TWO

Tensor Algebra:
Properties of a Tensor

"Continuum Mechanics may appear as a fortress surrounded by the walls of tensor notation" E. Tadmore, et. al.

MetaData

The prose, video, slides and the Q&A in this chapter are directed at scoring the following points:

1. The word "**Tensor**" applies to virtually all the quantities encountered in Engineering. Scalars and Vectors are zeroth and first order tensors. However, the word, with no prefix, refers to a second-order tensor.
2. For an object to be proved to be a tensor we need to show that it **transforms a vector** and its output is also a vector. Secondly, that transformation must be **linear**.
3. A tensor can be expressed in **Component Form**. The vector itself is more than its components as components presume reference to particular coordinate system. Outside that the numbers mean nothing.
4. For any tensor, certain scalar-valued functions are characteristic of the tensor, independent of coordinate systems. These are usually the targets of computations of any tensor. They are **Principal Invariants**.

5. Tensors can be **decomposed additively** or **multiplicatively** to simpler tensors. The goal is to make analysis easier and gain valuable insight by removing parts of the tensor not crucial to the problem.

What is a Tensor

It is a historical accident that the word *"Tensor"* first and foremost, refers, not to tensors generally, but to the **second-order tensor**. Strictly speaking, all the quantifiable objects we deal with are tensors. Scalars are known to be zeroth order tensors, vectors are first order tensors. We will later encounter third and fourth order tensors. It is assumed here that we already know what vectors and scalars are even though more time can still be spent to give more mathematically accurate definitions for each. We elect, initially, not to pursue that line. At the present moment, our principal concern is to define *second-order tensors* and understand their properties. Before we define these formally, we shall look at two familiar geometric occurences.

The Shadow or the Projection.

If you can identify four distinct directed arrows in figure 2.1, they are representing four vectors. The Brown arrow, (vector) labelled "$\mathbf{x}$" is lying "on the ground" and the rest are on a different plane. Of the remaining three, there are two copies of each. For each color or size, there is a free-standing copy, and another copy in a pile up. What we see as shadows are called projections in the mathematical sense.

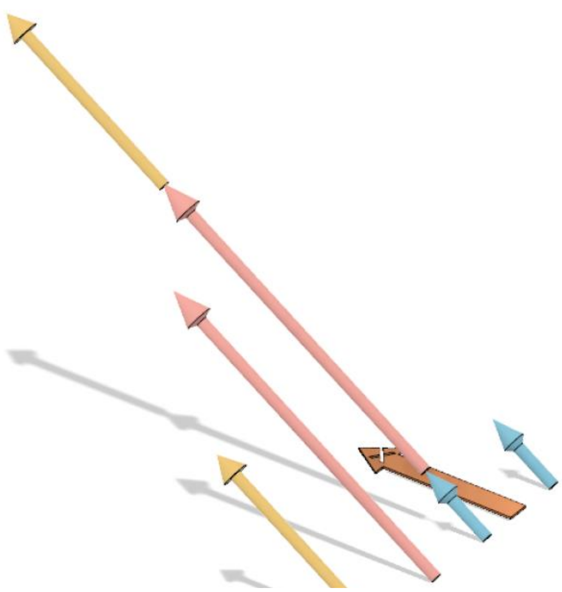


Figure 0-1 Projection Transformation

We define the projection, in the direction of $\mathbf{x} \in \mathbb{E}$, $\mathbf{P}_\mathbf{x}(\bullet)$ of $\mathbf{v} \in \mathbb{E}$ as follows:

$$\mathbf{P}_\mathbf{x}(\mathbf{v}) \equiv \mathbf{P}_\mathbf{x}\mathbf{v} = \left(\frac{1}{||\mathbf{x}||}\right)^2 (\mathbf{x} \otimes \mathbf{x})\mathbf{v}$$

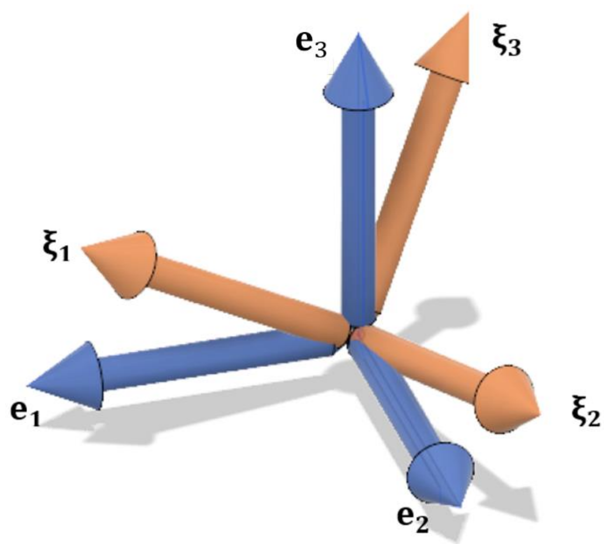
When it operates on any vector, it creates a vector on the plane of vector $\mathbf{x}$, and in the same direction as $\mathbf{x}$.

Observe that the projection of the pile up is equal to the sum of the individual projections; If, for example, we have a vector, twice the size of $\mathbf{v}$, Its projection will also be twice the size of the projection of $\mathbf{v}$.

In any case, the argument in the transformation is a vector, what you get out of it is also a vector. We call this the projection transformation. It is a transformation, whether we get it from the rays of the sun, shining on the arrows of from a *mathematical engine* as we have done here. The arrows have been transformed, linearly, to the plane containing vector $\mathbf{x}$; or, alternatively, they have been projected – "projection" being the name of the *transformation* we are dealing with here.

Coordinate Transformation

In figure 2.2, there are two sets of Ortho-Normal Basis ONB (unit magnitude for each base vector, and mutual orthogonality for any pair) vectors. One set has the basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ while the other is the set $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. We define the Coordinate Transformation of $\mathbf{v} \in \mathbb{E}$ defined in the coordinates with basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as the expression,



$$\mathbf{Cv} = (\boldsymbol{\xi}_i \otimes \mathbf{e}_i)\mathbf{v}$$

Suppose our parameter vector

$$\mathbf{v} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3 = \alpha_j \mathbf{e}_j$$

$$\begin{aligned}\mathbf{P}_{\mathbf{x}}(\mathbf{v}) = (\boldsymbol{\xi}_i \otimes \mathbf{e}_i)\alpha_j \mathbf{e}_j \\ = \alpha_j \boldsymbol{\xi}_i(\mathbf{e}_i \cdot \mathbf{e}_j) = \alpha_j \boldsymbol{\xi}_i \delta_{ij} \\ = \alpha_j \boldsymbol{\xi}_{\boldsymbol{j}} = 2\boldsymbol{\xi}_1 + 3\boldsymbol{\xi}_2 - \boldsymbol{\xi}_3\end{aligned}$$

So that the transformation takes any vector referred to the first set of coordinates and places them in exactly the same size, direction and location in the second coordinate system. It effects a change of coordinate system.

These transformations are only two examples of what a tensor does: It

1. takes one vector
2. produces another vector, and
3. performs the transformation linearly.

There is no doubt that in each of the transformations we have seen, the inputs (arguments) as well as the outputs (results) were vectors. What do we really mean by the term, "*transforms linearly*"?

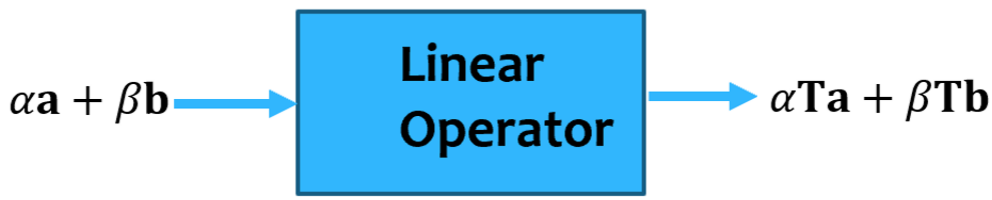
These two, $\mathbf{P}_{\mathbf{x}}(\bullet)$ as well as $\mathbf{C}(\bullet)$**,** transform in such a way that the transformation of a sum is the sum of the transformations; the transformation of a scalar multiple, is also a scalar multiple of the transformation; The transformation of a weighted sum is the weighted sum of the transformation.

Definition:

A *Second-Order Tensor* is the linear transformation of a vector into a vector.

Given that $\mathbf{a}, \mathbf{b} \in \mathbb{E}$ and $\alpha, \beta \in \mathbb{R}$, $\mathbf{T}$ is said to be a linear transformation if

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{T}\mathbf{a} + \beta\mathbf{T}\mathbf{b}$$



The projection as well as the coordinate transformations we have seen are *Linear Transformations*; and because they tansform from the vector space to the vector space, they are tensors of the second order. The projection transforms linearly because,

$$\begin{aligned}
\mathbf{C}(\alpha\mathbf{u} + \beta\mathbf{v}) &= (\boldsymbol{\xi}_i \otimes \mathbf{e}_i)(\alpha\mathbf{u} + \beta\mathbf{v}) \\
&= \boldsymbol{\xi}_i\big(\mathbf{e}_i \cdot (\alpha\mathbf{u} + \beta\mathbf{v})\big) = \boldsymbol{\xi}_i(\alpha\mathbf{e}_i \cdot \mathbf{u} + \beta\mathbf{e}_i \cdot \mathbf{v}) \\
&= \alpha\boldsymbol{\xi}_i(\mathbf{e}_i \cdot \mathbf{u}) + \beta\boldsymbol{\xi}_i(\mathbf{e}_i \cdot \mathbf{v}) \\
&= \alpha(\boldsymbol{\xi}_i \otimes \mathbf{e}_i)\mathbf{u} + \beta(\boldsymbol{\xi}_i \otimes \mathbf{e}_i)\mathbf{v} \\
&= \alpha\mathbf{C}\mathbf{u} + \beta\mathbf{C}\mathbf{v}
\end{aligned}$$

As shown in Figure___, the actual implementation of the operator in not the important thing. In the case of a shadow, the transformation could have come from the parallel rays of the sun or a mathematical factory like the tensor projector; for the coordinate transformation, again, it can be the mathematical formula given or that someone was sitting in a roller coaster and carrying the coordinate systems as well as the vector along. A linear transformation is defined, once the output relates linearly to the input. When such transforms a vector into another vector, it is a **second-order tensor**. Conventionally, the word “tensor” unqualified, refers to the second-order tensor. Usually, when we want to talk about tensors of other orders, an explicit reference to the order will be made unless the context already makes that clear.

Other Interesting Tensors

We adopt the convention that the set $\mathbb{L}$ is the set of all second-order tensors. Therefore, the statement, $\mathbf{T} \in \mathbb{L}$, literally that, $\mathbf{T}$ belongs to the set $\mathbb{L}$ means that $\mathbf{T}$ is a second order tensor. At this point, we have now met three tensors: The Dyad Product of two vectors, the Projection Tensor and the Coordinate Transformation Tensor.

$$\mathbf{u} \otimes \mathbf{v}, \mathbf{P}_{\mathbf{x}}, \mathbf{C}, \mathbf{S} \in \mathbb{L}$$

We now proceed to show the characteristics of other tensors that obey the same rules we have enunciated in the last section:

The Annihilator. We represent this tensor by the large size capital $\mathbf{O}$. It has this characteristic; for any vector $\mathbf{v}$**,**

$$\mathbf{O}\mathbf{v} = \mathbf{o}$$

yielding the zero vector as output (result), no matter the input (argument, operand).

The Identity & Spherical Tensors. The identity tensor is depicted by the large size capital bold $\mathbf{I}$. It has this characteristic; for any vector $\mathbf{v}$**,**

$$\mathbf{I}\mathbf{v} = \mathbf{v}$$

returning the input vector, no matter the input. Furthermore, $\forall \alpha \in \mathbb{R}$, the tensor, $\alpha\mathbf{I}$ is called a **Spherical Tensor**.

A spherical tensor is uniquely identified by the scalar multiplier of the identity tensor that produces it. It is therefore easy to misrepresent it as a scalar or a vector with equal components in the three directions. An example of this, as we shall see later, is *hydrostatic pressure*.

The Inverse. The identity tensor induces the concept of an inverse of a tensor. Given the fact that if $\mathbf{T} \in \mathbb{L}$ and $\mathbf{u} \in \mathbb{E}$, the mapping $\mathbf{w} \equiv \mathbf{T}\mathbf{u}$ produces a vector. Consider a linear mapping $\mathbf{Y}$, that, operating on $\mathbf{w}$, produces our original argument, $\mathbf{u}$, if we can find it:

$$\mathbf{Y}\mathbf{w} = \mathbf{u}$$

As a linear mapping, operating on a vector, clearly, $\mathbf{Y}$ is a tensor. It is called the inverse of $\mathbf{T}$ because,

$$\mathbf{Y}\mathbf{w} = \mathbf{Y}\mathbf{T}\mathbf{u} = \mathbf{u}$$

So that the composition (or product) $\mathbf{Y}\mathbf{T} = \mathbf{I}$, the identity mapping. For this reason, we write,

$$\mathbf{Y} = \mathbf{T}^{-1}$$

We now show that this relationship also implies that $\mathbf{TY} = \mathbf{I}$. Recall the vector defined by, $\mathbf{w} = \mathbf{Tu}$**.** Clearly,

$$\mathbf{TYTu} = \mathbf{TYw} = \mathbf{TIu} = \mathbf{Tu} = \mathbf{w}$$

(First equality by the definition of $\mathbf{w}$, second by the fact that $\mathbf{YT} = \mathbf{I}$). It is clear that

$$\mathbf{TYw} = \mathbf{w}$$

So that $\mathbf{TY} = \mathbf{YT} = \mathbf{I}$ as required.

Tensor Components

Tensors, just like vectors, can be expressed in component form with respect to a system of coordinates created with basis vectors. Using ONB, for a typical tensor $\mathbf{T}$, we can write,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

There are nine scalar components. These can be computed using the indicial notation and the summation convention. Accordingly,

$$\begin{aligned}\mathbf{T} = T_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + T_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + T_{13}\mathbf{e}_1 \otimes \mathbf{e}_3 + T_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + T_{22}\mathbf{e}_2 \otimes \mathbf{e}_2 + T_{23}\mathbf{e}_2 \otimes \mathbf{e}_3 \\ + T_{31}\mathbf{e}_3 \otimes \mathbf{e}_1 + T_{32}\mathbf{e}_3 \otimes \mathbf{e}_2 + T_{33}\mathbf{e}_3 \otimes \mathbf{e}_3\end{aligned}$$

We can find these components in terms of the tensor $\mathbf{T}$ in a way like the way we found the vector coefficients:

$$\begin{aligned}\mathbf{e}_\alpha \cdot \mathbf{T}\mathbf{e}_\beta &= T_{ij}\mathbf{e}_\alpha \cdot \left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_\beta \\ &= T_{ij}\mathbf{e}_\alpha \cdot \mathbf{e}_j\delta_{j\beta} = T_{ij}\delta_{i\alpha}\delta_{j\beta} \\ &= T_{\alpha\beta}\end{aligned}$$

Clearly,

$$\mathbf{T} = \left(\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j\right)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

In particular, $\mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j = \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$ so that the identity tensor has the representation,

$$\mathbf{I} = \left(\mathbf{e}_i \cdot \mathbf{I}\mathbf{e}_j\right)\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = \delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{e}_i \otimes \mathbf{e}_i$$

A third form for the component representation of the tensor can be found by observing that,

$$\mathbf{TI} = \mathbf{T}(\mathbf{e}_i \otimes \mathbf{e}_i) = (\mathbf{T}\mathbf{e}_i) \otimes \mathbf{e}_i = \mathbf{e}_i \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{e}_i)$$

In this section, we can see that the *Kronecker Deltas* introduced in the previous chapter are coefficients of the identity tensor as we can see in equation ___.

Components of a Composition.

A composition is a product of two transformations. Given that $\mathbf{S}$ and $\mathbf{T}$ are tensors, the application of $\mathbf{T}$ followed by the application of $\mathbf{S}$ to the result, is a composition tensor. Consider the composition of two tensors, $\mathbf{S}$ and $\mathbf{T}.$ Its action on an arbitrary vector $\mathbf{v} = v_k \mathbf{e}_k$ is,

$$\mathbf{STv} = \mathbf{S}(\mathbf{Tv}).$$

That is, $\mathbf{S}$ acts on the vector result of the action $\mathbf{Tv}.$ In order to find the component representation of this composition, it is useful to simplify the result dyad composition

$$(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)(\mathbf{e}_i \otimes \mathbf{e}_j)$$

We do this by finding its action on a typical vector.

$$\begin{aligned}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{u} &= (\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{u}) \\ &= \mathbf{e}_\alpha(\mathbf{e}_\beta \cdot \mathbf{e}_i)(\mathbf{e}_j \cdot \mathbf{u}) \\ &= (\mathbf{e}_\beta \cdot \mathbf{e}_i)(\mathbf{e}_\alpha \otimes \mathbf{e}_j)\mathbf{u}\end{aligned}$$

Showing that composing two dyads has the same effect as obtaining a dyad from the two extreme base vectors $\mathbf{e}_\alpha$ and $\mathbf{e}_j$ in this case, scaling the result by the dot product of the near vectors, $\mathbf{e}_\beta$ and $\mathbf{e}_i$. Clearly,

$$\begin{aligned}\mathbf{ST} &= (S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)(T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \\ &= S_{\alpha\beta}T_{ij}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)(\mathbf{e}_i \otimes \mathbf{e}_j) = S_{\alpha\beta}T_{ij}(\mathbf{e}_\beta \cdot \mathbf{e}_i)(\mathbf{e}_\alpha \otimes \mathbf{e}_j) \\ &= S_{\alpha\beta}T_{ij}\delta_{\beta i}(\mathbf{e}_\alpha \otimes \mathbf{e}_j) \\ &= S_{\alpha i}T_{ij}\mathbf{e}_\alpha \otimes \mathbf{e}_j = S_{ik}T_{kj}\mathbf{e}_i \otimes \mathbf{e}_j\end{aligned}$$

The result of the product of two dyads in this section can be generalized to a larger number of dyads. Given $\mathbf{a}_{i1}, \mathbf{a}_{i2} \dots \mathbf{a}_{in} \in \mathbb{E}$, the product

$$(\mathbf{a}_{i1} \otimes \mathbf{a}_{i2})(\mathbf{a}_{i3} \otimes \mathbf{a}_{i4}) \dots (\mathbf{a}_{i(n-1)} \otimes \mathbf{a}_{in})$$

can be shown to result in simply taking the first and the last of the vector operands and multiplying the that by the scalar products of all adjacent vectors:

$$\begin{aligned}&(\mathbf{a}_{i1} \otimes \mathbf{a}_{i2})(\mathbf{a}_{i3} \otimes \mathbf{a}_{i4}) \dots (\mathbf{a}_{i(n-1)} \otimes \mathbf{a}_{in}) \\ &\qquad = (\mathbf{a}_{i1} \otimes \mathbf{a}_{in})(\mathbf{a}_{i2} \cdot \mathbf{a}_{i3})(\mathbf{a}_{i4} \cdot \mathbf{a}_{i5}) \dots (\mathbf{a}_{i(n-2)} \cdot \mathbf{a}_{i(n-1)}).\end{aligned}$$

Reducing any tensor to a weighted sum of dyads is one way to simplify analyses as the dyads are much easier to deal with for this and several other reasons as we shall see.

Transpose of a Tensor, Symmetry

Given any two vectors, $\mathbf{u}$ and $\mathbf{v}$**,** and tensors $\mathbf{S}$ and $\mathbf{T}$, $\mathbf{S}$ is called the transpose of $\mathbf{T}$ if,

$$\mathbf{u}\cdot\mathbf{S}\mathbf{v} = \ \mathbf{v}\cdot\mathbf{T}\mathbf{u}$$

It is customary to use the same symbols for a tensor and its transpose. Accordingly, the transpose of $\mathbf{S}$ will be written as $\mathbf{S}^{\mathbf{T}}$. Furthermore, by virtue of the definition of transpose here, if $\mathbf{S}$ is the transpose of $\mathbf{T}$, then $\mathbf{T}$ is the transpose of $\mathbf{S}$

A tensor equal to its transpose is said to be **symmetrical**. Tensor $\mathbf{S}$ is symmetrical if,

$$\mathbf{S} = \mathbf{S}^{\mathbf{T}}$$

Given the dyad $\mathbf{a}\otimes\mathbf{b}$. For any two vectors, $\mathbf{u}$ and $\mathbf{v}$**,**

$$\mathbf{u}\cdot(\mathbf{a}\otimes\mathbf{b})\mathbf{v} = (\mathbf{u}\cdot\mathbf{a})(\mathbf{b}\cdot\mathbf{v}) = \mathbf{v}\cdot(\mathbf{b}\otimes\mathbf{a})\mathbf{u}$$

This shows that the transpose of a dyad is simply the swapping of its operands. Consequently, we find that, for $\mathbf{S} = \mathbf{S}^{\mathrm{T}}$**,**

$$\begin{aligned} S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j &= S_{ij}\left(\mathbf{e}_i\otimes\mathbf{e}_j\right)^{\mathrm{T}} \\ &= S_{ij}\mathbf{e}_j\otimes\mathbf{e}_i = S_{ji}\mathbf{e}_i\otimes\mathbf{e}_j \end{aligned}$$

So that, symmetry implies $S_{ij} = S_{ji}$. Furthermore, we find that $\mathbf{S}(\mathbf{u}\otimes\mathbf{v}) = \mathbf{S}\mathbf{u}\otimes\mathbf{v}$ because, for a vector $\mathbf{w}$,

$$\mathbf{S}(\mathbf{u}\otimes\mathbf{v})\mathbf{w} = \mathbf{S}\mathbf{u}(\mathbf{v}\cdot\mathbf{w}) = (\mathbf{S}\mathbf{u}\otimes\mathbf{v})\mathbf{w}$$

and,

$$\begin{aligned} (\mathbf{u}\otimes\mathbf{v})\mathbf{S}\mathbf{w} &= \mathbf{u}(\mathbf{v}\cdot\mathbf{S}\mathbf{w}) \\ &= \mathbf{u}(\mathbf{w}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{v}) = \mathbf{u}\Big((\mathbf{S}^{\mathrm{T}}\mathbf{v})\cdot\mathbf{w}\Big) \\ &= (\mathbf{u}\otimes\mathbf{S}^{\mathrm{T}}\mathbf{v})\mathbf{w} \end{aligned}$$

Tensor Invariants

Very often we are more interested, not in the tensors themselves but in scalar valued functions that take the tensor as argument. We will see several of these subsequently; perhaps the most important are the **principal invariants** of the tensor. For any three linearly independent vectors $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ and a tensor $\mathbf{T}$**,** it is can be shown (see Q&A) that, the three scalar valued functions $I_1(\mathbf{T}), I_2(\mathbf{T})$ and $I_3(\mathbf{T})$ defined below are independent of the choice of $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ and are therefore, intrinsic, or characteristic values of the tensor $\mathbf{T}$:

$$I_1(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \text{tr } \mathbf{T}$$

$$I_2(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \text{tr } \mathbf{T}^{\text{c}}, \text{and}$$

$$I_3(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{T}$$

Are respectively called the trace of $\mathbf{T}$, trace of its cofactor and determinant of $\mathbf{T}$ respectively. They are known as the principal invariants of the tensor. In particular, $I_1(\mathbf{T})$, trace of $\mathbf{T}$, is a linear operator because, given scalars α β as well as tensors $\mathbf{T}$ and $\mathbf{S}$,

$$\begin{aligned}
I_1(\alpha\mathbf{T} + \beta\mathbf{S}) &= \frac{[(\alpha\mathbf{T} + \beta\mathbf{S})\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, (\alpha\mathbf{T} + \beta\mathbf{S})\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, (\alpha\mathbf{T} + \beta\mathbf{S})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\
&= \frac{[(\alpha\mathbf{T})\mathbf{a}, \mathbf{b}, \mathbf{c}] + [(\beta\mathbf{S})\mathbf{a}, \mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \frac{[\mathbf{a}, (\alpha\mathbf{T})\mathbf{b}, \mathbf{c}] + [\mathbf{a}, (\beta\mathbf{S})\mathbf{b}, \mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\
&\quad + \frac{[\mathbf{a}, \mathbf{b}, (\alpha\mathbf{T})\mathbf{c}] + [\mathbf{a}, \mathbf{b}, (\beta\mathbf{S})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\
&= \alpha \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} + \beta \frac{[\mathbf{Sa}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Sb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Sc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\
&= \alpha I_1(\mathbf{T}) + \beta I_1(\mathbf{S}).
\end{aligned}$$

If $\mathbf{T}$ is a dyad, say, $\mathbf{T} = \mathbf{u} \otimes \mathbf{v}$**,** and we select the Cartesian ONB vectors as our linearly independent set, then,

$$\begin{aligned}
\text{tr } (\mathbf{u} \otimes \mathbf{v}) &\equiv I_1(\mathbf{u} \otimes \mathbf{v}) \\
&= \frac{[\{(\mathbf{u} \otimes \mathbf{v})\mathbf{e}_1\}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, \{(\mathbf{u} \otimes \mathbf{v})\ \mathbf{e}_2\}, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_2, \{(\mathbf{u} \otimes \mathbf{v})\mathbf{e}_3\}]}{[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]} \\
&= \frac{1}{1}\{[v_1\mathbf{u}, \mathbf{e}_2, \mathbf{e}_3] + [\mathbf{e}_1, v_2\mathbf{u}, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{e}_3, v_3\mathbf{u}]\} \\
&= \{(v_1\mathbf{u}) \cdot (e_{23i}\mathbf{e}_i) + (e_{31i}\mathbf{e}_i) \cdot (v_2\mathbf{u}) + (e_{12i}\mathbf{e}_i) \cdot (v_3\mathbf{u})\} \\
&= \{(v_1\mathbf{u}) \cdot (e_{231}\mathbf{e}_1) + (e_{312}\mathbf{e}_2) \cdot (v_2\mathbf{u}) + (e_{123}\mathbf{e}_3) \cdot (v_3\mathbf{u})\} = v_i u_i \\
&= \mathbf{u} \cdot \mathbf{v}
\end{aligned}$$

So that the trace of a dyad is simply the scalar product of its two vector operands. Note that we chose the base vectors in the above derivation since any three linearly independent vectors will be appropriate as the trace itself is independent of that choice.

It is easily shown, in this same way that, $I_2(\mathbf{u} \otimes \mathbf{v}) = I_3(\mathbf{u} \otimes \mathbf{v}) = 0$. Important to note that neither $I_2(\mathbf{T})$ nor $I_3(\mathbf{T})$ is linear for any tensor $\mathbf{T}$.

Other scalar invariants may be defined. Another set $\{J_1(\mathbf{T}) = \text{tr}\,\mathbf{T},\ J_2(\mathbf{T}) = \text{tr}\,\mathbf{T}^2,\ J_3(\mathbf{T}) = \text{tr}\,\mathbf{T}^3\}$ has been defined, all arising from equation ____ defining the trace.

We now show that the coefficient T_{ij} in the component representation,

$$\mathbf{T} = (\mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j)(\mathbf{e}_i \otimes \mathbf{e}_j) = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

of $\mathbf{T}$,

$$\begin{aligned} T_{ij} &= \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j = \text{tr}\left(\mathbf{T}(\mathbf{e}_j \otimes \mathbf{e}_i)\right) \\ &= \text{tr}\left(\mathbf{T}\mathbf{e}_j \otimes \mathbf{e}_i\right) = \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j \\ &= \text{tr}\left(\mathbf{T}^{\text{T}}(\mathbf{e}_i \otimes \mathbf{e}_j)\right) = \mathbf{T}^{\text{T}}\mathbf{e}_i \cdot \mathbf{e}_j \\ &= \mathbf{e}_i \cdot \mathbf{T}\mathbf{e}_j \end{aligned}$$

This leads to a definition:

The Inner Product of two tensors

The inner product of tensors $\mathbf{S}$ and $\mathbf{T}$ is the trace

$$\mathbf{S}:\mathbf{T} \equiv \text{tr}(\mathbf{S}^{\text{T}}\mathbf{T}) = \text{tr}(\mathbf{S}\mathbf{T}^{\text{T}})$$

From the above result, the scalar components of tensor $\mathbf{T}$ on the dyad bases $(\mathbf{e}_i \otimes \mathbf{e}_j)$ is given by,

$$T_{ij} = \mathbf{T}:(\mathbf{e}_i \otimes \mathbf{e}_j)$$

The Trace, $I_1(\mathbf{T}) = \textbf{tr}\,\mathbf{T}$

Beginning from the component representation,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

Taking the trace of both sides, we have,

$$\text{tr}\,\mathbf{T} = T_{ij}\,\text{tr}(\mathbf{e}_i \otimes \mathbf{e}_j) = T_{ij}\delta_{ij} = T_{ii}$$

as we have shown earlier that the trace of a dyad is the scalar product of its operands. We note that transposing a tensor does not alter its trace because,

$$\text{tr}\,\mathbf{T}^{\text{T}} = T_{ij}\,\text{tr}(\mathbf{e}_j \otimes \mathbf{e}_i) = T_{ij}\delta_{ji} = T_{ii} = \text{tr}\,\mathbf{T}.$$

Trace of the Cofactor, $\boldsymbol{I_2}(\mathbf{T}) = \mathbf{tr}\ \mathbf{T^c}$

The cofactor will be defined subsequently. Presently, we rely in the earlier definition of the second principal Invariant:

$$I_2(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \text{tr}\ \mathbf{T}^\text{c}$$

To express this in component form, we set our linearly independent set as the basis set, $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Note immediately that $[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] = 1$, so that,

$$\begin{aligned}
I_2(\mathbf{T}) &= [\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{e}_3] + [\mathbf{e}_1, \mathbf{Te}_2, \mathbf{Te}_3] + [\mathbf{Te}_1, \mathbf{e}_2, \mathbf{Te}_3] \\
&= \left[T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1, T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_2, \mathbf{e}_3\right] + \left[\mathbf{e}_1, T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_2, T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3\right] \\
&\qquad + \left[T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_1, \mathbf{e}_2, T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_3\right] \\
&= [T_{i1}\mathbf{e}_i, T_{\alpha 2}\mathbf{e}_\alpha, \mathbf{e}_3] + [\mathbf{e}_1, T_{\alpha 2}\mathbf{e}_\alpha, T_{i3}\mathbf{e}_i] + [T_{\alpha 1}\mathbf{e}_\alpha, \mathbf{e}_2, T_{i3}\mathbf{e}_i] \\
&= T_{i1}T_{\alpha 2}[\mathbf{e}_i, \mathbf{e}_\alpha, \mathbf{e}_3] + T_{\alpha 2}T_{i3}[\mathbf{e}_1, \mathbf{e}_\alpha, \mathbf{e}_i] + T_{\alpha 1}T_{i3}[\mathbf{e}_\alpha, \mathbf{e}_2, \mathbf{e}_i] \\
&= T_{i1}T_{\alpha 2}e_{i\alpha 3} + T_{\alpha 2}T_{i3}e_{1\alpha i} + T_{\alpha 1}T_{i3}e_{\alpha 2i} \\
&= T_{11}T_{22} - T_{21}T_{12} + T_{22}T_{33} - T_{32}T_{23} + T_{11}T_{33} - T_{31}T_{13} \\
&= \frac{1}{2}\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)
\end{aligned}$$

Half of the square of the trace minus the trace of the square. Note that this invariant, $I_2(\mathbf{T})$ is NOT linear in its argument $\mathbf{T}$.

Exercise. Show that the second invariant is independent of the set of linearly independent vectors chosen.

The Determinant, $\boldsymbol{I_3}(\mathbf{T}) = \mathbf{det}\,\mathbf{T}$

As previously observed, any three linearly independent vectors can be treated as the basis for defining the invariants. We select $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. For any tensor $\mathbf{T}$,

$$\begin{aligned}
I_3(\mathbf{T}) = [\mathbf{Te}_1, \mathbf{Te}_2, \mathbf{Te}_3] &= \left[T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{e}_1, T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_2, T_{rs}(\mathbf{e}_r \otimes \mathbf{e}_s)\mathbf{e}_3\right] \\
&= [T_{i1}\mathbf{e}_i, T_{\alpha 2}\mathbf{e}_\alpha, T_{r3}\mathbf{e}_r] = T_{i1}T_{\alpha 2}T_{r3}e_{i\alpha r} \\
&= T_{i1}T_{j2}T_{k3}e_{ijk} = \det \mathbf{T}
\end{aligned}$$

For the tensors $\mathbf{A}$ and $\mathbf{B}$, we use the definition of the determinant to show that $\det \mathbf{AB} = \det \mathbf{A} \times \det \mathbf{B}$:

Select linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. If $\mathbf{B}$ is non-singular, it is easy to show that $\mathbf{u}(= \mathbf{Ba}), \mathbf{v}(= \mathbf{Bb})$and $\mathbf{w}(= \mathbf{Bc})$ are also linearly independent. Now,

$$\det \mathbf{AB} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[\mathbf{ABa}, \mathbf{ABb}, \mathbf{ABc}]}{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$
$$= \frac{[\mathbf{Au}, \mathbf{Av}, \mathbf{Aw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ba}, \mathbf{Bb}, \mathbf{Bc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A} \times \det \mathbf{B}$$

Tensor Magnitude and Direction.

In the same way as vectors, the inner product of tensors induces the concept of magnitude and direction to tensors. Unlike vectors however, we do not have the same geometric interpretation in terms of the lengths of directed lines and their included angles. Inspired by the fact that $\mathbf{T}:\mathbf{T}$ is a scalar, we define he magnitude of a tensor

$$\|\mathbf{T}\| = \sqrt{\mathbf{T}:\mathbf{T}}$$

And, just like vectors, the angle between two tensors can be computed from,

$$\theta = \cos^{-1} \frac{\mathbf{S}:\mathbf{T}}{\|\mathbf{S}\| \|\mathbf{T}\|}.$$

With this definition, a second-order tensor fulfils all the stipulations necessary to be a Euclidean Vector Space $\mathbb{L}$:

1. **Addition operation** is **defined** and it is **commutative** and **associative** under $\mathbb{L}$: that is, $\mathbf{T} + \mathbf{S} \in \mathbb{L}$, $\mathbf{S} + \mathbf{T} = \mathbf{T} + \mathbf{S}$, $\mathbf{T} + (\mathbf{S} + \mathbf{V}) = (\mathbf{T} + \mathbf{S}) + \mathbf{V}$, $\forall\ \mathbf{T}, \mathbf{S}, \mathbf{V} \in \mathbb{L}$. Furthermore, $\mathbb{L}$ is **closed** under addition: That is, given that $\mathbf{T}, \mathbf{S} \in \mathbb{L}$, then $\mathbf{V} = \mathbf{T} + \mathbf{S} = \mathbf{S} + \mathbf{T}$, $\Rightarrow \mathbf{w} \in \mathbb{V}$.
2. $\mathbb{L}$ **contains** a **zero element** $\mathbf{O}$ such that $\mathbf{T} + \mathbf{O} = \mathbf{T}\ \forall\ \mathbf{T} \in \mathbb{L}$. For every $\mathbf{u} \in \mathbb{L}$, $\exists - \mathbf{T}: \mathbf{T} + (-\mathbf{T}) = \mathbf{O}$.
3. **Multiplication by a scalar**. For $\alpha, \beta \in \mathbb{R}$ and $\mathbf{T}, \mathbf{S} \in \mathbb{L}$, $\alpha\mathbf{T} \in \mathbb{L}$, $1\mathbf{T} = \mathbf{T}$, $\alpha(\beta\mathbf{T}) = (\alpha\beta)\mathbf{T}$, $(\alpha + \beta)\mathbf{T} = \alpha\mathbf{T} + \beta\mathbf{T}$, $\alpha(\mathbf{T} + \mathbf{S}) = \alpha\mathbf{T} + \alpha\mathbf{S}$.

Rule 1 is easily proven from linearity of transformations or using components. For example, consider the sum transformation on an arbitrary $\mathbf{u} \in \mathbb{V}$:

$$\begin{aligned}
(\mathbf{S} + \mathbf{T})\mathbf{u} &= \left(S_{ij}\mathbf{e}_i \otimes \mathbf{e}_j + T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j\right)u_\alpha \mathbf{e}_\alpha \\
&= \left(S_{ij} + T_{ij}\right)u_\alpha\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_\alpha \\
&= \left(S_{ij} + T_{ij}\right)u_\alpha \mathbf{e}_i \delta_{j\alpha} = \left(S_{ij} + T_{ij}\right)u_j \mathbf{e}_i \\
&= S_{ij}u_j\mathbf{e}_i + T_{ij}u_j\mathbf{e}_i
\end{aligned}$$

$$= \mathbf{Su} + \mathbf{Tu} = \mathbf{Tu} + \mathbf{Su}$$

The associative rules, as well as the zero-element rule (#2) are similarly established. The annihilator tensor fulfils this role.

Rule 3. Multiplication by a scalar. This is established by the linearity of the transformation for linearity for $\mathbf{T}$ stipulates that,

$$\mathbf{T}(\alpha\mathbf{a} + \beta\mathbf{b}) = \alpha\mathbf{Ta} + \beta\mathbf{Tb}$$

Let $\mathbf{a} = \mathbf{b} = \mathbf{u}$, then,

$$\begin{aligned} \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{u}) &= \alpha\mathbf{Tu} + \beta\mathbf{Tu} \\ &= (\alpha + \beta)\mathbf{Tu} \\ &= \big((\alpha + \beta)\mathbf{T}\big)\mathbf{u} \end{aligned}$$

Finally, scalar product between two tensors is defined. It naturally induces the concept of magnitude:

$$\|\mathbf{T}\| = \sqrt{\mathbf{T}:\mathbf{T}}$$

Higher-order tensors retain the same definition as second-order tensors. A fourth-order tensor transforms a second order tensor (a member of a vector space) to a second order tensor (vector space). A third order tensor transforms a vector to a second-order tensor (vector space), it also transforms a second-order tensor to a vector. In any case, the fact that a tensor transforms from a vector space to a vector space remains unchanged.

Additive Decompositions of Tensors

The definitions of the spherical tensor and the symmetric tensor induce two additive decompositions of tensors.

Spherical & Deviatoric Parts

Every tensor can be decomposed into a spherical and deviatoric parts. The spherical part of a tensor is obtained by dividing its trace by three and use the result to scale an identity tensor. For a tensor $\mathbf{S}$, the spherical part,

$$\operatorname{sph}\mathbf{S} = \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\mathbf{I} = \frac{1}{3}S_{kk}\delta_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

and the deviatoric part is what remains after removing the spherical part:

$$\operatorname{dev}\mathbf{S} = \mathbf{S} - \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\mathbf{I} = \left(S_{ij} - \frac{1}{3}S_{kk}\delta_{ij}\right)\mathbf{e}_i \otimes \mathbf{e}_j$$

The traces,

$$\operatorname{tr}(\operatorname{sph}\mathbf{S}) = \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\operatorname{tr}\mathbf{I} = \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)3 = \operatorname{tr}\mathbf{S}$$

$$\operatorname{tr}(\operatorname{dev}\mathbf{S}) = \operatorname{tr}\mathbf{S} - \left(\frac{1}{3}\operatorname{tr}\mathbf{S}\right)\operatorname{tr}\mathbf{I} = \operatorname{tr}\mathbf{S} - \operatorname{tr}\mathbf{S} = 0.$$

The deviatoric component has zero trace. It is traceless.

Symmetric and Skew Parts

We can also decompose a tensor $\mathbf{S}$, into symmetrical and anti-symmetrical parts. An anti-symmetric tensor, also called a skew tensor is defined as that which is the negative of its transpose. Hence the symmetric part

$$\operatorname{sym}\mathbf{S} = \frac{1}{2}(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) = \frac{1}{2}(S_{ij} + S_{ji})\mathbf{e}_i \otimes \mathbf{e}_j$$

and the skew part:

$$\operatorname{skw}\mathbf{S} = \frac{1}{2}(\mathbf{S} - \mathbf{S}^{\mathrm{T}}) = \frac{1}{2}(S_{ij} - S_{ji})\mathbf{e}_i \otimes \mathbf{e}_j$$

The transposes,

$$(\operatorname{sym}\mathbf{S})^{\mathrm{T}} = \left(\frac{1}{2}(\mathbf{S} + \mathbf{S}^{\mathrm{T}})\right)^{\mathrm{T}} = \frac{1}{2}(\mathbf{S}^{\mathrm{T}} + (\mathbf{S}^{\mathrm{T}})^{\mathrm{T}}) = \operatorname{sym}\mathbf{S}$$

$$(\operatorname{skw}\mathbf{S})^{\mathrm{T}} = \left(\frac{1}{2}(\mathbf{S} - \mathbf{S}^{\mathrm{T}})\right)^{\mathrm{T}} = \frac{1}{2}(\mathbf{S}^{\mathrm{T}} - (\mathbf{S}^{\mathrm{T}})^{\mathrm{T}}) = -\operatorname{skw}\mathbf{S}$$

These results can be established from the component representation as well. However, we shall opt for direct proofs anytime they are available. The proof from the components is left as an exercise.

$$\operatorname{tr}(\operatorname{sym}\mathbf{S}) = \frac{1}{2}\operatorname{tr}(\mathbf{S} + \mathbf{S}^{\mathrm{T}}) = \frac{1}{2}(\operatorname{tr}\mathbf{S} + \operatorname{tr}\mathbf{S})$$

commutative property of the scalar product makes the trace of a transpose the same as the trace of the tensor from which the transpose is obtained. It is easy to see, in the same way that the trace of a skew tensor also vanishes: $\operatorname{tr}(\operatorname{skw}\mathbf{S}) = 0$. The spherical part of a tensor is always symmetric. This symmetry is induced by that of the identity tensor as there is only a scaling between a spherical tensor and the identity. No judgement can be made on the deviatoric tensor however. Its symmetry wholly depends on the original tensor from which the deviatoric part is taken. If the latter is symmetric, so will the deviatoric part. If skew, so also will the deviatoric part. It is is quite possible that the deviatoric tensor is neither symmetric nor skew.

Axial Vector of a Skew Tensor & the Vector Cross

The Triad. In this section we introduce the triad: a third order tensor that can be created by the tensor product of three vectors. Just like the dyad, a triad is defined by its operation on a vector. Given vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ the triad produces a dyad as follows:

$$(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c})\mathbf{v} = (\mathbf{c} \cdot \mathbf{v})(\mathbf{a} \otimes \mathbf{b}).$$

The fact that the familiar alternating symbol, e_{ijk} are, for various combinations of its indices, components of a tensor, now becomes obvious. It will be introduced shortly in a computation to follow.

Recall that a skew tensor, the negative of its transpose, satisfies,

$$\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = -T_{ij}\mathbf{e}_j \otimes \mathbf{e}_i$$

There two immediate consequences of this:

1. For a skew tensor, $T_{ij} = -T_{ji}$

   The diagonal elements vanish; only three of the nine components are independent as the others are either zero or negatives of one of the three.
2. Following the above, ALL information contained in the tensor can be made into a vector. Such a vector exists for every antisymmetric tensor. It is called the Axial vector.

The converse of this is also true. Given any vector, we can construct a skew tensor based on the three components of the vector. Such a vector is called the Vector Cross (tensor) of the vector. The reason for such a name will become obvious shortly:

Given any vector $\mathbf{u} = u_\alpha \mathbf{e}_\alpha$ we can form the vector cross tensor by this formula:

$$\mathbf{\Omega} = (\mathbf{u}\,\times) \equiv -\mathbf{E}\mathbf{u}$$

where $\mathbf{E} \equiv e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ is the third order alternating tensor. (The operations of the triads, as we have seen, work similarly to that of the dyad). In component form,

$$\begin{aligned}\Omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j &= \left(-e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\right)(u_\alpha \mathbf{e}_\alpha)\\ &= \left(e_{ikj}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\right)(u_\alpha \mathbf{e}_\alpha)\end{aligned}$$

$$= e_{ikj}\mathbf{e}_i \otimes \mathbf{e}_j u_\alpha \delta_{k\alpha} = e_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

Note: The result of the transformation of any vector $\mathbf{v}$, by the **Vector Cross** of $\mathbf{u}$ is the same as performing a vector product between $\mathbf{u}$ and $\mathbf{v}$**:**

$$\begin{aligned}\boldsymbol{\Omega}\mathbf{v} &= (-\mathbf{Eu})\mathbf{v} \\ &= \left(e_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j\right)(v_k \mathbf{e}_k) \\ &= e_{i\alpha j} u_\alpha v_k \mathbf{e}_i \delta_{jk} \\ &= e_{i\alpha j} u_\alpha v_j \mathbf{e}_i \\ &= \mathbf{u} \times \mathbf{v}\end{aligned}$$

So that the effect of the operation of $\boldsymbol{\Omega}$ is the same as that of $(\mathbf{u}\,\times)$ on the same vector. Hence the name, vector cross.

Suppose we have been given a skew tensor,

$$\Omega_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = (\mathbf{u}\,\times) = e_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

We want to find the components u_α from the Ω_{ij}s.

Clearly,

$$\Omega_{ij} = e_{i\alpha j} u_\alpha$$

Multiplying both sides by e_{ijk}, we obtain,

$$e_{ijk}\Omega_{ij} = e_{ijk} e_{i\alpha j} u_\alpha = -2\delta_{k\alpha} u_\alpha = -2u_k$$

So that we can find the components of the dual vector from, $u_k = -\frac{1}{2} e_{ijk}\Omega_{ij}$; and if we are given the vector $\mathbf{u}$, we can find the vector cross from its components using, $\Omega_{ij} = e_{i\alpha j} u_\alpha$.

We can also define tensor products between objects other than vectors at this point. We can now rely more on the component form of these objects to arrive at consistent definitions as shown in the table below:

Given tensors $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, $\mathbf{S} = S_{lm}\mathbf{e}_l \otimes \mathbf{e}_m$ and vectors $\mathbf{u} = u_\alpha \mathbf{e}_\alpha$, $v = v_\beta \mathbf{e}_\beta$, the following tensor products can be taken:

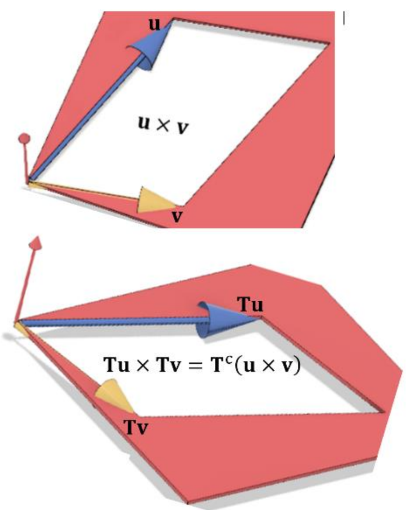
Product	Components	Operation on vector	Component form
$\mathbf{T} \otimes \mathbf{u}$	$T_{ij} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_\alpha$	$(\mathbf{T} \otimes \mathbf{u})\mathbf{v}$	$T_{ij} u_\alpha v_\beta \mathbf{e}_i \otimes \mathbf{e}_j \delta_{\alpha\beta}$ $= T_{ij} u_\alpha v_\alpha \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{T}(\mathbf{u} \cdot \mathbf{v})$

$\mathbf{u} \otimes \mathbf{T}$	$T_{ij}u_\alpha \mathbf{e}_\alpha \otimes \mathbf{e}_i \otimes \mathbf{e}_j$	$(\mathbf{u} \otimes \mathbf{T})\mathbf{v}$	$T_{ij}u_\alpha v_\beta \mathbf{e}_\alpha \otimes \mathbf{e}_i \delta_{j\beta} =$ $T_{ij}u_\alpha v_j \mathbf{e}_\alpha \otimes \mathbf{e}_i = \mathbf{u} \otimes \mathbf{T}\mathbf{v}$
$\mathbf{T} \otimes \mathbf{S}$	$T_{ij}S_{lm}\, \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_l \otimes \mathbf{e}_m$	$(\mathbf{T} \otimes \mathbf{S})\mathbf{u}$	$\mathbf{T} \otimes \mathbf{S}\mathbf{u}$
		$\mathbf{v}(\mathbf{T} \otimes \mathbf{S})$	$(\mathbf{T}^{\mathrm{T}}\mathbf{v}) \otimes \mathbf{S}$

The Cofactor of a Tensor

The cofactor of an invertible tensor is defined as $\mathrm{cof}\,\mathbf{T} \equiv \mathbf{T}^{\mathrm{c}} = \mathbf{T}^{-\mathrm{T}} \det \mathbf{T}$

Begin with a pair of linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$. Consider the parallelogram created by these vectors and the perpendicular to the parallelogram plane. The vector area shown is given by $\mathbf{u} \times \mathbf{v}$ and its direction is parallel to the shown normal. If the two vectors are transformed by a tensor $\mathbf{T}$, the transformed vectors create another parallelogram vector given by $\mathbf{T}\mathbf{u} \times \mathbf{T}\mathbf{v}$. the cofactor provides the relationship between these two vector areas:



$$\mathbf{T}\mathbf{u} \times \mathbf{T}\mathbf{v} = \mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$$

The **cofactor maps a vector area created by $\mathbf{u}$ and $\mathbf{v}$ into the vector area created by the transformed vectors $\mathbf{T}\mathbf{u}$ and $\mathbf{T}\mathbf{v}$**. This transformation is *bi-linear*. For example, given $\alpha, \beta \in \mathbb{R}$, linearity of tensor $\mathbf{T} \Rightarrow$

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{T}\mathbf{w} &= \alpha\mathbf{T}\mathbf{u} \times \mathbf{T}\mathbf{w} + \beta\mathbf{T}\mathbf{v} \times \mathbf{T}\mathbf{w} \\ &= \alpha\mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{w}) + \beta\mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{w}) \\ &= \mathbf{T}^{\mathrm{c}}\big((\alpha\mathbf{u} + \beta\mathbf{v}) \times \mathbf{w}\big)\end{aligned}$$

The last equality coming from the linearity of cofactor tensor $\mathbf{T}^{\mathrm{c}}$, and the distributive property of the vector product over addition.

In a deformation field, the changes in lengths, areas and volumes are highly variable – spatial and even temporal functions. Only small neighborhoods transform this way in the limit. In that case, the elements of areas are also transformed by the cofactor of the same tensor that transforms the element of length. If only elements of length are transformed by a tensorWe proceed now to obtain the components of the cofactor: If $\mathbf{T}^{\mathrm{c}} = T_{ij}^{c}\mathbf{e}_i \otimes \mathbf{e}_j$, then the inner product,

$$\begin{aligned}
\mathbf{T}^{\mathrm{c}}: \left(\mathbf{e}_i \otimes \mathbf{e}_j\right) = T_{ij}^{c} \\
&= \mathbf{e}_i \cdot \mathbf{T}^{\mathrm{c}}\mathbf{e}_j = \mathbf{e}_i \cdot \left[\mathbf{T}^{\mathrm{c}}\left(\frac{1}{2}e_{jmn}\mathbf{e}_m \times \mathbf{e}_n\right)\right] \\
&= \frac{1}{2}e_{jmn}\mathbf{e}_i \cdot (\mathbf{T}\mathbf{e}_m \times \mathbf{T}\mathbf{e}_n)
\end{aligned}$$

In the expression, $\mathbf{e}_i \cdot (\mathbf{T}\mathbf{e}_m \times \mathbf{T}\mathbf{e}_n)$ we seek the i^{th} component of $\mathbf{T}\mathbf{e}_m \times \mathbf{T}\mathbf{e}_n$.

$$\begin{aligned}
T_{ij}^{c} &= \frac{1}{2}e_{jmn}e_{i\alpha\beta}(\mathbf{e}_\alpha \cdot \mathbf{T}\mathbf{e}_m)\left(\mathbf{e}_\beta \cdot \mathbf{T}\mathbf{e}_n\right) \\
&= \frac{1}{2}e_{i\alpha\beta}e_{jmn}T_{\alpha m}T_{\beta n}
\end{aligned}$$

The cofactor,

$$\mathbf{T}^{\mathrm{c}} = \frac{1}{2}e_{i\alpha\beta}e_{jmn}T_{\alpha m}T_{\beta n}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)$$

The inverse tensor,

$$\begin{aligned}
\mathbf{T}^{-1} &= (\det \mathbf{T})^{-1}\mathbf{T}^{\mathrm{cT}} \\
&= \frac{(\det \mathbf{T})^{-1}}{2}e_{i\alpha\beta}e_{jmn}T_{\alpha m}T_{\beta n}\left(\mathbf{e}_j \otimes \mathbf{e}_i\right)
\end{aligned}$$

Second principal invariant of $\mathbf{T}$ is the trace of its cofactor, $\operatorname{tr} \mathbf{T}^{\mathrm{c}}$

$$\begin{aligned}
I_2(\mathbf{T}) &= \frac{1}{2}e_{i\alpha\beta}e_{jmn}T_{\alpha m}T_{\beta n}\left(\mathbf{e}_i \cdot \mathbf{e}_j\right) \\
&= \frac{1}{2}\left(\delta_{\alpha m}\delta_{\beta n} - \delta_{\alpha n}\,\delta_{\beta m}\right)T_{\alpha m}T_{\beta n} \\
&= \frac{1}{2}(T_{mm}T_{nn} - T_{mn}T_{nm}) \\
&= \frac{1}{2}(\operatorname{tr}^2 \mathbf{T} - \operatorname{tr} \mathbf{T}^2)
\end{aligned}$$

The Eigenvalue Problem

Vectors and tensors exist independently of the reference frame we use to characterize them. The values in the matrix part of a tensor only takes meaning from the coordinate frame whose basis vectors are weighted by those values. We know that vectors have magnitudes and directions intrinsic to each vector. So do tensors; there are characteristic values that pertain to the tensor that are not dependent on the coordinates to which we refer them. These are the eigen values and eigenvectors of the tensor. In order to discuss these quantities, we pose the fundamental eigenvalue problem:

Given that a second-order tensor transforms an input vector $\mathbf{u}$ to an output vector $\mathbf{v}$; ordinarily, we do not assume any relationship between $\mathbf{u}$ and $\mathbf{v}$. The eigenvalue problem is: What if the output vector is simply a scalar multiple of the input vector? To answer that question, we will need to solve the problem:

$$\mathbf{Tu} = \lambda\mathbf{u}$$

Where λ, if we can find it, is a scalar called eigenvalue, and $\mathbf{u}$ when it exists, is the corresponding eigenvector.

Eigenvalues and eigenvectors are essentially the fundamental quantities that engineers need in a typical tensor. Its physical interpretation is wide and diverse. From materials science where the eigenvalues represent principal stresses while eigenvectors represent principal surfaces, dynamics, where they are natural frequencies and mode shapes to electric circuits and several other applications. The importance of the eigenvalue problem cannot be overemphasized.

$$\begin{aligned}\mathbf{Tu} - \lambda\mathbf{u} &= T_{ij}\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_k)u_k - \lambda u_i\mathbf{e}_i \\ &= T_{ij}\mathbf{e}_i u_j - \lambda u_i\mathbf{e}_i \\ &= (T_{ij} - \lambda\delta_{ij})u_j\mathbf{e}_i = \mathbf{o}\end{aligned}$$

Which is possible only if the coefficient determinant, $|T_{ij} - \lambda\delta_{ij}|$, or $\det(\mathbf{T} - \lambda\mathbf{I})$ vanishes.

$$\det(\mathbf{T} - \lambda\mathbf{I}) = \frac{[(\mathbf{T} - \lambda\mathbf{I})\mathbf{a}, (\mathbf{T} - \lambda\mathbf{I})\mathbf{b}, (\mathbf{T} - \lambda\mathbf{I})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$\begin{aligned}&[(\mathbf{T} - \lambda\mathbf{I})\mathbf{a}, (\mathbf{T} - \lambda\mathbf{I})\mathbf{b}, (\mathbf{T} - \lambda\mathbf{I})\mathbf{c}] \\ &\quad = [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}] - ([\lambda\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \lambda\mathbf{b}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{Tb}, \lambda\mathbf{c}]) + [\mathbf{Ta}, \lambda\mathbf{b}, \lambda\mathbf{c}] \\ &\quad + [\lambda\mathbf{a}, \mathbf{Tb}, \lambda\mathbf{c}] + [\lambda\mathbf{a}, \lambda\mathbf{b}, \mathbf{Tc}] - [\lambda\mathbf{a}, \lambda\mathbf{b}, \lambda\mathbf{c}]\end{aligned}$$

Leading to the characteristic equation,

$$\begin{aligned}&[\mathbf{a}, \mathbf{b}, \mathbf{c}]\lambda^3 - ([\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}])\lambda^2 + ([\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}])\lambda \\ &\quad - [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}] = 0\end{aligned}$$

Or,

$$
\begin{aligned}
-\det(\mathbf{T}-\lambda\mathbf{I}) &= \lambda^3 - \frac{[\mathbf{Ta},\mathbf{b},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}\lambda^2 \\
&\quad + \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{Tc}]+[\mathbf{Ta},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}\lambda - \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} \\
&= \lambda^3 - I_1(\mathbf{T})\lambda^2 + I_2(\mathbf{T})\lambda - I_3(\mathbf{T}) = 0 \\
I_1(\mathbf{T}) &\equiv \frac{[\mathbf{Ta},\mathbf{b},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} \\
I_2(\mathbf{T}) &\equiv \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{Tc}]+[\mathbf{Ta},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \operatorname{tr}\mathbf{T}^{\mathrm{c}} \\
I_3(\mathbf{T}) &\equiv \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}.
\end{aligned}
$$

The selection of the linearly independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ is completely arbitrary. Consequently, the Principal Invariants are independent of the choice of these vectors and are intrinsic to the tensor $\mathbf{T}$.

Tensors in Spectral Form

One consequence of the eigenvalue problem is the possibility to present tensors in spectral form. The nine components of a regular tensor become six when the tensor is symmetric. In spectral form, the tensor is reduced to the eigenvalues – a much easier form.

A very important result that enable the reduction to spectral form is the Caley-Hamilton's Theorem:

We now state without proof (See Dill for proof) the important ***Caley-Hamilton*** theorem: Every tensor satisfies its own characteristic equation. That is, the characteristic equation not only applies to the eigenvalues but must be satisfied by the tensor $\mathbf{T}$ itself. This means,

$$\mathbf{T}^3 - I_1\mathbf{T}^2 + I_2\mathbf{T} - I_3\mathbf{I} = \mathbf{O}$$

is also valid. This fact is used in continuum mechanics to obtain the ***spectral decomposition*** of important material and spatial tensors.

It is easy to show that when the tensor is symmetric, its three eigenvalues are all real. When they are distinct, corresponding eigenvectors are orthogonal. It is therefore possible to create a basis for the tensor with an orthonormal system based on the normalized eigenvectors. This leads to what is called a *spectral decomposition* of a symmetric tensor in terms of a coordinate system formed by its eigenvectors:

$$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_i \otimes \mathbf{n}_i$$

Where $\mathbf{n}_i$ is the normalized eigenvector corresponding to the eigenvalue λ_i.

The above spectral decomposition is a special case where the eigenbasis forms an Orthonormal Basis. Clearly, all symmetric tensors are diagonalizable.

Multiplicity of roots, when it occurs robs this representation of its uniqueness because two or more coefficients of the eigenbasis are now the same.

The uniqueness is recoverable by the ingenious device of eigenprojection.

Case 1: All Roots equal.

The three orthonormal eigenvectors in an ONB obviously constitutes an Identity tensor $\mathbf{I}$. The unique spectral representation therefore becomes

$$\mathbf{T} = \sum_{i=1}^{3} \lambda_i \, \mathbf{n}_\mathrm{i} \otimes \mathbf{n}_\mathrm{i} = \lambda \sum_{i=1}^{3} \mathbf{n}_\mathrm{i} \otimes \mathbf{n}_\mathrm{i}$$

since $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ in this case.

Case 2: Two Roots equal:

$\lambda_1$unique while $\lambda_2 = \lambda_3$

In this case,

$$\mathbf{T} = \lambda_1 \mathbf{n}_1 \otimes \mathbf{n}_1 + \lambda_2 (\mathbf{I} - \mathbf{n}_1 \otimes \mathbf{n}_1)$$

since $\lambda_2 = \lambda_3$ in this case.

The eigenspace of the tensor is made up of the projectors:

$$\mathbf{P}_1 = \mathbf{n}_1 \otimes \mathbf{n}_1$$

and

$$\mathbf{P}_2 = \mathbf{I} - \mathbf{n}_2 \otimes \mathbf{n}_2$$

The eigen projectors in all cases are based on the normalized eigenvectors of the tensor. They constitute the eigenspace even in the case of repeated roots. They can be easily shown to be:

1. Idempotent: $\mathbf{P}_i \, \mathbf{P}_i = \mathbf{P}_i$ (no sums)
2. Orthogonal: $\mathbf{P}_i \, \mathbf{P}_j = \boldsymbol{O}$ (the anihilator)

3. Complete: $\sum_{i=1}^{n} \mathbf{P}_i = \mathbf{I}$ (the identity)

Orthogonal Tensors

Given a pair of vectors $\mathbf{a}$ and $\mathbf{b}$, an orthogonal tensor $\mathbf{Q}$ is said to be orthogonal if,

$$(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{a} \cdot \mathbf{b}$$

Specifically, we can allow $\mathbf{a} = \mathbf{b}$, so that

$$(\mathbf{Qa}) \cdot (\mathbf{Qa}) = \mathbf{a} \cdot \mathbf{a}$$

Or

$$||\mathbf{Qa}|| = ||\mathbf{a}||$$

In which case the mapping leaves the magnitude unaltered. Let $\mathbf{q} = \mathbf{Qa}$

$$(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{q} \cdot \mathbf{Qb} = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

By definition of the transpose, we have that,

$$\mathbf{q} \cdot \mathbf{Qb} = \mathbf{b} \cdot \mathbf{Q}^{\mathbf{T}}\mathbf{q} = \mathbf{b} \cdot \mathbf{Q}^{\mathbf{T}}\mathbf{Qa} = \mathbf{b} \cdot \mathbf{a}$$

Clearly, $\mathbf{Q}^{\mathbf{T}}\mathbf{Q} = \mathbf{I}.$ A condition necessary and sufficient for a tensor $\mathbf{Q}$ to be orthogonal is that $\mathbf{Q}$ be invertible and its inverse equal to its transpose.

Upon noting that the determinant of a product is the product of the determinants and that transposition does not alter a determinant, it is easy to conclude that,

$$\det(\mathbf{Q}^{\mathbf{T}}\mathbf{Q}) = (\det \mathbf{Q}^{\mathbf{T}})(\det \mathbf{Q}) = (\det \mathbf{Q})^2 = 1$$

Which clearly shows that

$$(\det \mathbf{Q}) = \pm 1$$

When the determinant of an orthogonal tensor is strictly positive, it is called "proper orthogonal". A rotation is a proper orthogonal tensor while a reflection is not.

Let $\boldsymbol{Q}$ be a rotation. For any pair of vectors $\mathbf{u}, \mathbf{v}$ show that $\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Qu}) \times (\mathbf{Qv})$

This question is the same as showing that the cofactor of $\mathbf{Q}$ is $\mathbf{Q}$ itself. That is that a rotation is self cofactor. We can write that

$$\mathbf{T}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Qu}) \times (\mathbf{Qv})$$

where

$$\mathbf{T} = \operatorname{cof}(\mathbf{Q}) = \det(\mathbf{Q})\, \mathbf{Q}^{-\mathrm{T}}$$

Now that $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = 1$, and

$$\mathbf{Q}^{-\mathrm{T}} = (\mathbf{Q}^{-1})^{\mathrm{T}} = (\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{Q}$$

This implies that $\mathbf{T} = \mathbf{Q}$ and consequently,

$$\mathbf{Q}(\mathbf{u} \times \mathbf{v}) = (\mathbf{Qu}) \times (\mathbf{Qv})$$

For a proper orthogonal tensor Q, show that the eigenvalue equation always yields an eigenvalue of +1. This means that there is always a solution for the equation,

$$\mathbf{Qu} = \mathbf{u}$$

For any invertible tensor,

$$\mathbf{S}^{\mathrm{C}} = (\det \mathbf{S})\mathbf{S}^{-\mathrm{T}}$$

For a proper orthogonal tensor $\boldsymbol{Q}$, $\det \mathbf{Q} = 1$. It therefore follows that,

$$\mathbf{Q}^{\mathrm{C}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$

It is easily shown that $\mathrm{tr}\mathbf{Q}^{\mathrm{C}} = I_2(\mathbf{Q})$

Characteristic equation for $\boldsymbol{Q}$ is,

$$\det(\mathbf{Q} - \lambda\mathbf{I}) = \lambda^3 - \lambda^2 Q_1 + \lambda Q_2 - Q_3 = 0$$

Or,

$$\lambda^3 - \lambda^2 Q_1 + \lambda Q_1 - 1 = 0$$

Which is obviously satisfied by $\lambda = 1$.

Examples

2.1	Show that for any tensor, $\mathbf{T} = (\mathbf{Te}_i) \otimes \mathbf{e}_i = \mathbf{e}_i \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{e}_i)$
	Consider an arbitrary tensor $\mathbf{v}$, $$\left((\mathbf{Te}_i) \otimes \mathbf{e}_i\right)\mathbf{v} = (\mathbf{Te}_i)v_i = \mathbf{Tv}$$ So that, $(\mathbf{Te}_i) \otimes \mathbf{e}_i = \mathbf{T}$ A more direct approach is to observe that, $$(\mathbf{Te}_i) \otimes \mathbf{e}_i = \mathbf{T}(\mathbf{e}_i \otimes \mathbf{e}_i) = \mathbf{TI}$$ $$\mathbf{e}_i \otimes (\mathbf{T}^{\mathrm{T}}\mathbf{e}_i) = (\mathbf{e}_i \otimes \mathbf{T}^{\mathrm{T}}\mathbf{e}_i) = (\mathbf{e}_i \otimes \mathbf{e}_i)\,\mathbf{T} = \mathbf{IT} = \mathbf{T}$$
2.2	Show that the transformation $\mathbf{Tv} = \frac{\mathbf{e}_1}{\|\mathbf{v}\|}$ is not a tensor.

$$\mathbf{T}(2\mathbf{v}) = \frac{\mathbf{e}_1}{\|2\mathbf{v}\|} = \frac{1}{2}\frac{\mathbf{e}_1}{\|\mathbf{v}\|} = \frac{1}{2}\mathbf{Tv}$$

In fact, for any non-zero scalar α,

$$\mathbf{T}(\alpha\mathbf{v}) = \frac{\mathbf{e}_1}{\|\alpha\mathbf{v}\|} = \frac{1}{\alpha}\mathbf{Tv}$$

It is clearly a nonlinear transformation.

2.3 Show that the transformation $\mathbf{Tv} = \|\mathbf{v}\|\mathbf{e}_1$ is not a tensor.

For any scalar α,

$$\mathbf{T}(\alpha\mathbf{v}) = \|\alpha\mathbf{v}\|\mathbf{e}_1 = \alpha\mathbf{Tv}$$

Now to a second test of linearity: How does it transform $\mathbf{u} + \mathbf{v}$?

$$\mathbf{T}(\mathbf{u} + \mathbf{v}) = \|\mathbf{u} + \mathbf{v}\|\mathbf{e}_1 = \sqrt{(\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v})}\mathbf{e}_1 \neq \mathbf{Tu} + \mathbf{Tv}$$

The transformation is not linear, hence not a tensor.

2.4 Show that the transformations (a) $\mathbf{Tv} = (\mathbf{a} \cdot \mathbf{v})\mathbf{e}_1$, and (b) $\mathbf{Tv} = \mathbf{a} \times \mathbf{v}$ are tensor transformations

a

$$\begin{aligned}\mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v}) &= \big(\mathbf{a} \cdot (\alpha\mathbf{u} + \beta\mathbf{v})\big)\mathbf{e}_1 = \big((\alpha\mathbf{a} \cdot \mathbf{u}) + \beta(\mathbf{a} \cdot \mathbf{v})\big)\mathbf{e}_1 \\ &= \alpha(\mathbf{a} \cdot \mathbf{u})\mathbf{e}_1 + \beta(\mathbf{a} \cdot \mathbf{v})\mathbf{e}_1 = \alpha\mathbf{Tu} + \beta\mathbf{Tv}\end{aligned}$$

The transformation is linear and thus a tensor since it is also a transformation whose input and output are both vectors.

b

$$\mathbf{Tv} = \mathbf{a} \times \mathbf{v} = (\mathbf{a}\,\times)\mathbf{v}$$

$\mathbf{T} = (\mathbf{a}\,\times)$ this is the Vector Cross of the vector **a.** This is a tensor whose component form is:

$(\mathbf{a}\,\times) = e_{ijk}a_j\mathbf{e}_i \otimes \mathbf{e}_k$. It is a transformation of a vector to a vector. Given $\alpha, \beta \in \mathbb{R}$, and $\mathbf{u}, \mathbf{v} \in \mathbb{E}$, its linearity is established because,

$$\begin{aligned}(\mathbf{a}\,\times)(\alpha\mathbf{u} + \beta\mathbf{v}) &= e_{ijk}a_j(\mathbf{e}_i \otimes \mathbf{e}_k)(\alpha\mathbf{u} + \beta\mathbf{v}) \\ &= e_{ijk}a_j\mathbf{e}_i[(\alpha\mathbf{u} + \beta\mathbf{v}) \cdot \mathbf{e}_k] \\ &= \alpha e_{ijk}a_j(\mathbf{e}_i \otimes \mathbf{e}_k)\mathbf{u} + \beta e_{ijk}a_j(\mathbf{e}_i \otimes \mathbf{e}_k)\mathbf{v} \\ &= \alpha\mathbf{Tu} + \beta\mathbf{Tv}\end{aligned}$$

2.5	(a) What tensor transforms the basis vectors as follows: $$\mathbf{Te}_1 = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$$ $$\mathbf{Te}_2 = -5\mathbf{e}_1 + 4\mathbf{e}_3$$ $$\mathbf{Te}_3 = 3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$$ (b) Find the Tensor that reverses this transformation. (c) Demonstrate the inverse transformation.
a	The tensor we seek is in the form, $$[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$ It operates on the vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ to obtain $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, $-5\mathbf{e}_1 + 4\mathbf{e}_3$ and $3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ respectively. Applying $\mathbf{Tv} = T_{ij}v_j\mathbf{e}_i$, we have, for $\mathbf{v} = \mathbf{e}_1$, $$\begin{aligned} \mathbf{Te}_1 &= (T_{11}v_1 + T_{12}v_2 + T_{13}v_3)\mathbf{e}_1 + (T_{21}v_1 + T_{22}v_2 + T_{23}v_3)\mathbf{e}_2 \\ &\quad + (T_{31}v_1 + T_{32}v_2 + T_{33}v_3)\mathbf{e}_3 \\ &= \big(T_{11}(1) + T_{12}(0) + T_{13}(0)\big)\mathbf{e}_1 + \big(T_{21}(1) + T_{22}(0) + T_{23}(0)\big)\mathbf{e}_2 \\ &\quad + \big(T_{31}(1) + T_{32}(0) + T_{33}(0)\big)\mathbf{e}_3 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 \\ &\quad = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \end{aligned}$$ So that $T_{11} = 1, T_{21} = 2, T_{31} = 3$. We avoid unnecessary computation in the following by observing that, for $\mathbf{v} = \mathbf{e}_2$, $v_1 = 0, v_2 = 1$ and $v_3 = 0$. The lengthy expression reduces to: $$\mathbf{Te}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 + T_{32}\mathbf{e}_3 = -5\mathbf{e}_1 + 4\mathbf{e}_3$$ So that $T_{12} = -5, T_{22} = 0, T_{32} = 4$. Lastly, like the above, for $\mathbf{v} = \mathbf{e}_3$, $v_1 = 0, v_2 = 0$ and $v_3 = 1$. $$\mathbf{Te}_3 = T_{13}\mathbf{e}_1 + T_{23}\mathbf{e}_2 + T_{33}\mathbf{e}_3 = 3\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$$ So that $T_{13} = 3, T_{23} = -1, T_{33} = -1$. The tensor we seek is, $$[\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} 1 & -5 & 3 \\ 2 & 0 & -1 \\ 3 & 4 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

B	The reverse transformation will do the opposite: It will take the vectors, $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$, $-5\mathbf{e}_1 + 4\mathbf{e}_3$ and $2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ and produce $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ respectively. By inverting the tensor, we obtain, $$\mathbf{T}^{-1} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} \frac{4}{25} & \frac{3}{25} & \frac{1}{5} \\ -\frac{1}{25} & -\frac{7}{25} & \frac{1}{5} \\ \frac{8}{25} & -\frac{19}{25} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$
C	$\mathbf{T}^{-1}(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)$ $$= (T_{11}^{-1}v_1 + T_{12}^{-1}v_2 + T_{13}^{-1}v_3)\mathbf{e}_1 + (T_{21}^{-1}v_1 + T_{22}^{-1}v_2 + T_{23}^{-1}v_3)\mathbf{e}_2 + (T_{31}^{-1}v_1 + T_{32}^{-1}v_2 + T_{33}^{-1}v_3)\mathbf{e}_3$$ $$= \left(\frac{4}{25}(1) + \frac{3}{25}(2) + \frac{1}{5}(3)\right)\mathbf{e}_1 + \left(-\frac{1}{25}(1) - \frac{7}{25}(2) + \frac{1}{5}(3)\right)\mathbf{e}_2 + \left(\frac{8}{25}(1) - \frac{19}{25}(2) + \frac{2}{5}(3)\right)\mathbf{e}_3 = \mathbf{e}_1$$ $\mathbf{T}^{-1}(-5\mathbf{e}_1 + 4\mathbf{e}_3)$ $$= (T_{11}^{-1}v_1 + T_{12}^{-1}v_2 + T_{13}^{-1}v_3)\mathbf{e}_1 + (T_{21}^{-1}v_1 + T_{22}^{-1}v_2 + T_{23}^{-1}v_3)\mathbf{e}_2 + (T_{31}^{-1}v_1 + T_{32}^{-1}v_2 + T_{33}^{-1}v_3)\mathbf{e}_3$$ $$= \left(\frac{4}{25}(-5) + \frac{3}{25}(0) + \frac{1}{5}(4)\right)\mathbf{e}_1 + \left(-\frac{1}{25}(-5) - \frac{7}{25}(0) + \frac{1}{5}(4)\right)\mathbf{e}_2 + \left(\frac{8}{25}(-5) - \frac{19}{25}(0) + \frac{2}{5}(4)\right)\mathbf{e}_3 = \mathbf{e}_2$$ $\mathbf{T}^{-1}(2\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)$ $$= (T_{11}^{-1}v_1 + T_{12}^{-1}v_2 + T_{13}^{-1}v_3)\mathbf{e}_1 + (T_{21}^{-1}v_1 + T_{22}^{-1}v_2 + T_{23}^{-1}v_3)\mathbf{e}_2 + (T_{31}^{-1}v_1 + T_{32}^{-1}v_2 + T_{33}^{-1}v_3)\mathbf{e}_3$$ $$= \left(\frac{4}{25}(2) + \frac{3}{25}(-1) + \frac{1}{5}(-1)\right)\mathbf{e}_1 + \left(-\frac{1}{25}(2) - \frac{7}{25}(-1) + \frac{1}{5}(-1)\right)\mathbf{e}_2 + \left(\frac{8}{25}(2) - \frac{19}{25}(-1) + \frac{2}{5}(-1)\right)\mathbf{e}_3 = \mathbf{e}_3$$

2.6	Given vectors $\mathbf{u}, \mathbf{v}$ Find the tensor that transforms any vector $\mathbf{a}$ to $\mathbf{u}(\mathbf{a} \cdot \mathbf{v})$
	Using the fact that the scalar product is commutative, we may see it more clearly by observing that, $$\mathbf{u}(\mathbf{a} \cdot \mathbf{v}) = \mathbf{u}(\mathbf{v} \cdot \mathbf{a}) = (\mathbf{u} \otimes \mathbf{v})\mathbf{a}$$ The tensor we seek is simply the dyad created by vectors $\mathbf{u}$ and $\mathbf{v}$.
	Given vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ show that the dyad product, $$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x}) = (\mathbf{u} \otimes \mathbf{x})(\mathbf{v} \cdot \mathbf{w})$$
	The proof is to show that both sides produce the same result when they act on the same vector. For arbitrary vector $\mathbf{y}$, observing that $(\mathbf{v} \cdot \mathbf{w})$ is scalar, the RHS is $$(\mathbf{u} \otimes \mathbf{x})(\mathbf{v} \cdot \mathbf{w})\mathbf{y} = \mathbf{u}(\mathbf{x} \cdot \mathbf{y})(\mathbf{v} \cdot \mathbf{w})$$ and the LHS applied to the same vector, becomes: $$(\mathbf{u} \otimes \mathbf{v})(\mathbf{w} \otimes \mathbf{x})\mathbf{y} = (\mathbf{u} \otimes \mathbf{v})[\mathbf{w}(\mathbf{x} \cdot \mathbf{y})] = \mathbf{u}(\mathbf{v} \cdot \mathbf{w})(\mathbf{x} \cdot \mathbf{y})$$ Which is obviously the result from the RHS also. It is easy to see, by a repeated application of this result, that the multiple product of dyads like $$(\mathbf{u}_1 \otimes \mathbf{v}_1)(\mathbf{u}_2 \otimes \mathbf{v}_2) \dots (\mathbf{u}_n \otimes \mathbf{v}_n) = (\mathbf{u}_1 \otimes \mathbf{v}_n)(\mathbf{v}_1 \cdot \mathbf{u}_2) \dots (\mathbf{v}_{n-1} \cdot \mathbf{u}_n)$$
2.7	For a tensor $\mathbf{S}$, given that $[(\mathbf{S}^{\mathrm{c}}\mathbf{u}) \times] = \mathbf{S}(\mathbf{u} \times)\mathbf{S}^{\mathrm{T}}$ for any two vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\big((\operatorname{cof}\mathbf{S})\mathbf{u} \times\big)\mathbf{v} = \mathbf{S}(\mathbf{u} \times \mathbf{S}^{\mathrm{T}}\mathbf{v})$
	The product of the given equation with the vector $\mathbf{v}$ immediately yields, $$[(\mathbf{S}^{\mathrm{c}}\mathbf{u}) \times]\mathbf{v} = \mathbf{S}(\mathbf{u} \times)\mathbf{S}^{\mathrm{T}}\mathbf{v}$$ $$\Rightarrow \big((\operatorname{cof}\mathbf{S})\mathbf{u} \times\big)\mathbf{v} = \mathbf{S}(\mathbf{u} \times \mathbf{S}^{\mathrm{T}}\mathbf{v})$$

2.8	Given that $\boldsymbol{\Omega}$ is a skew tensor with the corresponding axial vector $\boldsymbol{\omega}$. Given vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\boldsymbol{\Omega}\mathbf{u} \times \boldsymbol{\Omega}\mathbf{v} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})$ and, hence conclude that $\boldsymbol{\Omega}^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.

$$\begin{aligned}\boldsymbol{\Omega}\mathbf{u} \times \boldsymbol{\Omega}\mathbf{v} &= (\boldsymbol{\omega} \times \mathbf{u}) \times (\boldsymbol{\omega} \times \mathbf{v})\\ &= [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \mathbf{v}]\boldsymbol{\omega} - [(\boldsymbol{\omega} \times \mathbf{u}) \cdot \boldsymbol{\omega}]\mathbf{v}\\ &= [\boldsymbol{\omega} \cdot (\mathbf{u} \times \mathbf{v})]\boldsymbol{\omega} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{u} \times \mathbf{v})\end{aligned}$$

But by definition, the cofactor must satisfy,

$$\boldsymbol{\Omega}\mathbf{u} \times \boldsymbol{\Omega}\mathbf{v} = \boldsymbol{\Omega}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})$$

which compared with the previous equation yields the desired result that

$$\boldsymbol{\Omega}^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega}).$$

2.9	Show that $[(\mathbf{S}^{\mathrm{c}}\mathbf{u}) \times] = \mathbf{S}(\mathbf{u} \times)\mathbf{S}^{\mathrm{T}}$

The LHS can be written as:

$$[(\mathbf{S}^{\mathrm{c}}\mathbf{u}) \times] = e_{ijk}(\mathbf{S}^{\mathrm{c}}\mathbf{u})_j \mathbf{e}_i \otimes \mathbf{e}_k$$

where $\mathbf{S}^{\mathrm{c}} = \frac{1}{2} e_{jab} e_{\beta cd} S_{ac} S_{bd}\ \mathbf{e}_j \otimes \mathbf{e}_\beta$ so that

$$\begin{aligned}\mathbf{S}^{\mathrm{c}}\mathbf{u} &= \left(\frac{1}{2} e_{jab} e_{\beta cd} S_{ac} S_{bd} \mathbf{e}_j \otimes \mathbf{e}_\beta\right)(u_m \mathbf{e}_m) = \frac{1}{2} e_{jab} e_{\beta cd} S_{ac} S_{bd} \mathbf{e}_j \delta_{\beta m} u_m\\ &= \frac{1}{2} e_{jab} e_{\beta cd} u_\beta S_{ac} S_{bd} \mathbf{e}_j\end{aligned}$$

Consequently,

$$\begin{aligned}[(\mathbf{S}^{\mathrm{c}}\mathbf{u}) \times] &= \frac{1}{2} e_{ijk} e_{jab} e_{\beta cd} u_\beta S_{ac} S_{bd} \mathbf{e}_i \otimes \mathbf{e}_k\\ &= \frac{1}{2} e_{\beta cd}(\delta_{ka}\delta_{ib} - \delta_{kb}\delta_{ia}) u_\beta S_{ac} S_{bd} \mathbf{e}_i \otimes \mathbf{e}_k\\ &= \frac{1}{2} e_{\beta cd} u_\beta (S_{kc} S_{id} - S_{ic} S_{kd}) \mathbf{e}_i \otimes \mathbf{e}_k\\ &= \frac{1}{2} e_{\beta cd} u_\beta S_{kc} S_{id} \mathbf{e}_i \otimes \mathbf{e}_k - \frac{1}{2} e_{\beta cd} u_\beta S_{ic} S_{kd} \mathbf{e}_i \otimes \mathbf{e}_k\\ &= \frac{1}{2} e_{\beta cd} u_\beta S_{kc} S_{id} \mathbf{e}_i \otimes \mathbf{e}_k + \frac{1}{2} e_{\beta dc} u_\beta S_{ic} S_{kd} \mathbf{e}_i \otimes \mathbf{e}_k\\ &= e_{\beta cd} u_\beta S_{kc} S_{id} \mathbf{e}_i \otimes \mathbf{e}_k\end{aligned}$$

	On the RHS $$(\boldsymbol{u}\times)\boldsymbol{S}^{\mathrm{T}}=\left(e_{\alpha\beta\gamma}u_\beta\mathbf{e}_\alpha\otimes\mathbf{e}_\gamma\right)(S_{ki}\mathbf{e}_i\otimes\mathbf{e}_k)=e_{\alpha\beta\gamma}u_\beta S_{k\gamma}\mathbf{e}_\alpha\otimes\mathbf{e}_k.$$ We can therefore write, $$\mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}=(S_{ir}\mathbf{e}_i\otimes\mathbf{e}_r)\left(e_{\alpha\beta\gamma}u_\beta S_{k\gamma}\mathbf{e}_\alpha\otimes\mathbf{e}_k\right)=e_{\alpha\beta\gamma}u_\beta S_{i\alpha}S_{k\gamma}\mathbf{e}_i\otimes\mathbf{e}_k$$ which on $\alpha\to d,\gamma\to c$ is the same as the LHS $\Rightarrow$ $$[(\mathbf{S}^{\mathrm{c}}\mathbf{u})\times]=\mathbf{S}(\mathbf{u}\times)\mathbf{S}^{\mathrm{T}}$$ as required.
2.10	Show that for any invertible tensor $\mathbf{S}$ and any vector $\mathbf{u}$, $[(\mathbf{S}\mathbf{u})\times]=\mathbf{S}^{\mathrm{c}}(\mathbf{u}\times)\mathbf{S}^{-1}$ where $\mathbf{S}^{\mathrm{c}}$ and $\mathbf{S}^{-1}$ are the cofactor and inverse of $\mathbf{S}$ respectively.
	By definition, $$\mathbf{S}^{\mathbf{c}}=(\det\mathbf{S})\mathbf{S}^{-\mathrm{T}}$$ We are to prove that, $$[(\mathbf{S}\mathbf{u})\times]=\mathbf{S}^{\mathbf{c}}(\mathbf{u}\times)\mathbf{S}^{-\mathbf{1}}=(\det\mathbf{S})\mathbf{S}^{-\mathrm{T}}(\mathbf{u}\times)\mathbf{S}^{-\mathbf{1}}$$ or that, $$\mathbf{S}^{\mathrm{T}}[(\mathbf{S}\mathbf{u})\times]=(\mathbf{u}\times)(\det\mathbf{S})\mathbf{S}^{-\mathbf{1}}=(\mathbf{u}\times)(\mathbf{S}^{\mathbf{c}})^{\mathbf{T}}$$ On the RHS, the ij component of $\mathbf{u}\times$ is $$(\mathbf{u}\times)_{ij}=e_{i\alpha j}u_\alpha$$ which is exactly the same as writing, $(u\times)=e_{i\alpha l}u_\alpha\mathbf{e}_i\otimes\mathbf{e}_l$ in the invariant form. We now turn to the LHS; $$[(\mathbf{S}\mathbf{u})\times]=e_{l\alpha k}(\mathbf{S}\mathbf{u})_\alpha\mathbf{e}_l\otimes\mathbf{e}_k=e_{l\alpha k}S_{\alpha j}u_j\mathbf{e}_l\otimes\mathbf{e}_k$$ Now, $\mathbf{S}=S_{i\beta}\mathbf{e}_i\otimes\mathbf{e}_\beta$ so that its transpose, $\mathbf{S}^{\mathrm{T}}=S_{i\beta}\mathbf{e}_\beta\otimes\mathbf{e}_i$ so that $$\begin{aligned}\mathbf{S}^{\mathrm{T}}[(\mathbf{S}\mathbf{u})\times]&=e_{l\alpha k}S_{\alpha j}S_{i\beta}u_j\left(\mathbf{e}_\beta\otimes\mathbf{e}_i\right)(\mathbf{e}_l\otimes\mathbf{e}_k)\\&=e_{l\alpha k}S_{\alpha j}S_{i\beta}u_j\delta_{il}\left(\mathbf{e}_\beta\otimes\mathbf{e}_k\right)\\&=e_{l\alpha k}S_{\alpha j}S_{li}u_j(\mathbf{e}_i\otimes\mathbf{e}_k)\\&=e_{\beta\alpha k}S_{\alpha j}S_{\beta i}u_j(\mathbf{e}_i\otimes\mathbf{e}_k)\\&=(\mathbf{u}\times)(\mathbf{S}^{\mathrm{c}})^{\mathrm{T}}\end{aligned}$$
2.11	For any invertible tensor $\mathbf{S}$ show that $\mathbf{S}^{-\mathrm{c}}=(\det\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$, that is, the inverse of the cofactor is the transpose divided by the determinant.
	$$\mathbf{S}^{\mathrm{C}}=\det(\mathbf{S})\,\mathbf{S}^{-\mathrm{T}}$$

	Consequently, $$\mathbf{S}^{-\mathrm{c}} = (\det \mathbf{S})^{-1}(\mathbf{S}^{-\mathrm{T}})^{-1} = (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$$
2.12	For any invertible tensor $\mathbf{S}$ and a scalar α show that show that the cofactor of the product of α and $\mathbf{S}$ equals $\alpha^2 \times$ the cofactor of $\mathbf{S}$, that is, $(\alpha\mathbf{S})^{\mathrm{c}} = \alpha^2\mathbf{S}^{\mathrm{c}}$
	$$\begin{aligned}(\alpha\mathbf{S})^{c} &= (\det(\alpha\mathbf{S}))(\alpha\mathbf{S})^{-\mathrm{T}} \\ &= (\alpha^3 \det(\mathbf{S}))\alpha^{-1}\mathbf{S}^{-\mathrm{T}} \\ &= (\alpha^2 \det(\mathbf{S}))\mathbf{S}^{-\mathrm{T}} \\ &= \alpha^2\mathbf{S}^{\mathrm{c}}\end{aligned}$$
2.13	For any invertible tensor $\mathbf{S}$ show that $(\mathbf{S}^{-1})^{\mathrm{c}} = (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$
	$$\begin{aligned}(\mathbf{S}^{-1})^{\mathrm{c}} &= \det(\mathbf{S}^{-1})\,(\mathbf{S}^{-1})^{-\mathrm{T}} \\ &= (\det \mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}\end{aligned}$$
2.14	For any invertible tensor $\mathbf{S}$ show that $\det(\mathbf{S}^{\mathrm{c}}) = (\det \mathbf{S})^2$
	First note that the determinant of the product of a tensor C with a scalar α is, $$\det \alpha\mathbf{C} = e_{ijk}(\alpha C_{i1})(\alpha C_{j2})(\alpha C_{k3}) = \alpha^3 \det \mathbf{C}$$ The inverse of tensor $\boldsymbol{S}$, $$\mathbf{S}^{-1} = (\det \mathbf{S})^{-1}(\mathbf{S}^{\mathrm{cT}})$$ Let the scalar $\alpha = \det \mathbf{S}$. We can see clearly that, $$\mathbf{S}^{\mathrm{c}} = \alpha\mathbf{S}^{-\mathrm{T}}$$ Taking the determinant of this equation, we have, $$\det \mathbf{S}^{\mathrm{c}} = \alpha^3 \det \mathbf{S}^{-1} = \alpha^3 \det \mathbf{S}^{-1}$$ as the transpose operation has no effect on the value of a determinant. Noting that the determinant of an inverse is the inverse of the determinant, we have, $$\det \mathbf{S}^{\mathrm{c}} = \alpha^3 \det(\mathbf{S}^{-1}) = \frac{\alpha^3}{\alpha} = (\det \mathbf{S})^2$$

2.15	An orthogonal tensor $\mathbf{Q}$ is said to be "proper orthogonal" if its determinant $\lvert\mathbf{Q}\rvert = +1$. Show that a proper orthogonal tensor is the cofactor of itself. Show also that its first two invariants are equal.
	If $\mathbf{Q}$ is proper orthogonal, then $\det \mathbf{Q} = 1$ $$\mathrm{cof}\,\mathbf{Q} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = +1\,(\mathbf{Q}^{\mathrm{T}})^{-\mathbf{1}} = 1\,(\mathbf{Q}^{-1})^{-\mathbf{1}} = \mathbf{Q}$$ $$I_2(\mathbf{Q}) = I_1(\mathbf{Q}^{\mathrm{c}})$$ The second principal invariant for any tensor is equal to the first principal invariant of its co-factor. But we find here that $\mathbf{Q} = \mathbf{Q}^{\mathrm{c}}$**.** It follows that the first two invariants of a proper orthogonal tensor are equal. The third invariant, $I_3(\mathbf{Q}) = \det \mathbf{Q} = 1$. All essential information on an orthogonal tensor is known once we know its trace!
2.16	Given arbitrary vectors $\mathbf{a}$ and $\mathbf{b}$ the tensor $\mathbf{Q}$ is said to be orthogonal if $(\mathbf{Qa}) \cdot (\mathbf{Qb}) = \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ show that the inverse of $\mathbf{Q}$ is its transpose. and that $\mathbf{Q}$ is the cofactor of itself.
	Let $\mathbf{q} = \mathbf{Qa}$. By the definition of the transpose of a tensor, we have that, $$\mathbf{q} \cdot \mathbf{Qb} = \mathbf{b} \cdot \mathbf{Q}^{\mathrm{T}}\mathbf{q} = \mathbf{b} \cdot \mathbf{Q}^{\mathbf{T}}\mathbf{Qa} = \mathbf{b} \cdot \mathbf{a}$$ Clearly, $\mathbf{Q}^{\mathbf{T}}\mathbf{Q} = \mathbf{I}$ which makes the transpose the same as the inverse tensor. A condition necessary and sufficient for a tensor $\mathbf{Q}$ to be orthogonal is that $\mathbf{Q}$ be invertible and its inverse is equal to its transpose.
2.17	By appealing to the Cayley-Hamilton theorem, show that, that the inverse of an invertible tensor $\mathbf{S}$ can be written as $\mathbf{S}^{-1} = \frac{1}{\det \mathbf{S}}(\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})$
	The characteristic equation for $\mathbf{S}$ can be written as, $$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3\mathbf{I} = 0$$ Multiplying by the inverse, $\mathbf{S}^{-1}$, we have, $$\mathbf{S}^{\mathbf{2}} - I_1\mathbf{S} + I_2\mathbf{I} - I_3\mathbf{S}^{-1} = 0$$ from which the result,

	$$\mathbf{S}^{-1} = \frac{1}{\det \mathbf{S}}(\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})$$ immediately follows.
2.18	By direct notation and the relationship, $\mathbf{T}^{\mathrm{c}} = (\mathbf{T}^2 - I_1(\mathbf{T})\mathbf{T} + I_2(\mathbf{T})\mathbf{J})^{\mathrm{T}}$ show that the second invariant of a tensor is half the difference between of the square of its trace and the trace of its square.
	Take the trace of the given equation, $$\mathrm{tr}\,\mathbf{T}^{\mathrm{c}} = \mathrm{tr}\,\mathbf{T}^2 - I_1(\mathbf{T})I_1(\mathbf{T}) + 3I_2(\mathbf{T})$$ But recall that $\mathrm{tr}\,\mathbf{T}^{\mathrm{c}} = I_2(\mathbf{T})$. It therefore follows that, $$2I_2(\mathbf{T}) = I_1^2(\mathbf{T}) - \mathrm{tr}\,\mathbf{T}^2 = \mathrm{tr}^2\,\mathbf{T} - \mathrm{tr}\,\mathbf{T}^2$$ So that, $$I_2(\mathbf{T}) = \frac{1}{2}(\mathrm{tr}^2\,\mathbf{T} - \mathrm{tr}\,\mathbf{T}^2)$$

2.19	Show, using direct notation, that the cofactor of a tensor can be written as $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})^{\mathrm{T}}$ even if $\mathbf{S}$ is not invertible. I_1, I_2 are the first two invariants of $\mathbf{S}$.
	For any three linearly independent vectors, the trace of a tensor $\mathbf{T}$ $$\mathrm{tr}\,\mathbf{T} \equiv I_1(\mathbf{T}) = \frac{[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{g}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}$$ Replacing $\mathbf{g}_1$ by $\mathbf{Tg}_1$ in the above equation, we have, $$\mathrm{tr}\,\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] = [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3]$$ Or, upon rearrangement, $$[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3] = \mathrm{tr}\,\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

But, the second Invariant,

$$
\begin{aligned}
I_2(\mathbf{T}) &= \frac{[\mathbf{Tg}_1, \mathbf{Tg}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{Tg}_3] + [\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{\operatorname{tr}\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{g}_1, \mathbf{Tg}_2, \mathbf{Tg}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{\operatorname{tr}\mathbf{T}\,[\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + \mathbf{g}_1 \cdot \mathbf{T}^{\mathrm{c}}(\mathbf{g}_2 \times \mathbf{g}_3)}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]} \\
&= \frac{[(\operatorname{tr}\mathbf{T})\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}^{\mathrm{cT}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}{[\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]}
\end{aligned}
$$

so that,

$$[(I_2(\mathbf{T})\mathbf{I})\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] = [(\operatorname{tr}\mathbf{T})\mathbf{Tg}_1, \mathbf{g}_2, \mathbf{g}_3] - [\mathbf{T}^2\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3] + [\mathbf{T}^{\mathrm{cT}}\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3]$$

From which we can write that

$$\mathrm{I}_2(\mathbf{T})\mathbf{I} = (\operatorname{tr}\mathbf{T})\mathbf{T} - \mathbf{T}^2 + \mathbf{T}^{\mathrm{cT}}$$

or,

$$\mathbf{T}^{\mathrm{c}} = (\mathbf{T}^2 - I_1(\mathbf{T})\mathbf{T} + \mathrm{I}_2(\mathbf{T})\mathbf{I})^{\mathrm{T}}$$

2.20 Given an arbitrary tensor $\mathbf{T}$ a skew tensor $\mathbf{W}$ and a symmetric tensor $\mathbf{S}$. Show that

$$\mathbf{S}:\mathbf{T} = \mathbf{S}:\mathbf{T}^{\mathrm{T}} = \mathbf{S}:\operatorname{sym}\mathbf{T}$$

$$\mathbf{W}:\mathbf{T} = -\mathbf{W}:\mathbf{T}^{\mathrm{T}}$$

$$\mathbf{S}:\operatorname{skw}\mathbf{T} = \mathbf{S}:\mathbf{W} = 0$$

Note that $\mathbf{T} = \operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T}$, and $\mathbf{T}^{\mathrm{T}} = \operatorname{sym}\mathbf{T} - \operatorname{skw}\mathbf{T}$. Also note that the inner product between a skew and a symmetric tensor vanishes. Consequently,

$$
\begin{aligned}
\mathbf{S}:\mathbf{T} &= \mathbf{S}:(\operatorname{sym}\mathbf{T} + \operatorname{skw}\mathbf{T}) \\
&= \mathbf{S}:\operatorname{sym}\mathbf{T} + \mathbf{S}:\operatorname{skw}\mathbf{T} \\
&= \mathbf{S}:\operatorname{sym}\mathbf{T} \\
&= \mathbf{S}:\mathbf{T}^{\mathrm{T}}
\end{aligned}
$$

	$$\begin{aligned}\mathbf{W}:\mathbf{T} &= \mathbf{W}:(\operatorname{sym}\mathbf{T}+\operatorname{skw}\mathbf{T})\\ &= \mathbf{W}:\operatorname{sym}\mathbf{T}+\mathbf{W}:\operatorname{skw}\mathbf{T}\\ &= \mathbf{W}:\operatorname{skw}\mathbf{T} = \mathbf{W}:(\operatorname{sym}\mathbf{T}-\mathbf{T}^{\mathrm{T}}) = -\mathbf{W}:\mathbf{T}^{\mathrm{T}}\end{aligned}$$
	To show that $\mathbf{S}:\mathbf{W}=0$. Observe that, in component form, $\mathbf{S}=S_{ij}(\mathbf{e}_i\otimes\mathbf{e}_j)$, $\mathbf{W}=W_{\alpha\beta}(\mathbf{e}_\alpha\otimes\mathbf{e}_\beta)$. $$\begin{aligned}\mathbf{S}^{\mathrm{T}}\mathbf{W} &= S_{ij}W_{\alpha\beta}(\mathbf{e}_j\otimes\mathbf{e}_i)(\mathbf{e}_\alpha\otimes\mathbf{e}_\beta)\\ &= S_{ij}W_{\alpha\beta}(\mathbf{e}_j\otimes\mathbf{e}_\beta)\delta_{i\alpha} = S_{ij}W_{i\beta}\mathbf{e}_j\otimes\mathbf{e}_\beta\\ \operatorname{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{W}) &= S_{ij}W_{\alpha\beta}\operatorname{tr}(\mathbf{e}_j\otimes\mathbf{e}_\beta)\delta_{i\alpha} = S_{ij}W_{\alpha\beta}\delta_{j\beta}\delta_{i\alpha}\\ &= S_{ij}W_{ij} = \mathbf{S}:\mathbf{W}\\ &= S_{ji}W_{ij} = -S_{ji}W_{ji} = -S_{ij}W_{ij} = -\mathbf{S}:\mathbf{W}\end{aligned}$$ Which vanishes because it is equal to the negative of itself. $$\mathbf{S}:\operatorname{skw}\mathbf{T}=0$$ Because $\operatorname{skw}\mathbf{T}$ is a skew tensor. Hence, $$\mathbf{S}:\operatorname{skw}\mathbf{T}=\mathbf{S}:\mathbf{W}=0$$
2.21	Given that the trace of a dyad $\mathbf{a}\otimes\mathbf{b}$, $\operatorname{tr}(\mathbf{a}\otimes\mathbf{b})=\mathbf{a}\cdot\mathbf{b}$. By expressing the tensors $\mathbf{T}$ and $\mathbf{S}$ in component form, show that $\operatorname{tr}(\mathbf{ST})=\operatorname{tr}(\mathbf{TS})=\operatorname{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}})=\operatorname{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}})$
	In component form, $\mathbf{S}=S_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$, $\boldsymbol{T}=T_{ij}\mathbf{e}_i\otimes\mathbf{e}_j$. $$\mathbf{ST}=S_{ij}T_{\alpha\beta}(\mathbf{e}_i\otimes\mathbf{e}_j)(\mathbf{e}_\alpha\otimes\mathbf{e}_\beta)$$

$$\mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = S_{ij}T_{\alpha\beta}\delta_{i\beta}\delta_{j\alpha} = S_{ij}T_{ji} = \mathrm{tr}(\mathbf{ST})$$

Similar computations lead to the conclusion that

$$\mathrm{tr}(\mathbf{ST}) = \mathrm{tr}(\mathbf{TS}) = \mathrm{tr}(\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}})$$

2.22	For tensors $\mathbf{A}$ and $\mathbf{S}$ how that $\mathrm{sym}(\mathbf{A}^{\mathrm{T}}\mathbf{SA}) = \mathbf{A}^{\mathrm{T}}\mathrm{sym}(\mathbf{S})\mathbf{A}$

Clearly, $\mathrm{sym}(\mathbf{S}) = \frac{1}{2}(\mathbf{S} + \mathbf{S}^{\mathrm{T}})$

It also follows that,

$$\mathbf{A}^{\mathrm{T}}\mathrm{sym}(\mathbf{S})\mathbf{A} = \frac{1}{2}\mathbf{A}^{\mathrm{T}}(\mathbf{S} + \mathbf{S}^{\mathrm{T}})\mathbf{A}$$

$$= \frac{1}{2}(\mathbf{A}^{\mathrm{T}}\mathbf{SA} + \mathbf{A}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{A})$$

But $\mathrm{sym}(\mathbf{A}^{\mathrm{T}}\mathbf{SA}) = \frac{1}{2}(\mathbf{A}^{\mathrm{T}}\mathbf{SA} + \mathbf{A}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{A})$.

Hence $\mathrm{sym}(\mathbf{A}^{\mathrm{T}}\mathbf{SA}) = \mathbf{A}^{\mathrm{T}}\mathrm{sym}(\mathbf{S})\mathbf{A}$

2.23	Given three vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, (a) show that $(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$ and that for the unit vector $\mathbf{e}$, $[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] = [\mathbf{e}, \mathbf{u}, \mathbf{v}]$, (b) Using the result in (a), show that $[(\mathbf{u} \times \mathbf{v}), (\mathbf{v} \times \mathbf{w}), (\mathbf{w} \times \mathbf{u})] = [\mathbf{u}, \mathbf{v}, \mathbf{w}]^2$

a

$$(\mathbf{w} \times \mathbf{u}) \times (\mathbf{w} \times \mathbf{v}) = [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w} - [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{w}]\mathbf{v}$$

$$= [(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}]\mathbf{w}$$

$$= [(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}]\mathbf{w}$$

$$= (\mathbf{w} \otimes \mathbf{w})(\mathbf{u} \times \mathbf{v})$$

Consequently,

$$[\mathbf{e}, \mathbf{e} \times \mathbf{u}, \mathbf{e} \times \mathbf{v}] = \mathbf{e} \cdot [(\mathbf{e} \times \mathbf{u}) \times (\mathbf{e} \times \mathbf{v})]$$

$$= \mathbf{e} \cdot [(\mathbf{e} \otimes \mathbf{e})(\mathbf{u} \times \mathbf{v})]$$

$$= (\mathbf{u} \times \mathbf{v}) \cdot (\mathbf{e} \otimes \mathbf{e})\mathbf{e}$$

$$= (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{e} = [\mathbf{e}, \mathbf{u}, \mathbf{v}]$$

making use of the symmetry of $(\mathbf{e} \otimes \mathbf{e})$.

b	From the given result, $$\begin{aligned}[(\mathbf{u}\times\mathbf{v}),(\mathbf{v}\times\mathbf{w}),(\mathbf{w}\times\mathbf{u})] &= -(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\times\mathbf{v})\times(\mathbf{w}\times\mathbf{u})\\ &= -(\mathbf{u}\times\mathbf{v})\cdot(\mathbf{w}\otimes\mathbf{w})(\mathbf{v}\times\mathbf{u})\\ &= (\mathbf{u}\times\mathbf{v})\big((\mathbf{w}\cdot\mathbf{u}\times\mathbf{v})\mathbf{w}\big)\\ &= [\mathbf{u},\mathbf{v},\mathbf{w}]^2\end{aligned}$$
2.24	Given that vectors $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.
	If $\mathbf{T}$ is not singular, if $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly dependent, then $\exists\alpha$, and β both real such that $\alpha\mathbf{Tu}+\beta\mathbf{Tv}=\mathbf{o}$. But $\mathbf{u}$ and $\mathbf{v}$ are linearly independent. This means that $\alpha\mathbf{u}+\beta\mathbf{v}\neq\mathbf{o}$. $$\alpha\mathbf{Tu}+\beta\mathbf{Tv}=\mathbf{T}(\alpha\mathbf{u}+\beta\mathbf{v})=\mathbf{o}.$$ This means that $\alpha\mathbf{u}+\beta\mathbf{v}=\mathbf{o}$. This states that set of linearly independent vectors is linearly dependent! That is a contradiction!
	Alternative Proof If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore the cofactor, $\mathbf{T}^{\mathrm{c}}=\mathbf{T}^{-\mathrm{T}}\det\mathbf{T}\neq 0$ also exists and is non-zero. The linear independence of $\mathbf{u}$ and $\mathbf{v}$ means that the parallelogram formed by them has a non-trivial area $\mathbf{u}\times\mathbf{v}\neq 0$. Now, the parallelogram formed by $\mathbf{Tu}$ and $\mathbf{Tv}$ is also non zero because, $$\mathbf{Tu}\times\mathbf{Tv}=\mathbf{T}^{\mathrm{c}}(\mathbf{u}\times\mathbf{v})\neq 0$$ Hence $\mathbf{Tu}$ and $\mathbf{Tv}$ are also linearly independent.
2.25	Given that vectors $\mathbf{u},\mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu},\mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore,

$$\det \mathbf{T} = \frac{[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \neq 0$$

Consequently, $[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}] \neq 0$. Which shows that $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

Alternative Proof:

If $\mathbf{T}$ is not singular, if $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly dependent, then $\exists \alpha, \beta$ and γ all real such that $\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \mathbf{o}$. But $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent. This means that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} \neq \mathbf{o}$.

$$\alpha\mathbf{Tu} + \beta\mathbf{Tv} + \gamma\mathbf{Tw} = \mathbf{T}(\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w}) = \mathbf{o}.$$

This means that $\alpha\mathbf{u} + \beta\mathbf{v} + \gamma\mathbf{w} = \mathbf{o}$. This states that set of linearly independent vectors is linearly dependent! That is a contradiction!

2.26 Use the expressions $(\mathbf{S} + \mathbf{T})^{\mathrm{c}} = \mathbf{S}^{\mathrm{c}} + \mathbf{T}^{\mathrm{c}} + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$ and $\det(\mathbf{S} + \mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{\mathrm{c}}\mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$ to show that $\left(\mathbf{I} + (\boldsymbol{\omega}\times)\right)^{-1} = \frac{\mathbf{I} + \boldsymbol{\omega}\times + \boldsymbol{\omega}\otimes\boldsymbol{\omega}}{1 + \|\boldsymbol{\omega}\|^2}$

For any invertible tensor $\mathbf{T}$,

$$\mathbf{T}^{-1} = \frac{\mathbf{T}^{\mathrm{cT}}}{\det \mathbf{T}}$$

$$\det(\mathbf{I} + \boldsymbol{\omega}\times) = \det \mathbf{I} + \det(\boldsymbol{\omega}\times) + (\boldsymbol{\omega}\times)^{\mathrm{c}} : \mathbf{I} + \mathbf{I}^{\mathrm{c}} : (\boldsymbol{\omega}\times)$$

$$= 1 + 0 + |\omega|^2 + 0$$

$$\left(\mathbf{I} + (\boldsymbol{\omega}\times)\right)^{\mathrm{c}} = [(1 + \mathrm{tr}(\boldsymbol{\omega}\times))\mathbf{I} - (\boldsymbol{\omega}\times)^{\mathrm{T}} + (\boldsymbol{\omega}\times)^{\mathrm{c}}]$$

$$= \mathbf{I} + (\boldsymbol{\omega}\times) + \boldsymbol{\omega}\otimes\boldsymbol{\omega}$$

so that

$$\left(\mathbf{I} + (\boldsymbol{\omega}\times)\right)^{-1} = \frac{\mathbf{I} + \boldsymbol{\omega}\times + \boldsymbol{\omega}\otimes\boldsymbol{\omega}}{1 + \|\boldsymbol{\omega}\|^2}$$

2.27	Use the fact that $\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{c}\mathbf{S}^{T}) + \mathrm{tr}(\mathbf{S}^{c}\mathbf{T}^{T}) + \det(\mathbf{T})$ to show that $\det(\mathbf{S}+\mathbf{a}\otimes\mathbf{b}) = \det(\mathbf{S})\,(1+\mathbf{b}\cdot\mathbf{S}^{-1}\mathbf{a})$
	Note that $(\mathbf{a}\otimes\mathbf{b})^{c} = \mathbf{O}$ and that $\det(\mathbf{a}\otimes\mathbf{b}) = 0$ so that $$\begin{aligned}\det(\mathbf{S}+\mathbf{a}\otimes\mathbf{b}) &= \det(\mathbf{S}) + \mathrm{tr}\big(\mathbf{S}^{c}(\mathbf{b}\otimes\mathbf{a})\big)\\ &= \det(\mathbf{S})\,(1+\mathbf{a}\cdot\mathbf{S}^{-T}\mathbf{b})\\ &= \det(\mathbf{S})\,(1+\mathbf{b}\cdot\mathbf{S}^{-1}\mathbf{a})\end{aligned}$$
2.28	Use the expression $(\mathbf{S}+\mathbf{T})^{c} = \mathbf{S}^{c} + \mathbf{T}^{c} + \mathbf{T}^{T}\mathbf{S}^{T} + \mathbf{S}^{T}\mathbf{T}^{T} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{T} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{T} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{1}$ to show that for an arbitrary tensor **T**, $(\mathbf{I}+\mathbf{T})^{c} = \mathbf{T}^{c} + \mathbf{I}(1+\mathrm{tr}\,\mathbf{T}) - \mathbf{T}^{T}$ and that $(\mathbf{I}+\mathbf{u}\otimes\mathbf{v})^{c} = \mathbf{I}(1+\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\otimes\mathbf{u}$
	Substituting the identity tensor for **S** in the given expression, we have, $$\begin{aligned}(\mathbf{I}+\mathbf{T})^{c} &= \mathbf{T}^{c} + \mathbf{I} + (3\,\mathrm{tr}\,\mathbf{T} - \mathrm{tr}\,\mathbf{T})\mathbf{I} - 3\mathbf{T}^{T} - \mathbf{I}\,\mathrm{tr}\,\mathbf{T} + \mathbf{T}^{T} + \mathbf{T}^{T}\\ &= \mathbf{T}^{c} + \mathbf{I} + 2\,\mathrm{tr}\,\mathbf{T}\,\mathbf{I} - \mathrm{tr}\,\mathbf{T}\,\mathbf{I} - \mathbf{T}^{T}\\ &= \mathbf{T}^{c} + \mathbf{I}(1+\mathrm{tr}\,\mathbf{T}) - \mathbf{T}^{T}\end{aligned}$$ $$\begin{aligned}(\mathbf{I}+\mathbf{u}\otimes\mathbf{v})^{c} &= (\mathbf{u}\otimes\mathbf{v})^{c} + \mathbf{I}(1+\mathrm{tr}(\mathbf{u}\otimes\mathbf{v})) - (\mathbf{u}\otimes\mathbf{v})^{T}\\ &= \mathbf{0} + \mathbf{I}(1+\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\otimes\mathbf{u}\\ &= \mathbf{I}(1+\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\otimes\mathbf{u}\end{aligned}$$
2.29	Using direct notation and without going into components, show that the determinant of a vector cross is zero.
	Given basis vectors, $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$, the third invariant of $\boldsymbol{\omega}\times$, $$\begin{aligned}I_3(\boldsymbol{\omega}\times) &= \det(\boldsymbol{\omega}\times)\\ &= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, \boldsymbol{\omega}\times\mathbf{g}_2, \boldsymbol{\omega}\times\mathbf{g}_3]}{[\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3]}\\ &= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, (\boldsymbol{\omega}\times)^{c}(\mathbf{g}_2\times\mathbf{g}_3)]}{[\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3]}\\ &= \frac{[\boldsymbol{\omega}\times\mathbf{g}_1, (\boldsymbol{\omega}\otimes\boldsymbol{\omega})(\mathbf{g}_2\times\mathbf{g}_3)]}{[\mathbf{g}_1,\mathbf{g}_2,\mathbf{g}_3]}\end{aligned}$$

upon noting that the cofactor, $(\boldsymbol{\omega}\,\times)^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$.

And since $(\boldsymbol{\omega} \otimes \boldsymbol{\omega})$ is symmetric, the numerator above is,

$$(\boldsymbol{\omega} \times \mathbf{g}_1) \cdot (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\mathbf{g}_2 \times \mathbf{g}_3) = (\mathbf{g}_2 \times \mathbf{g}_3) \cdot (\boldsymbol{\omega} \otimes \boldsymbol{\omega})(\boldsymbol{\omega} \times \mathbf{g}_1)$$
$$= (\mathbf{g}_2 \times \mathbf{g}_3) \cdot [\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \mathbf{g}_1)]\boldsymbol{\omega} = 0$$

so that $I_3(\boldsymbol{\omega}\,\times) = \det(\boldsymbol{\omega}\,\times) = 0$.

Show that the trace of the cofactor, $\operatorname{tr}(\boldsymbol{\omega}\,\times)^{\mathrm{c}} = \|\boldsymbol{\omega}\|^2$

First note that $(\boldsymbol{\omega}\,\times)^{\mathrm{c}} = (\boldsymbol{\omega} \otimes \boldsymbol{\omega})$. Therefore,

$$\operatorname{tr}(\boldsymbol{\omega}\,\times)^{\mathrm{c}} = \operatorname{tr}(\boldsymbol{\omega} \otimes \boldsymbol{\omega})$$
$$= \boldsymbol{\omega} \cdot \boldsymbol{\omega} = \|\boldsymbol{\omega}\|^2$$

2.30 For an invertible tensor $\mathbf{S}$, Show that $\operatorname{cof}(\operatorname{cof}\mathbf{S}) = (\det\mathbf{S})\mathbf{S}$

$$\mathbf{S}^{\mathrm{c}} = \det(\mathbf{S})\,\mathbf{S}^{-\mathrm{T}}$$

So that,

$$\mathbf{S}^{\mathrm{cc}} \equiv \operatorname{cof}(\operatorname{cof}\mathbf{S}) = (\det\mathbf{S}^{\mathrm{c}})(\mathbf{S}^{\mathrm{c}})^{-\mathrm{T}}$$
$$= (\det\mathbf{S})^2[(\mathbf{S}^{\mathrm{c}})^{-1}]^{\mathrm{T}}$$
$$= (\det\mathbf{S})^2[(\det\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}]^{\mathrm{T}}$$
$$= (\det\mathbf{S})^2(\det\mathbf{S})^{-1}\mathbf{S}$$
$$= (\det\mathbf{S})\mathbf{S}$$

as required

2.31 Show that $(\mathbf{S}^{-1})^{\mathrm{c}} = (\det\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$

$$(\mathbf{S}^{-1})^{\mathrm{c}} = \det(\mathbf{S}^{-1})\,(\mathbf{S}^{-1})^{-\mathrm{T}}$$
$$= (\det\mathbf{S})^{-1}\mathbf{S}^{\mathrm{T}}$$

2.32 Show that for a scalar α and tensor $\mathbf{S}$, $(\alpha\mathbf{S})^{\mathrm{c}} = \alpha^2\mathbf{S}^{\mathrm{c}}$

$$(\alpha\mathbf{S})^{c} = (\det(\alpha\mathbf{S}))(\alpha\mathbf{S})^{-\mathrm{T}}$$
$$= (\alpha^3 \det(\mathbf{S}))\alpha^{-1}\mathbf{S}^{-\mathrm{T}}$$
$$= (\alpha^2 \det(\mathbf{S}))\mathbf{S}^{-\mathrm{T}} = \alpha^2\mathbf{S}^{c}$$

2.33	Using direct notation and without going into components, Find the cofactor of a vector cross $\boldsymbol{\omega} \times$

Given independent vectors $\mathbf{u}$ and $\mathbf{v}$, consider the product,

$$\big((\boldsymbol{\omega}\times)\,\mathbf{u}\big) \times \big((\boldsymbol{\omega}\times)\,\mathbf{v}\big) = (\boldsymbol{\omega}\times\ \mathbf{u}) \times (\boldsymbol{\omega}\times\ \mathbf{v})$$
$$= [(\boldsymbol{\omega}\times\ \mathbf{u})\cdot\mathbf{v}]\boldsymbol{\omega} - [(\boldsymbol{\omega}\times\ \mathbf{u})\cdot\boldsymbol{\omega}]\mathbf{v}$$
$$= [\boldsymbol{\omega}\cdot\ (\mathbf{u}\times\mathbf{v})]\boldsymbol{\omega}$$
$$= (\boldsymbol{\omega}\otimes\boldsymbol{\omega})(\mathbf{u}\times\ \mathbf{v})$$

Showing that the cofactor of $\boldsymbol{\omega}\times$ is the dyad $\boldsymbol{\omega}\otimes\boldsymbol{\omega}$**.**

2.34	Use the fact that the cofactor of any tensor can be written as $\mathbf{S}^{c} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{1})^{\mathrm{T}}$ to show that the cofactor of the sum of two tensors can be expressed in terms of the constituent tensors as, $(\mathbf{S}+\mathbf{T})^{c} = \mathbf{S}^{c} + \mathbf{T}^{c} + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} + \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$

Since $\mathbf{S}^{\mathbf{c}} = (\mathbf{S}^{\mathbf{2}} - I_1\mathbf{S} + I_2\mathbf{I})^{\mathbf{T}}$, let $\mathbf{S} \to \mathbf{S}+\mathbf{T}\ \Rightarrow$

$$(\mathbf{S}+\mathbf{T})^{\mathbf{c}} = \left\{(\mathbf{S}+\mathbf{T})^{\mathbf{2}} - \mathbf{tr}(\mathbf{S}+\mathbf{T})(\mathbf{S}+\mathbf{T}) + \frac{1}{2}[\mathrm{tr}^{\mathbf{2}}(\mathbf{S}+\mathbf{T}) - \mathrm{tr}(\mathbf{S}+\mathbf{T})^{\mathbf{2}}]\mathbf{I}\right\}^{\mathbf{T}}$$
$$= \Big\{(\mathbf{S}+\mathbf{T})^{\mathbf{2}} + (\mathbf{S}+\mathbf{T})^{\mathbf{2}} + \mathbf{TS} + \mathbf{ST} - \mathrm{tr}(\mathbf{S})\mathbf{S} - \mathrm{tr}(\mathbf{T})\mathbf{T} - \mathrm{tr}(\mathbf{S})\mathbf{T}$$
$$- \mathrm{tr}(\mathbf{T})\mathbf{S}$$
$$+ \frac{1}{2}[\mathrm{tr}(\mathbf{S}) + \mathrm{tr}(\mathbf{T}) + \mathbf{2}\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}^{\mathbf{2}}(\mathbf{S}) - \mathrm{tr}^{\mathbf{2}}(\mathbf{T}) - \mathrm{tr}(\mathbf{TS})$$
$$- \mathrm{tr}(\mathbf{ST})]\Big\}^{\mathbf{T}}$$

$$= \left(\mathbf{S}^2 - \mathrm{tr}(\mathbf{S})\mathbf{S} + \frac{1}{2}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S})^2]\mathbf{I}\right)^{\mathbf{T}}$$

$$+ \left(\mathbf{T}^2 - \mathrm{tr}(\mathbf{T})\mathbf{T} + \frac{1}{2}[\mathrm{tr}^2(\mathbf{T}) - \mathrm{tr}(\mathbf{T})^2]\mathbf{I}\right)^{\mathbf{T}} + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$$

$$- \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}} + [\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$$

$$= \mathbf{S}^{\mathrm{c}} + \mathbf{T}^{\mathrm{c}} + \mathbf{T}^{\mathrm{T}}\mathbf{S}^{\mathrm{T}} + \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} - \mathrm{tr}(\mathbf{T})\mathbf{S}^{\mathrm{T}} - \mathrm{tr}(\mathbf{S})\mathbf{T}^{\mathrm{T}}$$

$$+ [\mathrm{tr}(\mathbf{S})\mathbf{tr}(\mathbf{T}) - \mathrm{tr}(\mathbf{ST})]\mathbf{I}$$

2.35	Determinant is not a linear scalar-valued tensor function. For any two tensors **S** and **T**, use the direct method to show that the determinant of the sum $$\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^{\mathrm{c}}\mathbf{S}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$$

Given the set $\mathbf{a}, \mathbf{b}, \mathbf{c}$ of linearly independent vectors, the determinant of the sum,

$$\det(\mathbf{S}+\mathbf{T})$$

$$= \frac{\left((\mathbf{S}+\mathbf{T})\mathbf{a}, (\mathbf{S}+\mathbf{T})\mathbf{b}, (\mathbf{S}+\mathbf{T})\mathbf{c}\right)}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \det\mathbf{S} + \det\mathbf{T} + \frac{[\mathbf{Sa}, \mathbf{Sb}, \mathbf{Tc}] + [\mathbf{Sa}, \mathbf{Tb}, \mathbf{Sc}] + [\mathbf{Ta}, \mathbf{Sb}, \mathbf{Sc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \det \mathbf{S} + \det \mathbf{T} + \mathrm{tr}(\mathbf{T}\mathbf{S}^{c\mathrm{T}}) + \mathrm{tr}(\mathbf{S}\mathbf{T}^{c\mathrm{T}})$$

$$= \det \mathbf{S} + \det \mathbf{T} + \mathrm{tr}(\mathbf{S}^{c}\mathbf{T}^{\mathrm{T}}) + \mathrm{tr}(\mathbf{T}^{c}\mathbf{S}^{\mathrm{T}})$$

2.37 Given that vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are linearly independent, and that the tensor $\mathbf{T}$ is not singular, show that the set $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

If $\mathbf{T}$ is not singular, then its determinant exists and is not equal to zero. Therefore,

$$\det \mathbf{T} = \frac{[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \neq 0$$

Consequently, $[\mathbf{Tu}, \mathbf{Tv}, \mathbf{Tw}] \neq 0$. Which shows that $\mathbf{Tu}, \mathbf{Tv}$ and $\mathbf{Tw}$ are also linearly independent.

2.38 For the invertible tensor $\mathbf{T}$ and the tensors $\mathbf{F}, \mathbf{V}$ and $\mathbf{U}$, show that

$$(\mathbf{T} + \mathbf{UFV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$$

First consider the matrix $\begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ \mathbf{V} & \mathbf{F}^{-1} \end{pmatrix}$. Its inverse is obtained by solving the matrix equation,

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{pmatrix}\begin{pmatrix} \mathbf{T} & -\mathbf{U} \\ \mathbf{V} & \mathbf{F}^{-1} \end{pmatrix} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}$$

which yields,

$$\mathbf{A}\,\mathbf{T} + \mathbf{BV} = \mathbf{I}$$

$$-\mathbf{AU} + \mathbf{BF}^{-1} = \mathbf{0} \;\Rightarrow \mathbf{B} = \mathbf{AUF}$$

so that,

$$\mathbf{A}\,\mathbf{T} + \mathbf{AUFV} = \mathbf{A}(\mathbf{T} + \mathbf{UFV}) = \mathbf{I}$$

$$\Rightarrow \mathbf{A} = (\mathbf{T} + \mathbf{UFV})^{-1}$$

But $\mathbf{A} = \mathbf{T}^{-1} - \mathbf{BVT}^{-1}$ substituting in the second equation,

from which we can now write that $(\mathbf{T}^{-1} - \mathbf{BVT}^{-1})\mathbf{U} = \mathbf{BF}^{-1}$ so that

$$\mathbf{B} = \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}$$

$$\mathbf{A} = \mathbf{T}^{-1} - \mathbf{BVT}^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$$

Finally $\mathbf{A} = (\mathbf{T} + \mathbf{UFV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{F}^{-1} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$ as required

In the special case when $\mathbf{F}$ is the identity tensor, we have,

$$(\mathbf{T} + \mathbf{UV})^{-1} = \mathbf{T}^{-1} - \mathbf{T}^{-1}\mathbf{U}(\mathbf{I} + \mathbf{VT}^{-1}\mathbf{U})^{-1}\mathbf{VT}^{-1}$$

2.39 For any tensor $\mathbf{T}$, the arbitrary vector $\mathbf{u}$ and the scalar λ, show that the eigenvalue problem, $\mathbf{Tu} = \lambda\mathbf{u}$ leads to the characteristic equation, $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$, where $I_1 = I_1(\mathbf{T}), I_2 = I_2(\mathbf{T})$ and $I_3 = I_3(\mathbf{T})$ the first, second and third invariants of $\mathbf{T}$.

Writing the tensor and vector in component forms, we have

$$\mathbf{Tu} = T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j)u_k\mathbf{e}_k = \lambda\mathbf{u} = \lambda u_i\mathbf{e}_i$$

So that,

$$\begin{aligned}\mathbf{Tu} - \lambda\mathbf{u} &= T_{ij}\mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{e}_k)u_k - \lambda u_i\mathbf{e}_i \\ &= T_{ij}\mathbf{e}_i u_j - \lambda u_i\mathbf{e}_i \\ &= (T_{ij} - \lambda\delta_{ij})u_j\mathbf{e}_i = \mathbf{o}\end{aligned}$$

Which is possible only if the coefficient determinant, $|T_{ij} - \lambda\delta_{ij}|$ vanishes. Expanding, we find that,

$$\begin{aligned}&-T_{31}T_{22}T_{13} + T_{12}T_{23}T_{31} + T_{13}T_{31}T_{32} - T_{11}T_{23}T_{32} - T_{12}T_{21}T_{33} + T_{11}T_{22}T_{33} \\ &\quad + T_{12}T_{12}\lambda - T_{11}T_{22}\lambda + T_{13}T_{31}\lambda + T_{23}T_{32}\lambda - T_{13}T_{33}\lambda - T_{22}T_{33}\lambda \\ &\quad + T_{11}\lambda^2 + T_{22}\lambda^2 + T_{33}\lambda^2 - \lambda^3 \\ &= -T_{31}T_{22}T_{13} + T_{12}T_{23}T_{31} + T_{13}T_{31}T_{32} - T_{11}T_{23}T_{32} - T_{12}T_{21}T_{33} + T_{11}T_{22}T_{33} \\ &\quad + (T_{12}T_{12} - T_{11}T_{22} + T_{13}T_{31} + T_{23}T_{32} - T_{13}T_{33} - T_{22}T_{33})\lambda \\ &\quad + (T_{11} + T_{22} + T_{33})\lambda^2 - \lambda^3 = 0\end{aligned}$$

Or, $\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$, as required.

2.40	For linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, and a tensor $\mathbf{T}$ use the relationship, $[\lambda\mathbf{a} - \mathbf{Ta}, \lambda\mathbf{b} - \mathbf{Tb}, \lambda\mathbf{c} - \mathbf{Tc}] = \det(\lambda\mathbf{I} - \mathbf{T})[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ and the characteristic equation $\lambda^3 - I_1(\mathbf{T})\lambda^2 + I_2(\mathbf{T})\lambda - I_3(\mathbf{T}) = 0$ to find expressions for the for the invariants of $\mathbf{T}$.

The characteristic equation, $\det(\lambda\mathbf{I} - \mathbf{T}) = 0$ immediately implies that $[\lambda\mathbf{a} - \mathbf{Ta}, \lambda\mathbf{b} - \mathbf{Tb}, \lambda\mathbf{c} - \mathbf{Tc}] = 0$.

Expanding the scalar triple product, we have

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}]\lambda^3 - ([\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}])\lambda^2 \\ + ([\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}])\lambda - [\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}] = 0$$

From which we can see that,

$$I_1(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$I_2(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{c}] + [\mathbf{a}, \mathbf{Tb}, \mathbf{Tc}] + [\mathbf{Ta}, \mathbf{b}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \text{and}$$

$$I_3(\mathbf{T}) = \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

Assuming we have carefully chosen $[\mathbf{a}, \mathbf{b}, \mathbf{c}] \neq 0$.

2.41	Given that the cofactor $\mathbf{A}^c \equiv \operatorname{cof}\mathbf{A} = \mathbf{A}^{-\mathrm{T}} \det \mathbf{A}$ satisfies $\mathbf{Aa} \times \mathbf{Ab} = \mathbf{A}^c(\mathbf{a} \times \mathbf{b})$. Show by direct methods that transposing does not alter the determinant of a tensor.

$$\det \mathbf{A} = \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{\mathbf{Aa} \cdot \mathbf{Ab} \times \mathbf{Ac}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{\mathbf{Aa} \cdot \mathbf{A}^c(\mathbf{b} \times \mathbf{c})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$
$$= \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{A}^{c\mathrm{T}}\mathbf{Aa}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{A}^{-1} \det \mathbf{A}^{\mathrm{T}}\, \mathbf{Aa}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$
$$= \det \mathbf{A}^{\mathrm{T}} \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{A}^{-1}\mathbf{Aa}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det \mathbf{A}^{\mathrm{T}}$$

upon noting that $\mathbf{A}^{\mathrm{T}}\mathbf{Aa} = \mathbf{Ia} = \mathbf{a}$.

2.42	For a scalar α show that $\det \alpha\mathbf{A} = \alpha^3 \det \mathbf{A}$
	Given that $\det \mathbf{A} = \frac{[\mathbf{Aa},\mathbf{Ab},\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$, then $$\det \alpha\mathbf{A} = \frac{[\alpha\mathbf{Aa},\alpha\mathbf{Ab},\alpha\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \alpha^3 \frac{[\mathbf{Aa},\mathbf{Ab},\mathbf{Ac}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \alpha^3 \det \mathbf{A}$$
2.43	Define the inner product of tensors $\mathbf{T}$ and $\mathbf{S}$ as $\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^T\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^T)$ show that $I_1(\mathbf{T}) = \mathbf{T}:\mathbf{I}$
	$$\mathbf{T}:\mathbf{S} = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{S}) = \mathrm{tr}(\mathbf{T}\mathbf{S}^{\mathrm{T}})$$ Let $\mathbf{S} = \mathbf{I}$; $$\mathbf{T}:\mathbf{I} = \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\mathbf{I}) = \mathrm{tr}(\mathbf{T}\mathbf{I})$$ $$= \mathrm{tr}(\mathbf{T}) = I_1(\mathbf{T})$$
2.44	Show that every skew tensor is traceless.
a	In full component form, a skew tensor $\mathbf{W}$ can be written as: $$\mathbf{W} = W_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$ Once a tensor is in component form, its transpose is a reversal of its dyad bases. Consequently, $$\mathbf{W}^{\mathrm{T}} = W_{ij}\mathbf{e}_j \otimes \mathbf{e}_i = W_{ji}\mathbf{e}_i \otimes \mathbf{e}_j = -W_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$ The last equality arising from the fact that the transpose of a skew tensor is its opposite. The middle equality is the allowable reversal of roles for dummy variables. We can therefore write that, $$W_{ji}\mathbf{e}_i \otimes \mathbf{e}_j + W_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = (W_{ji} + W_{ij})\mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{0}$$ Which, taken component by component means, $W_{ji} = -W_{ij}$. In particular, $$W_{11} = -W_{11} = 0, W_{22} = -W_{22} = 0, \text{and } W_{33} = -W_{33} = 0$$ The trace of $\mathbf{W}$ is

$$\text{tr}\,\mathbf{W} = W_{ij}\mathbf{e}_i \cdot \mathbf{e}_j = W_{ij}\delta_{ij} = W_{ii} = W_{11} + W_{22} + W_{33} = 0.$$

b Or more elegantly:

$$\text{tr}\,\mathbf{W} = \mathbf{I}:\mathbf{W} = \mathbf{I}:\mathbf{W}^{\text{T}} = -\mathbf{I}:\mathbf{W} = 0$$

The second equality because the trace operation does not change with transposing. The third equation from the fact that the transpose of a skew tensor is its opposite. The result all comes out on one line with no appeal to components. Lastly, recall that trace is a linear operation. Hence,

$$\text{tr}\,\mathbf{W} = \text{tr}\,\mathbf{W}^{\text{T}} = -\text{tr}\,\mathbf{W} = 0$$

2.45 Define the cofactor of a tensor as, $\text{cof}\,\mathbf{T} \equiv \mathbf{T}^{\text{c}} \equiv \mathbf{T}^{-\mathbf{T}}\det\mathbf{T}$. Show that, for any pair of linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ the cofactor satisfies, $\mathbf{Tu} \times \mathbf{Tv} = \mathbf{T}^{\text{c}}(\mathbf{u} \times \mathbf{v})$

First note that if $\mathbf{T}$ is invertible, the independence of the vectors $\mathbf{u}$ and $\mathbf{v}$ implies the independence of vectors $\mathbf{Tu}$ and $\mathbf{Tv}$. Consequently, we can define the non-vanishing

$$\mathbf{n} \equiv \mathbf{Tu} \times \mathbf{Tv} \neq \mathbf{0}.$$

It follows that $\mathbf{n}$ must be on the perpendicular line to both $\mathbf{Tu}$ and $\mathbf{Tv}$. Therefore,

$$\mathbf{n} \cdot \mathbf{Tu} = \mathbf{n} \cdot \mathbf{Tv} = \mathbf{0}.$$

We can also take a transpose and write,

$$\mathbf{u} \cdot \mathbf{T}^{\mathbf{T}}\mathbf{n} = \mathbf{v} \cdot \mathbf{T}^{\mathbf{T}}\mathbf{n} = \mathbf{0}$$

Showing that the vector $\mathbf{T}^{\mathbf{T}}\mathbf{n}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$. It follows that $\exists\ \alpha \in \mathfrak{R}$ such that

$$\mathbf{T}^{\mathbf{T}}\mathbf{n} = \alpha(\mathbf{u} \times \mathbf{v})$$

Therefore, $\mathbf{T}^{\text{T}}(\mathbf{Tu} \times \mathbf{Tv}) = \alpha(\mathbf{u} \times \mathbf{v})$.

Let $\mathbf{w} = \mathbf{u} \times \mathbf{v}$ so that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, then we can take a scalar product of the above equation and obtain,

$$\mathbf{w}\cdot\mathbf{T}^{\mathrm{T}}(\mathbf{Tu}\times\mathbf{Tv})=\alpha(\mathbf{u}\times\mathbf{v}\cdot\mathbf{w})$$

The LHS is also $\mathbf{Tw}\cdot(\mathbf{Tu}\times\mathbf{Tv})=\mathbf{Tu}\times\mathbf{Tv}\cdot\mathbf{Tw}$. In the equation, $\mathbf{Tu}\times\mathbf{Tv}\cdot\ \mathbf{Tw}=\ \alpha(\mathbf{u}\times\mathbf{v}\cdot\mathbf{w})$, it is clear that

$$\alpha=\det\mathbf{T}$$

We have that, $\mathbf{Tu}\times\mathbf{Tv}=\mathbf{T}^{-\mathrm{T}}\det\mathbf{T}\,(\mathbf{u}\times\mathbf{v})$. And therefore, we have that,

$$\mathbf{Tu}\times\mathbf{Tv}=\mathbf{T}^{-\mathrm{T}}\det\mathbf{T}\,(\mathbf{u}\times\mathbf{v})=\mathbf{T}^{\mathrm{c}}(\mathbf{u}\times\mathbf{v}).$$

2.46 In component form, the third tensor invariant of a tensor $\mathbf{T}$, $I_3(\mathbf{T})=e_{\alpha\beta\gamma}T_{1\alpha}T_{2\beta}T_{3\gamma}=\det\mathbf{T}$. Show that $e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma}=e_{\alpha\beta\gamma}\det\mathbf{T}$

We do this by first establishing the fact that the LHS is completely antisymmetric in α, β and γ. We note that the indices i, j and k are dummy and therefore,

$$e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma}=-e_{kji}T_{i\alpha}T_{j\beta}T_{k\gamma}=-e_{kji}T_{k\gamma}T_{i\alpha}T_{j\beta}=-e_{ijk}T_{i\gamma}T_{k\alpha}T_{j\beta}$$

Showing that a simple swap of α and γ changes the sign. This is similarly true for the other pairs in the lower symbols. Thus we establish anti-symmetry in α, β and γ.

Noting that both sides of

$$e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma}=e_{\alpha\beta\gamma}\det\mathbf{T}$$

take the same values as the determinant of $\mathbf{T}$ when α, β and γ are equal to 1, 2 and 3 respectively. The arrangement of the indices makes this value positive or negative in the same antisymmetric way. This completes the proof

$$e_{ijk}T_{i\alpha}T_{j\beta}T_{k\gamma}=e_{\alpha\beta\gamma}\det\mathbf{T}$$

2.48 Given that $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\}$ are two orthonormal bases, show that the spectral form, $\mathbf{Q}=\boldsymbol{\xi}_1\otimes\mathbf{e}_1+\boldsymbol{\xi}_2\otimes\mathbf{e}_2+\boldsymbol{\xi}_3\otimes\mathbf{e}_3$ rotates $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1,\boldsymbol{\xi}_2,\boldsymbol{\xi}_3\}$

To prove that $\mathbf{Q}$ is a rotation, first observe that,

$$\mathbf{QQ}^{\mathrm{T}}=(\boldsymbol{\xi}_1\otimes\mathbf{e}_1+\boldsymbol{\xi}_2\otimes\mathbf{e}_2+\boldsymbol{\xi}_3\otimes\mathbf{e}_3)(\mathbf{e}_1\otimes\boldsymbol{\xi}_1+\mathbf{e}_2\otimes\boldsymbol{\xi}_2++\mathbf{e}_3\otimes\boldsymbol{\xi}_3)$$

$$= \boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_1 + \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_2 + \boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_3 = \mathbf{I}$$

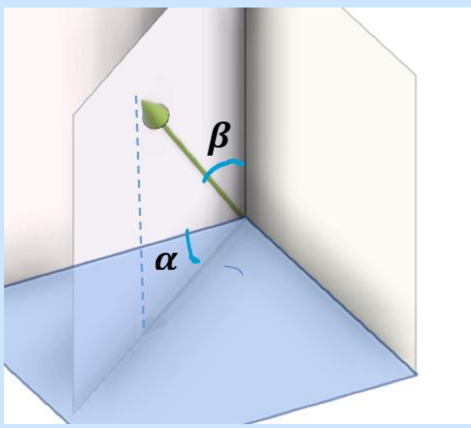
Furthermore,

$$\det \mathbf{Q} = [\mathbf{Q}\mathbf{e}_1, \mathbf{Q}\mathbf{e}_2, \mathbf{Q}\mathbf{e}_3] = [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = 1$$

since the set $\{\xi_1, \xi_2, \xi_3\}$ is orthonormal.

We have already seen that each coordinate vector in $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ rotates to $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$respectively because, $\mathbf{Q}\mathbf{e}_1 = (\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3)\mathbf{e}_1 = \boldsymbol{\xi}_1$, similarly, $\mathbf{Q}\mathbf{e}_2 = \boldsymbol{\xi}_2$ and $\mathbf{Q}\mathbf{e}_3 = \boldsymbol{\xi}_3$.

2.49 Find (a) the vector cross of a unit vector $\mathbf{u} = u_\alpha \mathbf{e}_\alpha$ If $\mathbf{u}$ is on a plane including $\mathbf{e}_\alpha$ inclined to that is oriented at an angle α to the $\mathbf{e}_1$ axis and making angle β with $\mathbf{e}_3$, find the components of $\mathbf{u}$ in terms of α and β.



$$(\mathbf{u}\times) = e_{i\alpha j} u_\alpha \mathbf{e}_i \otimes \mathbf{e}_j$$

$$= [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} 0 & -u_3 & u_2 \\ u_3 & 0 & -u_1 \\ -u_2 & u_1 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

From the figure, $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \sin\beta\cos\alpha \\ \sin\beta\sin\alpha \\ \cos\beta \end{pmatrix}$, so that

$$(\mathbf{u}\times) = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

2.50 For arbitrary, mutually orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\mathbf{u} \cdot \mathbf{T}\mathbf{v} = 0$ if and only if $\mathbf{T} = \lambda\mathbf{I}$.

	If $\mathbf{T} = \lambda\mathbf{I}$ then, $$\mathbf{u}\cdot\ \mathbf{Tv} = \lambda\mathbf{u}\cdot\ \mathbf{Iv} = \lambda(\mathbf{u}\cdot\mathbf{v}) = 0$$ from orthogonality of $\mathbf{u}, \mathbf{v}$. Now suppose $\mathbf{u}\cdot\ \mathbf{Tv} = 0$. Then, by the definition of the transpose, $$\mathbf{u}\cdot\ \mathbf{Tv} = \mathbf{v}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{u} = 0$$ We are given that $\mathbf{u}$ is orthogonal to $\mathbf{v}$. This can only be compatible with this scalar product if $\mathbf{T}^{\mathrm{T}}\mathbf{u}$ is parallel to $\mathbf{u}$. This happens only if $\mathbf{T}$ is a spherical tensor. That is, $\mathbf{T} = \lambda\mathbf{I}$.
2.51	Given that, $(\mathbf{u}\times)(\mathbf{v}\times) = \mathbf{v}\otimes\mathbf{u} - (\mathbf{u}\cdot\ \mathbf{v})\mathbf{I}$. If $\boldsymbol{\omega}_1$and $\boldsymbol{\omega}_2$ are the vector cross of the skew tensors $\mathbf{W}_1$ and $\mathbf{W}_2$ respectively, show that $\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1 = \boldsymbol{\omega}_2\otimes\boldsymbol{\omega}_1 - \boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_2$, $\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1 = (\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2)\times$.
	Using the given relationship, $$\begin{aligned}&(\boldsymbol{\omega}_1\times)(\boldsymbol{\omega}_2\times) - (\boldsymbol{\omega}_2\times)(\boldsymbol{\omega}_1\times)\\ &= (\boldsymbol{\omega}_2\cdot\boldsymbol{\omega}_1)\mathbf{I} - \boldsymbol{\omega}_2\otimes\boldsymbol{\omega}_1 - (\boldsymbol{\omega}_1\cdot\boldsymbol{\omega}_2)\mathbf{I} + \boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_2\\ &= \boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_2 - \boldsymbol{\omega}_2\otimes\boldsymbol{\omega}_1\end{aligned}$$ Clearly, $\mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1 = \boldsymbol{\omega}_2\otimes\boldsymbol{\omega}_1 - \boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_2$. The skew tensor, $$\begin{aligned}\big((\boldsymbol{\omega}_1\times\boldsymbol{\omega}_2)\times\big) &= -e_{i\alpha\beta}e_{ijk}(\omega_1)_j(\omega_2)_k\,\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\\ &= \big(\delta_{\alpha k}\delta_{\beta j} - \delta_{\alpha j}\delta_{\beta k}\big)(\omega_1)_j(\omega_2)_k\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\\ &= (\omega_1)_\beta(\omega_2)_\alpha\mathbf{e}_\alpha\otimes\mathbf{e}_\beta - (\omega_1)_\alpha(\omega_2)_\beta\mathbf{e}_\alpha\otimes\mathbf{e}_\beta\\ &= \boldsymbol{\omega}_2\otimes\boldsymbol{\omega}_1 - \boldsymbol{\omega}_1\otimes\boldsymbol{\omega}_2\\ &= \mathbf{W}_1\mathbf{W}_2 - \mathbf{W}_2\mathbf{W}_1\end{aligned}$$
2.52	For a tensor $\mathbf{A}$ with three eigenvalues λ_i, if $\boldsymbol{\gamma}_i$ are the corresponding normalized eigenvectors, find a spectral form for the tensor $\mathbf{A}$

For the eigenbasis, $\{\boldsymbol{\gamma}_i\}$ we have $\boldsymbol{\gamma}_i \cdot \boldsymbol{\gamma}_i = \delta_{ij}$. The components of $\mathbf{A}$ are evaluated as,

$$A_{ji} = \boldsymbol{\gamma}_j \cdot (\mathbf{A}\boldsymbol{\gamma}_i) = \lambda_i \boldsymbol{\gamma}_j \cdot \boldsymbol{\gamma}_i = \sum_{i=1}^{3} \lambda_i \delta_{ij} = \boldsymbol{\gamma}_i \cdot \left(\mathbf{A}^{\mathrm{T}} \boldsymbol{\gamma}_j\right)$$

We can therefore write

$$\mathbf{A} = A_{ji} \boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}_i = \sum_{i=1}^{3} \lambda_i \delta_{ij} \, \boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}_j = \sum_{i=1}^{3} \lambda_i \, \boldsymbol{\gamma}_i \otimes \boldsymbol{\gamma}_i$$

in which all the off-diagonal terms vanish.

2.53 Show that for two invertible tensors $\mathbf{T}$ and $\mathbf{S}$, $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\,\mathbf{T}^{-1}$

The inverse of the product $\mathbf{TS}$ contracted with $\mathbf{TS}$ yields the unit vector

$$(\mathbf{TS})^{-1}\,\mathbf{TS} = \mathbf{I}$$

Observe that $\mathbf{S}^{-1}\,\mathbf{T}^{-1}\,\mathbf{TS} = \mathbf{S}^{-1}\,\mathbf{IS} = \mathbf{I}$.

It follows immediately that $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\,\mathbf{T}^{-1}$.

2.54 Use the direct notation to show that the cofactor of the product of two tensors is the product of the cofactors, that is, for vectors $\mathbf{S}$ and $\mathbf{T}$, $(\mathbf{ST})^{\mathrm{c}} = \mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{c}}$

Consider vectors $\mathbf{u}$ and $\mathbf{v}$.

$$\begin{aligned}
\mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v}) &= \mathbf{Tu} \times \mathbf{Tv} \\
\mathbf{S}^{\mathrm{c}}[\mathbf{T}^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})] &= \mathbf{S}^{\mathrm{c}}\,[\mathbf{Tu} \times \mathbf{Tv}] \\
&= \mathbf{STu} \times \mathbf{STv} = (\mathbf{ST})^{\mathrm{c}}(\mathbf{u} \times \mathbf{v})
\end{aligned}$$

showing that $(\mathbf{ST})^{\mathrm{c}} = \mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{c}}$

2.55 Use the result $\det(\mathbf{S} + \mathbf{T}) = \det(\mathbf{S}) + \operatorname{tr}(\mathbf{T}^{\mathrm{c}}\mathbf{S}^{\mathrm{T}}) + \operatorname{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$ to show that $\det(\mathbf{S} + \mathbf{I}) = \det \mathbf{S} + \det \mathbf{S} \operatorname{tr}(\mathbf{S}^{-1}) + \operatorname{tr}(\mathbf{S}) + 1$.

$\mathbf{T} \to \mathbf{I}$ in the given identity and noting that the unit tensor is self cofactor $\Rightarrow$

$$\begin{aligned}\det(\mathbf{S}+\mathbf{I}) &= \det\mathbf{S} + \operatorname{tr}(\mathbf{1}\mathbf{S}^{\mathrm{T}}) + \operatorname{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{I}) + \det\mathbf{I}\\ &= \det\mathbf{S} + \operatorname{tr}(\mathbf{S}^{\mathrm{c}}) + \operatorname{tr}(\mathbf{S}) + 1\\ &= \det\mathbf{S} + \det\mathbf{S}\operatorname{tr}(\mathbf{S}^{-1}) + \operatorname{tr}\mathbf{S} + 1\end{aligned}$$

2.56 Tensors $\mathbf{S}$ and $\mathbf{T}$ are said to be similar if the invertible tensor exists such that $\mathbf{S} = \mathbf{B}\mathbf{T}\mathbf{B}^{-1}$. Show that $\mathbf{S}$ and $\mathbf{T}$ have the same eigenvalues as well as principal invariants.

The characteristic equation for $\mathbf{S}$ is,

$$\mathbf{S}\mathbf{v} = \lambda\mathbf{v}$$

where λ is the eigenvalue and $\mathbf{v}$ the eigenvector. But $\mathbf{S} = \mathbf{B}\mathbf{T}\mathbf{B}^{-1}$ substituting, we have,

$$\mathbf{B}\mathbf{T}\mathbf{B}^{-1}\mathbf{v} = \lambda\mathbf{v}$$

so that

$$\mathbf{T}\mathbf{B}^{-1}\mathbf{v} = \lambda\mathbf{B}^{-1}\mathbf{v}$$

If we define $\mathbf{v}_1 \equiv \mathbf{B}^{-1}\mathbf{v}$, we obtain,

$$\mathbf{T}\mathbf{v}_1 = \lambda\mathbf{v}_1$$

yielding the same characteristic equation as well as eigenvalues and principal invariants as $\mathbf{S}\mathbf{v} = \lambda\mathbf{v}$

2.56 a For arbitrary tensors $\mathbf{u}$ and $\mathbf{v}$ the dyad, $\mathbf{u}\otimes\mathbf{v}$. Use the result $\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \operatorname{tr}(\mathbf{T}^{\mathrm{c}}\mathbf{S}^{\mathrm{T}}) + \operatorname{tr}(\mathbf{S}^{\mathrm{c}}\mathbf{T}^{\mathrm{T}}) + \det(\mathbf{T})$ to show that $\det(\mathbf{S}+\mathbf{u}\otimes\mathbf{v}) = \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{v}):\mathbf{S}$.

$\mathbf{T} \to \mathbf{u}\otimes\mathbf{v}$, then $\mathbf{T}^{\mathrm{c}} = \mathbf{0}$ and $\mathbf{T}^{\mathrm{T}} = \mathbf{v}\otimes\mathbf{u}$ and $\det(\mathbf{u}\otimes\mathbf{v}) = 0$.

$$\begin{aligned}\det(\mathbf{S}+\mathbf{u}\otimes\mathbf{v}) &= \det\mathbf{S} + \mathbf{0} + \operatorname{tr}\big((\mathbf{v}\otimes\mathbf{u})\mathbf{S}^{\mathrm{c}}\big) + 0\\ &= \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{v}):\mathbf{S}^{\mathrm{c}}\end{aligned}$$

2.57	If $\mathbf{u}$ is perpendicular to $\mathbf{v}$ show that all the eigenvalues of the dyad $\mathbf{u} \otimes \mathbf{v}$ are zero.

For this tensor, $I_1 = \mathbf{u} \cdot \mathbf{v} = 0$ on account of $\mathbf{u}$ being perpendicular to $\mathbf{v}$. We now examine the other two invariants:

$$I_2 = \frac{[(\mathbf{u} \otimes \mathbf{v})\mathbf{a}, (\mathbf{u} \otimes \mathbf{v})\mathbf{b}, \mathbf{c}] + [\mathbf{a}, (\mathbf{u} \otimes \mathbf{v})\mathbf{b}, (\mathbf{u} \otimes \mathbf{v})\mathbf{c}] + [(\mathbf{u} \otimes \mathbf{v})\mathbf{a}, \mathbf{b}, (\mathbf{u} \otimes \mathbf{v})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

For linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$. Clearly,

$$I_2 = \frac{[(\mathbf{v} \cdot \mathbf{a})\mathbf{u}, (\mathbf{v} \cdot \mathbf{b})\mathbf{u}, \mathbf{c}] + [\mathbf{a}, (\mathbf{v} \cdot \mathbf{b})\mathbf{u}, (\mathbf{v} \cdot \mathbf{c})\mathbf{u}] + [(\mathbf{v} \cdot \mathbf{a})\mathbf{u}, \mathbf{b}, (\mathbf{v} \cdot \mathbf{c})\mathbf{u}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 0$$

on the collinearity of two vectors in each triple product.

$$I_3 = \frac{[(\mathbf{u} \otimes \mathbf{v})\mathbf{a}, (\mathbf{u} \otimes \mathbf{v})\mathbf{b}, (\mathbf{u} \otimes \mathbf{v})\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{[(\mathbf{v} \cdot \mathbf{a})\mathbf{u}, (\mathbf{v} \cdot \mathbf{b})\mathbf{u}, (\mathbf{v} \cdot \mathbf{c})\mathbf{u}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 0$$

The latter being the triple product of three parallel vectors. Hence we have a case of a tensor with three principal invariants vanishing. The characteristic equation becomes,

$$\lambda^3 - I_1\lambda^2 + I_2\lambda + I_3 = \lambda^3 = 0$$

Yielding three equal roots of zero. $\mathbf{u} \otimes \mathbf{v}$ is thus a non-zero tensor with zero eigenvalues.

2.58	Show that $(\mathbf{u} \times)(\mathbf{v} \times) = \mathbf{v} \otimes \mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{I}$ and that $\mathrm{tr}[(\mathbf{u} \times)(\mathbf{v} \times)] = -2(\mathbf{u} \cdot \mathbf{v})$

$$\begin{aligned}
(\mathbf{u} \times)(\mathbf{v} \times) &= e_{ijk}u_j e_{\alpha\beta\gamma}v_\beta (\mathbf{e}_i \otimes \mathbf{e}_k)(\mathbf{e}_\alpha \otimes \mathbf{e}_\gamma) \\
&= e_{ijk}u_j e_{k\beta\gamma}v_\beta (\mathbf{e}_i \otimes \mathbf{e}_\gamma) \\
&= (\delta_{i\beta}\delta_{j\gamma} - \delta_{i\gamma}\delta_{j\beta})u_j v_\beta (\mathbf{e}_i \otimes \mathbf{e}_\gamma) \\
&= u_j v_i (\mathbf{e}_i \otimes \mathbf{e}_j) - u_j v_j (\mathbf{e}_i \otimes \mathbf{e}_i) \\
&= \mathbf{v} \otimes \mathbf{u} - (\mathbf{u} \cdot \mathbf{v})\mathbf{I}
\end{aligned}$$

	Since tr $\mathbf{I} = 3$, the trace of this tensor is $-2(\mathbf{u} \cdot \mathbf{v})$
2.59	Show that $[\mathbf{u}, \mathbf{v}, \mathbf{w}] = -\mathrm{tr}[(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times)]$
	In the above we have shown that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = [\mathbf{v} \otimes (\mathbf{u} \times \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\times]$ Because the vector cross is traceless, the trace of $[(\mathbf{u} \cdot \mathbf{v})\mathbf{w}\times] = 0$. The trace of the first term, $\mathbf{v} \otimes (\mathbf{u} \times \mathbf{w})$ is obviously the same as $-[\mathbf{u}, \mathbf{v}, \mathbf{w}]$ which completes the proof.
2.60	For vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, show that $(\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) = \mathbf{v} \otimes (\mathbf{u} \times \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\times$.
	The tensor $(\mathbf{u}\times) = -e_{lmn}u_n\mathbf{e}_l \otimes \mathbf{e}_m$ similarly, $(\mathbf{v}\times) = -e_{\alpha\beta\gamma}v_\gamma\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ and $(\mathbf{w}\times) = -e_{ijk}w_k\mathbf{e}_i \otimes \mathbf{e}_j$. Clearly, $$\begin{aligned} (\mathbf{u}\times)(\mathbf{v}\times)(\mathbf{w}\times) &= -e_{lmn}e_{\alpha\beta\gamma}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_l \otimes \mathbf{e}_m)(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)(\mathbf{e}_i \otimes \mathbf{e}_j) \\ &= -e_{\alpha\beta\gamma}e_{lmn}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_l \otimes \mathbf{e}_j)\delta_{m\alpha}\delta_{\beta i} \\ &= -e_{mi\gamma}e_{lmn}e_{ijk}u_n v_\gamma w_k(\mathbf{e}_l \otimes \mathbf{e}_j) \\ &= (\delta_{il}\delta_{\gamma n} - \delta_{in}\delta_{\gamma l})e_{ijk}u_n v_\gamma w_k(\mathbf{e}_l \otimes \mathbf{e}_j) \\ &= e_{ljk}u_n v_n w_k(\mathbf{e}_l \otimes \mathbf{e}_j) - e_{ijk}u_i v_l w_k(\mathbf{e}_l \otimes \mathbf{e}_j) \\ &= e_{ljk}u_n v_n w_k(\mathbf{e}_l \otimes \mathbf{e}_j) - e_{ijk}u_i v_l w_k(\mathbf{e}_l \otimes \mathbf{e}_j) \\ &= [\mathbf{v} \otimes (\mathbf{u} \times \mathbf{w}) - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}\times] \end{aligned}$$
2.61	Show that the dyad $\mathbf{u} \otimes \mathbf{v}$ is NOT, in general symmetric: $\mathbf{u} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u} - (\mathbf{u} \times \mathbf{v})\times$

$$\mathbf{u}\times\mathbf{v} = e_{ijk}u_j v_k\,\mathbf{e}_i$$

$$\big((\mathbf{u}\times\mathbf{v})\times\big) = e_{\alpha i\beta}e_{ijk}u_j v_k\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$$

$$= -(\delta_{i\alpha}\delta_{k\beta} - \delta_{k\alpha}\delta_{i\beta})u_j v_k\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$$

$$= (-u_\alpha v_\beta + u_\beta v_\alpha)\mathbf{e}_\alpha\otimes\mathbf{e}_\beta$$

$$= \mathbf{v}\otimes\mathbf{u} - \mathbf{u}\otimes\mathbf{v}$$

2.62 If we can find α, β and γ such that the unit tensor, $\mathbf{I} = \alpha\mathbf{a}\otimes\mathbf{b} + \beta\mathbf{b}\otimes\mathbf{c} + \gamma\mathbf{c}\otimes\mathbf{a}$, show that unless $\mathbf{b}\cdot\mathbf{a} = \alpha^{-1}$ then $\mathbf{a},\mathbf{b}$ and $\mathbf{c}$ cannot be linearly independent.

In the expression,

$$\mathbf{I} = \alpha\mathbf{a}\otimes\mathbf{b} + \beta\mathbf{b}\otimes\mathbf{c} + \gamma\mathbf{c}\otimes\mathbf{a}$$

Take a product on the right with vector **a,**

$$\mathbf{Ia} = \alpha\mathbf{a}(\mathbf{b}\cdot\mathbf{a}) + \beta\mathbf{b}(\mathbf{c}\cdot\mathbf{a}) + \gamma\mathbf{c}(\mathbf{a}\cdot\mathbf{a})$$

$$\Rightarrow \mathbf{a}\big(1-\alpha(\mathbf{b}\cdot\mathbf{a})\big) = \beta\mathbf{b}(\mathbf{c}\cdot\mathbf{a}) + \gamma\mathbf{c}(\mathbf{a}\cdot\mathbf{a})$$

$$\mathbf{a} = \frac{\beta\mathbf{b}(\mathbf{c}\cdot\mathbf{a})}{1-\alpha(\mathbf{b}\cdot\mathbf{a})} + \frac{\gamma\mathbf{c}(\mathbf{a}\cdot\mathbf{a})}{\big(1-\alpha(\mathbf{b}\cdot\mathbf{a})\big)}$$

So that this expression enables us to write **a** in terms of **b** and **c.**

2.63 Show that if for every skew tensor $\mathbf{W}$, the inner product $\mathbf{S}:\mathbf{W} = 0$, it must follow that $\mathbf{S}$ is symmetric.

$$\mathbf{S}^{\mathrm{T}}\mathbf{W} = S_{ij}W_{\alpha\beta}(\mathbf{e}_j\otimes\mathbf{e}_i)(\mathbf{e}_\alpha\otimes\mathbf{e}_\beta)$$

$$= S_{ij}W_{\alpha\beta}\mathbf{e}_j\otimes\mathbf{e}_\beta\delta_{i\alpha}$$

$$\mathbf{S}:\mathbf{W} = \operatorname{tr}\mathbf{S}^{\mathrm{T}}\mathbf{W}$$

$$= S_{ij}W_{\alpha\beta}\delta_{j\beta}\delta_{i\alpha} = S_{ij}W_{ij}$$

$$= -S_{ij}W_{ji} = 0$$

Since all inner products $\mathbf{S}:\mathbf{W}=0$. But,

$$S_{ij}W_{ij}=-S_{ij}W_{ji}=-S_{ji}W_{ij}$$

So that $(S_{ij}-S_{ji})W_{ij}=0 \Rightarrow S_{ij}=S_{ji}$ Hence, $\mathbf{S}$ is symmetric.

2.64 Show that $I_1(\mathbf{T})\equiv \text{tr}(\mathbf{T})=\frac{[\mathbf{Ta},\mathbf{b},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$ is independent of the choice of the linearly independent vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Let us refer each vector to an ONB, $\mathbf{a}=a_i\mathbf{e}_i$, $\mathbf{b}=a_j\mathbf{e}_j$, and $\mathbf{c}=c_k\mathbf{e}_k$. Hence,

$$I_1(\mathbf{T})\equiv \text{tr}(\mathbf{T})=\frac{[\mathbf{T}(a_i\mathbf{e}_i),b_j\mathbf{e}_j,c_k\mathbf{e}_k]+[a_i\mathbf{e}_i,\mathbf{T}(b_j\mathbf{e}_j),c_k\mathbf{e}_k]+[a_i\mathbf{e}_i,b_j\mathbf{e}_j,\mathbf{T}(c_k\mathbf{e}_k)]}{[a_i\mathbf{e}_i,b_j\mathbf{e}_j,c_k\mathbf{e}_k]}$$

$$=\frac{a_ib_jc_k([\mathbf{Te}_i,\mathbf{e}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{Te}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{e}_j,\mathbf{Te}_k])}{a_ib_jc_k[\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k]}$$

But $[\mathbf{Te}_i,\mathbf{e}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{Te}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{e}_j,\mathbf{Te}_k]=T_{\alpha\alpha}e_{ijk}$. We have that

$$I_1(\mathbf{T})=\frac{a_ib_jc_kT_{\alpha\alpha}e_{ijk}}{a_ib_jc_ke_{ijk}}=T_{\alpha\alpha}$$

Which is obviously independent of the choice of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

2.65 Show that $[\mathbf{Te}_i,\mathbf{e}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{Te}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{e}_j,\mathbf{Te}_k]=T_{\alpha\alpha}e_{ijk}$.

$[\mathbf{Te}_i,\mathbf{e}_j,\mathbf{e}_k]=(T_{\alpha\beta}(\mathbf{e}_\alpha\otimes\mathbf{e}_\beta)\mathbf{e}_i)\cdot(e_{jk\gamma}\mathbf{e}_\gamma)=T_{\alpha\beta}\delta_{i\beta}\delta_{\alpha\gamma}e_{jk\gamma}=T_{\alpha i}e_{\alpha jk}$

Similarly, we have that $[\mathbf{e}_i,\mathbf{Te}_j,\mathbf{e}_k]=T_{\beta j}e_{i\beta k}$ and $[\mathbf{e}_i,\mathbf{e}_j,\mathbf{Te}_k]=T_{\gamma k}e_{ij\gamma}$. Expanding each term, we have,

$$T_{\alpha i}e_{\alpha jk}=T_{1i}e_{1jk}+T_{2j}e_{2jk}+T_{3k}e_{3jk}$$

$$T_{\beta j}e_{i\beta k}=T_{1j}e_{i1k}+T_{2j}e_{i2k}+T_{3j}e_{i3k}$$

$$T_{\gamma k}e_{ij\gamma}=T_{1k}e_{ij1}+T_{2k}e_{ij2}+T_{3k}e_{ij3}$$

Select $\{i,j,k\}$ as any combination of the possible values of 1,2,3, each time, the result is, $T_{\alpha i}e_{\alpha jk}+T_{\beta j}e_{i\beta k}+T_{\gamma k}e_{ij\gamma}=T_{\alpha\alpha}e_{ijk}$ using the expansion above.

2.66	Show that $I_2(\mathbf{T}) \equiv \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{Tc}]+[\mathbf{Ta},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$ is independent of the choice of the linearly independent vectors **a**, **b** and **c.**

Let us refer each vector to an ONB, $\mathbf{a} = a_i\mathbf{e}_i$, $\mathbf{b} = a_j\mathbf{e}_j$, and $\mathbf{c} = c_k\mathbf{e}_k$. Hence,

$$I_2(\mathbf{T}) \equiv \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{c}]+[\mathbf{a},\mathbf{Tb},\mathbf{Tc}]+[\mathbf{Ta},\mathbf{b},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

$$= \frac{[\mathbf{T}(a_i\mathbf{e}_i),\mathbf{T}(b_j\mathbf{e}_j),c_k\mathbf{e}_k]+[a_i\mathbf{e}_i,\mathbf{T}(b_j\mathbf{e}_j),\mathbf{T}(c_k\mathbf{e}_k)]+[\mathbf{T}(a_i\mathbf{e}_i),a_j\mathbf{e}_j,\mathbf{T}(c_k\mathbf{e}_k)]}{[a_i\mathbf{e}_i,b_j\mathbf{e}_j,c_k\mathbf{e}_k]}$$

$$= \frac{a_ib_jc_k([\mathbf{Te}_i,\mathbf{Te}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{Te}_j,\mathbf{Te}_k]+[\mathbf{Te}_i,\mathbf{e}_j,\mathbf{Te}_k])}{a_ib_jc_k[\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k]}$$

$$= \frac{a_ib_jc_k\left(\left(\mathbf{T}^{\mathrm{c}}(\mathbf{e}_i\times\mathbf{e}_j)\cdot\mathbf{e}_k\right)+\left(\mathbf{e}_i\cdot\mathbf{T}^{\mathrm{c}}(\mathbf{e}_j\times\mathbf{e}_k)\right)+\left(\mathbf{T}^{\mathrm{c}}(\mathbf{e}_k\times\mathbf{e}_i)\cdot\mathbf{e}_j\right)\right)}{a_ib_jc_k[\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k]}$$

$$= \mathbf{T}^{\mathrm{c}}\frac{a_ib_jc_k([\mathbf{e}_i,\mathbf{e}_j,\mathbf{T}^{\mathrm{cT}}\mathbf{e}_k]+[\mathbf{T}^{\mathrm{cT}}\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{T}^{\mathrm{cT}}\mathbf{e}_j,\mathbf{e}_k])}{a_ib_jc_k[\mathbf{e}_i,\mathbf{e}_j,\mathbf{e}_k]}$$

Where we used the fact that for vectors $\mathbf{u},\mathbf{v}$, the product $\mathbf{Tu}\times\mathbf{Tv} = \mathbf{T}^{\mathrm{c}}(\mathbf{u}\times\mathbf{v})$ followed by the definition of the transpose of a tensor. But $[\mathbf{e}_i,\mathbf{e}_j,\mathbf{T}^{\mathrm{cT}}\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{T}^{\mathrm{cT}}\mathbf{e}_j,\mathbf{e}_k]+[\mathbf{e}_i,\mathbf{e}_j,\mathbf{T}^{\mathrm{cT}}\mathbf{e}_k] = T^{\mathrm{c}}_{\alpha\alpha}e_{ijk}$. We therefore have that

$$I_2(\mathbf{T}) = \frac{a_ib_jc_kT^{\mathrm{c}}_{\alpha\alpha}e_{ijk}}{a_ib_jc_ke_{ijk}} = T^{\mathrm{c}}_{\alpha\alpha} = \operatorname{tr}(\mathbf{T}^{\mathrm{c}})$$

Which is obviously independent of the choice of **a**, **b** and **c**.

2.67	Show that $I_3(\mathbf{T}) = \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \det\mathbf{T}$ is independent of the choice of the linearly independent vectors **a**, **b** and **c.**

$$I_3(\mathbf{T}) = \frac{[\mathbf{Ta},\mathbf{Tb},\mathbf{Tc}]}{[\mathbf{a},\mathbf{b},\mathbf{c}]}$$

Let us refer each vector to an ONB, $\mathbf{a} = a_i\mathbf{e}_i$, $\mathbf{b} = b_j\mathbf{e}_j$, and $\mathbf{c} = c_k\mathbf{e}_k$. Hence,

$$I_3(\mathbf{T}) = \frac{[\mathbf{T}(a_i\mathbf{e}_i),\mathbf{T}(b_j\mathbf{e}_j),\mathbf{T}(c_k\mathbf{e}_k)]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} = \frac{a_ib_jc_k[\mathbf{Te}_i,\mathbf{Te}_j,\mathbf{Te}_k]}{e_{123}a_ib_jc_k}$$

Writing $\mathbf{T} = T_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ and substituting for each occurrence one by one,

$$I_3(\mathbf{T}) = \frac{a_i b_j c_k [T_{\alpha\beta}(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_i, \mathbf{T}\mathbf{e}_j, \mathbf{T}\mathbf{e}_k]}{e_{123} a_i b_j c_k} = \frac{a_i b_j c_k [T_{\alpha\beta}\mathbf{e}_\alpha \delta_{\beta i}, \mathbf{T}\mathbf{e}_j, \mathbf{T}\mathbf{e}_k]}{e_{123} a_i b_j c_k}$$

$$= \frac{a_i b_j c_k [T_{\alpha i}\mathbf{e}_\alpha, T_{\beta j}\mathbf{e}_\beta, T_{\gamma k}\mathbf{e}_\gamma]}{e_{ijk} a_i b_j c_k} = \frac{a_i b_j c_k T_{\alpha i} T_{\beta j} T_{\gamma k} [\mathbf{e}_\alpha, \mathbf{e}_\beta, \mathbf{e}_\gamma]}{e_{ijk} a_i b_j c_k} = \frac{a_i b_j c_k T_{\alpha i} T_{\beta j} T_{\gamma k} e_{\alpha\beta\gamma}}{e_{ijk} a_i b_j c_k}$$

$$= \frac{a_i b_j c_k T_{\alpha 1} T_{\beta 2} T_{\gamma 3} e_{\alpha\beta\gamma} e_{ijk}}{e_{ijk} a_i b_j c_k} = T_{\alpha 1} T_{\beta 2} T_{\gamma 3} e_{\alpha\beta\gamma} = \det \mathbf{T}$$

Which again is independent of the choice of **a**, **b** and **c**.

2.68 Let the spectral form of the tensor $\mathbf{S} = \lambda_1 \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2 \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal set. For a positive integer n, find the spectral form of $\mathbf{S}^n$ and that of $\mathbf{S}^{-1}$.

$$\begin{aligned}\mathbf{S}^2 &= (\lambda_1 \boldsymbol{e_1} \otimes \boldsymbol{e_1} + \lambda_2 \boldsymbol{e_2} \otimes \boldsymbol{e_2} + \lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)(\lambda_1 \boldsymbol{e_1} \otimes \boldsymbol{e_1} + \lambda_2 \boldsymbol{e_2} \otimes \boldsymbol{e_2} + \lambda_3 \boldsymbol{e}_3 \\ &\qquad \otimes \boldsymbol{e}_3) \\ &= (\lambda_1 \boldsymbol{e_1} \otimes \boldsymbol{e_1})(\lambda_1 \boldsymbol{e_1} \otimes \boldsymbol{e_1}) + (\lambda_1 \boldsymbol{e_1} \otimes \boldsymbol{e_1})(\lambda_2 \boldsymbol{e_2} \otimes \boldsymbol{e_2}) \\ &\qquad + (\lambda_1 \boldsymbol{e_1} \otimes \boldsymbol{e_1})(\lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) + \cdots + (\lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3)(\lambda_3 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3) \\ &= \lambda_1^2 \boldsymbol{e_1} \otimes \boldsymbol{e_1} + \lambda_2^2 \boldsymbol{e_2} \otimes \boldsymbol{e_2} + \lambda_3^2 \boldsymbol{e}_3 \otimes \boldsymbol{e}_3\end{aligned}$$

repeated multiplication leads to,

$$\mathbf{S}^n = \lambda_1^n \boldsymbol{e_1} \otimes \boldsymbol{e_1} + \lambda_2^n \boldsymbol{e_2} \otimes \boldsymbol{e_2} + \lambda_3^n \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$$

2.69 Given the unit vector, $\mathbf{w} = \sin\beta\cos\alpha\,\mathbf{e}_1 + \sin\beta\sin\alpha\,\mathbf{e}_2 + \cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W} \equiv (\mathbf{w}\times)$ and use the formula $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ to determine the general rotation through an angle θ.

$$\mathbf{W}(\alpha,\beta) = (\mathbf{w}\times) = \begin{pmatrix} 0 & -\cos\beta & \sin\beta\sin\alpha \\ \cos\beta & 0 & -\sin\beta\cos\alpha \\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0 \end{pmatrix}$$

$$\mathbf{Q}(\alpha,\beta,\theta) = \mathbf{I} + \mathbf{W}(\alpha,\beta)\sin\theta + \mathbf{W}^2(\alpha,\beta)(1-\cos\theta) =$$

$\mathbf{Q}(\alpha,\beta,\theta)$ Row 1:

	$\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))-\cos(\beta)\sin(\theta)$, $\sin(\alpha)\sin(\beta)\sin(\theta)+\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))\}$ $\mathbf{Q}(\alpha,\beta,\theta)$ Row 2: $\{\sin(\alpha)\cos(\alpha)\sin^2(\beta)(1-\cos(\theta))+\cos(\beta)\sin(\theta)$, $(1-\cos(\theta))\left(-\cos^2(\alpha)\sin^2(\beta)-\cos^2(\beta)\right)+1$, $\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\cos(\alpha)\sin(\beta)\sin(\theta)\}$ $\mathbf{Q}(\alpha,\beta,\theta)$ Row 3 $\{\cos(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))-\sin(\alpha)\sin(\beta)\sin(\theta)$, $\sin(\alpha)\sin(\beta)\cos(\beta)(1-\cos(\theta))+\cos(\alpha)\sin(\beta)\sin(\theta)$, $(1-\cos(\theta))(-\sin^2(\alpha)\sin^2(\beta)-\cos^2(\alpha)\sin^2(\beta))+1\}$
2.70	For a tensor **S** and scalar t, we define the exponential function, $$\exp(t\mathbf{S})=\mathbf{I}+(t\mathbf{S})+\frac{1}{2!}(t\mathbf{S})^2+\frac{1}{3!}(t\mathbf{S})^3+\cdots$$ Show that the transpose ofexp$(t\mathbf{S})$ equals $\exp(t\mathbf{S})^{\mathrm{T}}$ and that $\det(\exp(t\mathbf{S}))=\prod_{i=1}^{3}e^{(\lambda_i t)}=\exp(t\,\mathrm{tr}\,\mathbf{S})$.
	The transpose of the tensor equation can be found by transposing term-by term in $$(\exp(t\mathbf{S}))^{\mathrm{T}}=\mathbf{I}+(t\mathbf{S})^{\mathrm{T}}+\frac{1}{2!}(t\mathbf{S})^{2\mathrm{T}}+\frac{1}{3!}(t\mathbf{S})^{3\mathrm{T}}+\cdots$$ which is obviously the same as $\exp(t\mathbf{S})^{\mathrm{T}}$ by the given definition. Let the spectral form of **S** be such that, $$\mathbf{S}=\lambda_1(\mathbf{e}_1\otimes\mathbf{e}_1)+\lambda_2(\mathbf{e}_2\otimes\mathbf{e}_2)+\lambda_3(\mathbf{e}_3\otimes\mathbf{e}_3)$$ We can write the exponential series in terms of the spectral form so that,

	$$= \left(1 + \lambda_1 t + \frac{(\lambda_1 t)^2}{2!} + \cdots\right)(\mathbf{e}_1 \otimes \mathbf{e}_1)$$ $$+ \left(1 + \lambda_2 t + \frac{(\lambda_2 t)^2}{2!} + \cdots\right)(\mathbf{e}_2 \otimes \mathbf{e}_2)$$ $$+ \left(1 + \lambda_3 t + \frac{(\lambda_3 t)^2}{2!} + \cdots\right)(\mathbf{e}_3 \otimes \mathbf{e}_3)$$ $$= \mathrm{e}^{(\lambda_1 t)}\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathrm{e}^{(\lambda_2 t)}\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathrm{e}^{(\lambda_3 t)}\mathbf{e}_3 \otimes \mathbf{e}_3$$ so that the eigenvectors of $\exp(t\mathbf{S})$ are the same as those of $\mathbf{S}$ but the latter's eigenvalues are $\mathrm{e}^{(\lambda_1 t)}, \mathrm{e}^{(\lambda_2 t)}$ and $\mathrm{e}^{(\lambda_3 t)}$. Clearly, $$\det(\exp(t\mathbf{S})) = \prod_{i=1}^{3} \mathrm{e}^{(\lambda_i t)} = \exp\left(\sum_{i=1}^{3} t\lambda_i\right) = \exp(t \operatorname{tr} \mathbf{S}).$$
2.71	For an arbitrary vector, show that the cofactor of its vector cross is its tensor product with itself. That is $(\mathbf{u}\, \times)^{\mathrm{c}} = \mathbf{u} \otimes \mathbf{u}$

	First recall the result that for any tensor $\mathbf{S}$, the cofactor $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - \mathrm{I}_1\mathbf{S} + \mathrm{I}_2\mathbf{1})^{\mathrm{T}}$ $$(\mathbf{u}\times)^2 = (e_{i\alpha j}u_\alpha\mathbf{e}_i \otimes \mathbf{e}_j)(e_{l\beta m}u_\beta\mathbf{e}_l \otimes \mathbf{e}_m)$$ $$= e_{i\alpha j}e_{l\beta m}u_\alpha u_\beta(\mathbf{e}_i \otimes \mathbf{e}_m)\delta_{jl}$$ $$= e_{i\alpha j}e_{j\beta m}u_\alpha u_\beta(\mathbf{e}_i \otimes \mathbf{e}_m)$$ $$= (\delta_{i\beta}\delta_{\alpha m} - \delta_{im}\delta_{\alpha\beta})u_\alpha u^\beta(\mathbf{e}_i \otimes \mathbf{e}_m)$$ $$= (u_m u_i - \delta_{im}u_\alpha u_\alpha)\mathbf{e}_i \otimes \mathbf{e}_m$$ $$= \mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{I}$$ $$\mathrm{tr}[(\mathbf{u}\times)^2] = \mathbf{u}\cdot\mathbf{u} - 3\,\mathbf{u}\cdot\mathbf{u} = -\,2\,\mathbf{u}\cdot\mathbf{u}$$ $$\mathrm{tr}[(\mathbf{u}\times)] = 0$$ But from previous result, $$(\mathbf{u}\times)^{\mathrm{c}} = \left((\mathbf{u}\times)^2 - (\mathbf{u}\times)\mathrm{tr}(\mathbf{u}\times) + \frac{1}{2}[\mathrm{tr}^2(\mathbf{u}\times) - \mathrm{tr}((\mathbf{u}\times)^2)]\mathbf{1}\right)^{\mathrm{T}}$$ $$= \left(\mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{I} - 0 + \frac{1}{2}[0 + 2\,\mathbf{u}\cdot\mathbf{u}]\mathbf{I}\right)^{\mathrm{T}}$$ $$= (\mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{I} - 0 + [(\mathbf{u}\cdot\mathbf{u})]\mathbf{I})^{\mathrm{T}}$$ $$= \mathbf{u}\otimes\mathbf{u}$$
2.72	If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{I} + (1 - \cos(\theta))\mathbf{e}\otimes\mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Show that $\mathbf{Q}(\theta)(\mathbf{I} - \mathbf{e}\otimes\mathbf{e}) = \cos(\theta)(\mathbf{I} - \mathbf{e}\otimes\mathbf{e}) + \sin(\theta)(\mathbf{e}\times)$

We first observe that,

$$\mathbf{Q}(\theta)(\mathbf{e}\otimes\mathbf{e}) = \cos(\theta)(\mathbf{e}\otimes\mathbf{e}) + (1-\cos(\theta))\mathbf{e}\otimes\mathbf{e} + \sin(\theta)[\mathbf{e}\times(\mathbf{e}\otimes\mathbf{e})]$$

The last term vanishes immediately on account of the fact that $\mathbf{e}\otimes\mathbf{e}$ is a symmetric tensor. (The contraction of a symmetric and an antisymmetric tensor always vanishes). Consequently, we have,

$$\mathbf{Q}(\theta)(\mathbf{e}\otimes\mathbf{e}) = \cos(\theta)(\mathbf{e}\otimes\mathbf{e}) + (1-\cos(\theta))\mathbf{e}\otimes\mathbf{e} = \mathbf{e}\otimes\mathbf{e}$$

which again means that $\mathbf{Q}(\theta)$ has no effect on $\mathbf{e}\otimes\mathbf{e}$ so that,

$$\begin{aligned}\mathbf{Q}(\theta)(\mathbf{I}-\mathbf{e}\otimes\mathbf{e}) &= \cos(\theta)\mathbf{I} + \big(1-\cos(\theta)\big)\mathbf{e}\otimes\mathbf{e} + \sin(\theta)(\mathbf{e}\times) - \mathbf{e}\otimes\mathbf{e}\\ &= \cos(\theta)\,(\mathbf{I}-\mathbf{e}\otimes\mathbf{e}) + \sin(\theta)(\mathbf{e}\times)\end{aligned}$$

as required.

2.73 For an arbitrary unit vector $\mathbf{e}$, show that the skew tensor, $\mathbf{W} = (\mathbf{e}\times)$ is such that $\mathbf{W}^2 \equiv (\mathbf{e}\times)(\mathbf{e}\times) = (\mathbf{e}\otimes\mathbf{e}) - \mathbf{I}$

For an arbitrary vector $\mathbf{u}$,

$$\begin{aligned}(\mathbf{u}\times)^2 &= \big(e_{i\alpha j}u_\alpha \mathbf{e}_i\otimes\mathbf{e}_j\big)\big(e_{l\beta m}u^\beta \mathbf{e}_l\otimes\mathbf{e}_m\big)\\ &= e_{i\alpha j}e_{l\beta m}u_\alpha u_\beta(\mathbf{e}_i\otimes\mathbf{e}_m)\delta_{jl}\\ &= e_{i\alpha j}e_{j\beta m}u_\alpha u_\beta(\mathbf{e}_i\otimes\mathbf{e}_m)\\ &= \big(\delta_{i\beta}\delta_{\alpha m} - \delta_{im}\delta_{\alpha\beta}\big)u_\alpha u^\beta(\mathbf{e}_i\otimes\mathbf{e}_m)\\ &= (u_m u_i - \delta_{im}u_\alpha u_\alpha)\mathbf{e}_i\otimes\mathbf{e}_m\\ &= \mathbf{u}\otimes\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{I}\end{aligned}$$

If $\mathbf{u} = \mathbf{e}$, a unit vector, $\mathbf{u}\cdot\mathbf{u} = 1$ and $\mathbf{W} = (\mathbf{e}\times)$,

$$\mathbf{W}^2 \equiv (\mathbf{e}\times)(\mathbf{e}\times) = (\mathbf{e}\otimes\mathbf{e}) - \mathbf{I}$$

2.74 Find the rotation tensor around an axis parallel to the unit vector, $\mathbf{e} = \left\{-\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}$ through an angle $\frac{\pi}{3}$.

The skew tensor $(\mathbf{e}\times) = \mathbf{W} = \begin{pmatrix} 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}} \\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0 \end{pmatrix}$.

	$$(\mathbf{e}\times)^2=\mathbf{W}^2=\begin{pmatrix}0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}\\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0\end{pmatrix}\begin{pmatrix}0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}\\ \sqrt{\frac{2}{3}} & 0 & \frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & 0\end{pmatrix}$$ $$=\begin{pmatrix}-\frac{5}{6} & -\frac{1}{6} & -\frac{1}{3}\\ -\frac{1}{6} & -\frac{5}{6} & \frac{1}{3}\\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3}\end{pmatrix}$$ $$\mathbf{Q}\left(\frac{\pi}{6}\right)=\mathbf{I}+\mathbf{W}\sin\frac{\pi}{6}+\mathbf{W}^2\left(1-\cos\frac{\pi}{6}\right)$$ $$=\begin{pmatrix}1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & -\frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right)\\ \frac{1}{\sqrt{6}}+\frac{1}{6}\left(-1+\frac{\sqrt{3}}{2}\right) & 1-\frac{5}{6}\left(1-\frac{\sqrt{3}}{2}\right) & \frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right)\\ -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right) & -\frac{1}{2\sqrt{6}}+\frac{1}{3}\left(1-\frac{\sqrt{3}}{2}\right) & 1+\frac{1}{3}\left(-1+\frac{\sqrt{3}}{2}\right)\end{pmatrix}$$ $$=\begin{pmatrix}0.888354 & -0.430577 & 0.159465\\ 0.385919 & 0.888354 & 0.248782\\ -0.248782 & -0.159465 & 0.955341\end{pmatrix}$$ The inverse of this tensor is its transpose and its determinant is unity. Clearly, it is the rotation tensor we seek.
2.75	Given the unit vector, $\mathbf{w}=\sin\beta\cos\alpha\,\mathbf{e}_1+\sin\beta\sin\alpha\,\mathbf{e}_2+\cos\beta\,\mathbf{e}_3$. Find its vector cross, $\mathbf{W}\equiv(\mathbf{w}\times)$ and use the formula $\mathbf{Q}(\theta)=\mathbf{I}+\mathbf{W}\sin\theta+\mathbf{W}^2(1-\cos\theta)$ to determine the rotation tensor around the bisector of the $\mathbf{e}_1-\mathbf{e}_2$ axis through an angle θ.
	$$\mathbf{W}(\alpha,\beta)=(\mathbf{w}\times)=\begin{pmatrix}0 & -\cos\beta & \sin\beta\sin\alpha\\ \cos\beta & 0 & -\sin\beta\cos\alpha\\ -\sin\beta\sin\alpha & \sin\beta\cos\alpha & 0\end{pmatrix}$$

	Along the bisector of the $\mathbf{e}_1 - \mathbf{e}_2$ axis, $\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{2}$. Consequently, $\mathbf{w} = \frac{1}{\sqrt{2}}\mathbf{e}_1 + \frac{1}{\sqrt{2}}\mathbf{e}_2$. $\mathbf{W}\left(\frac{\pi}{4},\frac{\pi}{2}\right) = (\mathbf{w}\,\times) = \begin{pmatrix} 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}$, $\mathbf{W}^2\left(\frac{\pi}{4},\frac{\pi}{2}\right) = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix}$ And the rotation tensor for this axis is, $$\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta) = \begin{pmatrix} \frac{1}{2}(1+\cos\theta) & \frac{1}{2}(1-\cos\theta) & \frac{\sin\theta}{\sqrt{2}} \\ \frac{1}{2}(1-\cos\theta) & \frac{1}{2}(1+\cos\theta) & -\frac{\sin\theta}{\sqrt{2}} \\ -\frac{\sin\theta}{\sqrt{2}} & \frac{\sin\theta}{\sqrt{2}} & \cos\theta \end{pmatrix}$$
2.76	Show that $\mathbf{Q} = \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & 1/2 \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{-\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$ is an orthogonal tensor. Is it proper orthogonal? Compute the tensor $\mathbf{QFQ}^{\mathrm{T}}$ where $\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Find the eigenvectors and eigenvalues of the tensor $\mathbf{F}$.

```
Q = {{3/4, Sqrt[3]/4, 1/2}, {Sqrt[3]/4, 1/4, -Sqrt[3]/2},
  {-1/2, Sqrt[3]/2, 0}}
A = {{1, 0, 0}, {0, 2, 0}, {0, 0, 3}}
F = Transpose[Q]
```

$$\left\{\left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, \frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}\right\}, \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}\right\}$$

{{1, 0, 0}, {0, 2, 0}, {0, 0, 3}}

$$\left\{\left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}\right\}, \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}, \left\{\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0\right\}\right\}$$

```
Q.A.F
```

$$\left\{\left\{\frac{27}{16}, -\frac{7\sqrt{3}}{16}, \frac{3}{8}\right\}, \left\{-\frac{7\sqrt{3}}{16}, \frac{41}{16}, \frac{\sqrt{3}}{8}\right\}, \left\{\frac{3}{8}, \frac{\sqrt{3}}{8}, \frac{7}{4}\right\}\right\}$$

```
Eigenvalues [Q.A.F]
```

{3, 2, 1}

```
G = Q.A.F
```

$$\left\{\left\{\frac{27}{16}, -\frac{7\sqrt{3}}{16}, \frac{3}{8}\right\}, \left\{-\frac{7\sqrt{3}}{16}, \frac{41}{16}, \frac{\sqrt{3}}{8}\right\}, \left\{\frac{3}{8}, \frac{\sqrt{3}}{8}, \frac{7}{4}\right\}\right\}$$

```
Eigenvectors[G]
```

$$\left\{\left\{-\frac{1}{\sqrt{3}}, 1, 0\right\}, \left\{\frac{1}{2}, \frac{1}{2\sqrt{3}}, 1\right\}, \left\{-\frac{3}{2}, -\frac{\sqrt{3}}{2}, 1\right\}\right\}$$

2.77 Write the characteristic equation of the Tensor whose components are $\begin{pmatrix} 6 & 5 & 4 \\ 5 & 6 & 4 \\ 4 & 4 & 3 \end{pmatrix}$. Show that one of its eigenvalues equals 1 and there is a corresponding eigenvector, $\{1 \quad -1 \quad 0\}$

```
M = {{6, 5, 4}, {5, 6, 4}, {4, 4, 3}}
Eigenvalues[M]
Eigenvectors[M]
```

$$\{\{6,5,4\},\{5,6,4\},\{4,4,3\}\}$$

$$\{7+4\sqrt{3},\ 1,\ 7-4\sqrt{3}\}$$

$$\left\{\left\{-\frac{-9-5\sqrt{3}}{2\left(3+2\sqrt{3}\right)},\ -\frac{-9-5\sqrt{3}}{2\left(3+2\sqrt{3}\right)},\ 1\right\},\right.$$
$$\left.\{-1,1,0\},\ \left\{-\frac{9-5\sqrt{3}}{2\left(-3+2\sqrt{3}\right)},\ -\frac{9-5\sqrt{3}}{2\left(-3+2\sqrt{3}\right)},\ 1\right\}\right\}$$

2.78 Use the fact that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ where $\mathbf{W} \equiv (\mathbf{e}\,\times)$ - the vector cross of the unit vector, rotates every vector about the axis of $\mathbf{e}$ by the angle θ to find the tensor that rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_2, -\mathbf{e}_1, \mathbf{e}_3\}$.

Clearly, the rotation axis here is the unit vector $\boldsymbol{e}_3$ and the angle of rotation is $\frac{\pi}{2}$. Consequently, since $\mathbf{e}_3 = \{0,0,1\}$,

$\mathbf{W} \equiv (\mathbf{e}_3\,\times) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\mathbf{W}^2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$$\mathbf{Q}\left(\frac{\pi}{2}\right) = \mathbf{I} + \mathbf{W}\sin\frac{\pi}{2} + \mathbf{W}^2\left(1-\cos\frac{\pi}{2}\right)$$
$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This same tensor can be found directly by recognizing that the tensor, $\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ so that the tensor we seek is,

$$\mathbf{Q} = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

2.79	Given that $\mathbf{e}_1 = \{1,0,0\}, \mathbf{e}_2 = \{0,1,0\}, \mathbf{e}_3 = \{0,0,1\}, \mathbf{e}_4 = \left\{\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right\}, \mathbf{e}_5 = \left\{\frac{3}{4}, \frac{\sqrt{3}}{4}, -\frac{1}{2}\right\}, \mathbf{e}_6 = \left\{-\frac{1}{2}, \frac{\sqrt{3}}{2}, 0\right\}$, Find the tensor that transforms from $\{\mathbf{e}_2, \mathbf{e}_1, -\mathbf{e}_3\}$ to $\{\mathbf{e}_4, \mathbf{e}_5, \mathbf{e}_6\}$.
	Tensor, $\boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$ rotates $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$. The tensor we seek is, $$\mathbf{Q} = \mathbf{e}_4 \otimes \mathbf{e}_2 + \mathbf{e}_5 \otimes \mathbf{e}_1 - \mathbf{e}_6 \otimes \mathbf{e}_3$$ $$= \begin{pmatrix} \frac{3}{4} & \frac{\sqrt{3}}{4} & \frac{1}{2} \\ \frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}$$
2.80	If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{I} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times) \equiv \mathbf{W}$ is the vector cross of $\mathbf{e}$. Show that $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ [Note that $\mathbf{e} \otimes \mathbf{e} = \mathbf{W}^2 + \mathbf{I}$]
	Using the noted result, $$\mathbf{Q}(\theta) = \cos\theta\,\mathbf{I} + (1-\cos\theta)\mathbf{e}\otimes\mathbf{e} + \sin\theta\,(\mathbf{e}\times)$$ $$= \cos\theta\,\mathbf{I} + (1-\cos\theta)(\mathbf{W}^2 + \mathbf{I}) + \mathbf{W}\sin\theta$$ $$= \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$$

2.81	Let the spectral form of the tensor $\mathbf{S} = \lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}_3 \otimes \mathbf{e}_3$ where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form an orthonormal set. For a positive integer n, find the spectral form of $\mathbf{S}^n$ and that of $\mathbf{S}^{-1}$.
	$\mathbf{S}^2 = (\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}_3 \otimes \mathbf{e}_3)(\lambda_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3\mathbf{e}_3$

$$= (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1})(\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1}) + (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1})(\lambda_2 \mathbf{e_2} \otimes \mathbf{e_2})$$

$$+ (\lambda_1 \mathbf{e_1} \otimes \mathbf{e_1})(\lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3) + \cdots + (\lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3)(\lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3)$$

$$= \lambda_1^2 \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2^2 \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3^2 \mathbf{e}_3 \otimes \mathbf{e}_3$$

repeated multiplication leads to,

$$\mathbf{S}^n = \lambda_1^n \mathbf{e_1} \otimes \mathbf{e_1} + \lambda_2^n \mathbf{e_2} \otimes \mathbf{e_2} + \lambda_3^n \mathbf{e}_3 \otimes \mathbf{e}_3$$

2.82	Show that a cross product can only happen in three dimensions

Every vector has a skew tensor to which it is axial. A vector product is the same thing as the product of the axial vector. For both operations to be possible, the number of independent coefficients in both must equal:

The skew tensor expansion,

$$\mathbf{W} = \left(\boldsymbol{\xi}_i \cdot \mathbf{W}\boldsymbol{\xi}_j\right)\boldsymbol{\xi}_i \otimes \boldsymbol{\xi}_j$$

Gives $\frac{n(n-1)}{2}$ independent terms. A vector in n dimensional space is defined by n independent terms. An axial vector can only exist in a space where these are equal. We must solve the equation,

$$\frac{n(n-1)}{2} = n$$

Apart from the trivial solution zero, we have $n = 3$.

2.83	For two non-singular tensors, $\mathbf{S}$ and $\mathbf{T}$, show that $\det \mathbf{ST} = \det \mathbf{S} \det \mathbf{T}$

Given three linearly independent vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, since the scalar products are all scalar quantities, we can multiply or divide by them like regular scalars. Let the products, $\mathbf{Ta} = \mathbf{u}$, $\mathbf{Tb} = \mathbf{v}$ and $\mathbf{Tc} = \mathbf{w}$. It follows that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are also linearly independent since $\mathbf{T}$ is non-singular. Hence

$$\det \mathbf{ST} = \frac{[\mathbf{STa}, \mathbf{STb}, \mathbf{STc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{S}(\mathbf{Ta}), \mathbf{S}(\mathbf{Tb}), \mathbf{S}(\mathbf{Tc})]}{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]} \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \frac{[\mathbf{Su}, \mathbf{Sv}, \mathbf{Sw}]}{[\mathbf{u}, \mathbf{v}, \mathbf{w}]} \frac{[\mathbf{Ta}, \mathbf{Tb}, \mathbf{Tc}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

$$= \det \mathbf{S} \det \mathbf{T}$$

2.84	Given that the skew tensor $(\mathbf{e}\times) \equiv \mathbf{W}$, and that $\mathbf{Q}(\theta) \equiv \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ is the rotation along the axis $\mathbf{e}$ through the angle θ, Find out if the set $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent.
	First note that $\mathbf{W}$ is antisymmetric but $\mathbf{W}^2 = (\mathbf{e}\otimes\mathbf{e}) - \mathbf{I}$ is the linear combination of two symmetric tensors, and therefore symmetric. Assume that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ to be linearly dependent. It means we can find α, β and γ not all equal to zero such that $$\alpha\mathbf{I} + \beta\mathbf{W} + \gamma\mathbf{W}^2 = 0$$ Since α, β and γ are not all equal to zero, we assume in particular that $\beta \neq 0$. Consequently, we can write, $$\mathbf{W} = -\frac{\alpha}{\beta}\mathbf{I} - \frac{\gamma}{\beta}\mathbf{W}^2$$ In which we have expressed the anti-symmetric tensor $\mathbf{W}$ as a linear combination of two symmetric tensors! A contradiction! We can conclude that the set $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent.
2.85	Given that every rotation tensor $\mathbf{Q}$ can be expressed in terms of the skew tensor $\mathbf{W}(\equiv \mathbf{e}\times)$ as a function of the rotation angle θ: $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ and that $\{\mathbf{I}, \mathbf{W}, \mathbf{W}^2\}$ is linearly independent set of tensors, show that, $\{\mathbf{I}, \mathbf{Q}, \mathbf{Q}^\mathrm{T}\}$ is also a linearly independent set.
	Assume that the tensor set, $\{\mathbf{I}, \mathbf{Q}, \mathbf{Q}^\mathrm{T}\}$ is linearly dependent. It means we can find α, β and γ not all equal to zero such that $$\alpha\mathbf{I} + \beta\mathbf{Q} + \gamma\mathbf{Q}^\mathrm{T} = 0$$ Since $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$, we substitute and obtain, $$\begin{aligned} &\alpha\mathbf{I} + \beta\mathbf{Q} + \gamma\mathbf{Q}^\mathrm{T} = \\ &= \alpha\mathbf{I} + \beta\big(\mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\big) + \gamma\big(\mathbf{I} - \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\big) \\ &= (\alpha + \beta + \gamma)\mathbf{I} + (\beta - \gamma)\mathbf{W}\sin\theta + (\beta + \gamma)\mathbf{W}^2(1-\cos\theta) \\ &= a\mathbf{I} + b\mathbf{W} + c\mathbf{W}^2 = 0 \end{aligned}$$

	if we write $(\alpha+\beta+\gamma)=a,(\beta-\gamma)\sin\theta=b$ and $(\beta+\gamma)(1-\cos\theta)=c$ thereby contradicting the well-known fact that $\{\mathbf{I},\mathbf{W},\mathbf{W}^2\}$ is a linearly independent set.
2.86	If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta)=\cos(\theta)\mathbf{I}+(1-\cos(\theta))\mathbf{e}\otimes\mathbf{e}+\sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Given that for any vector $\mathbf{u}$, the vector $\mathbf{v}\equiv\mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$, and that, for any scalar α, $\mathbf{Q}(\theta)(\alpha\mathbf{e})=\alpha\mathbf{e}$, What is the physical meaning of $\mathbf{Q}(\theta)$?
	$\mathbf{Q}(\theta)$ is a rotation about the vector $\boldsymbol{e}$ counterclockwise through an angle θ. It therefore does not alter the magnitude or direction of any vector in the direction of $\boldsymbol{e}$; for any other vector, it has no effect on the magnitude but affects direction.
2.87	Define $\mathbf{Sym}$ as the set of all symmetric tensors. Show that $\mathbf{Sym}$ is invariant under $\mathbb{G}$ where $\mathbb{G}$ is the proper orthogonal group of all rotations, in the sense that for any tensor $\mathbf{A}\in\mathbf{Sym}$ every $\mathbf{Q}\in\mathbb{G}\Rightarrow\mathbf{QAQ^T}\in\mathbf{Sym}$.
	Since we are given that $\mathbf{A}\in\mathrm{Sym}$, we inspect the tensor $\mathbf{QAQ}^{\mathrm{T}}$. Its transpose is, $(\mathbf{QAQ}^{\mathrm{T}})^{\mathrm{T}}=(\mathbf{Q}^{\mathrm{T}})^{\mathrm{T}}\mathbf{AQ}^{\mathrm{T}}=\mathbf{QAQ}^{\mathrm{T}}$. So that $\mathbf{QAQ}^{\mathrm{T}}$ is symmetric and therefore $\mathbf{QAQ}^{\mathrm{T}}\in\mathrm{Sym}$. so that the transformation is invariant.
2.88	Define $\mathbb{L}^+$as the set of all tensors with a positive determinant. Show that $\mathbb{L}^+$ is invariant under G, the proper orthogonal group of all rotations, in the sense that for any tensor $\mathbf{A}\in\mathbb{L}^+$ $\mathbf{Q}\in\mathbf{G}\Rightarrow\mathbf{QAQ}^{\mathrm{T}}\in\mathbb{L}^+$.
	Since we are given that $\mathbf{A}\in\mathbb{L}^+$, the determinant of $\mathbf{A}$ is positive. Consider $\det(\mathbf{QAQ}^{\mathrm{T}})$. We observe the fact that the determinant of a product of tensors is the product of their determinants (proved above). We see clearly that,

	$\det(\mathbf{QAQ}^{\mathrm{T}}) = \det(\mathbf{Q}) \times \det(\mathbf{A}) \times \det(\mathbf{Q}^{\mathrm{T}})$. Since $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q}) = \det(\mathbf{Q}^{\mathrm{T}}) = 1$. Consequently, we see that, $$\det(\mathbf{QAQ}^{\mathrm{T}}) = \det(\mathbf{Q}) \times \det(\mathbf{A}) \times \det(\mathbf{Q}^{\mathrm{T}}) = \det(\mathbf{QAQ}^{\mathrm{T}}) = 1 \times \det(\mathbf{A}) \times 1 = \det(\mathbf{A})$$ Hence the determinant of $\mathbf{QAQ}^{\mathrm{T}}$ is also positive and therefore $\mathbf{QAQ}^{\mathrm{T}} \in \mathbb{L}^{+}$
2.89	If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{I} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e} \times)$ where $(\mathbf{e} \times)$ is the vector cross of $\mathbf{e}$. Show that for any vector $\mathbf{u}$, the vector $\mathbf{v} \equiv \mathbf{Q}(\theta)\,\mathbf{u}$ has the same magnitude as $\mathbf{u}$. What is the physical meaning of $\mathbf{Q}(\theta)$?
	Let the scalar $x \equiv \mathbf{e} \cdot \mathbf{u}$ be the projection of $\mathbf{u}$ onto the unit vector $\mathbf{e}$. The square of the magnitude of $\mathbf{v}$ is $\lvert\mathbf{v}\rvert^2$ $$= \mathbf{v} \cdot \mathbf{v} = \big(\cos\theta(\mathbf{1u}) + (1-\cos\theta)(\mathbf{e} \otimes \mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e} \times \mathbf{u})\big) \cdot \big(\cos\theta(\mathbf{1u}) + (1 - cos\theta)(\mathbf{e} \otimes \mathbf{e})\mathbf{u} + \sin\theta(\mathbf{e} \times \mathbf{u})\big) \cdot$$ $$= \big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e} \times \mathbf{u})\big)^2$$ $$= (\mathbf{u}\cos\theta) \cdot \big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e} \times \mathbf{u})\big)$$ $$+x\mathbf{e} \cdot \big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e} \times \mathbf{u})\big)(1-\cos\theta)$$ $$+(\mathbf{e} \times \mathbf{u}) \cdot \big(\mathbf{u}\cos\theta + (1-\cos\theta)x\mathbf{e} + \sin\theta(\mathbf{e} \times \mathbf{u})\big)\sin\theta$$ $$= \mathbf{u}^2\cos^2\theta + 2(\cos\theta - \cos^2\theta)x^2 + 2(\mathbf{e} \times \mathbf{u} \cdot \mathbf{u})\sin\theta\cos\theta + (1-\cos\theta)^2x^2 + 2x(\mathbf{e} \times \mathbf{u} \cdot \mathbf{e})(1-\cos\theta)\sin\theta + \sin^2\theta\,(\mathbf{e} \times \mathbf{u})^2$$ $$= \mathbf{u}^2\cos^2\theta + 2(\cos\theta - \cos^2\theta)\mathrm{x}^2 + 2(\mathbf{e} \times \mathbf{u} \cdot \mathbf{u})\sin\theta\cos\theta + (1-\cos\theta)^2\mathrm{x}^2 + 2x(\mathbf{e} \times \mathbf{u} \cdot \mathbf{e})(1-\cos\theta)\sin\theta + \sin^2\theta\,(\mathbf{u}^2 - x^2)$$ $$= \mathbf{u}^2(\cos^2\theta + \sin^2\theta) + x^2[2(\cos\theta - \cos^2\theta) + (1-\cos\theta)^2 - \sin^2\theta] = \mathbf{u}^2$$ As the term in square brackets vanish when expanded.

2.90	If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{I} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Show that for arbitrary $0 < \alpha, \beta \leq 2\pi$, that $\mathbf{Q}(\alpha+\beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$.

It is convenient to write $\mathbf{Q}(\alpha)$ and $\mathbf{Q}(\beta)$ in terms of their i, j components; we assume that the unit vector $\mathbf{e} = (x_1, x_2, x_3)$:

$$[\boldsymbol{Q}(\alpha)]_{ij} = \cos\alpha\, \delta_{ij} + (1-\cos\alpha)x_i x_j - \sin\alpha\, e_{ijk}x_k$$

Consequently, we can write for the product $\mathbf{Q}(\alpha)\mathbf{Q}(\beta)$,

$$[\boldsymbol{Q}(\alpha)\boldsymbol{Q}(\beta)]_{ij} = [\boldsymbol{Q}(\alpha)]_{ik}[\boldsymbol{Q}(\beta)]_{kj} =$$

$$= [\cos\alpha\,\delta_{ik} + (1-\cos\alpha)x_i x_k - \sin\alpha\,\epsilon_{ikl}x_l]\big[\cos\beta\,\delta_{kj} + (1-\cos\beta)x_k x_j - \sin\beta\,\epsilon_{kjn}x_n\big]$$

$$= \cos\alpha\cos\beta\,\delta_{ik}\delta_{kj} + \cos\alpha\,(1-\cos\beta)\delta_{ik}x_k x_j - \cos\alpha\sin\beta\,\delta_{ik}\epsilon_{kjn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_k\delta_{kj} + (1-\cos\alpha)(1-\cos\beta)x_i x_k^2 x_j - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} - \sin\alpha\cos\beta\,\epsilon_{ikl}x_l\delta_{kj} - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l$$

$$= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_j + (1-\cos\alpha)(1-\cos\beta)x_i x_j - (1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn} - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l - \sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j + \sin\alpha\sin\beta\,\epsilon_{ikl}\epsilon_{kjn}x_n x_l$$

$$= \cos\alpha\cos\beta\,\delta_{ij} + \cos\alpha\,(1-\cos\beta)x_i x_j - \cos\alpha\sin\beta\,\epsilon_{ijn}x_n + (1-\cos\alpha)\cos\beta\,x_i x_j + (1-\cos\alpha)(1-\cos\beta)x_i x_j - \boxed{(1-\cos\alpha)\sin\beta\,x_i x_k x_n\epsilon_{kjn}} - \sin\alpha\cos\beta\,\epsilon_{ijl}x_l - \boxed{\sin\alpha\,(1-\cos\beta)\epsilon_{ikl}x_l x_k x_j} + \sin\alpha\sin\beta\,\big(\delta_{lj}\delta_{in} - \delta_{ln}\delta_{ji}\big)x_n x_l$$

$$= (\cos\alpha\cos\beta - \sin\alpha\sin\beta)\delta_{ij} + [1 - (\cos\alpha\cos\beta - \sin\alpha\sin\beta)]x_i x_j - [(\cos\alpha\sin\beta - \sin\alpha\cos\beta)]\epsilon_{ijn}x_n$$

$$= [\mathbf{Q}(\alpha+\beta)]_{ij}$$

With the boxed terms vanishing on account of antisymmetric contraction with symmetric object.

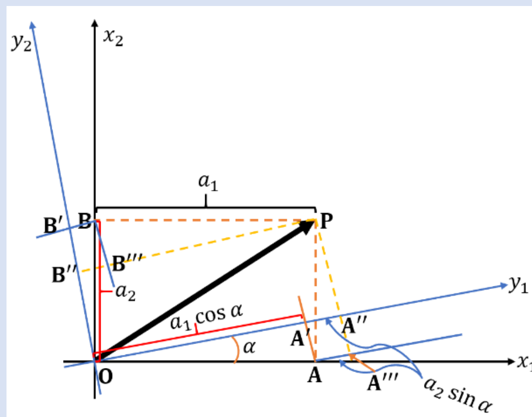
2.91	If for an arbitrary unit vector $\mathbf{e}$, the tensor, $\mathbf{Q}(\theta) = \cos(\theta)\mathbf{I} + (1 - \cos(\theta))\mathbf{e} \otimes \mathbf{e} + \sin(\theta)(\mathbf{e}\times)$ where $(\mathbf{e}\times)$ is the vector cross of $\mathbf{e}$. Show that . $\mathbf{Q}(\theta)$ is a periodic tensor function with period 2π. [Hint: $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$
	Since $\mathbf{Q}(\alpha + \beta) = \mathbf{Q}(\alpha)\mathbf{Q}(\beta)$ we can write that $\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi)$. But a direct substitution shows that, $\mathbf{Q}(0) = \mathbf{Q}(2\pi) = \mathbf{I}$. We therefore have that, $\mathbf{Q}(\alpha + 2\pi) = \mathbf{Q}(\alpha)\mathbf{Q}(2\pi) = \mathbf{Q}(\alpha)$ which completes the proof. The above results show that $\mathbf{Q}(\alpha)$ is a rotation along the unit vector $\mathbf{e}$ through an angle α.
2.92	Given that $\mathbf{Q}$ is an orthogonal tensor, show that the principal invariants of a tensor S satisfy $I_k(\mathbf{QSQ}^\mathrm{T}) = I_k(\mathbf{S}), k = 1,2, \text{or } 3$, that is, Rotations and orthogonal transformations do not change the Invariants.
	$$\begin{aligned} I_1(\mathbf{QSQ}^\mathrm{T}) &= \mathrm{tr}(\mathbf{QSQ}^\mathrm{T}) \\ &= \mathrm{tr}(\mathbf{Q}^\mathrm{T}\mathbf{QS}) = \mathrm{tr}(\mathbf{S}) \\ &= I_1(\mathbf{S}) \\ I_2(\mathbf{QSQ}^\mathrm{T}) &= \frac{1}{2}[\mathrm{tr}^2(\mathbf{QSQ}^\mathrm{T}) - \mathrm{tr}(\mathbf{QSQ}^\mathrm{T}\mathbf{QSQ}^\mathrm{T})] \\ &= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{QS^2Q}^\mathrm{T})] \\ &= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{Q}^\mathrm{T}\mathbf{QS}^2)] \\ &= \frac{1}{2}[I_1^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)] = I_2(\mathbf{S}) \\ I_3(\mathbf{QSQ}^\mathrm{T}) &= \det(\mathbf{QSQ}^\mathrm{T}) \\ &= \det(\mathbf{Q}^\mathrm{T}\mathbf{QS}) = \det(\mathbf{S}) \\ &= I_3(\mathbf{S}) \end{aligned}$$ Hence $I_k(\mathbf{QSQ}^\mathrm{T}) = I_k(\mathbf{S}),\ k = 1,2,\ or\ 3$
2.93	Show that if $\mathbf{\Omega}$ is skew, its axial vector $\boldsymbol{\omega}$ is such that $\lvert\mathbf{\Omega}\rvert^2 = 2\boldsymbol{\omega}^2$

	$$\begin{aligned}\lvert\boldsymbol{\Omega}\rvert^2 = \boldsymbol{\Omega}:\boldsymbol{\Omega} &= (\boldsymbol{\omega}\times):(\boldsymbol{\omega}\times)\\ &= \left(e_{ikj}\omega_k\mathbf{e}_i\times\mathbf{e}_j\right):\left(e_{\alpha\gamma\beta}\omega_\gamma\mathbf{e}_\alpha\times\mathbf{e}_{\boldsymbol{\beta}}\right)\\ &= e_{ikj}\omega_k e_{\alpha\gamma\beta}\omega_\gamma\delta_{i\alpha}\delta_{j\beta}\\ &= 2\delta_{k\gamma}\omega_k\omega_\gamma = 2\boldsymbol{\omega}^2\end{aligned}$$
2.94	For an arbitrary tensor $\mathbf{u}$, the vector cross is given as, $\mathbf{u}\times$. Use the result $\det(\mathbf{S}+\mathbf{T}) = \det(\mathbf{S}) + \mathrm{tr}(\mathbf{T}^c\mathbf{S}^T) + \mathrm{tr}(\mathbf{S}^c\mathbf{T}^T) + \det(\mathbf{T})$ to show that $\det(\mathbf{S}+\mathbf{u}\times) = \det\mathbf{S}\,(1+(\mathbf{u}\times)\mathbf{S}^{-1}) + (\mathbf{u}\otimes\mathbf{u}):\mathbf{S}$.
	$\mathbf{T}\to\mathbf{u}\times$, then $\mathbf{T}^c = \mathbf{u}\otimes\mathbf{u}$ and $\mathbf{T}^T = -\mathbf{u}\times$ and $\det(\mathbf{u}\times) = 0$. $$\begin{aligned}\det(\mathbf{S}+\mathbf{u}\times) &= \det\mathbf{S} + \mathrm{tr}\left((\mathbf{u}\otimes\mathbf{u})\mathbf{S}^T\right) - \mathrm{tr}\left(\mathbf{S}^c(\mathbf{u}\times)\right)\\ &= \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{u}):\mathbf{S}^T + (\mathbf{u}\times):\mathbf{S}^c\end{aligned}$$
2.95	Use the result $\det(\mathbf{S}+\mathbf{u}\times) = \det\mathbf{S} + (\mathbf{u}\otimes\mathbf{u}):\mathbf{S}^T + (\mathbf{u}\times):\mathbf{S}^c$ to show that for a skew tensor, $\det(\mathbf{I}+\boldsymbol{\Omega}) = 1+\frac{1}{2}\lvert\boldsymbol{\Omega}\rvert^2$
	Note that for any skew tensor, $\boldsymbol{\Omega}$ and its axial vector $\mathbf{u}$ $$\lvert\boldsymbol{\Omega}\rvert^2 = 2\mathbf{u}^2$$ In the given result, let $\mathbf{S}=\mathbf{I}$, and let $\mathbf{u}$ be the axial vector of $\boldsymbol{\Omega}$. Then, $$\det(\mathbf{I}+\boldsymbol{\Omega}) = \det\mathbf{I} + (\mathbf{u}\otimes\mathbf{u}):\mathbf{I} + \boldsymbol{\Omega}:\mathbf{I}$$ As the identity tensor is both self-transpose and self-cofactor. Simplifying, $$\begin{aligned}\det(\mathbf{I}+\boldsymbol{\Omega}) &= 1 + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}) + \mathrm{tr}\,\boldsymbol{\Omega}\\ &= 1 + \mathbf{u}\cdot\mathbf{u}\\ &= 1+\frac{1}{2}\lvert\boldsymbol{\Omega}\rvert^2\end{aligned}$$ Using the fact that a skew tensor is necessarily traceless.
2.96	Show that operating with the transpose of a rotation gives the coordinates of a fixed vector in rotated coordinates.

In the figure shown, Let the original coordinates be $\mathbf{O}\, x_1\, x_2\, x_3$ and imagine that we are leaving the vector $\mathbf{OP}$ which is presented as $\mathbf{v} = a_i \mathbf{e}_i$ where $\mathbf{e}_1$, $\mathbf{e}_2$and $\mathbf{e}_3$ are unit vectors along $\mathbf{O}\, x_1\, x_2\, x_3$ If the coordinates are

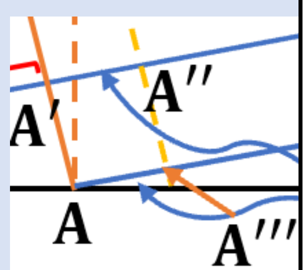
rotated to $\mathbf{O}\, y_1\, y_2\, y_3$ such that the same vector now becomes $\mathbf{v} = b_i \boldsymbol{\xi}_i$ where $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$and $\boldsymbol{\xi}_3$ are unit vectors along the $\mathbf{O}\, y_1\, y_2\, y_3$ system. These will be the new coordinates after the rotation of coordinates to this point.

Clearly, $\mathbf{OA} = a_1$ and $\mathbf{OB} = a_2$. We need to find the lengths , $\mathbf{OA}'' = b_1$ and $\mathbf{OB}'' = b_2$. We drop perpendicular lines to the lines $\mathbf{O}\, y_1$ and $\mathbf{O}\, y_2$ meeting them at $\mathbf{A}''$ and $\mathbf{B}''$respectively. It is clear that $\mathbf{OA}' = a_1 \cos\alpha$. Furthermore, $\mathbf{AA}''' = a_2 \sin\alpha$ because $\mathbf{PA}$ is the hypotenuse of a right angled triangle $\mathbf{APA}'''$with angle α at $\mathbf{APA}'''$ And it is easy to see that $\mathbf{AA}'\mathbf{A}''\mathbf{A}'''$ is a rectangle. Its opposite sides are equal, consequently, the length



$$\mathbf{OA}'' = b_1 = a_1 \cos\alpha + a_2 \sin\alpha.$$
$$= a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)$$

By the same arguments, noting that $\mathbf{BB}'\mathbf{B}''\mathbf{B}'''$ is also a rectangle. If we rotate from the coordinates $\mathbf{O}\, x_1\, x_2\, x_3$ to $\mathbf{O}\, y_1\, y_2\, y_3$, the rotation vector is:$\mathbf{R} = \boldsymbol{\xi}_i \otimes \mathbf{e}_i$. We take the transpose of this tensor and writing the unit vector with a prime because we are actually moving $\mathbf{O}\, x_1\, x_2\, x_3$ to $\mathbf{O}\, y_1\, y_2\, y_3$ while keeping the vector $\mathbf{v}$ unchanged. Hence, we have:



$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = (\mathbf{e}_j \otimes \boldsymbol{\xi}_j) a_i \mathbf{e}_i$$
$$= a_i \mathbf{e}_j (\boldsymbol{\xi}_j \cdot \mathbf{e}_i)$$

Expanding for this two-dimensional case, we have:

$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = \mathbf{e}_1\big(a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)\big) + \mathbf{e}_2\big(a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2)\big)$$

as expectedt.

2.97	Suppose that $\mathbf{U}$ and $\mathbf{C}$ are symmetric, positive-definite tensors with $\mathbf{U}^2 = \mathbf{C}$, write the invariants of $\mathbf{C}$ in terms of $\mathbf{U}$ $$I_1(\mathbf{C}) = \operatorname{tr}(\mathbf{U}^2) = I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})$$

By the Cayley-Hamilton theorem,

$$\mathbf{U}^3 - \mathrm{I}_1\mathbf{U}^2 + \mathrm{I}_2\mathbf{U} - \mathrm{I}_3\mathbf{I} = \mathbf{0}$$

which contracted with $\mathbf{U}$ gives,

$$\mathbf{U}^4 - \mathrm{I}_1\mathbf{U}^3 + \mathrm{I}_2\mathbf{U}^2 - \mathrm{I}_3\mathbf{U} = \mathbf{0}$$

so that,

$$\mathbf{U}^4 = \mathrm{I}_1\mathbf{U}^3 - \mathrm{I}_2\mathbf{U}^2 + \mathrm{I}_3\mathbf{U}$$

and

$$\begin{aligned}
\operatorname{tr}(\mathbf{U}^4) &= I_1\operatorname{tr}(\mathbf{U}^3) - I_2\operatorname{tr}(\mathbf{U}^2) + I_3\operatorname{tr}(\mathbf{U}) \\
&= I_1(\mathbf{U})\big(I_1^3(\mathbf{U}) - 3I_1(\mathbf{U})I_2(\mathbf{U}) + 3I_3(\mathbf{U})\big) - I_2(\mathbf{U})\big(I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})\big) \\
&\qquad + I_1(\mathbf{U})I_3(\mathbf{U}) \\
&= I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U}) + 4I_1(\mathbf{U})I_3(\mathbf{U}) + 2I_2^2(\mathbf{U})
\end{aligned}$$

But,

$$\begin{aligned}
I_2(\mathbf{C}) &= \frac{1}{2}[I_1^2(\mathbf{C}) - \operatorname{tr}(\mathbf{C}^2)] = \frac{1}{2}[I_1^2(\mathbf{U}^2) - \operatorname{tr}(\mathbf{U}^4)] = \frac{1}{2}[\operatorname{tr}^2(\mathbf{U}^2) - \operatorname{tr}(\mathbf{U}^4)] \\
&= \frac{1}{2}\Big[\big(I_1^2(\mathbf{U}) - 2I_2(\mathbf{U})\big)^2 - \operatorname{tr}(\mathbf{U}^4)\Big] \\
&= \frac{1}{2}\Big[\boxed{I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U})} + 4I_2^2(\mathbf{U}) \\
&\qquad - \Big(\boxed{I_1^4(\mathbf{U}) - 4I_1^2(\mathbf{U})I_2(\mathbf{U})} + 4I_1(\mathbf{U})I_3(\mathbf{U}) + 2I_2^2(\mathbf{U})\Big)\Big]
\end{aligned}$$

The boxed items cancel out so that,

$$I_2(\mathbf{C}) = I_2^2(\mathbf{U}) - 2I_1(\mathbf{U})I_3(\mathbf{U})$$

as required.

$$I_3(\mathbf{C}) = \det(\mathbf{C}) = \det(\mathbf{U}^2) = (\det(\mathbf{U}))^2 = I_3^2(\mathbf{U})$$

2.98	A tensor function $\mathbf{\Phi}(\mathbf{A})$ in the domain A $(ie, \mathbf{A} \in A)$ is said to be invariant under G if for every $\mathbf{Q} \in \mathrm{G}$, $\mathbf{\Phi}(\mathbf{Q}\mathbf{A}\mathbf{Q}^\mathrm{T}) = \mathbf{Q}\mathbf{\Phi}(\mathbf{A})\mathbf{Q}^\mathrm{T}$. Show

	that if $\mathbf{\Phi}_1(\mathbf{A})$ and $\mathbf{\Phi}_2(\mathbf{A})$ are both invariant under G, then the product function $\mathbf{\Phi}_1(\mathbf{A})\mathbf{\Phi}_2(\mathbf{A})$ is also invariant under G.
	$\mathbf{\Phi}_1(\mathbf{A})$ and $\mathbf{\Phi}_2(\mathbf{A})$ are both invariant under G, therefore, $\mathbf{\Phi}_1(\mathbf{QAQ}^{\mathrm{T}}) = \mathbf{Q\Phi}_1(\mathbf{A})\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{\Phi}_2(\mathbf{QAQ}^{\mathrm{T}}) = \mathbf{Q\Phi}_2(\mathbf{A})\mathbf{Q}^{\mathrm{T}}$. Clearly, $$\mathbf{\Phi}_1(\mathbf{QAQ}^{\mathrm{T}})\mathbf{\Phi}_2(\mathbf{QAQ}^{\mathrm{T}}) = \mathbf{Q\Phi}_1(\mathbf{A})\mathbf{Q}^{\mathrm{T}}\,\mathbf{Q\Phi}_2(\mathbf{A})\mathbf{Q}^{\mathrm{T}}$$ $$= \mathbf{Q\Phi}_1(\mathbf{A})\mathbf{\Phi}_2(\mathbf{A})\mathbf{Q}^{\mathrm{T}}$$ Which obviously shows that $\mathbf{\Phi}_1(\mathbf{A})\mathbf{\Phi}_2(\mathbf{A})$ satisfies the conditions for invariance under G.
2.99	Define Psym as the set of all symmetric, positive definite tensors. Show that Psym is invariant under $\mathbb{G}$ where is the proper orthogonal group of all rotations, in the sense that for any tensor $\mathrm{A} \in \mathrm{Psym}$ $\mathbf{Q} \in \mathrm{G}\mathbb{G} \Rightarrow \mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Psym}$. (G285)
	Since we are given that $\mathbf{A} \in \mathrm{Psym}$, it means its characteristic equation has roots that are all positive. This equation can be written as $$\lvert\mathbf{A} - \lambda\mathbf{I}\rvert = 0$$ The eigenvalues are the roots of the above equation. We now try to find the characteristic equation of the tensor $\mathbf{QAQ}^{\mathrm{T}}$. Following the above equation, if α is an eigenvalue of $\mathbf{QAQ}^{\mathrm{T}}$, then, $$\lvert\mathbf{QAQ}^{\mathrm{T}} - \alpha\mathbf{I}\rvert = \lvert\mathbf{QAQ}^{\mathrm{T}} - \alpha\mathbf{QIQ}^{\mathrm{T}}\rvert$$ $$= \lvert\mathbf{Q}(\mathbf{A} - \alpha\mathbf{I})\mathbf{Q}^{\mathrm{T}}\rvert$$ $$= \det(\mathbf{Q}) \times \det(\mathbf{A} - \alpha\mathbf{I}) \times \det(\mathbf{Q}^{\mathrm{T}})$$ $$= \det(\mathbf{A} - \alpha\mathbf{I}) = 0.$$ Clearly, $\mathbf{QAQ}^{\mathrm{T}}$ has the same characteristic equations as $\mathbf{A}$ and hence they have the same eigenvalues. Since $\mathbf{A} \in \mathrm{Psym}$ we have reached the same conclusion that $\mathbf{QAQ}^{\mathrm{T}} \in \mathrm{Psym}$.

2.100	

Higher-Order Tensors

Triads and tetrads define tensors of order three and four. This section formalizes certain issues on tensors of orders higher than two – beyond their basis tensors. Apart from tensors obtained from direct products of first and second order tensors, or from spatial derivatives (Frechet derivatives) perhaps the only important tensor of the third order is the alternating tensor we have accompanied in one way or another, from the d=beginning of the book. They were also useful in defining the curls of vectors and tensors.

Fourth Order Tensors

Definition

> *A fourth-order tensor is a linear transformation of a second-order tensor to a second-order tensor.*

End of Definition

Given a second order tensor $\mathbf{A}$, the transformation,

$$\mathbb{T}\mathbf{A} = \mathbf{B}$$

Such that $\mathbf{B}$ is also a second order tensor makes $\mathbb{T}$ a fourth-order tensor provided the transformation is linear; that is, for $\alpha, \beta \in \mathbb{R}$, and $\mathbf{A}$,$\mathbf{B} \in \mathbb{L}$,

$$\mathbb{T}(\alpha\mathbf{A} + \beta\mathbf{B}) = \alpha\mathbb{T}\mathbf{A} + \beta\mathbb{T}\mathbf{B}.$$

We can form second-order bases for fourth-order tensors like the bases for second-order tensors. In order to do that, we define covariant and contravariant second-order bases as follows: From the base vectors $\mathbf{g}^i$ and $\mathbf{g}^j$ we define the contravariant tensor bases, $\mathbf{G}^{ij} \equiv \mathbf{g}^i \otimes \mathbf{g}^j$. The covariant tensor bases are similarly defined from the covariant base vectors so that, $\mathbf{G}_{ij} \equiv \mathbf{g}_i \otimes \mathbf{g}_j$.

It is also necessary to define the behavior of familiar products as they apply to second order bases and tensors. Following the definition of the dyad product from its interactions with vectors, we introduce two products for second order tensors that create fourth order tensors: The **Dyadic Product**, $\otimes$**,** and the **Squared Times** product, $\boxtimes$**.** Define the dyadic and squared times products of tensors as, $(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B}:\mathbf{C})\mathbf{A}$ and $(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}}$ We proceed to show that $(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D}) = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}} \otimes \mathbf{D}$, for,

$$\begin{aligned}(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D})\mathbf{E} &= (\mathbf{A} \boxtimes \mathbf{B})(\mathbf{D}:\mathbf{E})\mathbf{C} \\ &= (\mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}})\mathbf{D}:\mathbf{E} \\ &= (\mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}} \otimes \mathbf{D})\mathbf{E}\end{aligned}$$

so that

$$(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{C} \otimes \mathbf{D}) = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}} \otimes \mathbf{D}$$

The following examples define the fourth-order identity tensors, Identity, symmetrizer, transposer and the anti-symmetrizer.

1. Define the dyadic and squared times products of tensors as, $(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B}:\mathbf{C})\mathbf{A}$ and $(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}}$ For vectors $\mathbf{a}$, $\mathbf{b}$ and tensors $\mathbf{A}, \mathbf{B}$ show that $(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{a} \otimes \mathbf{b}) = \mathbf{A}\mathbf{a} \otimes \mathbf{B}\mathbf{b}$.

$$(\mathbf{A} \boxtimes \mathbf{B})\,(\mathbf{a} \otimes \mathbf{b}) = \mathbf{A}(\mathbf{a} \otimes \mathbf{b})\mathbf{B}^{\mathrm{T}} = \mathbf{A}\mathbf{a} \otimes \mathbf{B}\mathbf{b}$$

2. Define the dyadic and squared times products of tensors as, $(\mathbf{A} \otimes \mathbf{B})\mathbf{C} = (\mathbf{B}:\mathbf{C})\mathbf{A}$ and $(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}}$ For vectors $\mathbf{a}$, $\mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ show that $(\mathbf{a} \otimes \mathbf{b}) \boxtimes (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \otimes \mathbf{c}) \otimes (\mathbf{b} \otimes \mathbf{d})$

For a tensor $\mathbf{E}$,

$$\begin{aligned}\big((\mathbf{a} \otimes \mathbf{b}) \boxtimes (\mathbf{c} \otimes \mathbf{d})\big)\mathbf{E} &= (\mathbf{a} \otimes \mathbf{b})\mathbf{E}(\mathbf{d} \otimes \mathbf{c}) \\ &= (\mathbf{a} \otimes \mathbf{c})[(\mathbf{E}^{\mathrm{T}}\mathbf{b}) \cdot \mathbf{d}] \\ &= (\mathbf{a} \otimes \mathbf{c})\,\mathrm{tr}\big((\mathbf{d} \otimes \mathbf{b})\mathbf{E}\big) \\ &= (\mathbf{a} \otimes \mathbf{c})[(\mathbf{b} \otimes \mathbf{d}):\mathbf{E}] \\ &= \big((\mathbf{a} \otimes \mathbf{c}) \otimes (\mathbf{b} \otimes \mathbf{d})\big)\mathbf{E}\end{aligned}$$

so that $(\mathbf{a} \otimes \mathbf{b}) \boxtimes (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \otimes \mathbf{c}) \otimes (\mathbf{b} \otimes \mathbf{d})$.

3. Define the tensor basis $\mathbf{G}^{ij} \equiv \mathbf{g}^i \otimes \mathbf{g}^j$, observe that unlike the scalar component g_{ij}, the tensor $\mathbf{G}^{ij}$ is not symmetrical in its indices; furthermore, show that $\mathbb{I} \equiv g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}$ is the fourth order identity tensor.

By the definition of $\mathbf{G}^{ij} \equiv \mathbf{g}^i \otimes \mathbf{g}^j$, It is immediately clear that $\mathbf{G}^{ij} \equiv \left[\mathbf{G}^{ji}\right]^T$. It is therefore not symmetric in its components. We further observe that $g_{ij}\mathbf{G}^{ij}$ is the component representation of the second-order unit tensor.

Lastly, $\mathbb{I}$ is the fourth-order identity tensor. This is evident because, given any second-order tensor $\mathbf{T}$, $\mathbb{I}\mathbf{T} = \mathbf{T}$. To show this to be true, take any component representation of $\mathbf{T}$ and expand $\mathbb{I}\boldsymbol{T}$:

$$\begin{aligned}
\mathbb{I}\mathbf{T} &= \left(g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}\right)\mathbf{T} \\
&= \left(g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}\right)T_{kl}\mathbf{g}^i \otimes \mathbf{g}^j \\
&= \left(g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta}\right)T_{kl}\mathbf{G}^{kl} \\
&= g_{ij}g_{\alpha\beta}T_{kl}\left(\mathbf{G}^{ij} \boxtimes \mathbf{G}^{\alpha\beta}\right)\mathbf{G}^{kl} \\
&= g_{ij}g_{\alpha\beta}T_{kl}\mathbf{G}^{ij}\mathbf{G}^{kl}\mathbf{G}^{\beta\alpha} \\
&= g_{ij}g_{\alpha\beta}T_{kl}g^{jk}g^{l\beta}\mathbf{G}^{i\alpha} \\
&= \delta_i^k\delta_\beta^l T_{kl}\mathbf{G}^{i\alpha} = T_{i\alpha}\mathbf{G}^{i\alpha} \\
&= \mathbf{T}
\end{aligned}$$

Showing that, $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$

4. Given that $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$, show that, $\mathbb{I} = g_{ij}g_{kl}\mathbf{G}^{ij} \boxtimes \mathbf{G}^{kl} = g_{ik}g_{jl}\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}$

The first expression is recognizable as $\mathbf{I} \boxtimes \mathbf{I}$ since

$$\begin{aligned}
\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I} &= g_{ij}\mathbf{G}^{ij} \boxtimes g_{\alpha\beta}\mathbf{G}^{\alpha\beta} \\
&= g_{ij}g_{\alpha\beta}\mathbf{G}^{ij} \boxtimes \mathbf{G}^{\alpha\beta}
\end{aligned}$$

Let us see how the second expression operates on a second-order tensor:

$$\begin{aligned}
g_{ik}g_{jl}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\mathbf{T} &= g_{ik}g_{jl}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)T_{\alpha\beta}\mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= g_{ik}g_{jl}T_{\alpha\beta}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\mathbf{g}^\alpha \otimes \mathbf{g}^\beta \\
&= g_{ik}g_{jl}T_{\alpha\beta}\mathbf{G}^{ij}\left(\mathbf{G}^{kl} : \left(\mathbf{g}^\alpha \otimes \mathbf{g}^\beta\right)\right) \\
&= g_{ik}g_{jl}T_{\alpha\beta}\mathbf{G}^{ij}g^{k\alpha}g^{l\beta} = \delta_i^\alpha\delta_j^\beta T_{\alpha\beta}\mathbf{G}^{ij}
\end{aligned}$$

$$= T_{ij}\mathbf{G}^{ij} = \mathbf{T}$$

confirming that $g_{ik}g_{jl}\mathbf{G}^{ij} \otimes \mathbf{G}^{kl} = \mathbb{I} = g_{ik}g_{jl}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}^k \otimes \mathbf{g}^l\right)$.

5. For a second-order tensor $\mathbf{A}$ show that $\mathbf{A}\mathbb{I} = \mathbb{I}\mathbf{A} = \mathbf{A}$ where $\mathbb{I}$ is the fourth-order unit tensor.

Note that $\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I}$. Therefore, $\mathbf{A}\mathbb{I} = \mathbf{A}(\mathbf{I} \boxtimes \mathbf{I}) = \mathbf{I}^{\mathrm{T}}\mathbf{A}\mathbf{I} = \mathbf{A}$. Similarly, $\mathbb{I}\mathbf{A} = (\mathbf{I} \boxtimes \mathbf{I})\mathbf{A} = \mathbf{I}\mathbf{A}\mathbf{I}^{\mathrm{T}} = \mathbf{A}$ since the identity tensor is symmetric and hence self-transpose.

6. The transposer tensor $\mathbb{T}$ turns a second-order tensor into its transpose: $\mathbb{T}\mathbf{S} = \mathbf{S}^{\mathrm{T}} = \mathbf{S}\mathbb{T}$; show that $\mathbb{T} = g_{il}g_{jk}\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}$

$$\begin{aligned}
\mathbb{T}\mathbf{S} &= g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\mathbf{S} \\
&= g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\left(S^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{ij}\left(\mathbf{G}^{kl}:\left(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{ij}\left(\mathbf{g}^k \cdot \mathbf{g}_\beta\right)\left(\mathbf{g}^l \cdot \mathbf{g}_\alpha\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{ij}\delta_\beta^k\delta_\alpha^l = S_{ji}\mathbf{G}^{ij} \\
&= \mathbf{S}^{\mathrm{T}} \\
\mathbf{S}\mathbb{T} &= \mathbf{S}g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) \\
&= \left(S^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right)g_{il}g_{jk}\left(\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\left(\left(\mathbf{g}_\alpha \otimes \mathbf{g}_\beta\right):\mathbf{G}^{ij}\right)\mathbf{G}^{kl} \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{kl}\left(\mathbf{g}^i \cdot \mathbf{g}_\alpha\right)\left(\mathbf{g}^j \cdot \mathbf{g}_\beta\right) \\
&= g_{il}g_{jk}S^{\alpha\beta}\mathbf{G}^{kl}\delta_\alpha^i\delta_\beta^j = S^{ij}\mathbf{G}_{ji} \\
&= \mathbf{S}^{\mathrm{T}}
\end{aligned}$$

7. Define the symmetrizer, $\mathbb{S}$ and anti symmetrizer, $\mathbb{W}$ tensors as fourth order tensors that return the symmetric and antisymmetric parts of a second-order tensor; show that $\mathbb{S} = \frac{1}{2}(\mathbb{I} + \mathbb{T})$ and $\mathbb{W} = \frac{1}{2}(\mathbb{I} - \mathbb{T})$.

Consider a tensor $\mathbf{A}$.

$$\mathbb{S}\mathbf{A} = \frac{1}{2}(\mathbb{I} + \mathbb{T})\mathbf{A} = \frac{1}{2}(\mathbb{I}\mathbf{A} + \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}}) = \operatorname{sym}\mathbf{A}$$

Similarly,

$$\mathbb{W}\mathbf{A} = \frac{1}{2}(\mathbb{I} - \mathbb{T})\mathbf{A} = \frac{1}{2}(\mathbb{I}\mathbf{A} - \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \operatorname{skw}\mathbf{A}$$

8. For any second-order tensor $\mathbf{A}$ Show that $\mathbb{S}\mathbf{A} = \mathbf{A}\mathbb{S}$, and that $\mathbb{W}\mathbf{A} = \mathbf{A}\mathbb{W}$ where $\mathbb{S}$ is the fourth-order symmetrizer tensor. [Hint: $\mathbf{A}\mathbb{I} = \mathbb{I}\mathbf{A}, \mathbb{T}\mathbf{S} = \mathbf{S}\mathbb{T}$]

Consider a tensor $\mathbf{A}$.

$$\begin{aligned}\mathbb{S}\mathbf{A} &= \frac{1}{2}(\mathbb{I} + \mathbb{T})\mathbf{A}\\ &= \frac{1}{2}(\mathbb{I}\mathbf{A} + \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^{\mathrm{T}})\\ &= \operatorname{sym}\mathbf{A}\end{aligned}$$

$$\begin{aligned}\mathbf{A}\mathbb{S} &= \mathbf{A}\left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\\ &= \frac{1}{2}(\mathbf{A}\mathbb{I} + \mathbf{A}\mathbb{T})\\ &= \frac{1}{2}(\mathbb{I}\mathbf{A} + \mathbb{T}\mathbf{A}) = \operatorname{sym}\mathbf{A}\end{aligned}$$

so that $\mathbb{S}\mathbf{A} = \mathbf{A}\mathbb{S} = \operatorname{sym}\mathbf{A}$. Similarly,

$$\begin{aligned}\mathbb{W}\mathbf{A} = \frac{1}{2}(\mathbb{I} - \mathbb{T})\mathbf{A} &= \frac{1}{2}(\mathbb{I}\mathbf{A} - \mathbb{T}\mathbf{A})\\ &= \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}}) = \operatorname{skw}\mathbf{A}\end{aligned}$$

$$\begin{aligned}\mathbf{A}\mathbb{W} = \mathbf{A}\left(\frac{1}{2}(\mathbb{I} - \mathbb{T})\right) &= \frac{1}{2}(\mathbf{A}\mathbb{I} - \mathbf{A}\mathbb{T})\\ &= \frac{1}{2}(\mathbb{I}\mathbf{A} - \mathbb{T}\mathbf{A}) = \frac{1}{2}(\mathbf{A} - \mathbf{A}^{\mathrm{T}})\\ &= \operatorname{skw}\mathbf{A}\end{aligned}$$

showing that $\mathbb{W}\mathbf{A} = \mathbf{A}\mathbb{W} = \operatorname{skw}\mathbf{A}$

9. For the fourth order tensors $\mathbb{S}, \mathbb{T}$, and $\mathbb{W}$ show that (a) $\mathbb{T}\mathbb{T} = \mathbb{I}$, (b) $\mathbb{T}\mathbb{S} = \mathbb{S}\mathbb{T}$, (c) $\mathbb{S}\mathbb{S} = \mathbb{S}$ (d) $\mathbb{W}\mathbb{W} = \mathbb{W}$ and (e) $\mathbb{S}\mathbb{W} = \mathbb{W}\mathbb{S} = \mathbb{O}$.

(a) An indicial proof $\mathbb{T}\mathbb{T} = \mathbb{I}$ is straightforward. A direct proof is however more illuminating:

Consider the double transpose:

$$\mathbb{T}\mathbb{T}\mathbf{A} = \mathbb{T}\mathbf{A}^{\mathrm{T}} = (\mathbf{A}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{A} = \mathbb{I}\mathbf{A}$$

showing clearly that $\mathbb{T}\mathbb{T} = \mathbb{I}$.

(b)

$$\mathbb{T}\mathbb{S} = \mathbb{T}\left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right) = \frac{1}{2}(\mathbb{T}\mathbb{I} + \mathbb{T}\mathbb{T}) = \frac{1}{2}(\mathbb{T} + \mathbb{I}) = \mathbb{S}$$

$$\mathbb{S}\mathbb{T} = \left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\mathbb{T} = \frac{1}{2}(\mathbb{I}\mathbb{T} + \mathbb{T}\mathbb{T}) = \frac{1}{2}(\mathbb{T} + \mathbb{I}) = \mathbb{S}$$

so that $\mathbb{T}\mathbb{S} = \mathbb{S}\mathbb{T} = \mathbb{S}$

(c) For a second-order tensor $\mathbf{A}$

$$\begin{aligned}\mathbb{S}\mathbb{S}\mathbf{A} &= \mathbb{S}(\operatorname{sym}\mathbf{A})\\ &= \left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\operatorname{sym}\mathbf{A}\\ &= \frac{1}{2}\operatorname{sym}\mathbf{A} + \frac{1}{2}\operatorname{sym}\mathbf{A}\\ &= \operatorname{sym}\mathbf{A} = \mathbb{S}\mathbf{A}\end{aligned}$$

so that $\mathbb{S}\mathbb{S} = \mathbb{S}$.

(d) For a second-order tensor $\boldsymbol{A}$

$$\begin{aligned}\mathbb{W}\mathbb{W}\mathbf{A} &= \mathbb{W}(\operatorname{skw}\mathbf{A})\\ &= \left(\frac{1}{2}(\mathbb{I} - \mathbb{T})\right)\operatorname{skw}\mathbf{A}\\ &= \frac{1}{2}\operatorname{skw}\mathbf{A} + \frac{1}{2}\operatorname{skw}\mathbf{A}\\ &= \operatorname{skw}\mathbf{A} = \mathbb{W}\mathbf{A}\end{aligned}$$

(e) For a second-order tensor $\mathbf{A}$

$$\begin{aligned}\mathbb{S}\mathbb{W}\mathbf{A} &= \mathbb{S}(\operatorname{skw}\mathbf{A})\\ &= \left(\frac{1}{2}(\mathbb{I} + \mathbb{T})\right)\operatorname{skw}\mathbf{A}\\ &= \frac{1}{2}\operatorname{skw}\mathbf{A} - \frac{1}{2}\operatorname{skw}\mathbf{A}\\ &= \mathbb{O}\mathbf{A} = \mathbf{O}\end{aligned}$$

Similarly,

$$\begin{aligned}\mathbb{W}\mathbb{S}\mathbf{A} &= \mathbb{W}\operatorname{sym}\mathbf{A}\\ &= \left(\frac{1}{2}(\mathbb{I} - \mathbb{T})\right)\operatorname{sym}\mathbf{A}\end{aligned}$$

$$= \frac{1}{2}\operatorname{sym}\mathbf{A} - \frac{1}{2}\operatorname{sym}\mathbf{A} = \mathbf{O}$$

showing that $\mathbb{SW} = \mathbb{WS} = \mathbb{O}$ the fourth-order zero tensor.

10. Given that, the transposer $\mathbb{T} = g_{il}g_{jk}\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}$, show that $\mathbb{TT} = \mathbb{I}$.

$$\begin{aligned}
\mathbb{TT} &= \left(g_{il}g_{jk}\mathbf{G}^{ij} \otimes \mathbf{G}^{kl}\right)\left(g^{\alpha\gamma}g^{\beta\delta}\boldsymbol{G}_{\alpha\beta} \otimes \mathbf{G}_{\delta\gamma}\right) \\
&= g_{il}g_{jk}g^{\alpha\gamma}g^{\beta\delta}\mathbf{G}^{ij} \otimes \mathbf{G}_{\delta\gamma}\left(\mathbf{G}^{kl}:\mathbf{G}_{\alpha\beta}\right) \\
&= g_{il}g_{jk}g^{\alpha\gamma}g^{\beta\delta}\mathbf{G}^{ij} \otimes \mathbf{G}_{\delta\gamma}\left(\delta^k_\alpha\delta^l_\beta\right) \\
&= g_{il}g_{jk}g^{k\gamma}g^{l\delta}\mathbf{G}^{ij} \otimes \mathbf{G}_{\delta\gamma} \\
&= g_{il}g_{jk}g^{k\gamma}g^{l\delta}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}_\delta \otimes \mathbf{g}_\gamma\right) \\
&= g_{il}g_{jk}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}^l \otimes \mathbf{g}^k\right) \\
&= g_{ik}g_{jl}\left(\mathbf{g}^i \otimes \mathbf{g}^j\right) \otimes \left(\mathbf{g}^k \otimes \mathbf{g}^l\right) \\
&= \mathbb{I}
\end{aligned}$$

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