but appropriately taking into account the powers defining Υ_p so that the resulting function to the power $1/p$ is a locally convex regularization for the nonconvex and non differentiable term, see Figure 8 below.

For $\gamma >> 1$, we define

$$
(4) \qquad h_{p,\gamma}(v) = \begin{cases} \frac{\gamma^{p-1}}{p}|v|^p, & \text{if } v \in [-\frac{1}{\gamma}, \frac{1}{\gamma}], \\ |v| + \frac{1}{\gamma}\frac{1-p}{p}, & \text{otherwise.} \end{cases}
$$

Remark 2. *The function $h_{p,\gamma}$ is a local regularization of the absolute value for different smoothing polynomial powers. In addition, notice that by construction, we have the relation*

$$
(5) \qquad h_{p,\gamma}(v) \leq |v|, \quad \forall v \in \mathbb{R}
$$

Now, we have the basic tool in order to formulate a regularized version of (P). We introduce the function $\Upsilon_{p,\gamma}$ defined by

$$
(6) \qquad u \mapsto \Upsilon_{p,\gamma}(u) := \int_\Omega h_{p,\gamma}(u(x))^{\frac{1}{p}}\, dx.
$$
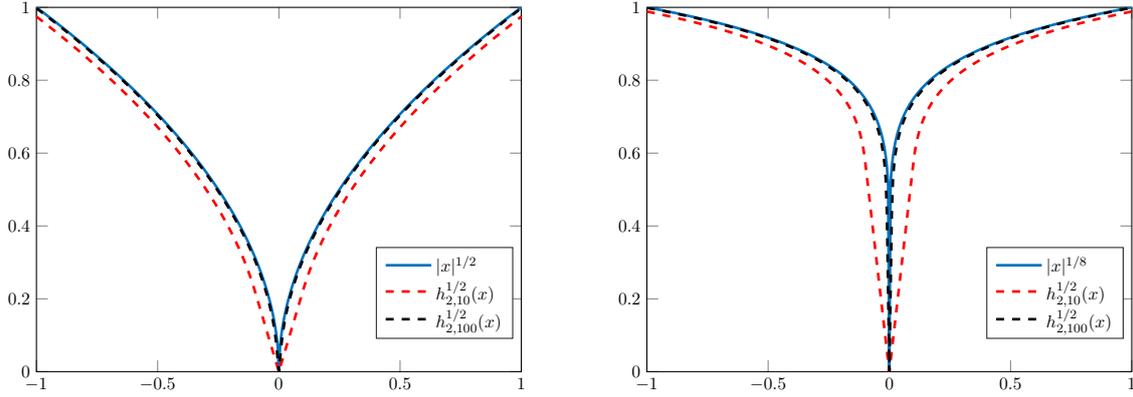
FIGURE 1. Exact and regularized penalizations (in dashed lines) for $\gamma = 10$ and $\gamma = 100$, for $p = 2$ (left) and $p = 8$ (right)

.

The regularized problem is obtained by replacing Υ_p by $\Upsilon_{p,\gamma}$. Therefore, the surrogate problem reads:

(P_γ)
$$
\begin{cases}
\min_{(y,u)} \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\Upsilon_{p,\gamma}(u) \\[2mm]
\text{subject to:} \\[1mm]
\qquad Ay = u + f \quad \text{in } \Omega, \\
\qquad\quad y = 0 \qquad \text{on } \Gamma.
\end{cases}
$$

Let us formulate the reduced optimal control problem from (P_γ) by replacing the control–to–state operator S. Let f be the regular part of the functional, which is $f(u) = \frac{1}{2}\|Su - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)}$. Now we have the reduced problem,

(7)
$$
\min_u J_\gamma(u) := f(u) + \beta\Upsilon_{p,\gamma}.
$$

From [15, Lemma 5.1] it is known that if a sequence (u_n) such that $u_n \to u$ in $L^1(\Omega)$ then $\Upsilon_p(u_n) \to \Upsilon_p(u)$ as $n \to \infty$. In the case of $\Upsilon_{\gamma,p}$ we have the following continuity property.

Lemma 2. *Let (u_n) be a sequence such that $u_n \to u$ in $L^1(\Omega)$, then*

$$
\Upsilon_{p,\gamma}(u_n) \to \Upsilon_{p,\gamma}(u), \quad \text{when } n \to \infty,
$$

for all $p > 1$ and all $\gamma > 0$.

Proof. Analogously to [15, Lemma 5] we define the following sets:

$$
\begin{aligned}
\Omega_1 &= \{x : |u(x)| \le \tfrac{1}{\gamma} \text{ and } |u_n(x)| \le \tfrac{1}{\gamma}\}, \\
\Omega_2 &= \{x : |u(x)| > \tfrac{1}{\gamma} \text{ and } |u_n(x)| > \tfrac{1}{\gamma}\}, \\
\Omega_3 &= \{x : |u(x)| \le \tfrac{1}{\gamma} \text{ and } |u_n(x)| > \tfrac{1}{\gamma}\} \cup \{x : |u(x)| > \tfrac{1}{\gamma} \text{ and } |u_n(x)| \le \tfrac{1}{\gamma}\},
\end{aligned}
$$

which we use to estimate $\left|\int_\Omega h_{p,\gamma}(u(x))^{\frac{1}{p}} - h_{p,\gamma}(u_n(x))^{\frac{1}{p}}\,dx\right|$ according to (4).

Since by our assumption $u_n \to u$ in $L^1(\Omega)$ whenever $n \to \infty$, in Ω_1 we have that

$$\left| \int_{\Omega_1} h_{p,\gamma}(u(x))^{\frac{1}{p}} - h_{p,\gamma}(u_n(x))^{\frac{1}{p}} \, dx \right| \leq \left(\frac{\gamma^{p-1}}{p} \right)^{\frac{1}{p}} \int_{\Omega_1} |\, |u(x)| - |u_n(x)| \,| \, dx,$$

$$(8) \hspace{4cm} \leq \left(\frac{\gamma^{p-1}}{p} \right)^{\frac{1}{p}} \int_\Omega |u(x) - u_n(x)| \, dx \to 0.$$

Now, in Ω_2 we can estimate

$$\left| \int_{\Omega_2} h_{p,\gamma}(u(x))^{\frac{1}{p}} - h_{p,\gamma}(u_n(x))^{\frac{1}{p}} \, dx \right| \leq \int_{\Omega_2} \left| \left(|u(x)| + \frac{1}{\gamma} \frac{1-p}{p} \right)^{\frac{1}{p}} - \left(|u_n(x)| + \frac{1}{\gamma} \frac{1-p}{p} \right)^{\frac{1}{p}} \right| \, dx$$

$$\leq \int_{\Omega_2} |\, |u(x)| - |u_n(x)| \,|^{\frac{1}{p}} \, dx,$$

$$\leq \int_{\Omega_2} |\, u(x) - u_n(x) \,|^{\frac{1}{p}} \, dx.$$

By applying Hölder inequality in the last integral, and by our convergence assumption, we have

$$\left| \int_{\Omega_2} h_{p,\gamma}(u(x))^{\frac{1}{p}} - h_{p,\gamma}(u_n(x))^{\frac{1}{p}} \, dx \right| \leq |\Omega|^{\frac{p}{p-1}} \left(\int_\Omega |u(x) - u_n(x)| \, dx \right)^{\frac{1}{p}}$$

$$(9) \hspace{5cm} \to 0.$$

Finally, we estimate in Ω_3. Without loss of generality we assume that $\{x : |u(x)| \leq \frac{1}{\gamma}$ and $|u_n(x)| > \frac{1}{\gamma}\}$. The neglected part can be argued in the same way by interchanging the role of $|u(x)|$ and $|u_n(x)|$. Taking into account that $|u(x)| \leq 1/\gamma < |u_n(x)|$ in Ω_3, it follows that

$$\left(\frac{\gamma^{p-1}}{p} \right) |u(x)|^p < |u_n(x)| + \frac{1}{\gamma} \frac{1-p}{p},$$

which implies

$$\left| \int_{\Omega_2} h_{p,\gamma}(u(x))^{\frac{1}{p}} - h_{p,\gamma}(u_n(x))^{\frac{1}{p}} \, dx \right| \leq \int_{\Omega_2} |h_{p,\gamma}(u(x)) - h_{p,\gamma}(u_n(x))|^{\frac{1}{p}} \, dx$$

$$= \int_{\Omega_2} \left| \left(\frac{\gamma^{p-1}}{p} \right) |u(x)|^p - |u_n(x)| - \frac{1}{\gamma} \frac{1-p}{p} \right|^{\frac{1}{p}} \, dx$$

$$(10) \hspace{3cm} = \int_{\Omega_2} \left(|u_n(x)| + \frac{1}{\gamma} \frac{1-p}{p} - \left(\frac{\gamma^{p-1}}{p} \right) |u(x)|^p \right)^{\frac{1}{p}} \, dx.$$

Furthermore, in Ω_3 we have that $\frac{1}{\gamma p} < |u_n(x)| + \frac{1}{\gamma} \frac{1-p}{p} < |u_n(x)|$ from which we obtain that

$$(11) \hspace{3cm} |u_n(x)| + \frac{1}{\gamma} \frac{1-p}{p} < |u_n(x)|^p \left(\frac{\gamma^{p-1}}{p} \right).$$

By replacing (11) in (10) we get the following relation

(12) $\qquad \left| \int_{\Omega_2} h_{p,\gamma}(u(x))^{\frac{1}{p}} - h_{p,\gamma}(u_n(x))^{\frac{1}{p}} \, dx \right|$

$$\leq \left(\frac{\gamma^{p-1}}{p} \right)^{\frac{1}{p}} \int_{\Omega_2} (|u_n(x)|^p - |u(x)|^p)^{\frac{1}{p}} \, dx$$

$$= \left(\frac{\gamma^{p-1}}{p} \right)^{\frac{1}{p}} \int_{\Omega_2} | \, |u_n(x)|^p - |u(x)|^p \, |^{\frac{1}{p}} \, dx$$

$$\leq \left(\frac{\gamma^{p-1}}{p} \right)^{\frac{1}{p}} \int_{\Omega} | \, |u_n(x)|^p - |u(x)|^p \, |^{\frac{1}{p}} \, dx.$$

The last term is familiar for the previous arguments, implying that the right–hand side of (12) tends to 0 as $n \to 0$. Finally, collecting the estimates (8), (9) and (12) the result of the lemma is proved. ∎

The next Theorem address to the question about existence of a solution of problem (P_γ).

Theorem 2. *There exists a solution $\bar{u}_\gamma \in L^2(\Omega)$ for the regularized problem (P_γ).*

Proof. The proof is analogous to Theorem (1). Following the same arguments we have a weakly convergent subsequence $(u_n)_{n \in \mathbb{N}}$ with weak limit $\bar{u}_\gamma \in L^2(\Omega)$ and $(y_n)_{n \in \mathbb{N}}$ converging to $\bar{y}$ strongly in $L^1(\Omega)$. Now, by Lemma 2 we known that $\Upsilon_{p,\gamma}$ is a continuous and quasiconvex function. Again, by [5, pg. 26] or [16, Lemma 2.1] it is weakly lower semi-continuous. Likewise, we estimate:

$$\inf J_\gamma(u) = \lim_{n \to \infty} J_\gamma(u_n) = \liminf_{n \to \infty} J_\gamma(u_n)$$
$$= \liminf_{n \to \infty} f(u_n) + \Upsilon_{p,\gamma}(u_n)$$
$$\geq \liminf_{n \to \infty} f(u_n) + \liminf_{n \to \infty} \Upsilon_{p,\gamma}(u_n)$$
$$\geq f(\bar{u}_\gamma) + \Upsilon_{p,\gamma}(\bar{u}_\gamma) = J_\gamma(\bar{u}_\gamma),$$

which implies that J_γ attains its minimum value at $\bar{u}_\gamma$ and thus $\bar{u}_\gamma$ is a solution for problem (7) and therefore it is an optimal control for (P_γ). ∎

3. Optimality Conditions of the regularized problem

Our aim in this section is to derive an optimality system for problem (P_γ) via a DC programming approach. The key idea is to introduce an L^1–norm penalization which allows us to formulate our problem as a difference of convex functions with functions G and H such that:

(13) $$J_\gamma(u) = G(u) - H(u).$$

A function that can be expressed in this form is known as a DC–function and several problems involving this type of functions had been analyzed and its theory can be found in the monograph of Hiriart Urruty [12] or [7]. Therefore, we investigate how to express the cost function of problem (P_γ) as a convenient difference of convex functions and then rely on the theory of DC programming.

We start by introducing the following quantity, which will be frequently used along this paper:

(14) $$\delta = \frac{\gamma^{\frac{p-1}{p}}}{p^{\frac{1}{p}}}.$$

The next step is to define G and H in (13) as follows:

(15)
$$
\begin{aligned}
G: \quad L^2(\Omega) &\rightarrow \mathbb{R} \\
u &\mapsto G(u) := \tfrac{1}{2}\|Su + Sf - y_d\|_{L^2(\Omega)}^2 + \alpha\|u\|_{L^2(\Omega)}^2 + \beta\delta\|u\|_{L^1(\Omega)}, \\
H: \quad L^2(\Omega) &\rightarrow \mathbb{R} \\
u &\mapsto H(u) := \beta\left(\delta\|u\|_{L^1(\Omega)} - \Upsilon_{p,\gamma}(u)\right).
\end{aligned}
$$

Lemma 3. *The real function $j : \mathbb{R} \rightarrow \mathbb{R}_+ \cup \{0\}$, defined by*

(16)
$$
j(z) = \begin{cases} \delta|z| - \left(|z| + \tfrac{1}{\gamma}\tfrac{1-p}{p}\right)^{\frac{1}{p}}, & \text{if } |z| > \tfrac{1}{\gamma} \\ 0, & \text{if } |z| \leq \tfrac{1}{\gamma}, \end{cases}
$$

is a nonegative, convex and continuously differentiable and its derivative is given by

(17)
$$
j'(z) = \begin{cases} \delta\operatorname{sign}(z) - \tfrac{1}{p}\left(|z| + \tfrac{1}{\gamma}\tfrac{1-p}{p}\right)^{\frac{1-p}{p}}\operatorname{sign}(z), & \text{if } |z| > \tfrac{1}{\gamma} \\ 0, & \text{if } |z| \leq \tfrac{1}{\gamma}. \end{cases}
$$

Proof. Let us first check differentiability. It is clear that j is differentiable if $|z| < \tfrac{1}{\gamma}$ or $|z| > \tfrac{1}{\gamma}$, where $j'(z) = 0$ and $j'(z) = \delta\operatorname{sign}(z) - \tfrac{1}{p}(|z| + \tfrac{1}{\gamma}\tfrac{1-p}{p})^{\frac{1-p}{p}}$. Therefore, we check differentiability at $z = \pm\tfrac{1}{\gamma}$. Consider $z = -\tfrac{1}{\gamma}$, since $j(\pm\tfrac{1}{\gamma}) = 0$ and $|\tfrac{1}{\gamma} + h| < \tfrac{1}{\gamma}$ for sufficiently small h, we have $\lim\limits_{h\to 0^+} \dfrac{j(z+h)-j(z)}{h} = \lim\limits_{h\to 0^+} \dfrac{j(-\tfrac{1}{\gamma}+h)}{h} = 0$. On the other hand, since $-\tfrac{1}{\gamma} + h < 0$ for sufficiently small h

$$
\lim_{h\to 0^-} \frac{j(z+h)-j(z)}{h} = \lim_{h\to 0^-} \frac{j(-\tfrac{1}{\gamma}+h)}{h}
$$

$$
= \lim_{h\to 0^-} \frac{\delta\left(\tfrac{1}{\gamma}-h\right) - \left(\tfrac{1}{\gamma}-h+\tfrac{1}{\gamma}\tfrac{1-p}{p}\right)^{\frac{1}{p}}}{h} = \lim_{h\to 0^-} \frac{\left(\tfrac{1}{\gamma p}\right)^{\frac{1}{p}} - \delta h - \left(\tfrac{1}{\gamma p}-h\right)^{\frac{1}{p}}}{h},
$$

where we apply the binomial theorem to get

$$
\lim_{h\to 0^-} \frac{\left(\tfrac{1}{\gamma p}\right)^{\frac{1}{p}} - \delta h - \left(\tfrac{1}{\gamma p}-h\right)^{\frac{1}{p}}}{h} = \lim_{h\to 0^-} \frac{\left(\tfrac{1}{\gamma p}\right)^{\frac{1}{p}} - \delta h - \left(\tfrac{1}{\gamma p}\right)^{\frac{1}{p}} - \tfrac{1}{p}\left(\tfrac{1}{\gamma p}\right)^{\frac{1-p}{p}}h + o(h)}{h}
$$

$$
= \lim_{h\to 0^-} \frac{o(h)}{h} = 0.
$$

Therefore $j'(-\tfrac{1}{\gamma}) = 0$. Analogously, it also follows that $j'(\tfrac{1}{\gamma}) = 0$ which implies formula (17). Moreover, an straightforward observation reveals that j' is continuous, therefore j is continuously differentiable.Convexity follows by noticing that function $\mathbb{R}_+ \ni z \mapsto (z + \tfrac{1}{\gamma}\tfrac{1-p}{p})^{1/p}$ is concave because it is the composition of an afine function and and concave function. Therefore, for $z > \tfrac{1}{\gamma}$, we find that the function

$$
\mathbb{R}_+ \ni z \mapsto \delta z - \left(z + \frac{1}{\gamma}\frac{1-p}{p}\right)^{1/p},
$$

is convex and monotone increasing which, by composition with the absolute value, implies the convexity of j. Finally, we make the simple but important observation that j vanishes in the interval $[-\tfrac{1}{\gamma}, -\tfrac{1}{\gamma}]$. This, together with convexity imply that j is nonnegative. ∎

By employing the function j we can write H as follows:

$$(18) \qquad \begin{aligned} H: \quad L^2(\Omega) &\rightarrow \quad \mathbb{R} \\ u &\mapsto \quad H(u) = \int_\Omega j(u)dx. \end{aligned}$$

Lemma 4. *The functions G and H defined in (15) are convex.*

Proof. Since $\alpha \geq 0$ and $\beta \geq 0$, it is clear that function G is strictly convex if $\beta + \alpha > 0$. In the case of H convexity follows from Lemma 3. ∎

With above definitions, it is clear that the representation (13) of J_γ has been set up. Therefore, J_γ is a DC-function and we can express optimality conditions in term of G and H by considering the following formulation for problem (7)

$$(19) \qquad \min_u J_\gamma(u) = G(u) - H(u),$$

Lemma 5. *The function H defined in (15) is Gâteaux differentiable, and its derivative $\delta H(\bar{u}; \cdot)$ is represented by $(\beta \bar{w}, \cdot)$, where $\bar{w} \in L^2(\Omega)$ depends on $\bar{u}$, p and γ, and it is given by*

$$(20) \qquad \bar{w}(x) := \begin{cases} \delta - \dfrac{1}{p}\left(|\bar{u}(x)| + \dfrac{1}{\gamma}\dfrac{1-p}{p}\right)^{\frac{1-p}{p}}, & \text{if } |\bar{u}(x)| \geq \frac{1}{\gamma}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. First, notice that $j'(z)$ given in (17) satisfies that

$$(21) \qquad 0 < |j'(z)| = \left| \delta - \frac{1}{p}\left(|z| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1-p}{p}} \right| < \delta, \quad \text{for } |z| > \frac{1}{\gamma}.$$

Therefore, by using (21) and the properties of j stablished in Lemma 3, we apply [1, Theorem 2.7, pg. 19] in order to deduce that superposition operator $u \mapsto j(u)$ is Gâteaux differentiable from $L^2(\Omega)$ into $L^2(\Omega)$, and its Gâteaux derivative in the direction v is given by $j'(u)v \in L^2(\Omega)$. Hence, Theorem 7.4-1 in [4] allows us to compute the Gâteaux derivative of H at u in any direction $v \in L^2(\Omega)$ given by

$$(22) \qquad \delta H(\bar{u}, v) = \int_\Omega j'(\bar{u}(x))v dx = (\bar{w}, v),$$

with $\bar{w}$ given by (20). ∎

3.1. **First–order necessary conditions.** We will derive an optimality system from the following optimality conditions of DC–programming theory. The conditions for local and global optimality can be found in [12, Proposition 3.1 and 3.2] or in [8]. From this theory we will use the following result which permits the characterization of local minima.

Proposition 1. *Let G and H, the convex functions defined in (15). If $\bar{u}$ is a local minimum of the DC–function $J_\gamma = G - H$, then*

$$(23) \qquad \partial H(\bar{u}) \subset \partial G(\bar{u}).$$

Next, we derive the optimality system with the help of the last proposition.

Theorem 3. *Let $\bar{u}$ a solution of (P_γ), then there exist $\bar{y} = S\bar{u}$ in $H_0^1(\Omega)$, an adjoint state $\phi \in H_0^1(\Omega)$, a multiplier $\zeta \in L^2(\Omega)$ and $\bar{w}$ given by (20) such that the following optimality system is satisfied:*

$$(24a) \qquad \begin{aligned} A\bar{y} &= \bar{u} + f, \quad \text{in } \Omega, \\ \bar{y} &= 0, \qquad\;\; \text{on } \Gamma, \end{aligned}$$

$$(24b) \qquad \begin{aligned} A^*\bar{\phi} &= \bar{y} - y_d, \quad \text{in } \Omega, \\ \bar{\phi} &= 0, \qquad\quad \text{on } \Gamma, \end{aligned}$$

$$(24c) \qquad \bar{\phi} + \alpha\bar{u} + \beta\left(\delta\,\zeta - \bar{w}\right) = 0,$$

$$(24d) \qquad \begin{aligned} \zeta(x) &= 1, \qquad \text{if } \bar{u}(x) > 0, \\ \zeta(x) &= -1, \quad\; \text{if } \bar{u}(x) < 0, \\ |\zeta(x)| &\leq 1, \qquad \text{if } \bar{u}(x) = 0, \end{aligned}$$

$$\text{for almost all } x \in \Omega.$$

Proof. Clearly, equation (24a) is equivalent to $S\bar{u} = \bar{y}$. By standard properties of the subdifferential calculus c.f.[13], the subdifferential of G at $\bar{u}$ is given by $\partial G(\bar{u}) = \nabla f(\bar{u}) + \partial\|\cdot\|_{L^1(\Omega)}(\bar{u})$. By Lemmas 2 and 5, it follows that $\partial H(\bar{u})$ consists in the singleton $\{\bar{w}\}$. Thus, condition (23) becomes

$$(25) \qquad \bar{w} \in \nabla f(\bar{u}) + \beta\delta\partial\|\cdot\|_{L^1(\Omega)}(\bar{u}).$$

Since S is a linear and continuous operator from $L^2(\Omega)$ into $L^2(\Omega)$, the computation of $\nabla f(\bar{u})$ is straightforward, see for instance [5]. Therefore, for $u \in L^2(\Omega)$ we have that

$$(26) \qquad \begin{aligned} \nabla f(\bar{u})u &= (Su, S\bar{u} + Sf - y_d)_{L^2(\Omega)} + \alpha(u, \bar{u})_{L^2(\Omega)} \\ &= (u, \alpha\bar{u} + S^*(\bar{y} - y_d))_{L^2(\Omega)}. \end{aligned}$$

Moreover, by introducing the adjoint state $\bar{\phi} \in H_0^1(\Omega)$ as the solution of the adjoint equation: (24b)

$$\begin{aligned} A^*\bar{\phi} &= \bar{y} - y_d \quad \text{in } \Omega, \\ \bar{\phi} &= 0 \qquad\quad \text{on } \Gamma, \end{aligned}$$

we are able to write $\bar{\phi} = S^*(\bar{y} - y_d)$ (S^* denoting the adjoint control–to–state operator). On the other hand, it is well known [14, Chapter 0.3.2], that any $\zeta \in \partial\|\cdot\|_{L^1(\Omega)}(\bar{u})$ is characterized by

$$(27) \qquad \zeta(x) \begin{cases} = 1, & \text{if } \bar{u}(x) > 0, \\ = -1, & \text{if } \bar{u}(x) < 0, \\ \in [-1, 1], & \text{if } \bar{u}(x) = 0. \end{cases}$$

In this way, from (26) we obtain that $\nabla f(\bar{u}) = \bar{\phi} + \alpha\bar{u}$ which together with (27) imply the existence of $\zeta \in \partial\|\cdot\|_{L^1(\Omega)}(\bar{u}) \subset L^\infty(\Omega)$ which allows us to write (25) in the form:

$$(28) \qquad \bar{\phi} + \alpha\bar{u} + \beta(\delta\zeta - \bar{w}) = 0.$$

∎

An important question regarding the regularized problem (P_γ) is about the convergence of the solutions of (P_γ) to a solution of the original problem (P) when $\gamma \to \infty$. We answer this question in the following Theorem.

Theorem 4. *Let $\bar{u}_\gamma$ be a solution of problem (P_γ) then the limit $u^* := \lim_{\gamma\to\infty} \bar{u}_\gamma$ is a solution for problem (P).*

Proof. We begin by noticing that the sequence $(\bar{u}_\gamma)_{\gamma>0}$ is bounded in $L^2(\Omega)$. Indeed, since $S0 = 0$, optimality of $\bar{u}_\gamma$ results in

$$\frac{\alpha}{2}\|u_\gamma\|_{L^2(\Omega)}^2 \leq J_\gamma(\bar{u}_\gamma) \leq J_\gamma(0) = \frac{1}{2}\|y_d\|_{L^2(\Omega)}^2,$$

which implies the boundedness of $(\bar{u}_\gamma)_{\gamma>0}$ in $L^2(\Omega)$ for $\alpha > 0$.

As usual, reflexivity of $L^2(\Omega)$ allows us to extract a weakly convergent subsequence, denoted again by $(\bar{u}_\gamma)_{\gamma>0}$ which has the limit $u^* \in L^2(\Omega)$. Arguing again that the optimality of $\bar{u}_\gamma$ implies that $J_\gamma(\bar{u}_\gamma) \leq J_\gamma(u)$ for any $u \in L^2(\Omega)$ and taking into account (5), it follows that

$$\begin{aligned} J_\gamma(\bar{u}_\gamma) &\leq J_\gamma(u) \\ &= \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\Upsilon_{p,\gamma}(u) \\ &\leq \frac{1}{2}\|y - y_d\|_{L^2(\Omega)}^2 + \frac{\alpha}{2}\|u\|_{L^2(\Omega)}^2 + \beta\Upsilon(u) = J(u), \end{aligned}$$

implying

$$\liminf_{\gamma\to 0} J_\gamma(\bar{u}_\gamma) \leq J(u). \tag{29}$$

On the other hand, we also have that

$$\begin{aligned} \liminf_{\gamma\to 0} J_\gamma(\bar{u}_\gamma) &= \liminf_{\gamma\to\infty}\left(f(\bar{u}_\gamma) + \Upsilon_{p,\gamma}(\bar{u}_\gamma)\right) \\ &\geq \liminf_{\gamma\to\infty} f(\bar{u}_\gamma) + \liminf_{\gamma\to\infty} \Upsilon_{p,\gamma}(\bar{u}_\gamma) \\ &\geq f(u^*) + \liminf_{\gamma\to\infty} \Upsilon_{p,\gamma}(\bar{u}_\gamma), \end{aligned} \tag{30}$$

where the last relation is obtained in view of the weakly lower semicontinuity of f. The next step is to consider the last term in (30). By definition of the Huber–type regularization (4) we find that

$$\begin{aligned} \liminf_{\gamma\to\infty} \Upsilon_{p,\gamma}(\bar{u}_\gamma) &= \liminf_{\gamma\to\infty} \int_\Omega (h_{p,\gamma}(\bar{u}_\gamma))^{\frac{1}{p}}\, dx \\ &\geq \liminf_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma} \frac{\gamma^{\frac{p-1}{p}}}{p^{\frac{1}{p}}}|\bar{u}_\gamma|\, dx + \liminf_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma^c}\left(|\bar{u}_\gamma| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1}{p}}\, dx. \end{aligned} \tag{31}$$

Since $\bar{u}_\gamma$ is bounded and $\chi_{\Omega_\gamma} \to 0$ as $\gamma \to \infty$ the first term in (31) vanishes, therefore, we drive our analysis to the last term in (31). Consider the function $g : \mathbb{R}_+ \cup \{0\} \to \mathbb{R}$ defined by $x \mapsto x^{\frac{1}{p}}$. It is clear that g is a concave differentiable function for $p > 1$. Furthermore, it is known that g satisfies the inequality

$$g(x + h) \geq g(x) - g'(x + h)h. \tag{32}$$

Applying this inequality we have

$$\begin{aligned} &\liminf_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma^c}\left(|\bar{u}_\gamma| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1}{p}}\, dx \\ &\geq \liminf_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma^c}\left(|\bar{u}_\gamma|^{\frac{1}{p}} - \frac{1}{\gamma}\frac{1-p}{p^2}\left(|\bar{u}_\gamma(x)| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1-p}{p}}\right)\, dx \\ &\geq \liminf_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma^c}|\bar{u}_\gamma|^{\frac{1}{p}} - \limsup_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma^c}\left(\frac{1}{\gamma}\frac{1-p}{p^2}\left(|\bar{u}_\gamma(x)| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1-p}{p}}\right)\, dx. \end{aligned}$$

Here, we apply Hölder inequality on the right term, hence

$$\liminf_{\gamma\to\infty} \int_\Omega \chi_{\Omega_\gamma^c} \left(|\bar{u}_\gamma| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1}{p}} dx$$

$$\geq \liminf_{\gamma\to\infty} \int_\Omega |\bar{u}_\gamma|^{\frac{1}{p}} - \limsup_{\gamma\to\infty} \|\chi_{\Omega_\gamma^c}\|_{L^\infty(\Omega)} \frac{1}{\gamma} \int_\Omega \left| \frac{1-p}{p^2} \left(|\bar{u}_\gamma(x)| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1-p}{p}} \right| dx$$

$$= \liminf_{\gamma\to\infty} \int_\Omega |\bar{u}_\gamma|^{\frac{1}{p}} dx - \limsup_{\gamma\to\infty} \frac{1}{\gamma} \int_\Omega \left| \frac{1-p}{p^2} \left(|\bar{u}_\gamma(x)| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1-p}{p}} \right| dx$$

$$(33) \qquad = \liminf_{\gamma\to\infty} \int_\Omega |\bar{u}_\gamma|^{\frac{1}{p}} dx,$$

where the last relation follows since the integrant of the right term is bounded. Taking into account (33) and (31) and the lower semicontinuity of $|\cdot|^{\frac{1}{p}}$ we arrive to the inequality:

$$(34) \qquad \liminf_{\gamma\to\infty} \Upsilon_{p,\gamma}(\bar{u}_\gamma) \geq \liminf_{\gamma\to\infty} \int_\Omega |\bar{u}_\gamma|^{\frac{1}{p}} dx = \int_\Omega |\bar{u}^*|^{\frac{1}{p}} dx,$$

which, combined with (30) implies that

$$(35) \qquad \liminf_{\gamma\to 0} J_\gamma(\bar{u}_\gamma) \geq f(u^*) + \int_\Omega |\bar{u}^*|^{\frac{1}{p}} dx = J(u^*).$$

Finally, (35) and (29) imply that $J(u^*) = \liminf_{\gamma\to\infty} J_\gamma(\bar{u}_\gamma)$ ∎

3.2. First–order necessary conditions with box–constraints.

Since box–constraints are importan in applications, we give a further discussion when they are included in our optimal control. Therefore, the set of feasible controls is given by

$$(36) \qquad U_{ad} = \{u \in L^2(\Omega) : u_a(x) \leq u(x) \leq u_b(x), \text{ a.a. } x \in \Omega\},$$

where u_a and u_b are given functions in $L^\infty(\Omega)$ satisfying $u_a(x) < u_b(x)$ a.a. $x \in \Omega$. Therefore, the control constrained optimal control problem reads:

$$(P\mathcal{C}) \qquad \begin{cases} \min_{(y,u)} \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\Upsilon_p(u) \\ \text{subject to:} \\ u \in U_{ad} \qquad \text{and} \qquad \begin{aligned} Ay &= u + f, &&\text{in } \Omega, \\ y &= 0, &&\text{on } \Gamma. \end{aligned} \end{cases}$$

Remark 3. *It follows by definition (36) that $U_{ad} \subset B_\infty(0, M)$ with $M = \max\{\|u_a\|_{L^\infty(\Omega)}, \|u_b\|_{L^\infty(\Omega)}\}$ Therefore, according to Lemma 1 if $\beta > \beta_0 = M^{\frac{p-1}{p}} \|S^*(Sf - y_d)\|_{L^\infty(\Omega)}$ then $\bar{u} = 0$ is solution of $(P\mathcal{C})$.*

Analogous to the unconstrained optimal control problem (P'), after introducing the control–to–state operator S and replacing Υ_p by $\Upsilon_{p,\gamma}$, we introduce the regularized control constrained problem

(PC_γ)
$$
\begin{cases}
\min_{(y,u)} \dfrac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta \Upsilon_{p,\gamma}(u) \\[2mm]
\text{subject to:} \\[2mm]
u \in U_{ad} \qquad \text{and} \qquad
\begin{aligned}
Ay &= u + f, &&\text{in } \Omega, \\
y &= 0, &&\text{on } \Gamma.
\end{aligned}
\end{cases}
$$

define a DC representation of the cost functional for problem (PC) by including the indicator function $\chi_{U_{ad}}$ for the admisible control set:

(37)
$$
\begin{aligned}
G: \quad L^2(\Omega) &\rightarrow \mathbb{R} \\
u &\mapsto G(u) := \tfrac{1}{2}\|Su + Sf - y_d\|^2_{L^2(\Omega)} + \alpha\|u\|^2_{L^2(\Omega)} + \beta\delta\|u\|_{L^1(\Omega)} + \chi_{U_{ad}}, \\
H: \quad L^2(\Omega) &\rightarrow \mathbb{R} \\
u &\mapsto H(u) := \beta\left(\delta\|u\|_{L^1(\Omega)} - \Upsilon_{p,\gamma}(u)\right).
\end{aligned}
$$

Thus, by similar arguments as in the unconstrained case and taking into account that $\partial\chi_{U_{ad}}(u)$ corresponds to the normal cone of U_{ad} at $\bar{u}$, we can derive an analogous optimality system.

Theorem 5. *Let $\bar{u}$ a solution of (P_γ), then there exist $\bar{y} = S\bar{u}$ in $H^1_0(\Omega)$, an adjoint state $\phi \in H^1_0(\Omega)$ and a multiplier $\zeta \in L^2(\Omega)$ and $\bar{w}$ given by (20) such that the following optimality system is satisfied :*

(38a)
$$
\begin{aligned}
A\bar{y} &= \bar{u} + f &&\text{in } \Omega, \\
\bar{y} &= 0 &&\text{on } \Gamma,
\end{aligned}
$$

(38b)
$$
\begin{aligned}
A^*\bar{\phi} &= \bar{y} - y_d &&\text{in } \Omega, \\
\bar{\phi} &= 0 &&\text{on } \Gamma,
\end{aligned}
$$

(38c)
$$
\langle \bar{\phi} + \alpha\bar{u} + \beta\left(\delta\zeta - \bar{w}\right), u - \bar{u}\rangle \geq 0, \quad \forall u \in U_{ad}
$$

(38d)
$$
\begin{aligned}
\zeta(x) &= 1, &&si\ \bar{u}(x) > 0, \\
\zeta(x) &= -1, &&si\ \bar{u}(x) < 0, \\
|\zeta(x)| &\leq 1, &&si\ \bar{u}(x) = 0, \\
&\text{for almost all } x \in \Omega.
\end{aligned}
$$

Moreover, there exist λ_a and λ_b in $L^2(\Omega)$ such that the the last optimality system can be written as a KKT optimality system:

(39a)
$$
\begin{aligned}
A\bar{y} &= \bar{u} + f &&\text{in } \Omega, \\
\bar{y} &= 0 &&\text{on } \Gamma,
\end{aligned}
$$

(39b)
$$
\begin{aligned}
A^*\bar{\phi} &= \bar{y} - y_d &&\text{in } \Omega, \\
\bar{\phi} &= 0 &&\text{on } \Gamma,
\end{aligned}
$$

(39c)
$$
\bar{\phi} + \alpha\bar{u} + \beta\left(\delta\zeta - \bar{w}\right) + \lambda_b - \lambda_a = 0
$$

(39d)
$$
\begin{aligned}
\lambda_a &\geq 0, &&\lambda_b \geq 0, \\
\lambda_a(\bar{u} - u_a) &= 0, &&\lambda_b(u_b - \bar{u}) = 0,
\end{aligned}
$$

(39e)
$$
\begin{aligned}
\zeta(x) &= 1 &&si\ \bar{u}(x) > 0, \\
\zeta(x) &= -1 &&si\ \bar{u}(x) < 0, \\
|\zeta(x)| &\leq 1 &&si\ \bar{u}(x) = 0,
\end{aligned}
$$

Proof. This theorem is proved by following the arguments of the proof of Theorem 3, where variational inequality (38c) follows by taking into consideration classical results on convex analysis and the fact that $\bar{w} \in \nabla f(\bar{u}) + \beta\delta \, \partial\| \cdot \|_{L^1(\Omega)}(\bar{u}) + \partial\chi_{U_{ad}}(\bar{u})$. ∎

In addition, by the usual projection operator $\mathcal{P}_{U_{ad}}$ (see [11, Lemma 1.11]) on the admissible control set, the variational inequality (38c) can be equivalently rewritten in equation form:

$$(40) \qquad \bar{u} = \mathcal{P}_{U_{ad}}\left[-\frac{1}{\alpha}\left(\bar{\phi} + \beta(\delta\zeta - \bar{w})\right)\right].$$

4. Numerical solution via the DC Algorithm (DCA)

In the former section we have derived necessary optimality conditions for problem (P_γ) which is suitable for applying the Semi-Smooth Newton method (SSN). However, SSN does not guarantee descend of the objective function.

By the nature of our problem we turn our attention to its numerical solution by first discretize–then–optimize approach by adapting the DC algorithm. The application of DC algorithm to our problem leads to a numerical scheme which relies on numerical methods for solving sparse L^1 optimal control problems, including SNN methods. Our method is completely determined by the formulation (19) which is a suitable DC–decomposition of the original optimal control problem. For simplicity, we present the algorithm in a functional setting, keeping in mind that there is an intermediary procedure for its discretization.

The DC–Algorithm is based on the fact that: if $\bar{u}$ is the solution of the primal problem (P') then $\partial H(\bar{u}) \subset \partial G(\bar{u})$ and conversely, if u^* is the solution of the dual problem denoted by (P'^*) we have the inclusion $\partial G^*(u^*) \subset \partial H^*(u^*)$, where H^* and G^* correspond to the dual functions of H and G respectively. This symmetry means that DC–Algorithm alternates in computing approximations of the solutions for the primal and the dual problems as follows:

$$(41) \qquad\qquad\qquad \text{First chose: } w_k \in \partial H(u_k),$$

$$(42) \qquad\qquad\qquad \text{then chose: } u_k \in \partial G^*(w_k).$$

DC-Algorithm provides a primal–dual updating procedure without need of the line–search step. This is and important feature in optimal control problems where the line–search step requires the evaluation of the cost function and its gradient requiring the computation of the state and adjoint equations, which usually are very expensive to solve numerically. A more detailed discussion on the DC method can be found in [7].

Since $\partial H(u_k) = \{w_k\}$, then formula (20) implies that w_k is given by

$$(43) \qquad w_k = \begin{cases} 0, & \text{if} \quad |u_k(x)| \leq \frac{1}{\gamma}, \\ \left[\delta - \frac{1}{p}\left(|u_k(x)| + \frac{1}{\gamma}\frac{1-p}{p}\right)^{\frac{1-p}{p}}\right]\text{sign}(u_k(x)), & \text{otherwise.} \end{cases}$$

For $g : \mathbb{R}^n \to \mathbb{R}$ convex and lower semi–continuous it follows that

$$g(x) = \sup\{\langle x, y\rangle - g^*(y) : y \in \mathbb{R}^n\}$$

Moreover, according to Rockafellar [18] the subgradients can be computed as:

$$(44) \qquad\qquad\qquad \partial G(y) = \text{argmax}_w\{\langle y, w\rangle - G^*(w)\},$$

$$(45) \qquad\qquad\qquad \partial G^*(w) = \text{argmax}_z\{\langle w, z\rangle - G(z)\},$$

therefore, according to (45), u_k can be obtained by solving the following optimal control problem

$$(46) \quad \min_{u_{k+1}} \frac{1}{2}\|Su_{k+1} + Sf - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_{k+1}\|^2_{L^2(\Omega)} + \delta\beta\|u_{k+1}\|_{L^1(\Omega)} - \int_\Omega w_k u_{k+1}\, dx.$$

Remark 4. *By the form of the DC splitting* (19) *we get rid of the problem* (44) *and we can compute w_k directly from* (43). *In addition, observe that problem* (46) *is a convex L^1–sparse optimal control problem with penalization parameter $\delta\beta$, for which it is known to have a unique solution for $\alpha > 0$ c.f.* [19]. *The case of $\alpha = 0$ with box–constraints is also possible. Moreover, this problem can be solved numerically in an efficient way. For example, it can be solved by semi–smooth Newton methods proposed in* [19] *or, it can be solved in the framework of sparse programming problems in finite dimensions after its discretization.*

In order to complete our algorithm, we now turn our attention to the following mechanism as stopping criterion. Looking at the gradient equation (24c) we could consider checking

$$(47) \quad \zeta_k = \frac{1}{\beta\delta}\left(w_k - \phi_k - \alpha u_k\right) \in \partial \,\|\cdot\|_{L^1(\Omega)}(u_k),$$

where u_k, ϕ_k, w_k represent the corresponding approximations of the optimal control, the adjoint state and the multipliers in the k–th iteration. In practice, a less sensitive stopping criterion gave us better results. This consists in checking the residual:

$$(48) \quad \|\zeta_{k+1} - \zeta_k\| \le tol,$$

where *tol* is a prescribed tolerance.

Algorithm 1 DCA for problem (P_γ)

1: Initialize u^0.
2: **while** stoping criteria is false **do**
3:　　Compute w_k given by (43)
4:　　Compute u_{k+1} by solving problem (46)
5:　　$k \leftarrow k + 1$.
6: **end while**

In case of the presence of box–constraints on the control, our formulation yields an box–constrained L^1 optimal control subproblem

$$(49) \quad \min_{u_k} \frac{1}{2}\|Su_k - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u_k\|^2_{L^2(\Omega)} + \delta\beta\|u_k\|_{L^1(\Omega)} - \int_\Omega w_k u_k\, dx.$$
$$\text{subject to:}$$
$$u_k \in U_{ad}.$$

Algorithm 2 DCA for problem $(P\mathcal{C}_\gamma)$

1: Initialize u^0.
2: **while** stoping criteria is false **do**
3:　　Compute w_k given by (43)
4:　　Compute u_{k+1} by solving problem (49)
5:　　$k \leftarrow k + 1$.
6: **end while**

---

5. Implementation aspects

5.1. **Approximation.** For simplicity, the approximation of problems (P) and (PC) is done by the finite–difference scheme, although any method of discretization can be considered such as finite elements. Uniform meshes are considered in the domain Ω with N internal nodes. The associated mesh parameter is given by $h = \frac{1}{N+1}$. Then, the state equation (3) is solved numerically with finite difference method, whereas the approximation of the integrals are computed accordingly using the following mid–point rule:

$$(50) \qquad \int_a^b \int_c^d u(x,y)dydx \approx \frac{1}{4}h^2\big\{u(a,c) + u(b,c) + u(a,d) + u(b,d)$$

$$+ 2\sum_{i=1}^{n-2} u(x_i,c) + 2\sum_{i=1}^{n-2} u(x_i,d) + 2\sum_{i=1}^{n-2} u(a,y_i)$$

$$+ 2\sum_{i=1}^{n-2} u(b,y_i) + 4\sum_{i=1}^{n-2}\sum_{j=1}^{n-2} u(x_i,y_i)\big\}.$$

Using this approximation, and reshaping the matrix $(u(x_i,y_j))_{i,j=1,\dots,N}$ as a vector $\mathbf{u} \in \mathbb{R}^{N^2}$ the L^1–norm is approximated by

$$(51) \qquad \|u\|_1 \approx \sum_{i=1}^{N^2} c_i|\mathbf{u}_i|,$$

where c_i are the corresponding coefficients given by (50).

5.2. **Auxiliar L^1–sparse optimal control problems.** DC algorithms 1 and 2 have a simple structure. However, they require to solve auxiliar L^1–norm optimal control problems (46) and (49) respectively. Clearly, the efficiency of the proposed algorithms strongly depend on the numerical methods applied for solving (46) and (49). As mentioned before, the numerical solution of the L^1–norm optimal control problems can be done by semi–smooth Newton methods proposed in [19]. Although our methodology was proposed for elliptic problems, it can be extended for parabolic problems or optimal control problems involving other equations, for which the semi–smooth system can be very large or not having an straightforward deduction. In contrast with descend methods, semi–smooth Newton methods do not guarantee reduction of the cost function. On the other hand, many methods for solving L^1–norm functionals are known to be of first order. We take advantage of the full second–order method OESOM which is suitable for this class of optimal control problems, see [6] for details and numerical evidence of the method.

The application of OESOM algorithm proposed in [6] is straightforward. Indeed, we only need to provide the cost function and the corresponding gradient which involves the computing of the adjoint state. The last one can be evaluated by means of the adjoint state (24b),(38b). The second order information is computed automatically by the BFGS method in the case of the smooth part of the cost function (alternatively Hessians can be also used), whereas approximated second order information of non differentiable term is calculated by the built–in enriched second order information constructed by the OESOM algorithm using the weak derivatives of the L^1–norm, see [6] for the details.

6. Numerical Evidence

In order to investigate the numerical performance of the proposed DC algorithm in Section 4 we have implemented Algorithm 1 and 2 using MATLAB. The associated sparse

L^1 subproblem was solved using the OESOM algorithm [6] by extending it to the box–constrained case with an additional projection step on the admissible control set. The OESOM algorithm is a second–order method for solving ℓ_1–norm penalized optimization problems which includes second order information hidden in the structure of the ℓ_1–norm. therefore, is an efficient method to solve the subproblem of the DC algorithms.

As illustrative examples, we consider the following tests defined on the domain $\Omega = (0,1) \times (0,1)$.

Example 1. *We consider the following problem:*

(E1)

$$
\begin{cases}
\min_{(y,u)} \ \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\Upsilon_2(u) \\
subject\ to \\
\qquad\quad -\Delta y = \ u, \quad in\ \Omega, \\
\qquad\qquad\ y = \ 0, \quad on\ \Gamma,
\end{cases}
$$

where we chose the desired state $y_d = e^{-\cos(2\pi xy)^2/0.1}$.

Performance of a single run. *We first solve this example fixing the values of* $\alpha = 1/4$ *and* $\beta = 7/10$. *Algorithm 2 gives an approximated solution after 19 iterations stopping when* $\|\zeta_{k+1} - \zeta_k\| < tol = 1e-5$. *The table and graphics below, show the performance and behavior of 2. We observe in Figure 2 the decreasing behavior of the objective function and the residual.*
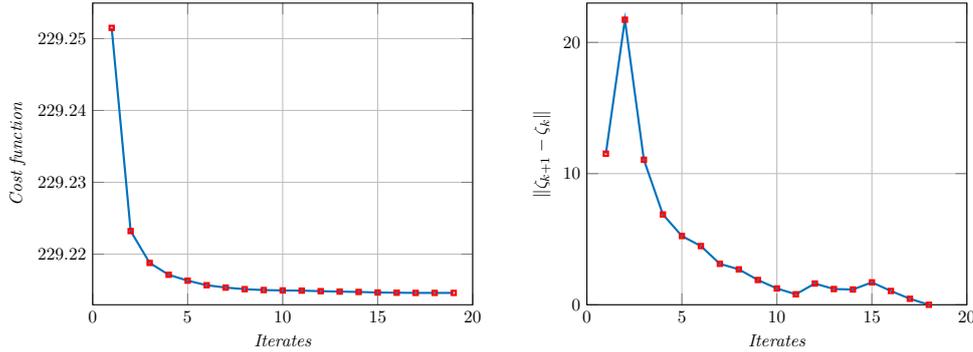
FIGURE 2. Cost function and residual of ζ at $\beta = 0.004$

Figure 3 depicts the evolution of stopping criteria, which is more erratic with a decreasing tendency. In each iteration new sparse components appear then, when comparing consecutive multipliers, they may differ from 0 to 1 in those components, causing oscillations on their difference. We also realize that the number of sparse components of the approximated solution is increasing at every iterate.

Varying the regularization parameter γ. *According to our theory, when* $\gamma \to \infty$ *the solution* $\bar{u}_\gamma \to \bar{u}$. *Here we solve (E1) for increasing values of* γ. *The numerical evidence of this convergence behavior is reflected in Table 2 where we observe optimal cost converges to a fixed value, whereas sparsity also stabilizes at 1525 null components of the solution.*

Varying the regularization parameter β. *Now we experiment with different values of* β, *which determines the sparsity inducting term* Υ. *Table 3 shows that a larger values of* β *results in sparser solutions until the solution vanishes which illustrates Lemma 1. As expected, it can also be observed that the optimal cost increases accordingly to the sparsity of the solution, reflected in smaller supports of the controls.*
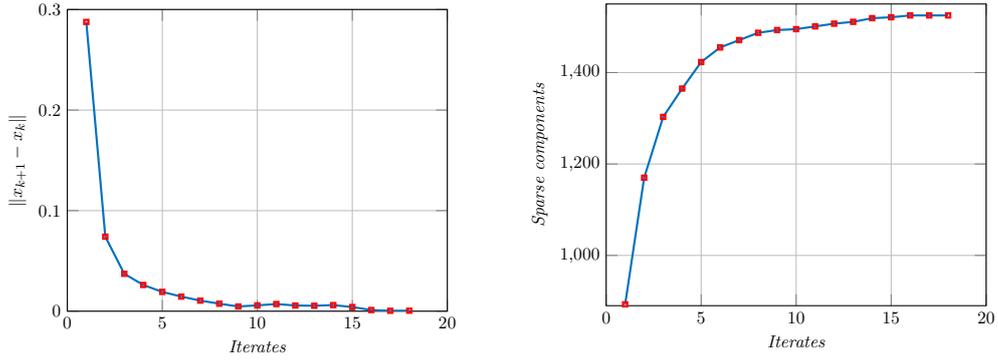
FIGURE 3. Residual of x_k and sparse components at $\beta = 0.004$

k	Cost	Residual	$\|\zeta_{k+1} - \zeta_k\|$	Null entries	OESOME iterations	Execution time
1	229.2515	63.2346	0.044062	42	4	1.4507
2	229.2233	0.28763	11.5196	893	11	6.1601
3	229.2188	0.074112	21.7329	1170	11	6.303
4	229.2172	0.037194	11.0554	1303	8	4.7229
5	229.2164	0.026078	6.8869	1365	11	5.7958
6	229.2157	0.019197	5.2423	1423	23	9.8522
7	229.2154	0.014521	4.4751	1455	6	3.697
8	229.2152	0.010528	3.1231	1471	6	3.3486
9	229.2151	0.0074617	2.6985	1487	5	3.0378
10	229.215	0.0046823	1.8893	1493	7	3.5644
11	229.215	0.0057956	1.2435	1495	7	3.1689
12	229.2149	0.0071222	0.80207	1501	6	3.573
13	229.2148	0.0056617	1.6193	1507	6	2.9722
14	229.2148	0.0054029	1.1993	1511	7	3.7972
15	229.2147	0.0060645	1.1654	1519	6	3.1879
16	229.2147	0.0041252	1.7088	1521	6	2.7979
17	229.2147	0.0011533	1.0534	1525	6	2.7833
18	229.2147	0.000409	0.45796	1525	5	2.0453
19	229.2147	0.00051555	8.3408e-07	1525	5	2.0342

TABLE 1. Performance data for DCA for Example 1
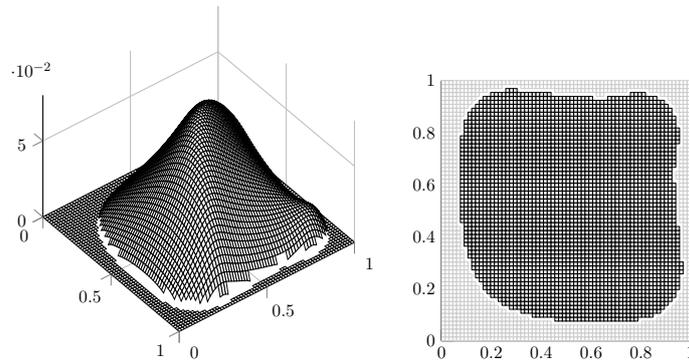
FIGURE 4. Optimal control and its support for $\beta = 0.0002$.

γ	Optimal Cost	Sparse components	DCA Iterations
100	229.219724	1080	17
200	229.214857	1499	24
500	229.214082	1582	18
1000	229.214356	1553	22
1500	229.214650	1525	15
2000	229.214651	1525	17
2500	229.214650	1525	20
3000	229.214650	1525	20
4000	229.214650	1525	22
5000	229.214650	1525	24

TABLE 2. Numerical convergence for increasing values of γ.

β	Optimal Cost	Sparse components	DCA Iterations
0.0002	229.1145	1034	25
0.0005	229.259	1729	30
0.0010	229.4327	2528	37
0.0015	229.5503	3004	30
0.0020	229.6252	3359	31
0.0025	229.6676	3631	40
0.0030	229.6849	3843	37

TABLE 3. Solutions become sparser as β increases.
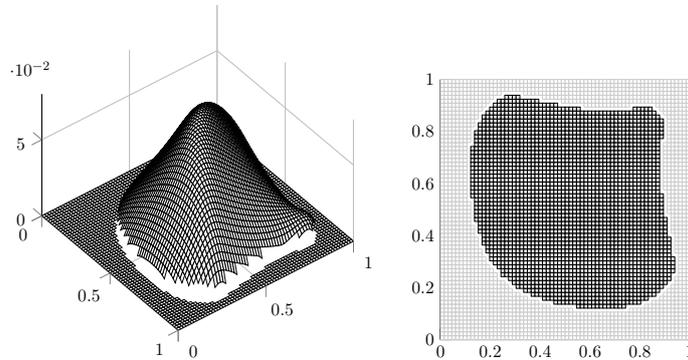
FIGURE 5. Optimal control and its support for $\beta = 0.001$.

Varying the fractional power p**.** *We finish this example with the variation of the fractional exponent $1/p$ which also plays a rol in the sparsity of the solution. In fact, p determines how expensive is an sparse control. Since for larger values of p the sparsity term tends to produce a volume constraint induced by the Donoho's counting norm c.f.[15]. However, the increment of p does not necessarily increase sparsity in the solution as we can see in Table 4.*
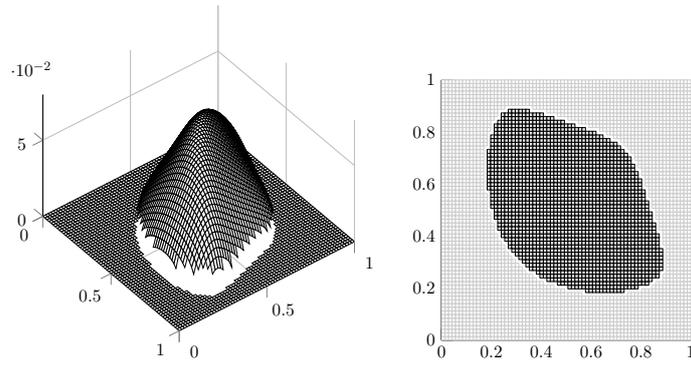
FIGURE 6. Optimal control and its support for $\beta = 0.002$.
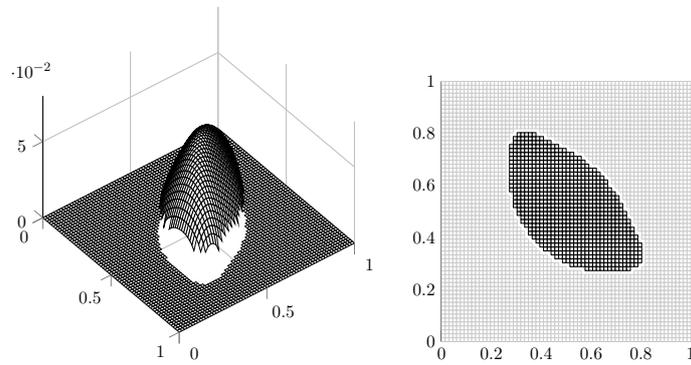
FIGURE 7. Optimal control and its support for $\beta = 0.003$.

p	Optimal Cost	Sparse components	DCA Iterations
1	229.2028	789	4
1.2	229.3232	1860	18
1.5	229.4736	2778	32
2	229.6256	3355	27
4	229.8814	3667	26
8	230.3485	3441	23
10	230.5699	3323	28
20	231.4921	2846	23

TABLE 4. Influence of the power parameter p in the sparsity of the solution.

Example 2. *This example consists in adding box–constraints and keeping the same parameters in Example 1. Therefore, we require in addition that*

$$u \in U_{ad} = \{u \in L^2(\Omega) : 0 \leq u \leq 0.035\}.$$

A similar performance results are observed in this case as depicted in Figure 8. The structure of the sparsity and the support of the optimal control is similar but now is also active on the prescribed bounds as observed in Figure 9.
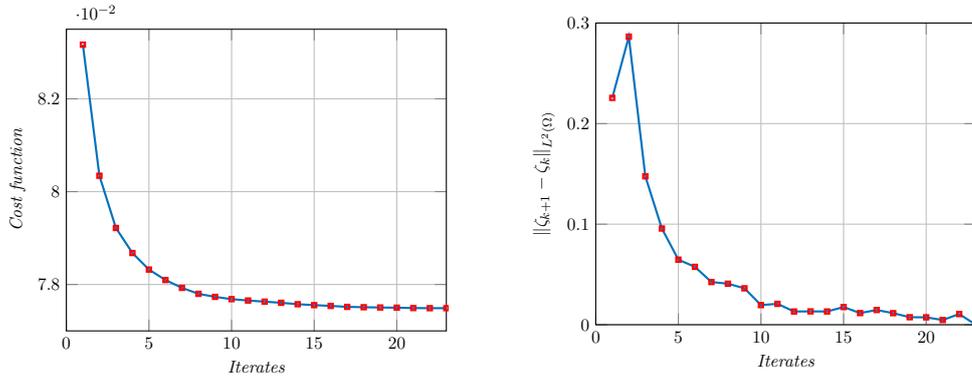
FIGURE 8. Cost function and residual of ζ.
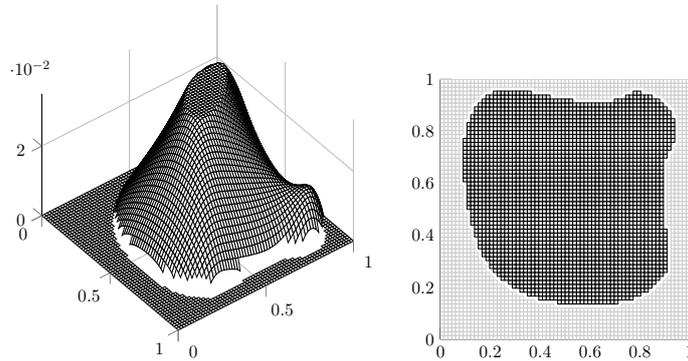
FIGURE 9. Box–constrained optimal control and its support.

Our final experiment consists in a box–constrained optimal control problem with L^q– term only ($\alpha = 0$). In this case we observe (c.f. Figure 10) a typical bang–bang optimal control shape.
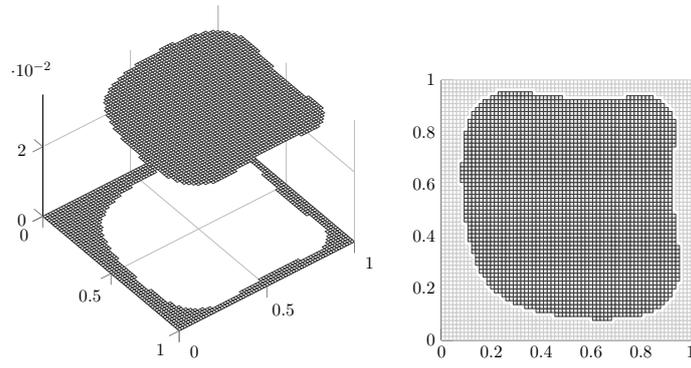
FIGURE 10. Box–constrained optimal control and its support for $\alpha = 0$.

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