On the Chvátal-rank of facets for the set covering polyhedron of circular matrices [1]

Graciela Nasini [2]

FCEIA, Universidad Nacional de Rosario, Rosario, Argentina

Luis M. Torres [3]

ModeMat, Escuela Politécnica Nacional, Quito, Ecuador

Hervé Kerivin and Annegret Wagler [4]

LIMOS, Université Clermont Auvergne, Clermont-Ferrand, France

Abstract

We study the family of minor related row family inequalities for the set covering polyhedron related to circular matrices introduced in [8]. We provide a construction to obtain facets with arbitrarily large coefficients. Moreover, we address the issue of generating these inequalities via the Chvátal-Gomory rounding procedure and provide examples of inequalities having Chvátal-rank strictly larger than one.

Keywords: polyhedral combinatorics, Chvátal-rank, set covering polyhedron, circulant matrices

[1] This research was supported by the MATH-AmSud cooperation program within the framework of project PACK-COVER.

[2] Email:nasini@fceia.unr.edu.ar

[3] Email:luis.torres@epn.edu.ec

[4] Email:[herve.kerivin,annegret.wagler]@uca.fr

1 Introduction

Given a $(m \times n)$-matrix A with $(0,1)$-entries and a cost vector $c \in \mathbb{Z}^n$, the *set covering problem (SCP)* can be stated as

$$\min\{c^T x : Ax \geq \mathbf{1}, x \in \mathbb{Z}^n_+\}.$$

It is a classic problem in combinatorial optimization with important practical applications, but well-known to be hard to solve in general. One established approach to tackle such problems is to study the polyhedral properties of their sets of feasible solutions. The *set covering polyhedron* $Q^*(A)$ is defined by the convex hull of all feasible solutions of SCP, i.e., of the incidence vectors of all *covers* of A. Its *fractional relaxation* $Q(A)$ is given by

$$Q(A) := \{x \in \mathbb{R}^n_+ : Ax \geq \mathbf{1}\}.$$

In general, we have $Q(A) \neq Q^*(A)$, even when A belongs to the particular class of circular matrices. For $n \in \mathbb{N}$, let $[n]$ denote the additive group defined on the set $\{1, \ldots, n\}$, with integer addition modulo n. We consider the columns (resp. rows) of A to be indexed by $[n]$ (resp. by $[m]$). A is said to be a *circular* matrix if its rows are the incidence vectors of a set I of cyclic intervals on $[n]$, with the property that no interval contains another one, i.e., A has no dominating rows. A square circular matrix is called a *circulant*. In this case, all intervals in I have the same number of elements k, and I contains all n possible intervals of this size. Thus, a circulant is completely defined by the two parameters n and k, and we shall denote it by C_n^k.

Valid and facet defining inequalities for $Q^*(C_n^k)$ have been studied for a long time. The *boolean facets* include the system of the inequalities $x \geq 0$ and $Ax \geq \mathbf{1}$ defining $Q(C_n^k)$, as well as the *rank constraint* $\mathbf{1}^T x \geq \lceil \frac{n}{k} \rceil$, which has been shown to be valid for $Q^*(C_n^k)$ and facet-defining if and only if n is not a multiple of k [6]. More recently, the class of *row family inequalities* was proposed in [3] and studied for certain minors of C_n^k.

Given $N \subset [n]$, the *minor of A obtained by contraction of N*, denoted by A/N, is the submatrix of A that results after removing all columns with indices in N and all dominating rows. A minor of a circular matrix A is called a *circulant minor* if it is equal to a circulant matrix $C_{n'}^{k'}$, up to permutation of rows and columns. We shall denote this by $A/N \approx C_{n'}^{k'}$.

Conditions for the existence of circulant minors of a circulant matrix have been studied in [1,5]. Let G_n^k be a directed graph having $[n]$ as the set of nodes and all arcs of the form $(j, j+k)$, $(j, j+k+1)$, for $1 \leq j \leq n$. Circulant minors can be characterized in terms of directed circuits in G_n^k.

Theorem 1.1 ([1]) *Assume $2 \le k \le n-1$, $2 \le n' < n$, $0 < k - k' < \min\{k, n-n'\}$. $C_n^k/N \approx C_{n'}^{k'}$ if and only if there exist $d = \gcd(n-n', k-k')$ disjoint simple directed circuits in G_n^k, $D_1, \ldots, D_d$, each having length $\frac{n-n'}{d}$, such that $N = \cup_{r=1}^d V(D_r)$.*

Circulant minors of C_n^k are known to induce valid (and in some cases facet-defining) inequalities for $Q^*(C_n^k)$. The class of *minor inequalities* was introduced in [2] and was further studied and generalized in [4,7]. In [8] it was observed that a circulant minor $C_n^k/N \approx C_{n'}^{k'}$ also induces a row family inequality that either is equivalent to or enhances the corresponding minor inequality. This *minor related row family inequality* (minor rfi) has the form

$$a \sum_{j \notin W} x_j + (a+1) \sum_{j \in W} x_j \ge a \left\lceil \frac{n'}{k'} \right\rceil,$$

where $a \in \{0, \ldots, k'-1\}$ with $a = n' \mod k'$, and $W \subset N$ is the set of nodes in the circuits D_r that are heads of arcs of the form $(j, j+k+1)$. In [4] it was conjectured that rank and $(1,2)$-valued minor inequalities suffice to describe $Q^*(C_n^k)$. In [7], a first example of a facet-defining $(2,3)$-valued minor inequality was presented. In this paper we show that there are circulant matrices such that $Q^*(C_n^k)$ has facet-defining minor related row family inequalities with *arbitrarily large* coefficients.

Moreover, we are interested in studying the difference between $Q(C_n^k)$ and $Q^*(C_n^k)$ in terms of the Chvátal-Gomory procedure. For given $a \in \mathbb{Z}^n$ and $b \notin \mathbb{Z}$, assume $a^T x \ge b$ is valid for $Q(A)$ and tight for some $x^* \in Q(A)$. Then the inequality $a^T x \ge \lceil b \rceil$ is valid for $Q^*(A)$, but violated by x^*. Such an inequality is called a *Chvátal-Gomory cut* for $Q(A)$ and the procedure for obtaining it is the *Chvátal-Gomory procedure*. The first Chvátal closure $Q'(A)$ is the set of points of $Q(A)$ satisfying all Chvátal-Gomory cuts. Let $Q^0 := Q(A)$ and $Q^t := (Q^{t-1})'$ for all $t \in \mathbb{N}$. Evidently, $Q^*(A) \subseteq Q^t \subseteq Q^{t-1}$ holds for every $t \in \mathbb{N}$. Moreover, it is known that there exists a finite $\hat{t} \in \mathbb{N}$ with $Q^{\hat{t}} = Q^*(A)$; the smallest such $\hat{t}$ is the *Chvátal-rank* of $Q(A)$. An inequality is said to have *Chvátal-depth* equal to t if it is valid for Q^t, but not valid for Q^{t-1}.

The Chvátal-rank of $Q(C_n^k)$ has been addressed in several previous works. Any minor inequality has Chvátal-depth at most one. On the other hand, it has been observed in [3] that this might not be the case for row family inequalities. In this paper, we provide examples of minor rfi's with Chvátal-depth strictly larger than one.

2 Facets with arbitrarily large coefficients

In the following we provide a construction to show that minor related row family inequalities with arbitrarily large coefficients can occur as facets of $Q^*(A)$, even in the particular case when A is a circulant matrix C_n^k.

Let $\alpha \in \mathbb{N}$ with $\alpha \geq 6$ and define $n := (\alpha - 1)(\alpha + 1)$, $k := \alpha$. Moreover, consider the finite sequences of natural numbers given by

$$n_a := (\alpha - 1)(\alpha - a) \qquad k_a := \alpha - a - 1,$$

where a takes values from the set $S := \left\{1, \ldots, \left\lfloor \frac{\alpha}{2} \right\rfloor - 1\right\}$. It is straightforward to verify that $C_{n_a}^{k_a}$ is a circulant minor of C_n^k, for all $a \in S$. Indeed, the conditions of Theorem 1.1 are satisfied as G_n^k contains $d = a + 1$ disjoint simple directed circuits, each one consisting of $\alpha - 1$ arcs of length $k + 1$. Let W_a denote the union of the sets of nodes of these circuits. Moreover, since $2a + 1 < \alpha$, it follows that $a < \alpha - a - 1$ and

$$\frac{n_a}{k_a} = \frac{(\alpha - 1)(\alpha - a)}{\alpha - a - 1} = \alpha + \frac{a}{\alpha - a - 1} < \alpha + 1.$$

Hence, $\left\lceil \frac{n_a}{k_a} \right\rceil = \alpha + 1$ and $n_a = a \mod k_a, \forall a \in S$. The minor related row family inequality of $Q^*(C_n^k)$ induced by $C_{n_a}^{k_a}$ is

$$a \sum_{j \notin W_a} x_j + (a + 1) \sum_{j \in W_a} x_j \geq a(\alpha + 1). \tag{1}$$

Theorem 2.1 *Inequality (1) defines a facet of $Q^*(C_n^k)$ if $\gcd(a, \alpha - 1) = 1$.*

In particular, if $\alpha - 1$ is a prime number then $Q^*(C_{\alpha^2-1}^{\alpha})$ has facets steming from minor rfi's with all possible coefficients a, $a + 1$, for $1 \leq a \leq \left\lfloor \frac{\alpha}{2} \right\rfloor - 1$.

Example 2.2 Choosing $\alpha = 8$, we obtain that C_{63}^8 contains all circulant minors of the form $C_{7(8-a)}^{8-a}$ with $a \in \{1, 2, 3\}$. As $\alpha - 1$ is prime, these minors C_{35}^4, C_{42}^5, C_{49}^6 induce $(a, a + 1)$-valued facets of $Q^*(C_{63}^8)$.

3 On the Chvátal-depth of minor rfi's

Here, we study the Chvátal-depth of the above defined minor rfi's and address both upper and lower bounds for their Chvátal-depth. Indeed, the construction of the previous section can be employed to illustrate a possible way to obtain minor related row family inequalities from inequalities with smaller coefficients by applying the Chvátal-Gomory rounding procedure.

Lemma 3.1 *For any $a \in \left\{2, \ldots, \left\lfloor \frac{\alpha}{2} \right\rfloor - 1\right\}$, the inequality (1) induced by a circulant minor isomoprhic to $C_{n_a}^{k_a}$ can be obtained from inequalities induced by circulant minors isomorphic to $C_{n_{a-1}}^{k_{a-1}}$ and from the rank inequality of C_n^k with a single application of the Chvátal-Gomory rounding procedure.*

If $a = 1$ then (1) is the minor inequality defined in [2]. This inequality is known to have Chvátal-depth at most one. The same holds for the rank inequality of C_n^k, which has rank equal to one if k does not divide n, and equal to zero otherwise. As a consequence, the next result follows.

Theorem 3.2 *The $(a, a+1)$-valued inequality (1) of $Q^*(C_n^k)$ induced by $C_{n_a}^{k_a}$ has Chvátal-depth at most a.*

Example 3.3 The minor inequality of $Q^*(C_{63}^8)$ induced by C_{35}^4 from Example 2.2 equals $3x(V - W_a) + 4x(W_a) \geq 27$. It has Chvátal-depth at most 3 since it can be obtained from the facets induced by C_{42}^5 which, in turn, can be generated by the facets induced by C_{49}^6 having Chvátal-depth 1.

On the other hand, the following lemma provides a necessary condition for a minor rfi to have Chvátal-depth strictly larger than one.

Lemma 3.4 *If $a^2 < (\alpha - a - 1)(a - 1)$ then the inequality (1) induced by $C_{n_a}^{k_a}$ cannot be obtained from the inequalities in the system defining $Q(C_n^k)$ by a single application of the Chvátal-Gomory rounding procedure.*

The last result does not necessarily imply that the inequality induced by the minor $C_{n_a}^{k_a}$ has Chvátal-depth larger than one, as it can still be obtained as a nonnegative combination of other inequalities with Chvátal-depth equal to one. However, this cannot be the case if the studied inequality defines a facet of $Q^*(C_n^k)$. Together with Theorem 2.1, this implies:

Theorem 3.5 *If $(\alpha - a - 1)(a - 1) > a^2$ and $\gcd(a, \alpha - 1) = 1$ then the inequality (1) induced by $C_{n_a}^{k_a}$ has Chvátal-depth strictly larger than one.*

In particular, choosing $a = 2$ it follows that $Q(C_{\alpha^2-1}^{\alpha})$ has Chvátal-rank strictly larger than one for all even $\alpha \geq 8$. The smallest such example with $a = 2$ and $\alpha = 8$ is $Q(C_{63}^8)$ as $Q^*(C_{63}^8)$ has a facet with Chvátal-depth larger than one induced by the minor C_{42}^5.

4 Concluding remarks

We provided a construction for facets of $Q^*(C_n^k)$ with arbitrarily large coefficients, belonging to the class of minor rfi's. Under the conditions in Theorem 3.5 these facets may have Chvátal-depth strictly larger than one. In this regard, minor rfi's differ from other previously described minor induced inequalities for $Q^*(C_n^k)$, which are known to have Chvátal-depth at most one. As future work, we intend to investigate whether larger lower bounds on the Chvátal-depth can be proven for inequalities with large coefficients.

Conversely, we have shown that a $(a, a+1)$-valued minor rfi constructed in the way proposed here cannot have Chvátal-depth larger than a. A subject of future research is to determine whether this upper bound holds for *any* $(a, a+1)$-valued minor related row family inequality.

Acknowledgement

Silvia Bianchi from Universidad Nacional de Rosario has independently found another construction to obtain facets of $Q^*(C_n^k)$ with arbitrarily large coefficients. We thank her for the fruitful discussions on the topic.

References

[1] Aguilera, N., *On packing and covering polyhedra of consecutive ones circulant clutters*, Discrete Applied Mathematics **158** (2009), pp. 1343–1356.

[2] Argiroffo, G. and S. Bianchi, *On the set covering polyhedron of circulant matrices*, Discrete Optimization **6** (2009), pp. 162–173.

[3] Argiroffo, G. and S. Bianchi, *Row family inequalities for the set covering polyhedron*, Electronic Notes in Discrete Mathematics **36** (2010), pp. 1169–1176.

[4] Bianchi, S., G. Nasini and P. Tolomei, *The set covering problem on circulant matrices: polynomial instances and the relation with the dominating set problem on webs*, Electronic Notes in Discrete Mathematics **36** (2010), pp. 1185–1192.

[5] G., G. C. and B. Novick, *Ideal* $0-1$ *matrices*, Journal of Combinatorial Theory (B) **60** (1994), pp. 145–157.

[6] Sassano, A., *On the facial structure of the set covering polytope*, Math. Prog. **44** (1989), pp. 181–202.

[7] Tolomei, P. and L. M. Torres, *Generalized minor inequalities for the set covering polyhedron related to circulant matrices*, Discrete Applied Mathematics **210** (2016), pp. 214–222.

[8] Torres, L. M., *Minor related row family inequalities for the set covering polyhedron of circulant matrices*, Electronic Notes in Discrete Mathematics **50** (2015), pp. 325–330.

Appendix

Here we provide the proofs of the previously presented results that have been omitted from the abstract due to space restrictions.

Proof. [of Theorem 2.1] Since k_a does not divide n_a, the rank inequality defines a facet of $Q^*(C_{n_a}^{k_a})$. Thus, there exist $n_a = (\alpha-1)(\alpha-a)$ covers of $C_{n_a}^{k_a}$ with affinely independent incidence vectors, each one having cardinality $\left\lceil \frac{n_a}{k_a} \right\rceil = \alpha + 1$. These covers trivially induce covers of C_n^k whose elements belong to $\overline{W}_a := [n] \setminus W_a$, and whose incidence vectors $x^1, \ldots, x^{n_a}$ are linearly independent roots of (1).

We construct $n - n_a$ further roots of this inequality. For this purpose, we consider a special embedding of W_a and $\overline{W}_a$ in $[n]$. Let $[n]$ be partitioned into $\alpha - 1$ blocks $B_1, \ldots, B_{\alpha-1}$, each one consisting of $\alpha+1$ consecutive elements. The first $\alpha - a$ elements of each block belong to $\overline{W}_a$ while the last $a+1$ elements belong to W_a (each of these to a different circuit in G_n^k). Denote by v_ℓ^i the i-th element of block B_ℓ, with $1 \le i \le \alpha+1$ and $1 \le \ell \le \alpha - 1$.

For $1 \le \ell \le \alpha - 1$, let $\overline{V}_\ell := \left\{ v_\ell^{\alpha-a} + t\alpha \,:\, 0 \le t < \alpha - a \right\}$. Observe that $\overline{V}_\ell \subset \overline{W}_a$, $\left|\overline{V}_\ell\right| = \alpha - a$, and that the last element of $\overline{V}_\ell$ is $v^1_{\ell+\alpha-a-1}$. Moreover consider the permutation of the elements of W_a given by:

$$
\begin{aligned}
V_\ell &:= (v_\ell^{\alpha-a+1}, v_{\ell-a+1}^{\alpha+1}, v_{\ell-a+2}^{\alpha}, \ldots, v_\ell^{\alpha-a+2}), \quad \forall 1 \le \ell \le \alpha - 1,\\
W_a &= (V_1, V_2, \ldots, V_{\alpha-1}).
\end{aligned}
$$

It can be verified that any cyclic interval C_j consisting of a consecutive elements from W_a intersects a consecutive blocks $B_\ell, \ldots, B_{\ell+a-1}$. Furthermore, the distance between two elements of C_j belonging to two consecutive blocks is either α or $\alpha - 1$. Thus, $C_j \cup \overline{V}_{\ell-\alpha+a+1}$ is a cover of C_n^k consisting of $\alpha - a$ nodes in $\overline{W}_a$ and a nodes in W_a. Its incidence vector $\overline{x}^j$ is a root of (1).

Consider the square matrix having as rows the n_a roots $x^1, \ldots, x^{n_a}$ and the $n - n_a$ roots $\overline{x}^1, \ldots, \overline{x}^{n-n_a}$. After adequate sorting of rows and columns, this matrix can be put in the form:

$$
\left(\begin{array}{c|c}
C_{n_a}^{\alpha+1} & O \\
\hline
M & C_{n-n_a}^{a}
\end{array}\right)
$$

This matrix is non sigular if and only if the matrix $C_{n-n_a}^a$ is non singular, which is the case if $\gcd(n - n_a, a) = 1$. Since $n - n_a = (\alpha-1)(a+1)$, the result follows. $\square$

Proof. [of Lemma 3.1] Consider again the set W_a inducing inequality (1). As observed above, W_a is the union of the node set of $a+1$ disjoint simple directed circuits $D^1, \ldots, D^{a+1}$ in G_n^k. Assume $a \geq 2$ and define $W_{a-1}^r := W_a \backslash V(D^r)$, for $1 \leq r \leq a+1$. Then C_n^k / W_{a-1}^r is a circulant minor of C_n^k and the corresponding row family inequality has the form

$$(a-1) \sum_{j \notin W_{a-1}^r} x_j + a \sum_{j \in W_{a-1}^r} x_j \geq (a-1)(\alpha+1).$$

Adding up all inequalities for $1 \leq r \leq a+1$ together with the rank inequality $\sum_{j=1}^n x_j \geq \alpha$ yields

$$[(a+1)(a-1)+1] \sum_{j \notin W_a} x_j + [a^2 + (a-1) + 1] \sum_{j \in W_a} x_j \geq (a^2-1)(\alpha+1) + \alpha$$
$$\Leftrightarrow \quad a^2 \sum_{j \notin W_a} x_j + a(a+1) \sum_{j \in W_a} x_j \geq a^2(\alpha+1) - 1.$$

Dividing the last inequality by a and rounding up the right-hand side, we obtain (1). □

Proof. [of Lemma 3.4] For simplicity in the notation, let $A := C_n^k$, $b := a(\alpha+1)$, and $c \in \mathbb{R}^n$ be the vector consisting of the left-hand side coefficients of the inequality, i.e.,

$$c_j := \begin{cases} a, & \text{if } j \notin W_a, \\ a+1, & \text{otherwise,} \end{cases}$$

The inequality $c^T x \geq b$ can be obtained by a single application of the Chvátal-Gomory rounding procedure if and only if there exists a vector of multiplicators $y \in \mathbb{R}^n$ such that

$$\text{(LP)} \begin{cases} A^T y \leq c, \\ \mathbf{1}^T y > b-1, \\ y \geq 0. \end{cases}$$

Consider the relaxation RLP of LP obtained by changing the strict inequality $\mathbf{1}^T y > b-1$ to $\mathbf{1}^T y \geq b-1$. Due to Farkas Lemma, RLP has no solution if and only if the following system of inequalities on the variables

$\lambda \in \mathbb{R}^n$, $\delta \in \mathbb{R}$ has a solution:

$$\text{(FLP)} \begin{cases} A\lambda - \delta\mathbf{1} \geq \mathbf{0}, \\ c^T\lambda - \delta(b-1) < 0, \\ \lambda, \delta \geq 0. \end{cases}$$

If $a^2 < (\alpha - a - 1)(a - 1)$, one feasible solution for FLP is given by

$$\lambda_j := \begin{cases} 1, \text{ if } j \notin W_a \\ 0, \text{ otherwise.} \end{cases} \qquad \delta := \alpha - a - 1.$$

Indeed, observe that computing $A\lambda$ results in adding all columns of A corresponding to the circulant minor $C_{n_a}^{k_a}$. On the rows correspondig to the minor, this sum is equal to $k_a = \alpha - a - 1$. All other rows where deleted during contraction and therefore must dominate a row in the circulant, so this sum is larger than or equal to k_a. Hence, $A\lambda - \delta\mathbf{1} \geq \mathbf{0}$. Additionally,

$$c^T\lambda = \sum_{j \notin W_a} c_j = an_a = a(\alpha - 1)(\alpha - a) = a\alpha(\alpha - a - 1) + a^2$$
$$< (\alpha - a - 1)(a(\alpha + 1) - 1) = \delta(b - 1),$$

which shows that the second inequality in FLP is also fulfilled. Hence, RLP and the more restricted system LP have no feasible solutions. □