# VaR at Its Extremes: Impossibilities and Conditions for One-Sided Random Variables

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**Abstract**

We investigate the extremal aggregation behavior of Value-at-Risk (VaR) – that is, its additivity properties across all probability levels – for sums of one-sided random variables. For risks supported on $[0, \infty)$, we show that VaR sub-additivity is impossible except in the degenerate case of exact additivity, which holds only under co-monotonicity. To characterize when VaR is instead fully super-additive, we introduce two structural conditions: negative simplex dependence (NSD) for the joint distribution and simplex dominance (SD) for a margin-dependent functional. Together, these conditions provide a unified and easily verifiable framework that accommodates non-identical margins, heavy-tailed laws, and a wide spectrum of negative dependence structures. All results extend to random variables with arbitrary finite lower or upper endpoints, yielding sharp constraints on when strict sub- or super-additivity can occur.

*Key words and phrases*: Value-at-Risk; VaR sub-additivity; VaR super-additivity; one-sided random variables; negative simplex dependence; simplex dominant functions

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# 1 Introduction

The study of quantiles has long been a cornerstone of mathematical and statistical research. Quantiles provide a fundamental link between abstract probability models and the observable outcomes they generate. In risk management, in particular, quantiles have been regarded as essential tools for assessing the riskiness of losses, asset prices, and other financial variables. Their most prominent manifestation is the Value-at-Risk (VaR) (Linsmeier & Pearson, 2000), which is interpreted as the minimum amount of capital a financial institution must hold so that losses exceed this level only with a small, pre-specified probability.

A variety of risk measures have been developed to refine or extend VaR. Among them, the conditional tail expectation – also known as expected shortfall – (Acerbi & Tasche, 2002; Tasche, 2002) which addresses several limitations of VaR, most notably the failure of sub-additivity. Nevertheless, interest in VaR has persisted, partly due to its ability to capture the opposite phenomenon of super-additivity, a feature that does not rely on the integrability requirements imposed by alternative risk measures. In this paper, we examine the extremal behaviors of VaR, focusing on the conditions under which it exhibits sub-additivity or super-additivity across all probability thresholds.

To formalize our analysis, we consider random vectors $\boldsymbol{X} = (X_1, \ldots, X_n)$, $n \in \mathbb{N}$, whose components are random variables representing, for example, asset prices or insurance losses. We focus in particular on their aggregate,

$$S = \sum_{i=1}^{n} X_i.$$

For any random variable or random vector, we denote its probability density function, cumulative distribution function (CDF), and decumulative (survival) function (DDF) by $f$, $F$, and $\overline{F}$, respectively, using subscripts to indicate the relevant variables. For example, $F_{\boldsymbol{X}}$ denotes the joint CDF of the random vector $\mathbf{X}$, while $F_{X_i}$ denotes the marginal CDF of $X_i$ for $i \in \{1, \ldots, n\}$. Unless explicitly stated, we impose no integrability assumptions on the random variables.

Throughout Sections 2 and 3 of the paper, we assume that each random variable $X_i$ has support with lower endpoint at zero, that is,

$$a_i = \sup\{x \in \mathbb{R} : F_{X_i}(x) \leq 0\} = 0, \qquad \forall i \in \{1, \ldots, n\}.$$

In the final section, Section 4, we show how this assumption can be relaxed. In particular, we extend all results to the setting where the lower endpoints $a_i$ are arbitrary but finite, and also to the complementary case in which the random variables are instead bounded above.

Finally, for any random variable $Z$, we define its VaR at confidence level $p \in (0, 1)$ as the left-quantile (left-inverse) of its distribution:

$$\mathrm{VaR}_p[Z] = \inf\{z \in \mathbb{R} : F_Z(z) \geq p\}.$$

Our primary objective is to investigate how the VaR of the sum, $\mathrm{VaR}_p[S]$, compares to the sum of the individual VaRs, $\sum_{i=1}^{n} \mathrm{VaR}_p[X_i]$ for all $p \in (0, 1)$. To provide a precise framework for this comparison, we introduce the following definitions.

**Definition 1.1.** *We say that $\boldsymbol{X}$ is VaR sub-additive (respectively, VaR super-additive) if*

$$\mathrm{VaR}_p[S] \leq (\geq) \sum_{i=1}^{n} \mathrm{VaR}_p[X_i], \quad \forall p \in (0, 1). \tag{1.1}$$

*In particular, $\boldsymbol{X}$ is called VaR additive if equality holds for all probability levels $p \in (0, 1)$.*

The remainder of the paper is organized as follows. Section 2 establishes an impossibility theorem for VaR sub-additivity, extending the recent findings of Imamura and Kato, 2025 and showing that sub-additivity can occur only in the degenerate case of VaR additivity. Section 3 develops a new and unified characterization of VaR super-additivity that encompasses most existing results in the literature while allowing for non-identically distributed margins and a wider range of dependence structures. In Section 4, we generalize these results to random variables with arbitrary finite lower or upper endpoints. Section 5 concludes the paper.

## 2 VaR Sub-additivity

VaR sub-additivity is widely regarded as a desirable property, as it reflects the risk-reducing effect of diversification. In pursuit of this property, numerous alternative risk measures have been introduced to guarantee it. The literature has examined VaR sub-additivity in various settings, including asymptotic regimes (Daníelsson et al., 2013) and classes of distributions such as the elliptical distributions where VaR is known to be sub-additive for confidence levels $p \geq \frac{1}{2}$ (McNeil et al., 2015). Nevertheless, when we examine VaR sub-additivity for right-sided random variables, the conclusion turns out to be remarkably simple. As we show in the next theorem, VaR sub-additivity cannot occur in our setting except in the degenerate case of exact additivity. Before presenting this main result, we recall the notion of co-monotonicity, which represents the extremal form of positive dependence.

**Definition 2.1.** *A random vector $\boldsymbol{X}$ is co-monotonic if its joint CDF $F_{\boldsymbol{X}}$ is the Fréchet upper bound*

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = \min \{F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\}.$$

**Theorem 2.2.** *$\boldsymbol{X}$ is VaR sub-additive if and only if $\boldsymbol{X}$ is VaR additive. In addition, $\boldsymbol{X}$ must be a co-monotonic vector.*

*Proof.* The reverse implication follows trivially from Definition 1.1.

For the 'only if' implication, suppose that $\boldsymbol{X}$ is VaR sub-additive. Fix any constant $k > 0$, and define the truncated random vector

$$\boldsymbol{X}_k = (X_{1,k}, \ldots, X_{n,k}), \qquad X_{i,k} \overset{d}{:=} X_i \mid S \leq k, \quad i \in \{1, \ldots, n\},$$

and let

$$S_k \overset{d}{:=} \sum_{i=1}^{n} X_{i,k}.$$

The conditional variables $X_{i,k}$ are well-defined since each $X_i$ has a zero lower endpoint and $k > 0$.

The CDFs of $X_{i,k}$ and $S_k$ can be written as

$$F_{X_{i,k}}(x_i) = \begin{cases} \dfrac{\mathbb{P}(X_i \leq x_i,\, S \leq k)}{F_S(k)}, & x_i < k, \\ 1, & x_i \geq k, \end{cases} \quad \text{and} \quad F_{S_k}(s) = \begin{cases} \dfrac{F_S(s)}{F_S(k)}, & s < k, \\ 1, & s \geq k. \end{cases}$$

Next, define random variables $(\widetilde{X}_{1,k}, \ldots, \widetilde{X}_{n,k})$ via

$$F_{\widetilde{X}_{i,k}}(x_i) = \begin{cases} \dfrac{F_{X_i}(x_i)}{F_S(k)}, & x_i < \mathrm{VaR}_{F_S(k)}[X_i], \\ 1, & x_i \geq \mathrm{VaR}_{F_S(k)}[X_i]. \end{cases}$$

For $x_i < \mathrm{VaR}_{F_S(k)}[X_i]$, we have $F_{X_i}(x_i) < F_S(k) \leq F_{X_i}(\mathrm{VaR}_{F_S(k)}[X_i])$. Then the ratio $\dfrac{F_{X_i}(x_i)}{F_S(k)}$ is strictly less than 1, and $x_i < k$, so the CDFs are well-defined.

From the definitions of $F_{X_{i,k}}$ and $F_{\widetilde{X}_{i,k}}$, we observe that

$$F_{X_{i,k}}(x_i) \leq F_{\widetilde{X}_{i,k}}(x_i), \qquad \forall x_i \in [0, \infty),$$

which implies

$$\mathrm{VaR}_p[\widetilde{X}_{i,k}] = \mathrm{VaR}_{pF_S(k)}[X_i] \leq \mathrm{VaR}_p[X_{i,k}], \qquad \forall p \in (0,1).$$

Similarly, by definition of $F_{S_k}$,

$$\mathrm{VaR}_p[S_k] = \mathrm{VaR}_{pF_S(k)}[S], \qquad \forall p \in (0,1).$$

Since $\boldsymbol{X}$ is VaR sub-additive,

$$\mathrm{VaR}_p[S_k] = \mathrm{VaR}_{pF_S(k)}[S] \leq \sum_{i=1}^{n} \mathrm{VaR}_{pF_S(k)}[X_i] \leq \sum_{i=1}^{n} \mathrm{VaR}_p[X_{i,k}],$$

and therefore,

$$\mathrm{VaR}_p[S_k] \leq \sum_{i=1}^{n} \mathrm{VaR}_p[X_{i,k}], \qquad \forall p \in (0,1).$$

Hence, $\boldsymbol{X}_k$ is also VaR sub-additive.

Since each $X_{i,k}$ has a finite expectation ($\mathbb{E}[X_{i,k}] \leq k < \infty$), Theorem 1 in Imamura and Kato, 2025 implies that $\boldsymbol{X}_k$ is co-monotonic and consequently VaR additive i.e.:

$$F_{\boldsymbol{X}_k}(x_1, \ldots, x_n) = \min\{F_{X_{1,k}}(x_1), \ldots, F_{X_{n,k}}(x_n)\},$$

and

$$\mathrm{VaR}_p[S_k] = \sum_{i=1}^{n} \mathrm{VaR}_p[X_{i,k}], \qquad \forall p \in (0,1).$$

Finally, by the monotone convergence of both the numerator $\mathbb{P}(X_i \leq x_i, S \leq k)$ and denominator $\mathbb{P}(S \leq k)$, each marginal CDF satisfies

$$F_{X_{i,k}}(x_i) \to F_{X_i}(x_i), \qquad \text{as } k \to \infty,$$

for every $x_i \in [0, \infty)$. Hence,

$$F_{\boldsymbol{X}_k}(x_1, \ldots, x_n) \to F_{\boldsymbol{X}}(x_1, \ldots, x_n), \qquad \forall (x_1, \ldots, x_n) \in [0, \infty)^n,$$

where

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) = \min\{F_{X_1}(x_1), \ldots, F_{X_n}(x_n)\}.$$

Thus, $\boldsymbol{X}$ is co-monotonic and VaR additive. □

Theorem 2.2 extends and strengthens the main result of Imamura and Kato, 2025, which relied on integrability assumptions. It reveals the rigid phenomenon of the impossibility of VaR sub-additivity for random variables supported on $[0, \infty)$. The sub-additivity inequality is satisfied only in the degenerate case where VaR is exactly additive, and such additivity occurs exclusively under co-monotonicity.

# 3 VaR super-additivity

Unlike the sub-additivity property of VaR, the opposite effect -— VaR super-additivity -— can in fact arise. For instance, consider the case of a counter-monotonic random vector.

$$\boldsymbol{X} = \left(X, \frac{1}{X}\right),$$

whose joint CDF is given by

$$F_{\boldsymbol{X}}(x_1, x_2) = \max\{F_X(x_1) + F_{1/X}(x_2) - 1,\ 0\},$$

where $X$ follows a Type II Pareto distribution with CDF

$$F_X(x) = 1 - \left(\frac{\theta}{\theta + x}\right)^{\alpha}, \qquad x \geq 0,\ \alpha, \theta > 0.$$

For simplicity, take $\alpha = \theta = 1$. The marginals are then identical Pareto(II) variables with

$$\mathrm{VaR}_p[X] = \frac{p}{1-p}.$$

A direct calculation shows that the VaR of the sum is

$$\mathrm{VaR}_p[X + 1/X] = \frac{2(1+p^2)}{1-p^2}.$$

Since

$$\frac{2(1+p^2)}{1-p^2} > \frac{2p}{1-p}, \qquad \forall\, p \in (0,1),$$

the vector $\boldsymbol{X}$ is VaR super-additive. This example demonstrates that VaR super-additivity can appear naturally from a suitable choice of dependence and margins.

Although several studies have constructed families of distributions that exhibit tail-level VaR super-additivity (Embrechts et al., 2008, 2009; Zhu et al., 2023), relatively few have examined this behavior across all probability levels $p$. Ibragimov (2009) analyzed full-range super-additivity in i.i.d. stable distributions, and Chen et al. (2025) introduced a class of random vectors with identically distributed, weakly negatively associated margins for which VaR is fully super-additive. The counter-monotonic vector presented above is a special case of this class.

Taken together, these results show that VaR super-additivity is not a pathological anomaly but occurs naturally under economically meaningful and probabilistically coherent conditions.

Although not stated in this form, Theorem 1 of Imamura and Kato, 2025 also implies a strong constraint on when super-additivity can happen. If all components $X_i$ are integrable i.e. $\mathbb{E}[|X_i|] < \infty$, then VaR super-additivity is equivalent to VaR additivity. Consequently, to construct examples of random vectors $\boldsymbol{X}$ that are genuinely VaR super-additive, at least one of the components must be non-integrable. Our counter-monotonic example above shows this implication as the two Pareto II variables have infinite means. The converse, however, does not necessarily hold. Non-integrability is intrinsically a tail property and does not by itself guarantee full super-additivity. The following example illustrates this point.

**Example 3.1.** *Let $\boldsymbol{X}$ be a bivariate random vector.*

*(1) Suppose $\boldsymbol{X}$ is counter-monotonic and defined by*

$$\boldsymbol{X} = \left(X,\ \frac{1}{1+X}\right),$$

*where $X$ follows a Pareto Type II distribution with unit scale and unit shape. Since*

$$\frac{1}{1+X} \sim \mathrm{Unif}(0,1),$$

*we obtain $\mathbb{E}\left[\frac{1}{1+X}\right] = \frac{1}{2}$, while $\mathbb{E}[X] = \infty$ for this Pareto distribution.*

*The marginal VaR functions are therefore*

$$\mathrm{VaR}_p[X] = \frac{p}{1-p}, \qquad \mathrm{VaR}_p\left[\frac{1}{1+X}\right] = p.$$

*For the sum* $S = X + \frac{1}{1+X}$, *one can show that*

$$\mathrm{VaR}_p[S] = \frac{p^2 - p + 1}{1 - p}.$$

*Comparing* $\mathrm{VaR}_p[S]$ *with* $\mathrm{VaR}_p[X] + \mathrm{VaR}_p\left[\frac{1}{1+X}\right]$ *reveals that VaR is super-additive for* $p \in \left(0, \frac{1}{2}\right]$, *and sub-additive for* $p \in \left[\frac{1}{2}, 1\right)$.

*(2) Consider now a bivariate random vector* $\boldsymbol{X} = (X_1, X_2)$ *with joint distribution function*

$$F_{\boldsymbol{X}}(x_1, x_2) = C(F_{X_1}(x_1),\, F_{X_2}(x_2)),$$

*where* $F_{X_1}$ *and* $F_{X_2}$ *are Pareto (II) marginal distributions with parameters* $\alpha_1 = \alpha_2 = \theta_1 = \theta_2 = 1$. *The copula* $C(u,v)$ *is an Ordinal Sum copula (see Example 3.4 in Nelsen, 2010) given by*

$$C(u,v) = \begin{cases} \max\left\{u + v - \frac{1}{2},\, 0\right\}, & (u,v) \in \left[0, \frac{1}{2}\right]^2, \\ \max\left\{u + v - 1,\, \frac{1}{2}\right\}, & (u,v) \in \left(\frac{1}{2}, 1\right]^2, \\ \min\{u, v\}, & \textit{otherwise.} \end{cases}$$

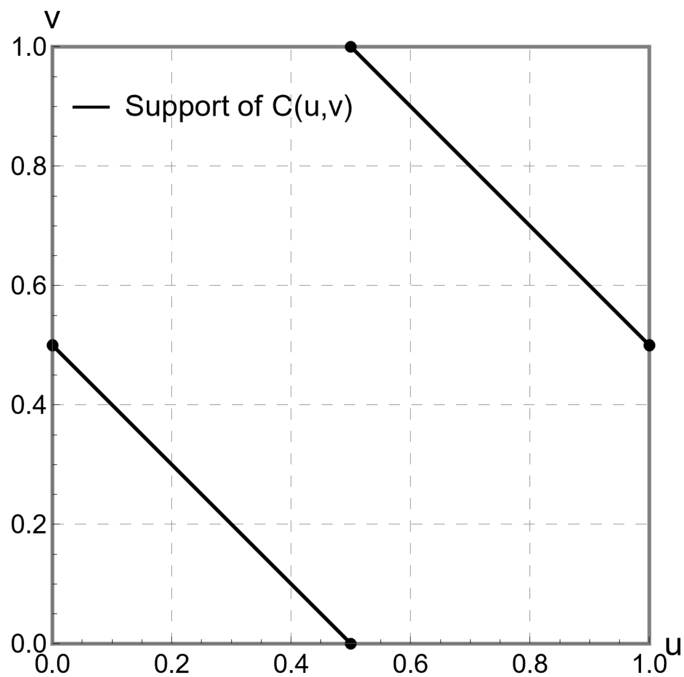



Figure 1: Support of the Ordinal Sum copula $C(u, v)$ on the unit square $[0, 1]^2$.

*Since both marginals are Pareto (II) with unit shape then their expectations are infinite. Their common VaR is*

$$\mathrm{VaR}_p[X_1] = \mathrm{VaR}_p[X_2] = \frac{p}{1-p}.$$

*For the sum $S = X_1 + X_2$, the VaR is piecewise and given by*

$$\mathrm{VaR}_p[S] = \begin{cases} \dfrac{6+8p^2}{9-4p^2}, & 0 < p \leq \frac{1}{2}, \\[2ex] \dfrac{2-2p(1-p)}{p(1-p)}, & \frac{1}{2} < p < 1. \end{cases}$$

*A direct comparison between $\mathrm{VaR}_p[S]$ and $\mathrm{VaR}_p[X_1] + \mathrm{VaR}_p[X_2]$ shows that $\mathrm{VaR}_p[S]$ is sub-additive whenever*

$$p \in \left[\frac{3-\sqrt{6}}{2}, \frac{1}{2}\right],$$

*and super-additive for all remaining values of $p$.*

Example 3.1 demonstrates that even in cases where we have (1) counter-monotonic dependence with one non-integrable margin, and (2) two non-counter-monotonic, non-integrable margins, the resulting dependence-margin combination may still exhibit intervals of VaR sub-additivity. Thus, neither a particular dependence structure nor the mere non-integrability of margins is sufficient on its own to guarantee VaR super-additivity. In fact, the negative dependence used in part (2) of the example is significantly weaker than full counter-monotonicity.

This indicates that VaR super-additivity cannot be deduced from dependence alone, nor from marginal tail behavior in isolation. Rather, it requires analyzing how the joint distribution interacts with the full set of marginal distributions. It is this interaction that determines whether a given random vector $\boldsymbol{X}$ belongs to a class for which VaR is guaranteed to be super-additive. Our objective, therefore, is to identify a dependence property together with a corresponding marginal behavior that, when combined, imply VaR super-additivity. Such a characterization must be sufficiently general to encompass all known results in the literature, yet specific enough to allow for straightforward verification.

Before presenting our main result, we introduce two key concepts.

**Definition 3.2.** *We say $\boldsymbol{X}$ is negative simplex dependent (NSD) if*

$$F_S(t) \leq \prod_{i=1}^{n} F_{X_i}(t), \qquad \forall t \in [0, \infty).$$

**Definition 3.3.** *A function $\Phi : [0, \infty)^n \to (-\infty, 0]$ is called simplex dominant (SD) if*

$$\Phi(x_1, \dots, x_n) \geq \Phi(s, \dots, s), \qquad s = \sum_{i=1}^{n} x_i, \quad \forall (x_1, \dots, x_n) \in [0, \infty)^n.$$

**Theorem 3.4.** *If $\boldsymbol{X}$ is NSD with continuous $F_{X_i}$, and*

$$\Phi(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i \log F_{X_i}(x_i), \tag{3.1}$$

*is SD then $\boldsymbol{X}$ is VaR super-additive.*

*Proof.* Since $F_S(\mathrm{VaR}_p[S]) \geq p$, then VaR super-additivity is equivalent to showing that

$$p \;\geq\; F_S\left(\sum_{i=1}^{n} \mathrm{VaR}_p[X_i]\right), \qquad \forall p \in (0,1).$$

For simplicity, let us use the notations

$$x_i(p) := \mathrm{VaR}_p[X_i], \qquad s(p) := \sum_{i=1}^{n} x_i(p).$$

First, since $\Phi$ is SD, setting each coordinate $x_i$ to $x_i(p)$ yields

$$\Phi\big(x_1(p), \dots, x_n(p)\big) \;\geq\; \Phi\big(s(p), \dots, s(p)\big),$$

hence

$$\sum_{i=1}^{n} x_i(p) \log F_{X_i}(x_i(p)) \;\geq\; s(p) \sum_{i=1}^{n} \log F_{X_i}(s(p)).$$

Second, because each $F_{X_i}$ is continuous, we have $F_{X_i}(x_i(p)) = p$, and therefore

$$\sum_{i=1}^{n} x_i(p) \log p \;\geq\; s(p) \sum_{i=1}^{n} \log F_{X_i}(s(p)),$$

$$\Longrightarrow\; s(p) \log p \;\geq\; s(p) \sum_{i=1}^{n} \log F_{X_i}(s(p)).$$

Given we have $s(p) > 0$, dividing by $s(p)$ yields

$$\log p \;\geq\; \sum_{i=1}^{n} \log F_{X_i}(s(p)).$$

Exponentiating gives

$$p \;\geq\; \prod_{i=1}^{n} F_{X_i}(s(p)).$$

Finally, for NSD vectors, the joint distribution satisfies

$$F_S(t) \;\leq\; \prod_{i=1}^{n} F_{X_i}(t), \qquad \forall t \in [0, \infty).$$

Thus at $t = s(p)$,

$$\prod_{i=1}^{n} F_{X_i}(s(p)) \;\geq\; F_S(s(p)).$$

Combining with the previous inequality gives

$$p \;\geq\; \prod_{i=1}^{n} F_{X_i}(s(p)) \;\geq\; F_S(s(p)) \;=\; F_S\left(\sum_{i=1}^{n} \mathrm{VaR}_p[X_i]\right), \qquad \forall p \in (0,1),$$

i.e.

$$p \;\geq\; F_S\left(\sum_{i=1}^{n} \mathrm{VaR}_p[X_i]\right), \qquad \forall p \in (0,1),$$

which is precisely VaR super-additivity. This completes the proof. □

The strength of Theorem 3.4 lies in its ability to encompass a broad class of dependence structures while permitting considerable flexibility in the choice of marginal distributions, which need not be identical.

The following two propositions provide sufficient conditions for establishing the NSD and SD properties.

**Proposition 3.5.** *When $\boldsymbol{X}$ is negative lower orthant dependent (NLOD) (Block et al., 1982;*

*Joe, 1997), that is*

$$F_{\boldsymbol{X}}(x_1, \ldots, x_n) \leq \prod_{i=1}^{n} F_{X_i}(x_i), \ \forall (x_1, \ldots, x_n) \in [0, \infty)^n,$$

*then* $\boldsymbol{X}$ *is NSD.*

*Proof.* The result can be easily deduced since the $n$-simplex lies inside the $n$-cube (as a corner of the $n$-cube) which gives

$$F_S(t) \leq F_{\boldsymbol{X}}(t, \ldots, t) \leq \prod_{i=1}^{n} F_{X_i}(t), \ \forall t \in [0, \infty).$$

In fact, to be NSD, $\boldsymbol{X}$ need only be NLOD along the diagonal $(t, \ldots, t), \ t \in [0, \infty)$, and not necessarily everywhere. □

**Proposition 3.6.** *If* $\Phi$ *is non-increasing in the sense that*

$$\Phi(x_1, \ldots, x_n) \geq \Phi(y_1, \ldots, y_n) \quad \text{whenever } x_i \leq y_i \text{ for all } i \in \{1, \ldots, n\},$$

*then* $\Phi$ *is SD. In particular, if* $\Phi$ *can be written as*

$$\Phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} \phi_i(x_i).$$

*Then* $\Phi$ *is non-increasing if and only if each* $\phi_i$ *is non-increasing. Consequently, if all* $\phi_i$ *are non-increasing then* $\Phi$ *is SD.*

*Proof.* First part. Fix $(x_1, \ldots, x_n) \in [0, \infty)^n$ and set

$$y_1 = \cdots = y_n = s := \sum_{i=1}^{n} x_i.$$

Since $x_i \le s$ for all $i$, the non-increasing property implies

$$\Phi(x_1, \dots, x_n) \ge \Phi(y_1, \dots, y_n) = \Phi(s, \dots, s),$$

and therefore $\Phi$ is SD.

Second part. Assume $\Phi$ can be written as

$$\Phi(x_1, \dots, x_n) = \sum_{i=1}^{n} \phi_i(x_i).$$

If every $\phi$ is non-increasing then the sum of non-increasing functions is non-increasing i.e. $\Phi$ is non-increasing.

Conversely, suppose that $\Phi$ is non-increasing. For each $i \in \{1, \dots, n\}$, take $x_i \le y_i$ and set $x_j = y_j = z$, $\forall j \ne i$, then

$$\Phi(x_1, \dots, x_n) = \phi_i(x_i) + \sum_{j \ne i} \phi_i(z) \ge \Phi(y_1, \dots, y_n) = \phi_i(y_i) + \sum_{j \ne i} \phi_i(z),$$

$$\implies \phi_i(x_i) \ge \phi_i(y_i),$$

i.e. each $\phi_i$ is non-increasing. Consequently, by the first part of the proof, $\Phi$ is SD whenever all $\phi_i$ are non-increasing. □

**Corollary 3.7.** *If $\boldsymbol{X}$ is NLOD with continuous $F_{X_i}$, and each $\phi_i(x_i) = x_i \log F_{X_i}(x_i)$ in Equation (3.1) is non-increasing, then $\boldsymbol{X}$ is VaR super-additive.*

The NSD property captures the dependence requirement on the joint distribution of $\boldsymbol{X}$ that ensures VaR super-additivity. We note, in passing, that the dependence structure used in part (2) of Example 3.1 is weaker than NSD, specifically it fails the NSD property at $t \in \left(\frac{7}{10}, 1 + \sqrt{2}\right)$. This contributed, though was not strictly required, to the failure of VaR super-additivity in that example. Nonetheless, by definition, NSD is a relatively weak form of negative dependence and is strictly implied by NLOD.

Below we provide an example of a VaR super-additive random vector $\boldsymbol{X}$ that satisfies NSD but not NLOD.

**Example 3.8.** *Consider the random vector*

$$\boldsymbol{X} = \left(X,\, X,\, \frac{1}{X}\right),$$

*where $X$ follows a unit-scale, unit-shape Pareto II distribution. Its joint distribution function is*

$$F_{\boldsymbol{X}}(x_1, x_2, x_3) = \begin{cases} 0, & \dfrac{1}{x_3} \geq \min\{x_1, x_2\}, \\ \dfrac{x_3}{1+x_3} - \dfrac{1}{1+\min\{x_1, x_2\}}, & \dfrac{1}{x_3} < \min\{x_1, x_2\}. \end{cases}$$

*The distribution of the sum $S = X + X + 1/X$ may be computed explicitly:*

$$F_S(s) = \begin{cases} 0, & s \leq 2\sqrt{2}, \\ \dfrac{\sqrt{s^2-8}}{s+3}, & s > 2\sqrt{2}. \end{cases}$$

*Each marginal distribution is identical:*

$$F_X(x) = 1 - \frac{1}{1+x}, \qquad x \geq 0.$$

*To verify that $\boldsymbol{X}$ is NSD, observe first that for $0 \leq t \leq 2\sqrt{2}$,*

$$F_S(t) = 0 \leq \left(\frac{t}{1+t}\right)^3 = F_X(t)^3.$$

*For $t > 2\sqrt{2}$, one checks analytically that*

$$F_S(t) = \frac{\sqrt{t^2-8}}{t+3} \;<\; \left(\frac{t}{1+t}\right)^3 = F_X(t)^3.$$

*Thus $F_S(t) \leq F_X(t)^3$ for all $t \geq 0$, proving that $\boldsymbol{X}$ is NSD.*

*Next we show that $\boldsymbol{X}$ is not NLOD. For $t > 1$,*

$$F_{\boldsymbol{X}}(t,t,t) - F_X(t)^3 = \frac{t-1}{1+t} - \frac{t^3}{(1+t)^3} = \frac{t^2-t-1}{(1+t)^3}.$$

*A simple algebraic check shows that $t^2 - t - 1 \geq 0$ whenever*

$$t \geq \frac{\sqrt{5}+1}{2}.$$

*Hence $F_{\boldsymbol{X}}(t,t,t) \geq F_X(t)^3$ for all such $t$, implying that $\boldsymbol{X}$ fails to be NLOD, even along the diagonal.*

*We now compare the associated VaRs. The marginal VaRs are*

$$\mathrm{VaR}_p[X] = \frac{p}{1-p},$$

*whereas for the sum we have*

$$\mathrm{VaR}_p[S] = \frac{3p^2 + \sqrt{p^2+8}}{1-p^2}.$$

*A direct algebraic comparison yields*

$$\mathrm{VaR}_p[S] = \frac{3p^2 + \sqrt{p^2+8}}{1-p^2} \;>\; \frac{3p}{1-p} = 3\,\mathrm{VaR}_p[X], \quad \forall p \in (0,1),$$

*so $\boldsymbol{X}$ is VaR super-additive.*

*This conclusion is an immediate consequence of Theorem 3.4: we have already shown that $\boldsymbol{X}$*

*is NSD, and for the chosen unit-shape Pareto margins the functions $\phi_i(x_i) = x_i \log F_{X_i}(x_i)$ in Equation (3.1) are non-increasing (as will be demonstrated in Example 3.9) and consequently SD by Proposition 3.6.*

The second part of Theorem 3.4 imposes structural conditions on the marginal distributions by specifying the behaviour of the function $\Phi$ in Equation (3.1). In practice, the SD property may not be straightforward to verify, so it is useful to rely on the non-increasing criteria. Applying the condition of Proposition 3.6 to the function $\Phi$ in Equation (3.1), i.e.

$$\Phi(x_1, \dots, x_n) = \sum_{i=1}^{n} \phi_i(x_i), \quad \text{where} \quad \phi_i(x_i) = x_i \log F_{X_i}(x_i),$$

it suffices to verify that each $\phi_i(x_i)$ is non-increasing on $[0, \infty)$. This is convenient, as it reduces the verification of SD to checking each margin separately. The next example lists several standard continuous marginal distributions $F_{X_i}$ for which the function $\phi_i$ indeed has the required monotonicity property.

**Example 3.9.** *We collect below several familiar continuous distributions whose associated functions $\phi_i$ are non-increasing.*

*(1)* ***Fréchet distribution.*** *The CDF is*

$$F_{X_i}(x_i) = \exp\left(-\left(\frac{x_i}{\theta_i}\right)^{-\alpha_i}\right), \qquad x_i \geq 0,\ \alpha_i, \theta_i > 0,$$

*which yields*

$$\begin{aligned}\phi_i(x_i) &= -x_i \left(\frac{x_i}{\theta_i}\right)^{-\alpha_i} \\ &= -\theta_i^{\alpha_i} x_i^{1-\alpha_i}.\end{aligned}$$

*This function is non-increasing precisely when $0 < \alpha_i \leq 1$.*

*(2)* ***Pareto(II)/Lomax distribution.*** *Here*

$$F_{X_i}(x_i) = 1 - \left(\frac{\theta_i}{\theta_i + x_i}\right)^{\alpha_i}, \qquad x_i \geq 0,\ \alpha_i, \theta_i > 0,$$

*and thus*

$$\phi_i(x_i) = x_i \log\left(1 - \left(\frac{\theta_i}{\theta_i + x_i}\right)^{\alpha_i}\right).$$

*The derivative becomes*

$$\begin{aligned}\phi_i'(x_i) &= \log F_{X_i}(x_i) + \frac{x_i f_{X_i}(x_i)}{F_{X_i}(x_i)} \\ &= \log F_{X_i}(x_i) + \frac{\alpha_i x_i \overline{F}_{X_i}(x_i)}{(\theta_i + x_i) F_{X_i}(x_i)}.\end{aligned}$$

***Claim.*** *$\phi_i'(x_i) \leq 0$ for all $x_i \in [0, \infty)$ if and only if $0 < \alpha_i \leq 1$.*

Necessity. *Assume $\phi_i'(x_i) \leq 0$ on $[0, \infty)$, and suppose $\alpha_i > 1$. Consider*

$$\lim_{x_i \to \infty} \frac{\phi_i'(x_i)}{\overline{F}_{X_i}(x_i)} = \alpha_i - 1.$$

*Since $\alpha_i > 1$, the ratio becomes positive for sufficiently large $x_i$, contradicting the non-positivity of $\phi_i'$. Thus, necessarily $0 < \alpha_i \leq 1$.*

Sufficiency. *Assume $0 < \alpha_i \leq 1$. Rewrite the derivative as*

$$\begin{aligned}\phi_i'(x_i) &= -\int_{F_{X_i}(x_i)}^{1} \frac{1}{w}\,\mathrm{d}w + \frac{\alpha_i x_i}{(\theta_i + x_i) F_{X_i}(x_i)} \int_{F_{X_i}(x_i)}^{1} \mathrm{d}w \\ &= -\int_{F_{X_i}(x_i)}^{1} \frac{(\theta_i + x_i) F_{X_i}(x_i) - \alpha_i x_i w}{w(\theta_i + x_i) F_{X_i}(x_i)}\,\mathrm{d}w.\end{aligned}$$

*Since $F_{X_i}(x_i) \le w \le 1$, a sufficient condition for the integrand to be non-negative is*

$$(\theta_i + x_i)F_{X_i}(x_i) - \alpha_i x_i \ge 0.$$

*Applying the mean value theorem to $t \mapsto t^{\alpha_i}$ on $\left[\frac{\theta_i}{\theta_i + x_i}, 1\right]$ yields the inequality and thus the desired non-positivity.*

*Therefore, $\phi_i$ is non-increasing if and only if $0 < \alpha_i \le 1$.*

(3) ***Lévy distribution.*** *With*

$$F_{X_i}(x_i) = \operatorname{erfc}\left(\sqrt{\frac{\theta_i}{2x_i}}\right), \qquad x_i \ge 0,\ \theta_i > 0,$$

*define*

$$\phi_i(x_i) = x_i \log\left(\operatorname{erfc}\left(\sqrt{\frac{\theta_i}{2x_i}}\right)\right).$$

*Differentiating gives*

$$\phi_i'(x_i) = \log\left(\operatorname{erfc}\left(\sqrt{\frac{\theta_i}{2x_i}}\right)\right) + \frac{\sqrt{\frac{\theta_i}{2\pi x_i}}\exp\left(-\frac{\theta_i}{2x_i}\right)}{\operatorname{erfc}\left(\sqrt{\frac{\theta_i}{2x_i}}\right)}.$$

*Let $t = \sqrt{\frac{\theta_i}{2x_i}}$. Then $\phi_i'(x_i) \le 0$ is equivalent to $\psi_i(t) \le 0$, where*

$$\psi_i(t) = \log(\operatorname{erfc}(t)) + \frac{t\exp(-t^2)}{\sqrt{\pi}\operatorname{erfc}(t)}.$$

*Since*

$$\lim_{t\to 0^+} \psi_i(t) = 0, \qquad \lim_{t\to\infty} \psi_i(t) = -\infty,$$

*it suffices to show* $\psi_i'(t) \leq 0$*. Differentiation leads to*

$$\psi_i'(t) = \frac{\exp(-2t^2)\left(2t - \sqrt{\pi}\,\exp(t^2)(2t^2+1)\mathrm{erfc}(t)\right)}{\pi\,\mathrm{erfc}(t)^2},$$

*which is non-positive whenever*

$$\frac{2t\exp(-t^2)}{\sqrt{\pi}(2t^2+1)} \leq \mathrm{erfc}(t),$$

*the classical Mills ratio bound (Mills, 1926). Thus* $\phi_i'(x_i) \leq 0$ *for all* $x_i \geq 0$ *i.e.* $\phi_i$ *is non-increasing on* $[0, \infty)$*.*

*(4)* ***One-parameter Beta Prime distribution.*** *With*

$$F_{X_i}(x_i) = \left(\frac{x_i}{1+x_i}\right)^{\alpha_i}, \qquad x_i \geq 0,\ \alpha_i > 0,$$

*we have*

$$\phi_i(x_i) = \alpha_i x_i \log\left(\frac{x_i}{1+x_i}\right).$$

*Differentiation gives*

$$\begin{aligned}
\phi_i'(x_i) &= \alpha_i \left(\log\left(\frac{x_i}{1+x_i}\right) + \frac{1}{1+x_i}\right) \\
&= \alpha_i \left(-\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{1}{1+x_i}\right)^k + \frac{1}{1+x_i}\right) \\
&\leq \alpha_i \left(-\frac{1}{1+x_i} + \frac{1}{1+x_i}\right) = 0.
\end{aligned}$$

*Hence* $\phi_i$ *is non-increasing for all* $\alpha_i > 0$*.*

*(5)* ***Log-hazard distribution.*** *If*

$$F_{X_i}(x_i) = \exp\left(-\frac{\log(1+x_i)^{\alpha_i}}{x_i}\right), \qquad x_i \geq 0,\ \alpha_i \in (-\infty, 1),$$

*then*

$$\phi_i(x_i) = -\log(1+x_i)^{\alpha_i}.$$

*This is non-increasing whenever* $\log(1+x_i)^{\alpha_i}$ *is non-decreasing, which occurs exactly when* $0 \leq \alpha_i < 1$.

*(6)* ***Log-Cauchy distribution.*** *The CDF is*

$$F_{X_i}(x_i) = \frac{1}{2} + \frac{1}{\pi}\arctan(\alpha_i \log(x_i)), \qquad x_i \geq 0,\ \alpha_i > 0.$$

*Hence*

$$\phi_i(x_i) = x_i \log\left(\frac{1}{2} + \frac{1}{\pi}\arctan(\alpha_i \log(x_i))\right).$$

*Differentiation yields*

$$\phi_i'(x_i) = \log F_{X_i}(x_i) + \frac{\alpha_i}{\pi(1 + (\alpha_i \log(x_i))^2)\, F_{X_i}(x_i)}.$$

*Introducing* $\theta = \arctan(\alpha_i \log(x_i))$ *gives*

$$\phi_i'(x_i) = \log F_{X_i}(x_i) + \frac{\alpha_i \cos^2\theta}{\pi F_{X_i}(x_i)} = \log F_{X_i}(x_i) + \frac{\alpha_i \sin^2(\pi F_{X_i}(x_i))}{\pi F_{X_i}(x_i)}.$$

*To test non-positivity, define*

$$\psi_i(u) = \log(u) + \frac{\alpha_i \sin^2(\pi u)}{\pi u}, \qquad u \in [0,1].$$

*Then $\phi_i'(x_i) \le 0$ holds for all $x_i \ge 0$ precisely when $\psi_i(u) \le 0$, $\forall u \in [0,1]$ or equivalently when*

$$0 < \alpha_i \le \inf_{u\in[0,1]} \frac{-\pi u \log(u)}{\sin^2(\pi u)} \approx 1.0568.$$

*Thus, $\phi_i$ is non-increasing on $[0,\infty)$ if and only if $0 < \alpha_i \le 1.0568$.*

*(7)* ***Inverse-Gamma distribution.*** *The CDF can be written as*

$$F_{X_i}(x_i) = \frac{1}{\Gamma(\alpha_i)}\Gamma\left(\alpha_i, \frac{\theta_i}{x_i}\right), \qquad x_i \ge 0,\ \alpha_i, \theta_i > 0,$$

*leading to*

$$\phi_i(x_i) = x_i \log\left(\frac{1}{\Gamma(\alpha_i)}\Gamma(\alpha_i, \tfrac{\theta_i}{x_i})\right).$$

*Differentiation gives*

$$\phi_i'(x_i) = \log F_{X_i}(x_i) + \frac{\left(\dfrac{\theta_i}{x_i}\right)^{\alpha_i} \exp(-\theta_i/x_i)}{\Gamma(\alpha_i)\, F_{X_i}(x_i)}.$$

*Let $t = \theta_i/x_i$. Then $\phi_i'(x_i) \le 0$ is equivalent to $\psi_i(t) \le 0$, where*

$$\psi_i(t) = \log\left(\frac{1}{\Gamma(\alpha_i)}\Gamma(\alpha_i, t)\right) + \frac{t^{\alpha_i}\exp(-t)}{\Gamma(\alpha_i, t)}.$$

***Claim.*** *$\psi_i(t) \le 0$ for all $t \ge 0$ if and only if $0 < \alpha_i \le 1$.*

Necessity. *Limits give*

$$\lim_{t\to 0^+} \psi_i(t) = 0, \qquad \lim_{t\to\infty} \psi_i(t) = \begin{cases} -\infty, & 0 < \alpha_i < 1, \\ 0, & \alpha_i = 1, \\ +\infty, & \alpha_i > 1, \end{cases}$$

*so non-positivity requires* $0 < \alpha_i \leq 1$.

Sufficiency. *If* $0 < \alpha_i \leq 1$, *then*

$$\psi_i'(t) = \frac{t^{\alpha_i - 1}\exp(-2t)\big(t^{\alpha_i} - \exp(t)(t + 1 - \alpha_i)\Gamma(\alpha_i, t)\big)}{\Gamma(\alpha_i, t)^2},$$

*which is non-positive whenever*

$$\frac{t^{\alpha_i}\exp(-t)}{t + 1 - \alpha_i} \leq \Gamma(\alpha_i, t),$$

*a Gautschi-type lower bound (Gautschi, 1959). Thus* $\psi_i$ *is non-increasing with* $\psi_i(0) = 0$, *proving* $\psi_i(t) \leq 0$ *on* $[0, \infty)$.

*Therefore,* $\phi_i$ *is non-increasing on* $[0, \infty)$ *if and only if* $0 < \alpha_i \leq 1$.

**Proposition 3.10.** *The following conditions are equivalent to the functions* $\phi_i(x_i) = x_i \log F_{X_i}(x_i)$ *being non-increasing on* $[0, \infty)$.

*(i) Suppose that* $F_{X_i}$ *is differentiable then* $\phi_i(x_i)$ *is non-increasing for all* $x_i \in [0, \infty)$ *if and only if*

$$x_i h_{X_i}(x_i) \leq \int_{x_i}^{\infty} h_{X_i}(w)\mathrm{d}w, \quad \forall x_i \in [0, \infty).$$

*Where* $h_{X_i}(x_i) = \dfrac{f_{X_i}(x_i)}{F_{X_i}(x_i)}$ *is the reverse hazard rate function (Block et al., 1998) of the random variable* $X_i$.

*(ii)* $\phi_i(x_i)$ *is non-increasing for all* $x_i \in [0, \infty)$ *if and only if the function* $G_i = \log \circ F_{X_i}$ *satisfies the scale-shrinking property, that is for any* $x_i \in [0, \infty)$:

$$G_i(\lambda x_i) \leq \frac{1}{\lambda} G(x_i), \qquad \forall \lambda \in [1, \infty).$$

*Proof.* We will prove each claim separately.

(i) Suppose $F_{X_i}$ is differentiable then:

$$\begin{aligned}\phi_i^{'}(x_i) &= \log F_{X_i}(x_i) + \frac{x_i f_{X_i}(x_i)}{F_{X_i}(x_i)},\\ &= -\int_{x_i}^{\infty} \frac{f_{X_i}(w)}{F_{X_i}(w)}\mathrm{d}w + \frac{x_i f_{X_i}(x_i)}{F_{X_i}(x_i)},\\ &= -\int_{x_i}^{\infty} h_{X_i}(w)\mathrm{d}w + x_i h_{X_i}(x_i).\end{aligned}$$

That means $\phi_i^{'}(x_i) \le 0,\ \forall x_i \in [0,\infty)$, i.e. $\phi_i(x_i)$ is non-increasing for all $x_i \in [0,\infty)$, if and only if

$$x_i h_{X_i}(x_i) \le \int_{x_i}^{\infty} h_{X_i}(w)\mathrm{d}w, \quad \forall x_i \in [0,\infty).$$

(ii) Pick any $x_i \le y_i$ such that $y_i = \lambda x_i$, $\lambda \ge 1$, then

$$\begin{aligned} y_i G_i(y_i) &\le x_i G_i(x_i),\\ \iff \lambda x_i G_i(\lambda x_i) &\le x_i G_i(x_i),\\ \iff G_i(\lambda x_i) &\le \frac{1}{\lambda} G_i(x_i).\end{aligned}$$

□

Although the non-increasing property is tractable, it is stronger than what is required for VaR super-additivity. The next example shows that $\Phi$ may be SD without, or equivalently without each $\phi_i$, being non-increasing.

**Example 3.11.** *Let* $\boldsymbol{X} = (X_1, X_2, X_3)$ *be an independent random vector (a special case of NSD). Assume that* $X_1$ *and* $X_2$ *are Fréchet distributed with unit scales and shape parameters* $\alpha_1 = \alpha_2 = \frac{1}{2}$*, while* $X_3$ *has a piecewise CDF composed of a power-law part followed by a Fréchet*

*CDF with* $\theta_3 = 1$ *and* $\alpha_3 = \frac{1}{2}$*. Explicitly,*

$$F_{X_1}(x) = F_{X_2}(x) = \exp\left(-\frac{1}{\sqrt{x}}\right),$$

$$F_{X_3}(x_3) = \begin{cases} \dfrac{x_3^2}{e}, & 0 \leq x_3 \leq 1, \\ \exp\left(-\dfrac{1}{\sqrt{x_3}}\right), & x_3 > 1. \end{cases}$$

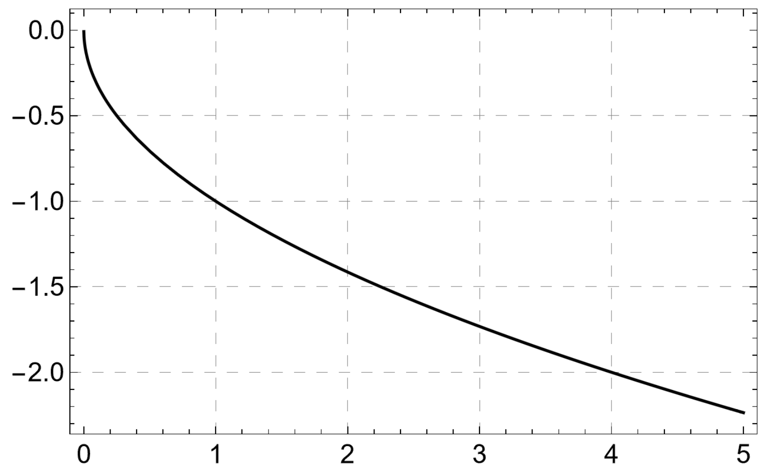
*The corresponding* $\phi_i$*-functions (as defined in Theorem 3.4) are*

$$\phi_1(x_1) = -\sqrt{x_1},$$

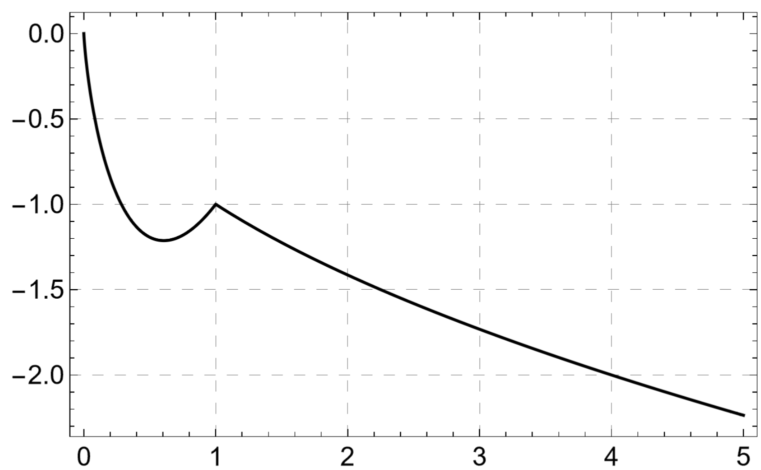
$$\phi_2(x_2) = -\sqrt{x_2},$$

$$\phi_3(x_3) = \begin{cases} x_3(2\log x_3 - 1), & 0 \leq x_3 \leq 1, \\ -\sqrt{x_3}, & x_3 > 1. \end{cases}$$

*It is clear that* $\phi_1$ *and* $\phi_2$ *are non-increasing, whereas* $\phi_3$ *fails to be non-increasing on the interval* $x_3 \in \left[\frac{1}{\sqrt{e}}, 1\right]$*. Figure 2 shows the graphs of these functions.*




(a) $\phi_1$ and $\phi_2$




(b) $\phi_3$

Figure 2: The marginal $\phi_i$ functions.

*By Proposition 3.6, this implies that* $\Phi$ *is not globally non-increasing. Nevertheless, we now verify that the SD condition for*

$$\Phi(x_1, x_2, x_3) = \phi_1(x_1) + \phi_2(x_2) + \phi_3(x_3)$$

*still holds. We claim that for all* $x_1, x_2, x_3 \geq 0$,

$$\Phi(x_1, x_2, x_3) \;\geq\; \Phi(s, s, s), \qquad s = x_1 + x_2 + x_3.$$

*Since* $\sqrt{x_1} + \sqrt{x_2} \leq \sqrt{2(x_1 + x_2)} = \sqrt{2(s - x_3)}$, *we obtain*

$$\Phi(x_1, x_2, x_3) = -\sqrt{x_1} - \sqrt{x_2} + \phi_3(x_3) \;\geq\; -\sqrt{2}\,\sqrt{s - x_3} + \phi_3(x_3).$$

*For fixed* $s$, *consider the function*

$$x_3 \mapsto 2\sqrt{s} - \sqrt{2}\sqrt{s - x_3} + \phi_3(x_3) - \phi_3(s).$$

*It is convex on each smooth piece of* $[0, s]$*; hence its minimum occurs at one of the points* $x_3 \in \{0, 1, s\}$*. Evaluating at these points yields nonnegative values:*

$$\begin{aligned}
&2\sqrt{s} - \sqrt{2}\sqrt{s - 0} + \phi_3(0) - \phi_3(s) \geq 0, \\
&2\sqrt{s} - \sqrt{2}\sqrt{s - 1} + \phi_3(1) - \phi_3(s) \geq 0, \\
&2\sqrt{s} - \sqrt{2}\sqrt{s - s} + \phi_3(s) - \phi_3(s) = 2\sqrt{s} \geq 0.
\end{aligned}$$

*Therefore,*

$$-\sqrt{2}\,\sqrt{s - x_3} + \phi_3(x_3) \;\geq\; -2\sqrt{s} + \phi_3(s) = \Phi(s, s, s),$$

*and the claim follows.*

*While the VaR of the sum* $S$ *can only be computed numerically, the VaRs of the marginals are given explicitly. For* $X_1$ *and* $X_2$,

$$\mathrm{VaR}_p[X_1] = \mathrm{VaR}_p[X_2] = \frac{1}{\log^2(1/p)},$$

*and for $X_3$,*

$$\mathrm{VaR}_p[X_3] = \begin{cases} \sqrt{e}\,\sqrt{p}, & 0 < p \le \dfrac{1}{e}, \\[2ex] \dfrac{1}{\log^2(1/p)}, & \dfrac{1}{e} < p < 1. \end{cases}$$

*Figure 3 compares* $\mathrm{VaR}_p[S]$ *with the sum of marginal VaRs.*

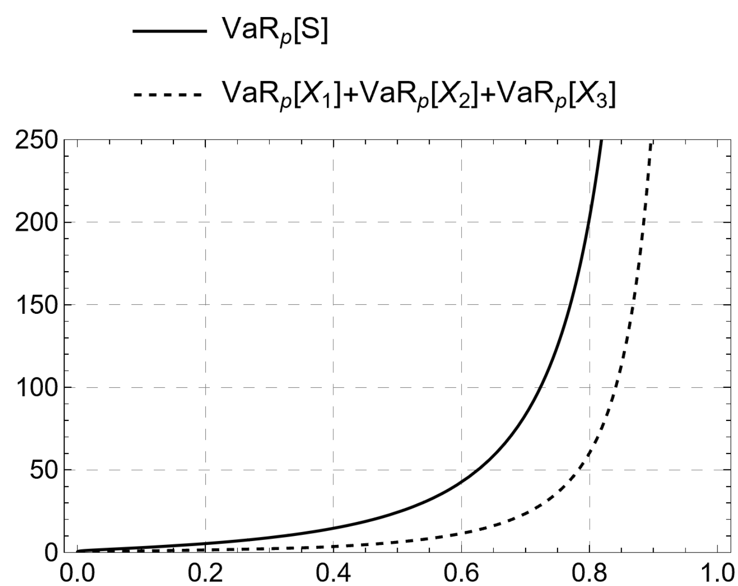



Figure 3: Comparison of $\mathrm{VaR}_p[S]$ and $\mathrm{VaR}_p[X_1] + \mathrm{VaR}_p[X_2] + \mathrm{VaR}_p[X_3]$.

*Figure 3 shows that* $\boldsymbol{X}$ *is VaR super-additive. This example illustrates that an NSD vector may have an SD aggregator* $\Phi$ *without* $\Phi$ *being globally non-increasing, while still exhibiting VaR super-additivity.*

A natural question that follows any characterization of a property for random vectors is: under what transformations does the property persist? In this spirit, we examine the conditions under which the transformed random vector

$$\widetilde{\boldsymbol{X}} = (\xi_1(X_1), \ldots, \xi_n(X_n)),$$

where each $\xi_i : [0, \infty) \to [0, \infty)$ is measurable, preserves the property of VaR super-additivity.

Specifically, we seek to identify assumptions on the functions $\xi_i$ that ensure $\widetilde{\boldsymbol{X}}$ remains VaR super-additive whenever the original vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is already VaR super-additive.

**Proposition 3.12.** *Let $\boldsymbol{X}$ be NLOD with continuous marginal distributions $F_{X_i}$, and suppose each $\phi_i$ in Equation (3.1) is non-increasing. Define*

$$\widetilde{\boldsymbol{X}} = (\widetilde{X}_1, \ldots, \widetilde{X}_n), \quad \text{where } \widetilde{X}_i = \xi_i(X_i).$$

*If each $\xi_i$ is strictly increasing, convex, and satisfies $\xi_i(0) = 0$, then $\widetilde{\boldsymbol{X}}$ is VaR super-additive.*

*Proof.* First, since $\boldsymbol{X}$ is NLOD and each $\xi_i$ is strictly increasing, we have

$$F_{\widetilde{\boldsymbol{X}}}(x_1, \ldots, x_n) = F_{\boldsymbol{X}}\left(\xi_1^{-1}(x_1), \ldots, \xi_n^{-1}(x_n)\right) \leq \prod_{i=1}^{n} F_{X_i}\left(\xi_i^{-1}(x_i)\right) = \prod_{i=1}^{n} F_{\widetilde{X}_i}(x_i),$$

which establishes that $\widetilde{\boldsymbol{X}}$ is NLOD.

Moreover, strict monotonicity and convexity of $\xi_i$ imply that $\xi_i^{-1}$ is continuous and strictly increasing. Combined with the continuity of $F_{X_i}$, this ensures that each marginal CDF

$$F_{\widetilde{X}_i} = F_{X_i} \circ \xi_i^{-1}$$

is continuous.

Second, define

$$\widetilde{\phi}_i(x_i) = x_i \log F_{\widetilde{X}_i}(x_i).$$

For $x_i < y_i$, we have

$$\begin{aligned} \widetilde{\phi}_i(y_i) &= y_i \log F_{\widetilde{X}_i}(y_i), \\ &= y_i \log F_{X_i}\left(\xi_i^{-1}(y_i)\right), \end{aligned}$$

$$\leq y_i \frac{\xi_i^{-1}(x_i)}{\xi_i^{-1}(y_i)} \log F_{X_i}\left(\xi_i^{-1}(x_i)\right),$$

where the last inequality follows by applying the non-increasing property of $\phi_i$ to the strictly increasing pair $\xi_i^{-1}(x_i) < \xi_i^{-1}(y_i)$.

By convexity of $\xi_i$ and the condition $\xi_i(0) = 0$, the secant slopes from the origin are non-decreasing: for $0 < u_i < v_i$,

$$\frac{\xi_i(u_i)}{u_i} \leq \frac{\xi_i(v_i)}{v_i}.$$

Setting $u_i = \xi_i^{-1}(x_i)$ and $v_i = \xi_i^{-1}(y_i)$ gives

$$\frac{x_i}{\xi_i^{-1}(x_i)} \leq \frac{y_i}{\xi_i^{-1}(y_i)} \quad \Longrightarrow \quad x_i \leq y_i \frac{\xi_i^{-1}(x_i)}{\xi_i^{-1}(y_i)}.$$

Combining these results, and noting that $\log \circ F_{X_i}$ is a negative function, we obtain

$$\widetilde{\phi}_i(y_i) \leq y_i \frac{\xi_i^{-1}(x_i)}{\xi_i^{-1}(y_i)} \log F_{X_i}\big(\xi_i^{-1}(x_i)\big) \leq x_i \log F_{X_i}\big(\xi_i^{-1}(x_i)\big) = \widetilde{\phi}_i(x_i),$$

so $\widetilde{\phi}_i(x_i)$ is non-increasing for all $x_i \in [0, \infty)$.

Applying Corollary 3.7, we conclude that $\widetilde{\boldsymbol{X}}$ is VaR super-additive. □

We conclude this section by noting that although the NSD and SD properties allow Theorem 3.4 to characterize a broad class of random vectors for which super-additivity is guaranteed, they are not the only indicators of VaR super-additivity. The following example illustrates situations in which VaR is super-additive even when neither NSD nor SD is satisfied.

**Example 3.13.** *Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate random vector.*

*(1) Assume that $\boldsymbol{X}$ follows a bivariate Pareto distribution of Type II with unit scale parame-*

*ters and shape* $0 < \alpha \leq 1$. *Its joint DDF is*

$$\overline{F}_{\boldsymbol{X}}(x_1, x_2) = (1 + x_1 + x_2)^{-\alpha}, \qquad x_1, x_2 \geq 0.$$

*Consequently,* $X_1$ *and* $X_2$ *have Pareto (II) marginal CDFs with the same shape parameter:*

$$F_{X_1}(x) = F_{X_2}(x) = 1 - (1 + x)^{-\alpha}, \qquad x \geq 0.$$

*A direct computation shows that the CDF of the sum* $S = X_1 + X_2$ *is*

$$F_S(s) = 1 - (1 + s)^{-\alpha-1}\big(1 + (\alpha + 1)s\big), \qquad s \geq 0.$$

*We now compare* $F_S(t)$ *with* $F_{X_1}(t)F_{X_2}(t)$. *Since* $0 < \alpha \leq 1$, *one checks that*

$$\frac{\alpha t}{1 + t} \leq 1 - (1 + t)^{-\alpha}, \qquad t \geq 0.$$

*Using this inequality, we rewrite* $F_S(t)$ *as*

$$\begin{aligned} F_S(t) &= 1 - (1 + t)^{-\alpha-1}(1 + (\alpha + 1)t) \\ &= 1 - (1 + t)^{-\alpha}\left(1 + \frac{\alpha t}{1 + t}\right) \\ &\geq 1 - (1 + t)^{-\alpha}\left(2 - (1 + t)^{-\alpha}\right) \\ &= 1 - 2(1 + t)^{-\alpha} + (1 + t)^{-2\alpha} \\ &= \big(1 - (1 + t)^{-\alpha}\big)^2 \\ &= F_{X_1}(t)\, F_{X_2}(t). \end{aligned}$$

*Hence* $F_S(t) \geq F_{X_1}(t)F_{X_2}(t)$ *for all* $t \geq 0$, *meaning that* $\boldsymbol{X}$ *is* not *NSD.*

*From Example 3.9, the functions* $\phi_i$ *are non-increasing for Pareto (II) marginals with* $0 < \alpha \leq 1$*, and therefore* $\Phi$ *is SD.*

*To compute VaRs, set* $\alpha = 1$ *for simplicity. Then*

$$\mathrm{VaR}_p[X_1] = \mathrm{VaR}_p[X_2] = \frac{p}{1-p}, \qquad \mathrm{VaR}_p[S] = \frac{p+\sqrt{p}}{1-p}.$$

*Since* $\sqrt{p} \geq p$*, we have*

$$\mathrm{VaR}_p[S] = \frac{p+\sqrt{p}}{1-p} \geq \frac{2p}{1-p} = \mathrm{VaR}_p[X_1] + \mathrm{VaR}_p[X_2],$$

*showing that* $\boldsymbol{X}$ *is VaR super-additive.*

(2) *Next, suppose that* $\boldsymbol{X}$ *is a mutually exclusive (Dhaene & Denuit, 1999) discrete vector supported on*

$$\big\{(2^{-k}, 0) : k \geq 1\big\} \;\cup\; \big\{(0, 2^{-k}) : k \geq 1\big\},$$

*with joint probability masses*

$$\mathbb{P}\big((X_1, X_2) = (2^{-k}, 0)\big) = \mathbb{P}\big((X_1, X_2) = (0, 2^{-k})\big) = \frac{1}{2^{\,k-1}}, \qquad k \geq 1.$$

*Using geometric-series identities, one obtains the marginal CDFs*

$$F_{X_1}(x) = F_{X_2}(x) = \begin{cases} 0, & x < 0, \\ \frac{1}{2}, & 0 \leq x < 2, \\ 1 - 2^{-(k+1)}, & 2^k \leq x < 2^{k+1}, \quad k \geq 1, \end{cases}$$

*and the CDF of the sum*

$$F_S(s) = \begin{cases} 0, & s < 2, \\ 1 - 2^{-k}, & 2^k \le s < 2^{k+1}, \quad k \ge 1. \end{cases}$$

*For $t < 2$, we immediately see that $F_S(t) < F_{X_1}(t)F_{X_2}(t)$. For $t \ge 2$ (with $2^k \le t < 2^{k+1}$), we have*

$$F_S(t) = 1 - 2^{-k} < 1 - 2^{-k} + 2^{-(2k+2)} = \left(1 - 2^{-(k+1)}\right)^2 = F_{X_1}(t)F_{X_2}(t),$$

*so $\boldsymbol{X}$ is NSD. This expected as the random vector $\boldsymbol{X}$ has a counter-monotonic joint law which belongs to the NSD class.*

*The marginal CDFs are discontinuous, and the functions $\phi_i(x) = x \log F_{X_i}(x)$ are not non-increasing. To see the latter, take $x = 2^k < y = 2^{k+1}$, $k \ge 1$. Then*

$$\begin{aligned} \phi_i(y) - \phi_i(x) &= 2^{k+1} \log\left(1 - 2^{-(k+2)}\right) - 2^k \log\left(1 - 2^{-(k+1)}\right) \\ &= 2^k \log\left(1 + \frac{1}{2^{k+3}} \frac{1}{2^{k+1} - 1}\right) > 0. \end{aligned}$$

*Thus $\phi_i$ is strictly increasing along the sequence $\{2^k\}$, and consequently $\Phi$ is not SD (take $x = y = 2^k$ and $s = x + y = 2^{k+1}$ then $\Phi(x, y) = \Phi(2^k, 2^k) < \Phi(2^{k+1}, 2^{k+1}) = \Phi(s, s)$).*

*Finally, the VaR functions are*

$$\mathrm{VaR}_p[X_1] = \mathrm{VaR}_p[X_2] = \begin{cases} 0, & 0 < p \le \frac{1}{2}, \\ 2^k, & 1 - 2^{-k} < p \le 1 - 2^{-(k+1)}, \quad k \ge 1, \end{cases}$$

$$\mathrm{VaR}_p[S] = 2^k, \qquad 1 - 2^{-(k-1)} < p \le 1 - 2^{-k}, \quad k \ge 1.$$

*Hence, for* $0 < p \leq \frac{1}{2}$,

$$\mathrm{VaR}_p[S] = 2 > 0 = \mathrm{VaR}_p[X_1] + \mathrm{VaR}_p[X_2],$$

*and for any* $k \geq 2$ *and the corresponding range of* $p$,

$$\mathrm{VaR}_p[S] = 2^k = 2^{k-1} + 2^{k-1} = \mathrm{VaR}_p[X_1] + \mathrm{VaR}_p[X_2].$$

*Therefore,* $\boldsymbol{X}$ *is VaR super-additive in this case as well.*

# 4 Further Generalizations and Remarks

The results established in Sections 2 and 3 extend naturally to random variables whose supports begin at arbitrary finite lower end-points

$$a_i = \sup\{x \in \mathbb{R} : F_{X_i}(x) \leq 0\} > -\infty, \qquad \forall i \in \{1, \ldots, n\}.$$

We denote the corresponding random vector by

$$\boldsymbol{X}^{\boldsymbol{a}} = (X_1^{a_1}, \ldots, X_n^{a_n}), \qquad \boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{R}^n,$$

with $\boldsymbol{X}^{\mathbf{0}} = \boldsymbol{X}$ representing the previously studied case of random variables supported on $[0, \infty)$.

Similarly, the theory extends to random variables possessing arbitrary finite upper end-points

$$b_i = \inf\{x \in \mathbb{R} : F_{X_i}(x) \geq 1\} < \infty, \qquad \forall i \in \{1, \ldots, n\},$$

for which we write

$$\boldsymbol{X}^{\boldsymbol{b}} = (X_1^{b_1}, \ldots, X_n^{b_n}), \qquad \boldsymbol{b} = (b_1, \ldots, b_n) \in \mathbb{R}^n.$$

For notational convenience, define the corresponding sum random variables by

$$S^{\boldsymbol{a}} = \sum_{i=1}^{n} X_i^{a_i}, \qquad S^{\boldsymbol{b}} = \sum_{i=1}^{n} X_i^{b_i}.$$

It is immediate that both transformed vectors admit the simple representations

$$\boldsymbol{X}^{\boldsymbol{a}} = \boldsymbol{a} + \boldsymbol{X}, \qquad \boldsymbol{X}^{\boldsymbol{b}} = \boldsymbol{b} - \boldsymbol{X},$$

and, due to the translation and scale equivariance of VaR, their components satisfy

$$\begin{aligned} \mathrm{VaR}_p[X_i^{a_i}] &= \mathrm{VaR}_p[a_i + X_i] = a_i + \mathrm{VaR}_p[X_i], \\ \mathrm{VaR}_p[X_i^{b_i}] &= \mathrm{VaR}_p[b_i - X_i] = b_i - \mathrm{VaR}_{1-p}[X_i]. \end{aligned}$$

Both facts will be crucial in what follows.

**Proposition 4.1.** *The following equivalences hold:*

*(i)* $\boldsymbol{X}^{\boldsymbol{a}}$ *is VaR sub-additive if and only if* $\boldsymbol{X}^{\boldsymbol{a}}$ *is VaR additive.*

*(ii)* $\boldsymbol{X}^{\boldsymbol{b}}$ *is VaR super-additive if and only if* $\boldsymbol{X}^{\boldsymbol{b}}$ *is VaR additive.*

*In both cases, the random vectors* $\boldsymbol{X}^{\boldsymbol{a}}$ *and* $\boldsymbol{X}^{\boldsymbol{b}}$ *must be co-monotonic.*

*Proof.* The proof in each case follows from the translation and scale equivariance properties of VaR.

(i) Using translation equivariance,

$$\boldsymbol{X}^{\boldsymbol{a}} \text{ is VaR sub-additive} \iff \boldsymbol{X} \text{ is VaR sub-additive.}$$

By Theorem 2.2,

$$\boldsymbol{X}^{\boldsymbol{a}} \text{ is VaR sub-additive} \iff \boldsymbol{X} \text{ is VaR additive.}$$

Applying translation equivariance once more yields

$$\boldsymbol{X}^{\boldsymbol{a}} \text{ is VaR sub-additive} \iff \boldsymbol{X}^{\boldsymbol{a}} \text{ is VaR additive.}$$

(ii) Using both scale and translation equivariance,

$$\boldsymbol{X}^{\boldsymbol{b}} \text{ is VaR super-additive} \iff \boldsymbol{X} \text{ is VaR sub-additive.}$$

Applying Theorem 2.2 again gives

$$\boldsymbol{X}^{\boldsymbol{b}} \text{ is VaR super-additive} \iff \boldsymbol{X} \text{ is VaR additive.}$$

Repeating the equivariance arguments leads to

$$\boldsymbol{X}^{\boldsymbol{b}} \text{ is VaR super-additive} \iff \boldsymbol{X}^{\boldsymbol{b}} \text{ is VaR additive.}$$

Finally, in both parts, co-monotonicity follows directly from Theorem 2.2. □

The preceding proposition highlights an important structural limitation: VaR sub-additivity cannot occur for random variables with finite lower end-points, while VaR super-additivity cannot occur for random variables with finite upper end-points.

**Corollary 4.2.** *For compactly supported random variables $\boldsymbol{X}^{\boldsymbol{a},\boldsymbol{b}}$, i.e. random variables possessing both finite lower and upper end-points, VaR sub-additivity and VaR super-additivity are each equivalent to VaR additivity. Consequently, such random variables can never exhibit strict VaR sub-additivity or strict VaR super-additivity.*

The limitations of VaR in the prior discussion motivates the search for conditions, analogous to those developed in Section 3, that permit the analysis of VaR super- and sub-additivity in more flexible settings. That prompts us to extend the general results of Section 3 to these shifted and scaled settings. In particular, the following proposition provides analogous conditions for VaR super-additivity of the shifted vector $\boldsymbol{X}^{\boldsymbol{a}}$ and VaR sub-additivity of the reflected and shifted vector $\boldsymbol{X}^{\boldsymbol{b}}$.

**Proposition 4.3.** *(i) Suppose $\boldsymbol{X}^{\boldsymbol{a}}$ has continuous marginal CDFs $F_{X_i^{a_i}}$ and satisfies*

$$F_{S^{\boldsymbol{a}}}(t+a_+) \leq \prod_{i=1}^{n} F_{X_i^{a_i}}(t+a_i), \qquad a_+ = \sum_{i=1}^{n} a_i, \quad \forall t \in [0,\infty), \tag{4.1}$$

*and that the function*

$$\Phi^{\boldsymbol{a}}(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i \log F_{X_i^{a_i}}(x_i+a_i), \qquad x_i \in [0,\infty), \tag{4.2}$$

*is SD. Then $\boldsymbol{X}^{\boldsymbol{a}}$ is VaR super-additive.*

*(ii) Suppose $\boldsymbol{X}^{\boldsymbol{b}}$ has continuous marginal DDFs $\overline{F}_{X_i^{b_i}}$ and satisfies*

$$\overline{F}_{S^{\boldsymbol{b}}}(b_+ - t) \leq \prod_{i=1}^{n} \overline{F}_{X_i^{b_i}}(b_i - t), \qquad b_+ = \sum_{i=1}^{n} b_i, \quad \forall t \in [0,\infty), \tag{4.3}$$

*and that the function*

$$\Phi^{\boldsymbol{b}}(x_1,\ldots,x_n) = \sum_{i=1}^{n} x_i \log \overline{F}_{X_i^{b_i}}(b_i - x_i), \qquad x_i \in [0,\infty), \tag{4.4}$$

*is SD. Then $\boldsymbol{X}^{\boldsymbol{b}}$ is VaR sub-additive.*

*Proof.* Continuity of each $F_{X_i}$ follows from the continuity of $F_{X_i^{a_i}}$ or of $\overline{F}_{X_i^{b_i}}$.

(i) Since $\boldsymbol{X}^{\boldsymbol{a}} = \boldsymbol{a} + \boldsymbol{X}$ and $S^{\boldsymbol{a}} = S + a_+$, we have

$$F_{S^{\boldsymbol{a}}}(t+a_+) = F_S(t), \qquad F_{X_i^{a_i}}(x_i+a_i) = F_{X_i}(x_i).$$

Thus the condition in Equation (4.1) implies

$$F_S(t) \leq \prod_{i=1}^{n} F_{X_i}(t), \qquad \forall t \in [0, \infty),$$

i.e. $\boldsymbol{X}$ is NSD. Moreover, if

$$\Phi^{\boldsymbol{a}}(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \log F_{X_i^{a_i}}(x_i + a_i)$$

is SD, then so is

$$\Phi(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i \log F_{X_i}(x_i).$$

By Theorem 3.4, $\boldsymbol{X}$ is VaR super-additive. Translation equivariance then gives that $\boldsymbol{X}^{\boldsymbol{a}}$ is VaR super-additive.

(ii) Since $\boldsymbol{X}^{\boldsymbol{b}} = \boldsymbol{b} - \boldsymbol{X}$ and $S^{\boldsymbol{b}} = b_+ - S$, we obtain

$$\overline{F}_{S^{\boldsymbol{b}}}(b_+ - t) = F_S(t), \qquad \overline{F}_{X_i^{b_i}}(b_i - x_i) = F_{X_i}(x_i).$$

Applying the same reasoning as in part (i), the given assumptions imply that $\boldsymbol{X}$ is VaR super-additive. Using both scale and translation equivariance, we conclude that $\boldsymbol{X}^{\boldsymbol{b}}$ is VaR sub-additive. □

Using the results we obtained in Proposition 4.3, we can now delineate the sufficient conditions that parallel those of Propositions 3.5 and 3.6. These conditions are easily verifiable and ensure that $\boldsymbol{X}^{\boldsymbol{a}}$ (resp. $\boldsymbol{X}^{\boldsymbol{b}}$) is VaR super-additive (resp. VaR sub-additive).

**Proposition 4.4.** *(i) If $\boldsymbol{X}^{\boldsymbol{a}}$ is NLOD with continuous $F_{X_i^{a_i}}$, and if each function appearing in Equation (4.2),*

$$\phi_i^{a_i}(x_i) = x_i \log F_{X_i^{a_i}}(x_i + a_i), \qquad x_i \in [0, \infty),$$

*is non-increasing, then* $\boldsymbol{X}^{\boldsymbol{a}}$ *is VaR super-additive.*

*(ii) If* $\boldsymbol{X}^{\boldsymbol{b}}$ *is NUOD (defined analogously to NLOD but with DDFs instead of CDFs) with continuous* $\overline{F}_{X_i^{b_i}}$*, and if each function appearing in Equation (4.4),*

$$\phi_i^{b_i}(x_i) = x_i \log \overline{F}_{X_i^{b_i}}(b_i - x_i), \qquad x_i \in [0, \infty),$$

*is non-increasing, then* $\boldsymbol{X}^{\boldsymbol{b}}$ *is VaR sub-additive.*

*Proof.* (i) We begin by verifying that the condition in Equation (4.1) holds. Since $\boldsymbol{X}^{\boldsymbol{a}}$ is NLOD, we have

$$F_{\boldsymbol{X}^{\boldsymbol{a}}}(x_1, \ldots, x_n) \leq \prod_{i=1}^{n} F_{X_i^{a_i}}(x_i), \qquad \forall x_i \in [a_i, \infty).$$

To relate this to the distribution of the shifted sum $S^{\boldsymbol{a}}$, observe that the $n$-box $[a_1, x_1] \times \cdots \times [a_n, x_n]$ contains the $n$-simplex with origin $(a_1, \ldots, a_n)$ and vertices

$$\{(x_1, a_2, \ldots, a_n), (a_1, x_2, \ldots, a_n), \ldots, (a_1, a_2, \ldots, x_n)\}.$$

Setting each $x_i = t + a_i$ with $t \in [0, \infty)$ ensures that this simplex lies inside the box, and therefore

$$F_{S^{\boldsymbol{a}}}(t + a_+) \leq F_{\boldsymbol{X}^{\boldsymbol{a}}}(t + a_1, \ldots, t + a_n) \leq \prod_{i=1}^{n} F_{X_i^{a_i}}(t + a_i), \qquad \forall t \in [0, \infty),$$

$$\implies \qquad F_{S^{\boldsymbol{a}}}(t + a_+) \leq \prod_{i=1}^{n} F_{X_i^{a_i}}(t + a_i).$$

Hence the requirement in Equation (4.1) is satisfied. As in Proposition 3.6, note that it actually suffices for $\boldsymbol{X}^{\boldsymbol{a}}$ to be NLOD only along the shifted diagonal $(t + a_1, \ldots, t + a_n)$, since this is the only region relevant for the comparison with $S^{\boldsymbol{a}}$.

Next, if each function $\phi_i^{a_i}$ is non-increasing, then by Proposition 3.6, the function $\Phi^{\boldsymbol{a}}$ is SD. Combining this property with the continuity of each $F_{X_i^{a_i}}$, we may invoke Proposition 4.3 to conclude that $\boldsymbol{X}^{\boldsymbol{a}}$ is VaR super-additive.

(ii) The proof mirrors that of part (i). Using the NUOD property of $\boldsymbol{X}^{\boldsymbol{b}}$, we obtain

$$\overline{F}_{\boldsymbol{X}^{\boldsymbol{b}}}(x_1, \dots, x_n) \le \prod_{i=1}^{n} \overline{F}_{X_i^{b_i}}(x_i), \qquad \forall x_i \in (-\infty, b_i].$$

In this setting, the $n$-box $[x_1, b_1] \times \cdots \times [x_n, b_n]$ contains a "reversed" $n$-simplex with origin $(b_1, \dots, b_n)$ and vertices

$$\{(x_1, b_2, \dots, b_n), (b_1, x_2, \dots, b_n), \dots, (b_1, b_2, \dots, x_n)\}.$$

Setting $x_i = b_i - t$ with $t \in [0, \infty)$ gives

$$\overline{F}_{S^{\boldsymbol{b}}}(b_+ - t) \le \overline{F}_{\boldsymbol{X}^{\boldsymbol{b}}}(b_1 - t, \dots, b_n - t) \le \prod_{i=1}^{n} \overline{F}_{X_i^{b_i}}(b_i - t), \qquad \forall t \in [0, \infty),$$

$$\Longrightarrow \qquad \overline{F}_{S^{\boldsymbol{b}}}(b_+ - t) \le \prod_{i=1}^{n} \overline{F}_{X_i^{b_i}}(b_i - t).$$

Thus the condition in Equation (4.3) holds. Again, as in part (i), it suffices that the NUOD property holds only along the shifted diagonal $(b_1 - t, \dots, b_n - t)$.

Finally, if each $\phi_i^{b_i}$ is non-increasing, then Proposition 3.6 guarantees that $\Phi^{\boldsymbol{b}}$ is SD. Together with continuity of each $\overline{F}_{X_i^{b_i}}$, Proposition 4.3 implies that $\boldsymbol{X}^{\boldsymbol{b}}$ is VaR sub-additive.

□

We end this section by investigating what happens if we take measurable functions of the components of $\boldsymbol{X}^{\boldsymbol{a}}$ (resp. $\boldsymbol{X}^{\boldsymbol{b}}$) when VaR super-additivity (resp. VaR sub-additivity) holds.

The results are direct extension of those in Proposition 3.12.

**Proposition 4.5.** *(i) Suppose $\boldsymbol{X}^{\boldsymbol{a}}$ is NLOD with continuous margins $F_{X_i^{a_i}}$, and assume that each $\phi_i^{a_i}$ in Equation (4.2) is non-increasing. Let*

$$\widetilde{\boldsymbol{X}^{\boldsymbol{a}}} = \left(\widetilde{X}_1^{a_1}, \ldots, \widetilde{X}_n^{a_n}\right),$$

*where $\widetilde{X}_i^{a_i} = \xi_i(X_i^{a_i})$ for $\xi_i : [a_i, \infty) \to [a_i, \infty)$. If each $\xi_i$ is strictly increasing, convex, and satisfies $\xi_i(a_i) = a_i$, then $\widetilde{\boldsymbol{X}^{\boldsymbol{a}}}$ is VaR super-additive.*

*(ii) Assume $\boldsymbol{X}^{\boldsymbol{b}}$ is NUOD with continuous margins $\overline{F}_{X_i^{b_i}}$, and suppose that each $\phi_i^{b_i}$ in Equation (4.4) is non-increasing. Define*

$$\widetilde{\boldsymbol{X}^{\boldsymbol{b}}} = \left(\widetilde{X}_1^{b_1}, \ldots, \widetilde{X}_n^{b_n}\right),$$

*where $\widetilde{X}_i^{b_i} = \xi_i(X_i^{b_i})$ for $\xi_i : (-\infty, b_i] \to (-\infty, b_i]$. If each $\xi_i$ is strictly increasing, convex, and satisfies $\xi_i(b_i) = b_i$, then $\widetilde{\boldsymbol{X}^{\boldsymbol{b}}}$ is VaR sub-additive.*

*Proof.* The argument follows the same structure as Proposition 3.12. Under the stated assumptions, two observations hold immediately:

- Since the margins $F_{X_i^{a_i}}$ and $\overline{F}_{X_i^{b_i}}$ are continuous and each $\xi_i$ is strictly increasing and convex, it follows that the transformed margins $F_{\widetilde{X}_i^{a_i}}$ and $\overline{F}_{\widetilde{X}_i^{b_i}}$ are also continuous.
- The strict monotonicity of the mappings $\xi_i$ ensures that the NLOD (resp. NUOD) property of $\boldsymbol{X}^{\boldsymbol{a}}$ (resp. $\boldsymbol{X}^{\boldsymbol{b}}$) is preserved by the coordinate-wise transformation, so $\widetilde{\boldsymbol{X}^{\boldsymbol{a}}}$ (resp. $\widetilde{\boldsymbol{X}^{\boldsymbol{b}}}$) is likewise NLOD (resp. NUOD).

Thus, it remains to verify that $\widetilde{\phi}_i^{a_i}$ and $\widetilde{\phi}_i^{b_i}$ are non-increasing.

(i) Case of $\widetilde{\phi}_i^{a_i}$: Fix $x_i < y_i$. Then,

$$\begin{aligned}\widetilde{\phi}_i^{a_i}(y_i) &= y_i \log F_{\widetilde{X}_i^{a_i}}(y_i + a_i)\\ &= y_i \log F_{X_i^{a_i}}\left(\xi_i^{-1}(y_i + a_i)\right).\end{aligned}$$

Applying the non-increasing property of $\phi_i^{a_i}$ to the strictly increasing pair

$$\xi_i^{-1}(x_i + a_i) - a_i < \xi_i^{-1}(y_i + a_i) - a_i$$

yields

$$\widetilde{\phi}_i^{a_i}(y_i) \le y_i \frac{\xi_i^{-1}(x_i + a_i) - a_i}{\xi_i^{-1}(y_i + a_i) - a_i} \log F_{X_i^{a_i}}(\xi_i^{-1}(x_i + a_i)).$$

Next, the convexity of $\xi_i$ and the condition $\xi_i(a_i) = a_i$ imply that the secant slopes from $a_i$ are non-decreasing: for all $a_i < u_i < v_i$,

$$\frac{\xi_i(u_i) - a_i}{u_i - a_i} \le \frac{\xi_i(v_i) - a_i}{v_i - a_i}.$$

With $u_i = \xi_i^{-1}(x_i + a_i)$ and $v_i = \xi_i^{-1}(y_i + a_i)$, this becomes

$$\frac{x_i}{\xi_i^{-1}(x_i + a_i) - a_i} \le \frac{y_i}{\xi_i^{-1}(y_i + a_i) - a_i},$$

which is equivalent to

$$x_i \le y_i \frac{\xi_i^{-1}(x_i + a_i) - a_i}{\xi_i^{-1}(y_i + a_i) - a_i}.$$

Since $\log \circ F_{X_i^{a_i}}$ is negative, combining the inequalities gives

$$\widetilde{\phi}_i^{a_i}(y_i) \le y_i \frac{\xi_i^{-1}(x_i + a_i) - a_i}{\xi_i^{-1}(y_i + a_i) - a_i} \log F_{X_i^{a_i}}(\xi_i^{-1}(x_i + a_i))$$

$$\leq x_i \log F_{X_i^{a_i}}(\xi_i^{-1}(x_i + a_i)) = \widetilde{\phi}_i^{a_i}(x_i).$$

Hence, $\widetilde{\phi}_i^{a_i}$ in (4.2) is non-increasing on $[0, \infty)$. By Proposition 4.4, we conclude that $\widetilde{\boldsymbol{X}}^{\boldsymbol{a}}$ is VaR super-additive.

(ii) Case of $\widetilde{\phi}_i^{b_i}$: An analogous argument applies. Let $x_i < y_i$. Then

$$\begin{aligned}\widetilde{\phi}_i^{b_i}(y_i) &= y_i \log \overline{F}_{\widetilde{X}_i^{b_i}}(b_i - y_i)\\ &= y_i \log \overline{F}_{X_i^{b_i}}\big(\xi_i^{-1}(b_i - y_i)\big).\end{aligned}$$

Applying the non-increasing property of $\phi_i^{b_i}$ to the strictly increasing pair

$$b_i - \xi_i^{-1}(b_i - x_i) < b_i - \xi_i^{-1}(b_i - y_i)$$

gives

$$\widetilde{\phi}_i^{b_i}(y_i) \leq y_i \frac{b_i - \xi_i^{-1}(b_i - x_i)}{b_i - \xi_i^{-1}(b_i - y_i)} \log \overline{F}_{X_i^{b_i}}\big(\xi_i^{-1}(b_i - x_i)\big).$$

Furthermore, convexity of $\xi_i$ and the constraint $\xi_i(b_i) = b_i$ imply that secant slopes from $b_i$ are non-decreasing: for $u_i < v_i < b_i$,

$$\frac{b_i - \xi_i(u_i)}{b_i - u_i} \leq \frac{b_i - \xi_i(v_i)}{b_i - v_i}.$$

Substituting $u_i = \xi_i^{-1}(b_i - x_i)$ and $v_i = \xi_i^{-1}(b_i - y_i)$ yields

$$\frac{x_i}{b_i - \xi_i^{-1}(b_i - x_i)} \leq \frac{y_i}{b_i - \xi_i^{-1}(b_i - y_i)},$$

which is equivalent to

$$x_i \leq y_i \frac{b_i - \xi_i^{-1}(b_i - x_i)}{b_i - \xi_i^{-1}(b_i - y_i)}.$$

Since $\log \circ \overline{F}_{X_i^{b_i}}$ is negative, we conclude

$$\begin{aligned}\widetilde{\phi}_i^{b_i}(y_i) &\leq y_i \frac{b_i - \xi_i^{-1}(b_i - x_i)}{b_i - \xi_i^{-1}(b_i - y_i)} \log \overline{F}_{X_i^{b_i}}\big(\xi_i^{-1}(b_i - x_i)\big) \\ &\leq x_i \log \overline{F}_{X_i^{b_i}}\big(\xi_i^{-1}(b_i - x_i)\big) = \widetilde{\phi}_i^{b_i}(x_i).\end{aligned}$$

Thus $\widetilde{\phi}_i^{b_i}$ in (4.4) is non-increasing on $[0, \infty)$, and by Proposition 4.4, $\widetilde{\boldsymbol{X}^{\boldsymbol{b}}}$ is VaR sub-additive.

□

# 5 Conclusions

This paper provides a comprehensive characterization of the extremal aggregation behavior of Value-at-Risk for sums of one-sided random variables. We first established an impossibility result: for risks supported on $[0, \infty)$, VaR sub-additivity can arise only through exact additivity – a phenomenon exclusive to co-monotonic vectors. On the opposite end of the spectrum, we developed a general and flexible framework for full VaR super-additivity. The key insight is that super-additivity does not follow from dependence or marginal structure in isolation, but from their joint interaction as captured by the NSD and SD conditions. These conditions unify and extend existing results in the literature, while accommodating non-identical margins and a diverse range of negative dependence structures.

We further showed that the theory remains robust under shifts, reflections, and monotone convex transformations of the components, and that analogous principles govern aggregation when the random variables have arbitrary finite endpoints. Taken together, the results reveal a sharp dichotomy: in lower-bounded settings, VaR is structurally incompatible with sub-additivity yet naturally exhibits super-additivity under suitable dependence–margin configurations, whereas the pattern is reversed in upper-bounded settings. This characterization not only clarifies the conditions under which VaR behaves as a diversification-averse or diversification-seeking risk measure, but also offers practical criteria for detecting such behavior in applications involving heavy tails or negatively dependent risks.

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