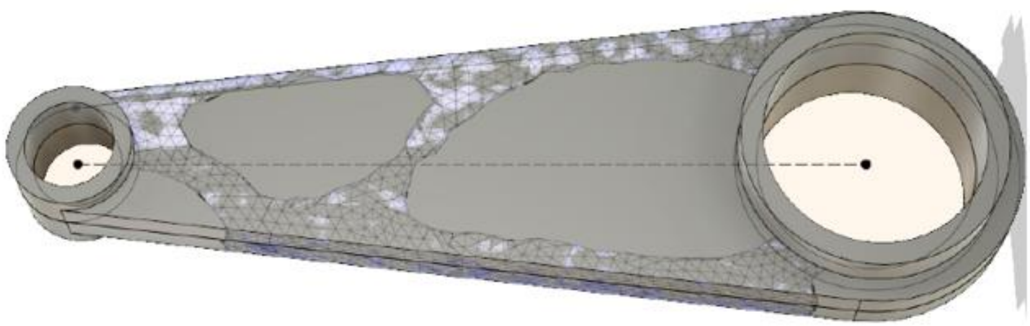
Continuum Mechanics

for Modeling Simulation & Design

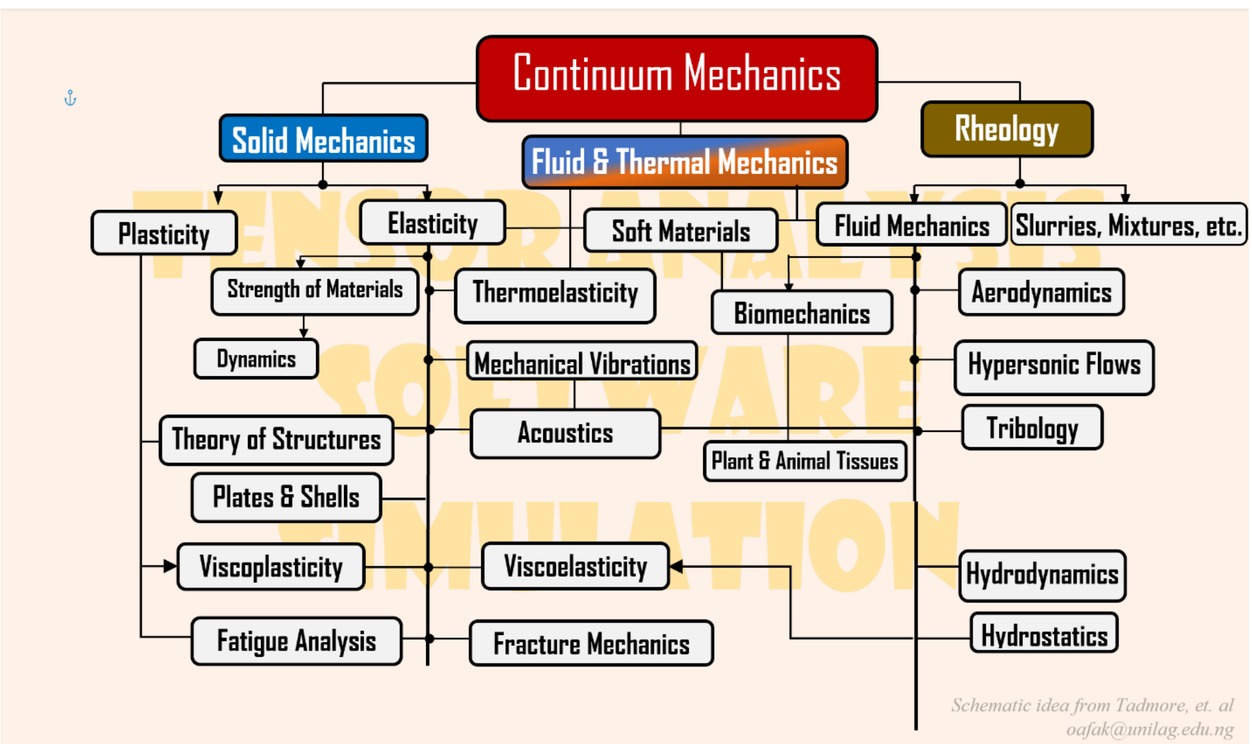


OA Fakinlede

The Course, Learning Resources & Your Responsibility

The Course, Continuum Mechanics, is being moved from a Post-Graduate level, to meet you at 300 Level. This is the first set at the University of Lagos to be taught this way. There may not be many universities starting this course at such a low undergraduate class. You are the first set to have this opportunity at 300 level. In that sense you are lucky.

Continuum Mechanics can be thought of as the grand unifying theory of engineering science. Many of the courses taught in an engineering curriculum are closely related and can be obtained as special cases of the general framework of continuum mechanics. The balance laws of mass, momentum and energy that are derived in the context of specific material constitutions are natural laws that are natural laws and independent of these contexts. This fact is easily lost on most undergraduate and even some graduate students.



From the picture above, you will see that many of the other things you will learn on your road to becoming an engineer are rooted in Continuum Mechanics. If you look carefully, you will see that these things are not written on a clean sheet of paper. A blank mind cannot absorb them. The language it speaks is called ***Tensor Analysis***, you need ***Software*** to practice and engage more challenging problems; then ***Simulation*** will help you deploy the knowledge gained to design virtually in order to save prototyping costs. In this set of courses, we take you through all these stages to enrich your knowledge. The approach here is to optimize your time so to learn things the shortest way and remain focused on **doing engineering with your knowledge**. Engineering is the *application of Science* to create technology **products and services**. It is rooted in theory. If you do not organize the learning of theory very well, you end up with full heads and no products as we have been doing. If you leave theory and simply do "practicals", you end up with half-baked crafts trade – again, no serious products. We are offering you an approach to avoid both extremes and learn, in order to do engineering correctly.

What you will learn here will alter your view about some of the other courses you will take on your way to a degree in engineering. If you do your part, you will be given skills, tools and knowledge that are directed at making you productive people that can change the narrative of dependency and hopelessness that has been Africa's story.

In my last year as dean of this faculty, I had the privilege to welcome your seniors when they started their program here. I quote from that welcome address:

> *"I want you to be ambitious. One of the biggest problems of the African mind is the absence of serious ambition. Once a Naija man can be a little better than his neighbor, he appears satisfied! If there is no electricity, and you get a small noise maker that helps you to watch Manchester United, you are already in heaven! You seem to forget that the same electricity is available to young people in Singapore 24 hours a day! And that there is absolutely no reason why Enyimba, the people's Elephant, cannot be more popular than Manchester United! What do they have? Football grass fields, one ball, 22 men and hundreds of thousands of passionate fans! With some clever marketing, this nets them more money than crude oil in its most comfortable price regimes, can get Nigeria. More depressing is that the City State of Singapore, smaller in population and size than Lagos, can actually consume over 60% of Nigeria's oil! That is the meaning of industrialization! It is lack of ambition that will cause a Minister of Aviation to steal two jeeps! Two jeeps! Even for all their rapacity, our thieves are not sufficiently ambitious! Why, for example, cannot the Minister of Aviation ensure than Nigeria can buy 100 of the latest wide body jets such as A380 or B777 and then steal two of them at the cost of nearly 1 billion dollars each! But once they can drive two jeeps in a convoy and use sirens to chase others from the road, even if they cannot comfortably get to where they are going, they are already satisfied!"*

This first chapter is an attempt, not just to **remind you** what you already know in vector analysis; it is designed to make you deepen your understanding of this most basic of tools in learning continuum mechanics, and prepare the way for tensor theory. If you understand vectors deeply, you have no problem in understanding tensors. When you are struggling along in the latter, the problem often comes from shallow understanding of vectors. Let your ambition be greater than just getting a good grade in this course. We have much more than that to offer you. You can be a real engineer that creates a future for yourself, make a comfortable living and help others to become successful. That is our goal. What is yours?

Learning Resources

The first resource you will meet is **your lecturer**. We have prepared for you. You will quickly see that this is not "just another course". We will listen to you; please ask us questions because we will answer you. When we cannot find the answer immediately, we will tell you. We don't know everything, but we know where we can find useful information to help you.

The book. There are seven chapters. You are only covering chapters 1-3 in 300 level. If you do it well, things get easier as you go along. If not, it will keep getting more difficult. Please start early and do not fall behind. This is a marathon!

Q&A. There are usually at least sixty problems in each chapter. Some have more than one hundred. ALL problems are solved. They are to give you practice and further elucidate the theory. Some of the questions are to be programmed. Our language and environment of choice is $Mathematica^{©}$. Do not fear, we will be gentle with you and you WILL see that programming is achievable and that you can thrive in it! Make an effort and ask questions when you get stuck. In this course, you are NEVER completely stuck unless you choose to remain so! Here are the levels of assistance you may use:

1. Your classmate that may be better in the topic or understands the material more than you or has already gone through the particular problem. That is the first line because she is the most easily accessible help for you.

2. Internet. It is a good idea to browse before throwing in the towel in surrender for any issue. There is scarcely anything you want to know that something has not been written about. You can become more knowledgeable each time you hit a roadblock.
3. Interactive materials. The book, the slides, Q&A, video, audio and other things that we shall deploy to teach this course are all available online. You can post questions and we will respond as appropriate. Please use this resource thoughtfully. You can reach your lecturer at her address she will give you. You can also ask me questions www.oafak.com specifying the issue that is problematic: page number or question number. If a matter is a general problem, we will do an addendum to explain it better.
4. Mathematica. You can get a lot of help on Mathematica from the installation itself. There is an enormous amount of teaching, examples and documentation once you have it installed. Again, you can ask us questions if you have difficulty using it. You can also join the Mathematica Stack Exchange Group on the Internet to post questions and read answers to other questions.

Feedback on the way you use the materials are also welcome as they help us improve. That can help the next set as we give them even better materials.

Your Responsibility

The first responsibility you have is to receive and act on instructions for this course. We do not like to say things more than once. They will be written, and you can go and be looking as many times as you like. Dates and times for assignments are given. Materials to cover before coming to class are specified. Please, make things easy for everybody, read and act on instructions.

The second is actually a superset of the first. **Be ambitious**! There are many people that have had good grades in university and still ended up in dependency upon their families and friends. Your case can be different, not just because you prayed for a miracle, but because you simply work hard! **Aim at understanding; passing and getting good grades will accompany that in this course**. Everything you are taught is geared towards design and product making. Be ambitious to be the best engineer you can be and make a difference.

Coming to class unprepared, coming late and disturbing the class with *hypocritical* greetings, are even worse than not coming at all! Come **to class**, come **on time**, come **prepared**! Come!!

… and, finally …

Welcome to the course on continuum mechanics. We are ready and have been waiting for you! We enjoy the course and we hope to infect you with our enthusiasm for it. We hope that, together, we shall have a good ride.

OA **Fakinlede**
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Lagos, July 8, 2019

About the Author

Omotayo Abayomi **Fakinlede** received BSc Eng at the University of Lagos in 1977 and PhD from the University of Alberta in 1985. Both degrees are in Mechanical Engineering. He taught at the Mechanical Engineering Department of University of Ilorin 1980-1998, was director of Energy Information Systems at the Energy Commission of Nigeria until his present stint at the University of Lagos where he has been head, Systems Engineering and more recently, dean of the Faculty of Engineering. He is currently Professor of Systems Engineering with interests in Mechanics, Design and Analysis.

ONE

Vectors: Elementary Principles & Computations Practicum

"*God made the integers, all else is the work of man.*" – Leopold Kronecker

MetaData

The prose, video, slides and the Q&A in this chapter are directed at scoring the following points:

1. A set of **linearly independent** vectors is a set where one member cannot be expressed as a linear combination of the others.
2. When you have the maximum number of such vectors in a set, all other vectors in that space can be expressed as linear combinations of the members of this set. The set of orthonormal vectors, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$, we are used to in the Cartesian form is only one kind of such a set. Occasions will arise that will make other linearly independent vectors useful to know.
3. When a set is complete – having the maximum number of linearly independent vectors, it is said to form **a basis** of the vector space that it **spans**. These words are codes to express the fact that they can be used to

represent any other vector in the space. All that will be needed is the set of scalar weights (or scaling factors) of the basis vectors will represent each vector.

4. These scalars are called **components** of the specific vectors represented. Once they are found, with the basis in mind, we use them instead of the vectors they represent because analyses are easier done with the components.
5. #4 above can lead to confusing the vector with its matrix representation. The components of a vector are meaningless unless we specify the basis vectors underlying the representation. This is where the vector, and as we shall see later, the tensor objects, significantly differ from the matrices they look like.
6. The number of vectors constituting a basis spanning the space is the **dimension** of that space.
7. We gain valuable compactness using the index notation and the **Summation Convention**. Mastering it early is a great advantage for later work.
8. Other topics treated include **Coordinate transformations**, **Dyads and Rotations**. **General Curvilinear coordinates** are introduced as an advanced topic that can be omitted at first reading.
9. The chapter ends with a brief introduction to Software (Mathematica) the we use to avoid tedium and helps to tackle more challenging problems than could be easily done manually or with a calculator.
10. Mathematica is one of two important software for the series of lectures and courses. Students that start early with the software will gain a lot of ground, will find that the subject helps to learn the software and the software makes learning the subject easier. There will be a lot of examples you CANNOT easily do manually. Those that postpone learning the software are already failing! The best time to learn is at the beginning! You gain a lot and should never be behind!

Notation

In this chapter, we shall be dealing with vectors and scalars. We adopt the following notation from elementary set theory:

Table 1. Notation

Notation	Meaning
$\alpha, \beta \in \mathbb{R}$	α and β belong to the space of real numbers. Or simply, α and β are real numbers.
$\mathbf{v}, \mathbf{w} \in \mathbb{E}$	$\mathbf{v}$ and $\mathbf{w}$ belong to (are members of) the Euclidean vector space $\mathbb{E}$ This is a set of vectors that allow the definition of the dot product. In three dimensions, it also allows the definition of the cross product.

δ_{ij}, δ^i_j	Kronecker Delta, Mixed Kronecker Delta; Coefficients of the Identity tensor
e_{ijk} $\epsilon^{ijk}, \epsilon_{ijk}$	Alternating, Levi-Civita Symbol. Also the coefficients of the Alternating tensor covariant and contravariant alternating tensor components
$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \dots$	ONB (Ortho-Normal Basis Coordinate System) Base Vectors
$\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3,$ $\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3$	Covariant Base vectors Contravariant Base Vectors
g_{ij}, g^{ij}	Covariant and contravariant metric tensors, Non-Cartesian identity tensor components
$\mathbb{R}$	Real space; Set of real numbers
$\mathbb{V}$ $\mathbb{V} \times \mathbb{V}$ $\mathbb{V} \times \mathbb{V} \times \mathbb{V}$	Real Vector Space Product Vector Space. Pick two vectors or one from each space, that is an element of the product space. For example, If $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, $\mathbf{u} \times \mathbf{v} = \mathbf{w}$ could be written as the transformation from the product space, $\mathbb{V} \times \mathbb{V} \to \mathbb{V}$ while $\mathbf{u} \cdot \mathbf{v} = \mathbf{w}$ is $\mathbb{V} \times \mathbb{V} \to \mathbb{R}$ from the same product space to the real space. An example of $\mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ is the scalar triple product. While $\mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ is typified by the vector triple product operation.
$\mathbb{E}$	Euclidean Vector Space. A real vector space in which the inner product is defined. In 3-D, a vector product can be defined.
$\in$	Belongs to, member of.
$\mathcal{E}$	Euclidean Point Space. Where we live. Where objects we are interested in physically reside. It is related to a vector space. It is NOT a vector space. Its elements are points.

$\forall$	Shorthand representing "for all … "
$\otimes$	Binary operator for Dyad or Tensor Product
$\exists$	There exists

Introduction

A vector, roughly speaking, is an abstract representation of quantities that have magnitude, direction and sense. In this chapter, we start by repeating, in a summary form, several of the established notions of what a vector is, and what operations are valid for them with one another and with scalars. These are essentially repetitions of the way you have been taught Vector Theory thus far. It will be seen later that there are other ways of learning the subject.

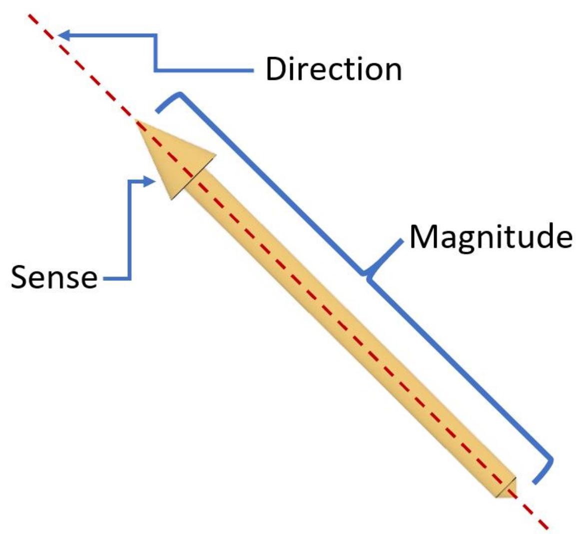


Figure 1. Vector magnitude, direction and sense

From our knowledge of Mechanics and similar subjects, vectors remind us of forces, velocities, moments, angular velocities, displacements and several quantities that have, in common, the fact that "magnitude" or "size" is not sufficient to quantify them; we must add direction and sense, for full characterization. A plane area, for example, can be thought of as a vector quantity if we add the outwardly drawn unit normal to its full description. In that case, the line of the normal is the direction, and the fact that it is "outward" is the sense. The inward normal is the opposite vector in the same direction.

This idea is widely applicable and props up in virtually everything we do. A more abstract – hence more widely applicable definition will be given later. It is very well and good to be clear on the meaning of the elementary notions at the outset. More accurate definitions will still include this as a special case as we shall see.

In figure 1, the length of the line gives us the magnitude of the vector; the direction of the line gives us the direction of the vector while the arrowhead indicates the sense of the vector. Furthermore, we assume that two vectors are equal if they have the same magnitude and are directed the same way.

Defined in this way, a vector may represent a force, an acceleration, a moment of an angular velocity. While these quantities are divers and represent vastly different things, in so far as each requires a magnitude, as well as a direction and a sense for full representation, the concept of a vector can be used to represent each; and we gain valuable analytical ability for doing so.

Vectors: Basic Properties

Equality of Vectors.

Two vectors are equal if their magnitudes, represented here by the lengths of the arrows, are equal, and they are pointing in the same direction. Accordingly, in figure 2,

$$\mathbf{a} = \mathbf{b} \neq \mathbf{c}$$

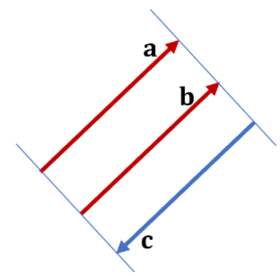


Figure 2. Equal, opposite vectors

While the three vectors are parallel, and of equal magnitude as they are contained within the same parallel lines, and they are not all pointing in the same direction. Two vectors that are equal and parallel but pointing in different directions are negatives of each other:

$$\mathbf{a} = \mathbf{b} = -\mathbf{c} \tag{1}$$

It also follows that, $-\mathbf{a} = -\mathbf{b} = \mathbf{c}$.

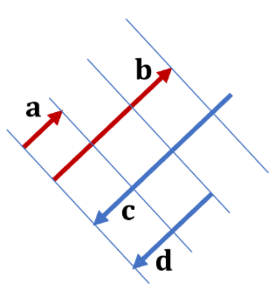


Figure 3. Vector Scaling

Vector Scaling.

We assume that the spacing lines in figure 3 are separated from each other by one unit as shown. Vector $\mathbf{a}$ has a magnitude of one unit. If the lengths of $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are 3, 3.5 and 2 respectively, then we have that, $\mathbf{b} = 3\mathbf{a}$; $\mathbf{c} = -3.5\mathbf{a}$; and $\mathbf{d} = -2\mathbf{a}$ – the signs being dictated by the sense in which they are pointing. These vectors are all scaled versions of $\mathbf{a}$. The **scaling factors** are real numbers and are therefore called “**scalars**” for this reason. One property of a vector we often take for granted is that it is something that can be **scaled**. These relationships are the operation of “multiplication by a scalar” or scaling of vectors. From this relationship, it is clear, for example, that,

$$\mathbf{b} = 3\mathbf{a} = -\frac{3}{3.5}\mathbf{c} \qquad (2)$$

in which case, $\mathbf{c}$ is a scaled version of $\mathbf{b}$, the scalar in this case being $-\frac{3}{3.5}$. This means they are not only scaled versions of $\mathbf{a}$ but also scaled versions of one another as the last example shows. The negative of a vector is simply the scaling of the same vector by a scalar value of -1. Scaling with a value of unity retains the original vector.

Vector Addition, Subtraction.

Vectors can be added or subtracted from each other. The parallelogram law of addition governs this operation:

In the two figures below, we have vectors $\mathbf{a}$ and $\mathbf{b}$. To effect the parallelogram law of addition, we place the tail of $\mathbf{a}$ at the tip of $\mathbf{b}$ or vice versa. In either case the resultant shown is the addition of the two vectors by the parallelogram rule.

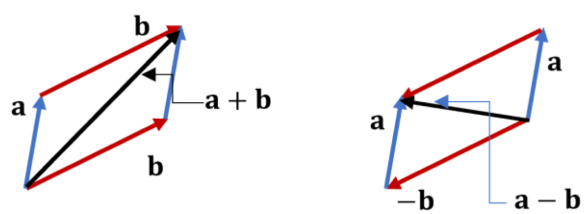


Figure 4. Parallelogram Rule

In Figure 4b, the same law is applied; this time to vectors $\mathbf{a}$ and $-\mathbf{b}$. Hence, subtracting vectors is simply effected by the addition of its negative to the other as shown.

Scaled Projections or Scalar Product.

As we have seen previously, we can scale a vector by simply multiplying it by a scalar. Another important operation for vectors is the scaled projection also called the “Scalar product”. Before we define this, observe that there is a fundamental difference between a scalar product on the one hand, and multiplication by a scalar on the other. Disambiguating these is very important and simple: Multiplication by a scalar takes place between a real number, (called a scalar or a scaling factor) and a vector; the result of the operation is a new vector, in the same direction as before with a sense dictated by the sign of the scalar multiplier. The scaling may increase or decrease magnitude, depending on the value of the scaling factor. In the multiplication by the scalar $\alpha \in \mathbb{R}$,

$$\mathbf{b} = \alpha\mathbf{a} \tag{3}$$

$\alpha = 1$ leaves the length unchanged, $0 < |\alpha| < 1$ creates a decrease in length, while $|\alpha| > 1$ increases the length.

On the other hand, scalar product is an operation that takes place between two vector operands: If $\mathbf{a}$ and $\mathbf{b}$ are both vectors, then the scalar quantity,

$$l = \mathbf{a} \cdot \mathbf{b} \tag{4}$$

Is the scalar product of $\mathbf{a}$ and $\mathbf{b}$. This product is also called the dot product on account of the operator "·" used to represent it. In order to define this product and give it a geometric meaning, consider vectors $\mathbf{a}$ and $\mathbf{b}$ in figure 5. Here, we project a line from the tip of vector $\mathbf{a}$ perpendicular to vector $\mathbf{b}$ as shown. There is another line from the tip of vector $\mathbf{b}$ to $\mathbf{a}$ (we needed to elongate $\mathbf{a}$ to make this possible). We examine the product of the projection of $\mathbf{a}$ on $\mathbf{b}$ and the magnitude of vector $\mathbf{b}$:

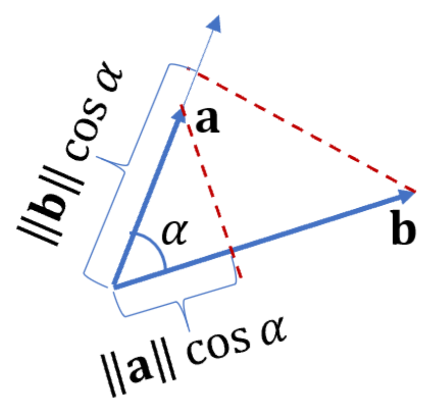


Figure 5. Components and Scalar product

$$||\mathbf{a}|| \cos\alpha \times ||\mathbf{b}||.$$

Comparing this with the product of the projection of $\mathbf{b}$ on $\mathbf{a}$ and the magnitude of vector**;** we find they are equal**:**

$$||\mathbf{a}|| \cos\alpha \times ||\mathbf{b}|| = ||\mathbf{b}|| \cos\alpha \times ||\mathbf{a}|| \tag{5}$$

The result on both sides of the equation is a scalar quantity. It is a product consisting of the two vector magnitudes and the angle between them, the largest value occurs when the angle is $zero$.

There is a shorthand for expressing this idea: It is called a **scalar product**. The scalar equality is defined as the scalar product of the two vectors, that is,

$$\mathbf{a} \cdot \mathbf{b} \equiv \|\mathbf{a}\|\|\mathbf{b}\| \cos \alpha \tag{6}$$

If $\|\mathbf{b}\| = 1$, that is, $\mathbf{b}$ is a vector of unit magnitude, also called a **Unit Vector**, then this quantity is the projection of vector $\mathbf{a}$ on the direction of $\mathbf{b}$. The converse is also true when $\mathbf{a}$ is a unit vector. This product is called a **Scalar Product** with the emphasis on the **scalar result** of the operation. A dot between the two vectors is the symbolic expression of a scalar product of two vectors. As a result of this, scalar products have the nickname, "**Dot Product**" reminding us of the fact that we let everybody know that the product we want is the one that produces scalar result and we use the dot to signify that intention.

Cross Product: Vector Area of a Parallelogram

Consider the rectangle, figure 6, whose base is vector $\mathbf{u}$ with height h as shown. Its area is obviously

$$A_r = base \times height = \|\mathbf{u}\|h.$$

Triangle $\mathbf{II}$ completes the parallelogram so that its slanting side is parallel to vector $\mathbf{v}$. Congruency of $\mathbf{I}$ and $\mathbf{II}$ is assured as they are both right angled triangles. Removing $\mathbf{I}$ gives the parallelogram, keeping it and removing $\mathbf{II}$ gives the rectangle. The rectangle is therefore of the same area as the parallelogram. But $h = \|\mathbf{v}\| \sin \theta$. Area of the parallelogram is therefore,

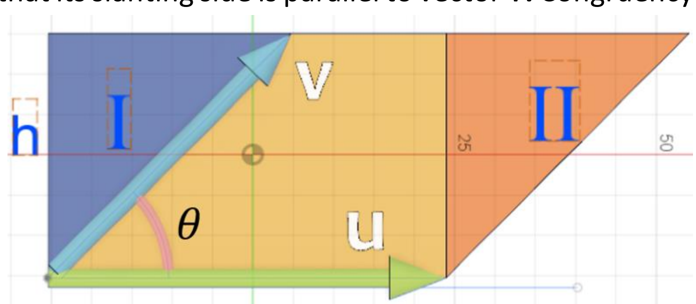


Figure 6. Area of a Parallelogram

$$A_p = A_r = \|\mathbf{u}\|\|\mathbf{v}\| \sin \theta \tag{7}$$

For two vectors, $\mathbf{u}$ and $\mathbf{v}$, we imbue the above scalar area with a direction. We choose this direction to be the outwardly drawn normal to the plane containing $\mathbf{u}$ and $\mathbf{v}$ which is also the direction of movement of a right threaded screw rotated from vector $\mathbf{u}$ to $\mathbf{v}$. Let the unit vector along this normal be $\mathbf{e}$. We define the cross product of $\mathbf{u}$ and $\mathbf{v}$ as the vector area of the parallelogram formed by the two vectors such that,

$$\mathbf{u} \times \mathbf{v} \equiv \mathbf{A} = A_p \mathbf{e} = \|\mathbf{u}\| \|\mathbf{v}\| \sin\theta \, \mathbf{e} \tag{8}$$

We have just completed the definition of the vector product between two vectors! As before, there is a special symbol for showing that you are carrying out a vector product between two vectors; it is the usual multiplication sign. We refer to it as the "cross symbol". It is therefore customary to give the vector product the name of the operator symbol used to signify it. You are free to call it **Vector Product** or **Cross Product**.

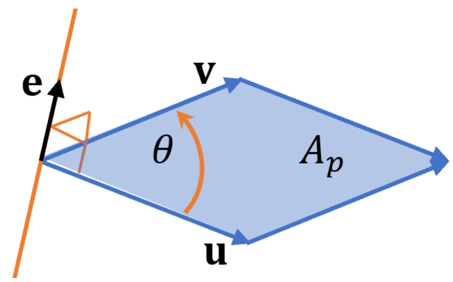


Figure 7. Direction of Unit Vector e

Note that we have defined three kinds of products. They are scaling, scalar product and vector product. They have other names. The set of names we have also introduced here are coined from the operator symbols to represent them. Given that α is a scalar, and that $\mathbf{a}$ and $\mathbf{b}$ are vectors, the following table depicts the operations we have defined thus far:

Table 2. Products with vectors

Product	Operation	Result	Other Names
Scalar Multiplication	$\mathbf{b} = \alpha\mathbf{a}$	Vector $\mathbf{b}$ in the same direction as $\mathbf{a}$. Scaled to the value of α. Sense depends on the sign of α	Scaling
Scalar Product	$\mathbf{a} \cdot \mathbf{b} \equiv \|\mathbf{a}\| \|\mathbf{b}\| \cos\alpha$	Result is a scalar value. Here α is the angle between the two vectors.	Dot Product, Inner Product
Vector Product	$\mathbf{u} \times \mathbf{v} \equiv \mathbf{A} = A_p\mathbf{e}$ $= \|\mathbf{u}\| \|\mathbf{v}\| \sin\alpha \, \mathbf{e}$	Result is a vector value. It's magnitude is the scalar area of the parallelogram formed by the vectors. Here α is the angle between the two vectors.	Cross Product

Observation.

The above table has one important implication: "product" or "multiplication" of vectors has at least three meanings. (A fourth meaning will be introduced later in this chapter). To simply say, take a product, when referring to vectors, is therefore ambiguous. It is important that a specific product be specified unless the context explicitly makes the product in question transparent. It is often necessary to therefore qualify which of the four products we have in mind explicitly. This is the reason why they have different operator symbols in the first place. Note that while other products take place between two vectors; the first, multiplication by a scalar, takes place between a vector and a scalar or vice versa.

Furthermore, while the other products listed are commutative, the vector product (and tensor product also as we shall see) is not commutative. The vector direction of the product reverses when the operands are swapped. More of this later.

Linear Independence, Basis Vectors

In the first instance, we further assume that this vector is contained in a plane. Suppose we introduce two new vectors $\mathbf{a}$ and $\mathbf{b}$. The only thing we require is that these two should not be collinear; their directions are different. We do not, for example, require these two new vectors to have unit magnitude; neither do we require them to be orthogonal, but they are not collinear.

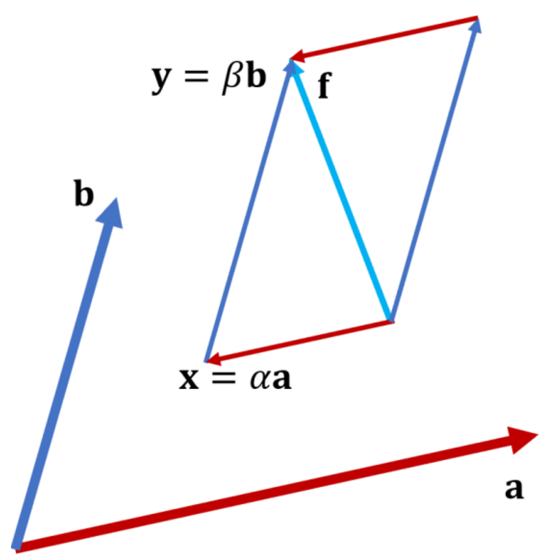


Figure 8. Representing a vector by Linearly Independent vectors

We will argue that these two vectors can be used to express any other vector on the plane in the sense that we only need two scaled versions of them to add up to any other vector. If we succeed in showing that, we then say that the two vectors span the space given by the plane. This idea of spanning comes from the fact that we can always select two scaling factors for the two vectors. With these, we can represent any vector as the weighted sum of the two vectors using the two scaling factors (or scalars)

At the tip of the vector $\mathbf{f}$, we draw a line parallel to $\mathbf{b}$. At the tail of the same vector, we draw another line parallel to $\mathbf{a}$. It is easy to see that the vectors $\mathbf{x}$ and $\mathbf{y}$, chosen along these lines are parallel to $\mathbf{a}$ and $\mathbf{b}$ respectively. Consequently, we may write that $\mathbf{x} = \alpha\mathbf{a}$; and $\mathbf{y} = \beta\mathbf{b}$ where α and β are the scaling factors (real numbers that can be positive or negative). From the forgoing, we see clearly that any vector $\mathbf{f}$ on this plane can be expressed as

$$\begin{aligned} \mathbf{f} &= \mathbf{x} + \mathbf{y} \\ &= \alpha\mathbf{a} + \beta\mathbf{b} \end{aligned} \tag{9}$$

$\alpha.\beta \in \mathbb{R}$. Where the above shorthand simply means that the scaling factors belong to the class of real numbers.

The proviso that the two vectors MUST not be collinear is paramount. If they were collinear, it would not be possible to guarantee that every vector in this plane can be so represented. We hereby conclude by this geometrical arrangement that in a 2-D plane, the maximum number of vectors that can be used in this way is two because a third vector can be expressed in terms of the other non-collinear two.

Another way of expressing the fact that these two vectors can be used, with appropriate scalars, in a weighted addition, to represent any other vector, is to say that the set $\{\mathbf{a}, \mathbf{b}\}$ forms a basis for the plane in question.

Notice that it is ***a basis***. There could be other pairs that can equally form a basis for this plane. One such famous pair is the coordinate unit vectors $\{\mathbf{i}, \mathbf{j}\}$ that have unit magnitude and are directed (orthogonal to each other) along the x and y −axes in a Cartesian system of coordinates when the plane in question is the $x - y$ plane. There are several ways you can obtain the vectors to form the basis in any plane. One thing they must have in common is that it MUST not be possible to express one as a scaled version of another. When that condition is satisfied, we say that the vectors are *Linearly Independent*. A set containing the maximum number of linearly independent vectors is what you need to form a basis in any situation.

A further observation about the basis vectors. It is possible to complete the parallelogram with the other two sides parallel to the basis vectors. One geometric way to check if the vectors truly form a basis (equivalently, are linearly independent), is that the parallelogram formed must have a non-zero area. Given that θ is the angle between the two vectors, the area of this parallelogram is given by the base times the perpendicular height,

$$A = \|\mathbf{a}\|\|\mathbf{b}\| \sin\theta = \|\mathbf{a} \times \mathbf{b}\| \tag{10}$$

Hence, we can say that any two vectors such that $\mathbf{a} \times \mathbf{b} \neq \mathbf{o}$ can be used as basis in a 2-D plane.

Linear Independence, 3-Dimensional Space

In three-dimensional space, we must require, in addition to the fact that our three vectors be non collinear, they must not all be contained in the same plane. If this condition is not satisfied, they will not be able to represent the vectors that are not contained in the plane. We hereby introduce the set of vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ that are not collinear and not coplanar as shown in figure 1.3 below.

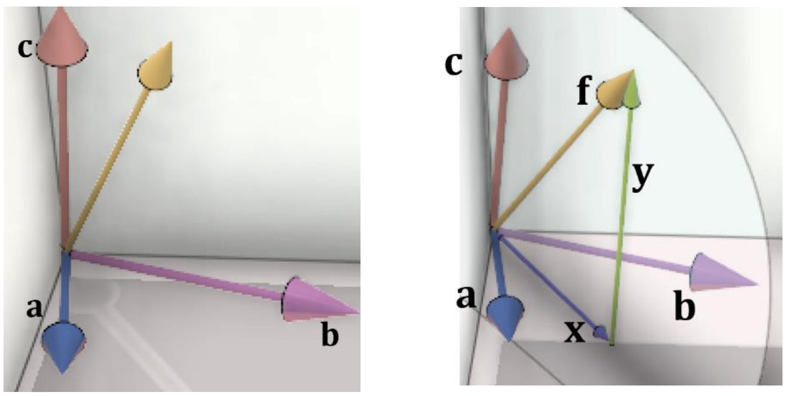


Figure 9. Linear Independence, 3D

The first two basis vectors $\{\mathbf{a}, \mathbf{b}\}$ are drawn on the $x - y$ plane. The third vector, $\mathbf{c}$ is shown in pink and drawn near the z −axis. A typical vector in this 3-D space can be constructed as shown in the directed line in yellow. In order to represent this vector in terms of the three basis vectors, construct the plane containing vectors , $\mathbf{c}$ and $\mathbf{f}$. Drop a line from the tip of $\mathbf{f}$ to the $x - y$ plane containing $\{\mathbf{a}, \mathbf{b}\}$parallel to vector $\mathbf{f}$. Call the vector image of $\mathbf{f}$ on the $x - y$ plane $\mathbf{x}$. The vector on this oblique plane, parallel to $\mathbf{c}$ is called $\mathbf{y}$. The fact that $\mathbf{x}$ on the same plane as $\mathbf{a}$ and $\mathbf{b}$ means we can, as we just did in the 2-D case represent it by the two basis vectors in that plane. Therefore, we can easily find $\alpha . \beta \in \mathbb{R}$ such that,

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} \tag{11}$$

We recall that $\mathbf{y}$ is parallel to $\mathbf{c}$, hence,$\exists \gamma \in \mathbb{R}$ such that, $\mathbf{y} = \gamma\mathbf{c}$. Consequently, any 3-D vector $\mathbf{f}$ can be expressed as,

$$\mathbf{f} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \tag{12}$$

The set, $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ forms a basis for the 3-D space. A tetrahedron formed by joining the tips of this set of basis vectors has a base that is half the size of the parallelogram base.

Dimensionality of Space

One Dimension

A **collection** or a bag full of vectors, which when each is uniquely identifiable is **all** we mean by a **set of vectors**. Somehow, some influential people feel we should call a set of vectors, a **vector space**. Consider the collection, or vector space in the picture below.

Figure 10. One Dimensional Vector Space

Imagine it goes on both sides such that we have many elements in the set. If we take one vector in the list, call it $\mathbf{a}$; any other vector $\mathbf{b}$ can be represented as a scalar multiple of $\mathbf{a}$. Alternatively. Given any other vector $\mathbf{b}$ in the vector space, the equation

$$\alpha\mathbf{a} + \beta\mathbf{b} = \mathbf{o} \tag{13}$$

can always be solved for $\mathbf{b}$ provided $\beta \neq 0$, this equation can be simplified to $\mathbf{b} = -\frac{\alpha}{\beta}\mathbf{a}$**.** In which case, once we have identified vector $\mathbf{a}$**,** all we need to represent any other vector in the space is the scalar $-\frac{\alpha}{\beta}$, which can take fractional, decimal, positive or negative values, as a multiple of $\mathbf{a}$. We express this fact by saying that "vector $\mathbf{a}$ spans this space". It forms a basis of this space from the fact that every other vector can be expressed by a scalar multiplier of $\mathbf{a}$; and the dimension of this space is one because only one vector is needed to span the space. It is just *a basis* because we could have chosen any other vector to perform this function. Therefore, there could be other bases. A different choice of basis leads to different set of choices for the scalar $-\frac{\alpha}{\beta}$ to define define each element in the new basis. The fact that any basis we correctly select contains only one vector what makes this a **one-dimensional vector space**.

Two-Dimensional Space

Consider another bag, collection or set of vectors (Figure 11) as shown below. Here, all the vectors are contained in a single flat plane. We showed earlier that any two non-collinear vectors, say $\mathbf{a}$ and $\mathbf{b}$ among these can be chosen in such a way that the other vectors in the vector space can be expressed in terms of scalar multiples of the two. We also showed further that once these are chosen, any other vector $\mathbf{x}$ can be expressed as a sum of scaled versions

$$\mathbf{x} = \alpha\mathbf{a} + \beta\mathbf{b} \tag{14}$$

of this two so that, $\mathbf{a}$ and $\mathbf{b}$ that have been so chosen have formed a basis of the vector space. Furthermore, given any vector c in the space, the equation,

$$\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = 0 \tag{15}$$

can be solved for $\mathbf{c}$ provided γ is not zero. This means that the maximum number of linearly independent vectors in this space is two. This makes the plane a **two-dimensional vector space**. In such a space, the maximum number of linearly independent vectors you can have is two. There is no uniqueness about the choice, as another two non collinear vectors may as well have been chosen. .

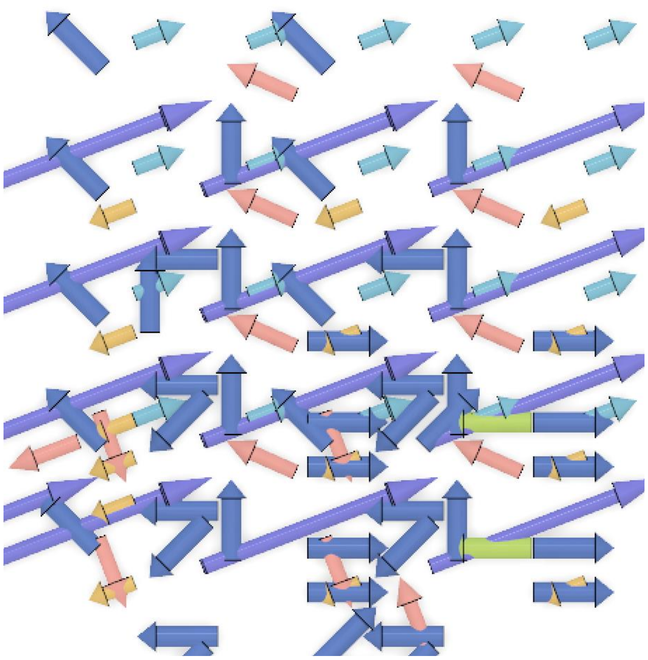


Figure 11. Two Dimensional Vectors

Three Dimensions

The arguments above can be carried to three dimensions. A geometric interpretation can be given. With a more accurate mathematical definition of vectors, we can even go to higher dimensions. Once we are past three dimensions, however, a geometric interpretation will no longer be possible, but the concept can remain useful for analytical purposes.

The maximum number of linearly independent vectors in a three-dimensional space is three. These will, in addition to not being collinear, they MUST NOT all be coplanar. That means that once you have four or more vectors, one will be expressible in terms of the other three.

Components in Different Bases

Up till this point, you may have taken for granted, the fact that you could express any vectors in terms of the basis vector set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ which are orthogonal unit vectors along the three coordinate axes in a Cartesian system of coordinates. Two properties of these vectors are that they are mutually orthogonal and that they have unit magnitudes are quite useful. They not only allow you to express any given vector in terms of these basis vectors, they also, by these attractive properties of normality (unit magnitude) and orthogonality (being at right angles bgto one another) make the computation of the coordinates along the basis vectors very simple.

Despite this, it is important to note that, we DO NOT have to require these properties in order to conclude that a set of vectors can form a basis. What we have proved here is that, in three dimensions, a set of linearly independent vectors (orthogonal or not, normalized or not) can form a basis set. Any other vector in the space, as we have shown above, can be expressed in terms of their components along these vectors. The method of computing their components along these axes may be more difficult; the fact remains they can be found.

It turns out that occasions will arise when we will no longer require our basis vectors to be orthonormal. However, the linear independence requirement will always be made because it is only linearly independent vectors that can form a basis for any space. Orthonormal sets form basis; not all basis vector sets are orthonormal; Orthonormal sets are linearly independent; not all linearly independent sets are orthonormal.

Volume of a Tetrahedron

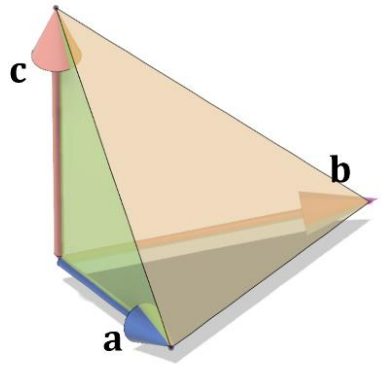


Figure 12. Volume of a Tetrahedron

The area of the triangular base of the tetrahedron (Figure 12) formed by three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ is half the parallelogram formed by the same vectors. Hence this base is $\frac{1}{2}\mathbf{a} \times \mathbf{b}$ with the vector area directed at the normal to this plane. If we take the dot product of this with vector $\mathbf{c}$**,** we have obtained the base times height. However, for a tetrahedron, or any volume obtained by a flat area lofted linearly to a single point is one third of this as we shall show (See Q&A 1.14). Consequently, a tetrahedron formed by the three vectors has the volume

$$V = \frac{1}{3}\left(\frac{1}{2}\mathbf{a} \times \mathbf{b}\right) \cdot \mathbf{c} = \frac{1}{6}|\mathbf{a} \times \mathbf{b} \cdot \mathbf{c}| = \frac{1}{6}|\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}| \tag{16}$$

As before, linear independence requires that the volume of this tetrahedron be nonzero. That means that no two of them can be colinear, and the three cannot be coplanar.

Volume of a Parallelepiped

A parallelepiped (Figure 13) with sides bound by vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ with $\mathbf{u}$ subtending an angle θ on the horizontal plane while $\mathbf{w}$ is inclined at angle α to the vertical axis. The base area

$$A = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta = \|\mathbf{u} \times \mathbf{v}\|$$

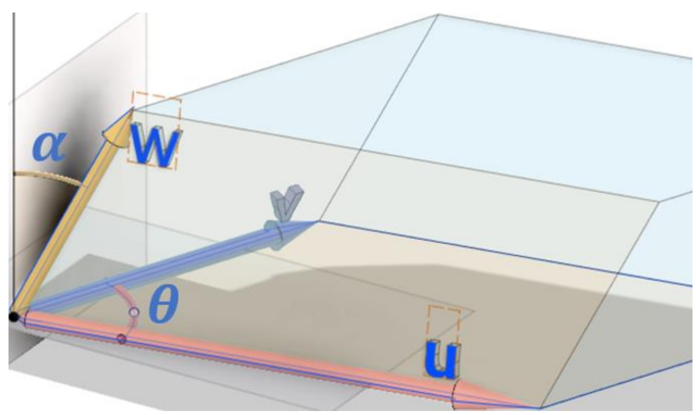


Figure 13. Volume of a Parallelepiped

Vertical height, h, of the object is $\|\mathbf{w}\|$ $\cos\alpha$. Volume therefore is

$$\begin{aligned} V &= Ah = \|\mathbf{u} \times \mathbf{v}\|\|\mathbf{w}\| \cos\alpha \\ &= |\mathbf{u} \times \mathbf{v} \cdot \mathbf{w}|. \end{aligned} \tag{17}$$

Orthonormal Basis (ONB) Vectors

It is often (not always) convenient to use the Cartesian System of coordinates. We can choose a convenient set of linearly independent vectors that are unit vectors and mutually orthogonal to one another. Instead of the calling this set $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ it is found more convenient to refer to them as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. In this case, $\mathbf{e}_1 \times \mathbf{e}_2 \cdot \mathbf{e}_3 = 1$. The base vectors of the coordinate system is now an indexed object. We can depict the as $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$. We could also have written, $\mathbf{e}_i, i = 1,2,3$ or $\mathbf{e}_i, i = 1, \dots ,3$.

If you are going to be severe and argue that this change in the method of representation does not amount to much; let us politely disagree: Imagine you have ten of them. In the earlier case you have to write, $\{\mathbf{i}, \mathbf{j}, \mathbf{k}, \mathbf{l}, \mathbf{m}, \mathbf{n}, \mathbf{o}, \mathbf{p}, \mathbf{q}, \mathbf{r}\ \}$. If there are 30, then you will run out of symbols and may need to look for another naming strategy. For indexed objects, the answer is very simple: $\mathbf{e}_i, i = 1, \dots ,10$, or $\mathbf{e}_i, i = 1, \dots ,30$ are equally easy! By the time we add the parsimony afforded by the summation convention, (next section) it will gradually become clear that there is no comparison in the ease of usage between indexed objects and regular symbol usage.

A typical vector $\mathbf{f}$ can be written in terms of the basis vectors as,

$$\mathbf{f} = \alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \quad (18)$$

The scalars a_1, a_2, a_3in this case are easily found by taking the dot product of the equation with $\mathbf{e}_1$,

$$\mathbf{f} \cdot \mathbf{e}_1 = a_1\mathbf{e}_1 \cdot \mathbf{e}_1 + a_2\mathbf{e}_2 \cdot \mathbf{e}_1 + a_3\mathbf{e}_3 \cdot \mathbf{e}_1 = a_1. \quad (19)$$

And we can similarly take products with $\mathbf{e}_2$ and $\mathbf{e}_3$ respectively and obtain that, $a_2 = \mathbf{f} \cdot \mathbf{e}_2$, and $a_3 = \mathbf{f} \cdot \mathbf{e}_3$.

The Einstein Summation Convention

We introduce an index notation to facilitate the expression of relationships in indexed objects. Whereas the components of a vector may be three different functions, indexing helps us to have a compact representation instead of using new symbols for each function, we simply index and achieve compactness in notation. As we later deal with higher ranked objects (for example, tensors), such notational conveniences become even more important. We shall often deal with coordinate transformations requiring such indexing.

When an index occurs twice on the same side of any equation, or term within an equation, it is understood to represent a summation on these repeated indices the summation being over the integer values specified by the range. A repeated index is called a summation index, while an unrepeated index is called a free index. The summation convention requires that one must never allow a summation index to appear more than twice in any given expression.

Consider the following set of transformation equations between variables sets, $\{x_1, x_2, x_3\}\ or\ x_j, j = 1, \dots, 3$ and $\{y_1, y_2, y_3\}\ or\ y_k, k = 1, \dots, 3$.

$$\begin{aligned} y_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \\ y_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \\ y_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \end{aligned} \tag{20}$$

We may write these equations using the summation symbols as:

$$\begin{aligned} y_1 &= \sum_{j=1}^{n} a_{1j}x_j \\ y_2 &= \sum_{j=1}^{n} a_{2j}x_j \\ y_3 &= \sum_{j=1}^{n} a_{3j}x_j \end{aligned} \tag{21}$$

In each of these, noting the repeated indices that can be made to signify summation, we can invoke the Einstein summation convention, and write that,

$$y_1 = a_{1j}x_j;\ y_2 = a_{2j}x_j;\ y_3 = a_{2j}x_j \tag{22}$$

Finally, we observe that y_1, y_2, and y_3 can be represented as we have been doing by $y_i,\ i = 1,2,3$ so that the three equations can be written more compactly as,

$$y_i = a_{ij}x_j, \quad i = 1,2,3 \tag{23}$$

Please note here that while j in each equation is a dummy index, i is not dummy as it occurs once on the left and in each expression on the right. We therefore cannot arbitrarily alter it on one side without matching that action on the other side. To do so will alter the equation. Again, if we are clear on the range of i, we may leave it out completely and write,

$$y_i = a_{ij}x_j \tag{24}$$

to represent, more compactly, the transformation equations above. It should be obvious there are as many equations as there are free indices.

If a_{ij} represents the components of a 3×3 matrix $\mathbf{A}$, we can show that,

$$a_{ij}a_{jk} = b_{ik} \tag{25}$$

where $\mathbf{B} = [b_{ij}]$ is the product matrix $\mathbf{AA}$.

To show this, apply summation convention and see that,

Table 3. Summation convention

i	k	$a_{ij}a_{jk}$	b_{ik}
1	1	$a_{11}a_{11} + a_{12}a_{21} + a_{13}a_{31}$	b_{11}
1	2	$a_{11}a_{12} + a_{12}a_{22} + a_{13}a_{32}$	b_{12}
1	3	$a_{11}a_{13} + a_{12}a_{23} + a_{13}a_{33}$	b_{13}
2	1	$a_{21}a_{11} + a_{22}a_{21} + a_{23}a_{31}$	b_{21}
2	2	$a_{21}a_{12} + a_{22}a_{22} + a_{23}a_{32}$	b_{22}
2	3	$a_{21}a_{13} + a_{22}a_{23} + a_{23}a_{33}$	b_{23}
3	1	$a_{31}a_{11} + a_{32}a_{21} + a_{33}a_{31}$	b_{31}
3	2	$a_{31}a_{12} + a_{32}a_{22} + a_{33}a_{32}$	b_{32}
3	3	$a_{31}a_{13} + a_{32}a_{23} + a_{33}a_{33}$	b_{33}

The above can easily be verified in matrix notation as,

$$\mathbf{AA} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} = \mathbf{B} \tag{26}$$

In this same way, we could have also proved that,

$$a_{ij}a_{kj} = b_{ik} \tag{27}$$

Where $\mathbf{B}$ is the product matrix $\mathbf{AA}^{\mathrm{T}}$. Note the arrangements could sometimes be counter intuitive.

Points to note:

1. An index must not be repeated more than once in any term. A repeated index is called a dummy index.
2. Dummy indices are mutable. Changing one pair to another pair, unused index, in the object does not change value. For example, $a_k a_{kj} = a_\alpha a_{\alpha j} = a_m a_{mj} = a_1 a_{1j} + a_2 a_{2j} + a_3 a_{3j}$
3. **IMPORTANT**: Because of #2, use a pair of new dummy variables to avoid situations that could have caused more repeats than allowed.

Also do not forget that the Einstein summation convention is a matter of *convenience*, allowing us to avoid writing too many summation symbols. The meaning of the expressions and equations are not affected by the correct use of this convention. A great deal of reduction in written terms can be achieved, nevertheless.

Orthonormal vector components again

In a previous section, we introduced the orthonormal basis vectors, $\mathbf{e}_i,\ i = 1,2,3$ With respect to this basis, we can express vectors $\mathbf{v}, \mathbf{w}$ in terms of the basis as, $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3 = v_i\mathbf{e}_i$, $\mathbf{w} = w_i\mathbf{e}_i$. The summation sign is no longer needed because of the summation convention. Each v_i is called the component of $\mathbf{v}$, while w_i is called the component of $\mathbf{w}$

The Kronecker Delta. δ_{ij}

The Kronecker delta is a symbol with two indices. The value attained depends on the values of the indices. In our case, each can assume values ranging from 1 to 3. The value of the symbol itself depends, not so much on the indices directly, but on their equality or non-equality. When the indices are equal, the Kronecker Delta takes the value of one; otherwise, its value is zero. Here are all possibilities:

$$\begin{aligned} &\delta_{11} = 1,\ \delta_{12} = 0,\ \delta_{13} = 0 \\ &\delta_{21} = 0,\ \delta_{22} = 1,\ \delta_{23} = 0 \\ &\delta_{31} = 0,\ \delta_{32} = 0,\ \delta_{33} = 1 \end{aligned} \tag{28}$$

These nine equations can be summarized in the simple form:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

The Kronecker Delta, for reasons that will later become obvious, is called the **substitution symbol.** We will later also see that they are the components of the **Identity Tensor** when referred to Cartesian coordinates.

Consider the scalar product of two Cartesian base vectors, $\mathbf{e}_i$and $\mathbf{e}_j$.

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases} \tag{30}$$

This is precisely the same as the definition of the Kronecker Delta! It is therefore clear that

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} \tag{31}$$

For any $\mathbf{v} \in \mathbb{V}$,

$$\mathbf{v} = v_i \mathbf{e}_i \tag{32}$$

is the vector expressed in component form using the summation convention. Taking the inner product of the above equation with the basis vector $\mathbf{e}_j$, we have

$$\begin{aligned} \mathbf{v} \cdot \mathbf{e}_j &= v_i \mathbf{e}_i \cdot \mathbf{e}_j = v_i \delta_{ij} \\ &= v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} \end{aligned} \tag{33}$$

We now examine the value of both sides for different values of j:

$j = 1, \mathbf{v} \cdot \mathbf{e}_1 = v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} = v_1 \delta_{11} + v_2 \delta_{21} + v_3 \delta_{31} = v_1$;

$j = 2, \mathbf{v} \cdot \mathbf{e}_2 = v_1 \delta_{1j} + v_2 \delta_{2j} + v_3 \delta_{3j} = v_1 \delta_{12} + v_2 \delta_{22} + v_3 \delta_{32} = v_2$ and

$j = 3, \mathbf{v} \cdot \mathbf{e}_3 = v_1 \delta_{13} + v_2 \delta_{23} + v_3 \delta_{33} = v_3$

In all cases, therefore,

$$\mathbf{v} \cdot \mathbf{e}_j = v_j \tag{34}$$

which contains the expressions for v_1, v_2, and v_3 as we allow $j = 1,2,3$ in the above equation.

Substitution Symbol

The epithet of "substitution symbol, as applied to the Kronecker Delta is the result of the above result: $v_i \delta_{ij} = v_j$! It is a general rule: When you have the product of the Kronecker Delta and another object with which it shares an index, the result of that product is to remove the Kronecker Delta and allow a substitution of the symbol that was not shared as in this expression. Look at the following examples:

Product with Kronecker Delta	Result
$S_{\alpha\beta}\delta_{i\alpha}$	$S_{i\beta}$
$T_{ijk}\delta_{j\alpha}$	$T_{i\alpha k}$
$\delta_{ij}\delta_{\alpha j}$	$\delta_{i\alpha}$
$\delta_{ij}\delta_{ij}$	$\delta_{ii} = \delta_{jj} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

$e_{ijk}\delta_{jk}$	$e_{ijj} = e_{ikk}$

The Alternating Levi-Civita Symbol.

Consider the following determinant of Kronecker Deltas,

$e_{ijk} \equiv \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$, where the indices i, j and k, varying *column* to *column*, can take the values 1,2 or 3. Clearly, the values $i = 1, j = 2$ and $k = 3$ gives the determinant,

$$e_{ijk} = e_{123} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \quad (35)$$

once we apply the definition of the Kronecker Deltas, it is clear that this is the determinant of the **Identity Tensor**. A simple check reveals the fact that

$$\begin{aligned} e_{123} = e_{231} = e_{312} = 1 \\ e_{132} = e_{321} = e_{213} = -1 \end{aligned} \quad (36)$$

and the value of this quantity is zero in every other case as can be checked by a simple determinant expansion. Those cases include situations when one or more of the indices is equal to another.

We can arrive at the same relationship if, going *row-wise*, we define

$$e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix} \quad (37)$$

Again, just as the previous case,

$$\begin{aligned} e_{123} = e_{231} = e_{312} = 1 \\ e_{132} = e_{321} = e_{213} = -1 \end{aligned} \quad (38)$$

with all the other cases returning zero. In either of these cases, the symbol, e_{ijk} or e_{rst} as we have defined it, is called the **Levi-Civita** or **Alternating Symbol**. An even permutation of its symbols retains sign while any odd permutation negates the sign. This behavior can be predicted from the knowledge of determinants. A row or column swap negates sign while two row or columns swaps becomes a double negation of sign and gives positive. Consequently, even permutations result in sign preservation while odd permutations negative. It is said to be perfectly anti-symmetric.

Continuing with the determinant interpretation, equality of the indices denotes a determinant with repeated rows or columns. Clearly, we have zero value for such a determinant.

Products of Alternating tensors

Consider the product, $e_{rst}e_{ijk}$ of the alternating symbols – the determinants we just defined. We will proceed to show that,

$$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix} \tag{39}$$

The definition of e_{ijk} and of δ_{ij} immediately shows that,

$e_{ijk} = \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$, and $e_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}$

The product,

$$\begin{aligned} e_{rst}e_{ijk} &= \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix} \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix} \\ &= \begin{vmatrix} \delta_{r1}\delta_{1i} + \delta_{r2}\delta_{2i} + \delta_{r3}\delta_{3i} & \delta_{r\alpha}\delta_{\alpha j} & \delta_{r\alpha}\delta_{\alpha k} \\ \delta_{s\alpha}\delta_{\alpha i} & \delta_{s\alpha}\delta_{\alpha j} & \delta_{s\alpha}\delta_{\alpha k} \\ \delta_{t\alpha}\delta_{\alpha i} & \delta_{t\alpha}\delta_{\alpha j} & \delta_{t\alpha}\delta_{\alpha k} \end{vmatrix} \\ &= \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix} \end{aligned}$$

(We showed the first working only; as an exercise, work the others out). We now consider a situation when one of the indices of the alternating symbols in a product are the same. To do this, we begin from the above result:

Given that

$e_{rst}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}$ we now show, by setting $t \to k$ in this expression, that

$$e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{rj}\delta_{si} \tag{40}$$

Clearly, not forgetting that repetition of an unknown index signifies a summation,

$$e_{rsk}e_{ijk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{kk} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & \delta_{11} + \delta_{22} + \delta_{33} \end{vmatrix} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$$

Expanding the equation, using the third row, we have:

$$\begin{aligned} e_{rsk}e_{ijk} &= \delta_{ki} \begin{vmatrix} \delta_{rj} & \delta_{rk} \\ \delta_{sj} & \delta_{sk} \end{vmatrix} - \delta_{kj} \begin{vmatrix} \delta_{ri} & \delta_{rk} \\ \delta_{si} & \delta_{sk} \end{vmatrix} + 3 \begin{vmatrix} \delta_{ri} & \delta_{rj} \\ \delta_{si} & \delta_{sj} \end{vmatrix} \\ &= \delta_{ki}\left(\delta_{rj}\delta_{sk} - \delta_{sj}\delta_{rk}\right) - \delta_{kj}(\delta_{ri}\delta_{sk} - \delta_{si}\delta_{rk}) \\ &\qquad + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) \\ &= \delta_{rj}\delta_{si} - \delta_{sj}\delta_{ri} - \delta_{ri}\delta_{sj} + \delta_{si}\delta_{rj} + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) \\ &= -2\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) + 3\left(\delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}\right) \\ &= \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj} \end{aligned}$$

It is instructive to observe the two terms in the last expression. Notice that there is a change in partners in the pairs. This observation, if we remember, means that once we can form one term, the other is a simply an index pairing exchange.

We now proceed to look at the example where two of the indices of the alternating symbols in the product are the same. Beginning from our most recent result, equation 15, that

$$e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj} \tag{411}$$

We proceed to show that $e_{rjk}e_{ijk} = 2\delta_{ri}$.

In the equation, $e_{rsk}e_{ijk} = \delta_{ri}\delta_{sj} - \delta_{si}\delta_{rj}$ set $s \to j$, we have,

$$\begin{aligned} e_{rjk}e_{ijk} &= \delta_{ri}\delta_{jj} - \delta_{ji}\delta_{rj} \\ &= 3\delta_{ri} - \delta_{ri} \\ &= 2\delta_{ri}. \end{aligned} \tag{42}$$

Component Form of Products of Vectors

Invoking the Einstein summation convention and using the Cartesian system of coordinates, we can write the component form of vectors $\mathbf{a} = a_i\mathbf{e}_i$, $\mathbf{b} = b_j\mathbf{e}_j$. We can go ahead to write the scalar and vector products in their component forms:

Scalar, Dot Product

To find the component form of the scalar product, let us remember the meaning of the scalar product as it applies to unit basis vectors. $\mathbf{e}_i \cdot \mathbf{e}_j$ is a projection of a vector to a direction perpendicular to it whenever $i \neq j$. This projection has the value of zero; it vanishes. When $i = j$, we are projecting a vector unto itself. This gives the value of unity since it is a unit vector that has been project. Clearly therefore, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$. Consequently,

$$\begin{aligned} \mathbf{a} \cdot \mathbf{b} &= (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} \\ &= a_i b_i \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned} \tag{43}$$

Which is the meaning of the compact form, $a_i b_i$. (**Note**: It is correct that $b_i \mathbf{e}_i = b_j \mathbf{e}_j$. Any dummy index would be ok. However, using the first would have led to $a_i b_i \mathbf{e}_i \cdot \mathbf{e}_i$ which would not only violate the summation convention ruled that no index be repeated more than once in any term. It would also have led to wrong results).

Vector, Cross Product.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (a_i \mathbf{e}_i) \times (b_j \mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \times \mathbf{e}_j \\ &= e_{ijk} a_i b_j \mathbf{e}_k \end{aligned} \tag{44}$$

The last step requires us to show that the cross product of the base vectors, $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk} \mathbf{e}_k$. This important result comes from a compendium of repeated application of the definition of the cross product as shown in the table below:

i	j	$e_i \times e_j$	$e_{ijk}\mathbf{e}_k$
1	3	$1 \times 1 \sin 90\, (-\boldsymbol{e_2})$	$e_{13k}\mathbf{e}_k = e_{131}\mathbf{e}_1 + e_{132}\mathbf{e}_2 + e_{133}\mathbf{e}_3 = -\mathbf{e}_2$
1	2	$\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$	$e_{12k}\mathbf{e}_k = e_{121}\mathbf{e}_1 + e_{122}\mathbf{e}_2 + e_{123}\mathbf{e}_3 = \mathbf{e}_3$
2	3	$\boldsymbol{e}_2 \times \boldsymbol{e}_3 = \boldsymbol{e}_1$	$e_{23k}\mathbf{e}_k = e_{231}\mathbf{e}_1 + e_{232}\mathbf{e}_2 + e_{233}\mathbf{e}_3 = \mathbf{e}_1$
3	1	$\boldsymbol{e}_3 \times \boldsymbol{e}_1 = \boldsymbol{e}_2$	$e_{31k}\mathbf{e}_k = e_{311}\mathbf{e}_1 + e_{312}\mathbf{e}_2 + e_{313}\mathbf{e}_3 = \mathbf{e}_2$

1	1	$\boldsymbol{e}_1 \times \boldsymbol{e}_1 = 0$	$e_{11k}\mathbf{e}_k = e_{111}\mathbf{e}_1 + e_{112}\mathbf{e}_2 + e_{113}\mathbf{e}_3 = 0$
2	2	$\boldsymbol{e}_2 \times \boldsymbol{e}_2 = 0$	$e_{22k}\mathbf{e}_k = e_{221}\mathbf{e}_1 + e_{222}\mathbf{e}_2 + e_{223}\mathbf{e}_3 = 0$
2	1	$\boldsymbol{e}_2 \times \boldsymbol{e}_1 = -\boldsymbol{e}_3$	$e_{21k}\mathbf{e}_k = e_{211}\mathbf{e}_1 + e_{212}\mathbf{e}_2 + e_{213}\mathbf{e}_3 = -\mathbf{e}_3$
3	2	$\boldsymbol{e}_3 \times \boldsymbol{e}_2 = -\boldsymbol{e}_1$	$e_{32k}\mathbf{e}_k = e_{321}\mathbf{e}_1 + e_{322}\mathbf{e}_2 + e_{323}\mathbf{e}_3 = -\mathbf{e}_1$
3	3	$\boldsymbol{e}_3 \times \boldsymbol{e}_3 = 0$	$e_{33k}\mathbf{e}_k = e_{331}\mathbf{e}_1 + e_{332}\mathbf{e}_2 + e_{333}\mathbf{e}_3 = 0$

Note that we only need to specify the i and j values as there is indexing into all the values of k because it is a dummy index in the above expression.

Expansion of the vector product is straightforward:

$$
\begin{aligned}
\mathbf{a} \times \mathbf{b} &= e_{ijk} a_i b_j \mathbf{e}_k \\
&= e_{123} a_1 b_2 \mathbf{e}_3 + e_{132} a_1 b_3 \mathbf{e}_2 + e_{231} a_2 b_3 \mathbf{e}_1 + e_{213} a_2 b_1 \mathbf{e}_3 + e_{312} a_3 b_1 \mathbf{e}_2 + e_{321} a_3 b_2 \mathbf{e}_1 \\
&= a_1 b_2 \mathbf{e}_3 - a_1 b_3 \mathbf{e}_2 + a_2 b_3 \mathbf{e}_1 - a_2 b_1 \mathbf{e}_3 + a_3 b_1 \mathbf{e}_2 - a_3 b_2 \mathbf{e}_1
\end{aligned}
$$

By avoiding repeated indices, we gain speed in ignoring zero elements in the expression.

You will see that only the six non-vanishing values of e_{ijk} appear in the expression here. We gain valuable time and avoid unnecessary evaluation by following a simple strategy:

1. Once the first index, $i = 1$, only two non-zero cases exist: $j = 2, k = 3$ and $j = 3, k = 2$
2. When $i = 2$, again, only two non-zero cases exist: $j = 3, k = 1$ and $j = 1, k = 3$
3. Lastly, when $i = 3$, again, only two non-zero cases exist: $j = 1, k = 2$ and $j = 2, k = 1$.

Using this approach, it becomes unnecessary to write 27 terms when 21 of them vanish. Instead, we can pick out only the six non-vanishing terms.

We can also make $\mathbf{e}_k$ the subject of the formula starting from the equation,

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k \tag{45}$$

Multiplying both sides by $e_{ij\alpha}$, we have,

$$e_{ij\alpha}\mathbf{e}_i \times \mathbf{e}_j = e_{ij\alpha} e_{ijk}\mathbf{e}_k = 2\delta_{k\alpha}\mathbf{e}_k = 2\mathbf{e}_\alpha$$

so that,

$$\mathbf{e}_\alpha = \frac{1}{2} e_{ij\alpha}\mathbf{e}_i \times \mathbf{e}_j = \frac{1}{2} e_{\alpha ij}\mathbf{e}_i \times \mathbf{e}_j \tag{46}$$

The Dyad

We are used to producing scalars or vectors by taking a product of two vectors. One exceedingly important object that you can also produce from taking such a binary product is a **Tensor**. Naturally, we shall call such a product a "Tensor Product".

Its symbol, $\otimes$, is not a dot or a cross. It is a symbol that may look strange. That symbol combines the product sign and a circle. It is called a dyad operator. Therefore, as before, a tensor product also has a nickname, "the Dyad", or a "Dyad Product".

The **dyad** is **defined** by the result of its action on a vector. Consider the dyad $\mathbf{a} \otimes \mathbf{b}$**.** Its action on a vector $\mathbf{c}$ is defined as follows:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a} \tag{47}$$

That is, it produces a vector in the direction of its first argument scaled by a factor of the scalar product of its second argument with the vector it acts upon. A dyad, as we shall see, is a tensor.

The most elementary tensor you can get is the dyad product of two base vectors: $\mathbf{e}_i \otimes \mathbf{e}_j$

The *tensor* product of two vectors can be expressed in terms of this dyad base:

$$\mathbf{a} \otimes \mathbf{b} = (a_i \mathbf{e}_i) \otimes (b_j \mathbf{e}_j) = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j \tag{48}$$

The summation convention still applies so that it is easy to see that the above expression contains nine components.

Observe immediately that, in 3D, just as you express a vector in terms of three basis vectors, there are nine base dyads for expressing every tensor: $\mathbf{e}_1 \otimes \mathbf{e}_1$, $\mathbf{e}_1 \otimes \mathbf{e}_2$, $\mathbf{e}_1 \otimes \mathbf{e}_3$, $\mathbf{e}_2 \otimes \mathbf{e}_1$, $\mathbf{e}_2 \otimes \mathbf{e}_2$, $\mathbf{e}_2 \otimes \mathbf{e}_3$, $\mathbf{e}_3 \otimes \mathbf{e}_1$, $\mathbf{e}_3 \otimes \mathbf{e}_2$, $\mathbf{e}_3 \otimes \mathbf{e}_3$

To find the components of a tensor is to find nine scalar coefficients to these base dyads. Just as a dot product is called an "**inner** product", a tensor product is called an "**outer** product" or a "**Kronecker** product".

Binary, Ternary Operations

We will introduce tensors more formally in the next chapter. For our purpose here, remember that with two vectors, we have defined three different products that may result. These are: scalar or dot product; vector or cross product; tensor or dyad product. This means that, unlike scalars, you DO NOT simply "multiply" two vectors. To say that creates an ambiguity because we have these three possible results: a scalar, a vector or a tensor. The specific product we have in mind MUST be specified. While the statement, "multiply two scalars" is sensible, the same statement, applied to two vectors, is ambiguous. When we are dealing with vector multiplication, we must disambiguate by being specific on which vector multiplication or product we have in mind. We do this in prose, we also do it in the equation in which vector products are involved. The disambiguation method is the sign, **dot**, **cross** or the dyad **circle on a product** sign that signifies a tensor product. It is therefore an incomplete specification of product, to simply concatenate two vectors, to signify a product, as you would be permitted to do when dealing with two scalar variables or numbers. Given that α and β are scalars, and that $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are vectors, the following table provides examples of products explaining why some may be ambiguous statements requiring more information to be correct:

Product	Right or wrong	Comments
$\alpha\mathbf{u}$	Correct	Scaling a vector, multiplication of a scalar and a vector; No explicit sign required
$\mathbf{u}\beta\mathbf{v}$	Error	$\mathbf{u}\beta$ is a scaled vector whose product with $\mathbf{v}$ is ambiguous. Possible additional information can make it $(\mathbf{u}\beta)\cdot\mathbf{v}$, $\mathbf{u}\times(\beta\mathbf{v})$, or $\mathbf{u}\otimes(\beta\mathbf{v})$. They have different meanings that cannot be reliable guessed unless you supply the needed information a priori.
$\beta\alpha$	Correct	Product of two scalars; No explicit sign required
$\mathbf{v}\mathbf{u}$	Error	Product of two vectors; $\mathbf{v}\cdot\mathbf{u}\neq\mathbf{v}\times\mathbf{u}\neq\mathbf{v}\otimes\mathbf{u}$ Explicit disambiguating sign required. We note here that certain authors imply this simple concatenation as the way they represent the tensor product, $\mathbf{v}\otimes\mathbf{u}$. In most current Literature on the subject, the tensor or dyad sign is the preferred way to represent this product. We retain that more popular convention here and subsequently.
$\beta(\mathbf{u}\times\mathbf{v})$	Correct	Vector product of two vectors gives a vector. Multiplying this result by a scalar does not require another sign. The order of the scaling is NOT important: $\beta(\mathbf{u}\times\mathbf{v})=\beta\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\beta\mathbf{v}=(\mathbf{u}\times\mathbf{v})\beta$ The order of the appearance of the vectors is inviolable:

		$\beta(\mathbf{u}\times\mathbf{v}) \neq \beta\mathbf{v}\times\mathbf{u} = \mathbf{v}\times\beta\mathbf{u} \neq (\mathbf{u}\times\mathbf{v})\beta$
$\mathbf{u}\cdot\mathbf{v}\alpha$	Correct	The dot product of a vector with a scaled vector. No ambiguity is created with the location of α; $\mathbf{u}\cdot\mathbf{v}\alpha$, $(\mathbf{u}\alpha)\cdot\mathbf{v}$, or $\alpha\mathbf{u}\cdot\mathbf{v}$ all mean the same thing.
$\beta\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}\alpha$	Correct	Scalar triple product with vector scaling along. Result is the same as $(\beta\alpha)\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = (\beta\alpha)\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$
$\beta\mathbf{u}\times\mathbf{v}\times\mathbf{w}$	Error	Vector triple product with vector scaling along. Vector product is not associative: $\beta\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \beta(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - \beta(\mathbf{u}\cdot\mathbf{v})\mathbf{w} \neq \beta(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \beta(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - \beta(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ Parentheses are required to show which product is intended.
$\mathbf{u}\cdot\mathbf{v}\otimes\mathbf{w}$	Error	$(\mathbf{v}\otimes\mathbf{w})\,\mathbf{u} \neq \mathbf{u}(\mathbf{v}\otimes\mathbf{w})$
$\mathbf{u}\times\mathbf{v}\otimes\mathbf{w}$	Correct	Treat the vector cross as a tensor, then obtain the LHS: $(\mathbf{u}\times\mathbf{v})\otimes\mathbf{w} = \mathbf{u}\times(\mathbf{v}\otimes\mathbf{w})$ The two different interpretations evaluate to the same value.

More on the Tensor Product

Given vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_j\mathbf{e}_j$, we may use matrix notation, in two different ways, and write,

$$\mathbf{a} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a_i\mathbf{e}_i$$
$$\mathbf{b} = [b_1, b_2, b_3]\begin{bmatrix}\mathbf{e}_1\\\mathbf{e}_2\\\mathbf{e}_3\end{bmatrix} = b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b_j\mathbf{e}_j \tag{49}$$

The dyad $\mathbf{a}\otimes\mathbf{b} = a_ib_j\mathbf{e}_i\otimes\mathbf{e}_j$ which can be given in its full component form as,

$$\mathbf{a}\otimes\mathbf{b} = a_ib_j\mathbf{e}_i\otimes\mathbf{e}_j \tag{50}$$

The matrices of scalars can cross the dyad sign because only one product is defined for scalars. For vectors, the case is different. Three different products are defined between two vectors. We must always be consistent with the product involved. The matrix for the dyad $\mathbf{a} \otimes \mathbf{b}$ is

$$[\mathbf{a} \otimes \mathbf{b}] = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{51}$$

The dyad itself is,

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{52}$$

or,

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \tag{53}$$

The matrix representation of the vector is $\mathbf{a}$ is

$$[\mathbf{a}] = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \; or \; [\mathbf{b}]^{\mathrm{T}} = [b_1, b_2, b_3]. \tag{54}$$

The vectors, in component form are expressed as,

$$\mathbf{a} = a_i \mathbf{e}_i = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } \mathbf{b} = b_j \mathbf{e}_j = [b_1, b_2, b_3] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}. \tag{55}$$

The matrix elements will change if we change the basis vectors to which the vector or dyad is referred. Again, as you can see, the matrix representations, in all cases, are not the same as the tensor or the vector.

Trace of a Dyad

A very important **linear operation** on a dyad is the trace operation. It turns a dyad into a scalar quantity. It is achieved by simply changing the dyad operator into a dot as follows:

$$\begin{aligned} \operatorname{tr}(\mathbf{a} \otimes \mathbf{b}) &= a_i b_j \operatorname{tr}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= a_i b_j \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) \\ &= a_i b_j \, \delta_{ij} \\ &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3. \end{aligned} \tag{56}$$

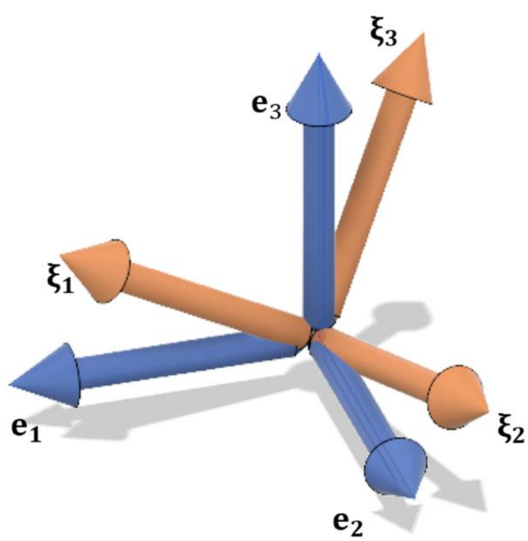
A simple observation will show that this is the sum of the diagonal elements

$$\begin{bmatrix} \boldsymbol{a_1 b_1} & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & \boldsymbol{a_2 b_2} & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & \boldsymbol{a_3 b_3} \end{bmatrix}$$

of the dyad matrix representation as shown above. There is more to say about linearity, linear operators and linear functions in the next chapter.

Coordinate Transformation

Consider a set of Cartesian coordinate orthonormal vectors, $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ shown in blue in figure 1.4. These vectors are position vectors at $\{1,0,0\}, \{0,1,0\}$ and $\{0,0,1\}$ respectively. Consider another orthonormal system, shown in pink, whose unit vectors are oriented as shown in the figure. Let these unit vectors be $\{\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, \boldsymbol{\xi_3}\}$. The set $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$, since they are orthonormal, are also linearly independent. Consequently, each member of the set, $\{\boldsymbol{\xi_1}, \boldsymbol{\xi_2}, \boldsymbol{\xi_3}\}$ can be expressed in terms of the basis vectors in $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$. (We note that the opposite is also possible: we could express the original vectors in terms of the rotated system). Taking these vectors one by one, we may write,



$$\boldsymbol{\xi_1} = \alpha_1 \mathbf{e_1} + \beta_1 \mathbf{e_2} + \gamma_1 \mathbf{e_3}$$
$$\boldsymbol{\xi}_2 = \alpha_2 \mathbf{e_1} + \beta_2 \mathbf{e_2} + \gamma_2 \mathbf{e_3}$$
$$\boldsymbol{\xi}_3 = \alpha_3 \mathbf{e_1} + \beta_3 \mathbf{e_2} + \gamma_3 \mathbf{e_3}$$

The coefficients can be found by taking the dot products as usual. Note that we can gain more compactness and use only one symbol for all the nine coefficients if we adopt this simple arrangement: Let $\alpha_i \equiv a_{i1}, \beta_i \equiv a_{i2}$, and $\gamma_i \equiv a_{i3}$. The three equations can therefore be written more compactly as,

$$\boldsymbol{\xi_i} = a_{ij} \mathbf{e_j} \tag{57}$$

We can find each of the nine coefficients by taking the scalar product of this equation with $\mathbf{e}_{\boldsymbol{\alpha}}$:

$$\boldsymbol{\xi_i} \cdot \mathbf{e}_{\boldsymbol{\alpha}} = a_{ij} \mathbf{e_j} \cdot \mathbf{e}_{\boldsymbol{\alpha}} = a_{ij} \delta_{j\alpha} = a_{i\alpha} \tag{58}$$

Or, $a_{ij} = \boldsymbol{\xi_i} \cdot \mathbf{e}_j$.These linear equations can always be inverted and we may have the converse:

$$\mathbf{e_j} = b_{jk} \boldsymbol{\xi}_k \tag{59}$$

$\mathbf{B} = [b_{ij}]$ is obviously the inverse of the coefficient matrix $\mathbf{A} = [a_{ij}]$. This inverse relationship can be obtained easily using the indicial notation. Starting with $\boldsymbol{\xi}_{\boldsymbol{i}} = a_{ij}\mathbf{e}_{\boldsymbol{j}}$, we could substitute for e_j and write,

$$\boldsymbol{\xi}_{\boldsymbol{i}} = a_{ij}\mathbf{e}_{\boldsymbol{j}} = a_{ij}b_{jk}\boldsymbol{\xi}_k \tag{60}$$

Taking scalar products again, we have,

$$\begin{aligned}\boldsymbol{\xi}_{\boldsymbol{i}} \cdot \boldsymbol{\xi}_\alpha = \delta_{i\alpha} &= a_{ij}b_{jk}\boldsymbol{\xi}_k \cdot \boldsymbol{\xi}_\alpha \\ &= a_{ij}b_{j\alpha} \\ &= (a_{ij}\mathbf{e}_{\boldsymbol{j}}) \cdot (a_{\alpha\beta}\mathbf{e}_{\boldsymbol{\beta}}) \\ &= a_{ij}a_{\alpha\beta}\delta_{j\beta} \\ &= a_{ij}a_{\alpha j}\end{aligned} \tag{61}$$

These equations in matrix form can be written as,

$$\mathbf{I} = \mathbf{AB} = \mathbf{AA}^{\mathrm{T}} \tag{62}$$

Showing that the inverse transformation matrix is the transpose of the original transformation. The inverse transformation can now be re-written, using this result:

$$\mathbf{e}_j = b_{jk}\boldsymbol{\xi}_k = a_{kj}\boldsymbol{\xi}_k \tag{63}$$

So that, in a transformation of from one orthonormal system to another, if

$$\boldsymbol{\xi}_i = a_{ij}\mathbf{e}_j,$$

then

$$\mathbf{e}_i = a_{ji}\boldsymbol{\xi}_j$$

because the inverse of the transformation is simply its transpose.

Example.

Show, in two dimensions that the rotation, $\mathbf{R}^{\mathrm{T}} = \mathbf{e}_j \otimes \boldsymbol{\xi}_j$ gives the coordinates of a fixed vector in rotated coordinates.

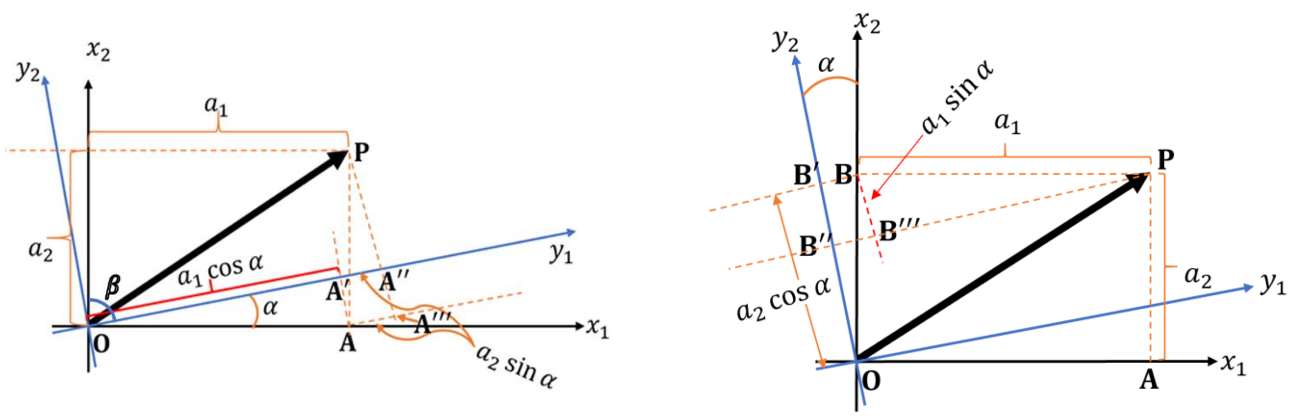


Figure 14. Vector rotation

Answer: In the figure 14, Let the original coordinates be $\mathbf{O}\ x_1\ x_2$ and imagine that we are leaving the vector $\mathbf{OP}$ which is presented as $\mathbf{v} = a_i \mathbf{e}_i$ where $\mathbf{e}_1$and $\mathbf{e}_2$ are unit vectors along $\mathbf{O}\ x_1\ x_2$ If the coordinates are rotated to $\mathbf{O}\ y_1\ y_2$ such that the same vector now becomes $\mathbf{v} = b_i \boldsymbol{\xi}_i$ where $\boldsymbol{\xi}_1$and $\boldsymbol{\xi}_2$ are unit vectors along the $\mathbf{O}\ y_1\ y_2$ system. These will be the new coordinates after the rotation of coordinates to this point.

Clearly, $\mathbf{OA} = a_1$ and $\mathbf{OB} = a_2$. We need to find the lengths , $\mathbf{OA}'' = b_1$ and $\mathbf{OB}'' = b_2$. We drop perpendicular lines to the lines $\mathbf{O}\ y_1$ and $\mathbf{O}\ y_2$ meeting them at $\mathbf{A}''$ and $\mathbf{B}''$respectively. It is clear that $\mathbf{OA}' = a_1 \cos\alpha$. Furthermore, $\mathbf{AA}''' = a_2 \sin\alpha$ because $\mathbf{PA}$ is the hypotenuse of a right angled triangle $\mathbf{APA}'''$with angle α at $\mathbf{APA}'''$ And it is easy to see that $\mathbf{AA}'\mathbf{A}''\mathbf{A}'''$ is a rectangle. Its opposite sides are equal, consequently, the length

$$\begin{aligned} \mathbf{OA}'' = b_1 &= a_1 \cos\alpha + a_2 \sin\alpha. \\ &= a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2) \end{aligned} \quad (64)$$

as we note that $\mathbf{e}_1$ is the unit vector along Ox_1while $\boldsymbol{\xi}_1$ is the unit vector along Oy_1 therefore, $\boldsymbol{\xi}_1 \cdot \mathbf{e}_1 = \|\boldsymbol{\xi}_1\|\|\mathbf{e}_1\| \cos\alpha = \cos\alpha$. Similarly, $\mathbf{e}_2$ is the unit vector along Ox_2 so that $\boldsymbol{\xi}_1 \cdot \mathbf{e}_2 = \|\boldsymbol{\xi}_1\|\|\mathbf{e}_2\| \cos\beta = \sin\alpha$. $\mathbf{B}''$ is the foot of the perpendicular from point $\mathbf{P}$ to the $\mathbf{O}\ y_2$-axis. $\mathbf{BB}'$is parallel to $\mathbf{PB}''$. $\mathbf{B}'''$ is the foot of the perpendicular from $\mathbf{B}$ to $\mathbf{PB}''$. By the same arguments as before, $\mathbf{BB}'\mathbf{B}''\mathbf{B}'''$ is also a rectangle. Clearly,

$$\begin{aligned} \mathbf{OB}'' = b_2 &= -a_1 \sin\alpha + a_2 \cos\alpha. \\ &= a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2) \end{aligned} \quad (65)$$

The rotation tensor is: $\mathbf{R}^{\mathrm{T}} = \mathbf{e}_j \otimes \boldsymbol{\xi}_j$. Hence, we have:

$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = (\mathbf{e}_j \otimes \boldsymbol{\xi}_j) a_i \mathbf{e}_i$$
$$= a_i \mathbf{e}_j (\boldsymbol{\xi}_j \cdot \mathbf{e}_i). \tag{66}$$

Expanding for this two-dimensional case, we have:

$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = \mathbf{e}_1\big(a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)\big) + \mathbf{e}_2\big(a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2)\big) \tag{67}$$

which is exactly what we have obtained by simple geometry.

The Euclidean Point Space

The 3D Euclidean Point Space we live in is where all engineering objects of interest to us reside. This space contains point locations that can be occupied by a location in an object at a particular time. It is often of interest to be able to do several things:

1. Locate the point in an unambiguous way,
2. Relate the point to one or more other points in its vicinity, and
3. Define quantities that take up values of interest at that point.

* Temperature map of this classroom (one thousand thermometers)
* Temperature distribution, Temperature field.
* Tensor Fields

Cartesian & Other Coordinate Systems

Our coordinate systems so far have very interesting features: They are based on spatially constant unit vectors orthogonal to each other. These are called Rectangular Cartesian or Orthonormal Base (ONB) Systems. We have seen that we are only required to have, for basis vector sets to span a space, that they are linearly independent.

ONBs are more than linearly independent; their orthonormal attributes make the computation of coordinates for any vector referred to them, very easy to obtain. There are other advantages:

* We can refer the room to a set of Cartesian coordinates (x, y, z).
* In this system, each location is represented by three ordered numbers. The first represents the x coordinate, the second the y coordinate, and the third, the z coordinate respectively.
* The basis vector set is $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ or $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. These are along the constant coordinate lines which are straight line intersections of the coordinate planes as shown below.

* Following $Mathematica^{®}$ code implements this idea (Type it and see for yourself).

In locating point $\mathbf{P}(x_1, y_1, z_1)$ above, we constructed three coordinate planes:

* A dark colored plane perpendicular to the x —axis,
* A purple plane perpendicular to the y —axis, and
* A purple plane perpendicular to the z —axis.

```
Cart1 = ParametricPlot3D[{1, y, z}, {y, 0, 1.4}, {z, 0, 1.4}, PlotStyle → Red];
Cart2 = ParametricPlot3D[{x, 1, z}, {x, 0, 1.4}, {z, 0, 1.4}, PlotStyle → Green];
Cart3 = ParametricPlot3D[{x, y, 1}, {x, 0, 1.4}, {y, 0, 1.4}, PlotStyle → Yellow];
Show[Cart1, Cart2, Cart3, PlotRange → {{0, 1.5}, {0, 1.5}, {0, 1.5}}, Ticks → None]
```

Position Vector

* Furthermore, we can define a vector for the point location $\boldsymbol{P}(x_1, y_1, z_1)$ .Such a vector is defined by joining the point $\boldsymbol{P}$ to the origin to form the vector **OP** represented by the line shown.
* The vector whose magnitude is defined by the length of **OP**, and whose direction is indicated by the direction of OP, a ***Position Vector.***
* We defined a vector (a member of the Euclidean Vector Space, that is now embedded in the Euclidean point space of our daily experience.
    * The latter contains just points, the former is a collection of objects that obey certain rules that make us label them "vectors".
    * This particular one is not just a vector, it is a position vector because it is the point

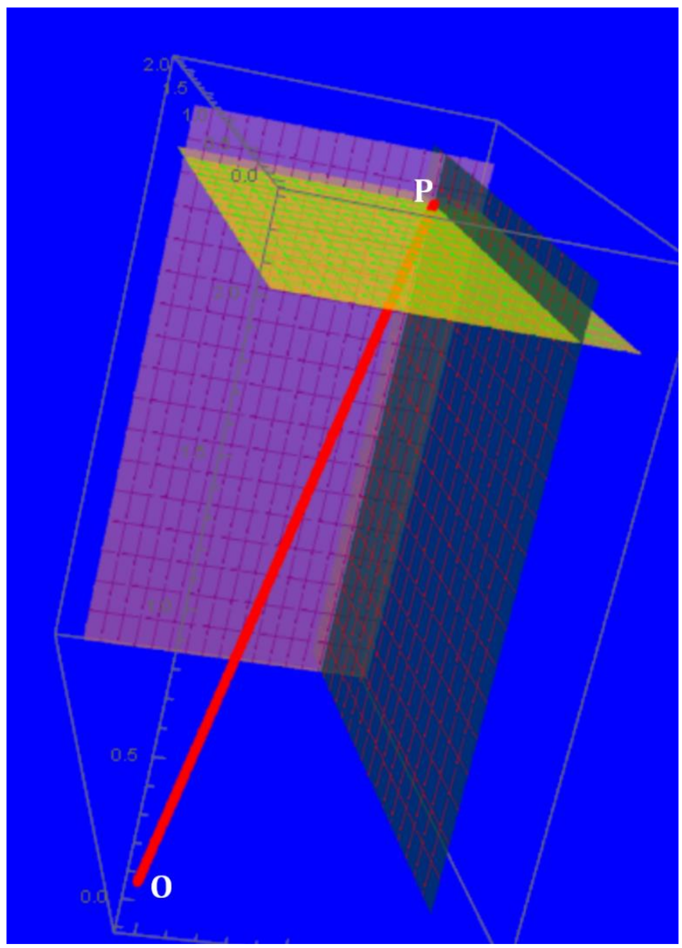


Figure 15. Cartesian Coordinate Surfaces

$\boldsymbol{P}(x_1, y_1, z_1)$ that gave birth to it. At any other point we define by three numbers, we can also get a position vector in this simple way.

Notice several things that are attractive in the Cartesian system we have described.

* Each coordinate surface is a plane. The three defined at a particular point are respectively parallel to the three you can define at any other point.
* Each coordinate lines: the intersection of these planes that are parallel to the axes are similarly parallel straight lines at all points in the system.
* The basis vectors – usually defined as unit vectors along the axes, are always the same at any point in the Cartesian system. It does not matter where the point P is located, the basis vectors are the same unit vectors we define as ($\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$) or ($\mathbf{e}_1, \mathbf{e}_2,$ and $\mathbf{e}_3$) along the coordinate lines at the origin.

These properties combine to make the Cartesian coordinate system very simple and easy to use. It is no wonder that it is the first coordinate system you get introduced to – for most people, as early as secondary school!

The first important advantage of the Cartesian system is the simplicity of the expression for a position vector. The position vector OP can be written simply as,

$$\mathbf{r} = x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k} \tag{68}$$

Or, more conveniently as,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \tag{69}$$

Where we have replaced (x_1, y_1, z_1) by (x_1, x_2, x_3) so we may benefit from the compactness of the Einstein's summation convention. This expression is linear in the coordinate variables. There are two other hidden reasons why this coordinate system is so simple and easy to use. It may not be obvious that the simple expression of the position vector we have here is possible only in the Cartesian system.

In other coordinate systems, the position vector is usually a much more complicated function of the coordinate variables and the basis vectors. In general, if we do not assume that we are using the Cartesian system,

$$\mathbf{r} = \mathbf{r}(\alpha_1, \alpha_2, \alpha_3, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \tag{70}$$

where $\alpha_i,\ i = 1,2,3$ are the coordinate variables and $\mathbf{g}_i,\ i = 1,2,3$ are the basis vectors. The simple linear form we have for the Cartesian case, as we shall see is a rare exception and a special case. The functional for of the position vectors can be complicated.

A second reason that the Cartesian system is so easy, useful and pervasive is the related fact of the constancy of the basis unit vectors. To illustrate this, imagine we continue with our thought experiment to get a temperature map for the room, then we have a scalar field $T(x_1, x_2, x_3)$. If we have a vector function defined at each point, then we get a vector field $\boldsymbol{v}(x_1, x_2, x_3)$ .We can easily write the vector field in terms of three scalar fields that we call its components; hence, we may write,

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3 \qquad (71)$$

Where $v_i(x_1, x_2, x_3),\ i = 1,2,3$ are the components of the velocity vector. The fact that the basis vectors $\mathbf{e}_i, i = 1,2,3$ neither varies temporally nor spatially means that differential and integral calculus with the Cartesian system take a particularly easy form. Differentiating the above equations, whether with respect to time or to space, we simply focus on the functions, $v_i(x_1, x_2, x_3)$ and ignore the constants $\mathbf{e}_i, i = 1,2,3$!

A third reason for the simplicity of the Cartesian system is in the fact that the three numbers representing the coordinates are of the same dimensionality.

The numbers, $x_1, x_2,$and x_3 (coefficients of the basis vectors) for the coordinates of $\mathbf{P}$ are all lengths. They are all the same dimension. There is nothing compelling you to use lengths for your coordinate variables in a coordinate system.

Observation: A partial differentiation of the position vector with respect to the coordinate variables yield the basis vectors for the coordinate system as shown here:

$$\begin{aligned} \mathbf{r} &= x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \\ \frac{\partial \mathbf{r}}{\partial x_i} &= \mathbf{e}_i, i = 1,2,3. \end{aligned} \qquad (72)$$

This applies to the other coordinate systems as well.

In fact, the two next most popular systems – the Spherical and Cylindrical systems use a combination of lengths and angles! If you are not careful, and you use these coordinate systems just the way you do the Cartesian, your first error might be that you are adding quantities of different dimensions and units in the same expression and will be guaranteed to obtain wrong results.

Coordinate Points & Coordinate Surfaces

In 3D Euclidean Point Space, each coordinate system is defined by three coordinate variables. (ξ_1, ξ_2, ξ_3). When each takes a value, say, $\xi_i = \alpha_i$ where each α_i is a real number, then we have the point $(\alpha_1, \alpha_2, \alpha_3)$. We can write this point in at least two other ways: $\xi_i = \alpha_i$, $i = 1, \dots, 3$ or as $(\xi_1 = \alpha_1, \xi_2 = \alpha_2, \xi_3 = \alpha_3)$. For each, $\xi_i = \alpha_i$, we have defined a coordinate surface. In the case of Cartesian coordinates, given any three $\alpha_i \in \mathbb{R}, i = 1,2,3,$ we have $x_1 = \alpha_1$, defining a plane with normal along the $\mathbf{e}_1$axis, $x_2 = \alpha_2$, defining a plane with normal along the $\mathbf{e}_2$axis and $x_3 = \alpha_3$, which is a plane with normal along the $\mathbf{e}_3$ axis. It is easy to see that at the point of intersection, these three planes meet at right angles. The coordinate system is, for this reason, orthogonal.

This coordinate system is also linear in the sense that the normal do not change as you change the values of α_i; that is, as you move from point to point, the normal to the coordinate planes remain the same vector as at any point.

The other coordinate systems we will look at are not linear in this sense. They are **CURVI**linear. We limit ourselves to curvilinear systems that remain orthogonal. In these systems, the following ideas remain unchanged:

1. For each $\xi_i = \alpha_i$, we define a coordinate surface;
2. The coordinate point, $\xi_i = \alpha_i$, $i = 1, \dots, 3$ is the intersection of the three surfaces;
3. The tangents to these surfaces are mutually orthogonal.

In contradistinction from the Cartesian system, these surfaces do not have constant normal as you move from point to point. This difference is NOT trivial, as we shall see. The first curvilinear system we shall consider is the Cylindrical Polar coordinate system as follows.

Cylindrical Polar Coordinates

In the cylindrical system, we select the three numbers that we shall use to represent a typical point P using a different strategy. We select two lengths and an angle. Since we already are quite used to the Cartesian system, let us first note that the third coordinate in the Cylindrical Polar

System is shared with the Cartesian. Even if we represent it with a different symbol, note that the z-coordinate as well as the $\mathbf{k}$, $\mathbf{e}_3$ or $\mathbf{e}_z$ essentially remain the same in both Cartesian and the Cylindrical Polar system.

Begin with our familiar Cartesian system of coordinates. We can represent the position of a point (position vector) with three coordinates x_1, x_2, x_3 $(\in R)$ such that,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i \tag{73}$$

That is, the choice of any three scalars can be used to locate a point. We now introduce a transformation (called a polar transformation) of $\{x_1, x_2\} \to \{r,\ \phi\}$ such that, $x_1 = r\cos\phi$, and $x_2 = r\sin\phi$ .Note also that this transformation is invertible: $r = \sqrt{x_1^2 + x_2^2}$,and $\phi = \tan^{-1}\frac{x_2}{x_1}$

With such a transformation, we can locate any point in the 3-D space with three scalars $\{r,\ \phi, z\}$ instead of our previous set $\{x_1, x_2, x_3\}$ .Our position vector is now,

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z \tag{74}$$

where we define $\boldsymbol{e}_r \equiv \cos\phi\,\boldsymbol{e}_1 + \sin\phi\,\boldsymbol{e}_2$,$\boldsymbol{e}_z$ is no different from $\mathbf{e}_3$ or $\mathbf{k}$. In order to complete our triad of basis vectors, we need a third vector, $\boldsymbol{e}_\phi$ .In selecting $\boldsymbol{e}_\phi$,we want it to be such that $\{\boldsymbol{e}_r, \boldsymbol{e}_\phi, \boldsymbol{e}_z\}$ can form an orthonormal (pairwise orthogonal and individually normalized) basis. Let

$$\mathbf{e}_\phi = \xi\mathbf{e}_1 + \eta\mathbf{e}_2 \tag{75}$$

To satisfy our conditions**,** $\mathbf{e}_\phi \cdot \mathbf{e}_r = 0$,$\mathbf{e}_\phi \cdot \mathbf{e}_z = 0$ (automatically satisfied by not choosing a different third coordinate) and $\sqrt{\xi^2 + \eta^2} = 1$.

It is easy to see that $\mathbf{e}_\phi \equiv -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$ satisfies these requirements. $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ forms an orthonormal (that is, each member has unit magnitude and they are pairwise orthogonal) triad just like $\mathbf{e}_i, i = 1,2,3$.

The transformation we have just described can be given a geometric interpretation. In either case, it is the definition of the **Cylindrical Polar coordinate system**.

Unlike our Cartesian system, we note that $\{\mathbf{e}_r(\phi), \mathbf{e}_\phi(\phi), \mathbf{e}_z\}$ as the first two of these are not constants but spatial variables dependent on angular orientation. $\mathbf{e}_z$ remains a constant vector as in the Cartesian case.

Geometric Interpretation

The coordinate system just described requires us, as before, to select three ordered numbers to uniquely represent a point in the Euclidean point space. The first is a length, r, the second, an angle ϕ, and the third, a length, z. These are the coordinate variables.

Recall that in the Cartesian case, the coordinate planes have equations, $x_1 = const$, $x_2 = const$, and $x_3 = const$ giving us three planes that intersect at the point defined by those three values of the constants used.

In a similar way, the coordinate planes in the Cylindrical Polar are: $(\xi_1 = \alpha_1)\ r = const$ describing a cylinder with the z-axis as its axis, $(\xi_2 = \alpha_2)\ \phi = const$ describing a plane through the axis and another plane, $(\xi_3 = \alpha_3)\ z = const$ describing a plane that is perpendicular to the cylinder axis. This is as shown in the figure 15.

```
c1 = ParametricPlot3D[{Sin[ϕ], Cos[ϕ], z}, {ϕ, 0, π}, {z, 1.5, 3.5}, PlotStyle → Red];
c2 = ParametricPlot3D[{r Sin[π / 3], r Cos[π / 3], z}, {r, 0, 2}, {z, 1.5, 3.5}, PlotStyle → Green];
c3 = ParametricPlot3D[{r Sin[ϕ], r Cos[ϕ], 2}, {ϕ, 0, 2 π}, {r, 0.5, 2.5}, PlotStyle → Yellow];
Show[c1, c2, c3, PlotRange -> {{0, 1.4}, {0, 1.5}, {1, 2.5}}, Ticks → None]
```

We can obtain the basis vectors by differentiation of the position vector:

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z$$
$$\frac{\partial \mathbf{r}}{\partial r} = \mathbf{e}_r;\ \frac{\partial \mathbf{r}}{\partial \phi} = r\mathbf{e}_\phi;\ \frac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z \tag{76}$$

The basis vectors obtain by differentiation also compels dimensional consistency, but they are no longer orthonormal even though they remain mutually orthogonal.

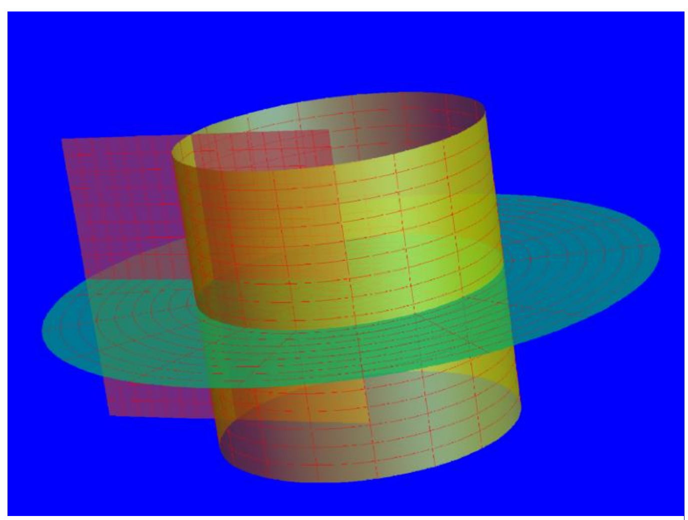


Figure 16. Cylindrical Polar Coordinate Surfaces

Mistakes to avoid

Two easy mistakes that can be made are:

1. **That the Cylindrical position vector is** $r\mathbf{e}_r(\phi) + \phi\boldsymbol{e}_\phi + z\boldsymbol{e}_z$ which is a simplistic copy of the Cartesian formula. This is wrong in at least two ways. For one thing, it is dimensionally incorrect because the

unit of the middle basis component is an angle while the other components are measuring lengths. Secondly, we cannot obtain the Cartesian result from this via a coordinate transformation.

2. **That the basis vectors are constants**. They are NOT all constants. $\mathbf{e}_r(\phi)$ and $\boldsymbol{e}_\phi(\phi)$ are both functions of ϕ unlike in the Cartesian case, but $\boldsymbol{e}_z$ is a constant like the Cartesian case.

Spherical coordinates

The spherical Polar coordinate system selects its three ordered triplets with yet another strategy. This can be explained by the same transformation route we started. Continuing further with our transformation, we may again introduce two new scalars such that $\{r,\ z\} \to \{\rho, \theta\}$ in such a way that the position vector,

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = \rho \sin\theta\, \mathbf{e}_r + \rho\cos\theta\, \mathbf{e}_z \equiv \rho\mathbf{e}_\rho \tag{77}$$

Here, $r = \rho\sin\theta,\ z = \rho\cos\theta$. As before, we can use three scalars, $\{\rho,\ \theta,\ \phi\}$ instead of $\{r,\ \phi, z\}$. In comparison to the original Cartesian system we began with, we have that,

$$\begin{aligned}\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = {} & \rho\sin\theta\,\mathbf{e}_r + \rho\cos\theta\,\mathbf{e}_z \\ & = \rho\sin\theta\,(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \rho\cos\theta\,\mathbf{k} \\ & = \rho\sin\theta\cos\phi\,\mathbf{i} + \rho\sin\theta\sin\phi\,\mathbf{j} + \rho\cos\theta\,\mathbf{k} \\ & \equiv \rho\mathbf{e}_\rho\end{aligned} \tag{78}$$

it is clear that the unit vector

$$\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}. \tag{79}$$

Again, we introduce the unit vector, $\mathbf{e}_\theta \equiv \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$ and retain $\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$ as before**.** It is easy to demonstrate the fact that these vectors constitute another orthonormal set. Combining the two transformations, we can move from $\{x, y, z\}$ system of coordinates to $\{\rho,\ \phi, \theta\}$ directly by the transformation equations, $x = \rho\sin\phi\cos\theta,\ y =$

$\rho \sin\phi \sin\theta$ and $z = \rho\cos\theta$ .The orthonormal set of basis for the $\{\rho,\ \theta,\ \phi\}$ system is $\{\mathbf{e}_\rho(\theta,\ \phi), \mathbf{e}_\theta(\theta,\ \phi), \mathbf{e}_\phi(\phi)\}$

$$\mathbf{r}(\rho,\theta,\ \phi) \equiv \rho\mathbf{e}_\rho(\theta,\ \phi)$$

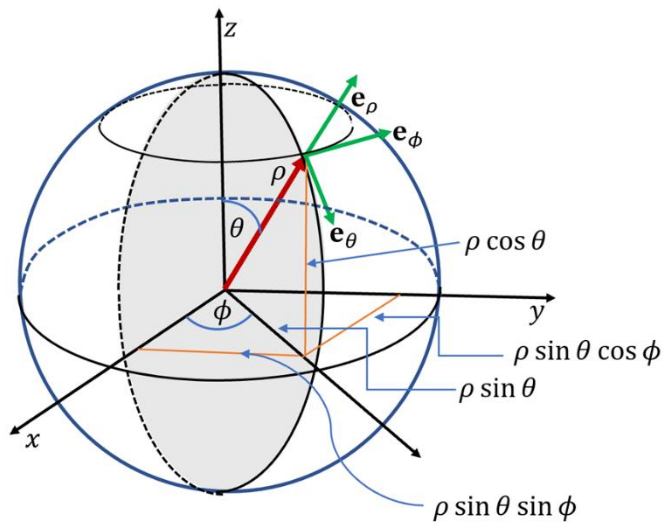


Figure 17. Computing Spherical Polar Components

Showing that the position vector depends on the three coordinate variables representing the radial distance, ρ, from the origin on the azimuthal (great circle, longitudinal) plane inclined at an angle θ to the meridian plane $(x - z)$, with a polar angle ϕ as shown below: the orthonormal basis vectors are shown at the point of interest. The projection of the radial distance to the "equatorial" plane is also shown

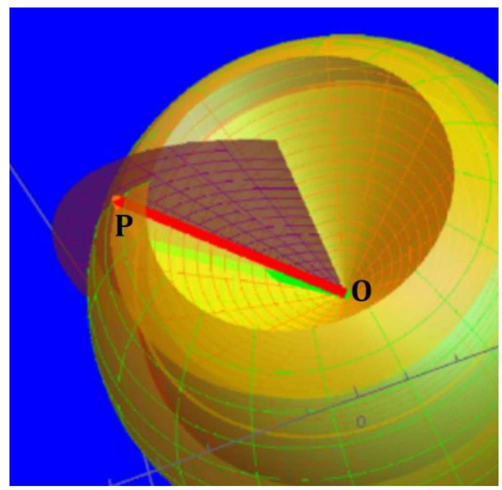


Figure 18. Spherical Coordinates

Coordinate Surfaces

In spherical coordinates, the point $\mathbf{P}$ lies at the intersection of a spherical surface, $\rho = const$, cone, $\theta = const$ and a plane, $\phi = const$. The cone and the sphere are both centered at the origin, $\mathbf{O}$, as shown in figure 17, and the position vector lies at the intersection of the cone and the plane, beginning from the origin and terminating at the point $\mathbf{P}$. Note that the plane $\phi = const$ passes through the same origin. As before, the coordinate surfaces are orthogonal as well as the tangents to the coordinate lines that are at the intersections of the coordinate planes. Just the same way we obtained the basis vectors by differentiation in the cylindrical system, we can obtain the same for the spherical:

For spherical polar,

$$\mathbf{r} = \rho\mathbf{e}_\rho(\theta,\ \phi)$$
$$\frac{\partial\mathbf{r}}{\partial\rho} = \mathbf{e}_\rho;\ \frac{\partial\mathbf{r}}{\partial\theta} = \rho\mathbf{e}_\theta;\ \frac{\partial\mathbf{r}}{\partial\phi} = \rho\ \sin\theta\mathbf{e}_\phi \tag{80}$$

The vectors $\mathbf{e}_\rho, \mathbf{e}_\theta$ and $\mathbf{e}_\phi$ are unit vectors. The multipliers in each case are the magnitudes of the basis vectors obtained from differentiation.

Other Coordinate Systems

There are many other ways of selecting three ordered scalars to create a coordinate system. The ones we have seen so far are all orthogonal coordinate systems because the coordinate planes meet at all points at right angles. Other orthogonal coordinate systems that have engineering significance include:

1. Parabolic and Parabolic Cylindric
2. Elliptic Cylinder, Elliptic, Bipolar,
3. Confocal,
4. Prolate and Oblate spheroidal, Toroidal

The strategy of definition is similar in each case. A few:

Parabolic Cylinder Coordinate System

Parabolic Cylindrical Coordinates are (ξ, η, z). Here the first two are square roots of length while the third scalar is length. Transformation equations are: $x_1 = \xi\eta, x_2 = \frac{1}{2}(\xi^2 - \eta^2)$ and $x_3 = z$. Substituting these in the Cartesian position vector,

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \xi\eta\mathbf{e}_1 + \frac{1}{2}(\xi^2 - \eta^2)\mathbf{e}_2 + z\mathbf{e}_3 \tag{81}$$

Again, by differentiating this with respect to the coordinate variables, ξ, η, z, we obtain the following basis vectors for the Parabolic Cylindrical System:

$$\eta\mathbf{e}_1 + \xi\mathbf{e}_2; \xi\mathbf{e}_1 + \eta\mathbf{e}_2; z\mathbf{e}_3 \tag{82}$$

The table below shows a summary of position and basis vectors for these and other coordinate systems

Table 4. Position & Basis Vectors for some Coordinate Systems

Coordinate System	Position Vector	Basis Vectors
Cartesian, $x_1 x_2 . x_3$	$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$	$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$
Cylindrical Polar r, ϕ, z	$r\mathbf{e}_r(\phi) + z\mathbf{e}_z$	$\mathbf{e}_r, r\mathbf{e}_\phi, \mathbf{e}_z$
Spherical Polar ρ, θ, ϕ	$\rho\mathbf{e}_\rho(\theta, \phi)$	$\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi$
Parabolic Cylindrical ξ, η, z	$\xi\eta\mathbf{e}_1 + \frac{1}{2}(\xi^2 - \eta^2)\mathbf{e}_2 + z\mathbf{e}_3$	$\eta\mathbf{e}_1 + \xi\mathbf{e}_2,$ $\xi\mathbf{e}_1 + \eta\mathbf{e}_2, z\mathbf{e}_3$
Parabolic ξ, η, ϕ	$\eta\xi\cos\phi\,\mathbf{e}_1 + \eta\xi\sin\phi\,\mathbf{e}_2 + \frac{1}{2}(\xi^2 - \eta^2)\mathbf{e}_3$	$\mathbf{e}_\xi = \eta\cos\phi\,\mathbf{e}_1 + \eta\sin\phi\,\mathbf{e}_2 + \xi\mathbf{e}_3,$ $\mathbf{e}_\eta = \xi\cos\phi\,\mathbf{e}_1 + \xi\sin\phi\,\mathbf{e}_2 - \eta\mathbf{e}_3,$ $\mathbf{e}_\phi = -\eta\xi\sin\phi\,\mathbf{e}_1 + \eta\xi\cos\phi\,\mathbf{e}_2$
Elliptic Cylindrical ξ, η, z	$\cosh\xi\cos\eta\,\mathbf{e}_1 + \sinh\xi\sin\eta\,\mathbf{e}_2 + z\mathbf{e}_3$	$\mathbf{e}_\xi = \sinh\xi\cos\eta\,\mathbf{e}_1 + \cosh\xi\sin\eta\,\mathbf{e}_2,$ $\mathbf{e}_\eta = -\cosh\xi\sin\eta\,\mathbf{e}_1 + \sinh\xi\cos\eta\,\mathbf{e}_2,$ $\mathbf{e}_z = \mathbf{e}_3$

Vector Spaces

We are now in a position to provide a more exact definition of what a vector really is. What you should observe in the following is that the definition is satisfied by our elementary notions about vectors. However, a vector is a more abstract object than we have been looking at. The abstraction is useful because it allows the analytical treatment of quantities that do not appear to be similar or related to the notions brought from elementary considerations.

We begin by assuming we have a bag containing real numbers. The numbers in this bag constitutes a *collection* – just like any collection of items like the dishes in your dining table, or shoes in your closet. We call this collection, the set $\mathbb{R}$. The set of real numbers we have just defined, is the foundation of our vector space. It is possible to build the vector space upon a different foundation, such as complex numbers, or rational numbers. For this reason, our definition has indicator, "real", in it.

Definition. A **real vector space** $\mathbb{V}$ is a set of elements (called vectors) such that,

1. **Addition operation** is **defined** and it is **commutative** and **associative** under $\mathbb{V}$: that is, $\mathbf{u}+\mathbf{v}\in\mathbb{V}$, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w}$, $\forall\,\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{V}$. Furthermore, $\mathbb{V}$ is **closed** under addition: That is, given that $\mathbf{u},\mathbf{v}\in\mathbb{V}$, then $\mathbf{w}=\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$, $\Rightarrow \mathbf{w}\in\mathbb{V}$.
2. $\mathbb{V}$ **contains** a **zero element** $\mathbf{o}$ such that $\mathbf{u}+\mathbf{o}=\mathbf{u}\ \forall\ \mathbf{u}\in\mathbb{V}$. For every $\mathbf{u}\in\mathbb{V}$, $\exists -\mathbf{u}:\mathbf{u}+(-\mathbf{u})=\mathbf{o}$.
3. **Multiplication by a scalar**. For $\alpha,\beta\in\mathbb{R}$ and $\mathbf{u},\mathbf{v}\in\mathbb{V}$, $\alpha\mathbf{u}\in\mathbb{V}$, $1\mathbf{u}=\mathbf{u}$, $\alpha(\beta\mathbf{u})=(\alpha\beta)\mathbf{u}$, $(\alpha+\beta)\mathbf{u}=\alpha\mathbf{u}+\beta\mathbf{u}$, $\alpha(\mathbf{u}+\mathbf{v})=\alpha\mathbf{u}+\alpha\mathbf{v}$.

End of definition

Note the following:

1. By "**under** $\mathbb{V}$", we mean, so long as you are only dealing with elements of the vector space $\mathbb{V}$.
2. The only multiplication needed to define a vector product is scaling. Not scalar, vector nor tensor products among vectors are needed to define a vector space. Consequently, there are several structures that would qualify as a vector space.
3. Our understanding of vectors thus far is admissible here. Condition #1 is satisfied by our parallelogram law of vector addition. The space is closed under addition because when you add two or more vectors (extending the parallelogram law to a polygon of vectors, the result you will get remains a vector, thus guaranteeing closure. Commutativity as well as associativity are straightforward when we try to add more than two vectors and find that the order of addition is immaterial.
4. For rule #2, note that a zero vector will be represented by a point; no length – resulting in a magnitude of zero. The negation of a vector is simply to retain the direction but change the sense of the arrow.

5. Rule three is merely a mathematical expression of the scaling process. It should also be handled by the addition law when applied to scaled vectors.

The Inner Product or Euclidean Vector Space

* An **Inner-Product** (also called a **Euclidean Vector**) **Space** $\mathbb{E}$ is a real vector space that defines, among its elements, the scalar product: for each pair $\mathbf{u}, \mathbf{v} \in \mathbb{E}, \exists\ l \in \mathbb{R}$ such that,

$$l = \mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u} \tag{83}$$

Further, $\mathbf{u} \cdot \mathbf{u} \geq 0$, the zero-value occurring only when $\mathbf{u} = 0$. It is called "Euclidean" because the laws of Euclidean geometry hold in such a space. "Euclidean Geometry" is the totality of the geometry you have done so far, including: Adding all angle of a triangle to 180 degrees, Parallel lines never meeting, Sum of two sides of triangle always larger than the third, etc. You will later get to know that there are other "geometries" where these things are not valid. These are non-Euclidean geometries.

* The inner product, because of its operational representation as a dot between two vector operands, is also called a dot product, is the mapping

$$" \cdot ": \mathbb{V} \times \mathbb{V} \to \mathbb{R} \tag{84}$$

from the product space to the real space. The notation here means nothing more than, first expressing the fact that the operational sign to denote the Inner Product is the dot, $" \cdot "$. The "product" $(\mathbb{V} \times \mathbb{V})$ is not the same meaning of multiplication of the type we are used to, but simply expressing the fact that we took one element of a vector space, and went back again to take another element of a vector space in order to perform the operation. And the right pointing arrow in the expression shows that the result of the operation is a member of the Real collection, or set: a complicated way of saying that it is a real number! If we had needed three element from the vector space, then we would have had, $\mathbb{V} \times \mathbb{V} \times \mathbb{V}$ for the scalar triple product. These operations will be written as,

$$" \cdot \times ": \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R} \tag{85}$$

We could also have written,

$$"[\ ,\ ,]": \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{R} \tag{86}$$

Because we may in fact prefer this notation as it emphasizes that only the ordering of the vectors is important, NOT the locations of the dot and the cross for the scalar triple product – showing that the symbolic representation of the operation which produces a scalar result but requires a dot and a cross, while,

$$" \times ": \mathbb{V} \times \mathbb{V} \times \mathbb{V} \to \mathbb{V} \tag{87}$$

will represent the vector triple product as it requires three vectors to produce a single vector. Our dyads require two vectors to produce a tensor. We can write,

$$" \otimes ": \mathbb{V} \times \mathbb{V} \to \mathbb{L} \tag{88}$$

if we represent the linear transformation that we call tensors by the symbol $\mathbb{L}$**.**

The inclusion of a definition for the Scalar product induces the concept of length. To make it easy, note that we have used spaces that have no concept of length – hence, it is not always necessary to include the concept into every structure we intend to develop. As a quick example, the thermodynamic plot of pressure to volume remains very useful even though the concept of distance between two arbitrary points is meaningless. In case of the vector space, for our use, the extension to the inclusion of the inner product as well as its induction of the length idea is, though not essential, is very useful indeed.

Magnitude The norm, length or magnitude of $\mathbf{u}$**,** denoted $\|\mathbf{u}\|$ is defined as the positive square root of $\mathbf{u} \cdot \mathbf{u} = \|\mathbf{u}\|^2$**.** When $\|\mathbf{u}\| = 1$, $\mathbf{u}$ is said to be a unit vector. When $\mathbf{u} \cdot \mathbf{v} = 0$, $\mathbf{u}$ and $\mathbf{v}$ are said to be orthogonal.

Direction Furthermore, for any two vectors $\mathbf{u}$ and $\mathbf{v}$, the angle between them is defined as,

$$\cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}\right) \tag{89}$$

The scalar **distance** $d \in \mathbb{R}$ between two position vectors $\mathbf{u}$ and $\mathbf{v}$

$$d = \|\mathbf{u} - \mathbf{v}\|. \tag{90}$$

Notice that we do not include the definition of the vector product in the definition of any vector space. The fact is that, once the concept of magnitude exists, we can define several other things on that basis. The vector product is just one of the many consequences of the scalar product. The latter being the more fundamental concept.

The Euclidean Point Space

It is a good thing to get a firm grasp of the **Euclidean Point Space**. It is NOT a vector space because its members are not vectors as we have defined them. There is a relationship between members of the Euclidean point space and vectors, as we shall see. The Euclidean Point Space is the ambient space in which **all** physical objects of interest reside. To make it simple, where you are sitting, or standing, reading this, is a Euclidean Point Space. It is made up of **points** rather than **vectors**.

What is a Point? On your graph paper from high school, you are used to locating points with an ordered pair of real numbers. These are the Cartesian coordinates of the point. We are also used to the extension of this concept to three dimensions. If $x = \{x_1, x_2, x_3\}, y = \{y_1, y_2, y_3\}$ and $z = \{z_1, z_2, z_3\}$ are three such points, we can define the vectors joining them to a given point

$$\mathrm{o} \equiv \{0,0,0\} \tag{91}$$

the origin of coordinates in $\mathcal{E}$. The Euclidean Point Space may also be referred to non-Cartesian systems. The three ordered numbers may no longer represent distances. They must be in correct order. The dimensionality of a space determines the number of elements contained in the description of a point in $\mathcal{E}$

Definition: The Euclidean Point Space, $\mathcal{E}$ is such that, for points x, y, z and an origin, if we represent the vector, $\mathbf{v}$ joining point x to point y as $\mathbf{v}(x, y) \in \mathbb{E}$, where $x, y \in \mathcal{E}$, then,

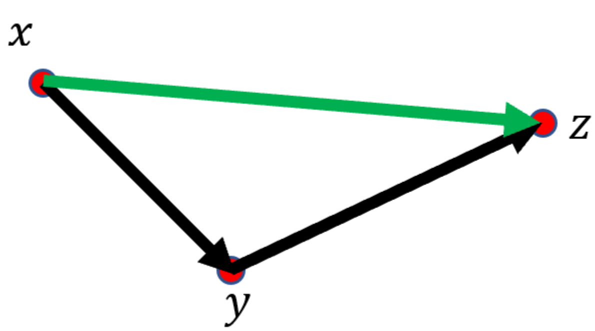


Figure 19. Euclidean Points Joined by Vectors

1. $\mathbf{v}(x, z) = \mathbf{v}(x, y) + \mathbf{v}(y, z)\ \forall\ x, y, z \in \mathcal{E}$, and
2. $\mathbf{v}(x, y) = \mathbf{v}(x, z) \Leftrightarrow y = z$ for each $x \in \mathcal{E}$

End of Definition

Consequences:

From Rule 1, we can see that, $\mathbf{v}(x, x) = \mathbf{v}(x, y) + \mathbf{v}(y, x) = \mathbf{o}$ as the vector joining a point to itself must necessarily be the zero vector. This neutral additive concept of a zero vector in the Euclidean POINT space leads to an additive inverse as the last equation immediately implies that,

$$\mathbf{v}(x, y) = -\mathbf{v}(y, x)\ \forall\ x, y\ \in \mathcal{E} \tag{92}$$

In simple terms, we are stating the geometrically obvious fact that the vector joining point x to point y is the negative of the one joining point y to point x.

The Position Vector

The question of the true nature of what is called a "position vector" can now be addressed. Remember, all points are resident in the Euclidean Point Space. A position vector joins a point to the origin of coordinates. It is a vector defined by the location of two points in $\mathcal{E}$. Consequently, we have,

$$\mathbf{v}(x) \equiv\ \mathbf{v}(x, o) = \mathbf{x}(o) = \mathbf{x} - \mathbf{o} \tag{93}$$

Where $\mathbf{x}(o), \mathbf{x}(y) \in \mathbb{E}$, that is they are vectors in the Euclidean Vector Space and we define

$$\mathbf{x}(y) \equiv \mathbf{x}(o) - \mathbf{y}(o) = \mathbf{x} - \mathbf{o} - (\mathbf{y} - \mathbf{o}). \tag{94}$$

The vector $\mathbf{x}(o) =\ \mathbf{x} - \mathbf{o}$ joining the point $x \in \mathcal{E}$ to the origin is called a **Position Vector**. The vector itself resides in the vector space, (in the sense that it takes its characteristics among vectors) the points defining it dwell in the Euclidean point space. Mathematically, this is called an *embedding* of a Vector Space (defining the Position Vectors) in the Euclidean Point Space (defining the points that create them).

The distance between two position vectors becomes sensible: It is the magnitude of the vector $\mathbf{v}(x, y)$ joining point x to point y in the Euclidean Point Space.

$$d(x - y) = \|\mathbf{v}(x, y)\| = \|\mathbf{x}(y)\| = \|\mathbf{x} - \mathbf{y}\| \tag{95}$$

Software

The most important thing about this course is NOT what you can know, but **WHAT YOU CAN DO WITH WHAT YOU KNOW.** Many engineering books contain computationally simple questions. They often do not reflect the reality of the kinds of real problems you come across. The reason for this is that it is assumed that you will do them manually with, at best, the use of a calculator. In this course, we are changing that assumption. We assume you want to use what you are learning. Therefore, you will use proper tools and can look at practical problems and go beyond simplistic problems couched to make manual solutions possible. One great enabler in doing things is **appropriate software**. In this course, we will be using two kinds of software. These are

Symbolic & Computations software, and Graphics, Simulations & Design software. We spend the rest of this chapter giving some guidelines to what we shall use.

For Symbolics & Computations, we will support $Mathematica^{©}$ by Wolfram Research. It is possible to survive the course without this software but, compared to someone who understands how to properly use it, you will be like a person walking from Lagos to Ibadan compared to someone travelling in a motor vehicle. Of course, if you do not get kidnapped on the way, you may eventually get there. But the difference is not trivial. Get yourself a copy of this software in order to do well in this course and get yourself ready for serious engineering computations in the modern way.

Licenses can be obtained for as low as two hundred US dollars ($200.00) per user if you try to get it as group of students. Those who have the single-board computer called Raspberry Pi are lucky because $Mathematica^{©}$ version 10 is already installed. Anything higher than version 9 is good enough for our use. Earlier versions are tolerable but get the more current one if you can.

Another software that can match $Mathematica^{©}$ in the ability for Symbolic Algebra is called Maple. We will later support Maple with the next version of this material. Unfortunately, we may not be able to help you directly in the sense that all the code examples we shall give will be in Mathematica. A good Maple user will have little difficulty translating or writing her own code.

For Graphics, Simulations and Design, we will use Fusion 360 by Autodesk. There are several competing software that can do similar things as Fusion 360. Our reasons for selecting this among others are as follows:

1. It is the most modern in the Autodesk stable for additive and subtractive manufacturing (3D Printing and CNC Machine support). We expect these courses to lead directly to product design and prototyping. It is better to become familiar with the current software as early in the process, as possible. Other Autodesk software such as Autodesk Inventor, AutoCAD, etc. are also OK. But we shall support Fusion 360 in the sense that we give examples and practical guides when necessary using that platform.
2. The second reason is that you are entitled to a fully licensed full version of the software if you remain a student. Go to their website and register as a student, they will direct you on how to get a copy.

3. We recommend that you get yourself a computer with at least 8GB RAM and, if possible, a graphics co-processor such as NVDIA series. Those who cannot afford top of the line computers need not despair. Autodesk allows you to run your simulations in the cloud. That means that your computer will mainly be used to do your design while the computationally intensive discretization or simulations will take place on their own computers using cloud credits. Again, these cloud credits are given to you free of charge. It is your responsibility to ensure you have enough data facility to use these credits.

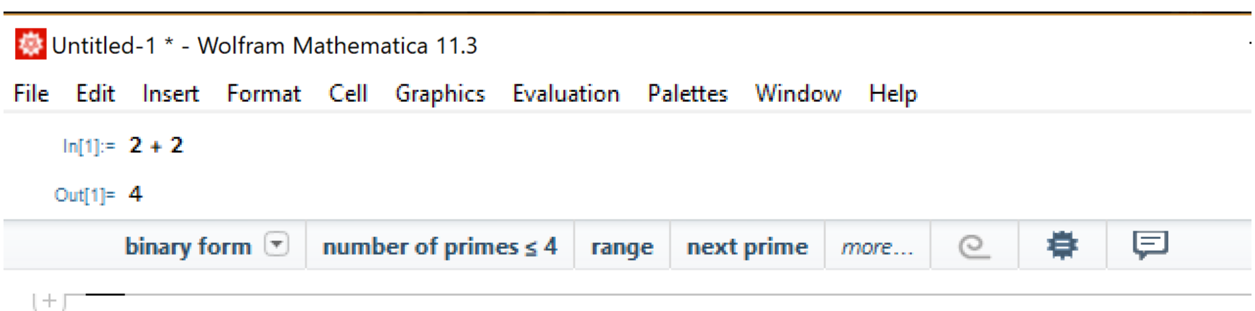
As in the case with the Symbolics and Computations software, there may be those that prefer other software. They are welcome. But **do not plan to go through this course manually**. If you are a poor student, go and sell all you have and get the correct equipment to move yourself out of poverty! Trying to get grades in this course without knowing what to do with its contents is a waste of your youth. Don't try it! It will not work!

We present a quick introduction to Mathematica next. It is NECESSARY to have the software running and not just read the notes. Each Mathematica Installation is loaded with enormous documentation and Help system. This vary from version to version. Our work requires version 9 and later. Current version at the time of writing this is 12.0.

Mathematica

Introduction

Perhaps the easiest way to begin using Mathematica is to start a new Notebook file. You can start using Mathematica Right away by typing



Upon launching the software, in the first line, you type 2*2. The Prefix: In[1]= is generated by the

Mathematica Notebook Environment. It tells you that this statement is an input made by you. In the example above, the next line is an Output: A Response by Mathematica. The happened because I held down the Shift Key and pressed Enter at the same time. That is the way to tell Mathematica to execute your input. Now take a look at another set of lines:

```
In[2]:= 2 + 2
        3 + 3
        5 * 5

Out[2]= 4

Out[3]= 6

Out[4]= 25
```

In this case, I typed 2+2 Pressed Enter, 3+3 Again Pressed Enter, 5*5 and at last pressed Shift and Enter together. It was only at the last point that Mathematica realized I wanted to execute the statements I had typed. It executed them one-by-one and gave me a list of results.

In the next example, I typed the same lines as above. At the end of each statement, I typed a semicolon. See what happens:

```
In[5]:= 2 + 2;
        3 + 3;
        5 * 5

Out[7]= 25
```

Looks like the earlier case, everything was executed but only the result of the last statement is shown. And even that is because I did not add a semicolon to it. This shows that we can suppress the result of statements with the semicolon. We could have placed the statements on the same line and get the same effect:

```
In[8]:= 2 + 2; 3 + 3; 5 * 5

Out[8]= 25
```

Functions & Conventions.

Mathematica contains all the elementary functions you are already familiar with: Trigonometric functions, Exponential and Logarithmic Functions, Hyperbolic functions. In addition, it contains, built in, perhaps all the special functions you will likely need: Gamma Function, Error function, etc. are all included. Its commands and operators including control structures are all available as functions. Before we look at specific examples, observe the important issue of notation:

ALL built in functions, constructs and structures are functions with capitalized first letters. Consequently, Mathematica does NOT recognize that you want the following trigonometric functions: $\sin x$, $\cos x$, $\arcsin \vartheta$. Instead, you will have to type

```
In[9]:= x = Pi / 3;

In[11]:= Sin[x]

Out[11]= √3
         ──
         2
```

In addition to functions, it also contains constants and other scientific quantities that you may need. The Greek symbol π, for example, can be invoked by typing Pi but the first letter MUST be capitalized. So MUST you capitalize the "S" in Sin as well as use SQUARE Brackets for collecting the function arguments. Mathematica is very insistent on the kind of bracket delimiters used. Only the Square brackets are recognized as function delimiters.

This has the consequence that you can define your variables using any names you want. You can be sure that Mathematica is not using the name if you start your own with a lowercase letter.

Another observation here is that, whenever possible, unless you override it, Mathematica will work in closed form and preserve full accuracy. In the above trigonometric example, we could force a decimal output by using the Numeric call as follows:

```
In[12]:= N[Sin[x], 6]

Out[12]= 0.866025

In[13]:= N[Sin[x], 60]

Out[13]= 0.866025403784438646763723170752936183471402626905190314027903
```

In the first call, we requested for a numerical rather than a symbolical output by using the $N[\blacksquare]$ function. This function can be called by one or two arguments. Here we called by two arguments: in the first case, we asked for six figures, in the second case we wanted 60 figures. Mathematica has no difficulties giving us any number of figures we request. In the next example, we request for the same operation numerically but for 600 figures:

```
In[14]:= N[Sin[x], 600]

Out[14]= 0.866025403784438646763723170752936183471402626905190314027903489725966508454400018540573093378624287837813070707703351514984972547499476239405827756047186824264046615951152791033987410050542337461632507656171633451661443325336127334460918985613523565830183930794009524993268689929694733825173753288025378309174064803050473801093595162541572914761979916498894912254144357231916458673612081992293927698833979031909176833055421586890447189158051044152762450835011760355572144347995478182898543584249036449746648242141510393204301994369348768791158658915697996491503919351438526956684781656051853632009625
```

The following are some functions with the way Mathematica interprets them:

Input	Interpretation	Comments
Sin [x]	$\sin x$	Ensure to Capitalize first letter and use square brackets
Integrate[a x^2,x]	$\int a\,x^2 dx$	Indefinite integral. It is still necessary to let Mathematica know which variable you are integrating with respect to after the comma.
Integrate[a x^2,$\{x, 0, 1\}$]	$\int_0^1 a\,x^2 dx$	Definite integral. The range is a list showing the variable of integration, beginning and end of domain.
Log[x,b]	$\log_x b$	
x y	$x \times y$	The space tells Mathematica you are multiplying the two variables whether you have declared them to be so or not.
TensorProduct[u,v]	$\mathbf{u} \otimes \mathbf{v}$	Tensor Product of two vectors. Mathematica expects the vectors to be defined as a list of numbers.

It may surprise those who already know how to program in a High-Level language that Mathematica does not insist that a variable be defines before usage. It can always treat it as an undefined symbol and do the necessary arithmetic in closed form whenever it can. The next two examples will demonstrate this:

```
In[15]:= Integrate[a y^2, y]

Out[15]= a y^3
         -----
           3

In[17]:= M = {{a, b}, {c, d}}

In[20]:= {{a, b}, {c, d}}
         Inverse[M] // MatrixForm

Out[20]= {{a, b}, {c, d}}

Out[21]//MatrixForm=
```

$$\begin{pmatrix} \frac{d}{-b\,c+a\,d} & -\frac{b}{-b\,c+a\,d} \\ -\frac{c}{-b\,c+a\,d} & \frac{a}{-b\,c+a\,d} \end{pmatrix}$$

In In[15], the indefinite integral, $\int a\, y^2 dy = \frac{ay^3}{3}$ as expected despite the fact that y is a variable throughout. This is a symbolic as opposed to a numerical operation. Most of the programming you have probably done were in numerical computations. Mathematica is capable of both numerical and symbolic computations. In In[20], we defined a matrix as a list of undefined variables, a, b, c and d. We then proceeded to invert this matrix. Again, Mathematica performed the symbolic operations as expected.

List Processing

We go a little bit further in list processing here. As we have already seen, matrices, tensors, tables, etc. are all treated as lists in Mathematica. When Mathematica is expecting more than one input as an argument or to be supplied to a process, the usual way is to represent these as lists. Arguments of functions may expect lists or single variables; the context and the particular function will determine that. Usually, a list is specified inside curly brace delimiters separated by commas. There can be a single list, there can be a lists of lists, etc.

```
In[22]:= M = {{1, 2, 3}, {4, 5, 6}, {1, -1, 7}}
         MatrixForm[M]
         M[[2]]
         M[[2]][[1]]
         Tr[M]

Out[22]= {{1, 2, 3}, {4, 5, 6}, {1, -1, 7}}
```

Here the matrix is typed in as a lists of lists. Each sub list is a row of elements, is separated by a comma just the same way as the scalar elements, the numbers, in each of the simple lists. Consider the outputs of the other commands:

Out[23]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & -1 & 7 \end{pmatrix}$$

```
Out[24]= {4, 5, 6}

Out[25]= 4

Out[26]= 13
```

The matrix form of the input matrix puts it in the familiar matrix format. We can index into the list of lists. The first index looks at the lists in the list, while the second index addresses the elements of the inner list. M[[2]][[1]] is for the first element in the second sub list as can be seen. The last function takes the trace of the trace of the square matrix. It is the sum of the diagonal elements. In this case $1 + 5 + 7 = 13$ as the answer given shows.

Assignment and Equality Signs

Mathematica treats the arithmetic equality "=" in the same way most other programming languages do. It is a good thing to note – especially if you have not programmed before that the expression,

$$a = b \tag{96}$$

in Mathematica, as it is in most programming languages **does NOT** mean the same thing that you have been used to in arithmetic or mathematics. It is NOT an equality; rather, it is an assignment. It would have been more proper to have allowed it to be (as in APL and some other matrix languages)

$$a \leftarrow b \tag{97}$$

The latter expresses the intention of the former in a clearer way: Look at the variables a and b as storage locations in the computer. What you are doing here is that an assignment in equation 25 makes a copy of what is in location identified by the left-hand side identifier, b, and places them, overwriting whatever was in a. Usually, whereas, the left hand of an assignment is a single identifier, as a is in this case, the right side could be anything that can result in a value: an expression, an identifier, a computation, a function returning a value. Whatever it is, once the expression on the RHS is computed, the result is assigned to a. Equation 26 expresses what is happening more clearly even though Equation 25 is the way Mathematica signifies an assignment.

How then do you write an equality sign? Simply with a double equality sign! Once you see "==", it is an equality sign carrying the same meaning as your "=" in usual mathematics and other subjects.

```
In[1]:= Solve[{a x + b y == 1, x - y == 2}, {x, y}]

Out[1]= {{x -> -(-1 - 2 b)/(a + b), y -> -(-1 + 2 a)/(a + b)}}
```

In[1] above is the Solve function. It is used to solve simple equations. Its arguments are lists: a list of the equations to be solved, followed by a list of variables you want to solve for. Here, as you can see, the equations are:

$$ax + by = 1, x - y = 2.$$

In the second list argument, you have a list of variables, here, x, y. The answer is provided by Mathematica in yet another list. Here we are given the values

$$x = \frac{1 + 2b}{a + b}, y = \frac{1 - 2a}{a + b}$$

It will be an error to have used the single "=" in the expression of the equations. Equality requires the double "==", while the single "=" is reserved for the assignment operation as we have explained.

Numerical Types

All numbers are not equal Programming languages distinguish between numbers for reasons of efficiency. Computations you want to carry out can have several possible ways. For example, compare the two computations:

$$2^3, 2^{3.1}$$

Will call for different methods. The way most programming languages work out which algorithm to call for is the kind of number the exponent here is. If it is an integer, or if it is a real number. The type of number also tells how big it is allowed to become in a program when variables take different values. It is therefore good to be clear which number to use.

Mathematica supports the following number types:

Integer, Real, Complex and Rational. Rational numbers are quotients (unevaluated) of integers. As much as possible, Mathematica will like to work at the highest precision and thus keep numbers as close to the highest possible precision they can get. Consider for example, this matrix:

```
In[21]:= M = {{1, 2}, {2, 3}};
         Eigenvalues[M]
         NM = {{1., 2.}, {2., 3.}};
         Eigenvalues[NM]

Out[22]= {2 + √5 , 2 - √5 }

Out[24]= {4.23607, -0.236068}
```

You can see that in the computations here, the same figures are used for the two matrices, while the first preserves the precision by using integers, the other gives the eigenvalues in real numbers. The function Head[], when used to pick the numbers show they are treated as integer and reals respectively as shown below:

```
In[27]:= Head[M[[1]][[2]]]
Out[27]= Integer

In[28]:= Head[NM[[1]][[2]]]
Out[28]= Real
```

Two points here:

1. The type of number you want can be known to Mathematica jut by the way you write them. For example, writing 2. instead of 2 tells Mathematica that the number is a real type even though it could have been represented by an integer. If a matrix list contains a real number, Mathematica can thenceforth use Real arithmetic instead of Integer to respond to actions on the matrix (Det[], Tr[], Eigenvalues[], etc.)
2. You can find the underlying data type that Mathematica is assuming by simply using the Head [] function with the variable concerned as argument.

```
In[1]:= 9 × 5 × 3; Times[9, 5, 3]
Out[1]= 135

In[2]:= Divide[Sin[x], x]
Out[2]= Sin[x]/x
```

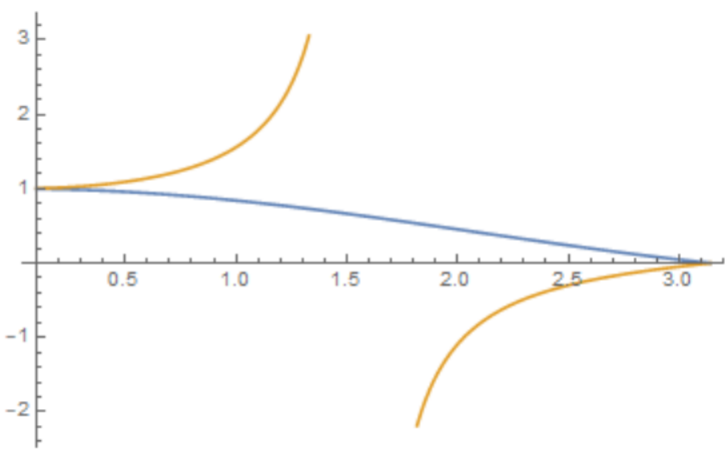
In[1] above is one way to multiply three numbers. This can be done either by typing the operator directly, or if you are using newer versions of the software, it can help you to add the times by itself. The second method is the function call. The function Times[], does multiplication. It can take many arguments as we can see. Division, addition, and virtually any regular operator connects to a specific function that provides an alternative way to get perform the operation.

Simple Graphing

Consider the Plot[] function. We pass a list of functions to be plotted and another list telling Plot the domain of interest. The latter contains the variable, start and end point:

```
In[3]:= Plot[{Sin[x] / x, Tan[x] / x}, {x, 0.1, Pi}]
```

Out[3]=



This shows that both $\frac{\sin x}{x}$ as well as $\frac{\tan x}{x}$ tend to unity as their arguments tend to zero. One comes from below, the other from above. We selected the initial points here to avoid a zero division; Mathematica does not care! It will automatically take the asymptotic values and it is too smart to divide by zero if there are superior interpretations such as limiting values as we have here.

Precision and accuracy

As we have already seen, symbolic computations keep exact values. Mathematica, whenever possible will work in this mode unless you deliberately override it. There are occasions when you want to specify the amount of precision you want. The precision of a computation is about the total number of digits in its decima representation; accuracy is about the number of places after the decimal. When we have converted a large number to the exponent form, we are dealing with accuracy even with the places after the decimal. The following code is instructive:

```
In[10]:= Table[With[{x = 10^(r / 3) + 3 / 53}, {N[x, 12], N[x, {Infinity, 10}]}], {r, 0, 6}] //
          TableForm

Out[10]//TableForm=
         1.05660377358    1.056603774
         2.21103846362    2.211038464
         4.69819260720    4.698192607
         10.0566037736    10.0566037736
         21.6009506739    21.600950674
         46.4724921097    46.472492110
         100.056603774    100.056603774
```

The Help System, Documentation Center.

Are you confused about the above use of Table? Or about anything else? The Mathematica Installation you are using has an elaborate help system that will require a whole book to explain. Keep matters simple: just begin to use the system! To get better assistance on any command, simply type double question marks and name the keyword as in:

```
In[11]:= ?? Table
```

Symbol

Out[11]=

Table[*expr*, *n*] generates a list of *n* copies of *expr*.

Table[*expr*, {*i*, i_{max}}] generates a list of the values of *expr* when *i* runs from 1 to i_{max}.

Table[*expr*, {*i*, i_{min}, i_{max}}] starts with $i = i_{min}$.

Table[*expr*, {*i*, i_{min}, i_{max}, *di*}] uses steps *di*.

Table[*expr*, {*i*, {i_1, i_2, ...}}] uses the successive values i_1, i_2, ...

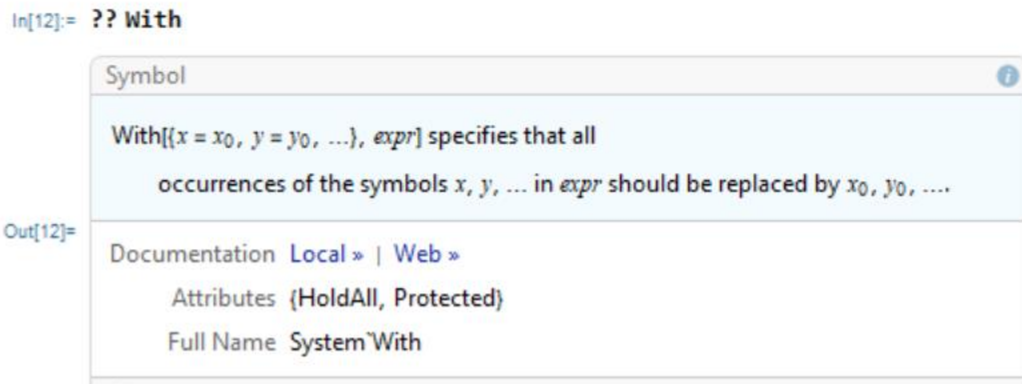
Table[*expr*, {*i*, i_{min}, i_{max}}, {*j*, j_{min}, j_{max}}, ...] gives a nested list. The list associated with *i* is outermost.

Documentation Local » | Web »

Attributes {HoldAll, Protected}

Full Name System`Table

This gives you a listing of the many ways you can call the function Table[]. The next listing helps with the function With[]:



Still not satisfied? Then go directly to the documentation Center. The last Drop Menu to your right is “Help”. It leads directly to the Documentation Center – the first among the options you will see. Type Table. You will be rewarded with a full screen of assistance. This includes full examples you can run directly, in situ! You don’t even need to copy them! You can even edit and run them as many times as you like.

The longer time you spend in the documentation Center, the more familiar and competent with the software environment you become. This is often superior to buying more books on the subject.

Numerical Example: Rotation

Given the vectors $\boldsymbol{\xi}_1 = 0.843394\mathbf{e}_1 + 0.389796\mathbf{e}_2 + 0.369791\mathbf{e}_3$, $\boldsymbol{\xi}_2 = -0.275206\mathbf{e}_1 + 0.904508\mathbf{e}_2 - 0.325769\mathbf{e}_3$, and $\boldsymbol{\xi}_3 = -0.461463\mathbf{e}_1 + 0.172983\mathbf{e}_2 + 0.870132\mathbf{e}_3$, where $\mathbf{e}_1 = \{1,0,0\}; \mathbf{e}_2 = \{0,1,0\}; \mathbf{e}_3 = \{0,0,1\}$ are the Cartesian basis vectors. Show that

(a) $\boldsymbol{\xi}_i, i = 1,2,3$ is an orthonormal, right-handed set of vectors.

(b) Show that of the matrix, tensor $\mathbf{Q} = \boldsymbol{\xi}_i \otimes \mathbf{e}_i$ rotates vectors along the direction of the vector, $\mathbf{v}_1 = 3\mathbf{e}_1 + 5\mathbf{e}_2 - 4\mathbf{e}_3$

(c) Use any two vectors to show that the rotation angle of this tensor is $\frac{\pi}{5}$

(d) Show that this result is independent of the vectors chosen. [Hint. Use another set of vectors and see the angle is the same – hence a property of the rotation tensor]

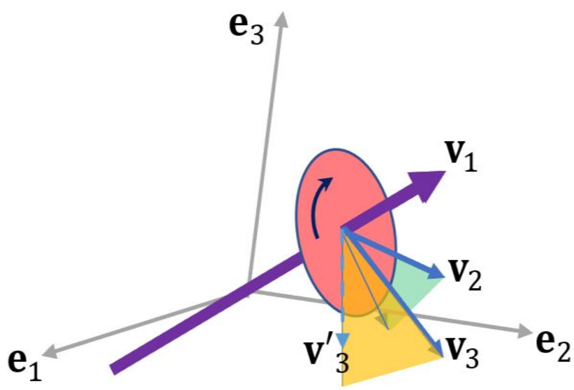


Figure 20. Rotation Tensor

```
(* Question Data Entry |
 Results Display Suppressed Using Semicolons *)
v1 := {3., 5., -4.};
e1 = {1, 0, 0}; e2 = {0, 1, 0}; e3 = {0, 0, 1};
ξ1 = {0.843394, 0.389796, 0.369791};
ξ2 = {-0.275206, 0.904508, -0.325769};
ξ3 = {-0.461463, 0.172983, 0.870132};
(* Two Arbitrary Vectors *)
v2 = {2, 1, -4};
v3 = {-10, 0, 15};
```

Solution.

Figure 14 shows two arbitrary vectors, $\mathbf{v}_2$ and $\mathbf{v}_3$ projected to a flat disc as $\mathbf{v}'_2$(not shown) and $\mathbf{v}'_3$. The vector $\mathbf{v}_1$ is along the normal to the disk. The solution to this problems rests on working with dyad products. We are going to use the result, that given any vector, $\mathbf{v}$

$$(\boldsymbol{\xi}_i \otimes \mathbf{e}_i)\mathbf{v} = (\mathbf{e}_i \cdot \mathbf{v})\boldsymbol{\xi}_i$$

Apart from that, much of what is left is computational tedium and drudgery. Here is where Mathematica comes in handy. If the rotation is along an axis, the vector along that axis will not be changed when this tensor is applied to it.

```
In[5]:= (* Test Autonormality *)
        Norm[ξ1]
        Norm[ξ2]
        Norm[ξ3]
        Dot[ξ1, ξ2]
        Dot[ξ2, ξ3]
        Dot[ξ3, ξ1]

In[11]:= (* Test Right Handedness *)
        Cross[ξ1, ξ2] (* Compare to ξ3 *)
        Cross[ξ2, ξ3] (* Compare to ξ1 *)
        Cross[ξ3, ξ1] (* Compare to ξ2 *)
```

Let $\mathbf{v}_2 = 2\mathbf{e}_1 + \mathbf{e}_2 - 4\mathbf{e}_3$ and $\mathbf{v}_3 = -10\mathbf{e}_1 + 15\mathbf{e}_3$. Let $\mathbf{v}_1$ be normal to the disc shown such that the projections of $\mathbf{v}_2$ and $\mathbf{v}_3$ are $\mathbf{v}'_2$ and $\mathbf{v}'_3$ respectively.

From In[5] we can see that $\boldsymbol{\xi}_i, i = 1,2,3$ is an orthonormal set because each is of unit magnitude and the fact that

$$\boldsymbol{\xi}_1 \cdot \boldsymbol{\xi}_2 = 0;\ \boldsymbol{\xi}_2 \cdot \boldsymbol{\xi}_3 = 0;\ \boldsymbol{\xi}_3 \cdot \boldsymbol{\xi}_1 = 0.$$

To demonstrate right handedness, look at the cross products and observe that,

$$\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2 = \boldsymbol{\xi}_3;\ \boldsymbol{\xi}_2 \times \boldsymbol{\xi}_3 = \boldsymbol{\xi}_1;\ \boldsymbol{\xi}_3 \times \boldsymbol{\xi}_1 = \boldsymbol{\xi}_2$$

The tensor products to derive $\mathbf{Q} = \boldsymbol{\xi}_i \otimes \mathbf{e}_i$ is coded as follows:

```
In[14]:= (* Test for Axis of Rotation *)
        Q = TensorProduct[ξ₁, e₁] + TensorProduct[ξ₂, e₂] + TensorProduct[ξ₃, e₃];
        Q.v₁
```

The fact that $\mathbf{Q}$, as defined, rotates around $\mathbf{v}_1$is shown by the fact that, $\mathbf{Q}\mathbf{v}_1 = \mathbf{v}_1$. It makes sense that any vector along the axis of rotation remains unchanged by the rotation.

```
In[16]:= (* Test for Angle of Rotation *)
        y₂ = Normalize[Cross[v₁, v₂]]; y₃ = Normalize[Cross[v₁, v₃]];
        (* Rotate y₂,y₃ *)
        w₂ = Q.y₂; w₃ = Q.y₃;
        α₁ = ArcCos[Dot[y₂, w₂]]
        α₂ = ArcCos[Dot[y₃, w₃]]
```

We cannot find the relevant angle of rotation of the vectors by examining $\mathbf{v}_2$ and $\mathbf{v}_3$. They are selected arbitrarily, therefore, are not guaranteed to be on the same plane and are not perpendicular to the axis of rotation. In order to find vectors along the plane of the disk shown, we observe that the normalized cross products:

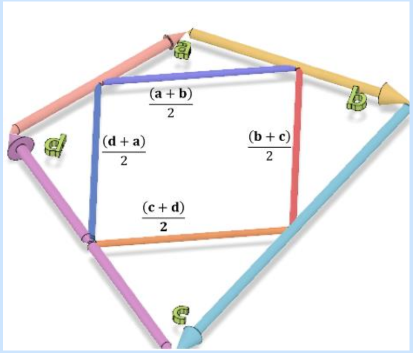
$$\mathbf{y}_2 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{\|\mathbf{v}_1\|\|\mathbf{v}_2\|}, and\ \mathbf{y}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_3}{\|\mathbf{v}_1\|\|\mathbf{v}_3\|}$$

must lie on the plane of the disk shown as they are perpendicular to $\mathbf{v}_2$ and $\mathbf{v}_1$, and $\mathbf{v}_3$ and $\mathbf{v}_1$respectively. We compute the rotation on the unit vectors $\mathbf{y}_2$ and $\mathbf{y}_3$ to obtain $\mathbf{w}_2$ and $\mathbf{w}_3$. Cosines of the angles between unit vectors $(\mathbf{y}_2, \mathbf{w}_2)$ and $(\mathbf{y}_3, \mathbf{w}_3)$, obtained by taking their scalar products, give us the angle of rotation. The value shown is the Cosine of $\frac{\pi}{5}$.

The last part is given as an exercise.

Solved Problems 1.1

1.1	Given that vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ form a closed circuit, show that the vectors joining their midpoints form a parallelogram



In the picture, $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}=0$. Clearly,

$$\frac{(\mathbf{a}+\mathbf{d})}{2}=-\frac{(\mathbf{b}+\mathbf{c})}{2}$$

also,

$$\frac{(\mathbf{a}+\mathbf{b})}{2}=-\frac{(\mathbf{c}+\mathbf{d})}{2}$$

Opposite sides of the lines joining the midpoints are parallel. This is a parallelogram.

1.2	Given that **a** and **b** are vectors, use indicial notation to show that $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$
	$\mathbf{a}\times\mathbf{b}=e_{ijk}a_j b_k \mathbf{e}_{\boldsymbol{i}}=-e_{ikj}b_k a_j \mathbf{e}_i=-\mathbf{b}\times\mathbf{a}$
1.3	Given that **a** and **b** are vectors, use indicial notation to show that $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\mathbf{0}$

$$\begin{aligned}\mathbf{a}\times\mathbf{b}&=e_{ijk}a_j b_k \mathbf{e}_k\\(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}&=\left(e_{ijk}a_j b_k \mathbf{e}_k\right)\cdot(a_\alpha \mathbf{e}_{\boldsymbol{\alpha}})\\&=e_{ijk}a_j b_k a_\alpha \mathbf{e}_{\boldsymbol{i}}\cdot\mathbf{e}_{\boldsymbol{\alpha}}\\&=e_{ijk}a_j b_k a_\alpha \delta_{i\alpha}\\&=e_{ijk}a_j b_k a_i\\&=-e_{jik}a_j b_k a_i\\&=-e_{ijk}a_j b_k a_i=0\end{aligned}$$

The expression in symmetrical in i and j, it is also anti-symmetrical in the same two indices at the same time. The same situation occurs on the RHS.

1.4	Show that (a) $(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b})=\|\mathbf{a}\|^2-\|\mathbf{b}\|^2$, and that $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b})=-\mathbf{2a}\times\mathbf{b}$

a	(a) Opening the parentheses,

(a) Opening the parentheses,

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\cdot\mathbf{a}-\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= \mathbf{a}\cdot\mathbf{a}-\mathbf{b}\cdot\mathbf{b}\\ &= ||\mathbf{a}||^2-||\mathbf{b}||^2\end{aligned}$$

(b) Similarly,

$$\begin{aligned}(\mathbf{a}+\mathbf{b})\times(\mathbf{a}-\mathbf{b}) &= \mathbf{a}\times\mathbf{a}-\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}-\mathbf{b}\times\mathbf{b}\\ &= -\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{a}=-\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{b}\\ &= -2\mathbf{a}\times\mathbf{b}\end{aligned}$$

1.5 Given that $\forall\ \mathbf{v}\in\mathbb{E}, \mathbf{a}\cdot\mathbf{v}=\mathbf{b}\cdot\mathbf{v},$ Show that $\mathbf{a}=\mathbf{b}$

We are given that $\forall\ \mathbf{v}\in\mathbb{E}, \mathbf{a}\cdot\mathbf{v}=\mathbf{b}\cdot\mathbf{v}$ this implies,

$$\mathbf{a}\cdot\mathbf{v}-\mathbf{b}\cdot\mathbf{v}=(\mathbf{a}-\mathbf{b})\cdot\mathbf{v}=0$$

Define the vector $\mathbf{c}\equiv\mathbf{a}-\mathbf{b}$. The equation becomes,

$$\mathbf{c}\cdot\mathbf{v}=||\mathbf{c}||||\mathbf{v}||\cos\theta=0.$$

Because v can be any vector, it does not have to be perpendicular to $\boldsymbol{c}$ and we can rule out the trivial case of its being the zero vector. This leaves us with the only choice that $||\mathbf{c}||=0$. And, the only vector that has zero magnitude is the zero vector. So that,

$\mathbf{c}\equiv\mathbf{a}-\mathbf{b}=\mathbf{o}$, or $\mathbf{a}=\mathbf{b}$.

1.6 Given that for any vector $\mathbf{v}, \mathbf{a}\times\mathbf{v}=\mathbf{b}\times\mathbf{v},$ Show that $\mathbf{a}=\mathbf{b}$

We are given that $\forall\ \mathbf{v}\in\mathbb{E}$, $\mathbf{a}\times\mathbf{v}=\mathbf{b}\times\mathbf{v}$, Now take a dot product with $\mathbf{a}$, we have that,

$$\mathbf{a}\cdot\mathbf{b}\times\mathbf{v}=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{v}=0=\mathbf{o}\cdot\mathbf{v}$$

for all $\mathbf{v}$ proving that $\mathbf{a}\times\mathbf{b}=\mathbf{o}$. This shows that $\mathbf{a}$ and $\mathbf{b}$ are collinear. We can therefore write that $\mathbf{b}=\alpha\mathbf{a}$

Hence, $\mathbf{a}\times\mathbf{v}=\mathbf{b}\times\mathbf{v}=\alpha\mathbf{a}\times\mathbf{v}$ where α is a scalar. So that

$$(\mathbf{a}\times\mathbf{v})(1-\alpha)=\boldsymbol{o}\Rightarrow 1=\alpha$$

showing that $\mathbf{a}=\mathbf{b}$ as was required.

1.7 Identify all the equations contained in the expression $e_{ijk}T_{jk}=0$

For each free index, there is an equation:

$$i = 1 \Rightarrow e_{1jk}T_{jk} = e_{123}T_{23} + e_{132}T_{32} = T_{23} - T_{32} = 0 \Rightarrow T_{23} = T_{32}$$

$$i = 2 \Rightarrow e_{2jk}T_{jk} = e_{213}T_{13} + e_{231}T_{31} = T_{13} - T_{31} = 0 \Rightarrow T_{13} = T_{31}$$

$$i = 3 \Rightarrow e_{3jk}T_{jk} = e_{312}T_{12} + e_{321}T_{21} = T_{12} - T_{21} = 0 \Rightarrow T_{12} = T_{21}$$

Notice that this same expression could have been written in the full invariant form:

$$e_{ijk}T_{jk}\mathbf{e}_i = \mathbf{o}$$

In this form, there is no free index, all indices are dummy. Notice that the RHS is a vector zero. Strictly speaking, this can be fully expanded to

$$\begin{aligned} e_{ijk}T_{jk}\mathbf{e}_i &= 0\mathbf{e}_1 + 0\mathbf{e}_2 + 0\mathbf{e}_3 = \mathbf{o} \\ &= e_{1jk}T_{jk}\mathbf{e}_1 + e_{2jk}T_{jk}\mathbf{e}_2 + e_{3jk}T_{jk}\mathbf{e}_3 \end{aligned}$$

which are three equations:

$$e_{1jk}T_{jk} = 0, e_{2jk}T_{jk} = 0, e_{3jk}T_{jk} = 0.$$

And this is the meaning of the single equation,

$$e_{ijk}T_{jk} = 0$$

where the free index facilitates the production of the three equations. It follows that either form of writing gives us the same set of equations if correctly interpreted. That is one reason why we should be careful to note whether the zero we are dealing with is a scalar zero, a vector zero or a tensor zero. It matters!

1.8	Given that $\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ Write the equation $\mathbf{AA}^{\mathrm{T}} = \mathbf{I}$ in indicial notation.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}\begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}a_{11} + a_{12}a_{12} + a_{13}a_{13} & a_{11}a_{21} + a_{12}a_{22} + a_{13}a_{23} & a_{11}a_{31} + a_{12}a_{32} + a_{13}a_{33} \\ a_{21}a_{11} + a_{22}a_{12} + a_{23}a_{13} & a_{21}a_{21} + a_{22}a_{22} + a_{23}a_{23} & a_{21}a_{31} + a_{22}a_{32} + a_{23}a_{33} \\ a_{31}a_{11} + a_{32}a_{12} + a_{33}a_{13} & a_{31}a_{21} + a_{32}a_{22} + a_{33}a_{23} & a_{31}a_{31} + a_{32}a_{32} + a_{33}a_{33} \end{pmatrix}$$

Observe that in each cell, every term maintains the cell's row and column numbers in their first term.

$$a_{(row\ no)1}a_{(col\ no)1} + a_{(row\ no)2}a_{(col\ no)2} + a_{(row\ no)3}a_{(col\ no)3}$$

Is true of EVERY cell in the above array! Look at it closely! It is also clear that we are summing over the second number in each term. Is it not clear that we can gain a significant amount of space if we simply writing, for the i^{th} and j^{th} column,

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3}$$

And is it not obvious that this can be written, using the summation convention as,

$$a_{i1}a_{j1} + a_{i2}a_{j2} + a_{i3}a_{j3} = a_{i\alpha}a_{j\alpha} = a_{ik}a_{jk}$$

On the right hand side, the identity matrix is,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}$$

Observe that anywhere the row and column numbers are the same, the value is 1. When they are not, the value is zero. So that the typical element is δ_{ij}. Hence, we can write the equation as,

$$a_{ik}a_{jk} = \delta_{ij}$$

1.9 Show that a transformation of every vector to the vector $4\mathbf{e}_2 + \mathbf{e}_1$cannot be a tensor.

Let us first transform an arbitrary vector $\mathbf{u}$;

$$\mathbf{Tu} = 4\mathbf{e}_2 + \mathbf{e}_1$$

For any scalar α, let us also transform $\alpha\mathbf{u}$, since $\alpha\mathbf{u}$ is a vector, this transformation, since it transforms every vector the same way, transforms to

$$\mathbf{T}(\alpha\mathbf{u}) = 4\mathbf{e}_2 + \mathbf{e}_1 = \mathbf{T} \neq \alpha\mathbf{Tu}$$

Hence, it is not a linear transformation. A tensor is a linear transformation of a vector to another vector. The transformation is NOT a tensor.

1.10 Can a transformation of every vector to the zero vector be a tensor? Why?

Let us transform an arbitrary vector $\mathbf{u}$;

$$\mathbf{Tu} = \mathbf{o}$$

For any scalar α, let us also transform $\alpha\mathbf{u}$, since $\alpha\mathbf{u}$ is a vector, this transformation, since it transforms every vector the same way, transforms to

$$\mathbf{T}(\alpha\mathbf{u}) = \mathbf{o} = \alpha\mathbf{Tu}$$

Hence this transformation is a tensor: It transforms linearly, and from tensor to tensor. It is the Annihilator Tensor.

1.11 Explain the terms, **Vector Cross**, Dual Vector, Deviatoric Tensor, Spherical Tensor.

Vector Cross is a tensor that operates on a vector, yielding the same vector result that would have been obtained were there to have been a cross product on that vector.

For any skew tensor, a Dual vector is a vector that is a vector cross of the tensor.

	A Deviatoric tensor is what remains after subtracting the spherical part of the tensor from the tensor A Spherical Tensor is a tensor that has the value zero in each non-diagonal element. The diagonal elements are of equal value. It follows that the tensor can be written in the form, $\alpha\mathbf{I}$ where α is a scalar, and $\mathbf{I}$ is the identity tensor.
1.12	Given that the vector cross formula is $(\mathbf{u}\,\times) = e_{ijk}u_j\mathbf{e}_i \otimes \mathbf{e}_k$, Find the vector cross of $\mathbf{u} = 4\mathbf{e}_2 + \mathbf{e}_1 - 3\mathbf{e}_3$. Is it a deviatoric tensor? Why?
	Let $\mathbf{\Omega} = (\mathbf{u}\,\times) = e_{ijk}u_j\mathbf{e}_i \otimes \mathbf{e}_k$ $i = 1, k = 2 \Rightarrow \Omega_{12} = e_{132}u_3 = -u_3 = 3$ $i = 1, k = 3 \Rightarrow \Omega_{13} = e_{123}u_2 = u_2 = 1$ $i = 2, k = 3 \Rightarrow \Omega_{23} = e_{213}u_1 = -u_1 = -4$ $[\Omega_{ij}] = \begin{pmatrix} 0 & 3 & 1 \\ -3 & 0 & -4 \\ -1 & 4 & 0 \end{pmatrix}$ The trace of a skew tensor is zero. It has no spherical part. Hence, the skew tensor is deviatoric.
1.13	Given $[S_{ij}] = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 3 & 0 & 3 \end{bmatrix}$ and $[a_i] = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ evaluate (a) S_{ii}, (b) $S_{ji}S_{ji}$, (c) $S_{jk}S_{kj}$, (d) $a_m a_m$, (e) $S_{mn}a_m a_n$, (f) $S_{nm}a_m a_n$
a	Because the subscript index is repeated, summation is implied for the full range of acceptable values – that is, 1,2 and 3; therefore, $S_{ii} = S_{11} + S_{22} + S_{33} = 1 + 1 + 3 = 5$.
b	In this case, two different indices are repeated. There is summation on both of them. To get it right, we must apply such one by one. We do it, starting with the first index, i, and later, after that is fully completed, we take the second index j, as follows: $S_{ij}S_{ij} = S_{1j}S_{1j} + S_{2j}S_{2j} + S_{3j}S_{3j} = S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{2j}S_{2j} + S_{3j}S_{3j}$ $= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} + S_{3j}S_{3j}$ $= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} + S_{31}S_{31} + S_{32}S_{32} + S_{33}S_{33}$ $= 1 \times 1 + 0 \times 0 + 2 \times 2 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 0 + 3 \times 3 = 28$
c	Proceeding as in the earlier two examples, we write, $S_{jk}S_{kj} = S_{1k}S_{k1} + S_{2k}S_{k2} + S_{3k}S_{k3}$

$$
\begin{aligned}
&= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{2k}S_{k2} + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{21} + S_{13}S_{31} + S_{21}S_{12} + S_{22}S_{22} + S_{23}S_{32} + S_{3k}S_{k3} \\
&= S_{11}S_{11} + S_{12}S_{12} + S_{13}S_{13} + S_{21}S_{21} + S_{22}S_{22} + S_{23}S_{23} + S_{31}S_{13} + S_{32}S_{23} \\
&\qquad + S_{33}S_{33} \\
&= 1 \times 1 + 0 \times 0 + 2 \times 3 + 0 \times 0 + 1 \times 1 + \cdots + 0 \times 2 + 3 \times 3 \\
&= 23
\end{aligned}
$$

d

Here, again, one index is repeated. A summation over all the allowable values of that index is implied. Accordingly,

$$a_m a_m = a_1 a_1 + a_2 a_2 + a_3 a_3 = 1 \times 1 + 2 \times 2 + 3 \times 3 = 14$$

e

This is another example of a double summation. We proceed as we have done previously:

$$
\begin{aligned}
S_{mn}a_m a_n &= S_{1n}a_1 a_n + S_{2n}a_2 a_n + S_{3n}a_3 a_n \\
&= S_{11}a_1 a_1 + S_{12}a_1 a_2 + S_{13}a_1 a_3 + S_{2n}a_2 a_n + S_{3n}a_3 a_n \\
&= S_{11}a_1 a_1 + S_{12}a_1 a_2 + S_{13}a_1 a_3 + S_{21}a_2 a_1 + S_{22}a_2 a_2 + S_{23}a_2 a_3 \\
&\qquad + S_{31}a_3 a_1 + S_{32}a_3 a_2 + S_{33}a_3 a_3 \\
&= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 2 \times 1 \times 3 + 0 \times 2 \times 1 + 1 \times 2 \times 2 \\
&\qquad + 2 \times 2 \times 3 + 3 \times 3 \times 1 + 0 \times 3 \times 2 + 3 \times 3 \times 3 \\
&= 59
\end{aligned}
$$

f

As in the above example, everything unchanged except that location of m and n indices in the first term are now revered. It is good to work this out fully manually and draw lessons from the result. This can have far reaching effects on your understanding of other materials later.

$$
\begin{aligned}
S_{nm}a_m a_n &= S_{n1}a_1 a_n + S_{n2}a_2 a_n + S_{n3}a_3 a_n \\
&= S_{11}a_1 a_1 + S_{21}a_1 a_2 + S_{31}a_1 a_3 + S_{n2}a_2 a_n + S_{n3}a_3 a_n \\
&= S_{11}a_1 a_1 + S_{21}a_1 a_2 + S_{31}a_1 a_3 + S_{12}a_2 a_1 + S_{22}a_2 a_2 + S_{32}a_2 a_3 + S_{13}a_3 a_1 \\
&\qquad + S_{23}a_3 a_2 + S_{33}a_3 a_3 \\
&= 1 \times 1 \times 1 + 0 \times 1 \times 2 + 3 \times 1 \times 3 + 0 \times 2 \times 1 + 1 \times 2 \times 2 + 0 \times 2 \times 3 \\
&\qquad + 2 \times 3 \times 1 + 2 \times 3 \times 2 + 3 \times 3 \times 3 \\
&= 59
\end{aligned}
$$

The fact that the last two examples gave the same answer is NOT a coincidence. This example may look like some easy problem that is merely tedious. However, it strikes at the very heart of understanding the skills involved in the summation convention. These skills are not elementary nor are they trivial. We pause a little moment to look again at the problems 2.1f and 2.1g. By the time it fully sinks in, you will see that it should not be necessary for you to do the tedious arithmetic to see that they MUST give the same answer. Here is the proof:

$$S_{ij}a_i a_j = S_{ij}a_j a_i$$

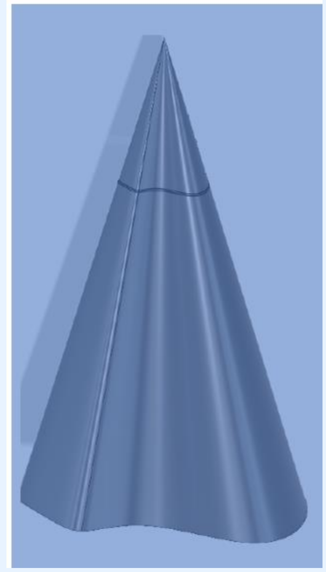
Is true for the simple fact that multiplying a_i and a_j will always give us the same answer no matter in what order the operands are given: multiplication is Commutative. The fact that i and j in the above equations are repeated means that they are dummy variables. They can therefore be exchanged for any other set of dummy variables provided we are consistent. Accordingly, replace i by m and j by n on the left-hand side, and replace i by n and j by m on the right-hand side, we obtain,

$$S_{mn}a_m a_n = S_{nm}a_m a_n$$

Never forget that we are only able to arbitrarily replace variables on a side without doing exactly the same at each object because we are here dealing with variables that have repeated themselves and are dummy variables.

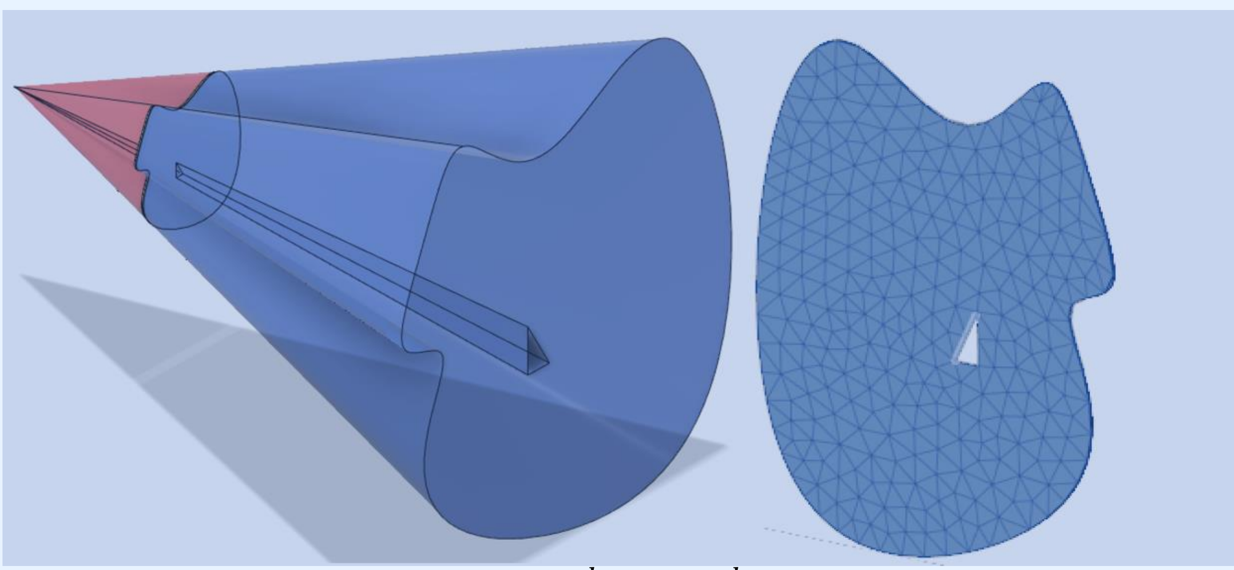
1.14	Show that the volume V of a pyramid, cone, tetrahedron or any other body that lofts from a flat surface to a point (Conramid) is such that $$V = \frac{1}{3} base \times height$$

Volume of a Conramid.



Definition: A ConRamid is an object with a flat base of any shape that tapers, linearly, to a point maintaining the same shape in any horizontal section. A cone, pyramid or tetrahedron are all special cases of a conramid.

Volume. We will show that the volume of a conramid is one third the height times the area of the base. In doing this, we shall first demonstrate that as the lengths vary linearly with distance from the tip, the elemental areas vary as the square of this distance. Let us assume that the base area is A, and the perpendicular distance between the base and the vertex is H. We consider an element at a distance x from the vertex. To make things easy we have selected a right angled triangle at the centerline – through the perpendicular. The breadth, b_{tx} and height, h_{tx} of this triangle, compared to the image (height h, breadth b) at the base is



$$b_{tx} = \frac{b}{H}x, h_{tx} = \frac{h}{H}x$$

. The area of this triangle is therefore,

$$A_{x_t} = \frac{1}{2}\left(\frac{b}{H}x\right)\left(\frac{h}{H}x\right) = \frac{bh}{2H^2}x^2 = \frac{1}{H^2}A_t x^2$$

We can easily mesh the entire disk in a set of triangles as shown. In this case, the total area of the disk at point x will be the sum of all the triangular areas:

The volume of the typical disk is,

$$\frac{1}{H^2}(A_1 + A_2 + \cdots + A_n)x^2 dx = \frac{1}{H^2}Ax^2 dx$$

where H is the height of the triangle at the base. The volume of the conramid is therefore,

$$V = \frac{A}{H^2}\int_0^H x^2 dx = \frac{1}{3}AH$$

Which is one third the area of the base times the height. This applies to a cone, a pyramid or a tetrahedron as we have assumed previously.

1.15	Evaluate $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d})$

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \big((\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\big) \cdot \mathbf{d} \\ &= \big((\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c}\,)\mathbf{a}\big) \cdot \mathbf{d} \\ &= (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c}\,)(\mathbf{a} \cdot \mathbf{d})\end{aligned}$$

1.16	Show that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})^2$

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) &= (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\mathbf{c} - (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{c}]\mathbf{a} \\ &= (\mathbf{a} \times \mathbf{b}) \cdot [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}]\mathbf{c} \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})\big((\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}\big) \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{b} \times \mathbf{c}\,) \\ &= (\mathbf{a} \times \mathbf{b} \cdot \mathbf{c})^2\end{aligned}$$

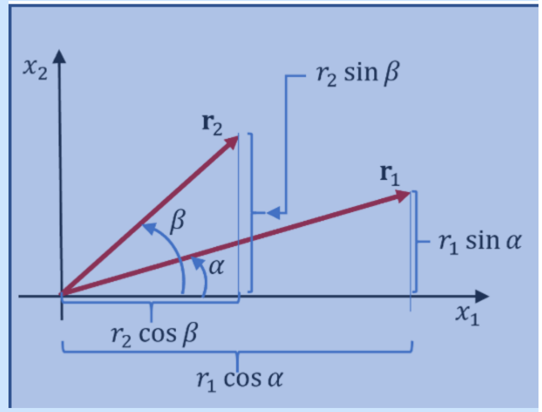
1.17	Given that position vectors $\mathbf{r}_1 and\ \mathbf{r}_2$ on the $x_1 - x_2$ plane are inclined at angles α, and β respectively to the x_1 axis. Find expressions for the component forms of these vectors and (a) use the dot product to show that, $\cos(\beta - \alpha) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$.(b) Use the cross product to show that $\sin(\beta - \alpha) = \cos\alpha - \sin\alpha \cos\beta$

(a) Writing $r_1 \equiv \|\mathbf{r}_1\|$, and $r_2 \equiv \|\mathbf{r}_2\|$ In the sketch below, let $\mathbf{e}_1$ and $\mathbf{e}_2$ be the unit coordinate vectors for x_1 and x_2 respectively. Clearly,

$$\mathbf{r}_1 = r_1 \cos\alpha\, \mathbf{e}_1 + r_1 \sin\alpha\, \mathbf{e}_2$$
$$\mathbf{r}_2 = r_2 \cos\beta\, \mathbf{e}_1 + r_2 \sin\beta\, \mathbf{e}_2$$
$$\mathbf{r}_1 \cdot \mathbf{r}_2 = \|\mathbf{r}_1\|\|\mathbf{r}_2\| \cos(\beta - \alpha)$$
$$= (r_1 \cos\alpha\, \mathbf{e}_1 + r_1 \sin\alpha\, \mathbf{e}_2) \cdot (r_2 \cos\beta\, \mathbf{e}_1 + r_2 \sin\beta\, \mathbf{e}_2)$$
$$= r_1 \cos\alpha\, r_2 \cos\beta + r_1 \sin\alpha\, r_2 \sin\beta$$

so that,

$$\cos(\beta - \alpha) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$



(b)

$$\mathbf{r}_1 \times \mathbf{r}_2 = \|\mathbf{r}_1\|\|\mathbf{r}_2\| \sin(\beta - \alpha)\, \mathbf{e}_3$$

$$= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ r_1 \cos\alpha & r_1 \sin\alpha & 0 \\ r_2 \cos\beta & r_2 \sin\beta & 0 \end{vmatrix}$$

$$= \mathbf{e}_3 r_1 r_2 (\cos\alpha \sin\beta - \sin\alpha \cos\beta)$$

$$\therefore \sin\,(\beta - \alpha) = \cos\alpha \sin\beta - \sin\alpha \cos\beta$$

1.18 The diagonals of a parallelogram are given by the vectors, $3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$. Find the area of the parallelogram.

a Assume vectors $\mathbf{D}_1$and $\mathbf{D}_2$ are the diagonals. The sides are $\frac{1}{2}(\mathbf{D}_1 + \mathbf{D}_2)$, and $\frac{1}{2}(\mathbf{D}_1 - \mathbf{D}_2)$. The required area is therefore,

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & -1 & 1 \\ 1 & 2 & -3 \end{vmatrix} = \mathbf{e}_1(3-2) - \mathbf{e}_2(-6-1) + \mathbf{e}_3(4+1)$$

The magnitude of this is $\sqrt{1 + 49 + 25} = 5\sqrt{3}$

It is a vector area with magnitude $5\sqrt{3}$.

```
In[17]:= a = {2, -1, 1}; b = {1, 2, -3};
         vArea = Cross[a, b];
         sArea = Norm[vArea]
         vArea // MatrixForm

Out[19]= 5 √3

Out[20]//MatrixForm=
  (1)
  (7)
  (5)
```

1.19 Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, using the result, $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$, show that $[(\mathbf{u} \times \mathbf{v}), (\mathbf{v} \times \mathbf{w}), (\mathbf{w} \times \mathbf{u})] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2$

a	From the given result, $$[(\mathbf{u}\times\mathbf{v}),(\mathbf{v}\times\mathbf{w}),(\mathbf{w}\times\mathbf{u})] = -(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{u}) = -(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\otimes\mathbf{w})(\mathbf{v}\times\mathbf{u}) = (\mathbf{u}\times\mathbf{v})\big((\mathbf{w}\cdot\mathbf{u}\times\mathbf{v})\mathbf{w}\big) = [\mathbf{u},\mathbf{v},\mathbf{w}]^2$$
1.20	Given three vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{w}\times\mathbf{u})\times(\mathbf{w}\times\mathbf{v}) = (\mathbf{w}\otimes\mathbf{w})(\mathbf{u}\times\mathbf{v})$ and that for the unit vector $\mathbf{e}$, $[\mathbf{e},\mathbf{e}\times\mathbf{u},\mathbf{e}\times\mathbf{v}] = [\mathbf{e},\mathbf{u},\mathbf{v}]$
	$$(\mathbf{w}\times\mathbf{u})\times(\mathbf{w}\times\mathbf{v}) = [(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}]\mathbf{w} - [(\mathbf{w}\times\mathbf{u})\cdot\mathbf{w}]\mathbf{v} = [(\mathbf{w}\times\mathbf{u})\cdot\mathbf{v}]\mathbf{w} = [(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}]\mathbf{w} = (\mathbf{w}\otimes\mathbf{w})(\mathbf{u}\times\mathbf{v})$$ Consequently, $$[\mathbf{e},\mathbf{e}\times\mathbf{u},\mathbf{e}\times\mathbf{v}] = \mathbf{e}\cdot[(\mathbf{e}\times\mathbf{u})\times(\mathbf{e}\times\mathbf{v})] = \mathbf{e}\cdot[(\mathbf{e}\otimes\mathbf{e})(\mathbf{u}\times\mathbf{v})] = (\mathbf{u}\times\mathbf{v})\cdot(\mathbf{e}\otimes\mathbf{e})\mathbf{e} = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{e} = [\mathbf{e},\mathbf{u},\mathbf{v}]$$ making use of the symmetry of $(\mathbf{e}\otimes\mathbf{e})$.
1.21	Given that $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ are vectors, find the values of scalars α and β in the equation, $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \alpha\mathbf{u}+\beta\mathbf{v}$

$$\mathbf{u} \times \mathbf{v} = e_{ijk} u_i v_j \mathbf{e}_k = s_k \mathbf{e}_k$$

Expanding the full equation, we have that

$$\begin{aligned}(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} &= e_{klm} s_k w_l \mathbf{e}_m \\ &= e_{klm} e_{ijk} u_i v_j w_l \mathbf{e}_m \\ &= (\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi}) u_i v_j w_l \mathbf{e}_m \\ &= e_{ijk} u_i v_j w_i \mathbf{e}_j - e_{ijk} u_i v_j w_j \mathbf{e}_i \\ &= (u_i w_i) v_j \mathbf{e}_j - (v_j w_j) u_i \mathbf{e}_i \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} = \alpha \mathbf{u} + \beta \mathbf{v}\end{aligned}$$

Clearly, $\alpha = -(\mathbf{v} \cdot \mathbf{w})$ and $\beta = (\mathbf{u} \cdot \mathbf{w})$

1.22 Given that $\mathbf{n}$ is a unit vector, use the fact that $\mathbf{n} \cdot \mathbf{u}$ is the projection of the vector $\mathbf{u}$ in the direction of $\mathbf{n}$ to represent $\mathbf{u}$ as $(\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$ or $(\mathbf{n} \otimes \mathbf{n})\mathbf{u} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})$.

By simple vector addition, we can represent $\mathbf{u}$ as $(\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n}$.

Since $\mathbf{n}$ is a unit vector, $\mathbf{n} \cdot \mathbf{n} = 1$. Therefore,

$$\begin{aligned}\mathbf{u} &= (\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n} \\ &= (\mathbf{n} \cdot \mathbf{u})\mathbf{n} + (\mathbf{n} \cdot \mathbf{n})\mathbf{u} - (\mathbf{n} \cdot \mathbf{u})\mathbf{n} \\ &= (\mathbf{n} \cdot \mathbf{u})\mathbf{n} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n}) \\ &= (\mathbf{n} \otimes \mathbf{n})\mathbf{u} + \mathbf{n} \times (\mathbf{u} \times \mathbf{n})\end{aligned}$$

1.23 Simplify the following by employing the substitution properties of the Kronecker Delta $(a) e_{ijk}\delta_{kn}$, $(b) e_{ijk}\delta_{is}\delta_{jm}$ $(c) e_{ijk}\delta_{is}\delta_{jm}$ $(d)\ a_{ij}\delta_{in}$ $(e)\delta_{ij}\delta_{jn}$ $(f)\delta_{ij}\delta_{jn}\delta_{ni}$

$(a)\ e_{ijn}$ $(b) e_{smk}$ $(c) e_{smk}$ $(d) a_{nj}$ $(e) \delta_{in}$ $(f) \delta_{ij}\delta_{ji} = \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$

1.24 Show (a) that the sum of triple products, (Jacobi's identity) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} + (\mathbf{v} \times \mathbf{w}) \times \mathbf{u} + (\mathbf{w} \times \mathbf{u}) \times \mathbf{v} = \mathbf{o}$, and (b) $(\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{w} \times \mathbf{y}) = (\mathbf{u} \cdot \mathbf{w})(\mathbf{v} \cdot \mathbf{y}) - (\mathbf{u} \cdot \mathbf{y})(\mathbf{v} \cdot \mathbf{w})$ (Lagrange Identity)

$$(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}$$
$$(\mathbf{v}\times\mathbf{w})\times\mathbf{u} = (\mathbf{v}\cdot\mathbf{u})\mathbf{w} - (\mathbf{w}\cdot\mathbf{u})\mathbf{v}$$
$$(\mathbf{w}\times\mathbf{u})\times\mathbf{v} = (\mathbf{w}\cdot\mathbf{v})\mathbf{u} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$$

Adding the three, we find that $(\mathbf{u}\times\mathbf{v})\times\mathbf{w} + (\mathbf{v}\times\mathbf{w})\times\mathbf{u} + (\mathbf{w}\times\mathbf{u})\times\mathbf{v} = \mathbf{o}$

This is the zero vector.

$$\begin{aligned}(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{y}) &= \big((\mathbf{u}\times\mathbf{v})\times\mathbf{w}\big)\cdot\mathbf{y}\\ &= \big((\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{v}\cdot\mathbf{w})\mathbf{u}\big)\cdot\mathbf{y}\\ &= (\mathbf{u}\cdot\mathbf{w})(\mathbf{v}\cdot\mathbf{y}) - (\mathbf{u}\cdot\mathbf{y})(\mathbf{v}\cdot\mathbf{w})\end{aligned}$$

as required.

1.25 Given that, $I_{ij} = \iiint_V \big(x_m x_m \delta_{ij} - x_i x_j\big)\,\rho(x_1,x_2,x_3)dx_1dx_2dx_3$ is the moment of inertia along the axis $i-j$ where $x = x_1, y = x_2, z = x_3$ and $\varrho(x_1,x_2,x_3)$ is scalar density of the material find all the components of the tensor.

$$\mathrm{I}_{11} = \iiint_V (\mathrm{y}^2+\mathrm{z}^2)\varrho(\mathrm{x,y,z})\mathrm{dxdydz},\quad \mathrm{I}_{21} = \mathrm{I}_{12} = \iiint_V \mathrm{xy}\varrho(\mathrm{x,y,z})\mathrm{dxdydz},$$
$$\mathrm{I}_{22} = \iiint_V (\mathrm{z}^2+\mathrm{x}^2)\varrho(\mathrm{x,y,z})\mathrm{dxdydz},\quad \mathrm{I}_{32} = \mathrm{I}_{23} = \iiint_V \mathrm{yz}\varrho(\mathrm{x,y,z})\mathrm{dxdydz},$$
$$\mathrm{I}_{31} = \mathrm{I}_{13} = \iiint_V \mathrm{xz}\varrho(\mathrm{x,y,z})\mathrm{dxdydz},\quad \mathrm{I}_{33} = \iiint_V (\mathrm{x}^2+\mathrm{y}^2)\varrho(\mathrm{x,y,z})\mathrm{dxdydz}$$

1.26 Write (a) in the long form,

$$a_i = \frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}$$

We can see that in this equation, the there is one free index, that is i and it occurs once in every term on both sides. There is a dummy index, that is, j appearing repeated in one term. Accordingly,

$$a_i = \frac{\partial v_i}{\partial t} + v_1\frac{\partial v_i}{\partial x_1} + v_2\frac{\partial v_i}{\partial x_2} + v_3\frac{\partial v_i}{\partial x_3}$$

Which are, indeed, three equations one each for $i = 1, i = 2$ and $i = 3$ as follows:

	$$a_2 = \frac{\partial v_2}{\partial t} + v_1 \frac{\partial v_2}{\partial x_1} + v_2 \frac{\partial v_2}{\partial x_2} + v_3 \frac{\partial v_2}{\partial x_3}$$ $$a_3 = \frac{\partial v_3}{\partial t} + v_1 \frac{\partial v_3}{\partial x_1} + v_2 \frac{\partial v_3}{\partial x_2} + v_3 \frac{\partial v_3}{\partial x_3}$$ The equation comes from the natural law of the indestructibility of masses – or mass balance. It is often inaccurately called a continuity equation in some texts.
1.27	Given that λ and μ are scalar constants, and that the identity tensor, $\mathbf{I} = \delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, $\mathbf{E} = E_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ and $\boldsymbol{\sigma} = \sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ write the equation, $\boldsymbol{\sigma} = \lambda\mathbf{I}\,\mathrm{tr}\,\mathbf{E} + 2\mu\mathbf{E}$ in component form.
a	$$\mathrm{tr}\,\mathbf{E} = E_{\alpha\beta}\mathrm{tr}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta) = E_{\alpha\beta}\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = E_{\alpha\beta}\delta_{\alpha\beta} = E_{\alpha\alpha}$$ The given equation, in component form can be written as, $$\sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \lambda\delta_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta E_{kk} + 2\mu E_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ Using the common bases, we can write this in terms of components only : $$\sigma_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = (\lambda\delta_{\alpha\beta}E_{kk} + 2\mu E_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$\sigma_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$$
1.28	If $\sigma_{ij} = \lambda\delta_{ij}E_{kk} + 2\mu E_{ij}$, show that, (a) $\sigma_{ij}E_{ij} = \lambda(E_{kk})^2 + 2\mu E_{ij}E_{ij}$ and (b) $\sigma_{ij}\sigma_{ij} = (E_{kk})^2(4\mu\lambda + 3\lambda^2) + 4\mu^2 E_{ij}E_{ij}$
a	Multiplying both sides by E_{ij} we have, $$\sigma_{ij}E_{ij} = \lambda\delta_{ij}E_{kk}E_{ij} + 2\mu E_{ij}E_{ij}$$ By the substitution nature of the Kronecker Delta, we have that, $\delta_{ij}E_{ij} = E_{jj} = E_{kk}$ because j as well as k are dummy indices here. Consequently, $$\sigma_{ij}E_{ij} = \lambda E_{kk}E_{jj} + 2\mu E_{ij}E_{ij} = \lambda(E_{kk})^2 + 2\mu E_{ij}E_{ij}$$ Squaring both sides of the equation, $$\sigma_{ij}\sigma_{ij} = (\lambda\delta_{ij}E_{kk} + 2\mu E_{ij})(\lambda\delta_{ij}E_{kk} + 2\mu E_{ij})$$ $$= \lambda^2\delta_{ii}(E_{kk})^2 + 2\lambda\mu E_{jj} + 2\lambda\mu\delta_{ij}E_{ij} + 4\mu^2 E_{ij}E_{ij}$$ $$= 3\lambda^2(E_{kk})^2 + 4\lambda\mu E_{jj} + 4\mu^2 E_{ij}E_{ij}$$ $$= (E_{kk})^2(4\mu\lambda + 3\lambda^2) + 4\mu^2 E_{ij}E_{ij}$$

1.29	Given that $a_{mn}x^m x^n = 0$. Show that $a_{mn}, m, n = 1,2,3$ is antisymmetric
a	Given that $a_{mn}x^m x^n = 0$ for arbitrary values of $x^n, n = 1,2,3$ then we can write, $$a_{mn}x^m x^n = -a_{mn}x^m x^n$$ because zero is also a negative of itself. Swapping the roles of x^m and x^n on the RHS of the above, we can write, $$\begin{aligned} a_{mn}x^m x^n &= -a_{mn}x^m x^n \\ &= -a_{mn}x^n x^m \\ &= -a_{nm}x^n x^m \end{aligned}$$ after swapping the roles of the two dummy indices. We therefore consolidate on the LHS by writing, $$a_{mn}x^m x^n + a_{nm}x^n x^m = 0$$ $$(a_{mn} + a_{nm})x^m x^n = 0$$ Notice that the quantity in the parenthesis is always symmetric. And also note the contraction of two symmetric tensors can only vanish if one or both tensors vanish. Here, $x^m x^n$ is a product of arbitrary tensors. We are left with the fact that $$a_{mn} + a_{nm} = 0$$ or, $$a_{mn} = -a_{nm}$$ which is the definition of anti-symmetry.
1.30	Given that, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that this product vanishes if the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are linearly dependent.
	Suppose it is possible to find scalars α and β such that, $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$. It therefore means that, $$\begin{aligned} [\mathbf{a}, \mathbf{b}, \mathbf{c}] &= e_{ijk}a_i b_j c_k = e_{ijk}(\alpha b_i + \beta c_i)b_j c_k \\ &= \alpha e_{ijk}b_i b_j c_k + \beta e_{ijk}c_i b_j c_k \\ &= 0 \end{aligned}$$

Note that $b_i b_j c_k$ is symmetric in i and j, $c_i b_j c_k$ is symmetric in i and k ande_{ijk} is antisymmetric in i, j and k Because each term is the product of a symmetric and an antisymmetric object which must vanish.

1.31 Show that the product of a symmetric and an antisymmetric object vanishes.

Consider the product sum, $e_{ijk} b_i b_j c_k$ in which $b_i b_j$ is symmetric in i and j and e_{ijk} is antisymmetric in i, j and k. Only the shared symmetrical and antisymmetrical indices i, j are relevant here.

$$e_{ijk} b_i b_j c_k = -e_{ijk} b_i b_j c_k = -e_{ijk} b_j b_i c_k = -e_{ijk} b_i b_j c_k = 0$$

The first equality on account of the antisymmetry of e_{ijk} in i, j; the second on the symmetry of $b_i b_j$ in i, j; the third on the fact that i, j are dummy indices. These vanish because a non-trivial scalar quantity cannot be the negative of itself.

This is a general rule and its observation makes a number of steps easy to see transparently. Watch out for it.

1.32 Define $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$

In component form,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = e_{ijk} a_i b_j c_k$$

Cyclic permutations of this, upon remembering that (i, j, k) are dummy indices, yield,

$$e_{ijk} b_j c_k a_i = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = e_{ijk} b_i c_j a_k$$

$$= e_{ijk} c_k a_i b_j = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = e_{ijk} c_i a_j b_k$$

$$[\mathbf{b}, \mathbf{c}, \mathbf{a}] = \mathbf{b} \cdot \mathbf{c} \times \mathbf{a} = -\mathbf{b} \cdot \mathbf{a} \times \mathbf{c} = -[\mathbf{b}, \mathbf{a}, \mathbf{c}]$$

In a similar way, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$, and $[\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}]$

1.33 Write in indicial notations (a) $s = A_1^2 + A_2^2 + A_3^2$ (b) $\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2}$

(a) In order to avoid confusion between squaring and simple indexing, we separate the two as follows:

$$s = A_1^2 + A_2^2 + A_3^2 = A_1A_1 + A_2A_2 + A_3A_3 = A_iA_i$$

In a similar way, the Laplacian operator for a scalar function can be expressed as follows:

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} + \frac{\partial^2 \phi}{\partial x_3^2} = \frac{\partial^2 \phi}{\partial x_1 \partial x_1} + \frac{\partial^2 \phi}{\partial x_2 \partial x_2} + \frac{\partial^2 \phi}{\partial x_3 \partial x_3} = \frac{\partial^2 \phi}{\partial x_i \partial x_i}$$

1.34 Given the vectors $\mathbf{a} = 3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$. Find the dyad $\mathbf{a} \otimes \mathbf{b}$, and (b) Find $\mathrm{tr}(\mathbf{a} \otimes \mathbf{b})$

$$\begin{aligned}
\mathbf{a} \otimes \mathbf{b} &= a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j \\
&= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \otimes [b_1, b_2, b_3] \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\
&= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} 3 \times 1 & 3 \times (-3) & 3 \times 4 \\ 1 & -3 & 4 \\ (-2) \times 1 & -2 \times (-3) & -2 \times 4 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix} \\
&= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} 3 & -9 & 12 \\ 1 & -3 & 4 \\ -2 & 6 & -8 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}
\end{aligned}$$

$$\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = a_i b_i = 3 - 3 - 8 = -8$$

1.35 Given that $\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$, Find expressions for $\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k$ and $\mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k$. Demonstrate the equality of the two expressions

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k = \mathbf{e}_i \times \mathbf{e}_j = e_{ij\alpha}\mathbf{e}_\alpha$$

Taking the scalar product of the above vector with $\mathbf{e}_k$,

$$\mathbf{e}_i \times \mathbf{e}_j \cdot \mathbf{e}_k = e_{ij\alpha}\mathbf{e}_\alpha \cdot \mathbf{e}_k = e_{ij\alpha}\delta_{\alpha k} = e_{ijk}$$

Starting with $\mathbf{e}_j \times \mathbf{e}_k = e_{jk\alpha}\mathbf{e}_\alpha$ we can also take the scalar product with $\mathbf{e}_i$ and write,

$$\mathbf{e}_i \cdot \mathbf{e}_j \times \mathbf{e}_k = \mathbf{e}_j \times \mathbf{e}_k \cdot \mathbf{e}_i = e_{jk\alpha}\mathbf{e}_\alpha \cdot \mathbf{e}_i = e_{jk\alpha}\delta_{\alpha i} = e_{jki} = e_{ijk}$$

As a double swap does not alter sign.

1.36 Show that Cylindrical Polar basis vectors, $\mathbf{e}_r(r,\phi), \mathbf{e}_\phi(r,\phi)$ and $\mathbf{e}_z$ constitute an orthonormal system. Hint: Show that they have unit magnitudes and are mutually orthogonal.

$$\|\mathbf{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$

$$\|\mathbf{e}_\phi\|^2 = \sin^2\phi + \cos^2\phi = 1$$

$$\|\mathbf{e}_z\|^2 = 1$$

They are individually normalized with each having a norm or magnitude of 1. Now lets take them in pairs:

$$\mathbf{e}_r \cdot \mathbf{e}_\phi = -\cos\phi\sin\phi + \cos\phi\sin\phi = 0$$

$$\mathbf{e}_\phi \cdot \mathbf{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 = 0$$

$$\mathbf{e}_z \cdot \mathbf{e}_r = \cos\phi \times 0 + \sin\phi \times 0 + 1 \times 0 = 0$$

So that they are pairwise orthogonal.

1.37 Show that Normalized Spherical Polar basis vectors,

$$\mathbf{e}_\rho(\rho,\theta,\phi) = \sin\theta\cos\phi\,\mathbf{e}_1 + \sin\theta\sin\phi\,\mathbf{e}_2 + \cos\theta\,\mathbf{e}_3$$

$$\mathbf{e}_\theta(\rho,\theta,\phi) = \cos\theta\cos\phi\,\mathbf{e}_1 + \cos\theta\sin\phi\,\mathbf{e}_2 - \sin\theta\,\mathbf{e}_3$$

$$\mathbf{e}_\phi(\rho,\theta,\phi) = -\sin\phi\,\mathbf{e}_1 + \cos\phi\,\mathbf{e}_2$$

constitute an orthonormal system. Hint: Show that they have unit magnitudes and are mutually orthogonal.

$$\|\mathbf{e}_\rho\|^2 = \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta$$

$$= \sin^2\theta\,(\cos^2\phi + \sin^2\phi) + \cos^2\theta = 1$$

$$\|\mathbf{e}_\theta\|^2 = \cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta$$

$$= \cos^2\theta\,(\cos^2\phi + \sin^2\phi) + \sin^2\theta = 1$$

$$\|\mathbf{e}_\phi\|^2 = \sin^2\phi + \cos^2\phi = 1$$

They are individually normalized with each having a norm or magnitude of 1. Now lets take them in pairs:

$$\mathbf{e}_\theta \cdot \mathbf{e}_\phi = -\sin\phi\cos\theta\cos\phi + \sin\phi\cos\theta\cos\phi + 0 = 0$$

So that they are pairwise orthogonal.

1.38	Begin with the Cartesian position vector, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, use the transformation equations to find the Spherical position vector, $\mathbf{R} = \rho\mathbf{e}_\rho(\theta, \phi)$,where $\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$. (b) By partial differentiation with respect to the coordinate variables, produce the set $\left\{\frac{\partial\mathbf{R}}{\partial\rho}, \frac{\partial\mathbf{R}}{\partial\theta}, \frac{\partial\mathbf{R}}{\partial\phi}\right\}$.and show that it is a set of orthogonal vectors.

$$\frac{\partial\mathbf{R}}{\partial\rho} = \mathbf{e}_\rho,$$

$$\frac{\partial\mathbf{R}}{\partial\theta} = \rho\frac{\partial\mathbf{e}_\rho}{\partial\theta} = \rho\frac{\partial}{\partial\theta}(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})$$

$$= \rho(\cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}) \equiv \rho\mathbf{e}_\theta.$$

$$\frac{\partial\mathbf{R}}{\partial\phi} = \rho\frac{\partial\mathbf{e}_\rho}{\partial\phi} = \rho\frac{\partial}{\partial\phi}(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})$$

$$= \rho(-\sin\theta\sin\phi\,\mathbf{i} + \sin\theta\cos\phi\,\mathbf{j})$$

$$\equiv \rho\sin\theta\,\mathbf{e}_\phi.$$

From these, we can see that $\left\{\frac{\partial\mathbf{R}}{\partial\rho}, \frac{\partial\mathbf{R}}{\partial\theta}, \frac{\partial\mathbf{R}}{\partial\phi}\right\} = \{\mathbf{e}_\rho, \rho\mathbf{e}_\theta, \rho\sin\theta\,\mathbf{e}_\phi\}$. Obviously, the magnitudes are $\{1, \rho, \rho\sin\theta\}$ respectively. Consequently, this basis set can be normalized to $\{\mathbf{e}_\rho, \mathbf{e}_\theta, \mathbf{e}_\phi\}$

1.39	The dot from the left. Given that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{E}$, Show that $\mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w}) = (\mathbf{w} \otimes \mathbf{v})\mathbf{u}$

The result on both sides is a vector. Testing a scalar product with $\mathbf{y} \in \mathbb{E}$: observe that

$$\mathbf{u} \cdot (\mathbf{v} \otimes \mathbf{w})\mathbf{y} = (\mathbf{u} \cdot \mathbf{v})(\mathbf{w} \cdot \mathbf{y})$$

and,

$$[(\mathbf{w} \otimes \mathbf{v})\mathbf{u}] \cdot \mathbf{y} = (\mathbf{w} \cdot \mathbf{y})(\mathbf{u} \cdot \mathbf{v})$$

The two are equal on account of the commutativity of the dot products.

This is a general result for tensors. A dot from the left gives the same result at contracting with the tensor transpose from the right as seen here.

1.40	Given that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{E}$, Show that $\mathbf{u} \times (\mathbf{v} \otimes \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}$

First observe that the result will be a tensor. Operating it on $\mathbf{y} \in \mathbb{E}$, we have,

$$[\mathbf{u} \times (\mathbf{v} \otimes \mathbf{w})]\, \mathbf{y} = (\mathbf{w} \cdot \mathbf{y})(\mathbf{u} \times \mathbf{v})$$

And on the right, we have

$$[(\mathbf{u} \times \mathbf{v}) \otimes \mathbf{w}]\mathbf{y} = (\mathbf{w} \cdot \mathbf{y})(\mathbf{u} \times \mathbf{v})$$

As was required.

1.41 Given the rotation tensor,

$$\mathbf{Q} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} 0.84339393538 & 0.389796451909 & 0.36979101642 \\ -0.2752066485 & 0.90450849718 & -0.325769364916 \\ -0.461462859127 & 0.172982960416 & 0.87013155617 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

Find the axis of rotation and the angle of rotation.

Hint: The cross product of any vector with the axis of rotation will be on the normal plane of rotation. The angle of rotation is between the rotated vectors on this plane.

Operate this rotation on any vector along the axis of rotation, there will be no change in the vector. For all such vectors, a rotation has the same effect as an Identity tensor. Let $\mathbf{v}_a = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2 + \gamma\mathbf{e}_3$ be the unit vector along the axis of rotation. Since $\mathbf{v}_a$ as a unit vector; its magnitude,

$$\|\mathbf{v}_a\| = \sqrt{\alpha^2 + \beta^2 + \gamma^2} = 1$$

```
Q = {{0.8433939353874569, 0.38979645190947854, 0.3697910164274408},
    {-0.275206648534447, 0.9045084971874737, -0.32576936491649305},
    {-0.46146285912746604, 0.17298296041645106, 0.8701315561749643}};
va = {α, β, γ}; u1 = {1, 0, 0}; u2 = {1, 1, -2};
w1 = Cross[va, u1]; w2 = Cross[va, u2];
y1 = Q.w1; y2 = Q.w2;
z1 = Transpose[Q].w1;

Solve[{Dot[y1, va] == 0, Dot[z1, va] == 0, α^2 + β^2 + γ^2 == 1}, {α, β, γ}]
{{α → 2.35908×10^7 - 2.45346×10^7 i, β → -2.55563×10^7 - 7.29829×10^6 i,
  γ → -1.42522×10^7 - 2.75238×10^7 i}, {α → 2.35908×10^7 + 2.45346×10^7 i,
  β → -2.55563×10^7 + 7.29829×10^6 i, γ → -1.42522×10^7 + 2.75238×10^7 i},
 {α → -0.424264, β → -0.707107, γ → 0.565685}, {α → -1., β → 0., γ → 0.},
 {α → 1., β → 0., γ → 0.}, {α → 0.424264, β → 0.707107, γ → -0.565685},
 {α → -2.35908×10^7 - 2.45346×10^7 i, β → 2.55563×10^7 - 7.29829×10^6 i,
  γ → 1.42522×10^7 - 2.75238×10^7 i}, {α → -2.35908×10^7 + 2.45346×10^7 i,
  β → 2.55563×10^7 + 7.29829×10^6 i, γ → 1.42522×10^7 + 2.75238×10^7 i}}

ArcCos[Normalize[w1].Normalize[y1]] /.
 {α → 0.4242640687119285`, β → 0.7071067811865477`, γ → -0.5656854249492378`}
ArcCos[Normalize[w2].Normalize[y2]] /.
 {α → 0.4242640687119285`, β → 0.7071067811865477`, γ → -0.5656854249492378`}
```

We select two vectors, $\mathbf{u}_1 = \mathbf{e}_1$ and $\mathbf{u}_2 = \mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$

$$\mathbf{w}_1 = \mathbf{v}_a \times \mathbf{u}_1 \text{ and } \mathbf{w}_2 = \mathbf{v}_a \times \mathbf{u}_2$$

Both lie on the normal plane to the axis of rotation. We now rotate these two vectors and obtain,

$$\mathbf{y}_1 = \mathbf{Q}\mathbf{w}_1, \mathbf{y}_2 = \mathbf{Q}\mathbf{w}_2, \text{ and}$$

$$\mathbf{z}_1 = \mathbf{Q}^{\mathrm{T}}\mathbf{w}_1$$

all of which lie on the normal plane to the axis of rotation. Clearly, each of these is inclined at right angles to $\mathbf{v}_a$. We use the condition on two of these with the constraint on the unit vector to find the unknown α, β and γ in the Solve[] function call shown.

There are many solutions – most involving complex numbers. We concentrate on the real solutions. The two non-trivial solutions give the same unit vector (same direction, opposite sense. We are now able to test the rotation angle of this tensor on each of the two vectors transformed to its plane: The angles between $(\mathbf{Q}\mathbf{w}_1, \mathbf{w}_1)$, as well as the angle between $(\mathbf{Q}\mathbf{w}_2, \mathbf{w}_2)$ are both equal to $\pi/5$ the angle of rotation of the tensor about $\mathbf{v}_a$.

Note: *The converse of this problem, to find a tensor that performs a given rotation about a given axis, is solved in* ***Q2.74*** *after a deeper treatment of rotation tensors.*

1.42	Begin with the Cartesian position vector, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$ use the transformation equations, $x = \xi\eta, y = \frac{1}{2}(\xi^2 - \eta^2)$ and $z = z$ to find the parabolic cylindrical position vector, $\mathbf{R} = \xi\eta\mathbf{e}_1 + \frac{1}{2}(\xi^2 - \eta^2)\mathbf{e}_2 + z\mathbf{e}_3$, (b) By a differentiating partially with respect to the coordinate variables obtain the set $\left\{\frac{\partial\mathbf{R}}{\partial\xi}, \frac{\partial\mathbf{R}}{\partial\eta}, \frac{\partial\mathbf{R}}{\partial z}\right\}$.and show that it is a set of orthogonal vectors, right-handed but not orthonormal.

The basis vectors are packed in columns of the gradient. We did not need to transpose here because in this case, there is symmetry.

$$\frac{\partial\mathbf{R}}{\partial\xi} = \mathbf{e}_\xi = \eta\mathbf{e}_1 + \xi\mathbf{e}_2$$

$$\frac{\partial\mathbf{R}}{\partial\eta} = \mathbf{e}_\eta = \xi\mathbf{e}_1 - \eta\mathbf{e}_2$$

$$\frac{\partial\mathbf{R}}{\partial z} = \mathbf{e}_z = \mathbf{e}_3$$

The norms of the vectors are:

$$\|\mathbf{e}_\xi\| = \|\mathbf{e}_\eta\| = \sqrt{\xi^2 + \eta^2}, \|\mathbf{e}_z\| = 1$$

The dot products yield pairwise zeros showing orthogonality. The magnitudes display no normality so the basis vectors are NOT orthonormal.

If these vectors are taken in the order, $\mathbf{e}_\eta$, $\mathbf{e}_\xi$, and $\mathbf{e}_z$, then the system is right=handed as,

```
In[1]:= R = {ξ η, (ξ^2 - η^2) / 2, z};
        cVec = Grad[R, {ξ, η, z}];

        cVec // MatrixForm;
```

$$\begin{pmatrix} \eta & \xi & 0 \\ \xi & -\eta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
        Dot[cVec[[1]], cVec[[2]]]
        Dot[cVec[[2]], cVec[[3]]]
        Dot[cVec[[3]], cVec[[1]]]

n[25]:= Norm[cVec[[1]]]
        Norm[cVec[[2]]]
        Norm[cVec[[3]]]

n[22]:= Cross[cVec[[2]], cVec[[1]]]
        Cross[cVec[[1]], cVec[[3]]]
        Cross[cVec[[3]], cVec[[2]]]
ut[22]= {0, 0, η^2 + ξ^2}

ut[23]= {ξ, -η, 0}

ut[24]= {η, ξ, 0}
```

$$\mathbf{e}_\eta \times \mathbf{e}_\xi = \mathbf{e}_z;\ \mathbf{e}_\xi \times \mathbf{e}_z = \mathbf{e}_\eta;\ \mathbf{e}_z \times \mathbf{e}_\eta = \mathbf{e}_\xi.$$

1.43	Create an orthonormal system out of the Parabolic Cylindrical coordinate bases and show the coordinate surfaces.

Dividing the right-handed orthonormal set with the respective magnitudes, we have the orthonormal set:

$$\left\{\frac{\xi\mathbf{e}_1-\eta\mathbf{e}_2}{\sqrt{\xi^2+\eta^2}},\frac{\xi\mathbf{e}_1+\eta\mathbf{e}_2}{\sqrt{\xi^2+\eta^2}},1\right\}$$

1.44 Begin with the Cartesian position vector, $\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, use the transformation equations, $x=\cosh\xi\cos\eta$, $y=\sinh\xi\sin\eta$ and $z=z$ to find the elliptic cylindrical position vector, $\mathbf{R}=\cosh\xi\cos\eta\,\mathbf{e}_1+\sinh\xi\sin\eta\,\mathbf{e}_2+z\mathbf{e}_3$, (b) By a differentiating partially with respect to the coordinate variables obtain the set $\left\{\frac{\partial\mathbf{R}}{\partial\xi},\frac{\partial\mathbf{R}}{\partial\eta},\frac{\partial\mathbf{R}}{\partial z}\right\}$.and show that it is a set of orthogonal vectors, right-handed but not orthonormal.

$$\mathbf{R}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$
$$=\cosh\xi\cos\eta\,\mathbf{e}_1+\sinh\xi\sin\eta\,\mathbf{e}_2+z\mathbf{e}_3$$

after substituting for x,y and z. Differentiating with respect to each coordinate variable is more easily accomplished by simply taking the gradient with respect to these variables. Mathematica populates the matrix with the basis vectors as rows of the transposed gradient in Out[11].

$$\frac{\partial\mathbf{R}}{\partial\xi}=\mathbf{e}_\xi=(\sinh\xi\cos\eta)\mathbf{e}_1+\cosh\xi\sin\eta\,\mathbf{e}_2$$

$$\frac{\partial\mathbf{R}}{\partial\eta}=\mathbf{e}_\eta=-\cosh\xi\sin\eta\,\mathbf{e}_1+\sinh\xi\cos\eta\,\mathbf{e}_2$$

$$\frac{\partial\mathbf{R}}{\partial z}=\mathbf{e}_z$$

The norms of the basis vectors are:

```
In[9]:= R = {a Cosh[ξ] Cos[η], a Sinh[ξ] Sin[η], z};
        cVec = Transpose[Grad[R, {ξ, η, z}]];

In[11]:= Simplify[cVec] // MatrixForm
Out[11]//MatrixForm=
 ( a Cos[η] Sinh[ξ]     a Cosh[ξ] Sin[η]   0 )
 ( -a Cosh[ξ] Sin[η]    a Cos[η] Sinh[ξ]   0 )
 ( 0                    0                  1 )

In[12]:=
 Dot[cVec[[1]], cVec[[2]]]
 Dot[cVec[[2]], cVec[[3]]]
 Dot[cVec[[3]], cVec[[1]]]

In[15]:= Norm[cVec[[1]]]
         Norm[cVec[[2]]]
         Norm[cVec[[3]]]

Out[15]= √(Abs[a Cosh[ξ] Sin[η]]^2 + Abs[a Cos[η] Sinh[ξ]]^2)

Out[16]= √(Abs[a Cosh[ξ] Sin[η]]^2 + Abs[a Cos[η] Sinh[ξ]]^2)

Out[17]= 1

In[21]:= Cross[cVec[[1]], cVec[[2]]]
         Cross[cVec[[2]], cVec[[3]]]
         Cross[cVec[[3]], cVec[[1]]]

Out[21]= {0, 0, a^2 Cosh[ξ]^2 Sin[η]^2 + a^2 Cos[η]^2 Sinh[ξ]^2}

Out[22]= {a Cos[η] Sinh[ξ], a Cosh[ξ] Sin[η], 0}

Out[23]= {-a Cosh[ξ] Sin[η], a Cos[η] Sinh[ξ], 0}
```

$$\|\mathbf{e}_\eta\| = \sqrt{(\cosh\xi \sin\eta)^2 + (\sinh\xi \cos\eta)^2}$$

$$\|\mathbf{e}_z\| = 1$$

The dot products yield pairwise zeros showing orthogonality. The magnitudes display no normality so the basis vectors are NOT orthonormal. If these vectors are taken in the order, $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, and $\mathbf{e}_z$, then the system is right-handed as,

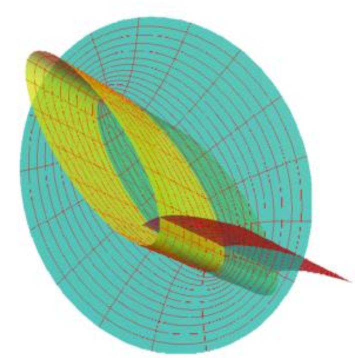
$$\mathbf{e}_\xi \times \mathbf{e}_\eta = \mathbf{e}_z;\ \mathbf{e}_\eta \times \mathbf{e}_z = \mathbf{e}_\xi;\ \mathbf{e}_z \times \mathbf{e}_\xi = \mathbf{e}_\eta.$$

1.45 Create an orthonormal system out of the Elliptic Cylindrical coordinate Bases and show the coordinate surfaces.

Dividing the right-handed orthogonal base vectors set with the respective magnitudes, we have the orthonormal set:

$$\frac{(\sinh\xi \cos\eta)\mathbf{e}_1 - \cosh\xi \sin\eta\,\mathbf{e}_2}{\sqrt{(\cosh\xi \sin\eta)^2 + (\sinh\xi \cos\eta)^2}}, \frac{\cosh\xi \sin\eta\,\mathbf{e}_1 + \sinh\xi \cos\eta\,\mathbf{e}_2}{\sqrt{(\cosh\xi \sin\eta)^2 + (\sinh\xi \cos\eta)^2}}, z$$

Coordinate surfaces. From a similar code to the one in the text, we have the coordinate surfaces shown.



1.46 Begin with the Cartesian position vector, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},$ use the transformation equations, $x = a\sinh\xi \sin\eta \cos\phi$, $y = a\sinh\xi \sin\eta \sin\phi$ and $z = \cosh\xi \cos\eta$ to (a) find the **prolate ellipsoidal** position vector,

$$\mathbf{R} = a\sinh\xi \sin\eta \cos\phi\, \mathbf{e}_1 + a\sinh\xi \sin\eta \sin\phi\, \mathbf{e}_2 + z\cosh\xi \cos\eta\, \mathbf{e}_3,$$

(b) By differentiating partially with respect to the coordinate variables obtain the set $\left\{\frac{\partial\mathbf{R}}{\partial\xi}, \frac{\partial\mathbf{R}}{\partial\eta}, \frac{\partial\mathbf{R}}{\partial z}\right\}$.and (c) show that the basis vectors form a set of orthogonal vectors

```
In[1]:= R = {a Sinh[ξ] Sin[η] Cos[ϕ], a Sinh[ξ] Sin[η] Sin[ϕ], a Cosh[ξ] Cos[η]};
        cVec = Transpose[Grad[R, {ξ, η, ϕ}]];

In[3]:= Simplify[cVec] // MatrixForm

Out[3]//MatrixForm=
 ( a sin(η) cosh(ξ) cos(ϕ)    a sin(η) cosh(ξ) sin(ϕ)    a cos(η) sinh(ξ)
   a cos(η) sinh(ξ) cos(ϕ)    a cos(η) sinh(ξ) sin(ϕ)   -a sin(η) cosh(ξ)
  -a sin(η) sinh(ξ) sin(ϕ)    a sin(η) sinh(ξ) cos(ϕ)    0 )

In[4]:=
        Dot[cVec[[1]], cVec[[2]]]
        Dot[cVec[[2]], cVec[[3]]]
        Dot[cVec[[3]], cVec[[1]]]

Out[4]= -a^2 Cos[η] Cosh[ξ] Sin[η] Sinh[ξ] +
         a^2 Cos[η] Cos[ϕ]^2 Cosh[ξ] Sin[η] Sinh[ξ] + a^2 Cos[η] Cosh[ξ] Sin[η] Sin[ϕ]^2 Sinh[ξ]

Out[5]= 0

Out[6]= 0

In[7]:= Simplify[%4]

Out[7]= 0

In[11]:= Norm[cVec[[1]]]
         Norm[cVec[[2]]]
         Norm[cVec[[3]]]
```

1.47 Obtain the results in Q1.38 by Mathematica code

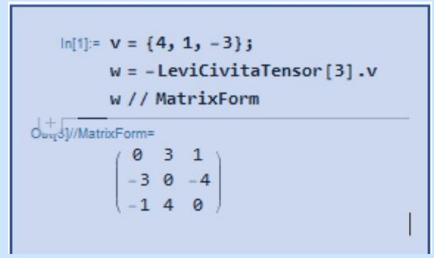
```
In[1]:= R = {ρ Sin[θ] Cos[ϕ], ρ Sin[θ] Sin[ϕ], ρ Cos[θ]};
        cVec = Transpose[Grad[R, {ρ, θ, ϕ}]];

In[3]:= cVec // MatrixForm

Out[3]//MatrixForm=
 (  Cos[ϕ] Sin[θ]        Sin[θ] Sin[ϕ]       Cos[θ]
    ρ Cos[θ] Cos[ϕ]      ρ Cos[θ] Sin[ϕ]    -ρ Sin[θ]
   -ρ Sin[θ] Sin[ϕ]      ρ Cos[ϕ] Sin[θ]     0 )

In[4]:=
        Dot[cVec[[1]], cVec[[2]]]
        Dot[cVec[[2]], cVec[[3]]]
        Dot[cVec[[3]], cVec[[1]]]

Out[4]= -ρ Cos[θ] Sin[θ] + ρ Cos[θ] Cos[ϕ]^2 Sin[θ] + ρ Cos[θ] Sin[θ] Sin[ϕ]^2

Out[5]= 0

Out[6]= 0

In[7]:= Simplify[%4]

Out[7]= 0
```

The basis vectors can be seen as the rows of the transpose of the gradient operation. Mutual orthogonality is demonstrated by taking the pairwise dot products and obtaining zero.

The result for the first two vectors did not initially appear to be zero. A Simplify function sets this right as can be seen in the code.

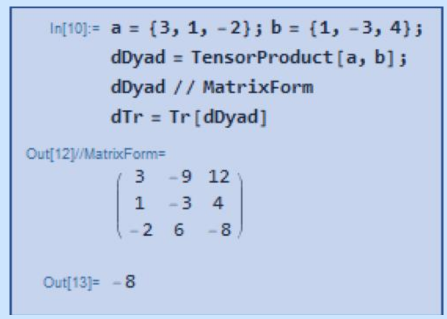
1.48	Obtain the result in 1.12 using Mathematica

The alternating tensor component, e_{ijk} is implemented as LeviCivitaTensor[]. The result here is the same as Q1.12 as expected.



1.49	Given the vectors $\mathbf{a} = 3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ and $\mathbf{b} = \mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$. Use Mathematica to find (a) the dyad $\mathbf{a} \otimes \mathbf{b}$, and (b) $\text{tr}(\mathbf{a} \otimes \mathbf{b})$

The functions TensorProduct[] and Tr[] perform the two operations directly. The result can be compared to Q1.34



1.50	Use the component form in equation 23 to show that the trace of a dyad is the scalar product of its operands.

Given vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_j\mathbf{e}_j$, From Equation 23, we find that the dyad product, written in component form is,

$$\mathbf{a} \otimes \mathbf{b} = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

The trace of this,

$$\begin{aligned} \text{tr}(\mathbf{a} \otimes \mathbf{b}) &= \text{tr}\left(a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= a_i b_j \,\text{tr}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= a_i b_j \,\text{tr}\left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = a_i b_j \delta_{ij} \\ &= a_i b_i \end{aligned}$$

Which is the scalar product of $\mathbf{a}$and $\mathbf{b}$.

1.51 Given the vectors $\mathbf{a} = 3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3, \mathbf{b} = \mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$ and $\mathbf{c} = -\mathbf{e}_1 - 4\mathbf{e}_2 + \mathbf{e}_3$. Use Mathematica to demonstrate that the dyad operation is NOT commutative by finding (a) $(\mathbf{a} \otimes \mathbf{b})\mathbf{c}$, and (b) $(\mathbf{b} \otimes \mathbf{a})\mathbf{c}$.

```
In[1]:= a = {3, 1, 2}; b = {1, -3, 4}; c = {-1, -4, 1};
        dDyad1 = TensorProduct[a, b];
        dDyad2 = TensorProduct[b, a];
        resOne = dDyad1.c
        resTwo = dDyad2.c

Out[4]= {45, 15, 30}

Out[5]= {-5, 15, -20}
```

The two operations as the code shows are not the same. This clearly demonstrates that, we cannot assume equality $\mathbf{b} \otimes \mathbf{a}$ does not produce the same result as $\mathbf{a} \otimes \mathbf{b}$ when they are both operating on the same vector. This means that, in general, $\mathbf{a} \otimes \mathbf{b} \neq \mathbf{b} \otimes \mathbf{a}$

1.52 Given the vectors $\mathbf{a} = 3\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3$ *and* $\mathbf{b} = \mathbf{e}_1 - 3\mathbf{e}_2 + 4\mathbf{e}_3$. Use Mathematica to find (a) $\mathbf{a} \otimes \mathbf{b}$ and (b) $\mathbf{b} \otimes \mathbf{a}$. Tell the relationship, if any, between $\mathbf{a} \otimes \mathbf{b}$ and $\mathbf{b} \otimes \mathbf{a}$

Looking at the results generated by $\mathbf{b} \otimes \mathbf{a}$ and $\mathbf{a} \otimes \mathbf{b}$ in the code here, it is clear that

$$\mathbf{a} \otimes \mathbf{b} = (\mathbf{b} \otimes \mathbf{a})^{\mathrm{T}}$$

```
a = {3, 1, 2}; b = {1, -3, 4};
dDyad1 = TensorProduct[a, b];
dDyad2 = TensorProduct[b, a];
MatrixForm[dDyad1]
MatrixForm[dDyad2]

Out[ ]//MatrixForm=
( 3  -9  12 )
( 1  -3   4 )
( 2  -6   8 )

Out[ ]//MatrixForm=
(  3   1   2 )
( -9  -3  -6 )
( 12   4   8 )
```

Consequent upon this observation, we can also see that the diagonal elements are the same. It also follows that

$$\mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) = \mathrm{tr}(\mathbf{b} \otimes \mathbf{a})^{\mathrm{T}}$$

So that the trace operation is Not affected by changing the operand order in the tensor product.

1.53 Begin with the Cartesian position vector, $\mathbf{R} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, use the transformation equations, $x = a\cosh\xi\cos\eta\cos\phi$, $y = a\cosh\xi\cos\eta\sin\phi$ and $z = \cosh\xi\cos\eta$ to (a) find the **oblate ellipsoidal** position vector,

$$\mathbf{R} = a\cosh\xi\cos\eta\cos\phi\,\mathbf{e}_1 + a\cosh\xi\cos\eta\sin\phi\,\mathbf{e}_2 + a\sinh\xi\sin\eta\,\mathbf{e}_3,$$

(b) By differentiating partially with respect to the coordinate variables obtain the set $\left\{\frac{\partial\mathbf{R}}{\partial\xi}, \frac{\partial\mathbf{R}}{\partial\eta}, \frac{\partial\mathbf{R}}{\partial z}\right\}$.and (c) show that the basis vectors form a set of orthogonal vectors

```
In[3]:= R = {a Cosh[ξ] Cos[η] Cos[ϕ], a Cosh[ξ] Cos[η] Sin[ϕ], a Sinh[ξ] Sin[η]};
        cVec = Transpose[Grad[R, {ξ, η, ϕ}]];

In[5]:= Simplify[cVec] // MatrixForm

Out[5]//MatrixForm=
  ( a Cos[η] Cos[ϕ] Sinh[ξ]     a Cos[η] Sin[ϕ] Sinh[ξ]     a Cosh[ξ] Sin[η] )
  ( -a Cos[ϕ] Cosh[ξ] Sin[η]    -a Cosh[ξ] Sin[η] Sin[ϕ]    a Cos[η] Sinh[ξ] )
  ( -a Cos[η] Cosh[ξ] Sin[ϕ]    a Cos[η] Cos[ϕ] Cosh[ξ]     0                )

In[6]:=
        Dot[cVec[[1]], cVec[[2]]]
        Dot[cVec[[2]], cVec[[3]]]
        Dot[cVec[[3]], cVec[[1]]]

Out[6]= a^2 Cos[η] Cosh[ξ] Sin[η] Sinh[ξ] -
         a^2 Cos[η] Cos[ϕ]^2 Cosh[ξ] Sin[η] Sinh[ξ] - a^2 Cos[η] Cosh[ξ] Sin[η] Sin[ϕ]^2 Sinh[ξ]

Out[7]= 0

Out[8]= 0

In[9]:= Simplify[%6]

Out[9]= 0
```

1.54 Given the rotation tensor,

$$\mathbf{Q} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix} 0.84339393538 & 0.389796451909 & 0.36979101642 \\ -0.2752066485 & 0.90450849718 & -0.325769364916 \\ -0.461462859127 & 0.172982960416 & 0.87013155617 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

Find the set of coordinate axes $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ that $\mathbf{Q}$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to. Demonstrate that $\mathbf{Q} = \boldsymbol{\xi}_i \otimes \mathbf{e}_i$.

	For each $\mathbf{e}_i$ the vector $$\boldsymbol{\xi}_i = \mathbf{Q}\mathbf{e}_i$$
1.55	Use Equation 31 to show that $e_{121} = 0$
	From equation 131, we have, $$e_{ijk} \equiv \begin{vmatrix} \delta_{1i} & \delta_{1j} & \delta_{1k} \\ \delta_{2i} & \delta_{2j} & \delta_{2k} \\ \delta_{3i} & \delta_{3j} & \delta_{3k} \end{vmatrix}$$ so that, substituting and swapping columns 1 and 3, $$e_{121} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{11} \\ \delta_{21} & \delta_{22} & \delta_{21} \\ \delta_{31} & \delta_{32} & \delta_{31} \end{vmatrix} = -\begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{11} \\ \delta_{21} & \delta_{22} & \delta_{21} \\ \delta_{31} & \delta_{32} & \delta_{31} \end{vmatrix}$$ Since a determinant changes sign when two columns are swapped. This fact is already obvious from the fact that two columns are identical. This is the same argument whenever any two indices in e_{ijk} coincide.
1.56	Find the parametric equation of a straight line in 3D. Hence draw the intersecting surfaces for the points $\mathbf{A}(1,2,1), \mathbf{B}(1,2,2)$ as well as the position vector $\mathbf{OB}$

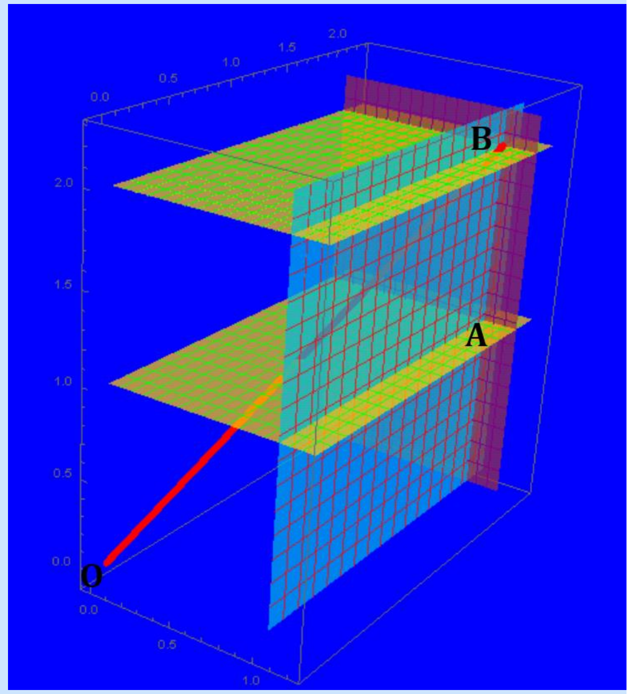
Point $\mathbf{A}(1,2,1)$ is the intersection of the planes, $x_1 = 1, x_2 = 2, x_3 = 1$; Point $\mathbf{B}(1,2,2)$ lies at the intersection of the planes, $x_1 = 1$, $x_2 = 2$ and $x_3 = 2$. Position vector $\mathbf{OB}$ joins $\mathbf{B}$ to the origin. This line has the equation,

$$\frac{x_1 - 0}{1 - 0} = \frac{x_2 - 0}{2 - 0} = \frac{x_3 - 0}{2 - 0}$$

Or,

$$x_1 = \frac{x_2}{2} = \frac{x_3}{2} = t$$

Where $t = 0$ is the origin, and $t = 1$ is point $\mathbf{B}$. At any other point on the line,



$x_1 = t$, $x_2 = 2t$, and $x_3 = 2t$. This line is drawn by the command,

```
ParametricPlot3D[{t, 2 t, 2 t}, {t, 0, 1}, PlotStyle → Directive[Red, Thickness[0.01]]
```

because $0 \leq t \leq 1$. This line and the four surfaces can be plotted as in the attached code:

```
plotStyle[color_RGBColor] := Directive[color, Opacity[0.7], Specularity[White, 20]];
x1 = ParametricPlot3D[{t, 2 t, 2 t}, {t, 0, 1}, PlotStyle → Directive[Red, Thickness[0.01]]];
x2 = ParametricPlot3D[{1, y, z}, {y, 0, 2.2}, {z, 0, 2.2}, Mesh → 16,
  MeshStyle → Directive[Opacity[.8], Thin, Red], ExclusionsStyle → {None, Red}, ImageSize → Large,
  PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Cyan]], SphericalRegion → True];
x3 = ParametricPlot3D[{x, 2, z}, {x, 0, 1.2}, {z, 0, 2.2}, Mesh → 16, MeshStyle → Directive[Thin, Purpl
   ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle
   SphericalRegion → True];
x4 = ParametricPlot3D[{x, y, 2}, {x, 0, 1.2}, {y, 0, 2.2}, Mesh → 16,
   MeshStyle → Directive[Opacity[1], Thin, Green], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
   SphericalRegion → True];
x4a = ParametricPlot3D[{x, y, 1}, {x, 0, 1.2}, {y, 0, 2.2}, Mesh → 16,
   MeshStyle → Directive[Opacity[.8], Thin, Green], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
   SphericalRegion → True];
Show[x2, x3, x4, x4a, x1, Background → Blue]
```

1.57 Given that the transformation equations for Spherical Polar Coordinates are$x_1 = \rho \sin\theta \cos\phi$ $x_2 = \rho \sin\theta \sin\phi$ and $x_3 = \cos\theta$ Find the spherical coordinates of the Cartesian points $\mathbf{A}(1,2,1), \mathbf{B}(1,2,2)$

We observe that the two points are both in the first octant with the Cartesian coordinate points all positive. We solve the sets of equations,

$$\rho \sin\theta \cos\phi = 1, \rho \sin\theta \sin\phi = 2, \rho \cos\theta = 1$$

and

$$\rho \sin\theta \cos\phi = 1, \rho \sin\theta \sin\phi = 2, \rho \cos\theta = 2$$

This Mathematica code,

```
Solve[{ρ Cos[ϕ] Sin[θ] == 1.0, ρ Sin[ϕ] Sin[θ] == 2.0, ρ Cos[θ] == 1.0}, {ρ, ϕ, θ}]
Solve[{ρ Cos[ϕ] Sin[θ] == 1.0, ρ Sin[ϕ] Sin[θ] == 2.0, ρ Cos[θ] == 2.0}, {ρ, ϕ, θ}]
··· Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete s
   information.
{{ρ → -2.44949, ϕ → -2.03444, θ → 1.99133}, {ρ → -2.44949, ϕ → 1.10715, θ → -1.99133},
 {ρ → 2.44949, ϕ → -2.03444, θ → -1.15026}, {ρ → 2.44949, ϕ → 1.10715, θ → 1.15026}}
··· Solve: Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete s
   information.
{{ρ → -3., ϕ → -2.03444, θ → 2.30052}, {ρ → -3., ϕ → 1.10715, θ → -2.30052},
 {ρ → 3., ϕ → -2.03444, θ → -0.841069}, {ρ → 3., ϕ → 1.10715, θ → 0.841069}}
```

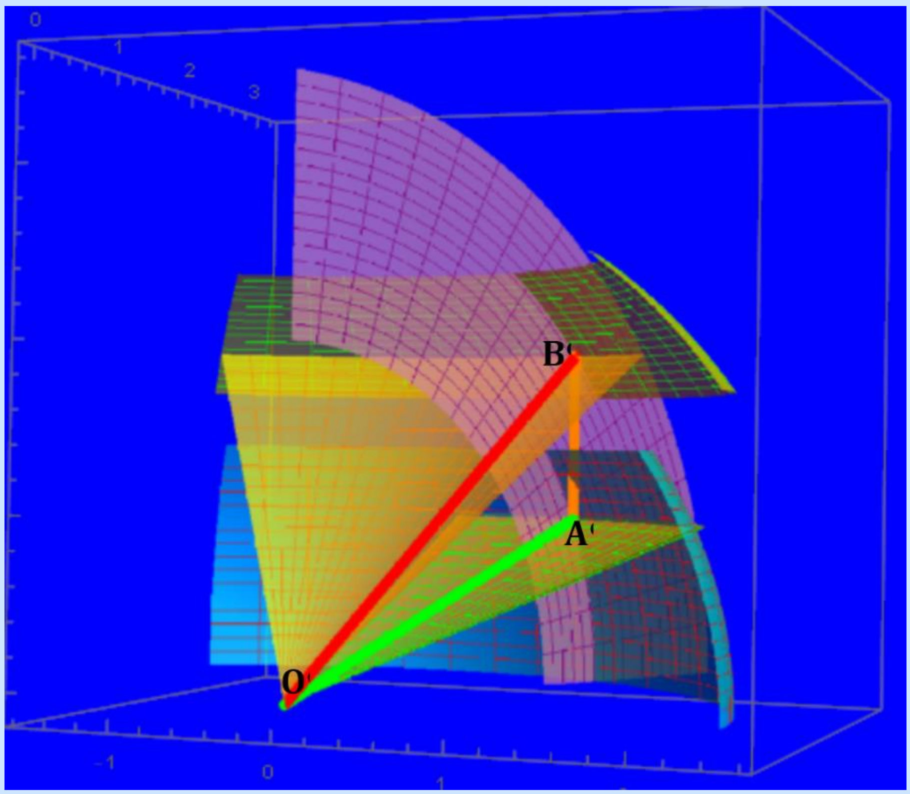
Gives, for points $\mathbf{A}(1,2,1), \mathbf{B}(1,2,2)$ respectively $\{\rho \to 2.449489742783178, \phi \to 1.1071487177940904, \theta \to 1.1502619915109316\}$, $\{\rho \to 3., \phi \to 1.1071487177940904, \theta \to 0.8410686705679302\}$. These are the solutions for the first octant.

1.58 Given that the transformation equations for Spherical Polar Coordinates are$x_1 = \rho \sin\theta \cos\phi$ $x_2 = \rho \sin\theta \sin\phi$ and $x_3 = \cos\theta$ Plot vectors $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{AB}$ as well as the Spherical coordinate surfaces meeting the Cartesian points $\mathbf{A}(1,2,1), \mathbf{B}(1,2,2)$. Explain how many parameters you need to use for a line and how many for a plane. Use the spherical transforms for $\mathbf{A}$ and $\mathbf{B}$ obtained in Q1.57.

Lines $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{AB}$ are shown in green, red and orange in the computed diagram shown. Observe that the two position vectors $\mathbf{OA}$ and $\mathbf{OB}$ lie in the intersections of cones $\theta = 1.1502, 0.841$ and the plane $\phi = 1.1071$. $\mathbf{AB}$ is the vertical line

$\{x = 1, y = 2, z = t; 1 \leq t \leq 2\}$

$\mathbf{A}$ lies on the sphere radius 2.449, while $\mathbf{B}$ lies in the sphere radius 3.0. Lines $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{AB}$ are all on the plane $\phi = 1.1071$ as can be seen by the constancy of that coordinate on all three lines. The origin lies on all ϕ-planes.

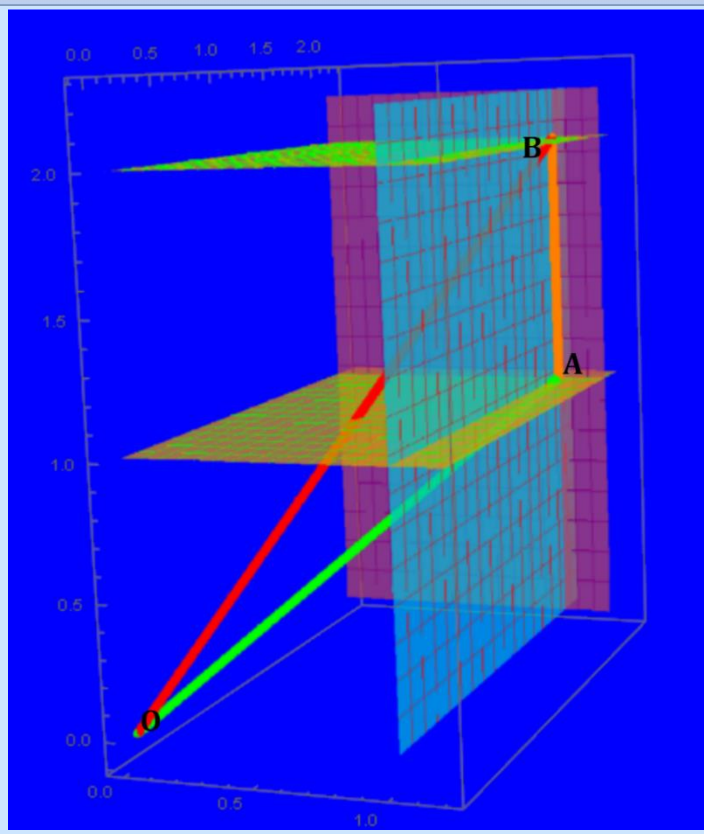


Note that parametric plots of surfaces require two parameters while that of a line – linear or curved, is governed by a single parameter. Again, confirming that a line is one dimensional while a surface is two dimensional.

```
plotStyle[color_RGBColor] := Directive[color, Opacity[0.7], Specularity[White, 20]];
x1 = ParametricPlot3D[{t, 2 t, 2 t}, {t, 0, 1}, PlotStyle → Directive[Red, Thickness[0.0
x1a = ParametricPlot3D[{ρ Sin[1.1502619915109316] Cos[1.1071487177940904],
    ρ  Sin[1.1502619915109316] Sin[1.1071487177940904], ρ Cos[1.1502619915109316]},
   {ρ, 0, 2.449489742783178}, PlotStyle → Directive[Green, Thickness[0.01]]];
x1b = ParametricPlot3D[{1, 2, t}, {t, 1, 2}, PlotStyle → Directive[Orange, Thickness[0.
x2 = ParametricPlot3D[{2.449489742783178 Sin[θ] Cos[ϕ], 2.449489742783178 Sin[θ] Sin[
    2.449489742783178 Cos[θ]}, {ϕ, 0, 2.2}, {θ, .95, Pi / 2}, Mesh → 16,
   MeshStyle → Directive[Opacity[.8], Thin, Red], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Cyan]],
   SphericalRegion → True];
x3 = ParametricPlot3D[{ρ Sin[θ] Cos[1.1071487177940904], ρ Sin[θ] Sin[1.107148717794090
     ρ Cos[θ]}, {ρ, 2.0, 3.5}, {θ, 0, π / 2}, Mesh → 16, MeshStyle → Directive[Thin, Purp
    ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64,
    PlotStyle → Evaluate[plotStyle[Pink]],
    SphericalRegion → True];
x4 = ParametricPlot3D[{ρ Sin[1.1502619915109316] Cos[ϕ], ρ Sin[1.1502619915109316] Sin
     ρ Cos[1.1502619915109316]}, {ρ, 0, 2.5}, {ϕ, 0, 1.5}, Mesh → 16,
    MeshStyle → Directive[Opacity[1], Thin, Green], ExclusionsStyle → {None, Red},
    ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
    SphericalRegion → True];
x4a = ParametricPlot3D[{3.0 Sin[θ] Cos[ϕ], 3.0 Sin[θ] Sin[ϕ], 3.0 Cos[θ]},
    {ϕ, 0, 2.2}, {θ, .6, .95}, Mesh → 16,
    MeshStyle → Directive[Opacity[.8], Thin, Green], ExclusionsStyle → {None, Red},
    ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
    SphericalRegion → True];
x4b = ParametricPlot3D[{ρ Sin[0.8410686705679302] Cos[ϕ], ρ Sin[0.8410686705679302] Si
     ρ Cos[0.8410686705679302]}, {ρ, 0, 3}, {ϕ, .8, 2.2}, Mesh → 16,
    MeshStyle → Directive[Opacity[.8], Thin, Orange], ExclusionsStyle → {None, Red},
    ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
    SphericalRegion → True];
Show[x2, x3, x4, x4a, x4b, x1, x1a, x1b, Background → Blue]
```

1.59	Plot vectors $\mathbf{OA}$, $\mathbf{OB}$ and $\mathbf{AB}$ as well as the coordinate surfaces meeting at the Cartesian points $\mathbf{A}(1,2,1)$, $\mathbf{B}(1,2,2)$. Parametrize lines by linear equations. .
	The attached code shows the parametrized lines using the Cartesian equation for lines. This and the rest of the implementation is as shown in the attached code:

```
plotStyle[color_RGBColor] := Directive[color, Opacity[0.7], Specularity[White, 20]];
x1 = ParametricPlot3D[{t, 2 t, 2 t}, {t, 0, 1}, PlotStyle → Directive[Red, Thickness[0.01]]];
x1b = ParametricPlot3D[{t, 2 t, t}, {t, 0, 1}, PlotStyle → Directive[Green, Thickness[0.01]]];
x1a = ParametricPlot3D[{1, 2, t}, {t, 1, 2}, PlotStyle → Directive[Orange, Thickness[0.01]]];
x2 = ParametricPlot3D[{1, y, z}, {y, 0, 2.2}, {z, 0, 2.2}, Mesh → 16,
  MeshStyle → Directive[Opacity[.8], Thin, Red], ExclusionsStyle → {None, Red}, ImageSize → Large,
  PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Cyan]], SphericalRegion → True];
x3 = ParametricPlot3D[{x, 2, z}, {x, 0, 1.2}, {z, 0, 2.2}, Mesh → 16, MeshStyle → Directive[Thin, Purple]
   ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Pi
   SphericalRegion → True];
x4 = ParametricPlot3D[{x, y, 2}, {x, 0, 1.2}, {y, 0, 2.2}, Mesh → 16,
   MeshStyle → Directive[Opacity[1], Thin, Green], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
   SphericalRegion → True];
x4a = ParametricPlot3D[{x, y, 1}, {x, 0, 1.2}, {y, 0, 2.2}, Mesh → 16,
   MeshStyle → Directive[Opacity[.8], Thin, Green], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Yellow]],
   SphericalRegion → True];
Show[x2, x3, x4, x4a, x1, x1b, x1a, Background → Blue]
```

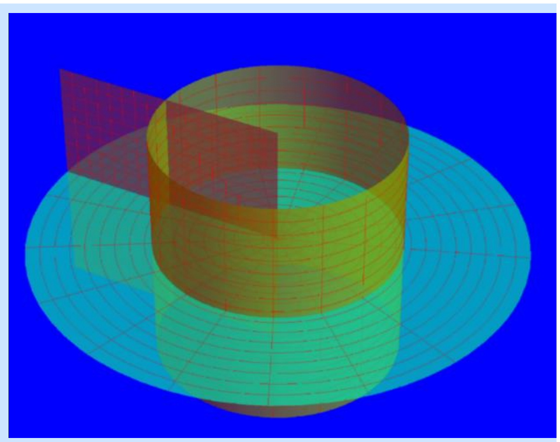


General Surfaces. The following code will be called to generate coordinate surfaces for different curvilinear systems in the questions that follow.

```
plotStyle[color_RGBColor] := Directive[
   color, Opacity[0.7], Specularity[White, 20]
  ];
plotSurface[eqns_List, rule_List, edge_List, form_List, color_RGBColor] :=
  ParametricPlot3D[
   eqns /. rule[[form[[1]]]],
   Evaluate[Sequence[edge[[form[[2]]]], edge[[form[[3]]]]]],
   Mesh → 16,
   MeshStyle → Directive[Opacity[.8], Thin, Red],
   (*BoundaryStyle→Directive[Red,Thick],*)
   ExclusionsStyle → {None, Red},
   ImageSize → Large,
   PlotPoints → 64,
   PlotStyle → Evaluate[plotStyle[color]],
   SphericalRegion → True
  ];
plotSurfaces[eqns_List, rule_List, edge_List] := Show[Table[
    plotSurface[eqns, rule, edge, RotateRight[{1, 2, 3}, iota],
     Evaluate[{Cyan, Pink, Yellow}[[iota]]]],
    {iota, 3}
   ], Axes → False, Boxed → False, ImageSize → Large, ImageResolution → 600,
   Background → Blue];
```

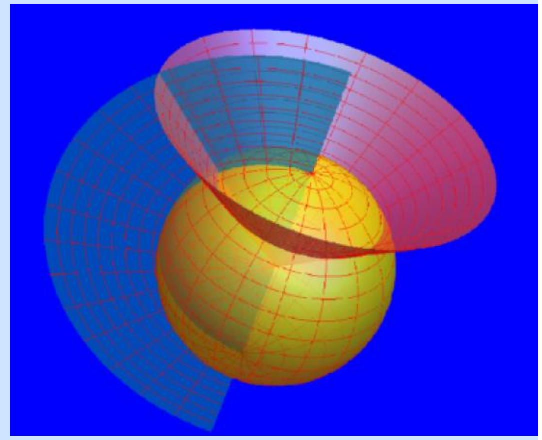
1.60 Generate the coordinate surfaces for the Cylindrical Polar System

```
eqns={r Cos[φ],r Sin[φ],z};
syms={r,φ,z};
rule={r->2.5,φ->π,z->2.5//N};
edge={{r,0,5},{φ,0,2π},{z,0,5}};
plotSurfaces[eqns,rule,edge]
```



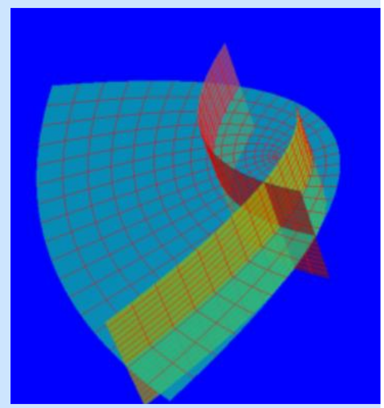
1.61	Generate the coordinate surfaces for the Spherical Polar System

```
seqns={ρ Sin[θ]Cos[ϕ],ρ Sin[θ]Sin[ϕ],ρ Cos[θ]};
ssyms={ρ,θ,ϕ};
srule={ρ->2.5,θ->π/4,ϕ->π};
sedge={{ρ,0,5},{θ,0,π},{ϕ,0,2π}};
plotSurfaces[seqns,srule,sedge]
```



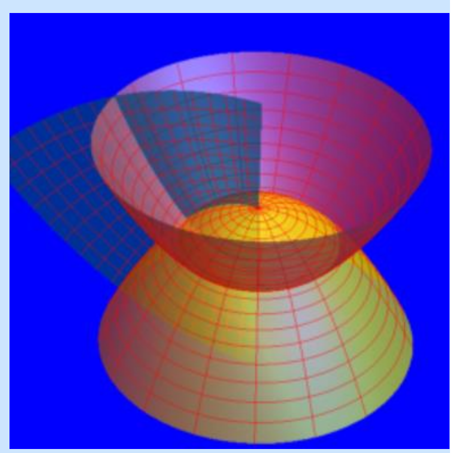
1.62	Generate the coordinate surfaces for the Parabolic Cylindrical System

```
pceqns={ξ η,1/2 (ξ^2-η^2),z};
pcsyms={ξ,η,z};
pcrule={{ξ->3π/4},{η->π},{z->0}};
pcedge={{ξ,-π,π},{η,0,2π},{z,-5,5}};
plotSurfaces[pceqns,pcrule,pcedge]
```



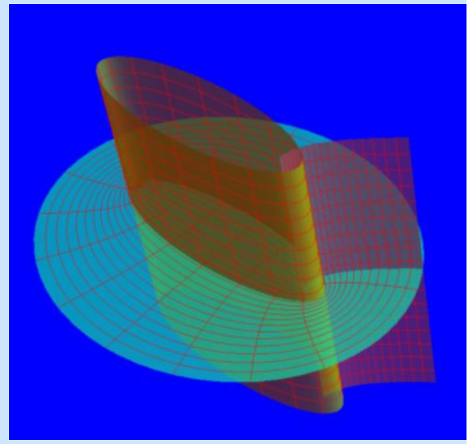
1.63	Generate the coordinate surfaces for the Cylindrical Polar System

```
peqns={ξ η Cos[φ],ξ η Sin[φ],1/2 (ξ^2-η^2)};
psyms={ξ,η,φ};
prule={ξ->2.5,η->2.5,φ->π};
pedge={{ξ,0,5},{η,0,5},{φ,0,2π}};
plotSurfaces[peqns,prule,pedge]
```



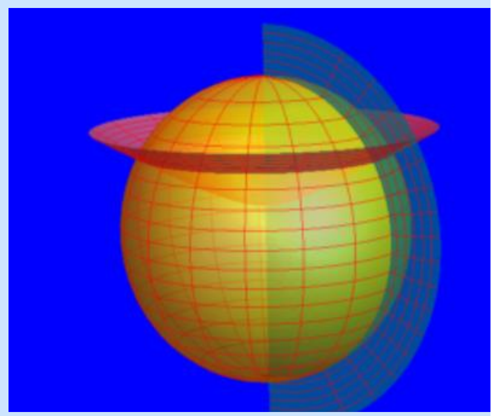
1.64 Generate the coordinate surfaces for the Elliptic Cylindrical System

```
eceqns={Cosh[ξ]Cos[ η],Sinh[ξ]Sin[ η],z};
ecsyms={ξ,η,z};
ecrule={ξ->0.3,η->π/6,z->1.25};
ecedge={{ξ,0,1.1},{η,0,2π},{z,0,2.5}};
plotSurfaces[eceqns,ecrule,ecedge]
```



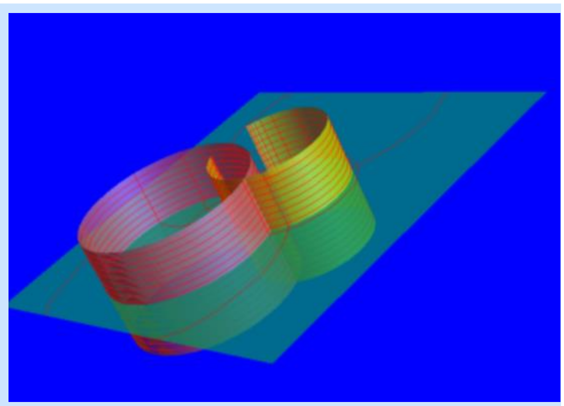
1.65 Generate the coordinate surfaces for the Elliptic System

```
eeqns={Cos[φ] Sqrt[(1-η^2) (ξ^2-1)],Sin[φ] Sqrt[(1-η^2) (ξ^2-1)],ξ η};
esyms={ξ,η,φ};
erule={ξ->3,η->-0.5,φ->π/8};
eedge={{ξ,1,4},{η,-1,1},{φ,0,2π}};
plotSurfaces[eeqns,erule,eedge]
```



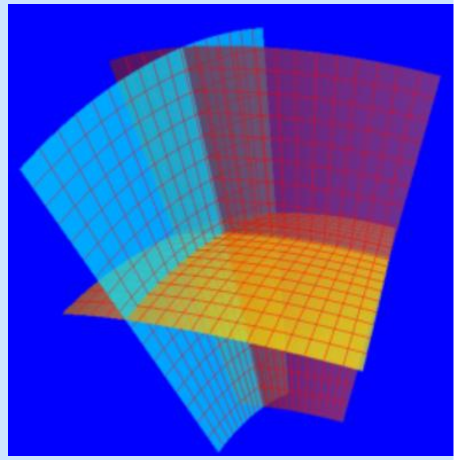
1.66 Generate the coordinate surfaces for the BiPolar System

```
beqns={a Sinh[v]/(Cosh[v]-Cos[u]),a Sin[u]/(Cosh[v]-Cos[u]),z};
bsyms={u,v,z};
brule={{u->π/8,a->2},{v->π/12,a->2},{z->0,a->2}};
bedge={{u,0,2π},{v,-π,π},{z,-5,5}};
plotSurfaces[beqns,brule,bedge]
```



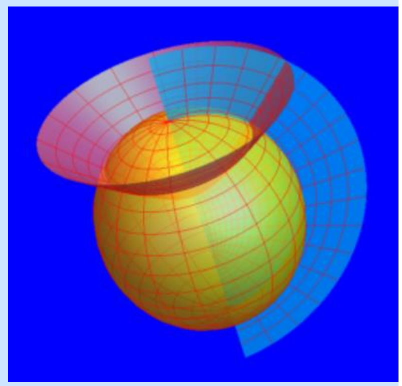
1.67 Generate the coordinate surfaces for the Conical System

```
beqns={a Sinh[v]/(Cosh[v]-Cos[u]),a Sin[u]/(Cosh[v]-Cos[u]),z};
bsyms={u,v,z};
brule={{u->π/8,a->2},{v->π/12,a->2},{z->0,a->2}};
bedge={{u,0,2π},{v,-π,π},{z,-5,5}};
plotSurfaces[beqns,brule,bedge]
```



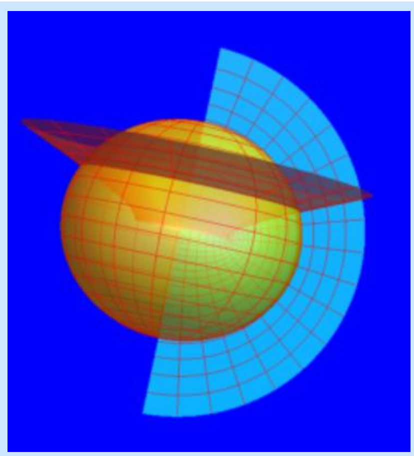
1.68 Generate the coordinate surfaces for the Prolate Spheroidal System

```
pseqns={a Sinh[ξ]Sin[η]Cos[φ],a Sinh[ξ]Sin[η]Sin[φ],a Cosh[ξ]Cos[η]};
pssyms={ξ,η,φ};
psrule={{ξ->1.5,a->.2},{η->π/4,a->.2},{φ->π/2,a->.2}};
psedge={{ξ,0,2},{η,0,π},{φ,0,2π}};
plotSurfaces[pseqns,psrule,psedge]
```



1.69 Generate the coordinate surfaces for the Oblate Spheroidal System

```
eqns={r Cos[ϕ],r Sin[ϕ],z};
syms={r,ϕ,z};
rule={r->2.5,ϕ->π,z->2.5//N};
edge={{r,0,5},{ϕ,0,2π},{z,0,5}};
plotSurfaces[eqns,rule,edge]
```



1.70 Parametrize and plot the line joining points $\mathbf{A}(1,2,-5)$ to $\mathbf{B}(3,3,0)$

The vector

$$\mathbf{v_{AB}} = (3-1)\mathbf{e}_1 + (3-2)\mathbf{e}_2 + \big(0-(-5)\big)\mathbf{e}_3$$
$$= 2\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3$$

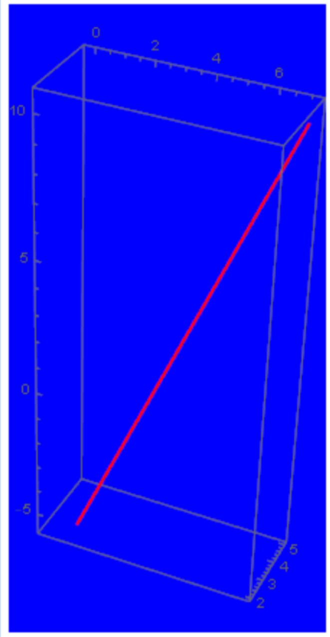
Consider a Cartesian point (x, y, z) on the line; position vector, of this point is $\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$. For $\mathbf{r}$ to be on this line, we must have

$$\mathbf{r} = \mathbf{r_A} + \alpha\mathbf{v_{AB}}$$

for scalar α since $\mathbf{v_{AB}} = \|\mathbf{v_{AB}}\|\mathbf{u}$ where $\mathbf{u}$ is the unit vector along $\mathbf{AB}$.

$$x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$
$$= \mathbf{e}_1 + 2\mathbf{e}_2 - 5\mathbf{e}_3$$
$$+ \alpha(2\mathbf{e}_1 + \mathbf{e}_2 + 5\mathbf{e}_3)$$

```
line = ParametricPlot3D[{2 α + 1, α + 2, 5 α - 5}, {α, 0, 3},
   PlotStyle → Directive[Thickness[0.01], Red]];
Show[line, Background → Blue]
```



From which we have,

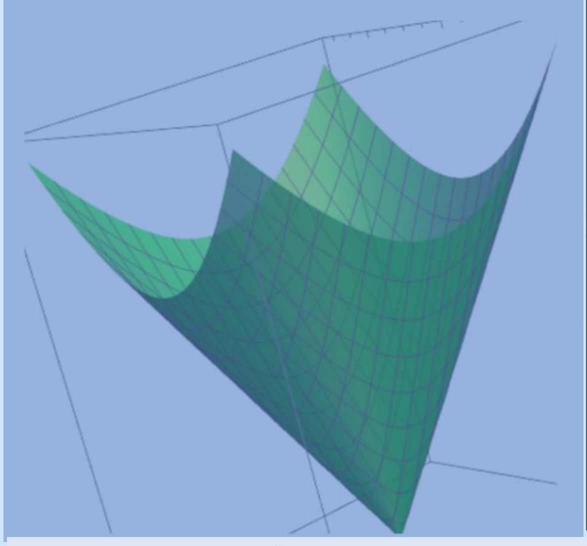
$$\alpha = \frac{x_1 - 1}{2} = x_2 - 2 = \frac{x_3 + 5}{5}$$

The equations, $x_1 = 2\alpha + 1$, $x_2 = \alpha + 2$ and $x_3 = 5\alpha - 5$ parametrize the line. A ParametricPlot3D[ ] function call is all that is needed to give effect to this.

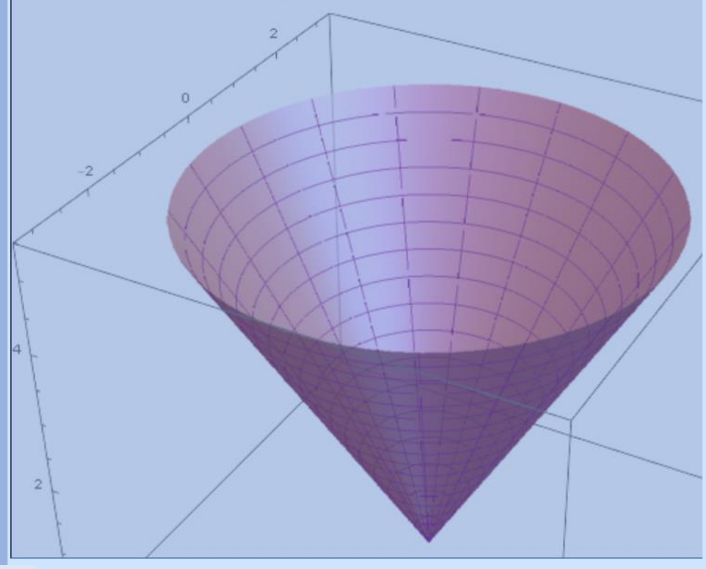
1.71 For an upward-facing cone with half-angle $\pi/6$, parametrize and plot using (a) Cartesian based equations in the range, $0 \le x_1 \le 3, 0 \le x_2 \le 3$ (b) Cylindrical Polar and (c) Spherical Polar Coordinates. (d) Explain any discrepancies.

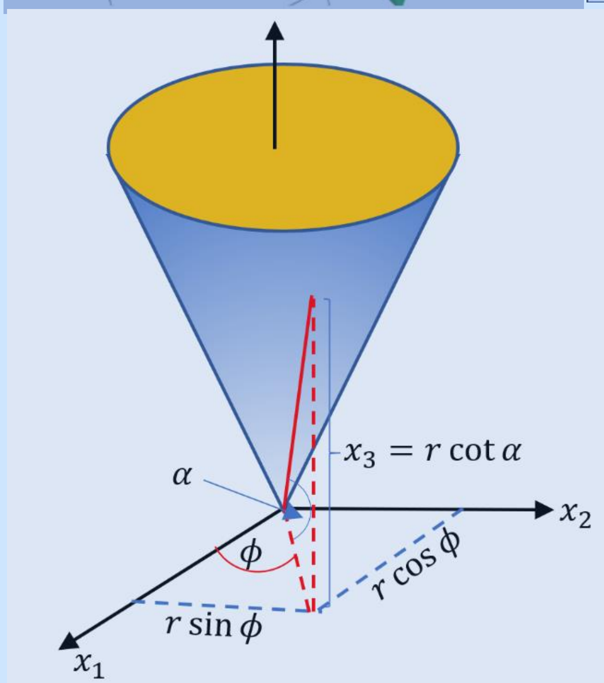
(a) For an inclined half angle α the z component of any point on the cone can be obtained from, $z^2 = (\cot^2\alpha)(x^2 + y^2)$. We can parametrize by the variables x and y such that $\sqrt{(\cot^2\alpha)(x^2 + y^2)}$ replaces z.

```
cone1 = ParametricPlot3D[{x, y, Cot[π / 6] Sqrt[x^2 + y^2]},
  {x, -3, 3}, {y, -3, 3}, Mesh → 16, MeshStyle →
   Directive[Thin, Purple], ExclusionsStyle → {None, Red},
  ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate
    [Directive[Green, Opacity[0.7], Specularity[White, 20]]]]
```



```
cone2 = ParametricPlot3D[{r Cos[ϕ], r Sin[ϕ], r Cot[π / 6]}, {ϕ, 0, 2 π},
  {r, 0, 3 }, Mesh → 16, MeshStyle → Directive[Thin, Purple],
  ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64,
  PlotStyle → Directive[Pink, Opacity[0.7], Specularity[White, 20]]]
```





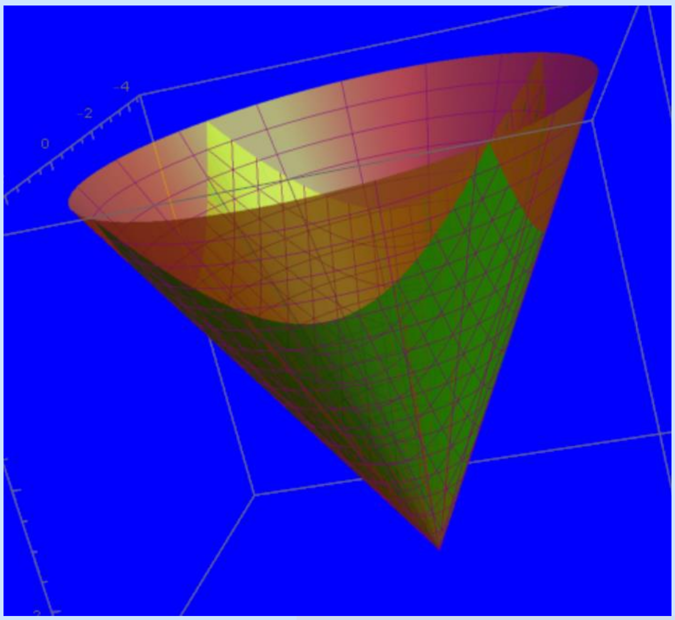
(b) In Cylindrical Polar coordinates, $x_1 =$
(c) $r\cos\phi$, $x_2 = r\sin\phi$, and $x_3 = r\cot\alpha$ for an included half-angle α. The ParametricPlot3D function call in the identifier cone3 shown above implements these parameters

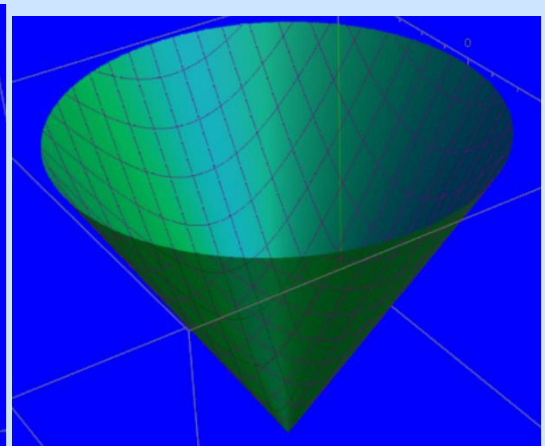
(d) In spherical coordinates,
$x_1 = \rho\sin\theta\cos\phi$, $x_2 = \rho\sin\theta\sin\phi$, and $x_3 = \rho\cos\theta$. The θ=const coordinate surface is a cone including a half-angle θ. Parametrization is therefore straightforward. This cone3 is exactly the same as cone2 for cylindrical polar. Hence a figure is not included for it.

```
cone3 = ParametricPlot3D[{ρ Cos[ϕ] Sin[π / 6], ρ Sin[ϕ] Sin[π / 6],
   ρ Cos[π / 6]}, {ϕ, 0, 2 π}, {ρ, 0, Sqrt[18 (1 + (Cot[π / 6]) ^2)] },
  Mesh → 16, MeshStyle → Directive[Thin, Purple], ExclusionsStyle
   → {None, Red}, ImageSize → Large, PlotPoints → 64, PlotStyle →
   Directive[Orange, Opacity[0.7], Specularity[White, 20]]]
```

(e) . The last figure in this answer is the combination of the two different cones. Observe that the first cone is clipped by the rectilinear coordinate planes of the Cartesian system. It is possible to correct this by adding more code. But as the expected cone results from the other two coordinate parametrizations, they are more suited to the construction and require less code lines to obtain the expected result. For completeness, here is the comparison:

The top of the Cartesian Parametrized cone can be clipped with plot option,





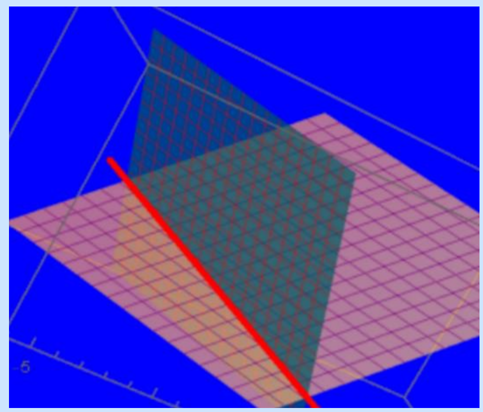
```
PlotRange → {{-3, 3}, {-3, 3}, {0, 5}}
```

1.72	Find three common points on the planes, $2x_1 + x_2 + 3x_3 = 10, and\ x_1 - x_2 + x_3 = 0.$
	Adding the two equations, we have, $$3x_1 + 4x_3 = 10$$ Any point on this line, satisfying the equation of both planes, must lie on both. Let $x_3 = 0$, $x_1 = \frac{10}{3}$. The point $\left(\frac{10}{3}, \frac{10}{3}, 0\right)$. Let $x_3 = 1$, $x_1 = 2$. Here $x_2 = x_1 + x_3 = 3$ so that $(2,3,1)$ also lies on the intersecting line. By testing $x_1 = 0$, we find that, $\left(0, \frac{5}{2}, \frac{5}{2}\right)$ also lies at the intersection.
1.73	Given that the point $(2,3,1)$ is a common point on the planes, $2x_1 + x_2 + 3x_3 = 10, and\ x_1 - x_2 + x_3 = 0$, (a) Find the line of intersection of the two planes, (b)

	parametrize and plot the intersection line and (c) parametrize and plot the two planes to demonstrate the answer to (a) above.

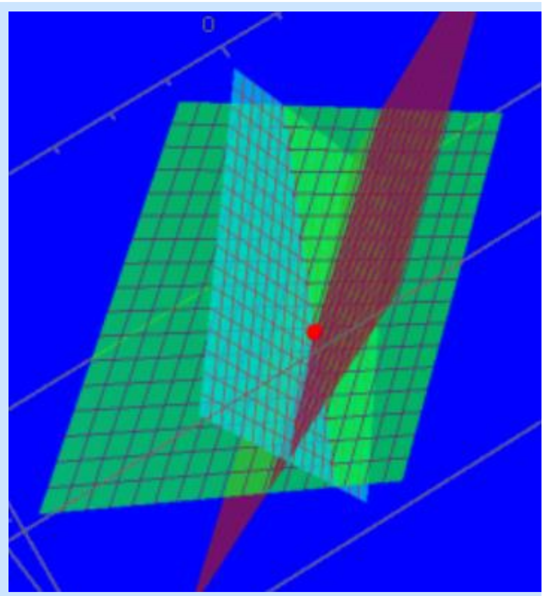
Clearly, the vector normals to the two planes are $2\mathbf{e}_1 + \mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$ respectively. The intersecting line is the line parallel to the cross product of these and passes through the common point $(2,1,3)$ that lies on both planes. The code below finds this vector and constructs the parametric equation of the line using it. Parametrizing each plane is done here using the values of x_1 and x_2 and evaluating x_3 for each plane:$\frac{1}{3}(10 - x_2 - 2x_1)$, $x_2 - x_1$ respectively. Each equation is therefore parametrized by the values of x_1 and x_2.

```
v1 = {2, 1, 3}; v2 = {1, -1, 1}; vL = Cross[v1, v2];
plotStyle[color_RGBColor] := Directive[color, Opacity[0.7], Specularity[White, 20]];
l1 = ParametricPlot3D[{2 + vL[[1]] t, 3 + vL[[2]] t, 1 + vL[[3]] t}, {t, -2, 1},
   PlotStyle → Directive[Red, Thickness[0.01]]];
p1 = ParametricPlot3D[{x1, x2, 1 / 3 (10 - x2 - 2 x1)}, {x1, -5, 5}, {x2, 0, 10}, Mesh → 16,
   MeshStyle → Directive[Opacity[.8], Thin, Red], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Cyan]],
   SphericalRegion → True];
p2 = ParametricPlot3D[{x1, x2, x2 - x1}, {x1, -5, 5}, {x2, 0, 10}, Mesh → 16,
   MeshStyle → Directive[Thin, Purple],
   ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64,
   PlotStyle → Evaluate[plotStyle[Pink]],
   SphericalRegion → True];
Show[l1, p1, p2, Background → Blue]
```



1.74	Show that the planes $2x_1 + x_2 + 3x_3 = 10, x_1 - x_2 + x_3 = 0\ and\ x_1 + x_2 + x_3 = 6$ meet at the point $(2,3,1)$. Demonstrate this by plotting the planes and the intersecting point.

To find the point of intersection of the three planes, we solve the three linear equations, $2x_1 + x_2 + 3x_3 = 10, x_1 - x_2 + x_3 = 0\ and\ x_1 + x_2 + x_3 = 6.$ The point of intersection is indeed the point $(2,3,1)$. In the code below, the parametric plotting of the three planes is quite straightforward. To plot the point, we use the same equation as if we were plotting a line. A dummy range is included to avoid error warning. The red dot is the plot of point $(2,3,1)$- the intersection of the three planes.



```
Solve[{2 x1 + x2 + 3 x3 == 10, x1 - x2 + x3 == 0, x1 + x2 + x3 == 6}];
plotStyle[color_RGBColor] := Directive[color, Opacity[0.7], Specularity[White, 20]];
point1 = ParametricPlot3D[{2, 3, 1}, {x1, 1, 2}, PlotStyle → Directive[Red, Thickness[0.02]]];
p1 = ParametricPlot3D[{x1, x2, 1 / 3 (10 - x2 - 2 x1)}, {x1, 0, 5}, {x2, 0, 5}, Mesh → 16,
   MeshStyle → Directive[Opacity[.8], Thin, Red], ExclusionsStyle → {None, Red},
   ImageSize → Large, PlotPoints → 64, PlotStyle → Evaluate[plotStyle[Cyan]],
   SphericalRegion → True];
p2 = ParametricPlot3D[{x1, x2, x2 - x1}, {x1, 0, 5}, {x2, 0, 5}, Mesh → 16,
  MeshStyle → Directive[Thin, Purple],
  ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64,
  PlotStyle → Evaluate[plotStyle[Pink]],
  SphericalRegion → True];
p3 = ParametricPlot3D[{x1, x2, 6 - x2 - x1}, {x1, 0, 5}, {x2, 0, 5}, Mesh → 16,
  MeshStyle → Directive[Thin, Purple],
  ExclusionsStyle → {None, Red}, ImageSize → Large, PlotPoints → 64,
  PlotStyle → Evaluate[plotStyle[Green]],
  SphericalRegion → True];
Show[point1, p1, p2, p3, Background → Blue]
```

General Curvilinear Systems

The Venerable Cartesian coordinate system has been our friend since high school. Its simplicity and accessibility at such an elementary level of our education rely on some properties that endow it with advantages over every other system of coordinates in use. Some of these are:

1. Cartesian coordinate surfaces are triplets of planes intersecting at the coordinate point. Moving from point to point, these triplets of planes remain parallel to the triplets planes at any other point: They therefore have the same normal vectors respectively.

2. As a consequence of #1 above, the coordinate curves, also meeting at the coordinate points, are straight lines that intersect orthogonally with one another.
3. When the base vectors are chosen, as they usually are, to be of unit magnitude, they remain unchanged from point to point. In Cartesian coordinates, the basis vectors, $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are orthonormal and constant.

What is a Coordinate System?

If at every point in the Euclidean point space, we define three continuously differentiable functions, $\xi^1(x_1, x_2, x_3)$, $\xi^2(x_1, x_2, x_3)$, $\xi^3(x_1, x_2, x_3)$, we can say that $\xi^i, i = 1, \dots, 3$ as a set, constitutes a coordinate system. For spherical polar coordinates, for example, the $\{\xi^1, \xi^2, \xi^3\}$ function set are $\{\rho, \phi, \theta\}$ where

$$\begin{aligned} \xi^1 &= \rho(x_1, x_2, x_3) = \sqrt{(x_1)^2 + (x_2)^2 + (x_3)^2} \\ \xi^2 &= \phi(x_1, x_2, x_3) = \tan^{-1}\frac{x_2}{x_1} \\ \xi^3 &= \theta(x_1, x_2, x_3) = \tan^{-1}\frac{\sqrt{(x_1)^2 + (x_2)^2}}{x_3} \end{aligned} \tag{98}$$

Of course, the origin of coordinates is excluded here as ξ^2 and ξ^3 are undefined at that point. We can obtain similar functions for the other coordinate systems we have defined. Using superscripts for the coordinate variables, we may write,

$$\xi^i(x_1, x_2, x_3), i = 1, \dots, 3 \tag{99}$$

are the coordinate functions, the values at each point, the coordinate variables, and while equations, $\xi^i(x_1, x_2, x_3) = const,\ i = 1, \dots, 3$ give us the coordinate surfaces which are spheres for $\rho(x_1, x_2, x_3) = const$, planes through the origin for $\phi(x_1, x_2, x_3) = const$ and conical surfaces with vertical axes ($z - axis$) and subtending half angles θ for $\theta(x_1, x_2, x_3) = const$. The intersections of pairs of planes give us the coordinate curves. The name, Curvilinear is a portmanteau from “curve-line” indicating that what used to be lines for Cartesian are now replaced by curves in these kinds of systems. Unlike the Cartesian systems, we now have,

1. curved surfaces for the coordinate surfaces. The curved surfaces are also in triplets. The normal on each surface is firstly a variable even on a particular sheet. Is also varies from sheet to sheet. The normal on each surface is a 3-D spatial field.

2. In the spherical coordinates example, that we have chosen, the surfaces as well as the coordinate curves meet at right angles. In general, curvilinear coordinates cannot be assumed to possess this orthogonal curvilinear quality. These base vectors can normalized even so, they still vary in direction from point to point.
3. As we shall see, the basis vectors, for general orthogonal systems, cannot be guaranteed to be unit vectors.

In a curvilinear coordinate system, we deal with curved coordinate surfaces instead of coordinate planes, curved coordinate lines and basis vectors that are not orthogonal, not normalized and not constant. It is no wonder that our interactions with such systems have been rather small until this level of education. Furthermore, it clear why we would never use such a system unless there are significant advantages to be gained by so doing. It turns out that there are problems that are unnecessarily complicated in their presentation unless we change to some curvilinear system. One example we already saw was the cone problem of Q1.74. The edge of the cone became jagged when we used Cartesian coordinates and we had to programmatically clip it off to make the parametric drawing there look like a cone! In many other cases, the difficulties become much more significant. When we look at the problems of bending of circular rods and beams, formulations in Cartesian coordinates are just not the way to go!

The rest of this chapter details the ingenuous ways by which, most of the results and ease we derive from working with simple orthonormal Cartesian systems accrue in such coordinate systems.

Base Vectors

Recall that to form a basis for any Euclidean space, you need three linearly independent vectors. In problems Q1.38, 1.42, 1.44, 1.46 and 1.47, we were able to generate basis vectors for different curvilinear systems by simply differentiating the position vector, $\mathbf{r}$. We generalize this procedure and find the respective basis vectors. We will also show that these vectors are tangents to the coordinate curves in any system curvilinear or not.

Consider neighboring points $\mathbf{P}(\xi^1, \xi^2, \xi^3)$ and $\mathbf{Q}(\xi^1 + \delta\xi^1, \xi^2, \xi^3)$ along ξ^1 on a coordinate surface describing a curvilinear system as shown in figure 20. Let point $\mathbf{O}$ be the origin so that $\mathbf{r} = \mathbf{OP}$, and $\mathbf{r} + \delta\mathbf{r} = \mathbf{OQ}$ so that $\delta\mathbf{r} = \mathbf{PQ}.$

Consider the quotient,

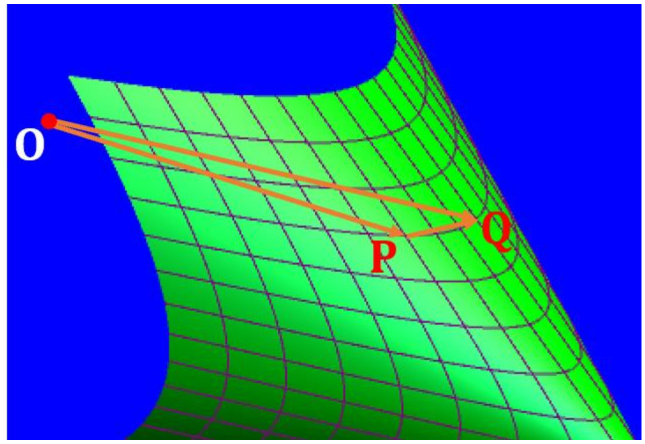


Figure 21. Curvilinear Basis

$$\frac{\mathbf{PQ}}{\|\mathbf{PQ}\|} \approx \frac{\delta \mathbf{r}}{\delta \xi^1}$$

which, in the limit,

$$\begin{aligned} \mathbf{g}_1(\xi^1, \xi^2, \xi^3) &\equiv \lim_{\delta \xi^1 \to 0} \frac{\mathbf{PQ}}{\|\mathbf{PQ}\|} \\ &= \lim_{\delta \xi^1 \to 0} \frac{\delta \mathbf{r}}{\delta \xi^1} \\ &= \frac{\partial \mathbf{r}}{\partial \xi^1} \end{aligned} \tag{100}$$

is clearly the tangent vector to the coordinate curve ξ^1 at the point $\mathbf{P}$. We can similarly form the other basis vectors and obtain,

$$\begin{aligned} \mathbf{g}_2(\xi^1, \xi^2, \xi^3) &\equiv \frac{\partial \mathbf{r}}{\partial \xi^2} \\ \mathbf{g}_3(\xi^1, \xi^2, \xi^3) &\equiv \frac{\partial \mathbf{r}}{\partial \xi^3} \end{aligned} \tag{101}$$

The basis vectors tangent to the remaining coordinate curves.

We do not yet specify the nature of the function, $\mathbf{r}(\xi^1, \xi^2, \xi^3)$. We only know that they are, in general, nonlinear. As we noted earlier, we are not able to assume the Cartesian relationship and write,

$$\mathbf{r}(\xi^1, \xi^2, \xi^3) \neq \xi^1 \mathbf{g}_1 + \xi^2 \mathbf{g}_2 + \xi^3 \mathbf{g}_3 = \xi^i \mathbf{g}_i \tag{102}$$

This linear expression for the position vector in terms of the coordinate functions only occurs in Cartesian systems. The position vector is only known to be a function of the coordinate variables. Using multivariate calculus, we can write the differential for the nonlinear function, $\mathbf{r}(\xi^1, \xi^2, \xi^3)$:

$$
\begin{aligned}
d\mathbf{r}(\xi^1, \xi^2, \xi^3) &= \frac{\partial \mathbf{r}}{\partial \xi^1} d\xi^1 + \frac{\partial \mathbf{r}}{\partial \xi^2} d\xi^2 + \frac{\partial \mathbf{r}}{\partial \xi^3} d\xi^3 \\
&= \mathbf{g}_1 d\xi^1 + \mathbf{g}_2 d\xi^2 + \mathbf{g}_3 d\xi^3 \\
&= \mathbf{g}_i d\xi^i.
\end{aligned} \tag{103}
$$

$\mathbf{g}_i, i = 1, \dots, 3$ vectors from the choice of $\xi^i, i = 1, \dots, 3$ as continuously differentiable functions, ξ^i are necessarily non-colinear and non-coplanar, so they form a proper basis for the curvilinear system.

Transformation Properties of $\boldsymbol{\xi}^{\boldsymbol{i}}$and $\mathbf{g}_{\boldsymbol{i}}$

Imagine we selected another set of functions, $\{\eta^1, \eta^2, \eta^3\}$. This could be a different transformation, say to oblate spheroidal coordinates. Since we have the six functions, $\xi^i(x_1, x_2, x_3), i = 1, \dots, 3$,and $\eta^j(x_1, x_2, x_3), j = 1, \dots, 3$, it should be possible to obtain either the one set of differentials if the other set is given; or other set of tangential basis vectors if the one is given. In order to do this, we need the transformation equations from one set to another. Let us begin with the coordinate differentials:

We first express one set of variables in terms of the other set. We can use the functional forms, which we assume to be invertible to achieve this, $\xi^i = \xi^i(\eta^1, \eta^2, \eta^3)$. From multivariate calculus,

$$
\begin{aligned}
d\xi^1 &= \frac{\partial \xi^1}{\partial \eta^1} d\eta^1 + \frac{\partial \xi^1}{\partial \eta^2} d\eta^2 + \frac{\partial \xi^1}{\partial \eta^3} d\eta^3 \\
d\xi^2 &= \frac{\partial \xi^2}{\partial \eta^1} d\eta^1 + \frac{\partial \xi^2}{\partial \eta^2} d\eta^2 + \frac{\partial \xi^2}{\partial \eta^3} d\eta^3 \\
d\xi^3 &= \frac{\partial \xi^3}{\partial \eta^1} d\eta^1 + \frac{\partial \xi^3}{\partial \eta^2} d\eta^2 + \frac{\partial \xi^3}{\partial \eta^3} d\eta^3
\end{aligned}
$$

Or, more compactly,

$$
d\xi^i = \frac{\partial \xi^i}{\partial \eta^j} d\eta^j \tag{104}
$$

Inverting the relationships, we have, $\eta^i = \eta^i(\xi^1, \xi^2, \xi^3)$, taking the differential as before, we can find that,

$$d\eta^j = \frac{\partial \eta^j}{\partial \xi^k} d\xi^k \tag{105}$$

Now, we look at the basis vectors. Recall from equation (103) that, $d\mathbf{r}(\xi^1, \xi^2, \xi^3) = \mathbf{g}_i d\xi^i$. If the basis vector obtained in the same way when coordinates are expressed in terms of set of functions, $\{\eta^1, \eta^2, \eta^3\}$ are $\boldsymbol{\gamma}_i$, $i = 1, \dots, 3$, then we can similarly write that,

$$\begin{aligned} d\mathbf{r}(\eta^1, \eta^2, \eta^3) &= \frac{\partial \mathbf{r}}{\partial \eta^1} d\eta^1 + \frac{\partial \mathbf{r}}{\partial \eta^2} d\eta^2 + \frac{\partial \mathbf{r}}{\partial \xi^3} d\eta^3 \\ &= \boldsymbol{\gamma}_j d\eta^j \end{aligned} \tag{106}$$

So that,

$$\begin{aligned} \mathbf{g}_i d\xi^i &= \boldsymbol{\gamma}_j d\eta^j \\ &= \boldsymbol{\gamma}_j \frac{\partial \eta^j}{\partial \xi^k} d\xi^k \end{aligned} \tag{107}$$

or,

$$\begin{aligned} \mathbf{g}_i(\xi^1, \xi^2, \xi^3) &= \frac{\partial \eta^j}{\partial \xi^i} \boldsymbol{\gamma}_j(\eta^1, \eta^2, \eta^3) \\ \boldsymbol{\gamma}_j(\eta^1, \eta^2, \eta^3) &= \frac{\partial \xi^i}{\partial \eta^j} \mathbf{g}_i(\xi^1, \xi^2, \xi^3) \end{aligned} \tag{108}$$

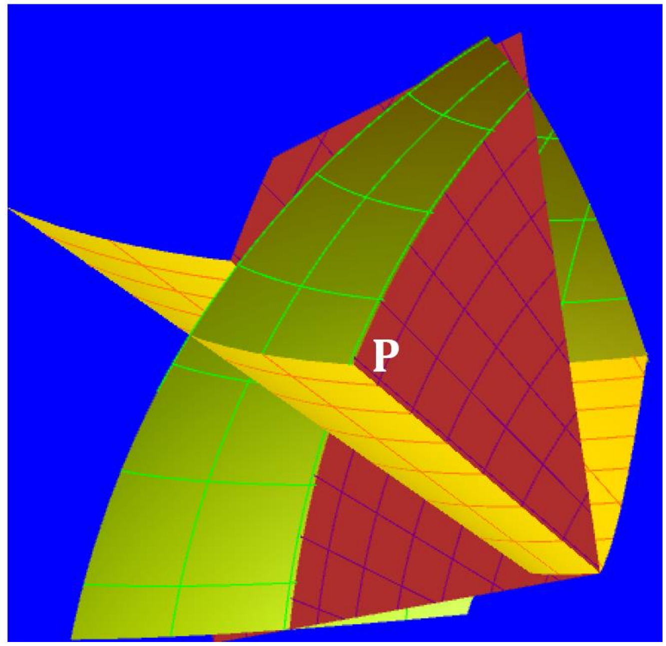
Combining equations (104) and (105),

$$\begin{aligned} d\xi^i &= \frac{\partial \xi^i}{\partial \eta^j} \frac{\partial \eta^j}{\partial \xi^k} d\xi^k \\ &= \delta^i_k d\xi^k \end{aligned} \tag{109}$$

showing that the transformation matrices are inverses of each other.

The transformation of the coordinate differentials is called contravariant while its inverse, for the basis vectors is called covariant. Mathematically covariant and contravariant transformations are as defined by their two different transformation equations. The fact that the two equations are different provides enough justification for naming them differently. There is an amount of discussion on the web about the deeper meaning of this choice of words “covariant” and “contravariant”. *** want to tackle this? ***

Figure 22. Intersecting curves and surfaces at a point



Another set of base vectors.

In the last section, we saw that the points in a curvilinear system meet at the intersection of three coordinate surfaces which are not necessarily planes. At this same point, there are also three coordinate curves, which are not necessarily straight. We showed that three linearly independent vectors,

$$\mathbf{g}_i(\xi^1, \xi^2, \xi^3) \equiv \frac{\partial \mathbf{r}}{\partial \xi^i} \tag{110}$$

which transform covariantly, form a basis for (or equivalently, spans) the space. Once we have an expression for the position vector, we can easily compute these basis vectors by differentiation. We also showed that these vectors are tangent to the coordinate curves at the point they intersect and define each point in the Euclidean point space. Figure 21 depicts the triplet of surfaces meeting to form the point $\mathbf{P}$; it also concurrently shows the triplet of lines also meeting to form the same point. We note again that the latter triplet are simply the intersecting curves of the pairs of surfaces at the same point as can be seen clearly in the diagram. The gradient of a surface is normal to the surface. If we can obtain the gradients at point $\mathbf{P}$ of the three surfaces meeting to define it, then we have another set of three vectors, non-colinear, non-coplanar, and therefore also spans the same vector space! Let us do the Math for this, shall we.

To make things very easy, we shall do using the Cartesian coordinate system as starting point. The only thing that will change here is that we shall use raised symbols for components (from the fact that we already know that their differentials transform contravarantly) and lowered indices for the base vectors (tangential to the coordinate curves, and therefore covariant). From equation (98) and (99), the three coordinate surfaces, $\xi^i(x_1, x_2, x_3)$, are functions of x_1, x_2 and x_3. From multivariate calculus, we can write,

$$\begin{aligned} d\xi^i &= \frac{\partial \xi^i}{\partial x^1} dx^1 + \frac{\partial \xi^i}{\partial x^2} dx^2 + \frac{\partial \xi^i}{\partial x^3} dx^3 \\ &= \frac{\partial \xi^i}{\partial x^j} dx^j \\ &= \left(\frac{\partial \xi^i}{\partial x^j} \mathbf{e}_j\right) \cdot (dx^\alpha \mathbf{e}_\alpha) \\ &= \left(\mathrm{grad}\, \xi^i\right) \cdot d\mathbf{r} \end{aligned} \tag{111}$$

We now invoke equation (103), and, using curvilinear coordinate system, substitute

$$d\mathbf{r}(\xi^1, \xi^2, \xi^3) = \mathbf{g}_i d\xi^i$$

so that,

$$\begin{aligned} d\xi^i &= \left(\mathrm{grad}\, \xi^i\right) \cdot d\mathbf{r} \\ &= \left(\mathrm{grad}\, \xi^i\right) \cdot \mathbf{g}_\alpha d\xi^\alpha \\ &= \delta^i_\alpha \, d\xi^\alpha \end{aligned} \tag{112}$$

where

$$\delta^i_\alpha = \begin{cases} 1, \text{if } i = \alpha \\ 0, \text{otherwise} \end{cases} \tag{113}$$

is the mixed Kronecker Delta. We now define the contravariant basis vector,

$$\mathbf{g}^i \equiv \mathrm{grad}\, \xi^i \,, i = 1, \dots, 3 \tag{114}$$

It is a straightforward matter to show that the transformation equations for this vector is contravariant in nature. To do this, let us, once again consider another set of curvilinear coordinates, $\{\eta^1, \eta^2, \eta^3\}$ which gives us two sets of triplet functions. the six functions, $\xi^i(x_1, x_2, x_3), i = 1, \dots, 3$,and $\eta^j(x_1, x_2, x_3), j = 1, \dots, 3$, it should be possible to obtain one set of gradients if the other is supplied. We shall do so at once. Start again from the coordinate transformation equations (105),

$$d\eta^i = \frac{\partial \eta^i}{\partial \xi^j} d\xi^j = \left(\operatorname{grad} \eta^i\right) \cdot d\mathbf{r}$$
$$= \frac{\partial \eta^i}{\partial \xi^j} \left(\operatorname{grad} \xi^j\right) \cdot d\mathbf{r} \tag{115}$$

from which it follows from the arbitrariness of $d\mathbf{r}$ that

$$\operatorname{grad} \eta^i = \frac{\partial \eta^i}{\partial \xi^j} \operatorname{grad} \xi^j \tag{116}$$

Showing, by their transformation equations, that the basis vectors formed from the surface normal vectors are themselves contravariant.

Reciprocal Basis Vectors.

The argument so far shows that for any curvilinear system of coordinates, there are at least two methodical ways to select our sets of basis vectors. Take the tangents of the coordinate curves at the point, we obtain a set of basis vectors,

$$\mathbf{g}_i(\xi^1, \xi^2, \xi^3) \equiv \frac{\partial \mathbf{r}}{\partial \xi^i}$$

or, we could take the normal to the surfaces at the point on interest, and obtain,

$$\mathbf{g}^i(\xi^1, \xi^2, \xi^3) \equiv \operatorname{grad} \xi^i$$

Which is another set of basis vectors. Equations (112) to (114) further show that the relationship,

$$\mathbf{g}^j \cdot \mathbf{g}_i = \mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \tag{117}$$

holds between the two sets of basis functions. This is called the **reciprocity relationship**. The two sets called reciprocal base vectors. The set obtained from tangents to the coordinate surfaces transforms covariantly, the other, obtained from the normals to the coordinate surfaces transforms contravariantly. This fact is sometimes used to refer to the reciprocal bases as ***covariant*** and ***contravariant*** bases respectively.

The arguments so far beg a question: Does the same duality occur only in Cartesian systems? Are there covariant and contravariant bases in the Cartesian system? The short answer is "yes", there are covariant and contravariant bases in any coordinate system you select. Remember that we have a simple rule to compute these vectors. In the one, we simply differentiate the position vector. For a Cartesian coordinate system, recall that for any point $\mathbf{P}(x_1, x_2, x_3)$, the position vector,

$$\mathbf{r} = x^1\mathbf{e}_1 + x^2\mathbf{e}_2 + \boldsymbol{x}^3\mathbf{e}_3 = x^i\mathbf{e}_i$$

Differentiating with respect to the coordinate variables,

$$\mathbf{g}_i(x^1, x^2, x^3) \equiv \frac{\partial \mathbf{r}}{\partial x^i} = \frac{\partial}{\partial x^i}\left(x^j\mathbf{e}_j\right) = \delta_i^j\mathbf{e}_j = \mathbf{e}_i \qquad (118)$$

Similarly, taking the gradients,

$$\mathbf{g}^i(x^1, x^2, x^3) \equiv \operatorname{grad} x^i = \left(\frac{\partial}{\partial x^j}x^i\right)\mathbf{e}_j = \delta_j^i\mathbf{e}_j = \mathbf{e}_i. \qquad (119b)$$

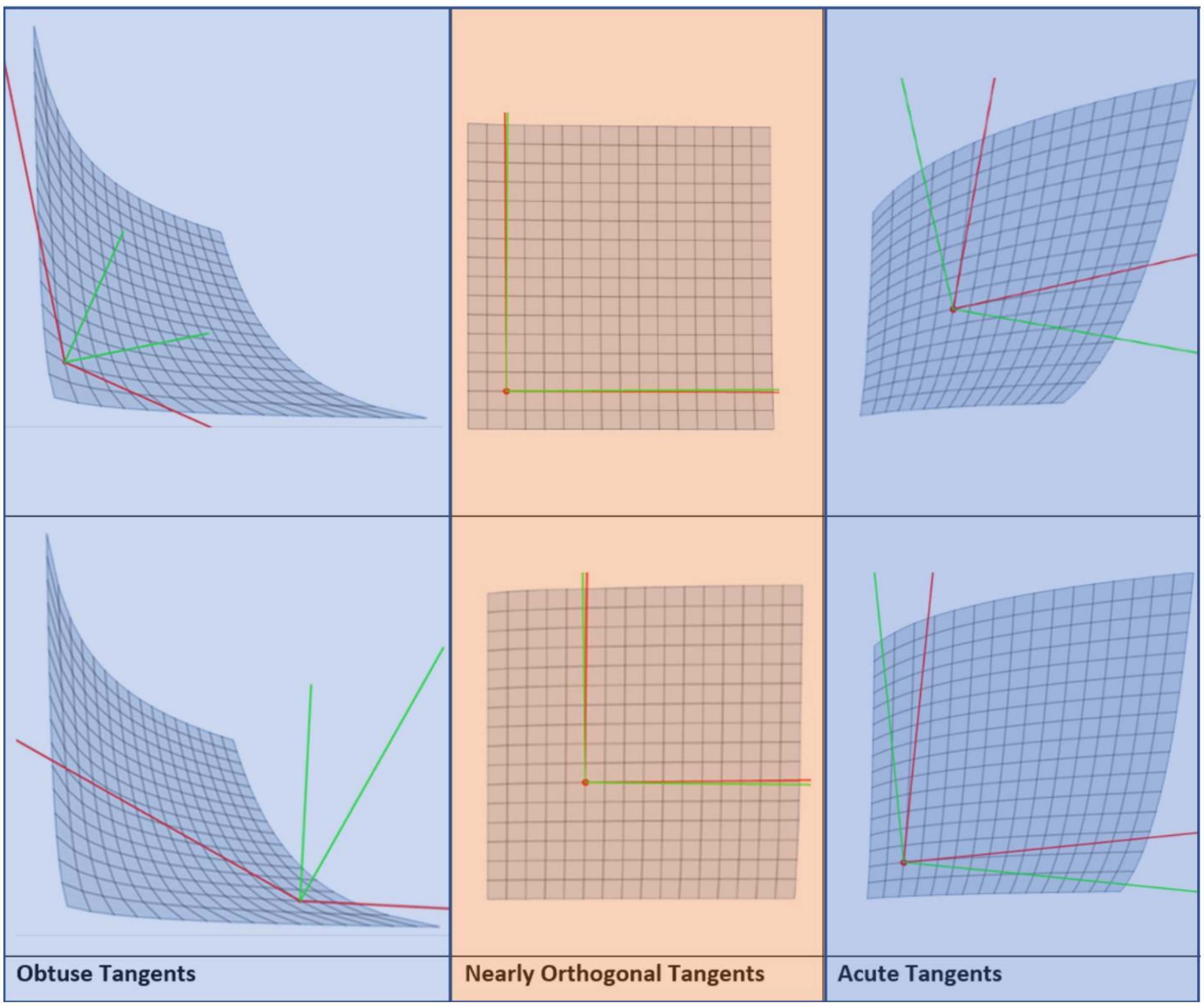


Figure 23 Covariant & Contravariant Basis Vectors

So we have the long answer to the question: There are covariant as well as contravariant bases in Cartesian. They coincide with each other and with the familiar orthonormal basis of the Cartesian system.

```
Manipulate[
 pos[n_] := 0.1 + n (.9/16);
 f[u_, v_, α_] := u v^α;
 g[u_, v_, α_] := v u^α;
 p[x_, y_, α_] := x^(-1/(α^2-1)) y^(α/(α^2-1));
 q[x_, y_, α_] := x^(α/(α^2-1)) y^(-1/(α^2-1));
 DynamicModule[{pt1 = {f[pos[uPos], pos[vPos], α], g[pos[uPos], pos[vPos], α]}},
  ParametricPlot[{f[u, v, α], g[u, v, α]}, {u, .1, 1}, {v, .1, 1}, PlotPoints → 15, Mesh → 15,
   ImageSize → Large, MeshStyle → Thick,
   Epilog -> {Red, Thick, Disk[pt1, .01],
     Line[{pt1, (({(D[f[u, v, α], v]), (D[g[u, v, α], v])}) /. {u → p[pt1[[1]], pt1[[2]], α],
           v → q[pt1[[1]], pt1[[2]], α]}) + pt1}],
     Line[{pt1, (({(D[f[u, v, α], u]), (D[g[u, v, α], u])}) /. {u → p[pt1[[1]], pt1[[2]], α],
           v → q[pt1[[1]], pt1[[2]], α]}) + pt1}],
     Green, Thick,
     Line[{pt1, (Take[Grad[Xx^(-1/(α^2-1)) Yy^(α/(α^2-1)), {Xx, Yy, Zz}], 2] /. {Xx → pt1[[1]], Yy → pt1[[2]]}) + pt1}],
     Line[{pt1, (Take[Grad[Xx^(α/(α^2-1)) Yy^(-1/(α^2-1)), {Xx, Yy, Zz}], 2] /. {Xx → pt1[[1]], Yy → pt1[[2]]}) + pt1}]}]]],
 {{α, -0.3}, -.7, .7}, {{uPos, 2}, 0, 16, 1}, {{vPos, 2}, 0, 16, 1}, ControlPlacement → Right]
```

Figure 24 Curvilinear basis animation code

The conclusion we draw from here is that, once the systems becomes curvilinear, there is a separation between the covariant and contravariant bases. A separation that cannot be seen in the Cartesian coordinate system. The following set of drawings (Try out the full Mathematica Demonstrations as shown in the code)

Reciprocity & Orthogonality

We can now address the issue of the angular inclinations of the basis vectors. Here we are completely free in our choice of three functions to form a coordinate system. All we must ensure is that the coordinate curves form linearly independent vectors when differentiated; and the normals to the coordinate planes are not co-planar nor co-linear. Nothing was said about the possibility of orthogonality among the triplets. To ensure linear independence, it can be proved that once the functions chosen are continuously differentiable, the linear independence, and hence spanning of the 3D Euclidean space is assured.

Irrespective of the angles between the triplets of vectors, based on tangents or normal, spanning the curvilinear system, the reciprocity relationship holds. There is always orthogonality between each covariant vector and its respective contravariant vector as follows:

$$\begin{aligned} &\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j \Rightarrow \\ &\mathbf{g}_1 \cdot \mathbf{g}^1 = 1;\ \mathbf{g}_1 \cdot \mathbf{g}^2 = 0;\ \mathbf{g}_1 \cdot \mathbf{g}^3 = 0; \\ &\mathbf{g}_2 \cdot \mathbf{g}^1 = 0;\ \mathbf{g}_2 \cdot \mathbf{g}^2 = 1;\ \mathbf{g}_2 \cdot \mathbf{g}^3 = 0; \\ &\mathbf{g}_3 \cdot \mathbf{g}^1 = 0;\ \mathbf{g}_3 \cdot \mathbf{g}^2 = 0;\ \mathbf{g}_3 \cdot \mathbf{g}^3 = 1; \end{aligned} \tag{120}$$

This fact is demonstrated in the pictures shown in figure 22. An animation of this is available in the code accompanying Figure (24)

In the first case, the coordinate basis chosen are inclined at an obtuse angle to each other as shown in red. Notice that these are tangential lines to the coordinate curves. In this 2D representation, the curves shown are also cross sections of the coordinate surfaces. The normal to these surfaces are shown in green. When the coordinate tangents are obtuse, the coordinate normal are acute. A closer look shows that there is pairwise orthogonality as predicted by the reciprocity relationship in Equation (117). As we move along any coordinate curve, these orientations continue to change while maintaining the reciprocity.

This situation subsists for the case when the tangent basis vectors are acute. Here the normal basis vectors are in an obtuse angle relationship. Again, as before, a pairwise orthogonality prevails between the covariant and contravariant basis vectors.

In the middle case, we look at two cases where the coordinate curves are approximately orthogonal. The two sets of basis vectors are almost indistinguishable from each other. When there is perfect orthogonality, the lines of action of these vectors are identical. If they are normalized, then the vectors themselves are identical. The Mathamatica code in figure 23 demonstrates this relationship fully.

In a curvilinear system, a vector now has two possible representations: One with covariant bases and contravariant components and the other with contravariant bases with covariant components. For any $\forall \mathbf{v} \in \mathbb{E}$,

$$\mathbf{v} = v^i \mathbf{g}_i = v_i \mathbf{g}^i \quad (121)$$

are two related representations in the reciprocal bases. Taking the inner product of the above equation with the basis vector $\mathbf{g}_j$, we have

$$\mathbf{v} \cdot \mathbf{g}_j = v^i \mathbf{g}_i \cdot \mathbf{g}_j = v_i \mathbf{g}^i \cdot \mathbf{g}_j \quad (122)$$

which gives us the *covariant* component,

$$\mathbf{v} \cdot \mathbf{g}_j = v^i g_{ij} = v_i \delta_j^i = v_j. \quad (123)$$

The substitution property of the mixed Kronecker delta remains the same as that of the Cartesian Kronecker Delta. In the same easy manner, we may evaluate the contravariant components of the same vector by taking the dot product of the same equation with the contravariant base vector $\mathbf{g}^j$:

$$\mathbf{v} \cdot \mathbf{g}^j = v^i \mathbf{g}_i \cdot \mathbf{g}^j = v_i \mathbf{g}^i \cdot \mathbf{g}^j \quad (124)$$

so that,

$$\mathbf{v} \cdot \mathbf{g}^j = v^i \delta_i^j = v_i g^{ij} = v^j \quad (125)$$

The nine scalar quantities, $g^{ij} \equiv \mathbf{g}^i \cdot \mathbf{g}^j$ as well as the nine related quantities $g_{ij} \equiv \mathbf{g}_i \cdot \mathbf{g}_j$ play important roles in the coordinate system spanned by these reciprocal sets of basis vectors as we shall see. They are called metric coefficients because they metrize the space defined by these bases by quantifying distances and angles.

Define

$$\sqrt{g} \equiv \mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k \quad (126)$$

and

$$\epsilon_{ijk} = \sqrt{g} e_{ijk} \quad (127)$$

We can obtain the relationship,

$$\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k \quad (128)$$

as follows:

Note that $\mathbf{g}^1$is orthogonal to $\mathbf{g}_2$ and $\mathbf{g}_3$. Therefore

$$\mathbf{g}_2 \times \mathbf{g}_3 = \alpha \mathbf{g}^1 \qquad (129)$$

as their cross product MUST lie parallel to $\mathbf{g}^1$ where α is a scalar to be found. Taking the scalar product of both sides, we have,

$$\sqrt{g} = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \alpha \mathbf{g}^1 \cdot \mathbf{g}_1 = \alpha \qquad (130)$$

Similarly,

$\mathbf{g}^2$is orthogonal to $\mathbf{g}_3$ and $\mathbf{g}_1$, and $\mathbf{g}^3$is orthogonal to $\mathbf{g}_1$ and $\mathbf{g}_2$, it follows that $\mathbf{g}_2 \times \mathbf{g}_3 = \mathbf{g}^1, \mathbf{g}_3 \times \mathbf{g}_1 = \mathbf{g}^2$, and $\mathbf{g}_1 \times \mathbf{g}_2 = \mathbf{g}^3$. Equation (128) captures these cases with the other six that vanish in a single expression as we saw in the ONB case.

Given that $g = \det g_{ij}$ of the covariant metric coefficients, it is not difficult to prove that

$$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} \equiv \sqrt{g} e_{ijk} \qquad (131)$$

The dual of the expression, the equivalent contravariant equivalent also follows from the fact that,

$$\mathbf{g}^i \times \mathbf{g}^j \cdot \mathbf{g}^k = \epsilon^{ijk} = \frac{1}{\sqrt{g}} . e^{ijk} \qquad (132)$$

Solved Problems 1.2

1.81	Given that, $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$ are three linearly independent vectors and satisfy $\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j$, show that $\mathbf{g}^1 = \frac{1}{V}\mathbf{g}_2 \times \mathbf{g}_3$, $\mathbf{g}^2 = \frac{1}{V}\mathbf{g}_3 \times \mathbf{g}_1$, and $\mathbf{g}^3 = \frac{1}{V}\mathbf{g}_1 \times \mathbf{g}_2$, where $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \sqrt{g}$.
a	It is clear, for example, that $\mathbf{g}^1$ is perpendicular to $\mathbf{g}_2$ as well as to $\mathbf{g}_3$ (an obvious fact because $\mathbf{g}^1 \cdot \mathbf{g}_2 = 0$ and $\mathbf{g}^1 \cdot \mathbf{g}_3 = 0$), we can say that the vector $\mathbf{g}^1$ must necessarily lie on the cross product $\mathbf{g}_2 \times \mathbf{g}_3$ of $\mathbf{g}_2$ and $\mathbf{g}_3$. It is therefore correct to write, $$\mathbf{g}^1 = \frac{1}{V}\mathbf{g}_2 \times \mathbf{g}_3$$ where $V^{-1}$is a constant we will now determine. We can do this right away by taking the dot product of both sides of the equation with $\mathbf{g}_1$ we immediately obtain, $$\mathbf{g}_1 \cdot \mathbf{g}^1 = V^{-1}\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = 1$$

So that, $V = \mathbf{g}_1 \cdot \ \mathbf{g}_2 \times \ \mathbf{g}_3$

the volume of the parallelepiped formed by the three vectors $\mathbf{g}_1, \mathbf{g}_2,$ and $\mathbf{g}_3$ when their origins are made to coincide.

Basis Vectors for Curvilinear & Cartesian Coordinates

The Cartesian basis vectors and the covariant curvilinear basis can represent any vectors in the 3D vector space. In particular, they can mutually form basis for each other. Accordingly,

$$\begin{aligned} \mathbf{e}_i &= \alpha_i^l \mathbf{g}_l \\ \mathbf{g}_i &= \beta_i^l \mathbf{e}_l \end{aligned} \tag{133}$$

Substituting for $\mathbf{g}_l$, we have,

$$\mathbf{e}_i = \alpha_i^j \beta_j^m \mathbf{e}_m = \delta_i^m \mathbf{e}_i \tag{134}$$

which gives,

$$\alpha_i^j \beta_j^m = \delta_i^m. \tag{135}$$

Furthermore,

$$\begin{aligned} \mathbf{g}_i \cdot \mathbf{g}_j &= \beta_i^l \mathbf{e}_l \cdot \beta_j^m \mathbf{e}_m \\ &= \beta_i^l \beta_j^m \delta_{lm} \\ &= \beta_i^l \beta_j^l \end{aligned} \tag{136}$$

Which means that the determinant of the matrix, .

$$[g_{ij}] \equiv [\mathbf{g}_i \cdot \mathbf{g}_j] = \det\left[\left(\beta_i^l\right)^2\right]. \tag{137}$$

The scalar triple product of the covariant base vectors, in terms of its components in the Cartesian basis vectors can be found:

$$\begin{aligned} \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 &= \left(\beta_1^l \mathbf{e}_l\right) \cdot (\beta_2^m \mathbf{e}_m) \times (\beta_i^n \mathbf{e}_n) \\ &= \beta_1^l \beta_2^m \beta_i^n \mathbf{e}_l \cdot \mathbf{e}_m \times \mathbf{e}_n \\ &= \beta_1^l \beta_2^m \beta_i^n e_{lmn} \\ &= \det\left[\beta_i^l\right] \\ &= \sqrt{g} \end{aligned} \tag{138}$$

as previously defined. Combining Equations (137) and (138) it is clear that,

$$\det[g_{ij}] = [\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3]^2 = g. \tag{139}$$

We further observe that

$$\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \sqrt{g} e_{ijk} = \epsilon_{ijk} \tag{140}$$

a fact that becomes clearer from the results of SP 1.85 below. For the contravariant base vectors, we have,

$$\mathbf{g}^i \cdot \mathbf{g}^j \times \mathbf{g}^k = \epsilon^{ijk} \tag{141}$$

When we are back to Cartesian coordinates, β_i^l of equation (137) becomes δ_i^l, $\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = 1$, and,

$$\mathbf{g}^i \cdot \mathbf{g}^j \times \mathbf{g}^k = \epsilon^{ijk} = \mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = e_{ijk} \tag{142}$$

as the distinction between covariant and contravariant basis vectors vanish. g_{ij}in Cartesian coordinates, this becomes the identity tensor, and its determinant,

$$\det[g_{ij}] = [\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3]^2 = g = 1. \tag{143}$$

Computation Method for Reciprocal Base Vectors

The nine reciprocity relationships in Equation (117) can be expressed in matrix form. If we pack the covariant base vectors, $\mathbf{g}_i, i = 1, \dots, 3$ into the columns of a matrix, in order for the reciprocity relationship to hold, the contravariant vectors $\mathbf{g}^j, j = 1, \dots, 3$ are the rows of its inverse. The converse is also true: If we compute the contravariant basis vectors and pack them into the columns of a matrix, the rows of its inverse are the covariant basis vectors.

This fact is used in Q1.93 and 1.95 to compute the reciprocal basis vectors once either of them is known. The accompanying code in those Q&A can be used for any coordinate system once we know the functional form of the position vector – which also defined the transformation equations.

Solved Problems 1.3

1.82	Show that $\mathbf{g}^j = g^{ij}\mathbf{g}_i = g^{ji}\mathbf{g}_i$ and establish the relation, $g_{ij}g^{jk} = \delta_i^k$
a	First expand $\mathbf{g}^j$ in terms of the $\mathbf{g}_i$s: $$\mathbf{g}^j = \alpha\mathbf{g}_1 + \beta\mathbf{g}_2 + \gamma\mathbf{g}_3$$

	Dotting with $\mathbf{g}^1 \Rightarrow \mathbf{g}^j \cdot \mathbf{g}^1 = \alpha\mathbf{g}_1 \cdot \mathbf{g}^1 + \beta\mathbf{g}_2 \cdot \mathbf{g}^1 + \gamma\mathbf{g}_3 \cdot \mathbf{g}^1 = \mathrm{g}^{j1} = \alpha$. In the same way we find that $\beta = \mathrm{g}^{j2}$ and $\gamma = \mathrm{g}^{j3}$so that, $$\mathbf{g}^j = g^{j1}\mathbf{g}_1 + g^{j2}\mathbf{g}_2 + g^{j3}\mathbf{g}_3 = g^{ji}\mathbf{g}_i.$$ Similarly, $\mathbf{g}_i = g_{i\alpha}\mathbf{g}^\alpha$. Recall the reciprocity relationship: $\mathbf{g}_i \cdot \mathbf{g}^k = \delta_i^k$. Using the above, we can write $$\mathbf{g}_i \cdot \mathbf{g}^k = (g_{i\alpha}\mathbf{g}^\alpha) \cdot (g^{k\beta}\mathbf{g}_\beta) = g_{i\alpha}g^{k\beta}\mathbf{g}^\alpha \cdot \mathbf{g}_\beta = g_{i\alpha}g^{k\beta}\delta_\beta^\alpha = \delta_i^k$$ which shows that $$g_{i\alpha}g^{k\alpha} = g_{ij}g^{jk} = \delta_i^k$$ As required. This shows that the tensor g_{ij} and g^{ij} are inverses of each other.
1.83	The trilinear mapping $\mathbb{E} \times \mathbb{E} \times \mathbb{E} \to \mathbb{R}$ from the product set of Euclidean vectors to the real space is defined by: $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$
a	In component form, $$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon^{ijk}a_i b_j c_k$$ Cyclic permutations of this, upon remembering that (i, j, k) are dummy indices, yield, $$\epsilon^{jki}b_j c_k a_i = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = \epsilon^{ijk}b_i c_j a_k$$ $$= \epsilon^{kij}c_k a_i b_j = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = \epsilon^{ijk}c_i a_j b_k$$ The other results follow from antisymmetric arrangements and the nature of ϵ^{ijk}.
1.84	Given that, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$. Show that this product vanishes if the vectors $(\mathbf{a}, \mathbf{b}, \mathbf{c})$ are linearly dependent.

Suppose it is possible to find scalars α and β such that, $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c}$. It therefore means that,

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \epsilon^{ijk} a_i b_j c_k = \epsilon^{ijk}(\alpha b_i + \beta c_i) b_j c_k$$
$$= \alpha\epsilon^{ijk} b_i b_j c_k + \beta\epsilon^{ijk} c_i b_j c_k$$
$$= 0$$

Note that $b_i b_j c_k$ is symmetric in i and j, $c_i b_j c_k$ is symmetric in i and k and ϵ^{ijk} is antisymmetric in i, j and k Because each term is the product of a symmetric and an antisymmetric object which must vanish.

1.85 For the basis vectors $\mathbf{g}_i, i = 1,2,3$ and their duals, $\mathbf{g}^j, j = 1,2,3$ If $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$ and $V = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3$, show that $vV = 1$.

Given $\mathbf{g}_i, i = 1,2,3$, note that $\mathbf{g}^1$ is perpendicular to $\mathbf{g}_2$ and to $\mathbf{g}_3$. It must be parallel to the vector $\mathbf{g}_2 \times \mathbf{g}_3$. A scalar constant V^{-1} must exist such that,

$$\mathbf{g}^1 = V^{-1}\, \mathbf{g}_2 \times \mathbf{g}_3$$
$$\mathbf{g}^2 = V^{-1}\, \mathbf{g}_3 \times \mathbf{g}_1$$
$$\mathbf{g}^3 = V^{-1}\, \mathbf{g}_1 \times \mathbf{g}_2$$

Since (dot the first with $\mathbf{g}_1$ to see) Now we are given that $v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3$. Using the above relations, we can write,

$$\mathbf{g}^2 \times \mathbf{g}^3 = (V^{-1}\, \mathbf{g}_3 \times \mathbf{g}_1) \times (V^{-1}\, \mathbf{g}_1 \times \mathbf{g}_2)$$
$$= V^{-2}[(\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_2)\mathbf{g}_1 - (\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_1)\mathbf{g}_2]$$
$$= V^{-2}(\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3)\mathbf{g}_1 = V^{-1}\mathbf{g}_1$$

We can now write,

$$v = \mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3 = \mathbf{g}^1 \cdot V^{-1}\mathbf{g}_1 = V^{-1}\mathbf{g}^1 \cdot \mathbf{g}_1 = V^{-1}$$

Showing that, $v\,V = 1$ as required. It is a trivial matter to show that $V = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3$, for, if we take a dot product of the equation, $\mathbf{g}^1 = V^{-1}\, \mathbf{g}_2 \times \mathbf{g}_3$, the result follows so that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \frac{1}{\mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3}$.

1.86	By transforming the position vector into the coordinate system spanned by the basis vectors, $\mathbf{g}_i$ defined by, $d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i} du^i \equiv \mathbf{g}_i du^i$, show that $V = \mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \sqrt{g}$, where g is the determinant $|g_{ij}|$.

Changing variables, we can write that,

$$\mathbf{r}(x, y, z) = x^i(u^1, u^2, u^3)\mathbf{e}_i = \mathbf{r}(u^1, u^2, u^3)$$

So that we have new coordinates $u^k, k = 1,2,3$. In this new system, the differential of the position vector $\mathbf{r}$ is,

$$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial u^i} du^i \equiv \mathbf{g}_i du^i$$

the above equation, as we shall soon show, defines the basis vectors in the new coordinate system. The vectors $\mathbf{g}_1, \mathbf{g}_2$ and $\mathbf{g}_3$ are not necessarily unit vectors but they form a basis of the new system provided,

$$V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 \neq 0$$

$$\mathbf{g}_i = \frac{\partial \mathbf{r}}{\partial u^i} = \frac{\partial x^k}{\partial u^i}\mathbf{e}_k$$

$$V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \begin{vmatrix} \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \\ \frac{\partial x^1}{\partial u^3} & \frac{\partial x^2}{\partial u^3} & \frac{\partial x^3}{\partial u^3} \end{vmatrix} = \left|\frac{\partial x^k}{\partial u^i}\right| \neq 0$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j = \frac{\partial \mathbf{r}}{\partial u^i} \cdot \frac{\partial \mathbf{r}}{\partial u^j} = \left(\frac{\partial x^k}{\partial u^i}\mathbf{e}_k\right) \cdot \left(\frac{\partial x^l}{\partial u^j}\mathbf{e}_l\right)$$

$$= \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\mathbf{e}_k \cdot \mathbf{e}_l = \frac{\partial x^k}{\partial u^i}\frac{\partial x^l}{\partial u^j}\delta_{kl} = \frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}$$

Clearly, the determinant of g_{ij} (we shall prove later that the determinant of a product of matrices is the product of the determinants)

$$g \equiv |g_{ij}| = \left|\frac{\partial x^k}{\partial u^i}\frac{\partial x^k}{\partial u^j}\right| = \left|\frac{\partial x^k}{\partial u^i}\right|^2 = V^2$$

This means, $V = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \left|\frac{\partial x^i}{\partial u^j}\right| = \sqrt{g}$. We can therefore write,

	$$\mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = e_{123}\sqrt{g}$$ Swapping indices 2 and 3, we have, $$\mathbf{g}_1 \cdot \mathbf{g}_3 \times \mathbf{g}_2 = -\sqrt{g} = e_{132}\sqrt{g} = \mathbf{g}_1 \times \mathbf{g}_3 \cdot \mathbf{g}_2$$ The second equality coming from the fact that swapping the cross with the dot changes nothing. Lastly, swapping 1 and 3 in the last equation shows that, $\mathbf{g}_3 \times \mathbf{g}_1 \cdot \mathbf{g}_2 = -(-\sqrt{g}) = e_{312}\sqrt{g}$. These three expressions together imply that, $\mathbf{g}_i \cdot \mathbf{g}_j \times \mathbf{g}_k = \epsilon_{ijk} = \sqrt{g}e_{ijk}$ as required.
1.87	Given that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \sqrt{g}$, where g is the determinant $\lvert g_{ij} \rvert$. Show that, $\mathbf{g}_i \times \mathbf{g}_j \cdot \mathbf{g}_k = \sqrt{g}e_{ijk} \equiv \epsilon_{ijk}$. Conclude further that $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{g}^k$
	Given that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \sqrt{g}$, the fact that the triple product obeys the rule, $[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}] = -[\mathbf{b}, \mathbf{a}, \mathbf{c}] = -[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{c}, \mathbf{b}]$, combined with the fact that the triple product vanishes when any two of its vectors are collinear allow us to write that $$\mathbf{g}_i \times \mathbf{g}_j \cdot \mathbf{g}_k = e_{ijk}\sqrt{g} \equiv \epsilon_{ijk}$$ By the reciprocity rule, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$, we have that, $\mathbf{g}_1 \cdot \mathbf{g}^1 = 1, \mathbf{g}_1 \cdot \mathbf{g}^2 = 0, \mathbf{g}_1 \cdot \mathbf{g}^3 = 0$. It follows that $\mathbf{g}^1$ must be perpendicular to the plane of $\mathbf{g}_2$and $\mathbf{g}_3$ making it parallel to $\mathbf{g}_2 \times \mathbf{g}_3$ A scalar constant α must exist such that, $\mathbf{g}^1 = \alpha\, \mathbf{g}_2 \times \mathbf{g}_3$. Dot product of both sides with $\mathbf{g}_1$ shows that $\alpha = 1/\sqrt{g}$. Therefore, $$\mathbf{g}_2 \times \mathbf{g}_3 = \sqrt{g}\mathbf{g}^1$$ $$\mathbf{g}_3 \times \mathbf{g}_1 = \sqrt{g}\mathbf{g}^2$$ $$\mathbf{g}_1 \times \mathbf{g}_2 = \sqrt{g}\mathbf{g}^3$$ These three results together can be expressed as, $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{g}^k$.
1.88	Use the reciprocity rule, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$ and the fact that $\mathbf{g}_1 \times \mathbf{g}_2 \cdot \mathbf{g}_3 = \frac{1}{\mathbf{g}^1 \cdot \mathbf{g}^2 \times \mathbf{g}^3}. = \sqrt{g}$ to show that $\mathbf{g}^i \times \mathbf{g}^j = \frac{1}{\sqrt{g}}e^{ijk}\mathbf{g}_k = \epsilon^{ijk}$.

	By the reciprocity rule, $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$, we have that, $\mathbf{g}_1 \cdot \mathbf{g}^1 = 1, \mathbf{g}_2 \cdot \mathbf{g}^1 = 0, \mathbf{g}_3 \cdot \mathbf{g}^1 = 0$. It follows that $\mathbf{g}_1$ must be perpendicular to the plane of $\mathbf{g}^2$ and $\mathbf{g}^3$ making it parallel to $\mathbf{g}^2 \times \mathbf{g}^3$. A scalar constant β must exist such that, $\mathbf{g}_1 = \beta \mathbf{g}^2 \times \mathbf{g}^3$. Dot product of both sides with $\mathbf{g}_1$ shows that $\beta = \sqrt{g}$. Therefore, $$\mathbf{g}^2 \times \mathbf{g}^3 = \frac{1}{\sqrt{g}}\mathbf{g}_1, \qquad \mathbf{g}^3 \times \mathbf{g}^1 = \frac{1}{\sqrt{g}}\mathbf{g}_2, \qquad \mathbf{g}^1 \times \mathbf{g}^2 = \frac{1}{\sqrt{g}}\mathbf{g}_3$$ These three results together can be written as, $\mathbf{g}^i \times \mathbf{g}^j = \frac{1}{\sqrt{g}} e^{ijk}\mathbf{g}_k = \epsilon^{ijk}$ if we write $\epsilon^{ijk} \equiv \frac{1}{\sqrt{g}} e^{ijk}$.
1.89	Given that $\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk}\mathbf{g}^k$, and that, Find an expression for $\mathbf{g}^k$ in terms of its dual vectors.
	Multiply both sides by $\epsilon^{ij\alpha}$ and find the expression for $\mathbf{g}^k$ $$\epsilon^{ij\alpha}\mathbf{g}_i \times \mathbf{g}_j = \epsilon^{ij\alpha}\epsilon_{ijk}\mathbf{g}^k = 2\delta_k^\alpha \mathbf{g}^k = 2\mathbf{g}^\alpha$$ So that $\mathbf{g}^i = \frac{1}{2}\epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$
1.90	Show that the cross product of vectors $\mathbf{a}$ and $\mathbf{b}$ in general coordinates is $a^i b^j \epsilon_{ijk}\mathbf{g}^k$ or $\epsilon^{ijk} a_i b_j \mathbf{g}_k$ where a^i, b^j are the respective contravariant components and a_i, b_j the covariant.
	Express vectors $\mathbf{a}$ and $\mathbf{b}$ as contravariant components: $\mathbf{a} = a^i\mathbf{g}_i$, and $\mathbf{b} = b^i\mathbf{g}_i$. Using the above result, we can write that, $$\mathbf{a} \times \mathbf{b} = (a^i\mathbf{g}_i) \times (b^j\mathbf{g}_j) = a^i b^j \mathbf{g}_i \times \mathbf{g}_j = a^i b^j \epsilon_{ijk}\mathbf{g}^k.$$ Express vectors $\mathbf{a}$ and $\mathbf{b}$ as covariant components: $\boldsymbol{a} = a_i\mathbf{g}^i$ and $\boldsymbol{b} = b_i\mathbf{g}^i$. Again, proceeding as before, we can write, $$\mathbf{a} \times \mathbf{b} = (a_i\mathbf{g}^i) \times (b_j\mathbf{g}^j) = \epsilon^{ijk} a_i b_j \mathbf{g}_k$$ Express vectors $\mathbf{a}$ as contravariant components: $\mathbf{a} = a^i\boldsymbol{g}_i$ and $\mathbf{b}$ as covariant components: $\mathbf{b} = b_i\mathbf{g}^i$

$$\mathbf{a} \times \mathbf{b} = (a^i \mathbf{g}_i) \times (b_j \mathbf{g}^j) = a^i b_j (\mathbf{g}_i \times \mathbf{g}^j)$$

1.91 Show that the cross product of vectors $\mathbf{a}$ and $\mathbf{b}$ in general coordinates is $a^i b^j \epsilon_{ijk} \mathbf{g}^k$ or $\epsilon^{ijk} a_i b_j \mathbf{g}_k$ where a^i, b^j are the respective contravariant components and a_i, b_j the covariant.

Express vectors $\mathbf{a}$ and $\mathbf{b}$ as contravariant components: $\mathbf{a} = a^i \mathbf{g}_i$, and $\mathbf{b} = b^i \mathbf{g}_i$. Using the above result, we can write that,

$$\mathbf{a} \times \mathbf{b} = (a^i \mathbf{g}_i) \times (b^j \mathbf{g}_j) = a^i b^j \mathbf{g}_i \times \mathbf{g}_j = a^i b^j \epsilon_{ijk} \mathbf{g}^k.$$

Express vectors $\mathbf{a}$ and $\mathbf{b}$ as covariant components: $\boldsymbol{a} = a_i \mathbf{g}^i$ and $\boldsymbol{b} = b_i \mathbf{g}^i$. Again, proceeding as before, we can write,

$$\mathbf{a} \times \mathbf{b} = (a_i \mathbf{g}^i) \times (b_j \mathbf{g}^j) = \epsilon^{ijk} a_i b_j \mathbf{g}_k$$

Express vectors $\mathbf{a}$ as contravariant components: $\mathbf{a} = a^i \mathbf{g}_i$ and $\mathbf{b}$ as covariant components: $\mathbf{b} = b_i \mathbf{g}^i$

$$\mathbf{a} \times \mathbf{b} = (a^i \mathbf{g}_i) \times (b_j \mathbf{g}^j) = a^i b_j (\mathbf{g}_i \times \mathbf{g}^j)$$

1.92 In the transformation from the (x_1, x_2, x_3) system to the (r, ϕ, Z) coordinate system, the position vector changed from $\mathbf{R} = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3$ to $\mathbf{R} = r\mathbf{e}_r(\phi) + z\mathbf{e}_z$ where $\mathbf{e}_r(\phi) = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi + 0\mathbf{e}_3$. Show by partial differentiation only, that the basis vectors in respective coordinates are $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{\mathbf{e}_r, \mathbf{e}_\phi, \mathbf{e}_z\}$ respectively, $\mathbf{e}_\phi(\phi) = r\frac{\partial \mathbf{e}_r(\phi)}{\partial \phi}$

$$\mathbf{R} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$$

Differentiating with respect to x_1, x_2, x_3 respectively from the columns of the matrix in the code, the Cartesian basis vectors are

$$\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$$

Similarly, $\mathbf{R} = r\mathbf{e}_r(\phi) + z\mathbf{e}_z$,

$$\frac{\partial \mathbf{R}}{\partial r} = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi + 0\mathbf{e}_3$$

$$= \mathbf{e}_r(\phi)$$

```
In[1]:= rCart = {x1, x2, x3};
        rCyl = {r Cos[ϕ], r Sin[ϕ], z};
        D[rCart, {{x1, x2, x3}}] // MatrixForm
Out[3]//MatrixForm=
  (1 0 0
   0 1 0
   0 0 1)

In[4]:= D[rCyl, {{r, ϕ, z}}] // MatrixForm
Out[4]//MatrixForm=
  (Cos[ϕ]  -r Sin[ϕ]  0
   Sin[ϕ]   r Cos[ϕ]  0
     0         0      1)
```

$$\frac{\partial \mathbf{R}}{\partial \phi} = -\mathbf{e}_1 r \sin\phi + \mathbf{e}_2 r \cos\phi + 0\mathbf{e}_3$$

$$= \mathbf{e}_\phi(\phi) = r\frac{\partial \mathbf{e}_r(\phi)}{\partial \phi}$$

and

$$\frac{\partial \mathbf{R}}{\partial z} = \mathbf{e}_z$$

1.93 In the transformation from the (x_1, x_2, x_3) system to the (r, ϕ, Z) coordinate system, the position vector changed from $\mathbf{R} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ to $\mathbf{R} = r\mathbf{e}_r(\phi) + z\mathbf{e}_z$ where $\mathbf{e}_r(\phi) = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi + z\mathbf{e}_3$. Find the reciprocal basis in the Cylindrical coordinate system $\mathbf{g}^j$, $j = 1,2,3$. [Hint: $2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$]

Mathematica code to make these differentiations easy as shown.

It is clear that,

$$\mathbf{g}_1 = \frac{\partial \mathbf{R}}{\partial r}$$

$$= \mathbf{e}_r(\phi) = \mathbf{e}_1 \cos\phi + \mathbf{e}_2 \sin\phi$$

$$\mathbf{g}_2 = \frac{\partial \mathbf{R}}{\partial \phi} = r\frac{\partial \mathbf{e}_r(\phi)}{\partial \phi}$$

$$\equiv \mathbf{e}_\phi = \mathbf{e}_2 r \cos\phi - \mathbf{e}_1 r \sin\phi$$

```
In[1]:= (* cCyl=Position Vector in Cylindrical Coordinates
           bVecs= columns contain the base vectors
            obtained by differentiating w.r.t
            coordinate variables
            bvecT= Transpose of bVec so that the rows of
             bVecT are the columns of bVec
             rootG= the scalar tripple product of base vectors
        *)
        rCyl = {r Cos[ϕ], r Sin[ϕ], z};
        bVecs = D[rCyl, {{r, ϕ, z}}];
        bVecsT = Transpose[bVecs];
        rootG = Dot[Cross[bVecsT[[1]], bVecsT[[2]]], bVecsT[[3]]];

In[5]:= bVecs // MatrixForm
Out[5]//MatrixForm=
  (Cos[ϕ]  -r Sin[ϕ]  0
   Sin[ϕ]   r Cos[ϕ]  0
     0         0      1)

In[6]:= Simplify[rootG]
Out[6]= r
```

and $\mathbf{g}_3 = \dfrac{\partial \mathbf{R}}{\partial \mathrm{z}} = \mathbf{e}_z = \mathbf{e}_3$

$$2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$$

so that

$$\mathbf{g}^1 = \frac{1}{2}\epsilon^{1jk}\mathbf{g}_j \times \mathbf{g}_k$$

$$\sqrt{g} = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = r$$

In this double sum, only two out of the nine terms survive; these are:

$$\begin{aligned}\mathbf{g}^1 &= \frac{1}{2}\epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3 + \frac{1}{2}\epsilon^{132}\mathbf{g}_3 \times \mathbf{g}_2 = \frac{1}{r}e_{123}\mathbf{g}_2 \times \mathbf{g}_3 \\ &= (\mathbf{e}_2 r\cos\phi - \mathbf{e}_1 r\sin\phi) \times \mathbf{e}_3 \\ &= \frac{1}{r}(\mathbf{e}_1\, r\cos\phi + \mathbf{e}_2 r\sin\phi) = \mathbf{e}_r(\phi)\end{aligned}$$

$$\begin{aligned}\mathbf{g}^2 &= \frac{1}{2}\epsilon^{231}\mathbf{g}_3 \times \mathbf{g}_1 + \frac{1}{2}\epsilon^{213}\mathbf{g}_1 \times \mathbf{g}_3 = \frac{1}{r}e_{123}\mathbf{g}_3 \times \mathbf{g}_1 \\ &= \frac{1}{r}\mathbf{e}_3 \times (\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi) = \frac{1}{r}(\mathbf{e}_2\cos\phi - \mathbf{e}_1\sin\phi) \\ &= \frac{1}{r}\mathbf{e}_\phi\end{aligned}$$

$$\begin{aligned}\mathbf{g}^3 &= \frac{1}{2}\epsilon^{312}\mathbf{g}_1 \times \mathbf{g}_2 + \frac{1}{2}\epsilon^{321}\mathbf{g}_2 \times \mathbf{g}_1 = \frac{1}{r}e_{123}\mathbf{g}_1 \times \mathbf{g}_2 \\ &= \frac{1}{r}(\mathbf{e}_1\cos\phi + \mathbf{e}_2\sin\phi) \times (\mathbf{e}_2 r\cos\phi - \mathbf{e}_1 r\sin\phi) = \mathbf{e}_z\end{aligned}$$

```
Cross[{Cos[ϕ], Sin[ϕ], 0}, {-r Sin[ϕ], r Cos[ϕ], 0}]
{0, 0, r Cos[ϕ]^2 + r Sin[ϕ]^2}
```

The reciprocal basis vectors of the cylindrical coordinate system is therefore, $\left\{\mathbf{e}_r(\phi), \frac{1}{r}\mathbf{e}_\phi, \mathbf{e}_z\right\}$.

What we have calculated manually are the rows of the matrix whose columns are the covariant bases. This is obtained directly by the Mathematica function call:

```
In[7]:= Simplify[Inverse[bVecs]] // MatrixForm
Out[7]//MatrixForm=
( Cos[ϕ]        Sin[ϕ]       0 )
( -Sin[ϕ]/r     Cos[ϕ]/r     0 )
( 0             0            1 )
```

1.94	Find the reciprocal bases for the Cartesian system.

To find the reciprocal or contravariant basis, we use the formula,

$$2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$$

Let us call the covariant basis (it is customary to label the basis obtained by direct differentiation covariant) of the Cartesian $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ and the contravariant basis $\{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$. Now, $\mathbf{g}^1 = \frac{1}{2}\epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3 + \frac{1}{2}\epsilon^{132}\mathbf{g}_3 \times \mathbf{g}_2 = \epsilon^{123}\mathbf{g}_2 \times \mathbf{g}_3$ so that, $\mathbf{I} = \mathbf{j} \times \mathbf{k} = \mathbf{i}$; $\mathbf{J} = \mathbf{k} \times \mathbf{i} = \mathbf{j}$; and $\mathbf{K} = \mathbf{i} \times \mathbf{j} = \mathbf{k}$. This shows that for the Cartesian system, the dual bases coincide and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$or $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\} = \{\mathbf{I}, \mathbf{J}, \mathbf{K}\}$ *or* $\{\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3\}$. Both systems are orthogonal and normalized. They also coincide.

1.95	In the transformation from the (x_1, x_2, x_3) system to the (ρ, θ, ϕ) coordinate system, the position vector changed from $\mathbf{R} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$ to $\mathbf{R} = \rho\mathbf{e}_\rho(\theta, \phi)$ where $\mathbf{e}_\rho(\theta, \phi) = \mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi + \rho\cos\theta\,\mathbf{e}_3$. Find the covariant bases as well as its reciprocal basis in the Cylindrical coordinate system $\mathbf{g}^j$, $j = 1,2,3$. [Hint: $2\mathbf{g}^i = \epsilon^{ijk}\mathbf{g}_j \times \mathbf{g}_k$]

The position vector for the spherical system,

$$\mathbf{rSph} = \mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi + \mathbf{e}_3\cos\theta$$

The covariant basis can be found by differentiating the position vector with respect to coordinate variables. We use the Mathematica code included and obtain,

$$\mathbf{g}_1 = \frac{\partial\mathbf{R}}{\partial\rho} = \mathbf{e}_\rho(\phi, \theta) = \mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2 \sin\theta\sin\phi + \mathbf{e}_3\cos\theta$$

$$\mathbf{g}_2 = \frac{\partial\mathbf{R}}{\partial\theta} = \mathbf{e}_\theta(\phi, \theta) = \mathbf{e}_1 \cos\theta\cos\phi + \mathbf{e}_2 \cos\theta\sin\phi - \mathbf{e}_3\sin\theta$$

$$= \rho\frac{\partial\mathbf{e}_\rho(\phi, \theta)}{\partial\theta}.$$

$$\mathbf{g}_3 = \frac{\partial\mathbf{R}}{\partial\phi} = \mathbf{e}_\phi(\phi, \theta)$$

$$= -\mathbf{e}_1 \sin\theta\cos\phi + \mathbf{e}_2\cos\theta\sin\phi$$

These are shown in the columns of **bVecs** in the attached code. The reciprocal basis vectors, as shown in Q1.93, are simply the rows of the inverse of the matrix of basis vectors **bVecs:**

Clearly, from the Mathematica code,

$\mathbf{g}_1 = \mathbf{e}_\rho(\phi, \theta)$,

$\mathbf{g}_2 = \frac{1}{\rho}\mathbf{e}_\theta(\phi, \theta)$, $\mathbf{g}_3 = \mathbf{e}_\phi(\phi, \theta)$,

and $\sqrt{g} = \mathbf{g}_1 \cdot \mathbf{g}_2 \times \mathbf{g}_3 = \rho^2 \sin\theta$

The last result is implemented in the variable **rootG**

```
In[1]:= (* cSph=Position Vector in Spherical Coordinates
          bVecs= columns contain the base vectors
           obtained by differentiating w.r.t
           coordinate variables
           bvecT= Transpose of bVec so that the rows of
            bVecT are the columns of bVec
            rootG= the scalar tripple product of base vectors
        *)
        rSph = {ρ Sin [θ] Cos[ϕ], ρ Sin [θ] Sin[ϕ], ρ Cos [θ]};
        bVecs = D[rSph, {{ρ, θ, ϕ}}];
        bVecsT = Transpose[bVecs];
        rootG = Dot[Cross[bVecsT[[1]], bVecsT[[2]]], bVecsT[[3]]];

In[5]:= bVecs // MatrixForm
Out[5]//MatrixForm=
        ( Cos[ϕ] Sin[θ]   ρ Cos[θ] Cos[ϕ]   -ρ Sin[θ] Sin[ϕ] )
        ( Sin[θ] Sin[ϕ]   ρ Cos[θ] Sin[ϕ]   ρ Cos[ϕ] Sin[θ]  )
        ( Cos[θ]          -ρ Sin[θ]         0                )

In[6]:= Simplify[rootG]
Out[6]= ρ^2 Sin[θ]

In[ ]:= Cross[{Cos[ϕ], Sin[ϕ], 0}, {-r Sin[ϕ], r Cos[ϕ], 0}]
Out[ ]= {0, 0, r Cos[ϕ]^2 + r Sin[ϕ]^2}

In[10]:= Simplify[Inverse[bVecs]] // MatrixForm
Out[10]//MatrixForm=
        ( Cos[ϕ] Sin[θ]              Sin[θ] Sin[ϕ]              Cos[θ]       )
        ( Cos[θ] Cos[ϕ]/ρ            Cos[θ] Sin[ϕ]/ρ            -Sin[θ]/ρ    )
        ( -Csc[θ] Sin[ϕ]/ρ           Cos[ϕ] Csc[θ]/ρ            0            )
```

1.96 Given that

$$\delta^{rst}_{ijk} \equiv e^{rst}e_{ijk} = \begin{vmatrix} \delta^r_i & \delta^r_j & \delta^r_k \\ \delta^s_i & \delta^s_j & \delta^s_k \\ \delta^t_i & \delta^t_j & \delta^t_k \end{vmatrix}$$ Show that $\delta^{rsk}_{ijk} = \delta^r_i\delta^s_j - \delta^s_i\delta^r_j$

Expanding the equation, we have:

$$e_{ijk}e^{rsk} = \delta^{rsk}_{ijk} = \delta^k_i \begin{vmatrix} \delta^r_j & \delta^r_k \\ \delta^s_j & \delta^s_k \end{vmatrix} - \delta^k_j \begin{vmatrix} \delta^r_i & \delta^r_k \\ \delta^s_i & \delta^s_k \end{vmatrix} + 3 \begin{vmatrix} \delta^r_i & \delta^r_j \\ \delta^s_i & \delta^s_j \end{vmatrix}$$

$$= \delta^k_i\left(\delta^r_j\delta^s_k - \delta^s_j\delta^r_k\right) - \delta^k_j(\delta^r_i\delta^s_k - \delta^s_i\delta^r_k) + 3\left(\delta^r_i\delta^s_j - \delta^s_i\delta^r_j\right) = \delta^r_j\delta^s_i$$

$$- \delta^s_j\delta^r_i - \delta^r_i\delta^s_j + \delta^s_i\delta^r_j + 3(\delta^r_i\delta^s_j$$

$$- \delta^s_i\delta^r_j) = -2(\delta^r_i\delta^s_j - \delta^s_i\delta^r_j) + 3(\delta^r_i\delta^s_j - \delta^s_i\delta^r_j) = \delta^r_i\delta^s_j - \delta^s_i\delta^r_j$$

1.97 Show that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

Let $\mathbf{z} = \mathbf{v} \times \mathbf{w} = \epsilon^{ijk} v_i w_j \mathbf{g}_k$. The triple product,

$$\begin{aligned}\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{u} \times \mathbf{z} &= \epsilon_{\alpha\beta\gamma} u^\alpha z^\beta \mathbf{g}^\gamma \\ &= \epsilon_{\alpha\beta\gamma} u^\alpha z^\beta \mathbf{g}^\gamma = \epsilon_{\alpha\beta\gamma} u^\alpha \epsilon^{ij\beta} v_i w_j \mathbf{g}^\gamma \\ &= \epsilon^{ij\beta} \epsilon_{\gamma\alpha\beta} u^\alpha v_i w_j \mathbf{g}^\gamma = \left(\delta^i_\gamma \delta^j_\alpha - \delta^i_\alpha \delta^j_\gamma\right) u^\alpha v_i w_j \mathbf{g}^\gamma \\ &= u^j v_\gamma w_j \mathbf{g}^\gamma - u^i v_i w_\gamma \mathbf{g}^\gamma \\ &= (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\end{aligned}$$

1.98 Given that, $\delta^{rsk}_{ijk} = \delta^r_i \delta^s_j - \delta^s_i \delta^r_j$, show that $\delta^{rjk}_{ijk} = 2\delta^r_i$.

Contracting one more index, we have:

$$e_{ijk} e^{rjk} = \delta^{rjk}_{ijk} = \delta^r_i \delta^j_j - \delta^j_i \delta^r_j = 3\delta^r_i - \delta^r_i = 2\delta^r_i$$

These results are useful in several situations.

1.99 Show that $g_{\gamma i} \epsilon^{\alpha\beta\gamma} \epsilon^{ijk} = g^{\alpha j} g^{\beta k} - g^{\alpha k} g^{\beta j}$

Note that

$$g_{\gamma i} \epsilon^{\alpha\beta\gamma} \epsilon^{ijk} = g_{\gamma i} \begin{vmatrix} g^{i\alpha} & g^{i\beta} & g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix} = \begin{vmatrix} g_{\gamma i} g^{i\alpha} & g_{\gamma i} g^{i\beta} & g_{\gamma i} g^{i\gamma} \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$

$$= \begin{vmatrix} \delta^\alpha_\gamma & \delta^\beta_\gamma & \delta^\gamma_\gamma \\ g^{j\alpha} & g^{j\beta} & g^{j\gamma} \\ g^{k\alpha} & g^{k\beta} & g^{k\gamma} \end{vmatrix}$$

$$= \delta^\alpha_\gamma \begin{vmatrix} g^{j\beta} & g^{j\gamma} \\ g^{k\beta} & g^{k\gamma} \end{vmatrix} - \delta^\beta_\gamma \begin{vmatrix} g^{j\alpha} & g^{j\gamma} \\ g^{k\alpha} & g^{k\gamma} \end{vmatrix} + \delta^\gamma_\gamma \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$

$$= \begin{vmatrix} g^{j\beta} & g^{j\alpha} \\ g^{k\beta} & g^{k\alpha} \end{vmatrix} - \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} + 3 \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix} = \begin{vmatrix} g^{j\alpha} & g^{j\beta} \\ g^{k\alpha} & g^{k\beta} \end{vmatrix}$$

$$= g^{\alpha j} g^{\beta k} - g^{\alpha k} g^{\beta j}$$

1.100 For vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = \mathbf{u} \otimes (\mathbf{v} \times \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\times$.

	The tensor $(\mathbf{u}\times) = -\epsilon_{lmn}u^n\mathbf{g}^l \otimes \mathbf{g}^m$ similarly, $(\mathbf{v}\times) = -\epsilon^{\alpha\beta\gamma}v_\gamma\mathbf{g}_\alpha \otimes \mathbf{g}_\beta$ and $(\mathbf{w}\times) = -\epsilon^{ijk}w_k\mathbf{g}_i \otimes \mathbf{g}_j$. Clearly, $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)(\mathbf{g}^l\otimes\mathbf{g}^m)(\mathbf{g}_i\otimes\mathbf{g}_j)$ $= -\epsilon^{\alpha\beta\gamma}\epsilon_{lmn}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)\delta^l_\beta\delta^m_i$ $= -\epsilon^{\alpha l\gamma}\epsilon_{lin}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)$ $= -\epsilon^{l\alpha\gamma}\epsilon_{lni}\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)$ $= -(\delta^\alpha_n\delta^\gamma_i - \delta^\alpha_i\delta^\gamma_n)\epsilon^{ijk}u^n v_\gamma w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j)$ $= -\epsilon^{ijk}u^\alpha v_i w_k(\mathbf{g}_\alpha\otimes\mathbf{g}_j) + \epsilon^{ijk}u^\gamma v_\gamma w_k(\mathbf{g}_i\otimes\mathbf{g}_j)$ $= [\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]$
1.101	Show that $[\mathbf{u},\mathbf{v},\mathbf{w}] = \text{tr}[(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times)]$
	In the above we have shown that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = [\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w}) - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times]$ Because the vector cross is traceless, the trace of $[(\mathbf{u}\cdot\mathbf{v})\mathbf{w}\times] = 0$. The trace of the first term, $\mathbf{u}\otimes(\mathbf{v}\times\mathbf{w})$ is obviously the same as $[\mathbf{u},\mathbf{v},\mathbf{w}]$ which completes the proof.
1.102	Show that $(\mathbf{u}\times)(\mathbf{v}\times) = (\mathbf{u}\cdot\mathbf{v})\mathbf{I} - \mathbf{u}\otimes\mathbf{v}$ and that $\text{tr}[(\mathbf{u}\times)(\mathbf{v}\times)] = 2(\mathbf{u}\cdot\mathbf{v})$
	$(\mathbf{u}\times)(\mathbf{v}\times) = -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}_\beta)(\mathbf{g}^l\otimes\mathbf{g}^m)$ $= -\epsilon_{lmn}\epsilon^{\alpha\beta\gamma}u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}^m)\delta^l_\beta = -\epsilon_{\beta mn}\epsilon^{\beta\gamma\alpha}u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}^m)$ $= [\delta^\gamma_n\delta^\alpha_m - \delta^\gamma_m\delta^\alpha_n]u^n v_\gamma(\mathbf{g}_\alpha\otimes\mathbf{g}^m)$ $= u^n v_n(\mathbf{g}_\alpha\otimes\mathbf{g}^\alpha) - u^n v_m(\mathbf{g}_n\otimes\mathbf{g}^m) = (\mathbf{u}\cdot\mathbf{v})\mathbf{I} - \mathbf{u}\otimes\mathbf{v}$ Obviously, the trace of this tensor is $2(\mathbf{u}\cdot\mathbf{v})$
1.103	
1.104	

1.105	
1.106	
1.107	
1.108	
1.110	
1.111	
1.112	

1.113	

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