Equilibrium Investment with Random Risk Aversion: (Non-)uniqueness, Optimality, and Comparative Statics

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Abstract

This paper investigates infinite-dimensional portfolio selection problem under a general distribution of the risk aversion parameter. We provide a complete characterization of all deterministic equilibrium investment strategies. Our results reveal that the solution structure depends critically on the distribution of risk aversion: the equilibrium is unique whenever it exists in the case of finite expected risk aversion, whereas an infinite expectation can lead to infinitely many equilibria or to a unique trivial one ($\bar{\pi} \equiv \mathbf{0}$). To address this multiplicity, we introduce three optimality criteria—optimal, uniformly optimal, and uniformly strictly optimal—and explicitly characterize the existence and uniqueness of the corresponding equilibria. Under the same necessary and sufficient condition, the optimal and uniformly optimal equilibria exist uniquely and coincide. Furthermore, assuming that the market price of risk is non-zero near the terminal time, we show that the optimal (and hence uniformly optimal) equilibrium is also uniformly strictly optimal. Finally, we perform comparative statics to demonstrate that a risk aversion distribution dominating another in the reverse hazard rate order leads to a less aggressive equilibrium strategy.

Keywords: Random Risk Aversion; Time-Inconsistency; Portfolio Selection; Intra-Personal Equilibrium; Multiple Equilibria; Comparative Statics

1 Introduction

The classical framework for portfolio selection, pioneered by Merton (1969, 1971) and Samuelson (1969), is based on the assumption that an investor's risk aversion can be captured by a single

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known constant. In practice, however, this coefficient is very difficult for an agent to specify with certainty, and a growing body of empirical evidence suggests that it varies dynamically with market conditions, often increasing during financial crises and bear markets, as shown in studies by Gordon and St-Amour (2000); Chetty (2006) and Guiso et al. (2018). These challenges have motivated a move towards models that incorporate random risk aversion. Within this paradigm, a more economically coherent approach, as advanced by Desmettre and Steffensen (2023), is to aggregate certainty equivalents over the distribution of the risk aversion parameter. While the random risk aversion in Desmettre and Steffensen (2023) provides a consistent performance measure in monetary units, it introduces a nonlinearity in expectation that naturally leads to the time inconsistency of the preference.

The first systematic treatment of time inconsistency was given by Strotz (1955), who proposed the notion of consistent planning, laying the foundation for the intra-personal game-theoretic framework. Building on this idea, Ekeland and Lazrak (2006) later developed a rigorous continuous-time formalization, which in turn inspired extensive research on continuous-time control with time-inconsistent preferences. For example, Bjork and Murgoci (2010) (published version Björk et al. (2017)) established a general theoretical framework by deriving an extended HJB equation and Hu, Jin, and Zhou (2012, 2017) investigated a time-inconsistent stochastic linear–quadratic control problem. For further discussions and related developments, see, e.g., Yan and Yong, He and Jiang (2021), Hernández and Possamaï (2023), and the references therein.

Within the Black-Scholes market model, Desmettre and Steffensen (2023) provide a verification theorem on the intra-personal equilibrium strategies for a CRRA utility function with a general random risk aversion and, in the case of a binomial random risk aversion, characterize the equilibrium by a three-dimensional ODE system without establishing the existence and uniqueness of the solution.[1] This technical gap is filled by Liang, Wang, and Xia (2025a). For general random risk aversion, Liang, Wang, and Xia (2025a) characterize intra-personal equilibrium strategies using an integral equation. They establish the existence and uniqueness of the solution to the integral equation under the assumptions that the variance of the random risk aversion is finite and that the function h defined in (2.6) below is bounded.

This paper aims to offer more insight into the portfolio selection problem with random risk aversion within the simplest model, in line with Desmettre and Steffensen (2023) and Liang, Wang, and Xia (2025a). Our main contributions are summarized as follows:

- **Complete characterization of all deterministic equilibria.** We relax the assumptions in the existing literature and derive a full characterization of all deterministic equilibrium

[1] This work has spurred further research extending the model to more complex settings. For example, Chen, Guan, and Liang (2025) incorporate a regime-switching framework in which market dynamics and preferences co-evolve; Wang and Jia (2025) study the management of the DC pension plan, providing semi-explicit solutions for a CRRA utility with a binomial random risk aversion; He, Jiang, and Xia (2025) establish the existence and uniqueness of the solution to an infinite-dimensional Riccati equation which characterizes the equilibrium strategies under Heston's SV model for a CRRA or CARA utility function with a general bounded random risk aversion. For an investigation on the corresponding pre-commitment problem, see, e.g., Xia (2024).

strategies. An important finding is that there may be infinitely many equilibria when the expectation of the random risk aversion is infinite. The uniqueness of the equilibrium strategy for the continuous-time time-inconsistent control problem is a very challenging problem (see, e.g., Hu, Jin, and Zhou (2021, Section 7)). This paper sheds light on this issue by demonstrating that equilibrium strategies, even within the class of deterministic strategies, may lack uniqueness. Furthermore, we provide a necessary and sufficient condition for the uniqueness of the deterministic equilibrium.

- **Resolution of equilibrium multiplicity via the optimality criterion.** A question then arises: which one should the agent choose from these infinitely many equilibrium strategies? We suggest that the agent should maximize the objective functional at the initial time over all available equilibria. This leads to the discussion of the so-called *optimal* equilibria. To our best knowledge, there is only one existing paper, Wei, Xia, and Zhao (2024), on optimal equilibria for a time-inconsistent *control* problem, and there are three papers, Huang and Zhou (2019, 2020) and Huang and Wang (2021), on optimal equilibria for time-inconsistent *stopping* problems. We give a necessary and sufficient condition for the existence of optimal equilibria and observe that the optimal equilibrium, if exists, is unique and is also *uniformly* optimal. Moreover, if the market price of risk remains non-zero near the terminal time, the optimal equilibrium is also *uniformly strictly* optimal.

- **Comparative statics on equilibrium strategies.** We examine how the distribution of the random risk aversion influences the equilibrium strategies. It is natural to conjecture that a "larger" random risk aversion would lead to less risky investments. We observe by counterexamples that the "largeness" in the sense of the first-order stochastic dominance is insufficient to validate the conjecture. However, a stronger stochastic order, the reverse hazard rate dominance, is sufficient. Thus we get an analogy with Borell (2007) and Xia (2011), where the comparative statics on the optimal portfolio with respect to risk aversion are carried out for the standard expected utility maximization with a general utility function and within the Black-Scholes market model (The corresponding comparative statics for a static model with a risk-free asset and a risky asset goes back to Arrow (1970) and Pratt (1964)).

The paper is organized as follows. Section 2 presents the model and derives the integral equation characterizing the equilibrium. Section 3 analyzes both the finite and infinite expectation cases, providing a complete characterization of all equilibria. Section 4 introduces three optimality criteria for equilibrium selection and establishes the necessary and sufficient conditions for their existence and uniqueness. Section 5 performs comparative statics to investigate the impact of risk aversion distributions on equilibrium strategies, establishing monotonicity under the reverse hazard rate order, and provides numerical illustrations that verify the theoretical results and present a counterexample under first-order stochastic dominance.[2]

[2] Herein, "increasing" means "non-decreasing" and "decreasing" means "non-increasing."

2 Problem Formulation

Let $T > 0$ be a finite time horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ a filtered complete probability space, where $\mathbb{F} = \{\mathcal{F}_t,\ 0 \le t \le T\}$ is the augmented natural filtration generated by a d-dimensional standard Brownian motion $\{B(t),\ 0 \le t \le T\}$ and $\mathcal{F} = \mathcal{F}_T$. The market consists of one risk-free asset (bond) and d risky assets (stocks). For simplicity, we assume that the interest rate of the bond is zero. The dynamics of the stock price processes S_i, $i = 1, \cdots, d$, are given by

$$dS_i(t) = S_i(t)[\mu_i(t)dt + \sigma_i(t)dB(t)], \quad t \in [0, T],$$

where the market coefficients $\mu : [0, T] \to \mathbb{R}^d$ and $\sigma : [0, T] \to \mathbb{R}^{d\times d}$ are right-continuous and deterministic (each σ_i denotes the i-th row of σ). We always assume that

$$\int_0^T |\mu(t)|dt + \sum_{i=1}^d \int_0^T |\sigma_i(t)|^2 dt < \infty$$

and $\sigma(t)$ is invertible for every $t \in [0, T]$. The market price of risk is $\lambda(t) \triangleq (\sigma(t))^{-1}\mu(t)$, $t \in [0, T]$. We also assume that $\int_0^T |\lambda(t)|^2 \mathrm{d}t < \infty$.

For any $m \ge 1$ and $\mathbb{S} \subset \mathbb{R}^m$, $L^0(\mathbb{F}, \mathbb{S})$ is the space of $\mathbb{S}$-valued, $\mathbb{F}$-progressively measurable processes. For each $t \in [0, T]$, $p \in [1, \infty]$, $L^p(\mathcal{F}_t, \mathbb{S})$ is the set of all L^p-integrable, $\mathbb{S}$-valued, and $\mathcal{F}_t$-measurable random variables.

A trading strategy is a process $\pi = \{\pi_t,\ t \in [0, T)\} \in L^0(\mathbb{F}, \mathbb{R}^d)$ such that

$$\int_0^T |\pi_t^\top \mu(t)|\mathrm{d}t + \int_0^T |\sigma^\top(t)\pi_t|^2 \mathrm{d}t < \infty \text{ a.s.},$$

where π_t represents the vector of portfolio weights determining the investment of wealth in the stocks at time t. For any $t \in [0, T)$ and any initial wealth level $w > 0$, let $W^{t,w,\pi} = \{W_s^{t,w,\pi},\ t \le s \le T\}$ denote the self-financing wealth process starting from w at time t and driven by a trading strategy π. It evolves according to the stochastic differential equation (SDE):

$$\begin{cases} \mathrm{d}W_s^{t,w,\pi} = W_s^{t,w,\pi}\pi_s^\top \mu(s)\mathrm{d}s + W_s^{t,w,\pi}\pi_s^\top \sigma(s)\mathrm{d}B(s), & s \ge t, \\ W_t^{t,w,\pi} = w. \end{cases} \tag{2.1}$$

Consider the constant relative risk aversion (CRRA) utility functions:

$$u^\gamma(x) = \begin{cases} \frac{x^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, x > 0, \\ \log x, & \gamma = 1, x > 0. \end{cases}$$

Under the expected utility theory, for each CRRA utility function u^γ, at time t, the certainty equivalent of $W_T^{t,w,\pi}$ is $(u^\gamma)^{-1}\left(\mathbb{E}_t\left[u^\gamma\left(W_T^{t,w,\pi}\right)\right]\right)$. Following Desmettre and Steffensen (2023), we assume that the objective functional of the agent at time t is the weighted average of the certainty equivalents:

$$J(\pi; t, w) = \int_{[0,\infty)} (u^\gamma)^{-1}\left(\mathbb{E}_t\left[u^\gamma\left(W_T^{t,w,\pi}\right)\right]\right) \mathrm{d}F_{\boldsymbol{R}}(\gamma), \tag{2.2}$$

where $F_{\boldsymbol{R}}$ is the probability distribution function of the *random risk aversion* $\boldsymbol{R}$, a non-negative random variable defined on another independent probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$. We always assume $F_{\boldsymbol{R}}(0) < 1$, which amounts to $\tilde{P}(\boldsymbol{R} > 0) > 0$.

A trading strategy π is called *admissible* if for all $t \in [0, T)$ and $w > 0$,

$$\begin{cases} \mathbb{E}_t\left[\left|u^{\gamma}\left(W_T^{t,w,\pi}\right)\right|\right] < \infty \ \ a.s. \quad \text{for all } \gamma \in \operatorname{supp}(F_{\boldsymbol{R}}), \\ \int_{[0,\infty)} \left|(u^{\gamma})^{-1}\left(\mathbb{E}_t\left[u^{\gamma}\left(W_T^{t,w,\pi}\right)\right]\right)\right| \mathrm{d}F_{\boldsymbol{R}}(\gamma) < \infty \quad a.s.. \end{cases}$$

Let Π denote the set of all admissible strategies.

Hereafter, we consider a fixed $\bar{\pi} \in \Pi$, which is a candidate equilibrium strategy. For any $t \in [0, T)$, $\varepsilon \in (0, T-t)$, and $k \in L^{\infty}(\mathcal{F}_t, \mathbb{R}^d)$, let $\pi^{t,\varepsilon,k}$ be defined by

$$\pi_s^{t,\varepsilon,k} \triangleq \begin{cases} \bar{\pi}_s + k, & s \in [t, t+\varepsilon), \\ \bar{\pi}_s, & s \notin [t, t+\varepsilon). \end{cases}$$

The strategy $\pi^{t,\varepsilon,k}$ serves as a perturbation of $\bar{\pi}$. Following Hu, Jin, and Zhou (2012, 2017) and Yan and Yong, we introduce the definition of equilibrium strategies as follows.

Definition 2.1. *A strategy $\bar{\pi}$ is called an equilibrium strategy if for any $t \in [0, T)$, $w > 0$, and $k \in L^{\infty}(\mathcal{F}_t, \mathbb{R}^d)$, whenever $\pi^{t,\varepsilon,k} \in \Pi$ for all sufficiently small $\varepsilon > 0$, and for any positive sequence $\{\varepsilon_n, n \geq 1\}$ satisfying $\lim_{n\to\infty} \varepsilon_n = 0$ we have*

$$\limsup_{n\to\infty} \frac{J(\pi^{t,\varepsilon_n,k}; t, w) - J(\bar{\pi}; t, w)}{\varepsilon_n} \leq 0 \quad a.s.. \tag{2.3}$$

Remark 2.2. *In literature, the inequality is usually*

$$\limsup_{\varepsilon\downarrow 0} \frac{J(\pi^{t,\varepsilon,k}; t, w) - J(\bar{\pi}; t, w)}{\varepsilon} \leq 0 \quad a.s..$$

However, the left-hand side of the inequality might be unmeasurable. To guarantee the measurability, we modify the definition of equilibrium accordingly.

Remark 2.3. *Note that*

$$J(\pi; t, w) = w \int_{[0,\infty)} (u^{\gamma})^{-1}\left(\mathbb{E}_t\left[u^{\gamma}\left(W_T^{t,w,\pi}/w\right)\right]\right) \mathrm{d}F_{\boldsymbol{R}}(\gamma) = wJ_0(t, \pi),$$

where

$$J_0(t, \pi) \triangleq \int_{[0,\infty)} (u^{\gamma})^{-1}\left(\tilde{\mathbb{E}}_t\left[u^{\gamma}\left(W_T^{\pi}/W_t^{\pi}\right)\right]\right) \mathrm{d}F_{\boldsymbol{R}}(\gamma)$$

is the objective functional used in Liang, Wang, and Xia (2025a) and W^{π} evolves according to

$$\mathrm{d}W_t^{\pi} = W_t^{\pi}\pi_t^{\top}\mu(t)\mathrm{d}t + W_t^{\pi}\pi_t^{\top}\sigma(t)\mathrm{d}B(t), \quad W_0^{\pi} = w_0 > 0.$$

Thus, under Definition 2.1, an equilibrium strategy for J is also an equilibrium strategy for J_0, and vice versa. We will frequently use J_0 in the subsequent analysis.

In our analysis, we follow the approach of Liang, Wang, and Xia (2025a) and focus on the equilibrium strategies given by $\bar{\pi} = (\sigma^\top)^{-1}a$, where a, referred to as the *risk exposure vector*, is a deterministic and right-continuous funcion in the L^2 space. Let $\Pi_d \subset \Pi$ denote the set of all such strategies. Because $k \in L^\infty(\mathcal{F}_t, \mathbb{R}^d)$, the perturbed strategy $\pi^{t,\varepsilon,k}$ can be random, even though $\bar{\pi} \in \Pi_d$ itself is deterministic. The wealth process $W^{t,w,\bar{\pi}}$ of the portfolio $\bar{\pi}$ satisfies the following SDE

$$\mathrm{d}W_s^{t,w,\bar{\pi}} = W_s^{t,w,\bar{\pi}}[a(s)^\top \lambda(s)\mathrm{d}s + a(s)^\top \mathrm{d}B(s)], \quad s \geq t,$$

and $|a|$ is referred to as the *risk exposure magnitude* (or local volatility).

For any $\bar{\pi} = (\sigma^\top)^{-1}a \in \Pi_d$, we introduce the following notations for ease of analysis:

$$v_a(t) \triangleq \int_t^T |a(s)|^2 ds, \quad y_a(t) \triangleq \int_t^T a^\top(s)\lambda(s)ds, \quad t \in [0,T].$$

Because the strategy is deterministic, the conditional distribution of the relative wealth $W_T^{t,w,\bar{\pi}}/w$ depends only on the deterministic quantities $v_a(t)$ and $y_a(t)$ and we have

$$J_0(t,\bar{\pi}) = \tilde{\mathbb{E}}\left[\mathrm{e}^{-\frac{1}{2}(\boldsymbol{R}v_a(t)-2y_a(t))}\right]. \tag{2.4}$$

The following theorem shows that $\pi \equiv \mathbf{0}$ is an equilibrium if $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$. (This result is natural: if the random risk aversion is very large, then the investor puts no money into the risky asset. However, in Theorem 3.2(2) we will see that $\pi \equiv \mathbf{0}$ is not necessarily the unique equilibrium.)

Theorem 2.4. *$\bar{\pi} \equiv \mathbf{0}$ is an equilibrium if $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$.*

Proof. Let $\bar{\pi} = \mathbf{0}$. For any $t \in [0,T)$, $\varepsilon \in (0, T-t)$, and $k \in L^\infty(\mathcal{F}_t, \mathbb{R}^d)$, we have $J_0(t,\bar{\pi}) = 1$ and

$$J_0(t,\pi^{t,\varepsilon,k}) = \mathrm{e}^{\int_t^{t+\varepsilon}\mu^\top(s)k\mathrm{d}s}\int_{[0,\infty)} \mathrm{e}^{-\frac{\gamma}{2}\int_t^{t+\varepsilon}|\sigma^\top(s)k|^2\mathrm{d}s}\mathrm{d}F_{\boldsymbol{R}}(\gamma) \triangleq \phi(\varepsilon)\psi(\varepsilon).$$

Noting that

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{\psi(\varepsilon)-1}{\varepsilon} &= -\frac{1}{2}|\sigma^\top(t)k|^2 \lim_{\varepsilon\to 0}\int_{[0,\infty)} \gamma \mathrm{e}^{-\frac{\gamma}{2}\int_t^{t+\varepsilon}|\sigma^\top(s)k|^2\mathrm{d}s}\mathrm{d}F_{\boldsymbol{R}}(\gamma)\\ &= -\frac{1}{2}|\sigma^\top(t)k|^2\int_{[0,\infty)}\gamma\mathrm{d}F_{\boldsymbol{R}}(\gamma),\end{aligned}$$

where the first equality uses the mean value theorem and the second the monotone convergence theorem. Thus, for any positive sequence $\{\varepsilon_n, n \geq 1\}$ satisfying $\lim_{n\to\infty}\varepsilon_n = 0$, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{J_0(t,\pi^{t,\varepsilon_n,k}) - J_0(t,\bar{\pi})}{\varepsilon_n} &= \lim_{n\to\infty}\frac{\phi(\varepsilon_n)\psi(\varepsilon_n)-1}{\varepsilon_n} = \lim_{n\to\infty}\frac{(\phi(\varepsilon_n)-1)\psi(\varepsilon_n)+\psi(\varepsilon_n)-1}{\varepsilon_n}\\ &= \left(\mu^\top(t)k - \frac{1}{2}|\sigma^\top(t)k|^2\int_{[0,\infty)}\gamma\mathrm{d}F_{\boldsymbol{R}}(\gamma)\right).\end{aligned}$$

Therefore, $\mathbf{0}$ is an equilibrium if $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$. □

The next theorem provides a necessary and sufficient condition for $\bar{\pi} = (\sigma^\top)^{-1}a \in \Pi_d$ to be an equilibrium.

Theorem 2.5. *$\bar{\pi} = (\sigma^\top)^{-1}a \in \Pi_d$ is an equilibrium if and only if*

$$a(t) = h(v_a(t))\lambda(t) \tag{2.5}$$

holds for any $t \in [0, T)$, where[3]

$$h(x) \triangleq \tilde{\mathbb{E}}\left[\mathrm{e}^{-\frac{1}{2}\boldsymbol{R}x}\right] \Big/ \tilde{\mathbb{E}}\left[\boldsymbol{R}\mathrm{e}^{-\frac{1}{2}\boldsymbol{R}x}\right], \quad x \in [0, \infty). \tag{2.6}$$

(Occasionally, we write $h_{\boldsymbol{R}}$ for h to emphasize its dependence on $\boldsymbol{R}$.)

Proof. It suffices to show that for any $t \in [0, T)$, (2.3) is equivalent to (2.5).

When $\tilde{\mathbb{E}}[\boldsymbol{R}] < \infty$, we have $h(0) = 1/\tilde{\mathbb{E}}[\boldsymbol{R}] \in (0, \infty)$ and $h \in C([0, \infty))$. In this case, an argument similar to the proof of Liang, Wang, and Xia (2025a, Theorem 3.4) yields the equivalence.

When $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$, we have $h(0) = 0$. In this case, similar to the case $\tilde{\mathbb{E}}[\boldsymbol{R}] < \infty$, (2.3) and (2.5) are equivalent for $t \in [0, T_0)$, where $T_0 = \inf\{t \in [0, T] : v_a(t) = 0\}$. Moreover, they are automatically satisfied for $t \in [T_0, T)$ based on Theorem 2.4 and $h(0) = 0$. □

Now we introduce a function $l : [0, \infty) \to (0, \infty)$, which is defined by

$$l(y) \triangleq \tilde{\mathbb{E}}[e^{-\boldsymbol{R}y}], \quad y \geq 0.$$

The function h defined in (2.6) plays a central role in our analysis. Here, we present two examples for which h can be explicitly worked out.

Example 2.6. *Consider the following two distributions for the random risk aversion $\boldsymbol{R}$.*

(1) *Suppose that $\boldsymbol{R}$ follows a Poisson distribution with parameter $\theta > 0$. In this case, $l(y) = e^{\theta(e^{-y}-1)}$ for any $y \geq 0$. According to (2.6), we obtain*

$$h(x) = -l(x/2)/l'(x/2) = \frac{1}{\theta}e^{x/2}, \quad x \geq 0.$$

(2) *Suppose that $\boldsymbol{R}$ follows a Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. Its probability density function is*

$$f_{\boldsymbol{R}}(r) = \frac{r^{\alpha-1}e^{-r/\beta}}{\Gamma(\alpha)\beta^\alpha}, \quad r > 0.$$

In this case, $l(y) = \tilde{\mathbb{E}}[e^{-\boldsymbol{R}y}] = (1 + \beta y)^{-\alpha}$ $(y \geq 0)$, which gives

$$h(x) = -\frac{l(\frac{x}{2})}{l'(\frac{x}{2})} = \frac{1}{\alpha\beta}\left(1 + \frac{\beta}{2}x\right), \quad x \geq 0.$$

In the two cases of the above example, the functions h are unbounded, violating the boundedness assumption on h made by Liang, Wang, and Xia (2025a). Moreover, as we will see in Example 3.5 below, the equilibrium strategies can be explicitly obtained for both cases. To accommodate such interesting cases, we will conduct a thorough analysis on Condition (2.5) and the equilibrium strategies under a general distribution of $\boldsymbol{R}$.

[3] This function h is slightly different from that in Liang, Wang, and Xia (2025a): $h(x)$ there amounts to $h(x^2)$ here. Nevertheless, the relevant properties, such as the boundedness of h, remain equivalent under the two formulations.

3 Equilibrium Strategy

We first transform Condition (2.5) into an ordinary differential equation (ODE) for v_a. Differentiating $v_a(t) = \int_t^T |a(s)|^2 ds$ with respect to t, we get from (2.5) that v_a satisfies the following ODE:

$$\begin{cases} v'(t) = -h^2(v(t))|\lambda(t)|^2, & t \in [0,T), \\ v(T) = 0. \end{cases} \tag{3.1}$$

Conversely, assume that v solves the ODE (3.1) and let $a(\cdot) = h(v(\cdot))\lambda(\cdot)$. Then a satisfies (2.5).

The ODE (3.1) is an equation with separated variables. We can obtain its solutions in closed form.

Now we introduce two functions. The first function Λ is defined by

$$\Lambda(t) \triangleq \int_t^T |\lambda(s)|^2 \mathrm{d}s, \quad t \in [0,T].$$

The second function H is defined by

$$H(y) \triangleq \int_0^y \frac{1}{h^2(x)} \mathrm{d}x = \int_0^y \left(\tilde{\mathbb{E}}\left[\boldsymbol{R} \mathrm{e}^{-\frac{1}{2}\boldsymbol{R}x} \right] \Big/ \tilde{\mathbb{E}}\left[\mathrm{e}^{-\frac{1}{2}\boldsymbol{R}x} \right] \right)^2 \mathrm{d}x, \quad y \in [0,\infty].$$

Occasionally, we write $H_{\boldsymbol{R}}$ for H to emphasize its dependence on $\boldsymbol{R}$.

If $\Lambda(0) = 0$, then $\lambda(\cdot) \equiv 0$ on $[0,T)$ and, in view of (2.5), there is a unique equilibrium $\pi \equiv \mathbf{0}$. This trivial case is excluded by making the following standing assumption.

Assumption 3.1. $\Lambda(0) \in (0,\infty)$.

The next theorem provides, in closed form, the solutions to the ODE (3.1) and the equilibrium strategies in Π_d .

Theorem 3.2. *Under Assumption 3.1, we have the following assertions.*

(1) *If $\tilde{\mathbb{E}}[\boldsymbol{R}] < \infty$, the ODE (3.1) admits a solution if and only if $H(\infty) > \Lambda(0)$. Under this condition, the solution is unique, given by $v(\cdot) = H^{-1}(\Lambda(\cdot))$, and the corresponding unique equilibrium is $(\sigma^\top(\cdot))^{-1} h(v(\cdot))\lambda(\cdot)$.*

(2) *If $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$ and $H(\varepsilon) < \infty$ for all $\varepsilon > 0$, the ODE (3.1) admits a one-parameter family of solutions, indexed by $T_0 \in \mathcal{T}$:*

$$v^{(T_0)}(t) \triangleq \begin{cases} H^{-1}\left(\int_t^{T_0} |\lambda(s)|^2 \mathrm{d}s \right), & 0 \le t < T_0, \\ 0, & T_0 \le t \le T, \end{cases} \tag{3.2}$$

where

$$\mathcal{T} \triangleq \left\{ T_0 \in [0,T] \,\middle|\, H(\infty) > \Lambda(0) - \Lambda(T_0) \text{ and } \Lambda(t) > \Lambda(T_0) \text{ for all } t \in [0,T_0) \right\}.$$

Then the set of all equilibrium strategies in Π_d is :

$$\Pi_d^e = \left\{ \pi^{(T_0)}(\cdot) \triangleq (\sigma^\top(\cdot))^{-1} h(v^{(T_0)}(\cdot))\lambda(\cdot) : T_0 \in \mathcal{T} \right\}.$$

(3) *If* $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$ *and* $H(\varepsilon) = \infty$ *for some* $\varepsilon > 0$*, the ODE* (3.1) *admits a unique solution* $v \equiv 0$ *and* $\pi \equiv \mathbf{0}$ *is the unique equilibrium in* Π_d*.*

Proof.

(1) Assume $\tilde{\mathbb{E}}[\boldsymbol{R}] < \infty$. In this case, $h(x) > 0$ for all $x \geq 0$. Suppose that $v(\cdot)$ is a solution to the ODE (3.1). Separating variables in (3.1) yields

$$-|\lambda(t)|^2 = \frac{v'(t)}{h^2(v(t))}, \quad t \in [0, T). \tag{3.3}$$

Integrating (3.3) from t to T yields

$$-\Lambda(t) = \int_t^T \frac{v'(s)}{h^2(v(s))} \mathrm{d}s = \int_{v(t)}^0 \frac{\mathrm{d}y}{h^2(y)} = -\int_0^{v(t)} \frac{\mathrm{d}y}{h^2(y)} = -H(v(t)),$$

where the second equality follows from the change of variables $y = v(s)$. Thus, $H(v(t)) = \Lambda(t)$ for all $t \in [0, T]$. As $h(0) = 1/\tilde{\mathbb{E}}[\boldsymbol{R}] > 0$, it follows that $H(0) = 0$ and $H(\cdot) \in C^1([0, +\infty))$ is strictly increasing. Thus, the inverse function H^{-1} is well-defined, continuous, and strictly increasing in $[0, H(\infty))$. Thus, $H(\infty) > H(v(0)) = \Lambda(0)$ and $v(t) = H^{-1}(\Lambda(t))$ for all $t \in [0, T]$, which yields the uniqueness of the solution. Conversely, if $H(\infty) > \Lambda(0)$, it is straightforward to verify that $v(\cdot) = H^{-1}(\Lambda(\cdot))$ indeed solves uniquely the ODE.

(2) Assume that $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$ and $H(\varepsilon) < \infty$ for all $\varepsilon > 0$. We can see that $H(0) = 0$, $H \in C^0([0, +\infty]) \cap C^1((0, +\infty))$, and H is strictly increasing. Suppose that $v(\cdot)$ is a solution to the ODE (3.1). Let $T_0 = \inf\{t \in [0, T] : v(t) = 0\}$. Then $v(t) = 0$ for any $t \in [T_0, T]$ and $v(t) > 0$ for any $t \in [0, T_0)$. For any $t \in [0, T_0)$, integrating (3.3) from t to T_0 yields

$$\Lambda(T_0) - \Lambda(t) = \int_t^{T_0} -|\lambda(s)|^2 \mathrm{d}s = \int_{v(t)}^{v(T_0)} \frac{\mathrm{d}y}{h^2(y)} = -H(v(t)).$$

Thus, $H(v(t)) = \Lambda(t) - \Lambda(T_0)$ for all $t \in [0, T_0)$. Therefore, $H(\infty) > H(v(0)) = \Lambda(0) - \Lambda(T_0)$ and $\Lambda(t) - \Lambda(T_0) = H(v(t)) > H(0) = 0$ for all $t \in [0, T_0)$. Consequently, $T_0 \in \mathcal{T}$. Moreover, for any $t < T_0$, we have

$$v(t) = H^{-1}(\Lambda(t) - \Lambda(T_0)) = H^{-1}\left(\int_t^{T_0} |\lambda(s)|^2 \mathrm{d}s\right),$$

which implies $v = v^{(T_0)}$. Conversely, a direct verification shows that (3.2) indeed solves the ODE.

(3) Assume that $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$ and $H(\varepsilon) = \infty$ for some $\varepsilon > 0$. This implies that $\int_0^y \frac{1}{h^2(x)} \mathrm{d}x = \infty$ for any $y > 0$. Clearly, $v \equiv 0$ is a solution (recall that $h(0) = 1/\tilde{\mathbb{E}}[\boldsymbol{R}] = 0$). Suppose, on the contrary, that there exists another solution $v(\cdot)$ with $v(t) > 0$ for some $t < T$. Let $t' \triangleq \inf\{s \in$

$(t,T] : v(s) = 0\}$. By continuity, we have $v(t') = 0$ and $v(s) > 0$ for any $s \in [t,t')$. Integrating the ODE (3.3) from t to t' yields

$$\Lambda(t') - \Lambda(t) = \int_t^{t'} -|\lambda(s)|^2 \mathrm{d}s = \int_{v(t)}^0 \frac{\mathrm{d}y}{h^2(y)} = -\infty,$$

which is impossible. Therefore, the ODE (3.1) admits the unique solution $v \equiv 0$. □

Assertion (2) of Theorem 3.2 characterizes the solutions to the ODE (3.1) via the parameter $T_0 \in \mathcal{T}$. The following lemma provides an alternative explicit characterization of the set $\mathcal{T}$ via an auxiliary mapping.

Lemma 3.3. *Let* $\mathcal{A} \triangleq [0, \Lambda(0)] \cap (\Lambda(0) - H(\infty), \Lambda(0)]$ *and*

$$\varphi(\eta) \triangleq \min\{t \in [0,T] \mid \Lambda(t) = \eta\}, \quad \eta \in [0, \Lambda(0)].$$

Then we have $\mathcal{T} = \{\varphi(\eta) \mid \eta \in \mathcal{A}\}$.

Proof. From the definition of $\mathcal{T}$, for any $T_0 \in \mathcal{T}$, we have $\Lambda(T_0) \in \mathcal{A}$ and

$$T_0 = \inf\{t \in [0,T] : \Lambda(t) = \Lambda(T_0)\} = \varphi(\Lambda(T_0)).$$

Thus, $\mathcal{T} \subset \{\varphi(\eta) \mid \eta \in \mathcal{A}\}$. Conversely, for any $\eta \in \mathcal{A}$, the continuity of Λ implies $\Lambda(\varphi(\eta)) = \eta \in \mathcal{A}$. Moreover, for any $t \in [0, \varphi(\eta))$, we have $\Lambda(t) > \Lambda(\varphi(\eta))$ by the definition of φ. Therefore, $\varphi(\eta) \in \mathcal{T}$ and hence $\{\varphi(\eta) \mid \eta \in \mathcal{A}\} \subset \mathcal{T}$. □

Remark 3.4. *Because φ is strictly decreasing, the sets $\mathcal{T}$ and $\mathcal{A}$ are in one-to-one correspondence. It is easily seen that $\mathcal{A}$ contains infinitely many elements. Consequently, under the conditions* $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$ *and* $H(\varepsilon) < \infty$ *for all* $\varepsilon > 0$*, we obtain infinitely many equilibria.*

It should be noted that the existing studies, such as Desmettre and Steffensen (2023) and Liang, Wang, and Xia (2025a), focus mainly on the specific cases covered by Theorem 3.2(1), under the additional assumptions that $h(\cdot)$ is bounded and $\boldsymbol{R}$ is square-integrable. Our result covers any general random risk aversion $\boldsymbol{R}$. In the following two examples, the functions h are unbounded and the equilibrium strategies are explicitly worked out.

Example 3.5. *This example derives the explicit equilibrium strategies for the two distributions discussed in Example 2.6.*

(1) ***Poisson distribution:*** *Recall that* $h(x) = \frac{1}{\theta} e^{\frac{x}{2}}$. *Then h is unbounded and*

$$H(y) = \int_0^y \frac{1}{h^2(x)} dx = \theta^2(1 - e^{-y}), \quad y \in [0, \infty),$$

with $H(\infty) = \theta^2$. *If* $\Lambda(0) < \theta^2$, *then* $v(t) = H^{-1}(\Lambda(t)) = -\ln\left(1 - \frac{\Lambda(t)}{\theta^2}\right)$. *In this case, the uinque equilibrium strategy is explicitly given by*

$$\bar{\pi}(t) = (\sigma^\top)(t)^{-1} a(t) = (\sigma^\top(t))^{-1} h(v(t)) \lambda(t) = \frac{(\sigma^\top(t))^{-1} \lambda(t)}{\sqrt{\theta^2 - \Lambda(t)}}, \quad t \in [0, T).$$

(2) ***Gamma distribution:*** *Recall that* $h(x) = \frac{1}{\alpha\beta}(1 + \frac{\beta}{2}x)$. *Then* h *is unbounded and*

$$H(y) = \int_0^y \frac{\alpha^2\beta^2}{(1+\frac{\beta}{2}x)^2}dx = 2\alpha^2\beta\left(1 - \frac{1}{1+\frac{\beta}{2}y}\right), \quad y \in [0,\infty),$$

with $H(\infty) = 2\alpha^2\beta$. *Provided that* $\Lambda(0) < 2\alpha^2\beta$, *we obtain* $v(t) = \frac{2}{\beta}\left[\left(1 - \frac{\Lambda(t)}{2\alpha^2\beta}\right)^{-1} - 1\right]$. *In this case, the unique equilibrium strategy is explicitly given by*

$$\bar{\pi}(t) = (\sigma^\top(t))^{-1}a(t) = \frac{(\sigma^\top(t))^{-1}\lambda(t)}{\alpha\beta - \frac{\Lambda(t)}{2\alpha}}, \quad t \in [0,T).$$

Example 3.6. *This example is provided for assertions (2) and (3) in Theorem 3.2. Let* $\boldsymbol{R}_\alpha \sim G_\alpha$, *where* $\alpha \in (0,1)$ *is a fixed parameter and* G_α *denotes the distribution function whose Laplace transform satisfies*[4]

$$\int_0^\infty \mathrm{e}^{-vx}\mathrm{d}G_\alpha(x) = \mathrm{e}^{-v^\alpha}, \quad v \geq 0.$$

Note that $\tilde{\mathbb{E}}[\boldsymbol{R}_\alpha] = \infty$ *for* $\alpha \in (0,1)$. *Moreover,* $h(x) = \frac{1}{\alpha}\left(\frac{x}{2}\right)^{1-\alpha}$, $x \geq 0$. *Obviously,* h *is unbounded. For simplicity, we assume that* $d = 1$ *and* $\sigma(\cdot) \equiv \sigma \neq 0$ *and* $\lambda(\cdot) \equiv \lambda \neq 0$ *are constants in* $\mathbb{R}$.

- *Let* $\alpha \in (0.5, 1)$. *By direct calculation, the solutions to the ODE (3.1) take the form*

$$v^{(T_0)}(t) = \begin{cases} \left(2^{2\alpha-2}\frac{1}{\alpha^2}(2\alpha-1)\lambda^2(T_0 - t)\right)^{\frac{1}{2\alpha-1}}, & t \in [0,T_0), \\ 0, & t \in [T_0,T], \end{cases}$$

  where $T_0 \in [0,T]$. *Consequently, the explicit expression for the equilibrium strategy* $\pi^{(T_0)}(t)$ *on* $[0,T)$ *is*

$$\begin{aligned} \pi^{(T_0)}(t) &= \begin{cases} (\sigma)^{-1}\frac{1}{\alpha}\left[\frac{1}{2}\left(2^{2\alpha-2}\frac{2\alpha-1}{\alpha^2}\lambda^2(T_0-t)\right)^{\frac{1}{2\alpha-1}}\right]^{1-\alpha}\lambda, & 0 \leq t < T_0, \\ 0, & T_0 \leq t < T, \end{cases} \\ &= \begin{cases} (\sigma)^{-1}\left(\frac{\lambda}{\alpha}\right)^{\frac{1}{2\alpha-1}}\left[\frac{2\alpha-1}{2}(T_0-t)\right]^{\frac{1-\alpha}{2\alpha-1}}, & 0 \leq t < T_0, \\ 0, & T_0 \leq t < T, \end{cases} \end{aligned}$$

  where $T_0 \in [0,T]$.

- *Let* $\alpha \in (0, 0.5]$. *Direct calculation shows that the ODE (3.1) admits only the trivial solution* $v \equiv 0$ *and therefore* $\pi = 0$ *is the unique equilibrium.*

Assume $\tilde{\mathbb{E}}[\boldsymbol{R}] < \infty$ and $H(\infty) > \Lambda(0)$. Assertion (1) of Theorem 3.2 shows that the ODE (3.1) admits a unique solution v and therefore there exists a unique function a such that (2.5) holds for all $t \in [0,T)$. To emphasize the dependence on $\boldsymbol{R}$, we occasionally write $v_{\boldsymbol{R}}$ and $a_{\boldsymbol{R}}$ for v and a, respectively.

[4] See Pollard (1946) and Feller (1971, Section XIII.6).

Remark 3.7. *The explicit characterization of the equilibrium strategy in Theorem 3.2(1) yields a clear structural decomposition of the risk exposure vector, given by*

$$a(t) = h(H^{-1}(\Lambda(t)))\lambda(t), \quad t \in [0, T). \tag{3.4}$$

Note that the definitions of the functions $h(\cdot)$ and $H(\cdot)$ depend solely on the distribution of the random risk aversion $\boldsymbol{R}$. The representation (3.4) shows that, at time t, the effects of three distinct factors on the risk exposure vector are cleanly separated: the instantaneous market opportunity $\lambda(t)$, the aggregate future market opportunities $\Lambda(t)$ over the remaining interval (t, T), and the random risk aversion preference itself.

More specifically, at time t, if $\Lambda(t)$ and $\boldsymbol{R}$ are fixed, a larger current risk premium $|\lambda(t)|$ leads to a larger $|a(t)|$. If $|\lambda(t)|$ and $\boldsymbol{R}$ are fixed, then a larger future market opportunity $\Lambda(t)$ implies a larger value of $H^{-1}(\Lambda(t))$, and hence a larger value of $h\big(H^{-1}(\Lambda(t))\big)$ (see Proposition 3.8 for a detailed discussion; see alos Lemma A.1 for the monotonicity of h). Finally, if both $\lambda(t)$ and $\Lambda(t)$ are fixed, one can study how $\boldsymbol{R}$ affects h and H, and thereby its impact on $a(t)$; see the comparative statics analysis in Section 5.

We emphasize that, in contrast to the classical CRRA utility, which generates a "myopic" strategy reacting primarily to the instantaneous market price of risk $\lambda(t)$, the equilibrium under random risk aversion is inherently forward-looking due to the dependence on $\Lambda(t)$. Finally, we refer to Liang et al. (2024) and Liang, Xia, and Yuan (2025b) for similar structural representations, although monotonicity properties and comparative statics are not investigated therein.

Next, we study the impact of changes in future market opportunities on the risk exposure vector.

Proposition 3.8. *Let $\boldsymbol{R}$ be a non-negative random variable with $\tilde{\mathbb{E}}[\boldsymbol{R}] \in (0, +\infty)$. Fix a vector $\lambda_0 \in \mathbb{R}^d$. Consider a sequence of market settings indexed by $n \in \mathbb{N}$, defined by the pairs $\{(\lambda_n(\cdot), T_n)\}_{n\geq 1}$, where $T_n > 0$ and $\lambda_n : [0, T_n] \to \mathbb{R}^d$ satisfying $\lambda_n(0) = \lambda_0$. Let $\Lambda_n \triangleq \int_0^{T_n} |\lambda_n(s)|^2 ds$. Let $a_n(0)$ denote the risk exposure vector at time 0 corresponding to the n-th market setting.*

(1) *If the sequence $\{\Lambda_n\}_{n\geq 1}$ is strictly increasing and $\Lambda_n \nearrow H(\infty)$, then*

$$\lim_{n\to\infty} a_n(0) = \frac{\lambda_0}{r_0} \quad \text{(if } r_0 > 0\text{)},$$

where $r_0 \triangleq \operatorname{essinf} \boldsymbol{R}$. Moreover, $|a_n(0)|$ converges monotonically increasing, i.e.,

$$|a_n(0)| \nearrow \frac{|\lambda_0|}{r_0}.$$

(2) *If the sequence $\{\Lambda_n\}_{n\geq 1}$ is strictly decreasing and $\Lambda_n \searrow 0$, then*

$$\lim_{n\to\infty} a_n(0) = \frac{\lambda_0}{\tilde{\mathbb{E}}[\boldsymbol{R}]}.$$

Moreover, $|a_n(0)|$ converges monotonically decreasing, i.e.,

$$|a_n(0)| \searrow \frac{|\lambda_0|}{\tilde{\mathbb{E}}[\boldsymbol{R}]}.$$

Here, we adopt the convention that $\frac{1}{0} = \infty$.

Proof. See Appendix B.1 $\square$

Remark 3.9. *The convergence results in Proposition 3.8 depend solely on the sequence of integral values* $\{\Lambda_n\}_{n\geq 1}$*, and no convergence of the parameter sequence* $\{(\lambda_n(\cdot), T_n)\}_{n\geq 1}$ *itself is required. Moreover, focusing on time* 0 *entails no loss of generality. For any arbitrary time* t*, analogous conclusions follow by a simple time translation, with* t *treated as the initial time.*

Remark 3.10. *In the single-stock case, assuming that* $\lambda > 0$ *and* $\sigma > 0$ *are constants, one can use* $\hat{\pi}(\cdot) = \frac{1}{\sigma} h(v(\cdot))\lambda$*, together with the monotonicity of* h *given in Lemma A.1, to establish a favorable property:* $\pi(\cdot)$ *is decreasing. See Section 4.3 of Desmettre and Steffensen (2023) for a detailed discussion. However, the results in Desmettre and Steffensen (2023) do not yield Proposition 3.8. The proof of Proposition 3.8 crucially relies on our explicit characterization of the risk exposure vector, given in* (3.4)*, which is not available in their framework.*

As the final part of this section, we use the explicit characterization of the equilibria in Theorem 3.2 to investigate the continuity properties of $a(\cdot)$ with respect to the distribution of the random risk aversion. More specifically, the following theorem shows that if a sequence of random risk aversion $\{R_n\}_{n\geq 1}$ converges in distribution to R, then the corresponding sequence $\{a_n\}_{n\geq 1}$ of volatility converges to a.

Theorem 3.11. *Let* $\{\boldsymbol{R}_n\}_{n\geq 1}$ *be a sequence of random risk aversions that converges in distribution to* $\boldsymbol{R}$ *and* $\lim_{n\to\infty} \tilde{\mathbb{E}}[\boldsymbol{R}_n] = \tilde{\mathbb{E}}[\boldsymbol{R}] \in (0, \infty)$*. Assume that* $H_n(\infty) > \Lambda(0)$ *for all* $n \geq 1$ *and* $H(\infty) > \Lambda(0)$*. Let* $a_n = a_{\boldsymbol{R}_n}$ *and* $a = a_{\boldsymbol{R}}$*. Then* $\lim_{n\to\infty} a_n(t) = a(t)$ *for all* $t \in [0, T)$*. Moreover, if* $\lambda \in L^\infty([0, T])$*, then the convergence is uniform on* $[0, T)$*.*

Proof. See Appendix B.2. $\square$

4 Optimal Equilibrium

Remark 3.4 shows that, if $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$ and $H(\varepsilon) < \infty$ for all $\varepsilon > 0$, then the set Π_d^e of the equilibria contains infinitely many elements. This multiplicity of equilibria raises the natural question: which element of Π_d^e should the agent choose? In this section, we propose that the agent choose the equilibrium π that maximizes objective functional $J(\pi; 0, w)$ at the initial time over all $\pi \in \Pi_d^e$. This leads to the following definition.

Definition 4.1. *An equilibrium* $\hat{\pi} \in \Pi_d^e$ *is called optimal if* $J(\hat{\pi}; 0, w) \geq J(\pi; 0, w)$ *for all* $\pi \in \Pi_d^e$ *and for all* $w > 0$*.*

Recalling the relationship $J(\pi; t, w) = wJ_0(t, \pi)$ in Remark 2.3, finding an optimal equilibrium in Π_d^e is equivalent to finding a $\hat{\pi} \in \Pi_d^e$ that maximizes $J_0(t, \pi)$.

Let $\pi \in \Pi_d^e$. We begin by computing $J_0(t,\pi)$. Using Theorem 3.2 (2), we have $\pi = \pi^{(T_0)}$ for some $T_0 \in \mathcal{T}$. Let $a^{(T_0)} \triangleq \sigma^\top \pi^{(T_0)}$ and $y^{(T_0)}(t) \triangleq \int_t^T (a_s^{(T_0)})^\top \lambda(s)\mathrm{d}s$. Then

$$
\begin{aligned}
-(\boldsymbol{R}v^{(T_0)}(t) - 2y^{(T_0)}(t)) &= -\boldsymbol{R}v^{(T_0)}(t) + 2\int_t^T (a_s^{(T_0)})^\top \lambda(s)\mathrm{d}s \\
&= -\boldsymbol{R}v^{(T_0)}(t) + 2\int_t^T h(v^{(T_0)}(s))|\lambda(s)|^2\mathrm{d}s \\
&= -\boldsymbol{R}v^{(T_0)}(t) + 2\int_0^{v^{(T_0)}(t)} \frac{1}{h(y)}\mathrm{d}y,
\end{aligned}
$$

where the third equality arises from the change of variable $y = v^{(T_0)}(s)$. Thus, we obtain

$$
J_0(t,\pi^{(T_0)}) = \exp\left\{\int_0^{v^{(T_0)}(t)} \frac{1}{h(y)}\mathrm{d}y\right\} l\left(\frac{v^{(T_0)}(t)}{2}\right).
$$

Let $\mathcal{L}(z) : [0,\infty) \to \mathbb{R}$ be defined by $\mathcal{L}(z) = \int_0^z \frac{1}{h(y)}\mathrm{d}y + \log l\left(\frac{z}{2}\right)$. Then we have

$$
J_0(t,\pi^{(T_0)}) = \exp\left(\mathcal{L}(v^{(T_0)}(t))\right). \tag{4.1}
$$

According to (4.1), in order to study the maximization of $J_0(t,\pi^{(T_0)})$ over $T_0 \in \mathcal{T}$, we need to analyze the monotonicity of $\mathcal{L}$ and the monotonicity of $v^{(T_0)}$ with respect to T_0. These properties are characterized in Lemma 4.2 and Lemma 4.3, respectively.

Lemma 4.2. *$\mathcal{L}(z)$ is strictly increasing in z.*

Proof. By direct differentiation, we have

$$
\frac{d\mathcal{L}}{dz} = \frac{1}{h(z)} - \frac{1}{2}\frac{\tilde{\mathbb{E}}\left[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}z}\right]}{\tilde{\mathbb{E}}\left[e^{-\frac{1}{2}\boldsymbol{R}z}\right]} = \frac{1}{2h(z)} > 0 \quad \text{for all } z > 0.
$$

As such, $\mathcal{L}(z)$ is strictly increasing in z. □

Lemma 4.3. *Under the conditions of Assertion (2) of Theorem 3.2, let $T_{0,1}, T_{0,2} \in \mathcal{T}$ with $T_{0,1} < T_{0,2}$. Then*

$$
v^{(T_{0,1})}(t) \begin{cases} < v^{(T_{0,2})}(t), & t \in [0, T_{0,2}), \\ = v^{(T_{0,2})}(t), & t \in [T_{0,2}, T]. \end{cases} \tag{4.2}
$$

In particular, if $H(\infty) > \Lambda(0)$ and $\Lambda(t) > 0$ for all $t \in [0,T)$, then $T \in \mathcal{T}$ and $v^{(T)}(t) > v^{(T_0)}(t)$ for any $T_0 \in \mathcal{T} \setminus \{T\}$ and any $t \in [0,T)$.

Proof. Let $T_{0,1}, T_{0,2} \in \mathcal{T}$ with $T_{0,1} < T_{0,2}$. For $t \in [T_{0,2}, T]$, we have $v^{(T_{0,1})}(t) = v^{(T_{0,2})}(t) = 0$ based on Definition of $\mathcal{T}$. For $t \in [T_{0,1}, T_{0,2})$, we have $v^{(T_{0,2})}(t) > 0 = v^{(T_{0,1})}(t)$. For $t \in [0, T_{0,1})$, we have

$$
v^{(T_{0,i})}(t) = H^{-1}\left(\int_t^{T_{0,i}} |\lambda(s)|^2\mathrm{d}s\right) = H^{-1}(\Lambda(t) - \Lambda(T_{0,i})), \quad i = 1, 2.
$$

By the definition of $\mathcal{T}$, $\Lambda(T_{0,1}) > \Lambda(T_{0,2})$. Because H^{-1} is strictly increasing, it follows that $v^{(T_{0,1})}(t) < v^{(T_{0,2})}(t)$ for $t \in [0, T_{0,1})$. Thus, (4.2) is proved.

Assume $H(\infty) > \Lambda(0)$ and $\Lambda(t) > 0$ for all $t \in [0, T)$. By the definition of $\mathcal{T}$, $T \in \mathcal{T}$. Plugging $T_{0,2} = T$ and $T_{0,1} = T_0$ into (4.2) yields $v^{(T)}(t) > v^{(T_0)}(t)$ for any $T_0 \in \mathcal{T} \backslash \{T\}$ and any $t \in [0, T)$. □

We are now ready to present the main result on the optimal equilibria.

Theorem 4.4. *Under the conditions of Assertion (2) of Theorem 3.2, we have the followings:*

(1) *The optimal strategy exists if and only if $H(\infty) > \Lambda(0)$. If it exists, it is unique and given by $\pi^{(\varphi(0))}(\cdot) = (\sigma^\top(\cdot))^{-1} h(v^{(\varphi(0))}(\cdot))\lambda(\cdot)$.*

(2) *$\pi^{(\varphi(0))}$ is also uniformly optimal:*

$$J(\pi^{(\varphi(0))}; t, w) \geq J(\pi; t, w) \text{ for all } t \in [0, T), w > 0, \text{ and } \pi \in \Pi_d^e.$$

(3) *$\pi^{(\varphi(0))}$ is uniformly strictly optimal, that is,*

$$J(\pi^{(\varphi(0))}; t, w) > J(\pi; t, w) \text{ for all } t \in [0, T), w > 0, \text{ and } \pi \in \Pi_d^e \setminus \{\pi^{(\varphi(0))}\},$$

if and only if

$$H(\infty) > \Lambda(0) \text{ and } \Lambda(t) > 0 \text{ for all } t \in [0, T).$$

If it is the case, then $\varphi(0) = T$.

Proof.

(1) Combining (4.1) with Lemma 4.2, we see that $J_0(0, \pi^{(T_0)})$ attains its maximum precisely when $v^{(T_0)}(0)$ is maximal. Moreover, by Lemma 4.3, $v^{(T_0)}(0)$ is strictly increasing in $T_0 \in \mathcal{T}$. Thus, finding the optimal equilibrium strategy is equivalent to identifying the maximal parameter $T_0 \in \mathcal{T}$. By Lemma 3.3, $\mathcal{T} = \{\varphi(\eta) \mid \eta \in \mathcal{A}\}$. Because φ is strictly decreasing on $\mathcal{A}$, the existence of a maximal element in $\mathcal{T}$ is equivalent to the existence of a minimal element in $\mathcal{A}$.

If $H(\infty) > \Lambda(0)$, then $\mathcal{A} = [0, \Lambda(0)]$ includes its minimum at 0, and hence a maximal strategy exists. Conversely, if $H(\infty) \leq \Lambda(0)$, then $\mathcal{A} = (\Lambda(0) - H(\infty), \Lambda(0)]$ is left-open and admits no minimum. Thus, $\mathcal{T}$ has a maximal element if and only if $H(\infty) > \Lambda(0)$. In conclusion, the optimal strategy exists if and only if $H(\infty) > \Lambda(0)$, and when it exists, it is given by $\pi^{(\varphi(0))}$.

(2) We now turn to the uniformly optimal equilibrium. In view of the conclusion in assertion (1), we restrict attention to the case $H(\infty) > \Lambda(0)$. Once this condition holds, and following the same line of reasoning as in assertion (1), we immediately obtain that $\pi^{(\varphi(0))}$ is also uniformly optimal, because (4.1) remains valid for every $t \in [0, T]$.

(3) Finally, we consider the uniformly strictly optimal equilibrium. If $\Lambda(t) = 0$ for some $t \in [0, T)$, then $v^{(T_0)}(t) \equiv 0$ for any $T_0 \in \mathcal{T}$, which implies $J(t, \pi^{(T_0)}) \equiv \exp(\mathcal{L}(0)) = 1$ for all $\pi^{(T_0)} \in \Pi_d^e$. Hence, no uniformly strictly optimal equilibrium exists in this case. Conversely, assume $\Lambda(t) > 0$ for all $t \in [0, T)$ and $H(\infty) > \Lambda(0)$. Because $\Lambda(t) > 0$ for all $t < T$, we have $\varphi(0) = T$. Let $T_0 \in \mathcal{T} \setminus \{T\}$, by Lemma 4.3, for any $t \in [0, T)$, it holds that

$$v^{(T)}(t) > v^{(T_0)}(t).$$

Because $\mathcal{L}(z)$ is strictly increasing in z, we obtain $J_0(t, \pi^{(T)}) > J_0(t, \pi^{(T_0)})$. Therefore, $\pi^{(\varphi(0))} = \pi^{(T)}$ is the unique uniformly strictly optimal equilibrium. □

Analogous to Proposition 3.8, we now investigate the impact of future market opportunities on the risk exposure vector in the case of $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$. Given the multiplicity of equilibria in this setting, we focus specifically on the optimal equilibrium characterized in Theorem 4.4. The following proposition presents the corresponding comparative statics results. We omit the proof as it follows a similar line of reasoning to that of Proposition 3.8.

Proposition 4.5. *Let $\boldsymbol{R}$ be a non-negative random variable with $\tilde{\mathbb{E}}[\boldsymbol{R}] = \infty$. Fix a vector $\lambda_0 \in \mathbb{R}^d$. Consider a sequence of market settings indexed by $n \in \mathbb{N}$, defined by the pairs $\{(\lambda_n(\cdot), T_n)\}_{n\geq 1}$, where $T_n > 0$ and $\lambda_n : [0, T_n] \to \mathbb{R}^d$ satisfy $\lambda_n(0) = \lambda_0$. Let $\Lambda_n \triangleq \int_0^{T_n} |\lambda_n(s)|^2 ds$ and assume $H(\infty) > \Lambda_n$ for all n. Let $a_n(0)$ denote the risk exposure vector at time 0 corresponding to the optimal equilibrium of the n-th market setting.*

(1) *If the sequence $\{\Lambda_n\}_{n\geq 1}$ is strictly increasing and $\Lambda_n \nearrow H(\infty)$, then*

$$\lim_{n\to\infty} a_n(0) = \frac{\lambda_0}{r_0},$$

where $r_0 \triangleq \operatorname{essinf} \boldsymbol{R}$. Moreover, $|a_n(0)|$ converges monotonically increasing, i.e.,

$$|a_n(0)| \nearrow \frac{|\lambda_0|}{r_0}.$$

(2) *If the sequence $\{\Lambda_n\}_{n\geq 1}$ is strictly decreasing and $\Lambda_n \searrow 0$, then*

$$\lim_{n\to\infty} a_n(0) = \mathbf{0}.$$

Moreover, $|a_n(0)|$ converges monotonically decreasing, i.e.,

$$|a_n(0)| \searrow 0.$$

Here, we adopt the convention that $\frac{1}{0} = \infty$.

5 Comparative Statics

In the previous sections, we fully characterized the equilibrium strategies associated with a given distribution of risk aversion $\boldsymbol{R}$ and discussed how to select the optimal equilibrium. We now turn to comparative statics and ask a natural economic question: how does a shift in the distribution of risk aversion affect the resulting equilibrium investment behavior? In this section, we only consider the case where the equilibrium exists.

To formalize such comparisons, we begin by recalling the notion of first-order stochastic dominance, which provides a standard criterion for ranking risk aversion distributions.

Definition 5.1. *Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be two random variables. We say that $\boldsymbol{R_1}$ dominates $\boldsymbol{R_2}$ in the sense of first-order stochastic dominance, denoted as $\boldsymbol{R_1} \succeq_1 \boldsymbol{R_2}$, if*

$$\tilde{\mathbb{P}}(\boldsymbol{R_1} \geq x) \geq \tilde{\mathbb{P}}(\boldsymbol{R_2} \geq x) \quad \textit{for all } x \in \mathbb{R}.$$

Intuitively, one might expect that if Investor 1 is "more risk-averse" than Investor 2 in the sense of first-order stochastic dominance, then Investor 1 should take a smaller position in the risky asset at all times. Surprisingly, this intuition turns out to be false, as the following numerical example shows.

Example 5.2. *Assume that there is one stock whose market price of risk is constant, $\lambda(\cdot) \equiv \lambda = 0.4$. The time horizon $T = 20$. The risk-aversion distributions of the two investors are:*

Investor 1: $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 1) = 0.9$, $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 3) = 0.1$.

Investor 2: $\tilde{\mathbb{P}}(\boldsymbol{R}_2 = 1) = 0.9$, $\tilde{\mathbb{P}}(\boldsymbol{R}_2 = 2) = 0.1$.

Obviously, $\boldsymbol{R}_1 \succeq_1 \boldsymbol{R}_2$. Figure 1 displays the comparative evolution of $|a_i|$, $i = 1, 2$.[5] *The orange-shaded Reversal Region highlights that the more risk-averse Investor 1 adopts a strictly more aggressive position than Investor 2 when the time to maturity is sufficiently long. This counterexample shows that under first-order stochastic dominance, the magnitudes of equilibrium investment strategies need not be monotone.*

[5] Because $\bar{\pi} = (\sigma^\top)^{-1}a$, in the one-dimensional case comparing $|a_i(t)|$ is equivalent to comparing $|\pi|_i$. More generally, $|a|$ represents the volatility of the investor's wealth process and thus captures the level of risk borne by the investor. Therefore, in what follows, we focus conceptually on comparing the risk exposure magnitude.

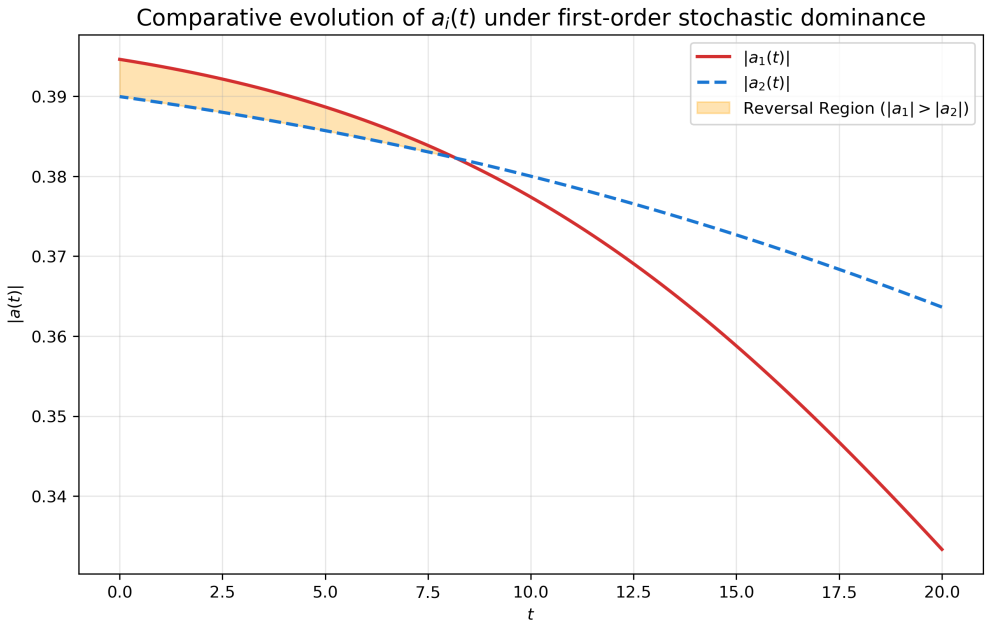

Figure 1: Comparative evolution of $a_i(t)$ under first-order stochastic dominance. The red solid line corresponds to Investor 1 (more risk-averse), with distribution $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 1) = 0.9$, $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 3) = 0.1$; the blue dashed line corresponds to Investor 2 (less risk-averse), with distribution $\tilde{\mathbb{P}}(\boldsymbol{R}_2 = 1) = 0.9$, $\tilde{\mathbb{P}}(\boldsymbol{R}_2 = 2) = 0.1$. The orange shaded region indicates where $|a_1(t)| > |a_2(t)|$. Parameters: $\lambda = 0.4$, horizon $T = 20$.

The appearance of this counterexample (Figure 1) may suggest that the search for a general comparative result under first-order stochastic dominance is fundamentally obstructed. In fact, the observed reversal is not an artifact of a particular parameter choice. We show below that such a reversal in investment magnitudes is unavoidable when the investment horizon T is sufficiently large, provided that the risk-aversion distributions satisfy the conditions stated in the following proposition.

In the following proposition, for convenience, we extend the market to an infinite horizon and assume that the market price of risk λ is defined on $[0, \infty)$. For every fixed $T > 0$, we require $\Lambda(0,T) \triangleq \int_0^T |\lambda(s)|^2 \mathrm{d}s < \infty$. In addition, we assume that $\lim_{T\to\infty} \Lambda(0,T) = \infty$ and $\lambda(0) \neq 0$. As discussed in the preceding sections, for each fixed T, we can define the equilibrium strategy $\{\pi_i(t,T) : t \in [0,T)\}$ associated with the distribution $\boldsymbol{R}_i$ (as used in the proposition below). The proposition shows that, as long as the distributions $\boldsymbol{R}_i$ satisfy certain conditions, the magnitudes $|\pi_i(0,T)|$ must exhibit a strict and deterministic ordering when the horizon T is sufficiently large.

Proposition 5.3. *Let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be two non-negative random variables with finite expectations i.e., $\tilde{\mathbb{E}}[\boldsymbol{R}_i] < \infty$, $i = 1,\ 2$. Assume that the following conditions are satisfied:*

(1) *Let $r_0 = \operatorname{essinf} \boldsymbol{R}_1 = \operatorname{essinf} \boldsymbol{R}_2 > 0$, and assume that there is a positive probability mass at r_0, i.e., $p_i = \tilde{\mathbb{P}}(\boldsymbol{R}_i = r_0) \in (0,1)$ for $i = 1, 2$.*

(2) *Define the infimum of the first support point of* $\boldsymbol{R}_i$ *strictly greater than* r_0 *as* $\tilde{r}_i = \inf\{r \in supp(\boldsymbol{R}_i) : r > r_0\}$*, and let* $\delta_i = \tilde{r}_i - r_0$ *with* $\delta_1 > \delta_2 > 0$.

Under these conditions, for a sufficiently large T*, we have*

$$|\pi_1(0,T)| > |\pi_2(0,T)|.$$

Proof. See Appendix B.3. □

As investigated by Desmettre and Steffensen (2023), we now focus on the two-point distribution, which represents the closest deviation from the standard expected utility model. This setting allows us to derive a single crossing property, rigorously characterizing the transition from short-term dominance to the long-term reversal observed in Proposition 5.3.

Proposition 5.4. *Under the conditions of Proposition 5.3, suppose further that* $\lambda(t) \neq 0$ *for all* $t \in [0,T)$*, and that* $\boldsymbol{R}_1$ *and* $\boldsymbol{R}_2$ *follow two-point distributions with* $\tilde{\mathbb{E}}[\boldsymbol{R}_1] > \tilde{\mathbb{E}}[\boldsymbol{R}_2]$*. Moreover, assume that* T *is sufficiently large so that* $|a_1(0)| > |a_2(0)|$*. Then* $|a_1(\cdot)|$ *and* $|a_2(\cdot)|$ *exhibit a single-crossing property: there exists a unique* $t^* \in [0,T]$ *such that* $|a_1(t^*)| = |a_2(t^*)|$*.*

Proof. See Appendix B.4. □

Having established the single crossing property, we investigate how the probability distribution drives the timing of this reversal. Intuitively, a larger probability assignment accelerates the strategy's convergence to the log-utility benchmark, leading to a faster intersection as time recedes from the horizon.

Proposition 5.5. *Under the conditions of Proposition 5.3 and Proposition 5.4, assume additionally that* $p_1 = p_2 = p$*. Let* t^* *be the intersection time where* $|a_1(t^*)| = |a_2(t^*)|$*. Then,* t^* *is strictly increasing with respect to* p*.*

Proof. Recalling that $Q_i(x) = \dfrac{\delta_i(1-p)e^{-\delta_i \frac{x}{2}}}{p + (1-p)e^{-\delta_i \frac{x}{2}}}$, we know that Q_i is strictly decreasing and satisfies $U_i = U_i(p) \triangleq Q_i(0) = \delta_i(1-p)$. Moreover, $\Lambda(t) = \int_0^{v_i(t)} (r_0 + Q_i(x))^2 \,\mathrm{d}x = 2\int_{Q_i(v_i(t))}^{U_i(p)} \dfrac{(r_0+q)^2}{q(\delta_i - q)} \,\mathrm{d}q \triangleq 2\int_{Q_i(v_i(t))}^{U_i(p)} G(q,\delta)\,\mathrm{d}q$ where the second equality arises from $q = Q_i(x)$, $\mathrm{d}x = \dfrac{2}{q(q-\delta_i)}\,\mathrm{d}q$ and $G(q,\delta) \triangleq \dfrac{(r_0+q)^2}{q(\delta - q)}$. At time t^*, let $Q = Q(p) \triangleq Q_1(v_1(t^*)) = Q_2(v_2(t^*))$ and $\Lambda^* = \Lambda^*(p) \triangleq \Lambda(t^*)$. The condition $|a_1(t^*)| = |a_2(t^*)|$ implies

$$\Lambda^* = 2\int_Q^{U_1(p)} G(q,\delta_1)\,dq = 2\int_Q^{U_2(p)} G(q,\delta_2)\,dq. \tag{5.1}$$

Differentiating (5.1) with respect to p and we have

$$\frac{1}{2}\frac{\mathrm{d}\Lambda^*}{\mathrm{d}p} = G(U_1,\delta_1)\frac{\mathrm{d}U_1}{\mathrm{d}p} - G(Q,\delta_1)\frac{\mathrm{d}Q}{\mathrm{d}p}$$

$$= -\delta_1 G(U_1, \delta_1) - G(Q, \delta_1)\frac{\mathrm{d}Q}{\mathrm{d}p}.$$

Similarly, for the second distribution, we have

$$\frac{1}{2}\frac{\mathrm{d}\Lambda^*}{\mathrm{d}p} = -\delta_2 G(U_2, \delta_2) - G(Q, \delta_2)\frac{\mathrm{d}Q}{\mathrm{d}p}.$$

Eliminating $\frac{\mathrm{d}Q}{\mathrm{d}p}$ from the system, we obtain

$$\frac{1}{2}\frac{\mathrm{d}\Lambda^*}{\mathrm{d}p} = \frac{\delta_2 G(U_2, \delta_2)G(Q, \delta_1) - \delta_1 G(U_1, \delta_1)G(Q, \delta_2)}{G(Q, \delta_2) - G(Q, \delta_1)}. \tag{5.2}$$

Substituting $U_i = \delta_i(1-p)$ into G, we have

$$\delta_i G(U_i, \delta_i) = \frac{(r_0 + \delta_i(1-p))^2}{p(1-p)\delta_i}.$$

Recalling the form of Q_i and the fact that Q_i is strictly decreasing, we have $\delta_1 > \delta_2 > U_2 > Q$, which implies $\delta_1 - Q > \delta_2 - Q > 0$ and $G(Q, \delta_1) < G(Q, \delta_2)$. Thus, the denominator of (5.2) is positive. For the numerator, we have

$$\frac{\delta_1 G(U_1, \delta_1)}{G(Q, \delta_1)} = \frac{(r_0 + \delta_1(1-p))^2}{p(1-p)\delta_1} \cdot \frac{Q(\delta_1 - Q)}{(r_0 + Q)^2} = \frac{Q}{p(1-p)(r_0+Q)^2}(r_0 + \delta_1(1-p))^2\left(1 - \frac{Q}{\delta_1}\right),$$
$$\frac{\delta_2 G(U_2, \delta_2)}{G(Q, \delta_2)} = \frac{Q}{p(1-p)(r_0+Q)^2}(r_0 + \delta_2(1-p))^2\left(1 - \frac{Q}{\delta_2}\right).$$

Because $\delta_1 > \delta_2$, it holds that $(r_0 + \delta_1(1-p))^2 > (r_0 + \delta_2(1-p))^2$ and $1 - \frac{Q}{\delta_1} > 1 - \frac{Q}{\delta_2}$. Therefore, $\frac{\delta_1 G(U_1, \delta_1)}{G(Q, \delta_1)} > \frac{\delta_2 G(U_2, \delta_2)}{G(Q, \delta_2)}$, which implies $\delta_2 G(U_2, \delta_2)G(Q, \delta_1) - \delta_1 G(U_1, \delta_1)G(Q, \delta_2) < 0$. Consequently, $\frac{\mathrm{d}\Lambda^*}{\mathrm{d}p} < 0$. Because $\Lambda(t)$ is a strictly decreasing function of t, a decrease in the required energy Λ^* corresponds to an increase in the intersection time t^*. □

Building on this analysis, we now relax the symmetry assumption to examine the individual effects. The following corollary presents the comparative statics of the intersection time with respect to the probability parameter of each investor.

Corollary 5.6. *Under the conditions of Proposition 5.3 and Proposition 5.4, we have the following assertions.*

(1) *If p_2 is fixed, t^* is strictly increasing with respect p_1.*

(2) *If p_1 is fixed, t^* is strictly decreasing with respect p_2*

Proof. We retain the notations from the proof of Proposition 5.4.

(1) Differentiating (5.1) with respect to p_1 yields

$$\frac{1}{2}\frac{\partial \Lambda^*}{\partial p_1} = -\delta_1 G(U_1, \delta_1) - G(Q, \delta_1)\frac{\partial Q}{\partial p_1},$$
$$\frac{1}{2}\frac{\partial \Lambda^*}{\partial p_1} = -G(Q, \delta_2)\frac{\partial Q}{\partial p_1}.$$

Eliminating $\frac{\partial Q}{\partial p_1}$, we obtain

$$\frac{1}{2}\frac{\partial \Lambda^*}{\partial p_1} = \frac{-\delta_1 G(U_1, \delta_1) G(Q, \delta_2)}{G(Q, \delta_2) - G(Q, \delta_1)}.$$

Because the denominator is positive and the numerator is negative, we have $\frac{\partial \Lambda^*}{\partial p_1} < 0$. Given that $\Lambda(t)$ is strictly decreasing, t^* must increase.

(2) Differentiating (5.1) with respect to p_2 yields

$$\frac{1}{2}\frac{\partial \Lambda^*}{\partial p_2} = -G(Q, \delta_1)\frac{\partial Q}{\partial p_2},$$
$$\frac{1}{2}\frac{\partial \Lambda^*}{\partial p_2} = -\delta_2 G(U_2, \delta_2) - G(Q, \delta_2)\frac{\partial Q}{\partial p_2}.$$

Eliminating $\frac{\partial Q}{\partial p_2}$, we have

$$\frac{1}{2}\frac{\partial \Lambda^*}{\partial p_2} = \frac{-\delta_2 G(U_2, \delta_2) G(Q, \delta_1)}{G(Q, \delta_1) - G(Q, \delta_2)}.$$

Because the denominator is negative while the numerator remains negative, we have $\frac{\partial \Lambda^*}{\partial p_2} > 0$, which implies that t^* must decrease. □

It is important to note that the conditions in Proposition 5.3 and Proposition 5.4 are compatible with first-order stochastic dominance. One can construct risk aversion distributions such that $\boldsymbol{R}_1 \succeq_1 \boldsymbol{R}_2$ holds while the conditions in Proposition 5.3 and 5.4 are simultaneously satisfied. For instance, the parameters in Figure 1 satisfy these conditions with $r_0 = 1$, $\delta_1 = 4$, $\delta_2 = 1$, $\tilde{\mathbb{E}}[\boldsymbol{R}_1] = 3 > 1.5 = \tilde{\mathbb{E}}[\boldsymbol{R}_2]$ and $\lambda \equiv 0.4$. Consequently, first-order stochastic dominance fails to rule out the reversal of investment magnitudes characterized by $|\pi_1(0,T)| > |\pi_2(0,T)|$ for a sufficiently large T and we can also observe the single crossing property.

However, we demonstrate that the desired monotonicity can be restored under a stronger stochastic order. Specifically, we introduce the concept of reverse hazard rate order and show that if $\boldsymbol{R_1}$ dominates $\boldsymbol{R_2}$ in this order, the intuition holds, i.e., $|\pi_1(t)| \leq |\pi_2(t)|$.

Definition 5.7. *Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be two random variables with cumulative distribution functions $F_{\boldsymbol{R_1}}$ and $F_{\boldsymbol{R_2}}$, respectively. We say that $\boldsymbol{R_1}$ dominates $\boldsymbol{R_2}$ in the reverse hazard rate order, denoted as $\boldsymbol{R_1} \succeq_{rh} \boldsymbol{R_2}$, if the ratio*

$$\frac{F_{\boldsymbol{R_2}}(\gamma)}{F_{\boldsymbol{R_1}}(\gamma)}$$

is decreasing in γ for all γ in the union of the supports of $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ where $F_{\boldsymbol{R_1}}(\gamma) > 0$.

The following proposition compares the equilibrium strategies of two investors ranked by the reverse hazard rate order.

Proposition 5.8. *Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be two non-negative random variables with $\tilde{\mathbb{E}}[\boldsymbol{R_i}] < +\infty$ ($i = 1, 2$), such that $\boldsymbol{R_1} \succeq_{rh} \boldsymbol{R_2}$ and their distributions are distinct. Let $\pi_i(t) = (\sigma^\top)^{-1} a_i(t)$ be the corresponding deterministic equilibrium strategies. For any $t \in [0, T)$, the magnitude of $a_i(t)$ satisfies*

$$|a_1(t)| \leq |a_2(t)|.$$

Moreover, if $\lambda(t) \neq 0$, the inequality is strict. As a result, the investor with higher risk aversion adopts a less aggressive equilibrium strategy. i.e., $|\pi_1(t)| \leq |\pi_2(t)|$.

Proof. Because $h_1(x) < h_2(x)$ for all positive s, it follows that $H_1(y) > H_2(y)$ for all $y > 0$. This implies that the inverse functions satisfy $H_2^{-1}(z) < H_2^{-1}(z)$ for all $z > 0$. From Theorem 3.2 (1), we have $v_i(t) = H_i^{-1}(\Lambda(t))$, where $\Lambda(t) = \int_t^T |\lambda(s)|^2 ds$ and $v_i(t) = \int_t^T |a_i(s)|^2 \mathrm{d}s$. Consequently, monotonicity of the inverse functions implies $v_1(t) \leq v_2(t)$. By (2.5), we have $|a_i(t)| = h(v_i(t), \boldsymbol{R_i})|\lambda(t)|$. From Proposition A.6, we know that $h_1(v_1(t)) < h_2(v_1(t))$. Because $v_1(t) \leq v_2(t)$, it holds that $h_2(v_1(t)) \leq h_2(v_2(t))$. Combining these relations yields $h_1(v_1(t)) < h_2(v_2(t))$. Multiplying by $|\lambda(t)|$, we obtain $|a_1(t)| \leq |a_2(t)|$. If $\lambda(t) \neq 0$, the strict inequality $|a_1(t)| < |a_2(t)|$ is preserved. □

For the case where the expected risk aversion is infinite, assume that the optimal equilibria exist for both $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$. Let $a_i(t)$ denote the deterministic coefficient corresponding to the optimal equilibrium strategy for $\boldsymbol{R}_i$ ($i = 1, 2$). Note that $\varphi(0) = \min\{t \in [0, T] \mid \Lambda(t) = 0\}$ is determined solely by the market parameters and is therefore identical for both investors. Consequently, we can establish the comparative result by examining the magnitudes of $a_i(t)$ on the intervals $[0, \varphi(0))$ and $[\varphi(0), T]$ separately. Specifically, on the interval $[0, \varphi(0))$, the analysis parallels the finite expectation case derived in Proposition 5.8, and thus the monotonicity of the equilibrium strategies is preserved. On the interval $[\varphi(0), T]$, the definition of $\varphi(0)$ implies that $\lambda(t) = 0$ almost everywhere. Consequently, the equilibrium strategies for both investors vanish, trivially satisfying the comparison inequality. Based on this decomposition, we have the following corollary.

Corollary 5.9. *Let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be two non-negative random variables with $\tilde{\mathbb{E}}[\boldsymbol{R}_i] = \infty$ ($i = 1, 2$), such that $\boldsymbol{R}_1 \succeq_{rh} \boldsymbol{R}_2$ and their distributions are distinct. Assume that the optimal equilibria exist for both investors (i.e., $H_i(\infty) > \Lambda(0)$, $i = 1, 2$). Let $\pi_i(t)$ be the corresponding deterministic optimal equilibrium strategies. Then, for any $t \in [0, T)$, the magnitude of $\pi_i(t)$ satisfies*

$$|\pi_1(t)| \leq |\pi_2(t)|.$$

Moreover, if $\lambda(t) \neq 0$, the inequality is strict.

While the previous results focus on comparing distinct distributions ranked by stochastic orders, it is also of theoretical interest to understand the effect of preference aggregation. We now investigate the equilibrium strategy when the risk aversion parameter is formed by a convex combination (i.e., a weighted average) of independent and identically distributed components.

Proposition 5.10. *Let $\boldsymbol{R}_1, \cdots, \boldsymbol{R}_n$ be i.i.d. non-negative random variables with $\tilde{\mathbb{E}}[\boldsymbol{R_i}] < +\infty$ $(i = 1, 2, \cdots, n)$ and $\boldsymbol{R} = \sum_{i=1}^{n} w_i \boldsymbol{R}_i$ be a convex combination with $w_i \in (0,1)$ and $\sum_{i=1}^{n} w_i = 1$. Let $a(t)$ and $a_i(t)$ denote the risk exposure vector associated to the risk aversion $\boldsymbol{R}$ and $\boldsymbol{R}_i$, respectively. Then we have*

$$|a(t)| \leq |a_1(t)| \quad \textit{for all } t \in [0, T).$$

If $\boldsymbol{R}_i$ is non-degenerate and $\lambda(t) \neq 0$, the inequality is strict.

Proof. Let $h(x)$ and $v(t)$ (resp. $h_i(x)$ and $v_i(t)$) be the functions associated to $\boldsymbol{R}$ (resp. $\boldsymbol{R_i}$). From the definition of $\boldsymbol{R}$, we have

$$\frac{1}{h(x)} = \sum_{j=1}^{n} w_j \frac{\tilde{\mathbb{E}}[\boldsymbol{R}_j e^{-\frac{1}{2} w_j \boldsymbol{R}_j x}]}{\tilde{\mathbb{E}}[e^{-\frac{1}{2} w_j \boldsymbol{R}_j x}]} \tag{5.3}$$

Because $\boldsymbol{R}_j$ are identically distributed to $\boldsymbol{R}_1$, each fraction in the sum is exactly $\frac{1}{h_1(w_j x)}$. Thus (5.3) is actually

$$\frac{1}{h(x)} = \sum_{j=1}^{n} w_j \frac{1}{h_1(w_j x)}$$

According to Proposition A.6, we know that $\frac{1}{h_1(x)}$ is decreasing in x. Because $w_j \in (0,1)$ and $x > 0$, we have $w_j x < x$ and $\frac{1}{h_1(w_j x)} \geq \frac{1}{h_1(x)}$ and

$$\frac{1}{h(x)} = \sum_{j=1}^{n} w_j \frac{1}{h_1(w_j x)} \geq \sum_{j=1}^{n} w_j \frac{1}{h_1(x)} = \frac{1}{h_1(x)}$$

This implies $h(x) \leq h_1(x)$. Analogous to the proof in Proposition 5.8, we have $v(t) \leq v_1(t)$. Finally, because $h(x)$ is increasing in x, we have

$$|a(t)| = h(v(t))|\lambda(t)| \leq h(v_1(t))|\lambda(t)| \leq h_1(v_1(t))|\lambda(t)| = |a_1(t)|$$

If $\boldsymbol{R}_i$ is non-degenerate and $\lambda(t) \neq 0$, h is strictly increasing, and all inequalities in the above proof become strict. Thus we have $|a(t)| < |a_1(t)|$ □

It is important to note that the inequality established above relies critically on the i.i.d. assumption. The following example demonstrates that if we retain independence but relax the identical distribution assumption, the equilibrium strategy for the convex combination need not be bounded by the strategies of the individual components.

Example 5.11. *Assume that there is one stock whose market price of risk is constant, $\lambda(\cdot) \equiv \lambda = 0.5$. The time horizon $T = 50$. The risk-aversion distributions of the two investors are:*

Investor 1: $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 0.1) = 0.2$, $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 8) = 0.8$.

Investor 2: $\tilde{\mathbb{P}}(\boldsymbol{R}_2 = 1.5) = 1$.

Assume that $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ are independent. We consider a composite investor with risk aversion $\boldsymbol{R} = 0.5\boldsymbol{R}_1 + 0.5\boldsymbol{R}_2$.

Figure 2 displays the comparative evolution of $|a(t)|$, $|a_1(t)|$ and $|a_2(t)|$. The red-shaded Reversal Region highlights that the investor with the mixed risk aversion $\boldsymbol{R}$ adopts a strictly more aggressive position than both Investor 1 and Investor 2 when the time to maturity is sufficiently long. This counterexample shows that without the i.i.d. assumption, the magnitudes of equilibrium investment strategies for a convex combination of risk aversions need not be bounded by the strategies of the individual components.

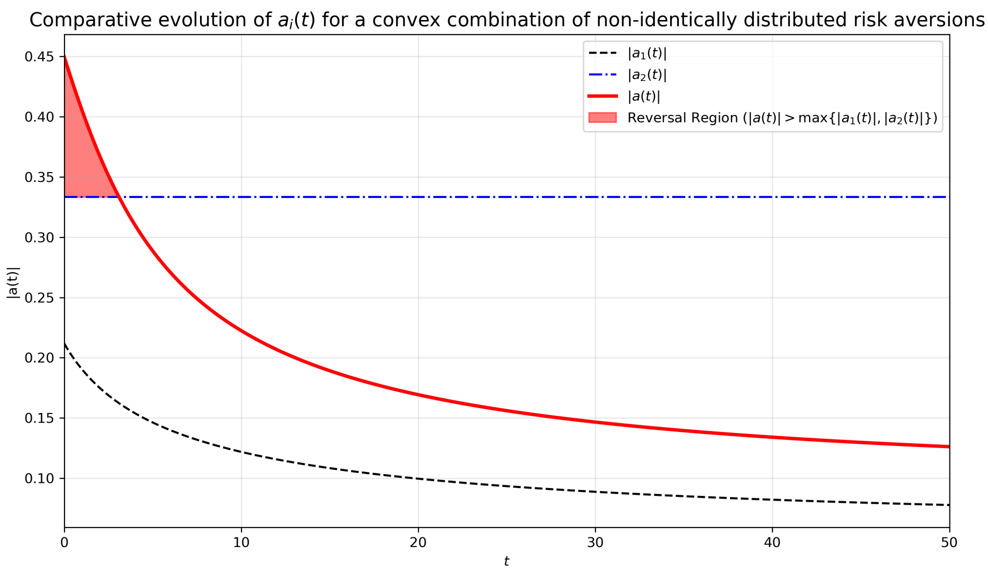

Figure 2: Comparative evolution of $a(t)$ for a convex combination of risk aversions. The black dashed line corresponds to Investor 1, with distribution $\tilde{\mathbb{P}}(\boldsymbol{R}_1 = 0.1) = 0.2, \tilde{\mathbb{P}}(\boldsymbol{R}_1 = 8.0) = 0.8$; the blue dash-dotted line corresponds to Investor 2, with degenerate distribution $\tilde{\mathbb{P}}(\boldsymbol{R}_2 = 1.5) = 1.0$. The red solid line corresponds to the mixed investor with risk aversion $\boldsymbol{R} = 0.5\boldsymbol{R}_1 + 0.5\boldsymbol{R}_2$. The red-shaded Reversal Region indicates where $|a(t)| > \max\{|a_1(t)|, |a_2(t)|\}$. Parameters: $\lambda = 0.5$, horizon $T = 50$.

We now specifically investigate the aggregation of i.i.d. risk aversions. Distinct from the general result above, the following proposition utilizes the i.i.d. property to prove that the equilibrium strategy converges monotonically to its limit.

Proposition 5.12. *Suppose that $\{\boldsymbol{R}_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed non-negative random variables with $\tilde{\mathbb{E}}[\boldsymbol{R}_i] = \mu \in (0, +\infty)$ $(i = 1, 2, \cdots,)$. Let $S_n = \dfrac{1}{n}\sum_{i=1}^{n} \boldsymbol{R}_i$*

denote the sample mean risk aversion and $a_n(t)$ denote the risk exposure vector associated to the risk aversion S_n. Then, as $n \to \infty$, we have

$$|a_n(t)| \searrow \frac{|\lambda(t)|}{\mu} \quad \textit{for all } t \in [0, T). \tag{5.4}$$

Proof. Let $h_n(x)$ and $v_n(t)$ be the functions associated to S_n. First, substituting $S_n = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{R}_i$ into the definition of $h_n(x)$ and utilizing the i.i.d. property of $\{\boldsymbol{R}_i\}$, we obtain

$$h_n(x) = \frac{\tilde{\mathbb{E}}\left[\prod_{i=1}^{n} e^{-\frac{x}{2n}\boldsymbol{R}_i}\right]}{\frac{1}{n}\sum_{j=1}^{n}\tilde{\mathbb{E}}\left[\boldsymbol{R}_j e^{-\frac{x}{2n}\boldsymbol{R}_j}\prod_{k\neq j} e^{-\frac{x}{2n}\boldsymbol{R}_k}\right]} = \frac{(\tilde{\mathbb{E}}\left[e^{-\frac{x}{2n}\boldsymbol{R}_1}\right])^n}{\tilde{\mathbb{E}}\left[\boldsymbol{R}_1 e^{-\frac{x}{2n}\boldsymbol{R}_1}\right](\tilde{\mathbb{E}}\left[e^{-\frac{x}{2n}\boldsymbol{R}_1}\right])^{n-1}}$$

$$= \frac{\tilde{\mathbb{E}}\left[e^{-\frac{1}{2}\boldsymbol{R}_1(\frac{x}{n})}\right]}{\tilde{\mathbb{E}}\left[\boldsymbol{R}_1 e^{-\frac{1}{2}\boldsymbol{R}_1(\frac{x}{n})}\right]} = h_1\left(\frac{x}{n}\right).$$

According to Proposition A.6, $h_1(x)$ is increasing. For any fixed $x \geq 0$, the sequence $\frac{x}{n}$ is decreasing in n, implying that $h_n(x)$ is decreasing in n. Furthermore, because $H_n(y) = \int_0^y h_n^{-2}(z)dz$ and $v_n(t) = H_n^{-1}(\Lambda(t))$, the monotonicity of h_n implies that $v_n(t)$ is strictly decreasing in n. Then we have

$$h_{n+1}(v_{n+1}(t)) \leq h_n(v_{n+1}(t)) \leq h_n(v_n(t)),$$

equivalently, we have

$$|a_{n+1}(t)| \leq |a_n(t)|, \quad \forall t \in [0, T).$$

Now we determine the limit of $|a_n(t)|$. By the Strong Law of Large Numbers, $S_n \to \mu$ almost surely as $n \to \infty$. This implies that S_n converges in distribution to the degenerate random variable $R \equiv \mu$. Moreover, because $\tilde{\mathbb{E}}[S_n] = \mu$ for all n, the assumptions of Proposition 3.8 are satisfied. Thus we conclude that

$$\lim_{n\to\infty} |a_n(t)| = h(v(t))|\lambda(t)| = \frac{|\lambda(t)|}{\mu}.$$

Combining this with the monotonicity established above, we obtain (5.4). □

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Appendix A Properties of the function h

In this appendix, we present some properties of the function h defined in (2.6).

Lemma A.1 (Monotonicity)**.** *The function h is increasing in $[0, \infty)$. The monotonicity is strict unless $\boldsymbol{R}$ is a constant.*

Proof. By direct differentiation, we have

$$h'(x) = \frac{1}{2} \frac{\tilde{\mathbb{E}}\left[e^{-\frac{1}{2}\boldsymbol{R}x}\right] \tilde{\mathbb{E}}\left[\boldsymbol{R}^2 e^{-\frac{1}{2}\boldsymbol{R}x}\right] - \left(\tilde{\mathbb{E}}\left[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}\right]\right)^2}{\left(\tilde{\mathbb{E}}\left[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}\right]\right)^2} \quad \text{for all } x > 0.$$

By Hölder's inequality,

$$\left(\tilde{\mathbb{E}}\left[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}\right]\right)^2 = \left(\tilde{\mathbb{E}}\left[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}\right] \cdot \tilde{\mathbb{E}}\left[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}\right]\right) \le \tilde{\mathbb{E}}\left[e^{-\frac{1}{2}\boldsymbol{R}x}\right] \tilde{\mathbb{E}}\left[\boldsymbol{R}^2 e^{-\frac{1}{2}\boldsymbol{R}x}\right]. \tag{A.1}$$

Hence, $h' \ge 0$ and h is increasing. The inequality in (A.1) becomes an equality if and only if $\tilde{\mathbb{E}}\left[e^{-\frac{1}{2}\boldsymbol{R}x}\right] \boldsymbol{R}^2 e^{-\frac{1}{2}\boldsymbol{R}x} = e^{-\frac{1}{2}\boldsymbol{R}x}\tilde{\mathbb{E}}\left[\boldsymbol{R}^2 e^{-\frac{1}{2}\boldsymbol{R}x}\right]$, which is equivalent to $\boldsymbol{R}$ following a single-point distribution. Therefore, h is strictly increasing unless $\boldsymbol{R}$ is a constant. ☐

Lemma A.2 (Asymptotic Analysis)**.** *Let $\boldsymbol{R}$ be a non-negative random variable, and let $r_0 \triangleq$ essinf $\boldsymbol{R}$. Then,*

$$\lim_{x \to \infty} h(x) = \frac{1}{r_0},$$

where we adopt the convention that $\frac{1}{0} = \infty$.

Proof. Define $\boldsymbol{X} \triangleq \boldsymbol{R} - r_0$, then

$$\frac{1}{h(x)} = \frac{\tilde{\mathbb{E}}[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}]}{\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}x}]} = \frac{\tilde{\mathbb{E}}[(r_0 + \boldsymbol{X})e^{-\frac{1}{2}(r_0+\boldsymbol{X})x}]}{\tilde{\mathbb{E}}[e^{-\frac{1}{2}(r_0+\boldsymbol{X})x}]} = r_0 + Q(x), \tag{A.2}$$

where $Q(x) \triangleq \frac{\tilde{\mathbb{E}}[\boldsymbol{X}e^{-\frac{1}{2}\boldsymbol{X}x}]}{\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{X}x}]}$. For any arbitrary $\varepsilon > 0$, because essinf $\boldsymbol{X} = 0$, there exists a probability mass $p_\varepsilon \triangleq \tilde{\mathbb{P}}(0 \le \boldsymbol{X} \le \varepsilon) > 0$. Therefore,

$$Q(x) \le \frac{\tilde{\mathbb{E}}[\boldsymbol{X}e^{-\frac{1}{2}\boldsymbol{X}x}\mathbb{I}_{\{\boldsymbol{X}\le\varepsilon\}} + \tilde{\mathbb{E}}[\boldsymbol{X}e^{-\frac{1}{2}\boldsymbol{X}x}\mathbb{I}_{\{\boldsymbol{X}>\varepsilon\}}]}{\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{X}x}\mathbb{I}_{\{\boldsymbol{X}\le\varepsilon\}}]} \le \varepsilon + \frac{\tilde{\mathbb{E}}[\boldsymbol{X}e^{-\frac{1}{2}\boldsymbol{X}x}\mathbb{I}_{\{\boldsymbol{X}>\varepsilon\}}]}{p_\varepsilon e^{-\frac{1}{2}\varepsilon x}} = \varepsilon + \frac{\tilde{\mathbb{E}}[\boldsymbol{X}e^{-\frac{1}{2}(\boldsymbol{X}-\varepsilon)x}\mathbb{I}_{\{\boldsymbol{X}>\varepsilon\}}]}{p_\varepsilon}.$$

Note that for any $x \ge 1$, $\boldsymbol{X}e^{-\frac{1}{2}(\boldsymbol{X}-\varepsilon)x}\mathbb{I}_{\{\boldsymbol{X}>\varepsilon\}}$ is dominated by $\boldsymbol{X}e^{-\frac{1}{2}(\boldsymbol{X}-\varepsilon)}\mathbb{I}_{\{\boldsymbol{X}>\varepsilon\}}$, which is integrable regardless of whether $\tilde{\mathbb{E}}[\boldsymbol{X}]$ is finite. Hence, by the dominated convergence theorem, $\limsup_{x\to\infty} Q(x) \le \varepsilon$. Because ε is arbitrary, we have $\lim_{x\to\infty} Q(x) = 0$. By (A.2), the convergence result for h follows immediately. ☐

Lemma A.3 (Uniform Convergence)**.** *Let $\{\boldsymbol{R}_n\}_{n\ge1}$ be a sequence of nonnegative random variables that converges in distribution to $\boldsymbol{R}$ and $\lim_{n\to\infty} \tilde{\mathbb{E}}[\boldsymbol{R}_n] = \tilde{\mathbb{E}}[\boldsymbol{R}] \in (0, \infty)$. Then, on every compact subset of $[0, \infty)$, the sequence $\{h_{\boldsymbol{R}_n}\}_{n\ge1}$ converges uniformly to $h_{\boldsymbol{R}}$.*

Proof. Fix a compact subset $\mathcal{K} \subset [0, \infty)$.

We first show that $\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}_n x}] \to \tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}x}]$ uniformly on $\mathcal{K}$. Observe that

$$\left| \frac{\mathrm{d}\left(\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}_n x}]\right)}{\mathrm{d}x} \right| = \left| -\frac{1}{2}\tilde{\mathbb{E}}[\boldsymbol{R}_n e^{-\frac{1}{2}\boldsymbol{R}_n x}] \right| \leq \frac{1}{2}\tilde{\mathbb{E}}[\boldsymbol{R}_n] \quad \text{for all } x \geq 0.$$

Because $\tilde{\mathbb{E}}[\boldsymbol{R}_n] \to \tilde{\mathbb{E}}[\boldsymbol{R}]$, the sequence $\{\tilde{\mathbb{E}}[\boldsymbol{R}_n]\}_{n\geq 1}$ is bounded. Hence, the family of functions $\{\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}_n x}]\}_{n\geq 1}$ is equicontinuous on $\mathcal{K}$. Moreover, because $\boldsymbol{R}_n \xrightarrow{d} \boldsymbol{R}$ and the function $r \mapsto e^{-\frac{1}{2}rx}$ is bounded and continuous, we have $\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}_n x}] \to \tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}x}]$ for any $x \geq 0$. It follows that $\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}_n x}] \to \tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{R}x}]$ uniformly on $\mathcal{K}$; see, e.g., Rudin (1976, Exercise 7.16).

Now we show the uniform convergence of $\{\tilde{\mathbb{E}}[\boldsymbol{R}_n e^{-\frac{1}{2}\boldsymbol{R}_n x}]\}_{n\geq 1}$ on $\mathcal{K}$. By $\boldsymbol{R}_n \xrightarrow{d} \boldsymbol{R}$ and $\tilde{\mathbb{E}}[\boldsymbol{R}_n] \to \tilde{\mathbb{E}}[\boldsymbol{R}] < \infty$, we know that the family $\{\boldsymbol{R}_n\}_{n\geq 1}$ is uniformly integrable. Consequently, for any $\varepsilon > 0$, there exists $M > 0$ such that $F_{\boldsymbol{R}}$ is continuous at M, $\sup_n \tilde{\mathbb{E}}[\boldsymbol{R}_n \mathbb{I}_{\{\boldsymbol{R}_n > M\}}] < \dfrac{\varepsilon}{3}$, and $\tilde{\mathbb{E}}[\boldsymbol{R}\mathbb{I}_{\{\boldsymbol{R}>M\}}] < \frac{\varepsilon}{3}$. Let $f_n^M(x) \triangleq \tilde{\mathbb{E}}[\boldsymbol{R}_n e^{-\frac{1}{2}\boldsymbol{R}_n x}\mathbb{I}_{\{\boldsymbol{R}_n \leq M\}}]$. Then f_n^M is continuously differentiable and its derivative is uniformly bounded by $\frac{1}{2}M^2$, which implies that the family $\{f_n^M\}_{n\geq 1}$ is equicontinuous on $\mathcal{K}$. Furthermore, because $\boldsymbol{R}_n \xrightarrow{d} \boldsymbol{R}$, we have the pointwise convergence $f_n^M(x) \to f^M(x) \triangleq \tilde{\mathbb{E}}[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}\mathbb{I}_{\{\boldsymbol{R}\leq M\}}]$; see, e.g., Föllmer and Schied (2016, Theorem A.43). As a result, f_n^M converges uniformly to f^M on $\mathcal{K}$. Hence, for all sufficiently large n,

$$\sup_{x\in\mathcal{K}} |f_n^M(x) - f^M(x)| < \frac{\varepsilon}{3}.$$

Observe that

$$|\tilde{\mathbb{E}}[\boldsymbol{R}_n e^{-\frac{1}{2}\boldsymbol{R}_n x}] - \tilde{\mathbb{E}}[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}]| \leq |f_n^M(x) - f^M(x)| + \tilde{\mathbb{E}}[\boldsymbol{R}_n \mathbb{I}_{\{\boldsymbol{R}_n > M\}}] + \tilde{\mathbb{E}}[\boldsymbol{R}\mathbb{I}_{\{\boldsymbol{R}>M\}}].$$

Therefore, for n large enough,

$$\sup_{x\in\mathcal{K}} \left| \tilde{\mathbb{E}}[\boldsymbol{R}_n e^{-\frac{1}{2}\boldsymbol{R}_n x}] - \tilde{\mathbb{E}}[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}] \right| < \varepsilon,$$

which implies the uniform convergence of $\{\tilde{\mathbb{E}}[\boldsymbol{R}_n e^{-\frac{1}{2}\boldsymbol{R}_n x}]\}_{n\geq 1}$ on $\mathcal{K}$.

Finally, because $\tilde{\mathbb{E}}[\boldsymbol{R}] > 0$, the function $\tilde{\mathbb{E}}[\boldsymbol{R}e^{-\frac{1}{2}\boldsymbol{R}x}]$ is bounded away from zero on $\mathcal{K}$. Therefore, $\{h_{\boldsymbol{R}_n}\}_{n\geq 1}$ converges uniformly to $h_{\boldsymbol{R}}$ on $\mathcal{K}$. ☐

The reverse hazard rate order can be characterized via expectation ratios and connected to first-order stochastic dominance under weighted measures, as established by Capéraà (1988) (see also Shaked and Shanthikumar (2007, Theorem 1.B.50)) and Wang and Lehrer (2024).

Lemma A.4 (Capéraà (1988))**.** *Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be two random variables. Then $R_1 \succeq_{rh} R_2$ if and only if*

$$\frac{\tilde{\mathbb{E}}\left[u(\boldsymbol{R_1})w(\boldsymbol{R_1})\right]}{\tilde{\mathbb{E}}[w(\boldsymbol{R_1})]} \geq \frac{\tilde{\mathbb{E}}[u(\boldsymbol{R_2})w(\boldsymbol{R_2})]}{\tilde{\mathbb{E}}[w(\boldsymbol{R_2})]},$$

for all functions u and w for which the expectations exist, such that w is positive, continuous, and decreasing, and u is continuous and strictly increasing.

Lemma A.5 (Wang and Lehrer (2024))**.** *Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be two random variables with cumulative distribution functions $F_{\boldsymbol{R_1}}$ and $F_{\boldsymbol{R_2}}$, respectively. Assume that $\boldsymbol{R_1} \succeq_{rh} \boldsymbol{R_2}$. Let $w(\cdot)$ be a positive, continuous, and strictly decreasing function. Define the weighted probability measures $\tilde{F}_i$ associated with $\boldsymbol{R}_i$ for $i = 1, 2$ by*

$$d\tilde{F}_i(x) = \frac{w(x)dF_{\boldsymbol{R_i}}}{\tilde{\mathbb{E}}\left[w(\boldsymbol{R_i})\right]}. \tag{A.3}$$

Then, $\boldsymbol{R}_1$ dominates $\boldsymbol{R}_2$ under the weighted measures in the sense of first-order stochastic dominance, i.e.,

$$\tilde{F}_1 \succeq_1 \tilde{F}_2.$$

We are now ready to establish the monotonicity of the function h defined in (2.6) with respect to the risk aversion distribution in the sense of the reverse hazard rate order.

Lemma A.6. *For fixed $x > 0$, $h(x)$ is strictly decreasing with respect to $\boldsymbol{R}$ in the sense of the reverse-hazard-rate order among distinct distributions.*

Proof. Fix $x > 0$. Let $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$ be two non-negative random variables such that $\boldsymbol{R_1} \succeq_{rh} \boldsymbol{R_2}$ and their distributions are distinct. For notational simplicity, in the sequel, let $h_i(\cdot)$ and $H_i(\cdot)$ denote the function $h(\cdot)$ and the function defined in Theorem 3.2 corresponding to the random variable $\boldsymbol{R}_i$ for $i = 1, 2$, respectively. According to Lemma A.4, let $u(\gamma) = \gamma$ and $w(\gamma) = \exp(-\frac{1}{2}\gamma x)$. Note that $u(\gamma)$ is a strictly increasing function, and for $x > 0$, $w(\gamma)$ is a positive, continuous, and strictly decreasing function of γ. Substituting them into this inequality, we have

$$\frac{\tilde{\mathbb{E}}[\boldsymbol{R_1}\exp(-\frac{1}{2}\boldsymbol{R_1}x)]}{\tilde{\mathbb{E}}[\exp(-\frac{1}{2}\boldsymbol{R_1}x)]} \geq \frac{\tilde{\mathbb{E}}[\boldsymbol{R_2}\exp(-\frac{1}{2}\boldsymbol{R_2}x)]}{\tilde{\mathbb{E}}[\exp(-\frac{1}{2}\boldsymbol{R_2}x)]}.$$

Moreover, assume that the equality holds. Define the weighted probability measures $\tilde{F}_i$ for $i = 1, 2$ by (A.3). The given condition is equivalent to $\tilde{\mathbb{E}}_{\tilde{F}_1}[\boldsymbol{R_1}] = \tilde{\mathbb{E}}_{\tilde{F}_2}[\boldsymbol{R_1}]$. From Lemma A.5, we have $\tilde{F}_1 \succeq_1 \tilde{F}_2$ and consequently, $d\tilde{F}_1(\gamma) = d\tilde{F}_2(\gamma)$. Given that $w(\gamma)$ is positive, this implies $\frac{d\tilde{F}_1}{\tilde{\mathbb{E}}\left[w(\boldsymbol{R_1})\right]} = \frac{d\tilde{F}_2}{\tilde{\mathbb{E}}\left[w(\boldsymbol{R_2})\right]}$. Integrating both sides over the entire support and using the property $\int d\tilde{F}_i(\gamma) = 1$, we obtain $1 = \frac{\tilde{\mathbb{E}}\left[w(\boldsymbol{R_1})\right]}{\tilde{\mathbb{E}}\left[w(\boldsymbol{R_2})\right]}$, which implies $\tilde{\mathbb{E}}\left[w(\boldsymbol{R_1})\right] = \tilde{\mathbb{E}}\left[w(\boldsymbol{R_2})\right]$. Substituting this back yields $dF_{\boldsymbol{R_1}}(\gamma) = dF_{\boldsymbol{R_2}}(\gamma)$, confirming that $F_{\boldsymbol{R_1}}(\gamma) = F_{\boldsymbol{R_2}}(\gamma)$. This contradicts the different distributions of $\boldsymbol{R_1}$ and $\boldsymbol{R_2}$. Thus, we have

$$\frac{\tilde{\mathbb{E}}[\boldsymbol{R_1}\exp(-\frac{1}{2}\boldsymbol{R_1}x)]}{\tilde{\mathbb{E}}[\exp(-\frac{1}{2}\boldsymbol{R_1}x)]} > \frac{\tilde{\mathbb{E}}[\boldsymbol{R_2}\exp(-\frac{1}{2}\boldsymbol{R_2}x)]}{\tilde{\mathbb{E}}[\exp(-\frac{1}{2}\boldsymbol{R_2}x)]},$$

and equivalently, $h_1(x) < h_2(x)$. □

Appendix B Proofs

B.1 Proof of Proposition 3.8

(1) Assume that the sequence $\{\Lambda_n\}_{n\geq 1}$ is strictly increasing and $\Lambda_n \nearrow H(\infty)$. Because the convergence is strictly increasing toward $H(\infty)$, we have $\Lambda_n < H(\infty)$ for all $n \geq 1$, ensuring the existence of the equilibrium. Thus, for each n, $v_n(0)$ is well-defined and uniquely determined by $v_n(0) = H^{-1}(\Lambda_n)$. Consequently, the convergence $\Lambda_n \to H(\infty)$ implies that $\lim_{n\to\infty} v_n(0) = \infty$. By Theorem 3.2, we have $a_n(0) = h(v_n(0))\lambda_0$. Using the asymptotic property of h established in Lemma A.2, namely $\lim_{x\to\infty} h(x) = \dfrac{1}{r_0}$, we obtain

$$\lim_{n\to\infty} a_n(0) = \lambda_0 \lim_{v_n(0)\to\infty} h(v_n(0)) = \frac{\lambda_0}{r_0}.$$

Furthermore, because $\{\Lambda_n\}$ is strictly increasing, $\{v_n(0)\}$ is strictly increasing. Combined with the monotonicity of h (see Lemma A.1), it follows that $|a_n(0)| = |\lambda_0| h(v_n(0))$ converges monotonically increasing to $\frac{|\lambda_0|}{r_0}$.

(2) Assume that the sequence $\{\Lambda_n\}_{n\geq 1}$ is strictly decreasing and $\Lambda_n \searrow 0$. Because $H(\infty) > 0$, and $\Lambda_n \searrow 0$, there exists an integer N such that for all $n > N$, $\Lambda_n < H(\infty)$. Thus, the equilibrium exists for all sufficiently large n. For such n, we have $v_n(0) = H^{-1}(\Lambda_n)$. Because $H(0) = 0$ and H^{-1} is continuous, we have

$$\lim_{n\to\infty} v_n(0) = H^{-1}(0) = 0.$$

Recall that $h(0) = \frac{1}{\tilde{\mathbb{E}}[\boldsymbol{R}]}$. Taking the limit of $a_n(0)$, we have

$$\lim_{n\to\infty} a_n(0) = \lambda_0 \lim_{v_n(0)\to 0} h(v_n(0)) = \frac{\lambda_0}{\tilde{\mathbb{E}}[\boldsymbol{R}]}.$$

Finally, because $\{\Lambda_n\}$ is strictly decreasing, $\{v_n(0)\}$ is strictly decreasing. The strict monotonicity of h implies that $|a_n(0)|$ converges monotonically decreasing to $\frac{|\lambda_0|}{\tilde{\mathbb{E}}[\boldsymbol{R}]}$.

B.2 Proof of Theorem 3.11

For notational simplicity, let $h_n = h_{\boldsymbol{R}_n}$, $H_n = H_{\boldsymbol{R}_n}$, and $v_n = v_{\boldsymbol{R}_n}$ $(n \geq 1)$. By Lemma A.3, the sequence $\{h_n\}_{n\geq 1}$ converges uniformly to h on every compact subset of $[0, \infty)$. Because the limit function h is strictly positive, it follows that $\{H_n\}_{n\geq 1}$ converges uniformly to H on every compact subset of $[0, \infty)$ as well. By Lemma B.1 below, the sequence of the inverse functions $\{H_n^{-1}\}_{n\geq 1}$ converge uniformly to H^{-1} on $[0, \Lambda(0)]$.

We now see the uniform convergence of $\{v_n\}_{n\geq 1}$ to v on $[0, T]$:

$$\sup_{t\in[0,T]} |v_n(t) - v(t)| = \sup_{t\in[0,T]} |H_n^{-1}(\Lambda(t)) - H^{-1}(\Lambda(t))| \leq \sup_{y\in[0,\Lambda(0)]} |H_n^{-1}(y) - H^{-1}(y)| \to 0.$$

Next, we show the uniform convergence of the composite sequence $h_n(v_n(\cdot))$. Indeed, by the triangle inequality,

$$\sup_{t\in[0,T]} |h_n(v_n(t)) - h(v(t))| \le \sup_{t\in[0,T]} |h_n(v_n(t)) - h(v_n(t))| + \sup_{t\in[0,T]} |h(v_n(t)) - h(v(t))|. \tag{B.1}$$

Let $\mathcal{K} \triangleq \left[0, \max_{t\in[0,T]} |v(t)| + 1\right]$. Then $v_n(t) \in \mathcal{K}$ for all $t \in [0,T]$ and all sufficiently large n. Therefore, the uniform convergence of $\{h_n\}_{n\ge1}$ on $\mathcal{K}$ implies $\sup_{t\in[0,T]} |h_n(v_n(t)) - h(v_n(t))| \to 0$. Moreover, h is continuous and therefore uniformly continuous on the compact set $\mathcal{K}$, which combined with the fact that $\sup_{t\in[0,T]} |v_n(t) - v(t)| \to 0$ implies $\sup_{t\in[0,T]} |h(v_n(t)) - h(v(t))| \to 0$. Consequently, by (B.1), we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} |h_n(v_n(t)) - h(v(t))| = 0. \tag{B.2}$$

Finally, recalling that $a_n(t) = h_n(v_n(t))\lambda(t)$ and $a(t) = h(v(t))\lambda(t)$, we get from (B.2) that $\lim_{n\to\infty} a_n(t) = a(t)$ for every $t \in [0,T)$. Furthermore, if $\lambda \in L^\infty([0,T])$, then the above convergence is uniform on $[0,T)$.

Lemma B.1. *Let H and H_n $(n \ge 1)$ be strictly increasing and continuous functions defined on $[0,\infty)$ with $H(0) = H_n(0) = 0$ for all $n \ge 1$. Suppose that $\{H_n\}_{n\ge1}$ converges uniformly to H on every compact subset of $[0,\infty)$. Then, for any bounded interval $[0,z] \subset [0, H(\infty))$, there exists an integer N such that for all $n \ge N_0$, the inverse function H_n^{-1} is well-defined on $[0,z]$, and the sequence $\{H_n^{-1}\}_{n\ge N_0}$ converges uniformly to H^{-1} on $[0,z]$.*

Proof. Fix $[0,z] \subset [0,H(\infty))$ and a constant $M > H^{-1}(z)$. By the uniform convergence of $\{H_n\}_{n\ge1}$ on $[0,M]$, there exists $N_0 \ge 1$ such that $H_n(M) > z$ for all $n \ge N_0$. Because each H_n is strictly increasing and continuous with $H_n(0) = 0$, it follows that $[0,z] \subset [0, H_n(M)] = H_n([0,M])$ for all $n \ge N_0$. Thus, for all $n \ge N_0$, the inverse function H_n^{-1} is well-defined on $[0,z]$, and moreover, for any $x \in [0,z]$, we have $H_n^{-1}(x) \in [0,M]$ and $H^{-1}(x) \in [0,M]$.

Next, because H^{-1} is continuous on the compact interval $[0, H(M)]$, it is uniformly continuous. Thus, for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x', x'' \in [0, H(M)]$ with $|x' - x''| < \delta$, we have

$$|H^{-1}(x') - H^{-1}(x'')| < \varepsilon. \tag{B.3}$$

By uniform convergence of $H_n \to H$ on $[0,M]$, there exists $N \ge N_0$ such that for all $n \ge N$ and all $y \in [0,M]$,

$$|H_n(y) - H(y)| < \delta.$$

For any $x \in [0,z]$, let $x_n = H_n^{-1}(x)$. Then, $x_n \in [0,M]$ and

$$|H(x_n) - x| = |H(x_n) - H_n(x_n)| < \delta.$$

Because $x, H(x_n) \in [0, H(M)]$ and $x_n = H^{-1}(H(x_n))$, inequality (B.3) yields

$$|H_n^{-1}(x) - H^{-1}(x)| = |x_n - H^{-1}(x)| = |H^{-1}(H(x_n)) - H^{-1}(x)| < \varepsilon.$$

As this holds uniformly for all $x \in [0,z]$, the convergence is uniform. ∎

B.3 Proof of Proposition 5.3

We first establish a technical lemma.

Lemma B.2. *Let $\boldsymbol{R}_1$ and $\boldsymbol{R}_2$ be two non-negative random variables satisfying the conditions (1) and (2) in Proposition 5.3. Recall the definition in Proposition 3.8 that $\boldsymbol{X}_i = \boldsymbol{R}_i - r_0$ and $Q_i(x) = \frac{\tilde{\mathbb{E}}[\boldsymbol{X}_i e^{-\frac{1}{2}\boldsymbol{X}_i x}]}{\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{X}_i x}]}$, $x \geq 0$. Then, the following assertions hold:*

(1) *For each $i \in \{1, 2\}$, there exist constant $M_i > 0$, such that $Q_i(x) \leq M_i e^{-\frac{1}{2}\delta_i x}$ for any $x \geq 0$. In particular, $Q_i(x) \to 0$ as $x \to \infty$.*

(2) *For each $i \in \{1, 2\}$, the integral $\int_0^\infty (h_i^{-2}(s) - r_0^2)ds$ converges to a finite constant K_i.*

(3) *For any $x_1, x_2 \geq 0$ and any $\varepsilon > 0$, it holds that $\dfrac{Q_1(x_1)}{Q_2(x_2)} \leq C \dfrac{e^{-\frac{1}{2}\delta_1 x_1}}{e^{-\frac{1}{2}(\delta_2+\varepsilon)x_2}}$, where C is a constant depending on ε.*

Proof.

(1) For $i \in \{1, 2\}$. By condition (1) in Proposition 5.3, we have $\mathbb{P}(\boldsymbol{X}_i = 0) = p_i$. The Dominated Convergence Theorem then implies $\lim_{x\to\infty} \tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{X}_i x}] = p_i \in (0, 1)$. Hence, there exists $m_i > 0$ such that $\tilde{\mathbb{E}}[e^{-\frac{1}{2}\boldsymbol{X}_i x}] \geq m_0$ for any $x \geq 0$. Moreover, by condition (2) in Proposition 5.3, the support of $\boldsymbol{X}_i$ on $(0, \infty)$ is contained in $[\delta_i, \infty)$. Therefore,

$$\tilde{\mathbb{E}}[\boldsymbol{X}_i e^{-\frac{1}{2}\boldsymbol{X}_i x}] = \int_{\delta_i}^\infty \gamma e^{-\frac{1}{2}\gamma x} \mathrm{d}F_{\boldsymbol{X}_i}(\gamma) \leq e^{-\frac{1}{2}\delta_i x} \int_{\delta_i}^\infty \gamma \mathrm{d}F_{\boldsymbol{X}_i}(\gamma) = e^{-\frac{1}{2}\delta_i x} \tilde{\mathbb{E}}[\boldsymbol{X}_i \mathbb{I}_{\{\boldsymbol{X}_i \geq \delta_i\}}].$$

The desired conclusion then follows by taking $M_i \triangleq \frac{\tilde{\mathbb{E}}[\boldsymbol{X}_i \mathbb{I}_{\{\boldsymbol{X}_i \geq \delta_i\}}]}{m_i}$.

(2) For $i \in \{1, 2\}$, based on (A.2), the conclusion follows immediately from assertion (1) and the fact that $\delta_i > 0$.

(3) For any $\varepsilon > 0$. By the definition of the support, we have $p_{\varepsilon,2} \triangleq P(\delta_2 \leq \boldsymbol{X_2} < \delta_2 + \varepsilon) > 0$. Restricting the integration to this interval yields

$$\tilde{\mathbb{E}}[\boldsymbol{X_2} e^{-\frac{1}{2}\boldsymbol{X_2} x}] \geq \int_{\delta_2}^{\delta_2+\varepsilon} x e^{-\frac{1}{2}x\gamma} \mathrm{d}F_{\boldsymbol{X_2}}(\gamma) \geq \delta_2 e^{-\frac{1}{2}(\delta_2+\varepsilon)x} p_{\varepsilon,2}.$$

Combining this with assertion (1) gives the desired bound with $C \triangleq \frac{M_1}{\delta_2 p_{\varepsilon,2}}$. □

Now we return to prove Proposition B.3.

By assertion (1) of Lemma B.2 and (A.2), we have $\lim_{x\to\infty} h_i(x) = \frac{1}{r_0}$. Because $r_0 > 0$, it follows that the integral $\int_0^\infty h_i^{-2}(x)\mathrm{d}x$ diverges, which implies $H_i(\infty) = \infty$. Consequently, Theorem 3.2 (1) ensures the existence of a unique equilibrium π_i for $\boldsymbol{R}_i$ for any $T > 0$. Based on the relation $\pi_i = (\sigma^\top)^{-1} a_i$, it suffices to compare a_i. By (2.5), we have $|a_i(0, T)| = h_i(v_i(0, T))|\lambda(0)|$.

First, we establish the asymptotic relationship between $v_1(0,T)$ and $v_2(0,T)$ as $T \to \infty$. By Theorem 3.2 (1), we have $v_i(0,T) = H_i^{-1}(\Lambda(0,T))$. It then follows that $\lim_{T\to\infty} v_i(0,T) = \infty$, because $H_i(\infty) = \infty$ and $\lim_{T\to\infty} \Lambda(0,T) = \infty$. Moreover, the relation $v_i(0,T) = H_i^{-1}(\Lambda(0,T))$ yields

$$\Lambda(0,T) = \int_0^{v_i(0,T)} (r_0^2 + (h_i^{-2}(x) - r_0^2))\mathrm{d}x = r_0^2 v_i(0,T) + \int_0^{v_i(0,T)} (h_i^{-2}(x) - r_0^2)\mathrm{d}x. \tag{B.4}$$

Define the tail integral $R_i(y) \triangleq \int_y^\infty (h_i^{-2}(x) - r_0^2)\mathrm{d}x$, which satisfies $\lim_{y\to\infty} R_i(y) = 0$ by Lemma B.2 (2). Then (B.4) can be rewritten as

$$\Lambda(0,T) = r_0^2 v_i(0,T) + K_i - R_i(v_i(0,T)). \tag{B.5}$$

Equating (B.5) for $i = 1$ and $i = 2$ gives

$$v_1(0,T) = v_2(0,T) + \frac{K_2 - K_1}{r_0^2} + \frac{R_2(v_2(0,T)) - R_1(v_1(0,T))}{r_0^2}.$$

Let $\Delta K = \dfrac{K_2 - K_1}{r_0^2}$ and $\xi(T) = \dfrac{R_2(v_2(0,T)) - R_1(v_1(0,T))}{r_0^2}$. Because $v_i(0,T) \to \infty$ as $T \to \infty$, the tail difference term $\xi(T)$ tends to 0. Consequently, we obtain the asymptotic relation $v_1(0,T) = v_2(0,T) + \Delta K + \xi(T)$.

Second, we compare $h_1(v_1(0,T))$ and $h_2(v_2(0,T))$. Because $h_i(s) = (r_0 + Q_i(s))^{-1}$, this reduces to comparing $Q_1(v_1(0,T))$ and $Q_2(v_2(0,T))$. Let $\varepsilon > 0$ be a constant satisfying $\delta_1 - \delta_2 - \varepsilon > 0$. Using the bound established in Lemma B.2 (3), we obtain

$$\frac{Q_1(v_1(0,T))}{Q_2(v_2(0,T))} \le C \frac{e^{-\frac{1}{2}\delta_1 v_1(0,T)}}{e^{-\frac{1}{2}(\delta_2+\varepsilon) v_2(0,T)}}.$$

Substituting $v_1(0,T) = v_2(0,T) + \Delta K + \xi(T)$ into the exponent yields

$$\frac{Q_1(v_1(0,T))}{Q_2(v_2(0,T))} \le C e^{-\frac{1}{2}\delta_1(\Delta K + \xi(T))} \cdot e^{-\frac{1}{2}(\delta_1 - \delta_2 - \varepsilon) v_2(0,T)}.$$

Because $\lim_{T\to\infty} \xi(T) = 0$, the first exponential term converges to a finite positive constant. Moreover, as $v_2(0,T) \to \infty$, the second term tends to 0, and hence $\frac{Q_1(v_1(0,T))}{Q_2(v_2(0,T))} \to 0$ as $T \to \infty$.

This implies for sufficiently large T, $Q_1(v_1(0,T)) < Q_2(v_2(0,T))$. Because $h_i(x) = (r_0 + Q_i(x))^{-1}$ is strictly decreasing in Q_i, we have $h_1(v_1(0,T)) > h_2(v_2(0,T))$. Consequently, $|a_1(0,T)| > |a_2(0,T)|$ because $\lambda(0) \neq 0$, and equivalently, $|\pi_1(0,T)| > |\pi_2(0,T)|$.

B.4 Proof of Proposition 5.4

For $i = 1, 2$, let $\mathbb{P}(\boldsymbol{R}_i = r_0) = p_i$. The function $Q_i(\cdot)$ is explicitly given by

$$Q_i(x) = \frac{\tilde{\mathbb{E}}[\boldsymbol{X}_i e^{-\boldsymbol{X}_i \frac{x}{2}}]}{\tilde{\mathbb{E}}[e^{-\boldsymbol{X}_i \frac{x}{2}}]} = \frac{\delta_i(1-p_i)e^{-\delta_i \frac{x}{2}}}{p_i + (1-p_i)e^{-\delta_i \frac{x}{2}}}, \quad x \in [0,\infty).$$

We first verify the existence of a solution to $|a_1(t)| = |a_2(t)|$. This equality is equivalent to $Q_1(v_1(t)) = Q_2(v_2(t))$, because $|a_i(t)| = |\lambda(t)|(r_0 + Q_i(v_i(t)))^{-1}$. Define $\mathcal{D}(t) \triangleq Q_1(v_1(t)) -$

$Q_2(v_2(t))$. At $t=0$, by assumption, $\mathcal{D}(0)<0$. At $t=T$, because $v_i(T)=0$ and $Q_i(0)=\tilde{\mathbb{E}}[\boldsymbol{R}_i]-r_0$, the condition $\tilde{\mathbb{E}}[\boldsymbol{R}_1]>\tilde{\mathbb{E}}[\boldsymbol{R}_2]$ implies $Q_1(0)>Q_2(0)$, and therefore $\mathcal{D}(T)>0$. Because $\mathcal{D}(t)$ is continuous, there exists at least one $t^*\in(0,T)$ such that $\mathcal{D}(t^*)=0$.

Next, we show uniqueness by differentiating $\mathcal{D}(\cdot)$. To this end, we compute $Q_i'(x)$:

$$\begin{aligned}
Q_i'(x) &= \frac{-\frac{\delta_i^2}{2}(1-p_i)e^{-\delta_i\frac{x}{2}}\left(p_i+(1-p_i)e^{-\delta_i\frac{x}{2}}\right)-\delta_i(1-p_i)e^{-\delta_i\frac{x}{2}}\left(-\frac{\delta_i}{2}(1-p_i)e^{-\delta_i\frac{x}{2}}\right)}{\left(p_i+(1-p_i)e^{-\delta_i\frac{x}{2}}\right)^2}\\
&= -\frac{\delta_i}{2}\left(\frac{\delta_i(1-p_i)e^{-\delta_i\frac{x}{2}}}{p_i+(1-p_i)e^{-\delta_i\frac{x}{2}}}\right)+\frac{1}{2}\left(\frac{\delta_i(1-p_i)e^{-\delta_i\frac{x}{2}}}{p_i+(1-p_i)e^{-\delta_i\frac{x}{2}}}\right)^2\\
&= -\frac{\delta_i}{2}Q_i(x)+\frac{1}{2}Q_i(x)^2\\
&= \frac{1}{2}Q_i(x)(Q_i(x)-\delta_i).
\end{aligned}$$

Differentiating $\mathcal{D}(t)$ and using $v_i'(t)=-|\lambda(t)|^2(r_0+Q_i(v_i(t)))^{-2}$ yields

$$\begin{aligned}
\mathcal{D}'(t) &= Q_1'(v_1)v_1'-Q_2'(v_2)v_2'\\
&= \frac{1}{2}Q_1(Q_1-\delta_1)\left(\frac{-|\lambda(t)|^2}{(r_0+Q_1)^2}\right)-\frac{1}{2}Q_2(Q_2-\delta_2)\left(\frac{-|\lambda(t)|^2}{(r_0+Q_2)^2}\right)\\
&= \frac{|\lambda(t)|^2}{2}\left[\frac{Q_2(Q_2-\delta_2)}{(r_0+Q_2)^2}-\frac{Q_1(Q_1-\delta_1)}{(r_0+Q_1)^2}\right].
\end{aligned}$$

At any solution t^* where $\mathcal{D}(t^*)=0$, we have $Q=Q_1(v_1(t^*))=Q_2(v_2(t^*))>0$. Substituting $Q_1=Q_2=Q$ yields

$$\mathcal{D}'(t^*)=\frac{|\lambda(t^*)|^2Q}{2(r_0+Q)^2}(\delta_1-\delta_2).$$

Because $\delta_1>\delta_2$ and $\lambda(t^*)\neq 0$, we have $\mathcal{D}'(t^*)>0$. Therefore, the solution t^* is unique.

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