The Moroccan Public Procurement Game

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Abstract

In this paper, we study the public procurement market through the lens of game theory by modeling it as a strategic game with discontinuous and non-quasiconcave payoffs. We first show that the game admits no Nash equilibrium in pure strategies. We then analyze the two-player case and derive two explicit mixed-strategy equilibria for the symmetric game and for the weighted $(p, 1-p)$ formulation. Finally, we establish the existence of a symmetric Nash equilibrium in the general N-player case by applying the diagonal disjoint payoff matching condition, which allows us to extend equilibrium existence to the mixed-strategy setting despite payoff discontinuities.

Keywords: Discontinuous games – Public procurement market – Nash equilibrium – Diagonal disjoint payoff matching

JEL Classification: C62 – C72 – D44

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1 Introduction

In March 2023, the Kingdom of Morocco implemented a significant reform of its public procurement rules for works and service contracts (under the previous system, contracts were awarded to the lowest bidder). According to the decree (Kingdom of Morocco, 2023), the *reference price* P is defined as:

$$P = \frac{E + \bar{x}}{2} \tag{1}$$

where E is the estimated cost established by the contracting authority, and $\bar{x}$ denotes the average of all submitted bids.

The contract is awarded according to the following rules:

(1) Bids must fall within an admissible interval $[A, B]$; otherwise, they are excluded.

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(2) The best offer is the one closest from below to the reference price P.

(3) If no bid lies below P, the winning offer is the closest to P from above.

(4) In the event of a tie, the winner is chosen by lot.

This rule induces a non-cooperative N-player game where each bidder chooses a strategy $x_i \in [A, B]$. Because the outcome depends discontinuously in an exotic way on the relative positions of the bids with respect to P, the resulting payoff functions are non-smooth, lack convexity, and violate the semi-continuity assumptions required for classical equilibrium existence theorems.

From an economic perspective, the new rule transforms the procurement process into a coordination contest similar to a Keynesian beauty contest: bidders attempt to anticipate the collective behavior of others rather than minimizing costs. This feature makes the game both aggregative and discontinuous.

Discontinuous games are ubiquitous in economics. In particular, models of oligopolistic competition, such as those of Bertrand and Cournot, exhibit discontinuities in the players' objective functions (Reny (2020)). Classical results such as Nash's theorem (Nash (1950)) or Glicksberg's theorem (Glicksberg (1952)) are of limited use in this context, and one must instead appeal to more equilibrium existence results developed for discontinuous games. For a recent comprehensive review of the literature on discontinuous games, one may consult Reny (2020).

One of the most important contributions to the study of discontinuous games is the paper by Partha Dasgupta and Eric Maskin (Dasgupta and Maskin (1986)). The significance of their result lies in showing how an infinite game can be approximated by a sequence of finite games, thereby providing a constructive method for approximating equilibria.

Significant progress has been made in the analysis of symmetric discontinuous games. The first major result was established by Baye et al. (1993), who proved the existence of equilibria for discontinuous and non-quasiconcave games satisfying the notions of diagonal transfer continuity and diagonal transfer quasiconcavity. Reny (1999) introduced a similar existence results for quasisymmetric, compact, diagonally quasiconcave games possessing the diagonal better-reply security property. A later and more general contribution was provided by Bich and Laraki (2012), who extended this existence result to games possessing the diagonal local better-reply–correspondence property.

In cases where pure strategy equilibria fail to exist, the existence of mixed strategy equilibria can still be established under certain conditions in discontinuous games, but proving properties like better-replay security becomes more challenging.

To establish these properties for the mixed extension, several authors have developed conditions under which they are inherited from the pure strategy game to its mixed strategy version. A remarkable contribution by Monteiro and Page Jr (2007) proves equilibrium existence via the notion of *uniform payoff security*, and a closely related concept, *uniformly diagonal security*, was later introduced by Prokopovych and Yannelis (2014). Although these techniques are often simpler to verify, they can still be too demanding for certain discontinuous games; our public procurement game is one such example.

A particularly useful result for verifying payoff security in mixed-strategy games is provided by Allison and Lepore (2014). The authors establish a powerful theorem for compact games showing that, if the game satisfies the *disjoint payoff matching* condition, then it is payoff secure. This result plays a crucial role in proving the existence of Nash equilibria in our game when the number of players exceeds two.

Despite their generality and theoretical depth, it is important to note that these results remain essentially non-constructive, as they do not yield explicit procedures for computing Nash equilibrium strategies.

A particularly insightful and constructive approach to proving the existence of equilibria and deriving explicit equilibrium strategies was introduced by Melvin Dresher in Dresher (1961) and employed for example in Hilhorst and Appert-Rolland (2018) to solve the N-player dual game. This method relies on a system of functional equations that leads to an explicit characterization of the optimal strategy, provided that a suitable form of independence is satisfied in the payoff functions.

By situating our problem in context, the Moroccan public procurement game reveals a distinctive structural feature: each player's payoff is a discontinuous non-quasiconcave function that depends not only on their individual strategy but also on an aggregate measure—specifically, the average—of all players' strategies, making the game a particular instance of an aggregative game.

According to the definition in Jensen (2018), a non-cooperative game $\{(S_i, g_i)_{i=1}^N\}$, where each strategy set $S_i \subset \mathbb{R}$, is called *aggregative* if there exists a continuous, additively separable function

$$\phi : \prod_{i=1}^{N} S_i \to \mathbb{R}$$

(called the *aggregator*) and functions

$$\Phi_i : S_i \times \mathbb{R} \to \mathbb{R}$$

(called the *reduced payoff functions*) such that, for each player $i = 1, \ldots, N$ and for all strategy profiles $x = (x_i, x_{-i}) \in \prod_{i=1}^{N} S_i$,

$$g_i(x_i, x_{-i}) = \Phi_i\big(x_i, \phi(x)\big)$$

The distinctive feature of our problem lies in the structure of the payoff function. Although the game is symmetric (see Alós-Ferrer and Ania (2005) for a detailed discussion on symmetric aggregative games), the dependence on the strategy profile is not limited to the individual strategy x_i and an aggregator; instead, it involves the full strategy profile x. More precisely, the reduced payoff function is instead of the form

$$\Phi_i : \prod_{i=1}^{N} S_i \times \mathbb{R} \to \mathbb{R},$$

so that the payoff for each player i can be written as

$$g_i(x_i, x_{-i}) = \Phi_i\big(x, \phi(x)\big).$$

We first examined the public procurement reform in Riane (2026) by modeling it as a Keynesian beauty contest, within the framework of cognitive theory and the common knowledge hypothesis. In that work, we statistically analyzed bidders' behavior using real data from the public procurement market.

In this paper, we investigate the theoretical aspect of the problem by formulating it as a strategic game. We study the existence of solutions in both its pure and mixed forms, and we aim to derive these solutions explicitly using constructive methods.

The remainder of the paper is organized as follows:

In Section 2, we analyze the game and show that it admits no Nash equilibrium in pure strategies.

Section 3 focuses on the two-player case, where we derive two distinct Nash equilibria together with their corresponding mixed strategies.

In Section 4, we extend the analysis to a non-symmetric two-player game, allowing for weighted averages.

Finally, in Section 5, we study the general symmetric N-player game and establish the existence of a symmetric Nash equilibrium.

2 The Moroccan Public Procurement Game in Pure Strategies

It is not difficult to show that the game, as presented in the introduction, does not admit a Nash equilibrium in pure strategies under the rules *(1)–(4)*. Suppose, for contradiction, that a pure-strategy Nash equilibrium exists, denoted by $X = (x_1, \ldots, x_N)$. If at least one player i is not a winner, that player has an incentive to deviate to win, exploiting the sensitivity of the mean to individual bids. More precisely, by playing

$$x^\star = \frac{\sum_{j \neq i} x_j + NE}{2N - 1} \tag{2}$$

the player i secure the winning position. In the case of a tie, he can adjust his position to be the unique winner by playing $x^\star - \varepsilon$ for some $\varepsilon > 0$. In particular, in this degenerate case where all the player play the same strategy E, the player can reduce his bid by some $\varepsilon > 0$ to become the closest to the reference price from below, thereby winning the contract under rule *(3)* and avoiding the tie-breaker in rule *(4)*.

Since the game is symmetric, and defining g_i to be the payoff function for player i, one can remark that

$$0 = \sup_{x_i \in [A,B]} \inf_{x^{-i} \in [A,B]^{N-1}} g_i(x_1, \ldots, x_N) < \inf_{x^{-i} \in [A,B]^{N-1}} \sup_{x_i \in [A,B]} g_i(x_1, \ldots, x_N) = 1, \tag{3}$$

3 The Two-Player Version

Does the game admit an equilibrium in mixed strategies? To investigate this, let us consider the two-player version of the game. The payoff function of player 1 is given by

$$g(x, y) = \mathbf{1}_{\{y < x \leqslant P\}} + \mathbf{1}_{\{P \leqslant x < y\}} + \mathbf{1}_{\{x \leqslant P < y\}} + \frac{1}{2}\mathbf{1}_{\{x = y\}}, \tag{4}$$

where $\mathbf{1}$ denotes the indicator function, x is player 1's bid, and y is player 2's bid. The winning region is illustrated in Figure 1.

Let μ and ν be arbitrary probability measures on $[A, B]$, and define $A_i = \dfrac{A + (3^i - 1)E}{3^i}$. The

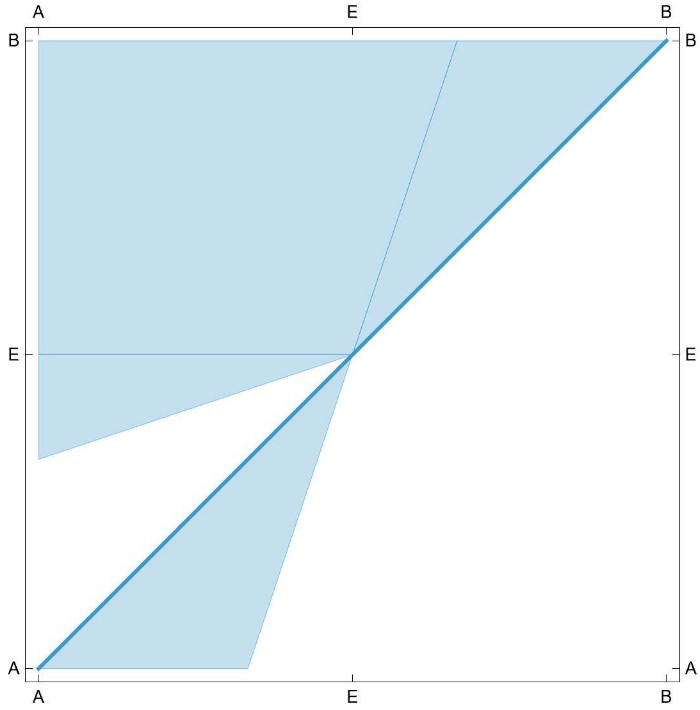


Figure 1: Payoff function g for two players. The blue region corresponds to a payoff of 1, the white region to 0, and the dark blue diagonal to $\frac{1}{2}$.

expected payoff for player 1 can be written in two equivalent forms

$$\begin{aligned} G(\mu,\nu) &= \iint g(x,y)\,d\mu(x)d\nu(y) \\ &= \int_A^E \mu\Big(\big(y, \tfrac{y+2E}{3}\big]\Big)\,d\nu(y) + \int_{A_1}^E \mu\big([A,\,3y-2E)\big)\,d\nu(y) \\ &\quad + \int_E^B \mu\big([A,y)\big)\,d\nu(y) - \mu\big([A,E)\big)\,\nu(\{E\}) + \frac{1}{2}\sum_{x\in[A,B]} \mu(\{x\})\,\nu(\{x\}). \end{aligned} \tag{5}$$

Alternatively, by changing the order of integration, we have

$$\begin{aligned} G(\mu,\nu) &= \iint g(x,y)\,d\mu(x)d\nu(y) \\ &= \int_{A_1}^E \nu\big([3x-2E,\,x)\big)\,d\mu(x) + \int_E^{B_1} \nu\big((x,B]\big)\,d\mu(x) \\ &\quad + \int_A^E \nu\Big(\big(\tfrac{x+2E}{3},\,B\big]\Big)\,d\mu(x) + \frac{1}{2}\sum_{x\in[A,B]} \mu(\{x\})\,\nu(\{x\}). \end{aligned} \tag{6}$$

In what follows, we present two methods to solve the game, each leading to a distinct mixed-strategy Nash equilibrium. The first method is inspired by the analysis of the counterexample given by Sion and Wolfe (1958), while the second applies the approach of Dresher (1961).

3.1 The Uniform-Based Strategy

We first compute the $\inf\sup G$. Let ν be an arbitrary probability measure on $[A,B]$.

1. If $\operatorname{supp}(\nu) \subseteq [E,B]$, choose $\mu = \delta_x$ for some $x \in [A,E[$, since the interval $[E,B]$ is strictly dominated by $[A,E[$.

2. If $\text{supp}(\nu) \cap [A, E[\neq \varnothing$, the mass of ν can be partitioned over the intervals $\{[A_i, A_{i+1}[\}_{i\in\mathbb{N}}$. Then:

   (a) If, for some $i \in \mathbb{N}$,
$$\nu([A_i, A_{i+1}[\cup[E, B]) \geqslant \frac{1}{2},$$
   choose $\mu = \delta_{A_{i+1}}$.

   (b) Otherwise:

      i. If $\nu(]A_1, B]) \geqslant \frac{1}{2}$, choose $\mu = \delta_A$.

      ii. If $\nu([A, A_1]) > \frac{1}{2}$ and $\nu([A, A_1[) < \frac{1}{2}$, choose $\mu = \delta_{A_1+\varepsilon}$ for some $\varepsilon > 0$; otherwise, choose $\mu = \delta_{A_1}$.

Hence,

$$\sup_{\mu\in\Delta([A,B])} G(\mu, \nu) \geqslant \frac{1}{2}. \tag{7}$$

Next, define

$$\nu^\star(y) = \frac{1}{2}\Big(U([A, A_1[) + U([A_1, A_2[)\Big)\,\mathcal{L}. \tag{8}$$

where U designates the uniform distribution and $\mathcal{L}$ the Lebesgue measure. Then, for all $x \in [A, B]$,

$$G(x, \nu^\star) = \frac{1}{2}\,\mathbf{1}_{[A,A_2]} + \frac{A + 26e - 27x}{4(E - A)}\mathbf{1}_{]A_2,A_3[} \leqslant \frac{1}{2}. \tag{9}$$

This implies that

$$\inf_{\nu\in\Delta([A,B])}\ \sup_{\mu\in\Delta([A,B])} G(\mu, \nu) = \frac{1}{2}. \tag{10}$$

One gets

$$G(\mu^\star, y) = \frac{1}{2}\,\mathbf{1}_{[A,A_2]} + \frac{27y - 5A - 22E}{4(E - A)}\mathbf{1}_{]A_2,A_3[} + \mathbf{1}_{[A_3,B]} \geqslant \frac{1}{2}. \tag{11}$$

Since the game is symmetric, by choosing

$$\mu^\star(y) = \frac{1}{2}\Big(U([A, A_1[) + U([A_1, A_2[)\Big)\,\mathcal{L}.$$

It follows by a similar argument that

$$\sup_{\mu\in\Delta([A,B])}\ \inf_{\nu\in\Delta([A,B])} G(\mu, \nu) = \frac{1}{2}.$$

Therefore, the game has a value in mixed strategies:

$$\sup_{\mu\in\Delta([A,B])}\ \inf_{\nu\in\Delta([A,B])} G(\mu, \nu) = \frac{1}{2} = \inf_{\nu\in\Delta([A,B])}\ \sup_{\mu\in\Delta([A,B])} G(\mu, \nu). \tag{12}$$

A Nash equilibrium is thus given by

$$\mu^\star = \nu^\star = \frac{1}{2}\Big(U([A, A_1[) + U([A_1, A_2[)\Big)\mathcal{L}. \tag{13}$$

3.2 The Functional Equation Solution

Suppose that player 2 adopts a strategy with a density f sufficiently smooth and supported on $[A,\, \tilde{A}]$. Integrating g with respect to f gives

$$\begin{aligned} \int_A^B g(x,y)\, f(y)\, dy = \mathbf{1}_{\{x<E\}} & \left(\int_{\frac{x+2E}{3}}^{E} f(y)\, dy + \int_{3x-2E}^{x} f(y)\, dy \right) \\ + \mathbf{1}_{\{E\leqslant x\}} & \left(\int_x^{3x-2E} f(y)\, dy + \int_{3x-2E}^{B} f(y)\, dy \right). \end{aligned} \tag{14}$$

By the optimality condition and symmetry we have

$$\begin{aligned} \int_{\frac{x+2E}{3}}^{E} f(y)\, dy + \int_A^x f(y)\, dy &= \frac{1}{2}, \quad A \leqslant x < A_1, \\ \int_{\frac{x+2E}{3}}^{E} f(y)\, dy + \int_{3x-2E}^{x} f(y)\, dy &= \frac{1}{2}, \quad A_1 \leqslant x < E, \\ \int_x^B f(y)\, dy &= \frac{1}{2}, \quad E \leqslant x \leqslant B. \end{aligned} \tag{15}$$

Differentiating these conditions with respect to x yields the following functional system

$$\begin{aligned} -\frac{1}{3} f\Big(\tfrac{x+2E}{3}\Big) + f(x) &= 0, \quad A \leqslant x < A_1, \\ -\frac{1}{3} f\Big(\tfrac{x+2E}{3}\Big) + f(x) - 3\, f(3x-2E) &= 0, \quad A_1 \leqslant x < E, \\ -f(x) &= 0, \quad E \leqslant x \leqslant B. \end{aligned} \tag{16}$$

The last equation implies that f is zero on $[E, B]$. Solving the first functional equation on $[A,\, A_1[$

$$f(x) = \frac{1}{3} f\Big(\tfrac{x+2E}{3}\Big). \tag{17}$$

Assume f takes the form $f(x) = C_0(x-E)^s$. Then

$$\frac{1}{3} f\Big(\tfrac{x+2E}{3}\Big) = C_0\, \frac{1}{3} \Big(\tfrac{x+2E}{3} - E\Big)^s = C_0\, \frac{1}{3^{s+1}} (x-E)^s. \tag{18}$$

Equating gives $s = -1$, so

$$f(x) = \frac{C_0}{x-E}. \tag{19}$$

Substituting back into the integral condition for $A \leqslant x < A_1$ and imposing an upper bound $\tilde{A}$ for the support of f (to ensure convergence) yields

$$\begin{aligned} C_0 &= \frac{1}{2} \left(\int_{\frac{x+2E}{3}}^{\tilde{A}} \frac{1}{y-E}\, dy + \int_A^x \frac{1}{y-E}\, dy \right)^{-1} \\ &= \frac{1}{2} \Big[\ln(E-\tilde{A}) - \ln(E-x) + \ln(3) + \ln(E-x) - \ln(E-A) \Big]^{-1} \\ &= \frac{1}{2} \ln\Big(3\, \frac{E-\tilde{A}}{E-A}\Big)^{-1}. \end{aligned} \tag{20}$$

Therefore,

$$f(x) = \frac{1}{2\, \ln\big(3\, \frac{E-\tilde{A}}{E-A}\big)\, (x-E)}. \tag{21}$$

Imposing the normalization condition

$$\int_A^{\tilde{A}} f(y)\,dy = 1, \tag{22}$$

we obtain:

$$\ln\Big(\frac{E-\tilde{A}}{E-A}\Big) = 2\,\ln\Big(3\,\frac{E-\tilde{A}}{E-A}\Big). \tag{23}$$

Solving gives:

$$\tilde{A} = A_2 = \frac{A+8E}{9}. \tag{24}$$

Knowing the support of f, we can verify the integral condition on $[A_1,\,E]$, which becomes

$$\int_{3x-2E}^{x} f(y)\,dy = \frac{1}{2}, \quad A_1 \leqslant x < A_2. \tag{25}$$

Differentiating again gives

$$f(x) - 3\,f(3x-2E) = 0, \tag{26}$$

which matches the solution of the first functional equation. Thus, we have identified another symmetric Nash equilibrium for the game, distinct from the uniform-based strategy in Section 3.1. It is given by $\mu = \nu = f\,\mathcal{L}$, where $\mathcal{L}$ denotes the Lebesgue measure and

$$f(x) = \frac{1}{\ln(9)\,(E-x)}\,\mathbf{1}_{[A,A_2]}. \tag{27}$$

One can check that, for all $y \in [A,B]$,

$$G(x,\nu^\star) = \frac{1}{2}\,\mathbf{1}_{[A,A_2]} + \frac{\ln\Big(\frac{27(E-x)}{E-A}\Big)}{2\ln(3)}\mathbf{1}_{]A_2,A_3[} \leqslant \frac{1}{2}.$$

and for all $y \in [A,B]$,

$$G(\mu^\star,y) = \frac{1}{2}\,\mathbf{1}_{[A,A_2[} + \frac{\ln\Big(\frac{E-A}{3(E-y)}\Big)}{\ln(9)}\mathbf{1}_{[A_2,A_3[} + \mathbf{1}_{[A_3,B]} \geqslant \frac{1}{2}.$$

4 The Two-Player $(p,\,1-p)$ Version

In this section, we generalize the symmetric two-player game by introducing an asymmetry in the definition of the reference price. This modification allows us to study how the equilibrium changes when the two bidders do not contribute equally to the determination of the reference value.

Let the reference price now depend asymmetrically on the two bids

$$P = \frac{p\,x + (1-p)\,y + E}{2}, \tag{28}$$

where $p \in [0,1]$ is a weight parameter representing the relative contribution of player 1's bid to the reference price. The original symmetric case corresponds to $p = \frac{1}{2}$. The payoff function of player 1 becomes

$$g_p(x,y) = \mathbf{1}_{\{\,y<x\leqslant P\}} + \mathbf{1}_{\{\,P\leqslant x<y\}} + \mathbf{1}_{\{\,x\leqslant P<y\}} + p\,\mathbf{1}_{\{\,x=y\}}. \tag{29}$$

The discontinuities in g_p are now asymmetric, since player 1's own bid influences the reference price more strongly when $p > \frac{1}{2}$, and less strongly when $p < \frac{1}{2}$. This asymmetric formulation induces a family of discontinuous zero-sum games parameterized by p, each of which has its own equilibrium structure and game value $v(p)$.

Before turning to explicit solutions, it is useful to interpret the parameter p:

1. When $p \to 0$: the reference price depends almost entirely on player 2's bid. Player 1 has negligible influence and thus a weaker strategic position.
2. When $p \to 1$: the situation reverses, and player 1 almost determines P by herself. The game becomes almost trivial, with player 1 effectively playing against her own benchmark.
3. When $p = \frac{1}{2}$: the game is perfectly symmetric, corresponding to the benchmark model analyzed in Section 3.

The parameter p therefore introduces an imbalance of influence over the reference price, leading to a family of games that transition from symmetric to asymmetric regimes.

Figure 2 illustrates the winning region for player 1 in the case $p = 0.1$.

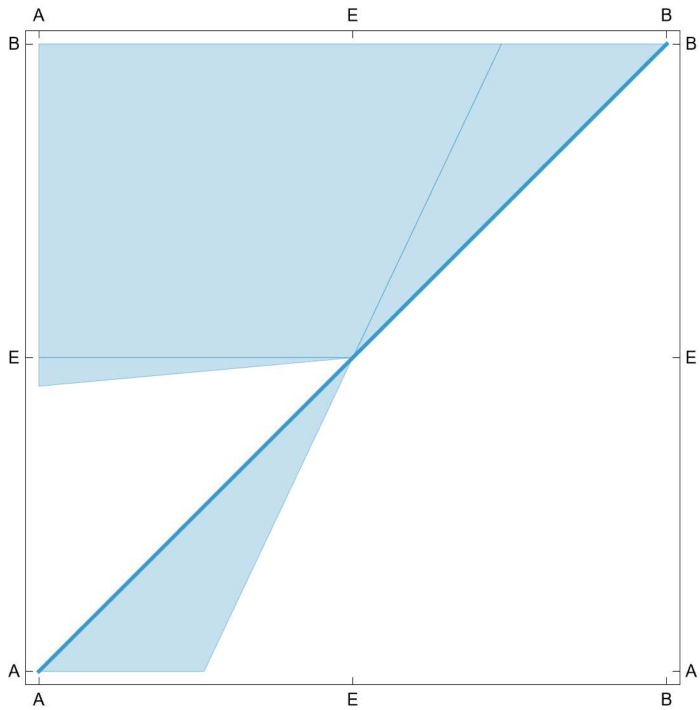


Figure 2: Payoff function g_p for $p = 0.1$. The blue region corresponds to a payoff of 1, the white region to 0, and the dark blue diagonal to p.

We introduce the maps

$$f_{1,p}(x) = \frac{p\,x + E}{p+1}, \quad f_{2,p}(x) = \frac{(1-p)\,x + E}{2-p},$$
$$h_{1,p}(x) = \frac{(2-p)\,x - E}{1-p}, \quad h_{2,p}(x) = \frac{(p+1)\,x - E}{p}. \tag{30}$$

Define the sequences

$$\begin{aligned}
\hat{A}_i &= f_{1,p}(\hat{A}_{i-1}), \quad \check{A}_i = f_{2,p}(\check{A}_{i-1}), \quad \hat{A}_0 = \check{A}_0 = A. \\
\hat{C}_i &= h_{1,p}(\hat{A}_i), \quad \check{C}_i = h_{2,p}(\check{A}_{i+1}), \\
\hat{D}_{i+1} &= f_{2,p}(\hat{D}_i), \quad \hat{D}_1 = \frac{(p+1)\,(1-p)^2\,A + [\,(5-3p)\,p-1\,]\,E}{p\,(2-p)^2}, \\
\check{D}_{i+1} &= f_{1,p}(\check{D}_i), \quad \check{D}_1 = \frac{2\,E + p\,(1-p)\,A}{(2-p)\,(p+1)}.
\end{aligned} \tag{31}$$

Note that for any actions x and y by players 1 and 2 respectively, the strict winning region for the player 1 is

$$\begin{aligned}
w(x) &= [\,h_{1,p}(x),\, x\,[\,\cup\,]\,f_{1,p}(x),\, B\,[, \\
w(y) &= [\,A, h_{2,p}(y)\,[\,\cup\,]\,y,\, f_{2,p}(y)[.
\end{aligned} \tag{32}$$

One should distinguish five cases:

1. $p = \frac{1}{2}$,
2. $\frac{1}{6}\big(5-\sqrt{13}\big) < p < \frac{1}{2}$,
3. $p = \frac{1}{6}\big(5-\sqrt{13}\big)$,
4. $0 < p < \frac{1}{6}\big(5-\sqrt{13}\big)$,
5. $p = 0$.

The threshold value $p = \frac{1}{6}(5-\sqrt{13})$ arises as the solution of the condition:

$$h_{2,p}\big(\check{A}_2\big) = A. \tag{33}$$

Finally, observe that for $p = 0$, the game admits a pure-strategy Nash equilibrium at $x = y = E$, with value 0.

4.0.1 The The Intermediate Regime $\frac{1}{6}(5-\sqrt{13}) < p < \frac{1}{2}$

For this range of p, the game is no longer symmetric. Let μ and ν be two arbitrary probability measures on $[A, B]$ and define

$$G_p(\mu,\nu) = \iint\limits_{[A,B]^2} g_p(x,y)\,d\mu(x)\,d\nu(y). \tag{34}$$

1. If $\mathrm{supp}(\nu) \subseteq [E, B]$, player 1 can choose $\mu = \delta_x$ for some $x \in [A, E[$.
2. If $\mathrm{supp}(\nu) \cap [A, E[\neq \varnothing$, then by choosing $\mu = \delta_A$, $\mu = \delta_{\hat{C}_1}$, or $\mu = \delta_{\hat{A}_1}$, player 1 guarantees at least $\frac{1}{3}$. Moreover, if we partition the interval $[A, \check{D}_1]$ into the five subintervals

$$[A, \check{C}_1[, \quad [\check{C}_1, \check{A}_1[, \quad [\check{A}_1, \check{C}_2[, \quad [\check{C}_2, \check{A}_2[, \quad [\check{A}_2, \check{D}_1],$$

then player 1 can secure a win in at most two out of five subintervals by selecting:

   (a) $\mu = \delta_{\check{D}_1}$: wins on $[\hat{A}_1, \check{A}_2[\subset [\check{C}_2, \check{A}_2[$ and $[\check{A}_2, \check{D}_1[$.

   (b) $\mu = \delta_{\check{C}_1}$: wins on $[A, \check{C}_1[$ and $[\check{A}_2, \check{D}_1[$.

   (c) $\mu = \delta_{\check{A}_1}$: wins on $[A, \check{C}_1[$ and $[\check{C}_1, \check{A}_1[$.

   (d) $\mu = \delta_{\check{C}_2}$: wins on $[\check{C}_1, \check{A}_1[$ and $[\check{A}_1, \check{C}_2[$.

(e) $\mu = \delta_{\check{A}_2}$: wins on $[\check{A}_1, \check{C}_2[$ and $[\check{C}_2, \check{A}_2[$.

Thus, the worst case for player 1 yields a payoff of at least $\frac{2}{5} - \varepsilon_p$ for some $\varepsilon_p > 0$.

Therefore,

$$\sup_{\mu \in \Delta([A,B])} G_p(\mu, \nu) \geqslant \frac{2}{5} - \varepsilon_p > \frac{2}{5}. \tag{35}$$

Next, define

$$\nu^\star = \Big(\check{p}_1\, U([A, \check{C}_1[) + \check{p}_2\, U([\check{C}_1, \check{A}_1[) + \check{p}_1\, U([\check{A}_1, \check{C}_2[) + \check{p}_2\, U([\check{C}_2, \check{A}_2[) + \check{p}_2\, U([\check{A}_2, \check{D}_1[)\Big)\mathcal{L}. \tag{36}$$

Then, for all $\mu \in \Delta([A, B])$,

$$G_p(\mu, \nu^\star) \leqslant \frac{2}{5} - \varepsilon_p. \tag{37}$$

Hence,

$$\inf_{\nu \in \Delta([A,B])} \sup_{\mu \in \Delta([A,B])} G_p(\mu, \nu) = \frac{2}{5} - \varepsilon_p. \tag{38}$$

We now proceed by reasoning in the reverse direction.

1. If $\operatorname{supp}(\mu) \subseteq [E, B]$, then player 2 can choose $\nu = \delta_x$ for some $x \in [A, E[$.
2. If $\operatorname{supp}(\mu) \cap [A, E[\neq \varnothing$, then by choosing $\nu = \delta_A$ or $\nu = \delta_{\check{A}_1}$, player 2 guarantees at least $\frac{1}{2}$. The worst case for player 2 occurs when $\operatorname{supp}(\mu) = [A, \check{D}_1]$. Partitioning $[A, \check{D}_1]$ into the same five subintervals as above, player 2 can secure wins in at least three out of five subintervals by selecting:
   (a) $\nu = \delta_{\check{D}_1}$: wins on $[\check{A}_1, \check{C}_2[$, $[\check{C}_2, \check{A}_2[$, and $[\check{A}_2, \check{D}_1[$.
   (b) $\nu = \delta_{\check{C}_1}$: wins on $[A, \check{C}_1[$, $[\check{C}_2, \check{A}_2[$, and $[\check{A}_2, \check{D}_1[$.
   (c) $\nu = \delta_{\check{A}_1}$: wins on $[A, \check{C}_1[$, $[\check{C}_1, \check{A}_1[$, and $[\check{A}_2, \check{D}_1[$.
   (d) $\nu = \delta_{\check{C}_2}$: wins on $[h_{2,p}(\check{C}_2), \check{C}_1[\subset [A, \check{C}_1[$, $[\check{C}_1, \check{A}_1[$, and $[\check{A}_1, \check{C}_2[$.
   (e) $\nu = \delta_{\check{A}_2}$: wins on $[\check{C}_1, \check{A}_1[$, $[\check{A}_1, \check{C}_2[$, and $[\check{C}_2, \check{A}_2[$.

   Therefore, the worst case for player 2 guarantees at least $\frac{3}{5} + \varepsilon_p$ for some $\varepsilon_p > 0$, so the corresponding payoff for player 1 is at most $\frac{2}{5} - \varepsilon_p$.

Hence,

$$\inf_{\nu \in \Delta([A,B])} G_p(\mu, \nu) \leqslant \frac{2}{5} - \varepsilon_p. \tag{39}$$

Now, define

$$\mu^\star = \Big(\hat{p}_1\, U([A, \check{C}_1[) + \hat{p}_2\, U([\check{C}_1, \check{A}_1[) + \hat{p}_1\, U([\check{A}_1, \check{C}_2[) + \hat{p}_2\, U([\check{C}_2, \check{A}_2[) + \hat{p}_2\, U([\check{A}_2, \check{D}_1[)\Big)\mathcal{L}. \tag{40}$$

Then, for all $\nu \in \Delta([A, B])$,

$$G_p(\mu^\star, \nu) \geqslant \frac{2}{5} - \varepsilon_p > \frac{2}{5}. \tag{41}$$

This shows that

$$\sup_{\mu \in \Delta([A,B])} \inf_{\nu \in \Delta([A,B])} G_p(\mu, \nu) = \frac{2}{5} - \varepsilon_p. \tag{42}$$

Therefore, we have established that the game has value $\frac{2}{5} - \varepsilon_p$ for some $\varepsilon_p > 0$.

4.0.2 The Critical Regime $p = \frac{1}{6}(5 - \sqrt{13})$

1. If $\mathrm{supp}(\nu) \subseteq [E, B]$, then player 1 can choose $\mu = \delta_x$ for some $x \in [A, E[$.
2. If $\mathrm{supp}(\nu) \cap [A, E[\neq \varnothing$, then by playing $\mu = \delta_A$, $\mu = \delta_{\check{A}_1}$, or $\mu = \delta_{\check{A}_2}$, player 1 guarantees at least $\frac{1}{3}$. More specifically, player 1 can:
   (a) win on $[\hat{A}_2, \check{A}_3[$ by choosing $\mu = \delta_{\check{A}_3}$,
   (b) win on $[A, \check{A}_1[$ by choosing $\mu = \delta_{\check{A}_1}$,
   (c) win on $[\check{A}_1, \check{A}_2]$ by choosing $\mu = \delta_{\check{A}_2}$.

Hence,

$$\sup_{\mu \in \Delta([A,B])} G_p(\mu, \nu) \geqslant \frac{1}{3}. \tag{43}$$

Define

$$\nu^\star = \frac{1}{3}\Big(U([A, \check{A}_1[) + U([\check{A}_1, \check{A}_2[) + U([\check{A}_2, \check{A}_3[)\Big)\mathcal{L}. \tag{44}$$

Then, for all $\mu \in \Delta([A, B])$,

$$G_p(\mu, \nu^\star) \leqslant \frac{1}{3}. \tag{45}$$

We conclude that

$$\inf_{\nu \in \Delta([A,B])} \sup_{\mu \in \Delta([A,B])} G_p(\mu, \nu) = \frac{1}{3}. \tag{46}$$

We now consider the sup inf:

1. If $\mathrm{supp}(\mu) \subseteq [E, B]$, then player 2 can choose $\nu = \delta_x$ for some $x \in [A, E[$.
2. If $\mathrm{supp}(\mu) \cap [A, E[\neq \varnothing$, then by choosing $\nu = \delta_A$, $\nu = \delta_{\check{A}_1}$, or $\nu = \delta_{\check{A}_2}$, player 2 guarantees at least $\frac{2}{3}$. More precisely, player 2 can:
   (a) win on $[\check{A}_1, \check{A}_3[$ by choosing $\nu = \delta_{\check{A}_3}$,
   (b) win on $[A, \check{A}_1[$ and $[\check{A}_2, \check{A}_3[$ by choosing $\nu = \delta_{\check{A}_1}$,
   (c) win on $[A, \check{A}_2[$ by choosing $\nu = \delta_{\check{A}_2}$.

   Therefore, the worst-case payoff for player 1 is at most $\frac{1}{3}$.

Hence,

$$\inf_{\nu \in \Delta([A,B])} G_p(\mu, \nu) \leqslant \frac{1}{3}. \tag{47}$$

Now, define

$$\mu^\star = \frac{1}{3}\Big(U([A, \check{A}_1[) + U([\check{A}_1, \check{A}_2[) + U([\check{A}_2, \check{A}_3[)\Big)\mathcal{L}. \tag{48}$$

Then, for all $\nu \in \Delta([A, B])$,

$$G_p(\mu^\star, \nu) \geqslant \frac{1}{3}. \tag{49}$$

This shows that

$$\sup_{\mu \in \Delta([A,B])} \inf_{\nu \in \Delta([A,B])} G_p(\mu, \nu) = \frac{1}{3}. \tag{50}$$

Therefore, the value of the game in this case is $\frac{1}{3}$.

4.0.3 The Low-p Regime $0 < p < \frac{1}{6}(5 - \sqrt{13})$

Define

$$m = \arg \max_{\{ i \in \mathbb{N} \colon \check{A}_{2+i} \leqslant \check{D}_1 \}} i, \quad \text{where} \quad m \geqslant 1.$$

Let's calculate the inf sup:

1. If $\mathrm{supp}(\nu) \subseteq [E, B]$, then player 1 can choose $\mu = \delta_x$ for some $x \in [A, E[$.
2. If $\mathrm{supp}(\nu) \cap [A, E[\neq \varnothing$, then by playing $\mu = \delta_{\check{A}_i}$ for $1 \leqslant i \leqslant m+2$ or $\mu = \delta_{\check{D}_1}$, player 1 guarantees at least $\frac{1}{m+2}$. Specifically:
   (a) player 1 wins on $[\hat{A}_1, \check{A}_{m+2}[\subset [\check{A}_{m+1}, \check{A}_{m+2}[$ and $[\check{A}_{m+2}, \check{D}_1[$ by choosing $\mu = \delta_{\check{D}_1}$,
   (b) player 1 wins on $[\check{A}_i, \check{A}_{i+1}[$ by choosing $\mu = \delta_{\check{A}_{i+1}}$ for $0 \leqslant i \leqslant m+1$.

Hence,

$$\sup_{\mu \in \Delta([A,B])} G_p(\mu, \nu) \geqslant \frac{1}{m+2} + \varepsilon_p. \tag{51}$$

Define

$$\nu^\star = \Big(\hat{p}_1 \sum_{\substack{i=0 \\ i \text{ odd}}}^{m+1} U([\check{A}_i,\, \hat{D}_{m+1+i}[) + \hat{p}_2 \sum_{\substack{i=0 \\ i \text{ even}}}^{m+1} U([\hat{D}_{m+1+i},\, \check{A}_{i+1}[) + \hat{p}_1 U([\check{A}_{m+2},\, \check{D}_1[)\Big)\mathcal{L}. \tag{52}$$

Then, for all $\mu \in \Delta([A, B])$,

$$G_p(\mu,\, \nu^\star) \leqslant \frac{1}{m+2} + \varepsilon_p. \tag{53}$$

This shows

$$\inf_{\nu \in \Delta([A,B])} \sup_{\mu \in \Delta([A,B])} G_p(\mu, \nu) = \frac{1}{m+2} + \varepsilon_p. \tag{54}$$

We calculate now the sup inf:

1. If $\mathrm{supp}(\mu) \subseteq [E, B]$, then player 2 can choose $\nu = \delta_x$ for some $x \in [A, E[$.
2. If $\mathrm{supp}(\mu) \cap [A, E[\neq \varnothing$, then by playing $\nu = \delta_{\check{A}_i}$ for $1 \leqslant i \leqslant m+2$ or $\nu = \delta_{\check{D}_1}$, player 2 guarantees at most $\frac{m+1}{m+2}$. Specifically:
   (a) player 2 wins on $[\check{A}_1, \check{D}_1[$ by choosing $\nu = \delta_{\check{D}_1}$,
   (b) player 2 wins on $[\check{A}_1, \check{D}_1[\setminus [\check{A}_{i-1}, \check{A}_i[$ by choosing $\nu = \delta_{\check{A}_i}$ for $0 \leqslant i \leqslant m+1$,
   (c) player 2 wins on $[\check{A}_1, \check{D}_1[\setminus \big([A,\, h_{2,p}(\check{A}_{m+2})[\cup [\check{A}_{m+2},\, \check{D}_1[\big)$ by choosing $\nu = \delta_{\check{A}_{m+2}}$.

Hence,

$$\inf_{\nu \in \Delta([A,B])} G_p(\mu, \nu) \leqslant \frac{1}{m+2} + \varepsilon_p. \tag{55}$$

Define

$$\mu^\star = \Big(\hat{p}_1 \sum_{\substack{i=0 \\ i \text{ odd}}}^{m+1} U([\check{A}_i,\, \hat{D}_{m+1+i}[) + \hat{p}_2 \sum_{\substack{i=0 \\ i \text{ even}}}^{m+1} U([\hat{D}_{m+1+i},\, \check{A}_{i+1}[) + \hat{p}_1 U([\check{A}_{m+2},\, \check{D}_1[)\Big)\mathcal{L}. \tag{56}$$

For all $\nu \in \Delta([A,B])$,

$$G_p(\mu^\star,\, \nu) \geqslant \frac{1}{m+2} + \varepsilon_p. \tag{57}$$

This implies

$$\sup_{\mu\in\Delta([A,B])}\ \inf_{\nu\in\Delta([A,B])} G_p(\mu,\nu) = \frac{1}{m+2} + \varepsilon_p. \tag{58}$$

Therefore, the game has value $\frac{1}{m+2} + \varepsilon_p$ for some $\varepsilon_p > 0$.

4.1 Explicit Solution

One limitation of the previous section's analysis is the lack of an explicit closed form for the game value, particularly the term ε_p, which we compute numerically. In this section, we present an alternative approach to obtain an explicit expression for the value using a functional system.

Suppose that players 1 and 2 adopt mixed strategies with densities f_1 and f_2, respectively. Integrating g_p with respect to f_2 gives

$$\begin{aligned}\int g_p(x,y)\, f_2(y)\, dy = \mathbf{1}_{\{x<E\}}\Big(\int_{\frac{px+E}{p+1}}^{E} f_2(y)\, dy\Big) + \mathbf{1}_{\{x<E\}}\Big(\int_{\frac{(p+1)x-E}{p}}^{x} f_2(y)\, dy\Big)\\ + \mathbf{1}_{\{E\leqslant x\}}\Big(\int_{x}^{\frac{(p+1)x-E}{p}} f_2(y)\, dy\Big) + \mathbf{1}_{\{E\leqslant x\}}\Big(\int_{\frac{(p+1)x-E}{p}}^{B} f_2(y)\, dy\Big).\end{aligned} \tag{59}$$

Applying the same reasoning as in Section 3.2, for x in $\mathrm{supp}(f_1)$, we have

$$\begin{aligned}\int_{\frac{px+E}{p+1}}^{E} f_2(y)\, dy + \int_{A}^{x} f_2(y)\, dy &= v(p), \quad A \leqslant x < \hat{A}_1,\\ \int_{\frac{px+E}{p+1}}^{E} f_2(y)\, dy + \int_{\frac{(p+1)x-E}{p}}^{x} f_2(y)\, dy &= v(p), \quad \hat{A}_1 \leqslant x < E,\\ \int_{x}^{B} f_2(y)\, dy &= v(p), \quad E \leqslant x \leqslant B.\end{aligned} \tag{60}$$

Differentiating with respect to x yields

$$\begin{aligned}-\frac{p}{p+1}\, f_2\Big(\tfrac{px+E}{p+1}\Big) + f_2(x) &= 0, \quad A \leqslant x < \hat{A}_1,\\ -\frac{p}{p+1}\, f_2\Big(\tfrac{px+E}{p+1}\Big) + f_2(x) - \frac{p+1}{p}\, f_2\Big(\tfrac{(p+1)x-E}{p}\Big) &= 0, \quad \hat{A}_1 \leqslant x < E,\\ -f_2(x) &= 0, \quad E \leqslant x \leqslant B.\end{aligned} \tag{61}$$

The last condition shows that f_2 vanishes on $[E,B]$. Solving on $[A,\, \hat{A}_1[$

$$f_2(x) = \frac{p}{p+1}\, f_2\Big(\tfrac{px+E}{p+1}\Big), \tag{62}$$

whose solution is

$$f_2(x) = C_2\, \frac{1}{x-E}. \tag{63}$$

A similar argument for player 1 gives

$$\begin{aligned}\int g_p(x,y)\, f_1(x)\, dx = \mathbf{1}_{\{\, y<E\}}\Big(\int_y^{\frac{(1-p)\,y+E}{2-p}} f_1(x)\, dx\Big) + \mathbf{1}_{\{\, y<E\}}\Big(\int_A^{\frac{(1+p)y-E}{p}} f_1(x)\, dx\Big) \\ + \mathbf{1}_{\{\, E\leqslant y\}}\Big(\int_{\frac{(1-p)\,y+E}{2-p}}^{y} f_1(x)\, dx\Big) + \mathbf{1}_{\{\, E\leqslant y\}}\Big(\int_E^{\frac{(1-p)\,y+E}{2-p}} f_1(x)\, dx\Big).\end{aligned} \tag{64}$$

Then, for y in $\mathrm{supp}(f_2)$

$$\begin{aligned}\int_y^{\frac{(1-p)\,y+E}{2-p}} f_1(x)\, dx &= v(p), \quad A \leqslant y < \hat{A}_1, \\ \int_y^{\frac{(1-p)\,y+E}{2-p}} f_1(x)\, dx + \int_A^{\frac{(1+p)y-E}{p}} f_1(x)\, dx &= v(p), \quad \hat{A}_1 \leqslant y < E, \\ \int_E^{y} f_1(x)\, dx &= v(p), \quad E \leqslant y \leqslant B.\end{aligned} \tag{65}$$

Differentiating gives

$$\begin{aligned}\frac{1-p}{2-p}\, f_1\Big(\tfrac{(1-p)\,y+E}{2-p}\Big) - f_1(y) &= 0, \quad A \leqslant y < \hat{A}_1, \\ \frac{1-p}{2-p}\, f_1\Big(\tfrac{(1-p)\,y+E}{2-p}\Big) - f_1(y) + \frac{1+p}{p}\, f_1\Big(\tfrac{(1+p)\,y-E}{p}\Big) &= 0, \quad \hat{A}_1 \leqslant y < E, \\ f_1(y) &= 0, \quad E \leqslant y \leqslant B.\end{aligned} \tag{66}$$

Thus, f_1 is also zero on $[E, B]$. Solving on $[A,\, \hat{A}_1[$

$$f_1(y) = \frac{1-p}{2-p}\, f_1\Big(\tfrac{(1-p)\,y+E}{2-p}\Big), \tag{67}$$

which implies

$$f_1(x) = C_1\, \frac{1}{x-E}. \tag{68}$$

Imposing that the supports of f_1 and f_2 are $[A,\, \tilde{A}]$ with

$$\tilde{A} = \check{D}_1 = \frac{2E + p(1-p)\, A}{(2-p)(p+1)}, \tag{69}$$

as determined in the previous analysis, and enforcing

$$\int_A^{\tilde{A}} f_1(x)\, dx = 1, \qquad \int_A^{\tilde{A}} f_2(y)\, dy = 1, \tag{70}$$

we find

$$C_1 = C_2 = \Big[\ln\Big(\frac{E-\tilde{A}}{E-A}\Big)\Big]^{-1}. \tag{71}$$

Finally, integrating g_p yields the explicit game value

$$\begin{aligned}v(p) &= \iint\limits_{[A,\tilde{A}]^2} g_p(x,y)\, f_1(x)\, f_2(y)\, dx\, dy \\ &= \frac{\ln\left(\frac{1-p}{2-p}\right)}{\ln\left(\frac{p(1-p)}{(2-p)(p+1)}\right)}.\end{aligned} \tag{72}$$

One can check that by taking

$$f^\star(x) = \frac{1}{\left[\ln\left(\frac{E-\tilde{A}}{E-A}\right)\right](x-E)}\mathbf{1}_{[A,\tilde{A}]}. \tag{73}$$

that

$$\int_{[A,B]} g_p(x,y)\, f^\star(x)\, dx = \frac{\ln\left(\frac{1-p}{2-p}\right)}{\ln\left(\frac{p(p-1)}{(p-2)(p+1)}\right)}\mathbf{1}_{[A,\tilde{A}]} + \frac{\ln\left(\frac{(p+1)(E-y)}{p(E-A)}\right)}{\ln\left(\frac{p(p-1)}{(p-2)(p+1)}\right)}\mathbf{1}_{[\tilde{A},\tilde{A}_2[} + \mathbf{1}_{[\tilde{A}_2,B]}. \tag{74}$$

and

$$\int_{[A,B]} g_p(x,y)\, f^\star(y)\, dy = \frac{\ln\left(\frac{1-p}{2-p}\right)}{\ln\left(\frac{p(p-1)}{(p-2)(p+1)}\right)}\mathbf{1}_{[A,\tilde{A}]} + \frac{\ln\left(\frac{p(p-1)^2(E-A)}{(p-2)^2(p+1)(E-x)}\right)}{\ln\left(\frac{p(p-1)}{(p-2)(p+1)}\right)}\mathbf{1}_{[\tilde{A},\tilde{A}_3[}. \tag{75}$$

where $\tilde{A}_2 = \frac{A(p-1)p^2+E(p(p-3)-2)}{(p-2)(p+1)^2}$ and $\tilde{A}_3 = \frac{A\,p(p-1)^2-E\left(p^2+p-4\right)}{(p-2)^2(p+1)}$.

5 The N-players version

When N players ($N \geqslant 3$) participate in the Moroccan public procurement game, the situation becomes considerably more complex. To generalize the analysis of Section 3 to the N-player case, let us denote

$$N^- = \{1 \leqslant j \leqslant N \mid x_j < P\} \quad , \quad N^0 = \{1 \leqslant j \leqslant N \mid x_j = P\} \quad , \quad N^+ = \{1 \leqslant j \leqslant N \mid x_j > P\} \tag{76}$$

The payoff function of player i is given by the discontinuous function

$$g_i(x_1,\ldots,x_N) = \sum_{n=0}^{N-1}\sum_{J_n(i)}\frac{1}{n+1}\prod_{k\in J_n(i)}\mathbf{1}_{\{x_i=x_k\}}\left(\mathbf{1}_{\{x_i\leqslant P\}}\prod_{j\neq i,\, j\in N^-\setminus J_n(i)}\mathbf{1}_{\{x_j<x_i\}} + \mathbf{1}_{\{x_i>P\}}\prod_{j\neq i,\, j\notin J_n(i)}\mathbf{1}_{\{x_i<x_j\}}\right) \tag{77}$$

where $J_n(i)$ is a subset of n elements of $\{1,\ldots,i-1,i+1,\ldots,N\}$ and

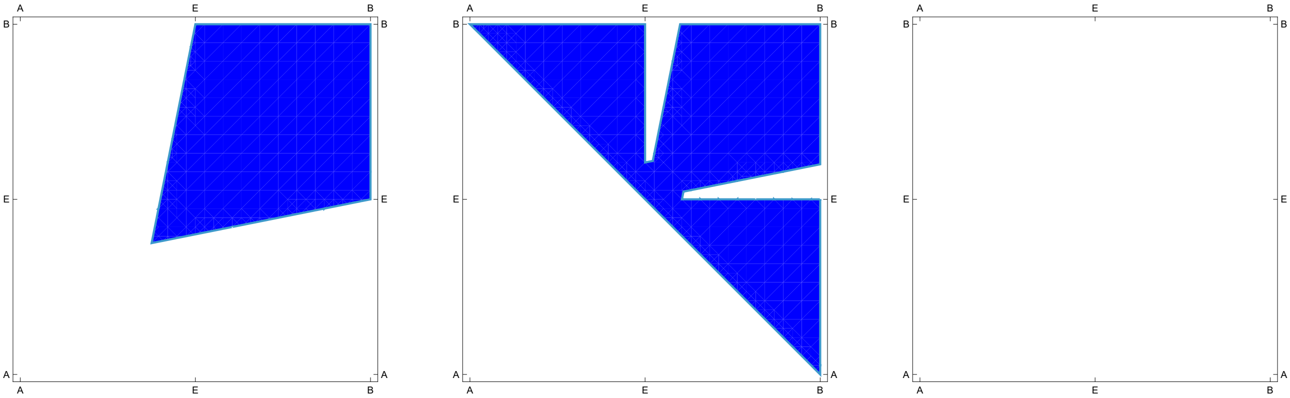
$$P(x_1,\ldots,x_N) = \frac{\sum_{i=1}^N x_i + N\,E}{2N} \tag{78}$$

The situation resembles a class of aggregative games, with the additional feature that the payoff functions depend not only on the player's own strategy and the aggregator, but also on the strategies of the other players. Moreover, the direct approach fails in this setting for $N \geqslant 3$, making functional derivation of the solution impracticable.

5.1 Illustration of the case $N = 3$

Let us illustrate the specific case of three players. In this setting, the payoff function of player 1 is given by

$$\begin{aligned} g(x,y,z) = {} & \mathbf{1}_{\{z\leqslant y<x\leqslant P\}} + \mathbf{1}_{\{y<z<x\leqslant P\}} + \mathbf{1}_{\{y<x\leqslant P<z\}} + \mathbf{1}_{\{z<x\leqslant P<y\}} \\ & + \mathbf{1}_{\{x\leqslant P<z\leqslant y\}} + \mathbf{1}_{\{x\leqslant P<y<z\}} + \mathbf{1}_{\{P<x<z\leqslant y\}} + \mathbf{1}_{\{P<x<y<z\}} \\ & + \frac{1}{2}\left(\mathbf{1}_{\{z<y=x\leqslant P\}} + \mathbf{1}_{\{y<z=x\leqslant P\}} + \mathbf{1}_{\{y=x\leqslant P<z\}} + \mathbf{1}_{\{z=x\leqslant P<y\}} + \mathbf{1}_{\{P\leqslant y=x<z\}} + \mathbf{1}_{\{P\leqslant z=x<y\}}\right) \\ & + \frac{1}{3}\mathbf{1}_{\{x=y=z\}} \end{aligned} \tag{79}$$



$x = A$. $x = E$. $x = B$.

Figure 3: The winning region for different values of x.

given the bids x, y, and z of the three players, respectively. We can represent the winning region ($g(x, y, z) = 1$) of player 1 for different values of x (see Figure 3):

We will show that g has a finite number of jump points and can be expressed accordingly. Rearranging the inequality $x \leqslant P$ gives

$$x \leqslant \frac{x + y + z + 3E}{6} \iff x \leqslant \frac{y + z + 3E}{5}.$$

Define the opponents-only threshold

$$t(y, z) := \frac{y + z + 3E}{5}. \tag{80}$$

For fixed (y, z), ties of the form $x = t(y, z) = y$ or $x = t(y, z) = z$ introduce two additional x-values at which the ordering of $\{x, y, z, t(y, z)\}$ changes. Solving

$$P(x, y, z) = y \iff x = 5y - 3E - z =: p_y(y, z), \quad P(x, y, z) = z \iff x = 5z - 3E - y =: p_z(y, z), \tag{81}$$

We obtain two more opponent-dependent cutpoints. Thus for fixed (y, z) the function $x \mapsto g(x, y, z)$ depends only on the ordering of $\{\, y,\ z,\ t(y, z),\ p_y(y, z),\ p_z(y, z) \,\}$ and is piecewise constant with possible discontinuities only at these five points.

Fix (y, z). For each $s \in \mathcal{S}(y, z) := \{\, y,\ z,\ t(y, z),\ p_y(y, z),\ p_z(y, z) \,\}$ define the one-sided interval limits

$$g_1(s^-, y, z) := \lim_{x \uparrow s} g_1(x, y, z), \qquad g_1(s^+, y, z) := \lim_{x \downarrow s} g_1(x, y, z).$$

These one-sided limits are values taken on open intervals adjoining s, hence belong to $\{0, 1\}$. Define the interval-jump

$$\Delta_s(y, z) := g_1(s^+, y, z) - g_1(s^-, y, z) \in \{-1, 0, 1\}. \tag{82}$$

The five cutpoints satisfy the linear relations

$$p_y - y = 5(y - t), \qquad p_z - z = 5(z - t), \qquad p_y - p_z = 6(y - z). \tag{S}$$

Hence p_y lies on the same side of t as y, p_z lies on the same side of t as z, and $p_y < p_z \iff y < z$. Only four geometric orderings (up to symmetry $y \leftrightarrow z$) are compatible with (S):

Case I: $y < t$ and $z < t$. Two orderings are possible:

$$(O_1) \quad p_y < p_z < y < z < t, \qquad (O_2) \quad p_y < y < p_z < z < t.$$

Case II: $y < t < z$. A single ordering is possible:

$$(O_3) \quad p_y < y < t < z < p_z.$$

Case III: $t < y < z$. Two orderings are possible:

$$(O_4) \quad t < y < z < p_y < p_z. \qquad (O_5) \quad t < y < p_y < z < p_z.$$

These are the *only* admissible relative orders of $\{p_y, p_z, y, z, t\}$.

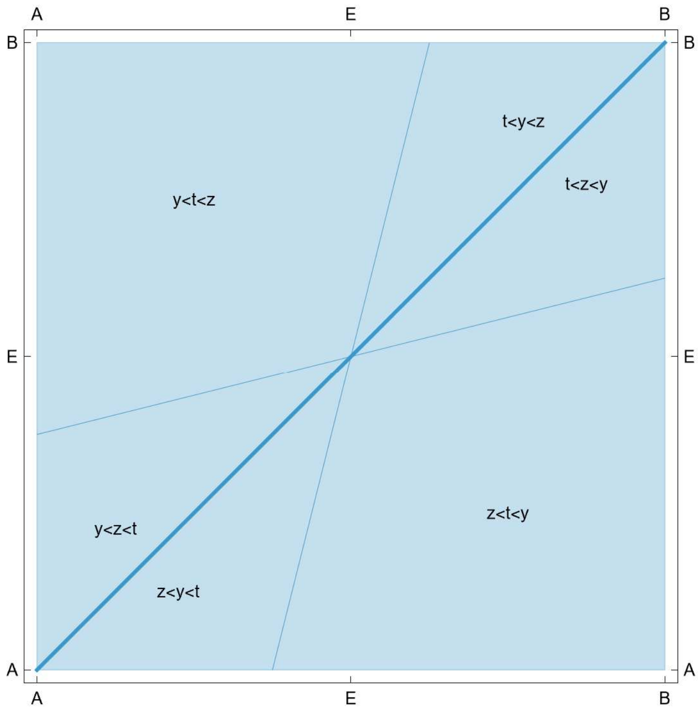


Figure 4: The six regions of the $y - z$ plane.

We can summarize the sign of Δ_s in the following table:

5.2 Existence result for $N \geqslant 3$

Let's start by noticing that the game is symmetric, since

$$g_{\pi(i)}(x_{\pi(1)}, \ldots, x_{\pi(N)}) = g_i(x_1, \ldots, x_N) \tag{83}$$

for any permutation π and the set of actions is $[A, B]$ for all i. Moreover, the function g_i is measurable and bounded between 0 and 1, but it is neither quasiconcave nor continuous. The strategy set, is

O_k / s	y	p_y	z	p_z	t
O_1	0	-1	+1	0	-1
O_2	+1	-1	+1	-1	-1
O_3	+1	-1	0	0	-1
O_4	-1	0	0	0	0
O_5	-1	0	0	0	0

Table 1: Interval-jump signs Δ_s for each ordering cell O_k.

convex and compact.

Next, we study the set of discontinuities of g_i, denoted by $\mathcal{D}_i$. It is formed by the following disjoints hypersurfaces:

1. **Hypersurfaces of ties:**

$$\mathcal{T}_i = \left\{(x_1, \dots, x_N) \,\middle|\, \exists n,\, 1 \leqslant n \leqslant N,\, x_i = x_k \text{ for } k \in J_n(i), \text{ and } g_i(x_1, \dots, x_N) = \frac{1}{n}\right\} \tag{84}$$

2. **Hypersurfaces of fixed points:**

$$\mathcal{F}_i = \left\{(x_1, \dots, x_N) \,\middle|\, x_i = \frac{\sum_{j \neq i} x_j + NE}{2N-1}, \quad g_i(x_1, \dots, x_N) = 1\right\} \tag{85}$$

3. **Hypersurfaces of transition points:** define $\underline{j} = \arg\max_{1 \leqslant j \leqslant N} \left\{x_j \,\middle|\, x_j \leqslant \frac{\sum_{j \neq i} x_j + NE}{2N-1}\right\}$

$$\mathcal{P}_i = \left\{(x_1, \dots, x_N) \,\middle|\, x_i = (2N-1)x_{\underline{j}} - \sum_{j \neq i, \underline{j}} x_j - NE, \quad g_i(x_1, \dots, x_N) = 0\right\} \tag{86}$$

The set $\mathcal{D}_i$ is closed and has zero N-dimensional Lebesgue measure, i.e., $\mathcal{L}^N(\mathcal{D}_i) = 0$.

The function g_i is neither lower semi-continuous nor upper semi-continuous. Counterexamples can be found at diagonal points $(x, \dots, x) \in \mathcal{T}_i$ for $x \in [A, B]$.

The function g_i is, however, upper semi-continuous on $[A, B]^N \backslash \mathcal{T}_i \bigcup \mathcal{P}_i$. Indeed, for all $i \in \{1, \dots, N\}$, for any $\mathbf{x} = (x_1, \dots, x_N) \in \mathcal{F}_i$, and for any sequence $(\mathbf{x}_n)$ such that $\lim_{n \to \infty} \mathbf{x}_n = \mathbf{x}$, we have

$$\lim_{n \to \infty} \sup g_i(\mathbf{x}_n) \leqslant g_i(\mathbf{x}) = 1. \tag{87}$$

One can verify that $g_i(x_i, \mathbf{x}_{-i})$ is weakly lower semi-continuous in x_i for all i in the sens of Dasgupta and Maskin (1986). Indeed, for $\mathbf{x} \in \mathcal{T}_i$ for some i, g_i is easily seen to be either left- or right-lower semi-continuous. If $\mathbf{x} \in \mathcal{F}_i$ (the single-winner situation), the same reasoning applies. If $\mathbf{x} \in \mathcal{P}_i$, one can check that the function is lower semi-continuous: for any sequence $(\mathbf{x}_n)$ such that $\lim_{n \to \infty} \mathbf{x}_n = \mathbf{x}$, we have

$$\lim_{n \to \infty} \inf g_i(\mathbf{x}_n) \geqslant g_i(\mathbf{x}) = 0. \tag{88}$$

Moreover, the sum $\sum_{i=1}^N g_i$ is upper semi-continuous, since the sum always equals 1. Unfortunately, the discontinuity set cannot be defined by a one-to-one correspondence, and Lemma 3 of Dasgupta and Maskin (1986) fails to apply as soon as $N \geqslant 3$.

The upper semi-continuity of the game implies its reciprocal upper semi-continuity introduced in Reny (1999), while one can prove that the game is payoff secure since, for all i and all $\mathbf{x}_{-i} \in [A,B]^{N-1}$,

$$\sup_{x_i \in [A,B]} g_i(x_i, \mathbf{x}_{-i}) = \sup_{x_i \in [A,B]} \underline{g_i}(x_i, \mathbf{x}_{-i}) = 1$$

where $\underline{g_i}$ represents the lower semi-continuous regularization of g_i (obtained by replacing its values on $\mathcal{T}_i$ and $\mathcal{F}_i$ with zero). Indeed, one can use the alternative definition of payoff security (Reny (1999)) to prove the following: For all $\mathbf{x}_{-i} \in S_{-i}$ define

$$t(\mathbf{x}_{-i}) = \frac{\sum_{j \neq i} x_j + NE}{2N-1} \tag{89}$$

We distinguish two cases:

1. $\nexists j\,:\, x_j = t(\mathbf{x}_{-i})$: Let $\underline{j} = \arg\max_{1 \leqslant j \leqslant N} \{x_j \,|\, x_j < t(\mathbf{x}_{-i})\}$ and $\overline{j} = \arg\min_{1 \leqslant j \leqslant N} \{x_j \,|\, x_j > t(\mathbf{x}_{-i})\}$, then define

$$\begin{aligned} \underline{\varepsilon} &= \mid t(\mathbf{x}_{-i}) - x_{\underline{j}} \mid \\ \overline{\varepsilon} &= \mid t(\mathbf{x}_{-i}) - x_{\overline{j}} \mid \end{aligned}$$

   Then fix $\tilde{\varepsilon} = \varepsilon < \dfrac{\min(\underline{\varepsilon}, \overline{\varepsilon})}{2}$ and define

$$\begin{aligned} \bar{x}_i &= t(\mathbf{x}_{-i}) - \varepsilon \\ N_{\tilde{\varepsilon}}(\mathbf{x}_{-i}) &= \{\mathbf{y}_{-i} \in S_{-i} : \|\mathbf{y}_{-i} - \mathbf{x}_{-i}\| < \tilde{\varepsilon}\}\,. \end{aligned}$$

2. $\exists j\,:\, x_j = t(\mathbf{x}_{-i})$: Let $\underline{j} = \arg\max_{1 \leqslant j \leqslant N} \{x_j \,|\, x_j < t(\mathbf{x}_{-i})\}$ and $\overline{j} = \arg\min_{1 \leqslant j \leqslant N} \{x_j \,|\, x_j > t(\mathbf{x}_{-i})\}$, then define

$$\begin{aligned} \underline{\varepsilon} &= \mid t(\mathbf{x}_{-i}) - x_{\underline{j}} \mid \\ \overline{\varepsilon} &= \mid t(\mathbf{x}_{-i}) - x_{\overline{j}} \mid \end{aligned}$$

   Then fix $\varepsilon < \dfrac{\min(\underline{\varepsilon}, \overline{\varepsilon})}{2}$ and $\tilde{\varepsilon} < \frac{\varepsilon}{2N}$ and define

$$\begin{aligned} \bar{x}_i &= t(\mathbf{x}_{-i}) - \varepsilon \\ N_{\tilde{\varepsilon}}(\mathbf{x}_{-i}) &= \{\mathbf{y}_{-i} \in S_{-i} : \|\mathbf{y}_{-i} - \mathbf{x}_{-i}\| < \tilde{\varepsilon}\}\,. \end{aligned}$$

3. $\forall j\,:\, x_j = t(\mathbf{x}_{-i})$: Fix $\varepsilon > 0$ and $\tilde{\varepsilon} < \frac{\varepsilon}{2N}$ and define

$$\begin{aligned} \bar{x}_i &= t(\mathbf{x}_{-i}) - \varepsilon \\ N_{\tilde{\varepsilon}}(\mathbf{x}_{-i}) &= \{\mathbf{y}_{-i} \in S_{-i} : \|\mathbf{y}_{-i} - \mathbf{x}_{-i}\| < \tilde{\varepsilon}\}\,. \end{aligned}$$

In all the cases one can check that

$$\mathbf{y}_{-i} \in N_{\tilde{\varepsilon}}(\mathbf{x}_{-i}) \implies g_i(\bar{x}_i, \mathbf{y}_{-i}) = 1 \geqslant g_i(x_i, \mathbf{x}_{-i}). \tag{90}$$

Thus, the game is better-reply secure (Reny, 1999, Prop. 3.2). However, the game is not quasiconcave which prevent the application of Reny's fundamental result about equilibrium existence in pure strategies.

As pointed out by Reny (1999), when moving to mixed strategies, securing a payoff becomes in certain respects easier, yet also more difficult. In fact, establishing payoff security in mixed strategies is considerably harder than in the pure-strategy case. To facilitate this verification, several authors have proposed conditions on the underlying pure-strategy game that allow one to conclude about payoff security of its mixed extension. A first such condition is the notion of *uniformly payoff secure games*, introduced by Monteiro and Page Jr (2007).

Definition 5.1. Uniformly payoff security (Monteiro and Page Jr (2007))
A game $\{(g_i, S_i)_{i=1}^N\}$ is *uniformly payoff secure* if for every $x_i \in S_i$ and every $\varepsilon > 0$ there exists for each player i a strategy $\bar{x}_i \in S_i$ such that for every $\mathbf{x}_{-i} \in S_{-i}$ there exists a neighborhood $N(\mathbf{x}_{-i})$ of $\mathbf{x}_{-i}$ with the property that

$$\mathbf{y}_{-i} \in N(\mathbf{x}_{-i}) \implies g_i(\bar{x}_i, \mathbf{y}_{-i}) \geqslant g_i(x_i, \mathbf{x}_{-i}) - \varepsilon. \tag{91}$$

On can check that our game is not uniformly payoff secure. Take for example $N = 2$, and choose $x_i < E$, then

1. If $x_i < \bar{x}_i$, choose y s.t. $y < x_i < t < \bar{x}_i$.
2. If $x_i > \bar{x}_i$, choose y s.t. $\bar{x}_i < y < x_i < t$.
3. If $x_i = \bar{x}_i$, choose $y = x_i = \bar{x}_i$.

The game is not uniformly diagonally secure in the sens of Prokopovych and Yannelis (2014) neither.

A less demanding condition is introduced by Allison and Lepore (2014) and can check directly the payoff security in the mixed extension. The property to be tested is called *disjoint payoff matching*, and it is particularly well suited to our problem.

Definition 5.2. Disjoint Payoff Matching (Allison and Lepore (2014))
A game $\{(g_i, S_i)_{i=1}^N\}$ satisfies *disjoint payoff matching* if for all $x_i \in S_i$, there exists a sequence of deviations $(x_i^k)_{k\geqslant 1} \subset S_i$ such that the following hold:

$$\begin{aligned} &1. \quad \liminf_{k\to\infty} g_i(x_i^k, \mathbf{x}_{-i}) \geqslant g_i(x_i, \mathbf{x}_{-i}) \qquad \forall\, \mathbf{x}_{-i} \in S_{-i}, \\ &2. \quad \limsup_{k\to\infty} \mathbb{D}_i(x_i^k) = \varnothing. \end{aligned} \tag{92}$$

where

$$\mathbb{D}_i(x_i) = \{\, \mathbf{x}_{-i} \in S_{-i} \;:\; g_i \text{ is discontinuous in } \mathbf{x}_{-i} \text{ at } (x_i, \mathbf{x}_{-i})\} \,. \tag{93}$$

Theorem 5.1. ***(Allison and Lepore (2014))***
Let $\{(g_i, S_i)_{i=1}^N\}$ be a compact game and satisfies the disjoint payoff matching condition. Then its mixed extension is payoff secure.

This result, together with the upper semicontinuity of the game, ensures that the mixed extension is better-reply secure, since upper semicontinuity of the pure strategy game is inherited by its mixed extension.

Theorem 5.2. ***(Reny, 1999, Corollary 5.2)***
Suppose that $\{(g_i, S_i)_{i=1}^N\}$ is a compact, Hausdorff game. Then it possesses a mixed–strategy Nash equilibrium if its mixed extension is better–reply secure. Moreover, its mixed extension is better–reply secure if it is both reciprocally upper semicontinuous and payoff secure.

We will now prove that our game possesses a symmetric Nash equilibrium. The idea relies on diagonally better–reply security in quasi–symmetric games.

Definition 5.3. (Reny (1999))
A game $\{(g_i, S_i)_{i=1}^N\}$ is *diagonally payoff secure* if for every $x \in S_i$ and every $\varepsilon > 0$, each player i has a strategy $\bar{x}_i \in S_i$ such that

$$g_i\big(y, \ldots, \bar{x}_i, \ldots, y\big) \geqslant g_i(x, \ldots, x) - \varepsilon \tag{94}$$

for all $(y, \ldots, y)$ in some open neighborhood of $(x, \ldots, x)$.

An alternative way to prove the game is diagonally better-reply secure is to define a diagonal version of the disjoint payoff matching condition.

Definition 5.4. Diagonal Disjoint Payoff Matching (DDPM)
The quasi-symmetric game $\{(g_i, S_i)_{i=1}^N\}$ satisfies *diagonal disjoint payoff matching* if for all $y \in S_i$, there exists a sequence of deviations $(x^k)_{k\geqslant 1} \subset S_i$ such that the following hold:

$$\begin{aligned} &1. \qquad \liminf_{k\to\infty} g_i(x, \ldots, x^k, \ldots, x) \;\geqslant\; g_i(x, \ldots, y, \ldots, x) \qquad \forall\, (x, \ldots, x) \in \prod_{i\neq j}^{N} S_i, \\ &2. \qquad \limsup_{k\to\infty} \mathbb{D}_i(x^k) = \varnothing. \end{aligned} \tag{95}$$

where

$$\mathbb{D}_i(y) = \{x \in S_i \;:\; g_i \text{ is discontinuous in at } (x, \ldots, y, \ldots, x)\}\,. \tag{96}$$

We now introduce a diagonal version of Theorem 1 in Allison and Lepore (2014); its proof follows the same reasoning as the original, and is provided in the appendix.

Theorem 5.3.
Let $\{(g_i, S_i)_{i=1}^N\}$ be a quasi-symmetric compact game and satisfies the diagonal disjoint payoff matching condition. Then its mixed extension is diagonally payoff secure.

Let's construct such a sequence of deviations (x_i^k) for our game $\{(g_i, [A, B])_{i=1}^N\}$. For any $x_i \in [A, B]$, the set of discontinuities $\mathbb{D}_i(x_i)$ is the union of the following hypersurfaces:

1. **Hypersurfaces of ties:**

$$\mathbb{T}_i(x_i) = \left\{(x, \ldots, x) \;\middle|\; x_i = x, \text{ and } g_i(x, \ldots, x_i, \ldots, x) = \frac{1}{N}\right\} \tag{97}$$

2. **Hypersurfaces of fixed points:**

$$\mathbb{F}_i(x_i) = \left\{(x, \ldots, x) \;\middle|\; x = \frac{(2N-1)x_i - NE}{N-1}, \quad g_i(x, \ldots, x_i, \ldots, x) = 1\right\} \tag{98}$$

3. **Hypersurfaces of transition points:**

$$\mathbb{P}_i(x_i) = \left\{ (x, \ldots, x) \;\middle|\; x = \frac{x_i + NE}{N+1}, \quad g_i(x, \ldots, x_i, \ldots, x) = 0 \right\} \tag{99}$$

Since $\mathbb{T}_i(x_i)$, $\mathbb{F}_i(x_i)$ and $\mathbb{P}_i(x_i)$ are mutually disjoint, we define the sequence of deviations (x_i^k) as follows.

1. For $x_i \in]A, B]$, choose a strictly increasing sequence $x_i^k \uparrow x_i$ in a neighborhood of x_i, then:
   (a) If $(x_i, \mathbf{x}_{-i})$ is a continuity point, the inferior limit condition is trivially satisfied.
   (b) If $\mathbf{x}_{-i} \in \mathbb{F}_i(x_i)$, one can verify that $g_i(\cdot, \mathbf{x}_{-i})$ in a neighborhood of x_i decreases drastically from 1 to 0 for $x > x_i$.
   (c) If $\mathbf{x}_{-i} \in \mathbb{P}_i(x_i)$, one can verify that $g_i(\cdot, \mathbf{x}_{-i})$ in a neighborhood of x_i takes the value 1 below x_i and becomes 0 for $x \geqslant x_i$.
2. For $x_i = A$, choose $x_i^k \downarrow A$.

In addition to the fact that the maps $s_1(x_j) = x_j$, $s_2(\mathbf{x}_{-i}) = \sum_{j \neq i} x_j$ and $s_3(\mathbf{x}_{-i}) = (2N-1)x_{\underline{j}} - \sum_{j \neq i, \underline{j}} x_j$ are single–valued functions, we conclude that $\mathbb{D}_i(x_i^k) \cap \mathbb{D}_i(x_i^l) = \varnothing$ for $k \neq l$.

However, we get a problem at ties situation: If $\mathbf{x}_{-i} \in \mathbb{T}_i(x_i)$, one can check $x_i^k \uparrow x_i$ is not a good choice when $x_i < E$. To bypass this obstacle, one could define the game $\{(\tilde{g}_i, [A, B])_{i=1}^N\}$ where each payoff function $\tilde{g}_i$ has zero value on the ties. One can check for $x_i^k \downarrow x_i$ and $\mathbf{x}_{-i} \in \mathbb{T}_i(x_i)$ that

1. If $x_i < E$, $\tilde{g}_i(\cdot, \mathbf{x}_{-i})$ in a neighborhood of x_i is null for $x \leqslant x_i$ then jump to 1 for $x > x_i$.
2. If $x_i \geqslant E$, $\tilde{g}_i(\cdot, \mathbf{x}_{-i})$ in a neighborhood of x_i is 1 for $x < x_i$ and takes the value zero for $x \geqslant x_i$.

The DDPM condition is satisfied for the new game $\{(\tilde{g}_i, [A, B])_{i=1}^N\}$, and by Theorem 5.3, its mixed extension is diagonally payoff secure.

Theorem 5.4.
The game $\{(\tilde{g}_i, [A, B])_{i=1}^N\}$ is diagonally payoff secure.

Remark 5.1.
Using an analogous argument, one verifies that the modified game $\{(\tilde{g}_i, [A, B])_{i=1}^N\}$ satisfies the disjoint payoff matching condition, and hence it is payoff secure.

Diagonal payoff security together with reciprocal upper semicontinuity implies diagonal better–reply security (Reny, 1999, Prop. 4.2). And one can try to deduce the existence of a symmetric Nash equilibrium in mixed strategies for this version of the game. However, the game $\{(\tilde{g}_i, S_i)_{i=1}^N\}$ is not reciprocal upper semicontinuous.

The auxiliary game $\{(\tilde{g}_i, S_i)_{i=1}^N\}$ resolves ties by assigning a null payoff at tie profiles. This corresponds to an *exogenously specified* tie-breaking rule, introduced solely for analytical convenience and without affecting payoffs outside a null set of strategy profiles.

A more general approach to payoff discontinuities induced by ties consists in allowing *endogenous tie-breaking rules*, whereby the resolution of ties is not fixed ex ante but is instead selected as part of the equilibrium outcome. Existence results for such games are provided by **?**, who study games with endogenous sharing rules and establish equilibrium existence under weak continuity requirements.

Theorem 5.5. *(**?**)*
Every compact metric game admits a mixed sharing rule solution.

Appendix

Theorem 5.6.
Let $\{(g_i, S_i)_{i=1}^N\}$ be a quasi-symmetric compact game and satisfies the diagonal disjoint payoff matching condition. Then its mixed extension is diagonally payoff secure.

Lemma 5.7.
Suppose that the quasi-symmetric compact game $\{(g_i, S_i)_{i=1}^N\}$ satisfies diagonal disjoint payoff matching. Set $S_{-i} = \prod_{j \neq i} S_j$. Then for all $\varepsilon > 0$, $y \in S_i$, $\mu \in \Delta(S_i)$ and $\boldsymbol{\mu} = \underbrace{(\mu, \ldots, \mu)}_{N \text{ times}}$, there exists a deviation $\tilde{x} \in S_i$ and a compact set $K \subset S_{-i} \backslash \mathbb{D}_i(\tilde{x})$ such that:

1.
$$g_i(x, \ldots, \tilde{x}, \ldots, x) \;>\; g_i(x, \ldots, y, \ldots, x) - \varepsilon \qquad \forall\, (x, \ldots, x) \in K,$$

2.
$$\boldsymbol{\mu}_{-i}(S_{-i} \backslash K) < \varepsilon.$$

Proof. of Lemma 5.7
Assume that $\{(g_i, S_i)_{i=1}^N\}$ satisfies diagonal disjoint payoff matching and consider any player i, any $\varepsilon > 0$, and any $\mu \in \Delta(S_i)$. Let $\{x^k\}$ be a defection sequence from the definition of diagonal disjoint payoff matching. Define the collection of sets

$$E_k^i = \Big\{ x \in S_i \;:\; g_i(x, \ldots, x^k, \ldots, x) \;>\; g_i(x, \ldots, y, \ldots, x) - \varepsilon \Big\}.$$
$$E_k = \prod_{j \neq i} E_k^i$$

Notice that $\liminf_k E_k = S_{-i}$, so $\boldsymbol{\mu}_{-i}(\liminf_k E_k) = 1$. Further, $\limsup_k \mathbb{D}_i(x^k) = \varnothing$, so

$$\boldsymbol{\mu}_{-i}(\limsup_k \mathbb{D}_i(x^k)) = 0$$

By statement (5) in Section 9 of Halmos (Halmos (1950)),

$$\boldsymbol{\mu}_{-i}(\liminf_k E_k) \leqslant \liminf_k \boldsymbol{\mu}_{-i}(E_k) \quad \text{and} \quad \boldsymbol{\mu}_{-i}(\limsup_k \mathbb{D}_i(x^k)) \geqslant \limsup_k \boldsymbol{\mu}_{-i}(\mathbb{D}_i(x^k)),$$

and so

$$\lim_k \boldsymbol{\mu}_{-i}(E_k) = 1 \qquad \text{and} \qquad \lim_k \boldsymbol{\mu}_{-i}(\mathbb{D}_i(x^k)) = 0.$$

It follows that there exists a k such that

$$\boldsymbol{\mu}_{-i}(E_k) > 1 - \frac{\varepsilon}{3} \qquad \text{and} \qquad \boldsymbol{\mu}_{-i}\big(\mathbb{D}_i(x^k)\big) < \frac{\varepsilon}{3}.$$

Choose such a k. By regularity of $\boldsymbol{\mu}_{-i}$, we may choose a closed (and thus compact) subset

$$K \subset E_k \backslash \mathbb{D}_i(x^k)$$

such that

$$\boldsymbol{\mu}_{-i}(K) > \boldsymbol{\mu}_{-i}\Big(E_k \backslash \mathbb{D}_i(x^k)\Big) - \frac{\varepsilon}{3}.$$

It follows that

$$\boldsymbol{\mu}_{-i}(S_{-i} \backslash K) < \varepsilon.$$

This completes the proof. □

Proof. of Theorem 5.6
Let $\varepsilon > 0$ and suppose that $\boldsymbol{\mu} = (\mu, \dots, \mu) \in \prod_{i=1}^N \Delta(S_i)$. Note that for each player i there exists some strategy y in the support of μ_i such that

$$\int g_i(x, \dots, y, \dots, x)\, d\boldsymbol{\mu}_{-i} \;\geqslant\; \int g_i(x, \dots, x)\, d\boldsymbol{\mu}.$$

From diagonal disjoint payoff matching and Lemma 5.7, there exists a deviation $\tilde{x}$ and a set $K(\varepsilon) \subset S_{-i} \backslash \mathbb{D}_i(\tilde{x})$ such that

$$g_i(x, \dots, \tilde{x}, \dots, x) \;>\; g_i(x, \dots, y, \dots, x) - \frac{\varepsilon}{6} \qquad \forall\, (x, \dots, x) \in K(\varepsilon),$$

and

$$\boldsymbol{\mu}_{-i}(X_{-i} \backslash K(\varepsilon)) < \frac{\varepsilon}{6M}, \qquad M \equiv \sup |g_i|.$$

It follows that

$$\int_{K(\varepsilon)} g_i(x, \dots, \tilde{x}, \dots, x)\, d\boldsymbol{\mu}_{-i} \;>\; \int_{K(\varepsilon)} g_i(x, \dots, y, \dots, x)\, d\boldsymbol{\mu}_{-i} \;-\; \frac{\varepsilon}{6}.$$

Further, we have

$$\begin{aligned}
&\int_{X_{-i} \backslash K(\varepsilon)} g_i(x, \dots, \tilde{x}_i, \dots, x)\, d\boldsymbol{\mu}_{-i} - \int_{X_{-i} \backslash K(\varepsilon)} g_i(x, \dots, y, \dots, x)\, d\boldsymbol{\mu}_{-i} \\
&> - \int_{X_{-i} \backslash K(\varepsilon)} \big|g_i(x, \dots, \tilde{x}, \dots, x)\big| + \big|g_i(x, \dots, y, \dots, x)\big|\, d\boldsymbol{\mu}_{-i} \\
&> -2 \sup |g_i| \cdot \boldsymbol{\mu}_{-i}(S_{-i} \backslash K(\varepsilon)) > -\frac{2\varepsilon}{6}.
\end{aligned}$$

Combining those two results yields

$$\int g_i(x, \dots, \tilde{x}, \dots, x)\, d\boldsymbol{\mu}_{-i} \;>\; \int g_i(x, \dots, y, \dots, x)\, d\boldsymbol{\mu}_{-i} - \frac{\varepsilon}{2}.$$

Define

$$\underline{g_i}(x, \dots, \tilde{x}, \dots, x) = \sup_{V \ni (x, \dots, x)} \; \inf_{(x', \dots, x') \in V} g_i(x', \dots, \tilde{x}, \dots, x'),$$

where the supremum is taken over all neighborhoods V of $(x, \dots, x)$. As noted by Reny (Reny (1999)), $\underline{g_i}(x, \dots, x)$ is lower semicontinuous. From Reny's proof of Proposition 5.1 (Reny (1999)), it follows that $\int \underline{g_i}(x, \dots, x)\, d\boldsymbol{\mu}_{-i}$ is lower semicontinuous in $\boldsymbol{\mu}_{-i}$. Thus there exists a neighborhood $N(\mu)$ such that for all $\lambda \in N(\mu)$, $\boldsymbol{\lambda}_{-i} = \underbrace{(\lambda, \dots, \lambda)}_{(N-1)\text{ times}}$,

$$\int \underline{g_i}(x, \dots, x)\, d\boldsymbol{\lambda}_{-i} \;>\; \int \underline{g_i}(x, \dots, x)\, d\boldsymbol{\mu}_{-i} - \frac{\varepsilon}{6}.$$

Since M bounds both $\underline{g_i}$ and g_i, we have

$$\int_{X_{-i} \backslash K(\varepsilon)} \Big(\underline{g_i}(x, \dots, x) - g_i(x, \dots, \tilde{x}, \dots, x)\Big)\, d\boldsymbol{\mu}_{-i} \geqslant - \int_{X_{-i} \backslash K(\varepsilon)} \big|\underline{g_i}(x, \dots, x)\big| + \big|g_i(x, \dots, \tilde{x}, \dots, x)\big|\, d\boldsymbol{\mu}_{-i} > -\frac{2\varepsilon}{6}.$$

Further, since $g_i(x, \dots, \tilde{x}, \dots, x)$ is continuous in $(x, \dots, x)$ at all $(x, \dots, x) \in K(\varepsilon)$, we have $g_i(x, \dots, x) = g_i(x, \dots, \tilde{x}, \dots, x)$ on $K(\varepsilon)$. Therefore,

$$\begin{aligned}
\int \underline{g_i}(x, \dots, x)\, d\boldsymbol{\mu}_{-i} &= \int g_i(x, \dots, \tilde{x}, \dots, x)\, d\boldsymbol{\mu}_{-i} + \int_{X_{-i} \backslash K(\varepsilon)} \Big(\underline{g_i}(x, \dots, x) - g_i(x, \dots, \tilde{x}, \dots, x)\Big) d\boldsymbol{\mu}_{-i} \\
&> \int g_i(x, \dots, \tilde{x}, \dots, x)\, d\boldsymbol{\mu}_{-i} - \frac{2\varepsilon}{6}.
\end{aligned}$$

Using $g_i(x, \ldots, \tilde{x}, \ldots, x) \geqslant \underline{g_i}(x, \ldots, x)$ and combining the two precedent inequalities we have for all $\lambda \in N(\mu)$,

$$\begin{aligned}\int g_i(x, \ldots, \tilde{x}, \ldots, x)\, d\boldsymbol{\lambda}_{-i} &\geqslant \int \underline{g_i}(x, \ldots, x)\, d\boldsymbol{\lambda}_{-i} \\ &> \int \underline{g_i}(x, \ldots, x)\, d\boldsymbol{\mu}_{-i} - \frac{\varepsilon}{6} \\ &> \int g_i(x, \ldots, \tilde{x}, \ldots, x)\, d\boldsymbol{\mu}_{-i} - \frac{\varepsilon}{2}.\end{aligned}$$

Finally, we find that for all $\lambda \in N(\mu)$,

$$\begin{aligned}\int g_i(x, \ldots, \tilde{x}, \ldots, x)\, d\boldsymbol{\lambda}_{-i} &> \int g_i(x, \ldots, \tilde{x}, \ldots, x)\, d\boldsymbol{\mu}_{-i} - \frac{\varepsilon}{2} \\ &> \int g_i(x, \ldots, y, \ldots, x)\, d\boldsymbol{\mu}_{-i} - \varepsilon \\ &\geqslant \int g_i\, d\boldsymbol{\mu} - \varepsilon.\end{aligned}$$

Therefore, the mixed extension of $\{(g_i, S_i)_{i=1}^N\}$ is diagonal payoff secure. □

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Conflict of interest

The author declares that they have no conflict of interest.

Authors' contributions

The author carried out the study conception and design. Material preparation, data collection and analysis were performed by the author. The manuscript was written by the author.

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During the preparation of this work the author used ChatGPT in order to improve the text. After using this tool/service, the author reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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