THREE

Tensor Analysis: Differential and Integral Calculus with Tensors

"We do not fuss over smoothness assumptions: Functions and boundaries of regions are presumed to have continuity and differentiability properties sufficient to make meaningful underlying analysis…" Morton Gurtin, et al.

MetaData

1. The first chapter provided the necessary background to study tensors. Chapter 2 explored the properties of tensors.
2. This third chapter begins the applications by introducing the concept of differentiation for objects larger than scalars.

3. The gradient of scalar, vector or tensor is connected to the extension of the concept of directional derivative called the **Gateaux differential**.
4. The complexity in differentiation of large objects comes, not from the objects themselves, but from the **domain** of differentiation. Differentiation rules with respect to scalar arguments follows closely the similar laws for scalar fields.
5. Integral theorems of **Stokes**, Green and **Gauss** are introduced with computational examples.
6. Orthogonal **Curvilinear** systems are dealt with in the addendum. Important results and the use of **Computer Algebra** are given.

In the calculus we are embarking upon, there will be tensor valued functions and tensor domains. In the most complex cases, the domains and function values are tensors. Most common is the case where the domain itself is the three-dimensional Euclidean Point Space. Even though this space embeds the vectors containing the position vectors, they are not tensors. Tensor-valued functions defined in such domains are called tensor fields.

Differentiation & Large Objects

Differentiating with Respect to Scalar Arguments

We are already familiar with the techniques of differentiation of simple objects like scalars with respect to other scalars. These scalars are defined in scalar domains. In the simplest case, $x, h \in \mathbb{R}$ the derivative, $f'(x)$, of the function, $f: \mathbb{R} \to \mathbb{R}$, is defined as,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

Things become more complex when we handle tensors. A second-order tensor, as we have seen, when expressed in the component form, contains nine scalars. A vector – a first-order tensor, as we know, is made of three scalar members.

The complication does not actually arise from the size of the objects themselves. In fact, the derivation of tensor objects with respect to scalar domains, with some adjustments, basically conforms to the same rules as the above derivation of scalars:

Division of a tensor by a scalar is accomplished by multiplying the tensor by the inverse of the scalar. This operation is defined in all vector spaces to which our vectors and tensors belong. Consequently, the derivative of the tensor $\mathbf{T}(t)$, with respect to a scalar argument, such as time, for example, can be defined as,

$$\frac{d}{dt}\mathbf{T}(t) = \lim_{t\to 0} \frac{\mathbf{T}(t+h) - \mathbf{T}(t)}{t} \equiv \lim_{t\to 0} \frac{1}{t}\big(\mathbf{T}(t+h) - \mathbf{T}(t)\big)$$

If $\alpha(t) \in \mathbb{R}$, and tensor , $\mathbf{T}(t) \in \mathbb{L}$ are both functions of time $t \in \mathbb{R}$, we find,

$$\begin{aligned}\frac{d}{dt}(\alpha\mathbf{T}) &= \lim_{h\to 0} \frac{\alpha(t+h)\mathbf{T}(t+h) - \alpha(t)\mathbf{T}(t)}{t} \\ &= \lim_{h\to 0} \frac{\alpha(t+h)\mathbf{T}(t+h) - \alpha(t)\mathbf{T}(t+h) + \alpha(t)\mathbf{T}(t+h) - \alpha(t)\mathbf{T}(t)}{t}\end{aligned}$$

$$= \lim_{h\to 0} \frac{\alpha(t+h)\mathbf{T}(t+h) - \alpha(t)\mathbf{T}(t+h)}{t} + \lim_{h\to 0} \frac{\alpha(t)\mathbf{T}(t+h) - \alpha(t)\mathbf{T}(t)}{t}$$

$$= \left(\lim_{h\to 0} \frac{\alpha(t+h) - \alpha(t)}{t}\right)\left(\lim_{h\to 0} \mathbf{T}(t+h)\right) + \alpha(t) \lim_{h\to 0} \frac{\mathbf{T}(t+h) - \mathbf{T}(t)}{t}$$

$$= \frac{d}{dt}(\alpha\mathbf{T}) = \alpha\frac{d\mathbf{T}}{dt} + \frac{d\alpha}{dt}\mathbf{T}$$

showing that the familiar product rule applies as expected.

Proceeding in a similar fashion, for $\alpha(t) \in \mathbb{R}, \mathbf{u}(t), \mathbf{v}(t) \in \mathbb{E}$, and $\mathbf{S}(t), \mathbf{T}(t) \in \mathbb{L}$, all being functions of a scalar variable t, the following results hold as expected.

Expression	**Note**
$\frac{d}{dt}(\alpha\mathbf{u}) = \alpha\frac{d\mathbf{u}}{dt} + \frac{d\alpha}{dt}\mathbf{u}$	Each term on the RHS retains the commutative property of multiplication by a scalar.
$\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}$	Each term on the RHS retains the commutative property of the scalar product
$\frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt}$	Original product order must be maintained
$\frac{d}{dt}(\mathbf{u}\otimes\mathbf{v}) = \frac{d\mathbf{u}}{dt}\otimes\mathbf{v} + \mathbf{u}\otimes\frac{d\mathbf{v}}{dt}$	Original product order must be maintained
$\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}$	Sum of tensors
$\frac{d}{dt}\mathbf{TS} = \frac{d\mathbf{T}}{dt}\mathbf{S} + \mathbf{T}\frac{d\mathbf{S}}{dt}$	Product of tensors. Note that we must maintain the order of the product as shown. $\mathbf{T}\frac{d\mathbf{S}}{dt} \neq \frac{d\mathbf{S}}{dt}\mathbf{T}$
$\frac{d}{dt}\mathbf{T}:\mathbf{S} = \frac{d\mathbf{T}}{dt}:\mathbf{S} + \mathbf{T}:\frac{d\mathbf{S}}{dt}$	Scalar Product of tensors. the order is not important here $\mathbf{T}:\frac{d\mathbf{S}}{dt} = \frac{d\mathbf{S}}{dt}:\mathbf{T}$
$\frac{d}{dt}(\alpha\mathbf{T}) = \alpha\frac{d\mathbf{T}}{dt} + \frac{d\alpha}{dt}\mathbf{T}$	Scalar product of a tensor. Order is not important in multiplication by a scalar.
$\frac{d}{dt}(\mathbf{Tu}) = \frac{d\mathbf{T}}{dt}\mathbf{u} + \mathbf{T}\frac{d\mathbf{u}}{dt}$	The original operation order must be maintained in each term of the derivatives.
$\frac{d}{dt}\mathbf{S}^{\mathrm{T}} = \left(\frac{d\mathbf{S}}{dt}\right)^{\mathrm{T}}$	Derivative of a transpose is the transpose of the derivative

Scalar Argument: Time Derivatives of Tensors

1. Consequences of Differentiating the Identity Tensor

Differentiating a tensor with respect to time, or any other scalar parameter follows the expected pattern with little modification. Here are some further results.

The Inverse. Certain important results come from the obvious fact that the identity tensor is a constant so that

$$\frac{d\mathbf{I}}{dt} = \mathbf{O}.$$

We recognize the fact that the derivative of the tensor with respect to a scalar must give a tensor. The value here is the annihilator tensor, $\mathbf{O}$**.** For any invertible tensor valued scalar function, $\mathbf{S}(t)$, we differentiate the equation, $\mathbf{S}^{-\mathbf{1}}(t)\ \mathbf{S}(t) = \mathbf{I}$ to obtain,

$$\frac{d\mathbf{S}^{-\mathbf{1}}}{dt}\ \mathbf{S} + \mathbf{S}^{-1}\frac{d\mathbf{S}}{dt} = \mathbf{O}$$

$$\Rightarrow \frac{d\mathbf{S}^{-\mathbf{1}}}{dt} = -\mathbf{S}^{-\mathbf{1}}\frac{d\mathbf{S}}{dt}\mathbf{S}^{-\mathbf{1}}$$

If we post multiply both sides by $\mathbf{S}^{-\mathbf{1}}$, the following important expression results for the derivative of the inverse tensor with respect to a scalar parameter, in terms of the derivative of the original tensor function:

$$\frac{d\mathbf{S}^{-\mathbf{1}}}{dt} = -\mathbf{S}^{-\mathbf{1}}\frac{d\mathbf{S}}{dt}\mathbf{S}^{-\mathbf{1}}$$

Conversely,

$$\frac{d\mathbf{S}}{dt} = -\mathbf{S}\frac{d\mathbf{S}^{-\mathbf{1}}}{dt}\mathbf{S}$$

Orthogonal Tensors. An orthogonal tensor as well as its transpose can each be functions of a scalar parameter. They still must equal the identity tensor at any value of the argument, so that $\mathbf{Q}(t)\mathbf{Q}^{\mathrm{T}}(t) = \mathbf{I}$. One consequence of this relationship is that the tensor valued function, $\mathbf{\Omega}(t) \equiv$

$\frac{d\mathbf{Q}(t)}{dt}\mathbf{Q}^{\mathrm{T}}(t)$ of the same scalar parameter must be skew. This is another result from differentiating the identity tensor $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$,

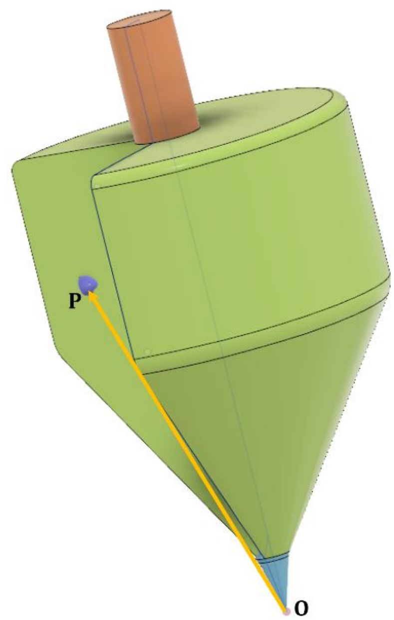
$$\frac{d}{dt}\left(\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\right) = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = \frac{d\mathbf{I}}{dt} = \mathbf{O}$$

Consequently,

$$\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = -\left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}}$$

So we have that the tensor $\mathbf{\Omega} = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}$ is negative of its own transpose, hence it is skew.

Angular Velocity. Consider a rigid body fixed at one end $\mathbf{O}$ – for example a spinning top shown. It is given a rotation $\mathbf{R}(t)$ from rest so that each point $\mathbf{P}$ is at a position vector $\mathbf{r}(t)$ at a time t, related to the original position $\mathbf{r}_o$ by the equation,



$$\mathbf{r}(t) = \mathbf{R}(t)\mathbf{r}_o$$

We can find the velocity by differentiating the position vector,

$$\frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{r}_o = \frac{d\mathbf{R}}{dt}\mathbf{R}^{-1}\mathbf{r}$$

And the rotation is an orthogonal tensor, hence its inverse is its transpose, so that,

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{R}}{dt}\mathbf{R}^{\mathrm{T}}\mathbf{r} = \mathbf{\Omega}\mathbf{r}$$

And, $\mathbf{\Omega}$ as we have seen above, is a skew tensor (every rotation is a proper orthogonal tensor) hence it is associated with an axial vector $(\boldsymbol{\omega}\,\times) = \mathbf{\Omega}$. From this fact we can see that every point in the body has an angular velocity, $\boldsymbol{\omega}$, such that,

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}$$

$\boldsymbol{\omega}$, defined by this expression, the axial vector of the $\frac{d\mathbf{R}}{dt}\mathbf{R}^{\mathrm{T}}$ where $\mathbf{R}(t)$ is the rotation function, is called the angular velocity.

2. The Magnitude of a Tensor.

We saw in the previous chapter that the tensor belongs to its own Euclidean vector space which is equipped with a scalar product, and consequently, a scalar magnitude. Consider the magnitude,

$$\phi(t) = \sqrt{\mathbf{A}(t) : \mathbf{A}(t)} = \|\mathbf{A}(t)\|$$

so that, $\phi^2 = \mathbf{A}:\mathbf{A}$. Differentiating this scalar equation, and remembering that the scalar operand here is just a product, we have,

$$\frac{d}{dt}\phi^2 = 2\phi\frac{d\phi}{dt} = \frac{d\mathbf{A}}{dt}:\mathbf{A} + \mathbf{A}:\frac{d\mathbf{A}}{dt} = 2\frac{d\mathbf{A}}{dt}:\mathbf{A}.$$

This simplifies to

$$\begin{aligned}\frac{d\phi}{dt} &= \frac{d}{dt}|\mathbf{A}(t)| \\ &= \frac{d\mathbf{A}(t)}{dt}:\frac{\mathbf{A}(t)}{|\mathbf{A}(t)|}\end{aligned}$$

3. Tensor Invariants.

Consider as before, that tensor $\mathbf{A}(t)$ varies with a scalar parameter t. What are the responses of the three invariants of $\mathbf{A}$? The invariants, as was shown in chapter 2 are the Trace, $\mathrm{tr}\,\mathbf{A}$, Trace of the cofactor, $\mathrm{tr}\,\mathbf{A}^{\mathrm{c}}$ and the Determinant, $\det \mathbf{A}$.

Trace is a linear operator. It follows immediately that

$$\frac{d}{dt}\,\mathrm{tr}\,\mathbf{A} = \mathrm{tr}\,\frac{d\mathbf{A}}{dt}$$

- A fact that can also be established easily by observing that,

$$\begin{aligned}\frac{d}{dt}I_1(\mathbf{A}) = \frac{d}{dt}\mathrm{tr}\,\mathbf{A} &= \frac{\left[\frac{d\mathbf{A}}{dt}\mathbf{a},\mathbf{b},\mathbf{c}\right] + \left[\mathbf{a},\frac{d\mathbf{A}}{dt}\mathbf{b},\mathbf{c}\right] + \left[\mathbf{a},\mathbf{b},\frac{d\mathbf{A}}{dt}\mathbf{c}\right]}{[\mathbf{a},\mathbf{b},\mathbf{c}]} \\ &= \mathrm{tr}\,\frac{d\mathbf{A}}{dt}\end{aligned}$$

The second invariant is NOT a linear scalar valued function of its tensor argument. However, we have the expression,

$$\mathbf{A}^{\mathrm{c}} = \mathbf{A}^{-\mathrm{T}}\det\mathbf{A} \Rightarrow \mathrm{tr}\mathbf{A}^{\mathrm{c}} = \mathrm{tr}(\mathbf{A}^{-\mathrm{T}}\det\mathbf{A})$$

Differentiating with respect to t,

$$\frac{d}{dt}\,\mathrm{tr}\mathbf{A}^{\mathrm{c}} = \mathrm{tr}\frac{d}{dt}\left(\mathbf{A}^{-\mathrm{T}}\det\mathbf{A}\right)$$

Not a very useful quantity. The *derivative of the third invariant* with respect to a scalar argument is of *momentous importance*. It is the basis of **Liouville's** theorem and is fundamental to the study of continuum flow in general. Just like the second invariant, the third invariant is not a linear function of its tensor argument.

$$I_3(\mathbf{A}) = \frac{[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \det\mathbf{A}$$

so that, $[\mathbf{a}, \mathbf{b}, \mathbf{c}]\det\mathbf{A} = [\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]$. Differentiating, we have,

$$\begin{aligned}
[\mathbf{a}, \mathbf{b}, \mathbf{c}]\frac{d}{dt}\det\mathbf{A} &= \left[\frac{d\mathbf{A}}{dt}\mathbf{a}, \mathbf{Ab}, \mathbf{Ac}\right] + \left[\mathbf{Aa}, \frac{d\mathbf{A}}{dt}\mathbf{b}, \mathbf{Ac}\right] + \left[\mathbf{Aa}, \mathbf{Ab}, \frac{d\mathbf{A}}{dt}\mathbf{c}\right] \\
&= \left[\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}\right] + \left[\mathbf{Aa}, \frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}\mathbf{Ab}, \mathbf{Ac}\right] + \left[\mathbf{Aa}, \mathbf{Ab}, \frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}\mathbf{Ac}\right] \\
&= \mathrm{tr}\left(\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}\right)[\mathbf{Aa}, \mathbf{Ab}, \mathbf{Ac}]
\end{aligned}$$

So that, $\frac{d}{dt}\det\mathbf{A} = \mathrm{tr}\left(\frac{d\mathbf{A}}{dt}\mathbf{A}^{-1}\right)\det\mathbf{A}$.

Differentiation: Vector & Tensor Arguments

So far, the domain of differentiation has remained in the real space. The objects we have differentiated were larger in the sense that we dealt with vectors and tensors. The evaluation of derivatives of larger objects presented a small difficulty but remained essentially the same so long as the domain remained real. When the domain of differentiation itself is a made up of large objects, the task of differentiation becomes a little more demanding. Such problems are standard in Continuum mechanics. For example, the Energy function is a scalar, yet we can obtain the strains from it by differentiating with respect to the stress. We are dealing there with the differentiation of a scalar function of a tensor: stress. We may want to compute the velocity gradient from the velocity function. Here, we are differentiating a vector field defined on the Euclidean point space, $\mathcal{E}$, with respect to the position vector of the points in $\mathcal{E}$. In these and several other derivatives of interest, the domains are no longer in the simple real space.

The approach to this challenge is twofold:

1. Recognize that the vectors and tensors live in their respective Euclidean **VECTOR** spaces where the concept of length is already defined.
2. Use the above to extend the concept of directional derivative to include the derivative of any object from a given Euclidean space with respect to objects from another.

Such a generalization is in the Gateaux differential. Consider a map,

$$\mathbf{F}\colon \mathbf{V} \to \mathbb{W}$$

This maps from the domain $\mathbf{V}$ to $\mathbb{W}$ both of which are Euclidean vector spaces. The concepts of limit and continuity carries naturally from the real space to any Euclidean vector space.

Let $\boldsymbol{v}_0 \in \mathbf{V}$ and $\boldsymbol{w}_0 \in \mathbb{W}$, as usual we can say that the limit

$$\lim_{\boldsymbol{v}\to \boldsymbol{v}_0} \mathbf{F}(\mathbf{v}) = \mathbf{w}_0$$

if for any pre-assigned real number $\epsilon > 0$, no matter how small, we can always find a real number $\delta > 0$ such that $|\mathbf{F}(\mathbf{v}) - \mathbf{w}_0| \leq \epsilon$ whenever $|\mathbf{v} - \mathbf{v}_0| < \delta$. The function is said to be continuous at $\mathbf{v}_0$ if $\mathbf{F}(\mathbf{v}_0)$ exists and $\mathbf{F}(\mathbf{v}_0) = \mathbf{w}_0$

The Gateaux Differential

Specifically, for $\alpha \in \mathbb{R}$ let this map be:

$$D\mathbf{F}(\mathbf{x},\mathbf{h}) \equiv \lim_{\alpha\to 0}\frac{\mathbf{F}(\mathbf{x}+\alpha\mathbf{h}) - \mathbf{F}(\mathbf{x})}{\alpha} = \frac{d}{d\alpha}\mathbf{F}(\mathbf{x}+\alpha\mathbf{h})\bigg|_{\alpha=0}$$

We focus attention on the second variable h while we allow the dependency on $\boldsymbol{x}$ to be as general as possible. We shall show that while the above function can be any given function of $\mathbf{x}$ (linear or nonlinear), the above map is always linear in $\mathbf{h}$ irrespective of what kind of Euclidean space we are mapping from or into. It is called the ***Gateaux Differential***.

Real functions in Real Domains. Let us make the Gateaux differential a little more familiar in real space in two steps: First, we move to the real space and allow $h \to dx$ and we obtain,

$$DF(x,dx) = \lim_{\alpha\to 0}\frac{F(x+\alpha dx) - F(x)}{\alpha} = \frac{d}{d\alpha}F(x+\alpha dx)\bigg|_{\alpha=0}$$

And let $\alpha dx \to \Delta x$, the middle term becomes,

$$\lim_{\Delta x\to 0}\frac{F(x+\Delta x) - F(x)}{\Delta x}dx = \frac{dF}{dx}dx$$

from which it is obvious that the Gateaux derivative is a generalization of the well-known differential from elementary calculus. The Gateaux differential helps to compute a local linear approximation of any function (linear or nonlinear).

Linearity. It is easily shown that the Gateaux differential is linear in its second argument, ie, for $\alpha \in \mathbf{R}$. It is easily shown that,

$$D\mathbf{F}(\mathbf{x}, \alpha\mathbf{h}) = \alpha D\mathbf{F}(\mathbf{x}, \mathbf{h})$$

Furthermore,

$$\begin{aligned} D\mathbf{F}(\mathbf{x}, \mathbf{g}+\mathbf{h}) &= \lim_{\alpha\to 0} \frac{\mathbf{F}\big(\mathbf{x}, \alpha(\mathbf{g}+\mathbf{h})\big) - \mathbf{F}(\mathbf{x})}{\alpha} \\ &= \lim_{\alpha\to 0} \frac{\mathbf{F}\big(\mathbf{x}, \alpha(\mathbf{g}+\mathbf{h})\big) - \mathbf{F}(\mathbf{x}, \alpha\mathbf{g}) + \mathbf{F}(\mathbf{x}, \alpha\mathbf{g}) + \mathbf{F}(\mathbf{x})}{\alpha} \\ &= \lim_{\alpha\to 0} \frac{\mathbf{F}(\mathbf{y}+\alpha\mathbf{h}) - \mathbf{F}(\mathbf{y})}{\alpha} + \lim_{\alpha\to 0} \frac{\mathbf{F}(\mathbf{x}, \alpha\mathbf{g}) + \mathbf{F}(\mathbf{x})}{\alpha} \\ &= D\mathbf{F}(\mathbf{x}, \mathbf{h}) + D\mathbf{F}(\mathbf{x}, \mathbf{g}) \end{aligned}$$

as the variable $\mathbf{y} \equiv \mathbf{x} + \alpha\mathbf{g} \to \mathbf{x}$ as $\alpha \to 0$; For $\alpha, \beta \in \mathbf{R}$, using similar arguments, we can also show that,

$$D\mathbf{F}(\mathbf{x}, \alpha\mathbf{g} + \beta\mathbf{h}) = \alpha D\mathbf{F}(\mathbf{x}, \mathbf{g}) + \beta D\mathbf{F}(\mathbf{x}, \mathbf{h})$$

Points to Note:

- The Gateaux differential is not unique to the point of evaluation. Rather, at each point $\mathbf{x}$ there is a Gateaux differential for each "vector" $\mathbf{h}$. If the domain is a vector space, then we have a Gateaux differential for each of the infinitely many directions at each point. In two of more dimensions, there are infinitely many Gateaux differentials at each point! Remember however, that $\mathbf{h}$ may not even be a vector, but second- or higher-order tensor. It does not matter, as the tensors themselves are in a Euclidean space that define magnitude and direction as a result of the embedded inner product.
- The Gateaux differential is a one-dimensional calculation along a specified direction $\mathbf{h}$. Because it's one-dimensional, you can use ordinary one-dimensional calculus to compute it. Product rules and other constructs for the differentiation in real domains apply.

Gradient or Fréchet Derivatives

A scalar, vector or tensor valued function in a scalar, vector or tensor valued domain is said to be Fréchet differentiable if a subdomain exists in which we can find grad $\mathbf{F}(\mathbf{x})$ such that,

$$(\text{grad}\ \mathbf{F}(\mathbf{x})) \bullet \mathbf{h} = D\mathbf{F}(\mathbf{x}, \mathbf{h})$$

This equation defines the gradient of a function in terms of its operating on the domain object to obtain the Gateaux differential.

The nature of, grad $\mathbf{F}(\mathbf{x})$ as well as the kind of product, " $\bullet$ ", between the gradient and the differential depend on the value type of the function and the type of argument involved. In the simple case of a scalar valued function of a scalar argument, we are back to the regular derivative as can be seen in the first row of the table, and the product involved is simply multiplication of two scalars. The Gateaux differential here is your regular differential.

For a scalar valued function, when the argument type is a vector, we get, for the Fréchet derivative, the familiar gradient operation, grad $\phi(\mathbf{x})$. The Gateaux differential here is the directional derivative, $(\text{grad}\ \phi(\mathbf{x})) \cdot d\mathbf{x}$, in the direction given by the differential, $d\mathbf{x}$. Notice two things here:

1. The function value is a scalar;
2. The function differential, $D\phi(\mathbf{x}, d\mathbf{x}) = (\text{grad}\ \phi(\mathbf{x})) \cdot d\mathbf{x}$ is also a scalar.

The product between grad $\phi(\mathbf{x})$ and the vector differential is a scalar product so that the gradient of a scalar valued function of a vector argument is itself a vector. In the table, product of the Fréchet derivative in column 2 with the argument in column 3 is a scalar hence the correct product here is the scalar product.

No	grad $\mathbf{F}(\mathbf{x})$	Argument	Product $\bullet$	$\mathbf{F}(\mathbf{x})$	Gateaux-Fréchet Example
1	Scalar	Scalar	Multiply	Scalar	$DF(x) = \frac{dF(x)}{dx} dx$
2	Vector	Vector	Scalar product	Scalar	$D\phi(\mathbf{x}, d\mathbf{x}) = (\text{grad}\ \phi(\mathbf{x})) \cdot d\mathbf{x}$
3	Tensor	Tensor	Scalar product	Scalar	$Df(\mathbf{T}, d\mathbf{T}) = \frac{df(\mathbf{T})}{d\mathbf{T}} : d\mathbf{T}$
4	Tensor	Vector	Contraction	Vector	$D\boldsymbol{\psi}(\mathbf{x}, d\mathbf{x}) = (\text{grad}\ \boldsymbol{\psi}(\mathbf{x}))d\mathbf{x}$
5	Tensor	Scalar	Scalar multiply	Tensor	$D\mathbf{T}(x) = \frac{d\mathbf{T}(x)}{dx} dx$

6	Tensor (3)	Vector	Contraction	Tensor	$D\mathbf{F}(\mathbf{x}, d\mathbf{x}) = (\text{grad } \mathbf{F}(\mathbf{x}))d\mathbf{x}$
7	Tensor (4)	Tensor	Contraction	Tensor	$D\mathbf{F}(\mathbf{S}, d\mathbf{S}) = (\text{grad } \mathbf{F}(\mathbf{S}))d\mathbf{S}$

For a scalar valued function of a tensor argument, row 3, Gateaux differential is a scalar – notice it will *necessarily be the same type* as the *function value* as it is simply a change in value: Change in a scalar is scalar, change in vector is vector and change in tensor is tensor. The gradient here is a second-order tensor. The proper product to recover the scalar value from the product of these tensors is the tensor scalar product. On rows six and seven, the tensor order for the Fréchet derivative is higher than two and so stated.

We will rely on a set of worked examples to demonstrate these.

Scalar-valued function of Tensors

Some of the most important functions you will differentiate are scalar-valued functions that take tensor arguments. Here are some examples:

1. Principal Invariants of Tensors and related functions. This could include invariants of other tensors such as the deviatoric parts of the original tensor, etc.
2. Traces of powers of the tensor. Traces of products and transposes, etc. These results are powerful because we often able to convert scalar valued functions to the sums and products of traces and their powers.
3. Magnitudes of tensors.

(a) Direct Application of Gateaux Differential

Given any scalar $k > 0$, for the scalar-valued tensor function, $\frac{d}{d\mathbf{S}} \text{tr } \mathbf{S} = \mathbf{I}$, show that, and $\frac{d}{d\mathbf{S}} \text{tr } \mathbf{S}^2 = \mathbf{S}^\text{T}$.

Compute the Gateaux differential directly here:

$$\begin{aligned} Df(\mathbf{S}, d\mathbf{S}) &= \frac{d}{d\alpha} f(\mathbf{S} + \alpha d\mathbf{S})\bigg|_{\alpha=0} \\ &= \frac{d}{d\alpha} \text{tr}(\mathbf{S} + \alpha d\mathbf{S})\bigg|_{\alpha=0} = \text{tr}(\mathbf{I}\, d\mathbf{S}) \\ &= \mathbf{I}: d\mathbf{S} = \frac{df(\mathbf{S})}{d\mathbf{S}}: d\mathbf{S} \end{aligned}$$

So that, we have found a function that, multiplies the differential argument to give us the Gateaux differential. That is the Fréchet derivative, or gradient. As you can see here, it is the Identity tensor:

$$\frac{d}{d\mathbf{S}}\mathrm{tr}(\mathbf{S}) = \mathbf{I}.$$

When $k = 2$, The Gateaux differential in this case,

$$\begin{aligned}
Df(\mathbf{S}, d\mathbf{S}) &= \frac{d}{d\alpha} f(\mathbf{S} + \alpha d\mathbf{S})\Big|_{\alpha=0} = \frac{d}{d\alpha}\mathrm{tr}(\mathbf{S} + \alpha d\mathbf{S})^2\Big|_{\alpha=0} \\
&= \frac{d}{d\alpha}\mathrm{tr}\{(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})\}\Big|_{\alpha=0} \\
&= \mathrm{tr}\left[\frac{d}{d\alpha}(\mathbf{S} + \alpha d\mathbf{S})(\mathbf{S} + \alpha d\mathbf{S})\right]\Big|_{\alpha=0} \\
&= \mathrm{tr}[d\mathbf{S}(\mathbf{S} + \alpha d\mathbf{S}) + (\mathbf{S} + \alpha d\mathbf{S})d\mathbf{S}]|_{\alpha=0} \\
&= \mathrm{tr}[d\mathbf{S}\,\mathbf{S} + \mathbf{S}\,d\mathbf{S}] = 2\mathbf{S}^{\mathrm{T}}: d\mathbf{S} \\
&= \frac{df(\mathbf{S})}{d\mathbf{S}}: d\mathbf{S}
\end{aligned}$$

So that,

$$\frac{d}{d\mathbf{S}}\mathrm{tr}\,\mathbf{S}^2 = \mathbf{S}^{\mathrm{T}}$$

Using these two results and the linearity of the trace operation, we can proceed to find the derivative of the second principal invariant of the tensor $\mathbf{S}$:

$$\begin{aligned}
\frac{d}{d\mathbf{S}}\, I_2(\mathbf{S}) &= \frac{1}{2}\frac{d}{d\mathbf{S}}[\mathrm{tr}^2(\mathbf{S}) - \mathrm{tr}(\mathbf{S}^2)] \\
&= \frac{1}{2}[2\mathrm{tr}(\mathbf{S})\mathbf{I} - 2\,\mathbf{S}^{\mathrm{T}}] \\
&= \mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}}
\end{aligned}$$

using the fact that differentiating $\mathrm{tr}^2(\mathbf{S})$ with respect to $\mathrm{tr}(\mathbf{S})$ is a scalar derivative of a scalar argument. To find the derivative of the third principal invariant of the tensor $\mathbf{S}$, we appeal to the Cayley-Hamilton theorem, which expresses the determinant in terms of traces only,

$$I_3(\mathbf{S}) = \frac{1}{6}[\mathrm{tr}^3(\mathbf{S}) - 3\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{S}^2) + 2\mathrm{tr}(\mathbf{S}^3)]$$

$$\frac{d}{d\mathbf{S}}\, I_3(\mathbf{S}) = \frac{1}{6}\frac{d}{d\mathbf{S}}[\mathrm{tr}^3(\mathbf{S}) - 3\mathrm{tr}(\mathbf{S})\mathrm{tr}(\mathbf{S}^2) + 2\mathrm{tr}(\mathbf{S}^3)]$$

$$= \frac{1}{6}[3\text{tr}^2(\mathbf{S})\mathbf{I} - 3\text{tr}(\mathbf{S}^2)\mathbf{I} - 3\text{tr}(\mathbf{S})2\mathbf{S}^{\text{T}} + 2 \times 3(\mathbf{S}^2)^{\text{T}}]$$

$$= I_2\mathbf{I} - I_1(\mathbf{S})\mathbf{S}^{\text{T}} + \mathbf{S}^{2\text{T}}.$$

Given that $\mathbf{A}$ is a constant tensor, show that $\frac{\partial}{\partial \mathbf{S}}\text{tr}(\mathbf{AS}) = \mathbf{A}^{\text{T}}$.

For this scalar-valued function, the Gateaux differential,

$$Df(\mathbf{S}, d\mathbf{S}) = \frac{d}{d\alpha}\text{tr}(\mathbf{AS} + \alpha\mathbf{A}d\mathbf{S})\Big|_{\alpha=0} = \frac{df(\mathbf{S})}{d\mathbf{S}} : d\mathbf{S}$$

$$= \frac{d}{d\alpha}\ \text{tr}(\mathbf{AS})\Big|_{\alpha=0} + \frac{d}{d\alpha}\ \alpha\text{tr}(\mathbf{A}d\mathbf{S})\Big|_{\alpha=0}$$

$$= \text{tr}(\mathbf{A}d\mathbf{S}) = \mathbf{A}^{\mathbf{T}} : d\mathbf{S}$$

Giving us,

$$\frac{df(\mathbf{S})}{d\mathbf{S}} = \mathbf{A}^{\mathbf{T}}.$$

Finally, we look at the derivative of magnitude. We have found this by a simpler means earlier. It is instructive to look more rigorously, using the Gateaux differential. Given a scalar α variable, the derivative of a scalar function of a tensor $f(\mathbf{A})$ is

$$\frac{\partial f(\mathbf{A})}{\partial \mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} f(\mathbf{A} + \alpha\mathbf{B})$$

for any arbitrary tensor $\mathbf{B}$. In the case of $f(\mathbf{A}) = |\mathbf{A}|$,

$$\frac{\partial|\mathbf{A}|}{\partial \mathbf{A}} : \mathbf{B} = \lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} |\mathbf{A} + \alpha\mathbf{B}|$$

$$|\mathbf{A} + \alpha\mathbf{B}| = \sqrt{\text{tr}(\mathbf{A} + \alpha\mathbf{B})(\mathbf{A} + \alpha\mathbf{B})^T} = \sqrt{\text{tr}(\mathbf{AA}^{\text{T}} + \alpha\mathbf{BA}^{\text{T}} + \alpha\mathbf{AB}^{\text{T}} + \alpha^2\mathbf{BB}^{\text{T}})}$$

Note that everything under the root sign here is scalar and that the trace operation is linear. Consequently, we can write,

$$\lim_{\alpha \to 0} \frac{\partial}{\partial \alpha} |\mathbf{A} + \alpha\mathbf{B}| = \lim_{\alpha \to 0} \frac{\text{tr}\ (\mathbf{BA}^{\text{T}}) + \text{tr}\ (\mathbf{AB}^{\text{T}}) + 2\alpha\ \text{tr}\ (\mathbf{BB}^{\text{T}})}{2\sqrt{\text{tr}(\mathbf{AA}^{\text{T}} + \alpha\mathbf{BA}^{\text{T}} + \alpha\mathbf{AB}^{\text{T}} + \alpha^2\mathbf{BB}^{\text{T}})}} = \frac{2\mathbf{A} : \mathbf{B}}{2\sqrt{\mathbf{A} : \mathbf{A}}} = \frac{\mathbf{A}}{|\mathbf{A}|} : \mathbf{B}$$

So that,

$$\frac{\partial|\mathbf{A}|}{\partial \mathbf{A}} : \mathbf{B} = \frac{\mathbf{A}}{|\mathbf{A}|} : \mathbf{B}$$

or,

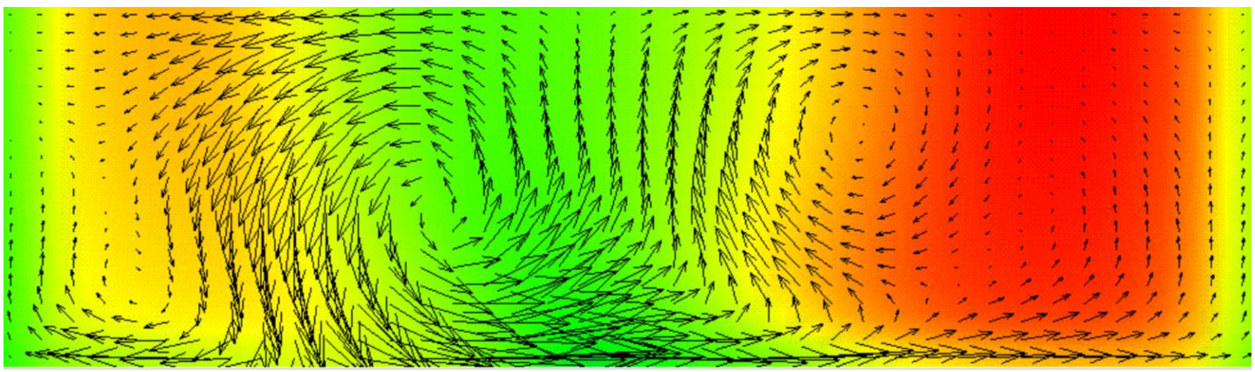
$$\frac{\partial|\mathbf{A}|}{\partial \mathbf{A}} = \frac{\mathbf{A}}{|\mathbf{A}|}$$

as required since $\mathbf{B}$ is arbitrary.

Differentiation of Fields

Euclidean Point Space in Concrete Terms

In Continuum Mechanics, we are often concerned with quantities that are scalars (density, mass, temperature, etc.,) vectors (e.g. displacements, velocities, etc.,) or second-order tensors (e.g., stresses, strains, etc.). There are other things that are derivatives, in space, of these basic quantities. Some of these are temperature gradients, velocity gradients, deformation gradient



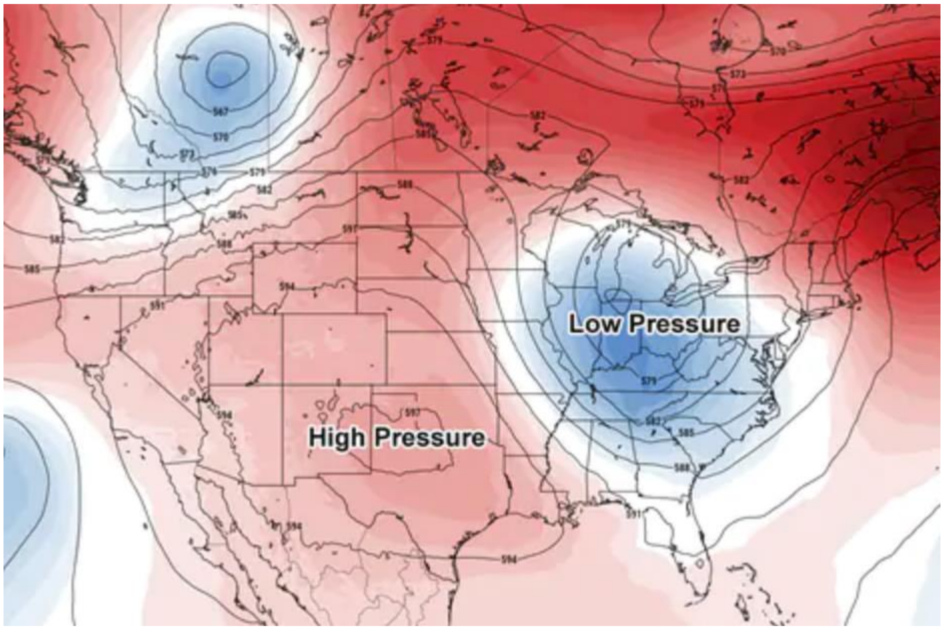
(Which, some have argued, is not even a gradient in the strict sense).

One other fact is that these quantities of interest are often associated with points in space. A set of examples will clarify this before we provide an abstract definition:

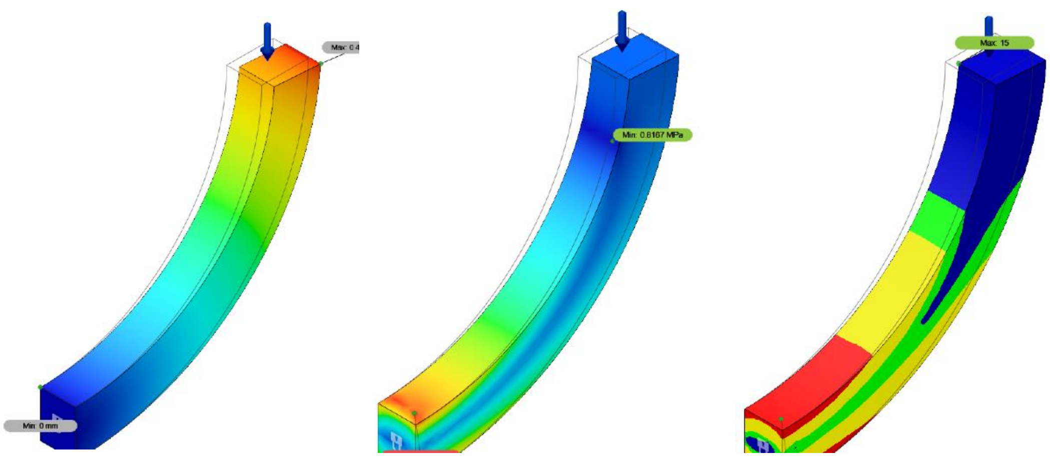
1. A Velocity Map

   The arrows here show the velocity at each point. The color gives other information. About the distribution.

2. Weather Map



The meteorologist has a lot of ways to tell about the weather and predicting it. There are radar maps that give the pressure distribution, wind velocities, temperature, precipitation, etc.:



3. Simulation Maps

The diagram below shows field functions, displacement, stress and factor of safety displayed in the simulation of a vertically loaded curved bar that is supported at one end.

These and many others maps in our daily experience are concrete examples of point functions or field functions. The domains here are a flow field $\mathcal{F}$, A geographical region $\mathcal{G}$ and the body of a curved bar $\mathcal{B}$. Each represents a subset of the Euclidean point space in concrete terms. More precisely, $\mathcal{F}, \mathcal{G}, \mathcal{B} \subset \mathcal{E}$.

$$\mathbf{v}: \boldsymbol{\mathcal{F}} \to \mathbb{E}, p: \mathcal{G} \to \mathbb{R}, \mathbf{T}: \boldsymbol{\mathcal{B}} \to \mathbb{L}$$

Expressing the facts that the velocity $\mathbf{v}(x)$ was defined at the subset $\mathcal{F}$ in the Euclidean point Space and a vector object is created at each point. Those objects themselves reside in the vector space which we already understand how to deal with in that the structure and rules governing them are well defined. The same goes for the pressure $p(x)$**,** and the stress $\mathbf{T}(x)$ as they have all been defined in the respective subsets.

When we talk about temperature, density, strain functions as fields, these are different kinds of objects that share the common feature of being defined in the Euclidean Point Space.

Furthermore, there are other functions that are derived from these base functions. Some examples such derived fields are Temperature Gradient, Velocity Gradient and the Deformation Gradient. In this section, it will be clear that these derived point functions are one rank higher than the ones they are derived from. Temperature Gradient, for example is a vector quantity derived from scalar temperature while the velocity gradient is a second-order tensor derived from the Fréchet derivative of the vector velocity.

Components of Gradient in ONB Systems

We now express the gradients measuring changes in the fields beginning with

$$D\phi(\mathbf{x}, d\mathbf{x}) = (\text{grad } \phi(\mathbf{x})) \cdot d\mathbf{x}$$

With a scalar-valued function with vector arguments that are now the position vectors in the Euclidean Point space. Given a parameter t, for the point $\{x_1, x_2, x_3\}$ consider the neighboring point is at $\{x_1 + a_1 t, x_2 + ta_2, x_3 + ta_3\}$

$$\begin{aligned} D\phi(\mathbf{x}, d\mathbf{x}) &= \lim_{\alpha \to 0} \frac{\phi(\mathbf{x} + \alpha d\mathbf{x}) - \phi(\mathbf{x})}{\alpha} \\ &= \left(\frac{\partial \phi(\mathbf{x})}{\partial x_1} \mathbf{e}_1 + \frac{\partial \phi(\mathbf{x})}{\partial x_2} \mathbf{e}_2 + \frac{\partial \phi(\mathbf{x})}{\partial x_3} \mathbf{e}_3 \right) \cdot \, d\mathbf{x} \end{aligned}$$

So that, for a scalar-valued function, in an Orthonormal system,

$$\text{grad } \phi(\mathbf{x}) = \left(\frac{\partial \phi(\mathbf{x})}{\partial x_1} \mathbf{e}_1 + \frac{\partial \phi(\mathbf{x})}{\partial x_2} \mathbf{e}_2 + \frac{\partial \phi(\mathbf{x})}{\partial x_3} \mathbf{e}_3 \right).$$

Without further ado, we shall generalize this expression and this provide the results for more general cases. We use the comma notation to denote partial derivatives and apply it post so that we can write,

$$\text{grad}\ \phi(\mathbf{x}) = \frac{\partial \phi(\mathbf{x})}{\partial x_i}\mathbf{e}_i = \phi_{,i}\,\mathbf{e}_i$$

Addition, product and other rules apply to Gradient in the comma notation as follows:

$$\begin{aligned}\text{grad}\left(\phi(\mathbf{x})\psi(\mathbf{x})\right) &= (\phi\psi)_{,i}\,\mathbf{e}_i \\ &= (\phi_{,i}\,\psi + \phi\psi_{,i}\,)\mathbf{e}_i \\ &= \psi\ \text{grad}\ \phi(\mathbf{x}) + \phi\ \text{grad}\ \phi(\mathbf{x})\end{aligned}$$

If we define the gradient operator as a post fix operator,

$$\text{grad}\ (\blacksquare) = (\blacksquare)_{,\alpha} \otimes \mathbf{e}_\alpha$$

Where (as long as the coordinate system of reference is ONB) the comma signifies partial derivative, with respect to the α coordinate. The tensor product applying in all cases except for the scalar function where there is no existing basis vector to take the product with. It therefore stands for ordinary product in this case.

Gradient of a Vector Function.

Consider a vector function, $\mathbf{v}(x)$, defined in a Euclidean space spanned by ONB. This can be written in terms of its components,

$\mathbf{v}(x) = v_i(x)\mathbf{e}_{\boldsymbol{i}}$

Then,

$$\begin{aligned}\text{grad}\ \mathbf{v}(\mathbf{x}) &= (v_i(x)\mathbf{e}_{\boldsymbol{i}})_{,\alpha} \otimes \mathbf{e}_\alpha \\ &= v_{i,\alpha}\,(x)\mathbf{e}_{\boldsymbol{i}} \otimes \mathbf{e}_\alpha \\ &= [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3]\begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}\end{aligned}$$

For a tensor field $\mathbf{T}(\mathbf{x})$, the gradient can be obtained in the same way:

$$\begin{aligned}\text{grad}\ \mathbf{T}(\mathbf{x}) &= \left(T_{ij}(\mathbf{x})\mathbf{e}_{\boldsymbol{i}} \otimes \mathbf{e}_j\right)_{,\alpha} \otimes \mathbf{e}_\alpha \\ &= T_{ij,\alpha}\,(\mathbf{x})\mathbf{e}_{\boldsymbol{i}} \otimes \mathbf{e}_j \otimes \mathbf{e}_\alpha\end{aligned}$$

which is a third-order tensor containing 27 terms. Each set of nine terms can be written out as we have done for the tensor gradient of a vector-valued field above.

Now, a temperature field is a scalar field. The forgoing shows that the gradient of such a filed is a vector field on its own. A velocity field is a vector field. Gradient of such a field is a second-order tensor. This is a well-known field called Velocity Gradient.

The Divergence

Gradients of objects larger than scalar are at least second-order tensors. Such derived fields can be contracted in the following way by taking the trace of the last two bases (when they are more than two.) Such a contraction is called the divergence, not of the derived field, but of the original field.

Temperature and other scalar fields cannot have divergence because their gradients are vectors and therefore cannot be contracted (you cannot take the trace). The gradient of a vector field such as the Velocity Gradient, can only be contracted in one way (has only one possible trace). Gradients of larger objects such as tensor fields can be contracted (traced) in more than one way: In that case, the disambiguation rule for contraction to obtain a divergence is to contract with the basis that came from the derivative. For example,

$\text{grad}\, \mathbf{v}(\mathbf{x}) = v_{i},_{\alpha}(x)\mathbf{e}_{\boldsymbol{i}} \otimes \mathbf{e}_{\alpha}$. The divergence of the same field is

$$\begin{aligned}
\text{div}\, \mathbf{v}(\mathbf{x}) &= \text{tr grad}\, \mathbf{v}(\mathbf{x}) \\
&= v_{i},_{\alpha}(x)\mathbf{e}_{\boldsymbol{i}} \cdot \mathbf{e}_{\alpha} = v_{i},_{\alpha}(x)\delta_{i\alpha} \\
&= v_{i},_{i}(x) = \frac{\partial v_i(\mathbf{x})}{\partial x_i} \\
&= \frac{\partial v_1(\mathbf{x})}{\partial x_1} + \frac{\partial v_2(\mathbf{x})}{\partial x_2} + \frac{\partial v_3(\mathbf{x})}{\partial x_3}
\end{aligned}$$

Similarly, for the tensor, $\mathbf{T}(\mathbf{x})$, the divergence,

$$\begin{aligned}
\text{div}\, \mathbf{T}(\mathbf{x}) &= \text{tr grad}\, \mathbf{T}(\mathbf{x}) \\
&= T_{ij},_{\alpha}(\mathbf{x})\mathbf{e}_{\boldsymbol{i}}(\mathbf{e}_j \cdot \mathbf{e}_{\alpha}) \\
&= T_{ij},_{\alpha}(\mathbf{x})\mathbf{e}_{\boldsymbol{i}}\delta_{j\alpha} = T_{ij},_{j}(\mathbf{x})\mathbf{e}_i \\
&= \left(\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3}\right)\mathbf{e}_2 + \left(\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3}\right)\mathbf{e}_3
\end{aligned}$$

The Laplacian or "grad squared".

If we take the gradient of a scalar twice, we obtain a second order tensor. The trace of this field creates the well know Laplacian operator wrongly called grad square. The square of the gradient of a scalar is the tensor of order 2 from which the trace is taken. The Laplacian is a scalar operator that appears frequently in the governing equations of many systems. Let us find the expression for this for a typical scalar field.

You will recall that, for a scalar field, $\phi(\mathbf{x})$,

$$\text{grad}\,\phi(\mathbf{x}) = \frac{\partial\phi(\mathbf{x})}{\partial x_i}\mathbf{e}_i = \phi_{,i}\,\mathbf{e}_i.$$

Taking the gradient of this expression a second time, we have,

$$\text{grad grad}\,\phi(\mathbf{x}) = (\phi_{,i}\,\mathbf{e}_i)_{,\boldsymbol{j}} \otimes \mathbf{e}_{\boldsymbol{j}} = \phi_{,ij}\,\mathbf{e}_i \otimes \mathbf{e}_{\boldsymbol{j}}$$

The trace of this tensor can be found by contracting the dyad bases:

$$\begin{aligned}\Delta\phi(\mathbf{x}) &= \text{tr grad grad}\,\phi(\mathbf{x}) = \phi_{,ij}\,\mathbf{e}_i \cdot \mathbf{e}_{\boldsymbol{j}} \\ &= \phi_{,ij}\,\delta_{ij} = \phi_{,ii} \\ &= \frac{\partial^2\phi(\mathbf{x})}{\partial x_1^2} + \frac{\partial^2\phi(\mathbf{x})}{\partial x_2^2} + \frac{\partial^2\phi(\mathbf{x})}{\partial x_3^2}.\end{aligned}$$

The trace of the double gradient of any field can be taken in this way. The observation one can make here is that while gradient raise the order of its operand by one, divergence decreases its own by one while the Laplacian operation makes no order change. As an example, grad $\mathbf{v}(\mathbf{x}) = v_{i,j}\,(x)\mathbf{e}_{\boldsymbol{i}} \otimes \mathbf{e}_j$ so that,

$$\text{grad grad}\,\mathbf{v}(\mathbf{x}) = v_{i,jk}\,(x)\mathbf{e}_{\boldsymbol{i}} \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

so that, the Laplacian of $\mathbf{v}(\mathbf{x})$ is,

$$\begin{aligned}\Delta\mathbf{v}(\mathbf{x}) &= \text{tr grad grad}\,\mathbf{v}(\mathbf{x}) \\ &= v_{i,jk}\,(x)\mathbf{e}_{\boldsymbol{i}}\delta_{jk} = v_{i,jj}\,(x)\mathbf{e}_{\boldsymbol{i}} \\ &= \left(\frac{\partial^2 v_1}{\partial x_1^2} + \frac{\partial^2 v_1}{\partial x_2^2} + \frac{\partial^2 v_1}{\partial x_3^2}\right)\mathbf{e}_1 + \left(\frac{\partial^2 v_2}{\partial x_1^2} + \frac{\partial^2 v_2}{\partial x_2^2} + \frac{\partial^2 v_2}{\partial x_3^2}\right)\mathbf{e}_2 + \left(\frac{\partial^2 v_3}{\partial x_1^2} + \frac{\partial^2 v_3}{\partial x_2^2} + \frac{\partial^2 v_3}{\partial x_3^2}\right)\mathbf{e}_3\end{aligned}$$

We can continue and find the trace of grad grad∙ of a tensor object. The process is similar and the result of the trace, to give the Laplacian, will also be a tensor of the same order.

Curl of Vector and Tensor Fields.

The third-order alternating tensor, $\boldsymbol{\mathcal{E}} \equiv e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ was introduced in the last chapter. Compositions of this tensor with vectors and other tensors yield some useful constructs in Continuum Mechanics. We have already seen its action resulting in the axial vector for a skew tensor. We are about see that the well-known curl of vectors and tensors can be neatly defined by the divergence of products with this tensor.

1. **Curl of a vector**. Given any vector $\mathbf{u}(x) = u_\alpha(x)\mathbf{e}_\alpha$, the second-order tensor, the composition $\boldsymbol{\mathcal{E}}\mathbf{u}$ is skew and is the transpose of the vector cross of $\mathbf{u}$.

$$\begin{aligned}\boldsymbol{\mathcal{E}}\mathbf{u} &= e_{ijk}u_\alpha(x)\big(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\big)\mathbf{e}_\alpha\\ &= e_{ijk}u_\alpha(x)\big(\mathbf{e}_i \otimes \mathbf{e}_j\big)\mathbf{e}_k \cdot \mathbf{e}_\alpha\\ &= e_{ijk}u_\alpha(x)\big(\mathbf{e}_i \otimes \mathbf{e}_j\big)\delta_{k\alpha} = e_{ijk}u_k(x)\big(\mathbf{e}_i \otimes \mathbf{e}_j\big)\end{aligned}$$

Gradient of $\boldsymbol{\mathcal{E}}\mathbf{u}$ is,

$$\mathrm{grad}(\boldsymbol{\mathcal{E}}\mathbf{u}) = e_{ijk}u_{k,\alpha}(x)\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_\alpha$$

And the trace gives us the divergence,

$$\begin{aligned}\mathrm{curl}\ \mathbf{u} \equiv \mathrm{div}(\boldsymbol{\mathcal{E}}\mathbf{u}) &= \mathrm{tr}\ \mathrm{grad}(\boldsymbol{\mathcal{E}}\mathbf{u})\\ &= e_{ijk}u_{k,\alpha}\,\mathbf{e}_i\big(\mathbf{e}_j \cdot \mathbf{e}_\alpha\big)\\ &= e_{ijk}u_{k,j}\,\mathbf{e}_i\\ &= \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3}\\ u_1 & u_2 & u_3 \end{pmatrix}\end{aligned}$$

which is the curl of the vector field $\mathbf{u}(x)$**.** The curl of a vector has zero divergence:

$$\mathrm{grad\ curl}\ \mathbf{u} = e_{ijk}u_{k,jl}\,\mathbf{e}_i \otimes \mathbf{e}_l$$

The trace of this expression,

$$\begin{aligned}\mathrm{div\ curl}\ \mathbf{u} = \mathrm{tr\ grad\ curl}\ \mathbf{u} &= e_{ijk}u_{k,jl}\,\mathbf{e}_i \cdot \mathbf{e}_l\\ &= e_{ijk}u_{k,jl}\,\delta_{il} = e_{ijk}u_{k,ji}\\ &= 0\end{aligned}$$

2. **Curl of a Tensor Field**. Given a tensor $\mathbf{T}(\mathbf{x}) = \big(T_{ij}(\mathbf{x})\mathbf{e}_i \otimes \mathbf{e}_j\big)$, the composition,

$$\begin{aligned}\mathbf{T}\boldsymbol{\mathcal{E}} &= \big(T_{\alpha\beta}(\mathbf{x})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta\big)\big(e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k\big)\\ &= \big(T_{\alpha\beta}(\mathbf{x})e_{ijk}\mathbf{e}_\alpha \otimes \mathbf{e}_j \otimes \mathbf{e}_k\big)\delta_{\beta i}\end{aligned}$$

$$= T_{\alpha i} e_{ijk} \mathbf{e}_\alpha \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$

This third-order tensor, $T_{\alpha i} e_{ijk} \mathbf{e}_\alpha \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ is skew in two of its indices j and k. The transpose of the divergence of this tensor is

$$\text{curl } \mathbf{T}(\mathbf{x}) = (\text{div } \mathbf{T}\boldsymbol{\mathcal{E}})^{\mathrm{T}}$$

$$= e_{ijk} T_{\alpha k},_j \mathbf{e}_i \otimes \mathbf{e}_\alpha$$

$$= (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3) \begin{pmatrix} \dfrac{\partial T_{13}}{\partial x_2} - \dfrac{\partial T_{12}}{\partial x_3} & \dfrac{\partial T_{23}}{\partial x_2} - \dfrac{\partial T_{22}}{\partial x_3} & \dfrac{\partial T_{33}}{\partial x_2} - \dfrac{\partial T_{32}}{\partial x_3} \\ \dfrac{\partial T_{11}}{\partial x_3} - \dfrac{\partial T_{13}}{\partial x_1} & \dfrac{\partial T_{21}}{\partial x_3} - \dfrac{\partial T_{23}}{\partial x_1} & \dfrac{\partial T_{31}}{\partial x_3} - \dfrac{\partial T_{33}}{\partial x_1} \\ \dfrac{\partial T_{12}}{\partial x_1} - \dfrac{\partial T_{11}}{\partial x_2} & \dfrac{\partial T_{22}}{\partial x_1} - \dfrac{\partial T_{21}}{\partial x_2} & \dfrac{\partial T_{32}}{\partial x_1} - \dfrac{\partial T_{31}}{\partial x_2} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

The curl of a symmetric tensor is traceless:

$$\text{tr curl } \mathbf{T} = e_{ijk} T_{\alpha k},_j \mathbf{e}_i \cdot \mathbf{e}_\alpha = e_{ijk} T_{ik},_j$$

which vanishes when $\mathbf{T}$ is symmetric.

Show that for a constant vector $\mathbf{s}$, $\text{curl } \mathbf{T}^{\mathrm{T}}\mathbf{s} = (\text{curl } \mathbf{T})\mathbf{s}$

$$\mathbf{T}(\mathbf{x}) = T_{ij}(\mathbf{x}) \mathbf{e}_i \otimes \mathbf{e}_j$$

$$\mathbf{T}^{\mathrm{T}}\mathbf{s} = \left(T_{ji} \mathbf{e}_i \otimes \mathbf{e}_j\right) s_\alpha \mathbf{e}_\alpha = T_{ji} s_j \mathbf{e}_i = T_{\alpha k} s_\alpha \mathbf{e}_k$$

Curl of this vector is,

$$e_{ijk}(T_{\alpha k} s_\alpha),_j \mathbf{e}_i = e_{ijk} T_{\alpha k},_j s_\alpha \mathbf{e}_i$$

since $\mathbf{s}$ is a constant vector. For the tensor $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ itself, we have that, $\text{curl } \mathbf{T} = e_{ijk} T_{\alpha k},_j \mathbf{e}_i \otimes \mathbf{e}_\alpha$. Hence,

$$(\text{curl } \mathbf{T})\mathbf{s} = \left(e_{ijk} T_{\alpha k},_j \mathbf{e}_i \otimes \mathbf{e}_\alpha\right) s_\beta \mathbf{e}_\beta$$

$$= e_{ijk} T_{\alpha k},_j s_\alpha \mathbf{e}_i$$

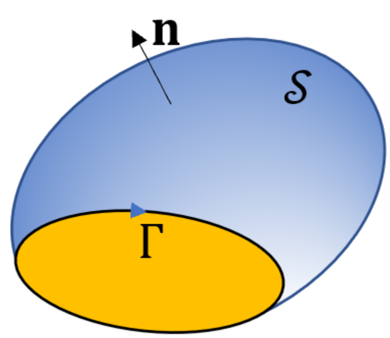
So that $\text{curl } \mathbf{T}^{\mathrm{T}}\mathbf{s} = (\text{curl } \mathbf{T})\mathbf{s}$ as required. Note that this relationship is often used as the definition of the curl of a second-order tensor.

Integral Field Theorems

Integral field theorems are needed to derive the governing equations of continua. The most relevant results, known by several names in the Literature, are often associated with the names of Stokes, Green and Gauss. These theorems are related to one another in several ways. Our practical uses are emphasized here rather than establishment of proofs that are available in mathematical texts. We begin with Stokes Theorem:

Stokes' Theorem

Consider a surface with a boundary curve as shown in the picture. The outward drawn normal, $\mathbf{n}$, is in such a way that a traversal of the boundary curve in the direction shown keeps the surface on the left-hand side. Such a surface is said to be positively oriented. Stokes theorem relates the integral of the curl of a vector or tensor field over this surface $\mathcal{S}$ in the Euclidean Point Space to the line integral of the vector field $\mathbf{v}(\mathbf{x}, t)$ over its boundary curve Γ. In this case, Stokes theorem states that, given a positively oriented surface $\mathcal{S}$, and bounded as shown by a path Γ, the line integral,



$$\int_\Gamma \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = \iint_\mathcal{S} (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{s}.$$

That is, the line integral taken over the path shown is equal to the surface integral of the curl of the same field over the entire surface. For tensor fields, the theorem states:

$$\int_\Gamma \mathbf{T}(\mathbf{x}, t) \cdot d\mathbf{x} = \iint_\mathcal{S} (\operatorname{curl} \mathbf{T})^{\mathrm{T}} d\mathbf{s}$$

$$= \iint_\mathcal{S} \operatorname{div} \mathbf{T}\boldsymbol{\mathcal{E}}\, d\mathbf{s}$$

where $\boldsymbol{\mathcal{E}} = e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$. Notice that the transpose of the curl is taken in the case of the tensor field. The curl of a vector field is a vector and admits no transpose. It is instructive to demonstrate the meaning of these integrals by a simple computation, aided by Mathematica.

Computation Issues

The line integral in equations___ and ___ are easily computed by taking a parametrized position vector, bound to the integration path, $\mathbf{x} = \boldsymbol{\gamma}(\theta)$ along the path. We therefore have,

$$\int_\Gamma \mathbf{v}(\mathbf{x}, t) \cdot d\boldsymbol{\gamma} = \int_{\theta_B}^{\theta_E} \mathbf{v}(\boldsymbol{\gamma}(\theta), t) \cdot \frac{d\boldsymbol{\gamma}}{d\theta} d\theta$$

As the path is traversed beginning with θ_B and ending with θ_E. $\frac{d\boldsymbol{\gamma}}{d\theta}$ is the vector pointing along the tangent to the path.

On the RHS, we are dealing with a surface integral. For a surface parametrized by the scalars, ξ_1, ξ_2 if the position vector $\boldsymbol{\psi}(\xi_1, \xi_2)$ spanning the surface as attain all possible values is differentiated with respect to each variable, again we obtain the surface base vectors. The angle between these vectors are NOT guaranteed to be orthogonal to each other, but their cross product,

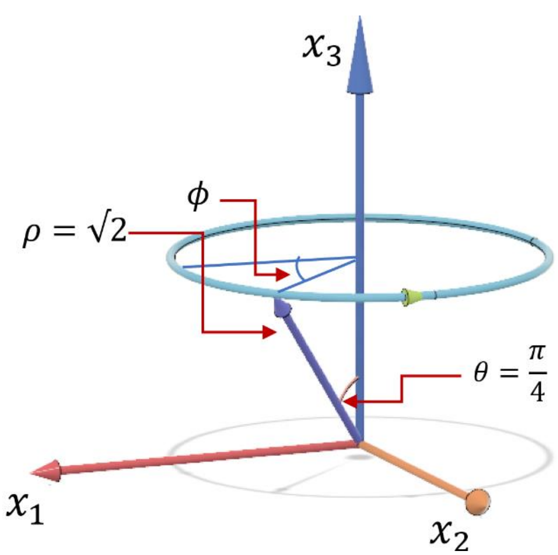
$$\frac{\partial \boldsymbol{\psi}(\xi_1, \xi_2)}{\partial \xi_1} \times \frac{\partial \boldsymbol{\psi}(\xi_1, \xi_2)}{\partial \xi_2}$$

gives the vector in the direction normal to the surface since the two vectors, defined by the derivatives, lie on the surface. The orientation of the surface must be checked to see if a traversal of the path puts the right handed system is such a way that the surface normal is outward. If not, a reversal of the cross product.

The surface integral then becomes,

$$\iint_{\mathcal{S}} (\text{curl } \mathbf{v}) \cdot d\mathbf{s} = \int_{\xi_{2B}}^{\xi_{2E}} \int_{\xi_{1B}}^{\xi_{1E}} (\text{curl } \mathbf{v}) \cdot \left(\frac{\partial \boldsymbol{\psi}(\xi_1, \xi_2)}{\partial \xi_1} \times \frac{\partial \boldsymbol{\psi}(\xi_1, \xi_2)}{\partial \xi_2} \right) d\xi_1 d\xi_2$$

Instead of offering a proof of this important result, we shall proceed to a simple example of a demonstration of it involving a computation of both sides of the above expressions for the simple vector case. Consider a closed counter-clockwise circular path, centered at the x_3 -axis and perpendicular to it on the plane $x_3 = 1$, so that a typical position vector on it is inclined at angle



θ to the vertical as shown in figure ___. We parametrize the path with the familiar spherical coordinates, as $x = \rho \sin\theta \cos\phi$, $y = \rho \sin\theta \sin\phi$ and $z = \rho\cos\theta$. On the circle, only ϕ varies. If we take $\{\rho = \sqrt{2}, \theta = \frac{\pi}{4}\}$, the parameters are that of a unit circle at 1 unit distance along the $x_3$axis.

The line integral of a vector field, according to Stokes, equals that of the surface integral on a surface with this curve as its edge.

The line integral can be computed as stated earlier;

$$\boldsymbol{\gamma}(\theta) = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathrm{e}_2 + \mathrm{e}_3$$

which is the position vector for all the points on this path as $\theta_B = 0; \theta_E = 2\pi$. Differentiating this,

$$d\boldsymbol{\gamma}(\theta) = \frac{d\boldsymbol{\gamma}(\theta)}{d\theta} d\theta = (-\sin\theta\, \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2) d\theta$$

For any prescribed vector field, $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\boldsymbol{\gamma}(\theta), t)$. A $Mathematica^{®}$ implementation of this integral with the function,

$$\mathbf{v}(\mathbf{x}) = \sin x\, \mathbf{e}_1 + \left(\cos y + x + \frac{x^2}{2} + \frac{x^3}{3}\right) \mathbf{e}_2 + z \cos x \cos y\, \mathbf{e}_3$$

is given here:

Stokes Consolidated.nb * - Wolfram Mathematica 11.3

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```
In[1]:= (*Define input vector function and line integration path.
          v=f1e1+f2e2+f3e3
          γ is the position vector locked on the path as θ varies. Path
          tangent is the derivative of this w.r.t θ
          v(θ) is the vector field v evaluated along the path.
           IL is the line integral v(x).dγ along the circular path
        *)
        f1[x_, y_, z_] := Sin[x]
        f2[x_, y_, z_] := Cos[y] + x + x^2/2 + x^3/3
        f3[x_, y_, z_] := z Cos [x] Cos [y]
        γ[α_] := {Cos[α], Sin[α], 1}
        v[θ_] := {f1[ Cos[θ], Sin[θ], 1], f2[ Cos[θ], Sin[θ], 1],
          f3[ Cos[θ], Sin[θ], 1]}
        IL = Integrate[(v[θ].D[γ[θ], θ]), {θ, 0, 2 π}]

Out[6]= 5 π
        ---
         4
```

There at least three easy ways we can realize such a surface:

1. A flat circular region on the same plane as the path. This is the region, $x_1^2 + x_2^2 \le 1$, on the plane at $x_3 = 1$. It has the parametric equations,

$$x_1 = r\cos\phi,\ x_2 = r\sin\phi,\ x_3 = 1$$

By inspection, we can see that the normal to this plane surface is the $\mathbf{e}_3$ axis. However, this can be computed directly by using equation 1.__ giving us two vectors on the coordinate lines on the plane:

In the plane region,

$$\boldsymbol{\psi}(r,\phi) = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2$$

With $r \le 1; 0 \le \phi \le 2\pi$ covers the region. And the normal to the surface is

$$\frac{\partial\boldsymbol{\psi}(r,\phi)}{\partial r} \times \frac{\partial\boldsymbol{\psi}(r,\phi)}{\partial\phi}$$

Since both derivatives represent vectors lying on the surface. We can now integrate the curl of any vector field with the domain set to this plane surface.

Stokes Consolidated.nb * - Wolfram Mathematica 11.3

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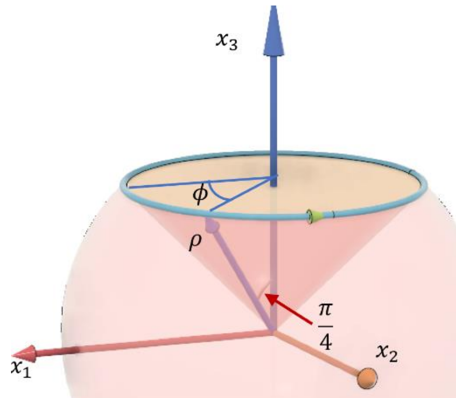
```
In[∘]:= (*Define input vector function and line integration path.
          v1f=Curl v(x)
         ϕ(r,θ) is the radial vector bound to the surface as the
         parameters r and θ varies. A derivative wrt either variable
         gives the surface base vector along the respective coordinate
         path. -
           nVecf, the cross product of these two provides the surface
        normal the direction of the unit normal to the surface.
         v2f holds the resut intermediate to avoid problems with
        integration variable .
         I_D is the line integral on the plane of disk.
        *)
        v1f[x_, y_, z_] := Curl[{f1[x, y, z], f2[x, y, z], f3[x, y, z]},
          {x, y, z}]
        v2f[x_, y_, z_] = v1f[x, y, z]
        ϕ[r_, θ_] := {r Cos[θ], r Sin[θ], 0}
        nVecf = Cross[D[ϕ[r, θ], r], D[ϕ[r, θ], θ]]
        I_D = Integrate[Integrate[(v2f[r Cos[θ], r Sin[θ], 1].nVecf),
           {θ, 0, 2 π}], {r, 0, 1}]

Out[∘]= {-z Cos[x] Sin[y], z Cos[y] Sin[x], 1 + x + x^2}

Out[∘]= {0, 0, r Cos[θ]^2 + r Sin[θ]^2}

Out[∘]= 5 π / 4
```

2. The cone surface, $x_3^2 = x_1^2 + x_2^2$ given that $x_1^2 + x_2^2 \leq 1$. The radius vectors constrained to this region is,

$$\boldsymbol{\psi}(\rho, \phi) = \rho \sin\frac{\pi}{4} \cos\phi \;\mathbf{e}_1 + \rho \sin\frac{\pi}{4} \sin\phi \,\mathbf{e}_2 + \rho \cos\frac{\pi}{4} \;\mathbf{e}_3$$



Contained within the surface for $0 \leq \rho \leq \sqrt{2}$, $0 \leq \phi \leq 2\pi$. Here the two surface vectors are: $\frac{\partial \boldsymbol{\psi}(\rho,\phi)}{\partial \phi}$ and $\frac{\partial \boldsymbol{\psi}(\rho,\phi)}{\partial \phi}$

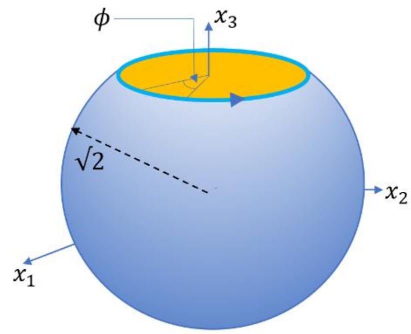
The cross products of which give the normal to the surface.

Stokes Consolidated.nb * - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[•]:= (*Define input vector function and line integration path.
           v1f=Curl v(x)
          ϕ(r,θ) is the radial vector bound to the surface as the
          parameters r and θ varies. A derivative wrt either variable
          gives the surface base vector along the respective coordinate
          path. -
            nVecf, the cross product of these two provides the surface
         normal the direction of the unit normal to the surface.
          v2f holds the resut intermediate to avoid problems with
         integration variable .
          I_D is the line integral on the plane of disk.
        *)
        v1f[x_, y_, z_] := Curl[{f1[x, y, z], f2[x, y, z], f3[x, y, z]},
          {x, y, z}]
        v2f[x_, y_, z_] = v1f[x, y, z]
        ϕ[r_, θ_] := {r Cos[θ], r Sin[θ], 0}
        nVecf = Cross[D[ϕ[r, θ], r], D[ϕ[r, θ], θ]]
        I_D = Integrate[Integrate[(v2f[r Cos[θ], r Sin[θ], 1].nVecf),
            {θ, 0, 2 π}], {r, 0, 1}]

Out[•]= {-z Cos[x] Sin[y], z Cos[y] Sin[x], 1 + x + x^2}

Out[•]= {0, 0, r Cos[θ]^2 + r Sin[θ]^2}

Out[•]= 5 π / 4
```

3. Lastly, we can take the sphere centered at the origin with radius $\sqrt{2}$ as shown in Figure ___. The radius vector constrained to the surface is,



$$\boldsymbol{\psi}(\theta, \phi) = \sqrt{2} \sin\theta \cos\phi \ \mathbf{e}_1 + \sqrt{2} \sin\theta \sin\phi \, \mathbf{e}_2 + \sqrt{2} \cos\theta \, \mathbf{e}_3$$

With $\frac{\pi}{4} \leq \theta \leq \pi; 0 \leq \phi \leq 2\pi$. Surface normal can be evaluated from $\frac{\partial \boldsymbol{\psi}(\theta,\phi)}{\partial \theta} \times \frac{\partial \boldsymbol{\psi}(\theta \ \phi)}{\partial \phi}$. In the *Mathematica*® code below,

Computations using the three surfaces give the same value of $\frac{5\pi}{4}$ which is the same as the field integrated over the closed path shown earlier. For hand calculations, simpler functions may be

```
v1c[x_, y_, z_] := Curl[{f1[x, y, z], f2[x, y, z], f3[x, y, z]}, {x, y, z}]
v2c[x_, y_, z_] = v1c[x, y, z]
ψ[ρ_, ϕ_] := {ρ Sin[π / 4] Cos[ϕ], ρ Sin[π / 4] Sin[ϕ], ρ Cos[π / 4]}
nVecc = Cross[D[ψ[ρ, ϕ], ρ], D[ψ[ρ, ϕ], ϕ]]
IC =
 Integrate[
  Integrate[
   (v2c[ρ Sin[π / 4] Cos[ϕ], ρ Sin[π / 4] Sin[ϕ], ρ Cos[π / 4]].nVecc),
   {ϕ, 0, 2 π}], {ρ, 0, Sqrt[2]}]
]= {-z Cos[x] Sin[y], z Cos[y] Sin[x], 1 + x + x^2}
```

used.

Green's Theorem

Green's theorem can be viewed as a two-dimensional version of the 3-D Stokes Theorem:

Given two scalar fields $P_1(x_1, x_2), P_2(x_1, x_2)$, functions defined over a two dimensional Euclidean Point Space, we can form the vector field,

$$\mathbf{v}(\mathbf{x}) = P_1(x_1, x_2)\mathbf{e}_1 + P_2(x_1, x_2)\mathbf{e}_2$$

$$\operatorname{curl} \mathbf{v}(\mathbf{x}) = \begin{pmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ P_1(x_1, x_2) & P_2(x_1, x_2) & 0 \end{pmatrix}$$

$$= \frac{\partial P_1}{\partial x_2}\mathbf{e}_1 - \frac{\partial P_2}{\partial x_1}\mathbf{e}_2$$

From Stokes Theorem,

$$\oint_{\Gamma} \mathbf{v}(\mathbf{x}) \cdot d\boldsymbol{\gamma} = \iint_{\mathcal{S}} (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{s}$$

But $d\mathbf{s} = dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2$ so that,

$$\begin{aligned}\oint_{\Gamma} (P_1(x_1,x_2)\mathbf{e}_1 + P_2(x_1,x_2)\mathbf{e}_2) \cdot d\boldsymbol{\gamma} &= \iint_{\mathcal{S}} \left(\frac{\partial P_1}{\partial x_2}\mathbf{e}_1 - \frac{\partial P_2}{\partial x_1}\mathbf{e}_2\right) \cdot (dx_1\mathbf{e}_1 + dx_2\mathbf{e}_2) \\ &= \iint_{\mathcal{S}} \left(\frac{\partial P_1}{\partial x_2}dx_1 - \frac{\partial P_2}{\partial x_1}dx_2\right)\end{aligned}$$

Gauss Divergence Theorem

The divergence theorem is central to several other results in Continuum Mechanics. It related the volume integral of a field to the surface integral of the tensor product of the field and unit normal. We present here a generalized form [Ogden] which states that, **Gauss Divergence Theorem**

For a tensor field $\boldsymbol{\Xi}$, The volume integral in the region $\Omega \subset \mathcal{E}$,

$$\int_{\Omega} (\operatorname{grad} \boldsymbol{\Xi})\, dv = \int_{\partial\Omega} \boldsymbol{\Xi} \otimes \mathbf{n}\, ds$$

where $\mathbf{n}$ is the outward drawn normal to $\partial\Omega$ – the boundary of Ω. The tensor product applies when the field is of order one or higher: vectors, second and higher-order tensor. For scalars, it degenerates, as expected, to a simple multiplication by a scalar. We will now derive several more familiar statements of this theorem from equation ____ above:

Vector field.

Replacing the tensor with the vector field $\mathbf{f}$ and taking the trace on both sides, remembering that the integral operator is linear and therefore the trace of an integral equals the integral of its trace, we have,

$$\operatorname{tr} \int_{\Omega} (\operatorname{grad} \mathbf{f})\, dv = \operatorname{tr} \int_{\partial\Omega} \mathbf{f} \otimes \mathbf{n}\, ds$$

so that,

$$\int_{\Omega} (\operatorname{div} \mathbf{f})\, dv = \int_{\partial\Omega} \mathbf{f} \cdot \mathbf{n}\, ds$$

Stating that the integral of the divergence of a vector field over the volume equals the integral over the surface of the same function with the outwardly drawn normal. Which is the usual form of the Gauss theorem. We can proceed immediately to the form of this theorem that applies to a scalar function:

Scalar Fields

Consider a scalar field ϕ and a constant vector $\mathbf{a}$**.** We apply the vector form of the divergence theorem to the product $\phi\mathbf{a}$**:**

$$\int_\Omega (\mathrm{div}[\phi\boldsymbol{a}])\,dv = \int_{\partial\Omega} \phi\mathbf{a}\cdot\mathbf{n}\,ds = \mathbf{a}\cdot\int_{\partial\Omega}\phi\mathbf{n}\,ds$$

For the LHS, note that, $\mathrm{div}[\phi\boldsymbol{a}] = \mathrm{tr}\,\mathrm{grad}[\phi\boldsymbol{a}]$

$$\mathrm{grad}[\phi\boldsymbol{a}] = (\phi a_i),_j\,\mathbf{e}_i\otimes\mathbf{e}_j = a_i\phi,_j\,\mathbf{e}_i\otimes\mathbf{e}_j$$

The trace of which is,

$$a_i\phi,_j\,\mathbf{e}_i\cdot\mathbf{e}_j = a_i\phi,_j\,\delta_{ij} = a_i\phi,_i = \mathbf{a}\cdot\mathrm{grad}\,\phi$$

For the arbitrary constant vector $\mathbf{a}$, we therefore have that,

$$\int_\Omega (\mathrm{div}[\phi\boldsymbol{a}])\,dv = \mathbf{a}\cdot\int_\Omega \mathrm{grad}\,\phi\;\,dv = \mathbf{a}\cdot\int_{\partial\Omega}\phi\mathbf{n}\,ds$$

$$\int_\Omega \mathrm{grad}\,\phi\;\,dv = \int_{\partial\Omega}\phi\mathbf{n}\,ds$$

Second-Order Tensors

We can apply the general form above to a second-order tensor field, $\mathbf{T}(\mathbf{x}) = T_{ij}(\mathbf{x})\mathbf{e}_{\boldsymbol{i}}\otimes\mathbf{e}_j$. Applying equation ___ as before,

$$\int_\Omega T_{ij},_k\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k dv = \int_{\partial\Omega} T_{ij}n_k\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\,ds$$

Contracting, we have

$$\int_\Omega T_{ij},_k\,(\mathbf{e}_i\otimes\mathbf{e}_j)\mathbf{e}_k dv = \int_{\partial\Omega} T_{ij}n_k(\mathbf{e}_i\otimes\mathbf{e}_j)\mathbf{e}_k\,ds$$

$$\int_\Omega T_{ij},_k\,\delta_{kj}\mathbf{e}_i\,dv = \int_{\partial\Omega} T_{ij}n_k\delta_{kj}\mathbf{e}_i\,ds$$

$$\int_\Omega T_{ij},_j\,\mathbf{e}_i\,dv = \int_{\partial\Omega} T_{ij}n_j\mathbf{e}_i\,ds$$

Which is the same as,

$$\int_{\Omega} (\mathrm{div}\mathbf{T})\, dv = \int_{\partial\Omega} \mathbf{Tn}\, ds$$

Relating the volume integral of the divergence of a tensor field to the same tensor field integrated over the surface after being contracted with the unit normal to the boundary of the same volume.

Curvilinear Coordinates

Tensors provide a unified way to express most of the quantities and objects we come across in Continuum Mechanics. The material objects we deal with abide in the Euclidean 3D space. For computation purposes, we refer to this space using a global chart or a coordinate syste. Apart from those that have ventured into the advanced topics section in each chapter, we have restricted consideration so far to the simplest of such systems: the Cartesian System with Orthonormal Basis Vectors (ONB).

There are oblique Cartesian coordinates that are the same as the ONB but their axes do not necessarily meet at right angles. These oblique systems are not our main concern. The Cartesian system is useful and easy to use. Reasons for this ease of use include

1. Base vectors meet at right angles. They are not only mutually independent as base vectors must be, by they also do not have zero components in other base vector directions: their scalar products with one another vanish.
2. The basis vectors are spatial constants. Remain normal to each other and always point in the same directions at any point.
3. Because of these, derivatives of tensors expressed in terms of such bases only concern the coefficients as the base vectors are always constants.
4. In addition to these, the position vectors in Cartesian coordinates ale linear in the coordinate variables.

In curvilinear coordinate systems, the basis vectors are spatial variables, they are not necessarily unit vectors and they are not necessarily orthogonal.

For the purpose of this work, we restrict attention to coordinate systems that are orthogonal. There are very useful and practical systems that meet this criterion, for example, the Cylindrical, spherical and spheroidal systems we have already seen are curvilinear. While they may not necessarily meet the normality criterion, they are orthogonal. Yet we have to cope with the fact that the basis vectors are variables.

In the next section, we derive the derivatives of tensor fields with such varying bases. However, it no longer necessary to go through the tedium to doing such computations unless specific applications compels one to go through the labor.

In this section, we shall show some of the principal results and principles for working with not Cartesian systems that obviate the necessity of going through the computation of the derivatives with varying bases vectors.

The Covariant Derivative

We defined the gradient, in Cartesian systems as,

$$\mathrm{grad}\,(\blacksquare) = (\blacksquare)_{,\alpha} \otimes \mathbf{e}_\alpha$$

The comma in the above equation has been interpreted as a partial derivative thus far. The fact is that this equation is actually valid in every coordinate system. That includes the orthogonal curvilinear coordinates we are considering! All we will need to do is to simply change our interpretation of this equation.

From now on, when we are referring to coordinate systems other than Cartesian, the comma operator will be interpreted as **Covariant** Derivatives. The full definition of covariant derivative is in the advanced section. However, the explanation given here should suffice:

A covariant derivative is simply one where all the terms that would have arisen from the spatial variability of the basis vectors are already added. Consequently, when we write,

$$\mathrm{grad}\,(\blacksquare) = (\blacksquare)_{,\alpha} \otimes \mathbf{g}_\alpha$$

we have an expression that is valid everywhere. Specifically we can represent a second-order tensor and write,

$$\begin{aligned}\mathrm{grad}\,\mathbf{T} &= \left(T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j\right)_{,\alpha} \otimes \mathbf{g}_\alpha = \left(T^i{}_j\mathbf{g}_i \otimes \mathbf{g}^j\right)_{,\alpha} \otimes \mathbf{g}_\alpha = \left(T_i{}^j\mathbf{g}^i \otimes \mathbf{g}_j\right)_{,\alpha} \otimes \mathbf{g}_\alpha \\ &= \left(T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j\right)_{,\alpha} \otimes \mathbf{g}_\alpha\end{aligned}$$

which are four different ways of expressing the same gradient in curvilinear systems, they work exactly in the same way as the ordinary Cartesian experience with fixed vectors. The covariant derivative has so taken care of the variability of base issues that the vector bases behave like constants under covariant differentiation! It does it by computing extra terms, called Christoffel symbols, that add more terms to the expression to account for the variation of the basis vectors. Just like in the Cartesian system, therefore, there are no further terms to worry about.

The heavy lifing therefore transfers to the computation of the covariant derivative and evaluating the physical components of the tensor components. We do not emphasize the computations of these things as they are easily programmed. One of the advantages of a Symbolic Algebra System is that it facilitates such computation just for the asking! You do not need to do much programming! A few examples will be given shortly. Remember, all you need to worry about are the equations as they exist in Cartesian systems. With correct interpretations, they are valid in every system. And, when you need to compute actual values for those coordinate systems, a symbolic algebra system like $Mathematica^{©}$ delivers them easily.

The consequence of these is that many textbooks, including modern books devote a large number of pages on these items that are done with completeness by software to an extent far beyond what can reasonable be placed in textbooks.

If there is any need (very doubtful) to compute the Christoffel symbols, they can be done by programming also – no need for manual computations. If you want manual computations despite all this, then read the advanced part of this chapter, the manual computations are explained there.

Gradient Divergence and Curl in Curvilinear Coordinates.

In this section we will show how to find these differential operations in the common systems of Cylindrical, Spherical and Oblate Spheroidal systems. The extension to any popular coordinate system is immediate. Furthermore, you will see that Mathematica allows you to define a completely new coordinate system by yourself. And, if your definition is correctly specified to it, it will compute all the differential operators you need for you!

Gradient is easy to compute manually and we will do so right away:

For a scalar field, $\psi(x_1, x_2, x_3)$, in Cartesian coordinates, we already know that,

$$\text{grad}\,\psi = \frac{\partial\psi}{\partial x_1}\mathbf{e}_1 + \frac{\partial\psi}{\partial x_2}\mathbf{e}_2 + \frac{\partial\psi}{\partial x_3}\mathbf{e}_3$$

The same expression can be obtained in cylindrical coordinates by observing that, in that system, the position vector is,

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z$$

$$\mathbf{g_1} = \frac{\partial \mathbf{r}}{\partial r} = \mathbf{e}_r;\ \mathbf{g_2} = \frac{\partial \mathbf{r}}{\partial \phi} = r\mathbf{e}_\phi;\ \mathbf{g_3} = \frac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z$$

The natural basis vectors,

$$\mathbf{g}_{\boldsymbol{i}} \equiv \frac{\partial \mathbf{r}}{\partial \xi_i}$$

Obtained by differentiating the position vector with respect to the coordinates ξ_i in this case, $\xi_1 = r, \xi_2 = \theta$ and $\xi_3 = z$, are all unit vectors except the middle one. Suppose we did not know that. What we will need to do for each system, to get the physically consistent gradient in physical components, we need to divide each term by the magnitude of each basis vector. In this case therefore,

$$h_1 = \sqrt{\mathbf{g}_1 \cdot \mathbf{g}_1} = 1$$
$$h_2 = \sqrt{\mathbf{g}_2 \cdot \mathbf{g}_2} = r\ and$$
$$h_2 = \sqrt{\mathbf{g_3} \cdot \mathbf{g}_3} = 1$$

If we write, $\mathbf{g}'_\alpha, \alpha = 1,2,3$ as the normalized basis vectors in the curvilinear system, the gradient is therefore,

$$\begin{aligned}\operatorname{grad}(\psi) &= \sum_{\alpha=1}^{3} \frac{1}{h_\alpha}(\psi)_{,\alpha} \otimes \mathbf{g}'_\alpha \\ &= \frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1}\,\mathbf{g}'_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial \xi_2}\,\mathbf{g}'_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial \xi_3}\,\mathbf{g}'_3 \\ &= \frac{\partial \psi}{\partial r}\,\mathbf{e}_r + \frac{1}{r}\frac{\partial \psi}{\partial r}\,\mathbf{e}_\theta + \frac{\partial \psi}{\partial \xi_3}\,\mathbf{e}_z.\end{aligned}$$

Notice again, the summation convention cannot be used here, hence we revert to the regular nomenclature and use summation directly. This is the gradient in the normalized natural bases. Observe that we have re-introduces the summation symbol in the above computation to avoid breaking the summation convention. Furthermore, we have just dealt with a scalar field. For such a field, the covariant derivative coincides with the partial derivatives. Hence all we need do here is to normalize the basis vectors and divide by the magnitude of the basis vectors to obtain the physically consistent tensor expression.

In Spherical coordinates, with no further ado, We can write:

$$\text{grad}\ (\psi) = \frac{1}{h_1}\frac{\partial \psi}{\partial \xi_1}\ \mathbf{g}'_1 + \frac{1}{h_2}\frac{\partial \psi}{\partial \xi_2}\ \mathbf{g}'_2 + \frac{1}{h_3}\frac{\partial \psi}{\partial \xi_3}\ \mathbf{g}'_3$$
$$= \frac{\partial \psi}{\partial \rho}\ \mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial \psi}{\partial \theta}\ \mathbf{e}_\theta + \frac{1}{\rho \sin\theta}\frac{\partial \psi}{\partial \phi}\ \mathbf{e}_\phi.$$

To find the gradient in any coordinate systems is simply to invoke the command as follows:

```
Grad[ψ[r, θ, z], {r, θ, z}, "Cylindrical"]
```

$$\left\{\psi^{(1,0,0)}(r, \theta, z), \frac{\psi^{(0,1,0)}(r, \theta, z)}{r}, \psi^{(0,0,1)}(r, \theta, z)\right\}$$

```
Grad[ψ[ξ, η, ϕ], {ξ, η, ϕ}, "OblateSpheroidal"]
```

$$\left\{\frac{\sqrt{2}\ \psi^{(1,0,0)}(\xi, \eta, \phi)}{a\sqrt{\cos(2\eta)+\cosh(2\xi)}}, \frac{\sqrt{2}\ \psi^{(0,1,0)}(\xi, \eta, \phi)}{a\sqrt{\cos(2\eta)+\cosh(2\xi)}}, \frac{\csc(\eta)\,\text{sech}(\xi)\,\psi^{(0,0,1)}(\xi, \eta, \phi)}{a}\right\}$$

Comparing the above output to the previous result, it is obvious that, in Cylindrical coordinates,

$$\frac{\partial \psi}{\partial r} = \psi^{(1,0,0)}(r,\theta,z), \qquad \frac{\partial \psi}{\partial \theta} = \psi^{(0,1,0)}(r,\theta,z), \qquad \frac{\partial \psi}{\partial z} = \psi^{(0,0,1)}(r,\theta,z)$$

And that the result is given in lists with the understanding that we are referring to the normalized basis vectors in each case.

Gradient of a vector

The gradient f a vector is a tensor. To compute it manually, we will have to carry out the covariant derivative with the attendant Christoffel symbols. How about doing it by simply writing the function on the Mathematica Notebook as follows?

```
Grad[{f₁[r, θ, z], f₂[r, θ, z], f₃[r, θ, z]}, {r, θ, z}, "Cylindrical"]
```

$$\begin{pmatrix} f_1^{(1,0,0)}(r, \theta, z) & \frac{f_1^{(0,1,0)}(r,\theta,z) - f_2(r,\theta,z)}{r} & f_1^{(0,0,1)}(r, \theta, z) \\ f_2^{(1,0,0)}(r, \theta, z) & \frac{f_1(r,\theta,z) + f_2^{(0,1,0)}(r,\theta,z)}{r} & f_2^{(0,0,1)}(r, \theta, z) \\ f_3^{(1,0,0)}(r, \theta, z) & \frac{f_3^{(0,1,0)}(r,\theta,z)}{r} & f_3^{(0,0,1)}(r, \theta, z) \end{pmatrix}$$

You are able to get any vector operation on a tensor function with such simple commands.

Examples

3.1	Given that $\alpha(t) \in \mathbb{R}, \mathbf{u}(t), \mathbf{v}(t) \in \mathbb{E}$, and $\mathbf{S}(t), \mathbf{T}(t) \in \mathbb{L}$, are all functions of a scalar variable t, show that $(a) \frac{d}{dt}(\alpha\mathbf{u}) = \alpha\frac{d\mathbf{u}}{dt} + \frac{d\alpha}{dt}\mathbf{u}, (b)\ \frac{d}{dt}(\mathbf{u}\cdot\mathbf{v}) = \frac{d\mathbf{u}}{dt}\cdot\mathbf{v} + \mathbf{u}\cdot\frac{d\mathbf{v}}{dt}, (c) \frac{d}{dt}(\mathbf{u}\times\mathbf{v}) = \frac{d\mathbf{u}}{dt}\times\mathbf{v} + \mathbf{u}\times\frac{d\mathbf{v}}{dt},$ *and* $(d) \frac{d}{dt}(\mathbf{u}\otimes\mathbf{v}) = \frac{d\mathbf{u}}{dt}\otimes\mathbf{v} + \mathbf{u}\otimes\frac{d\mathbf{v}}{dt}$
	$\frac{d}{dt}(\mathbf{u}\otimes\mathbf{v}) = \lim_{h\to 0}\frac{\mathbf{u}(t+h)\otimes\mathbf{v}(t+h) - \mathbf{u}(t)\otimes\mathbf{v}(t)}{t}$ $= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\otimes\mathbf{v}(t+h) - \mathbf{u}(t)\otimes\mathbf{v}(t+h) + \mathbf{u}(t)\otimes\mathbf{v}(t+h) - \mathbf{u}(t)\otimes\mathbf{v}(t)}{t}$ $= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\otimes\mathbf{v}(t+h) - \mathbf{u}(t)\otimes\mathbf{v}(t+h)}{t}$ $+ \lim_{h\to 0}\frac{\mathbf{u}(t)\otimes\mathbf{v}(t+h) - \mathbf{u}(t)\otimes\mathbf{v}(t)}{t}$ $= \left(\lim_{h\to 0}\frac{\mathbf{u}(t+h) - \mathbf{u}(t)}{t}\right)\otimes\left(\lim_{h\to 0}\mathbf{v}(t+h)\right) + \mathbf{u}(t)$ $\otimes \lim_{h\to 0}\frac{\mathbf{v}(t+h) -\otimes\,\mathbf{v}(t)}{t}$ $= \frac{d\mathbf{u}}{dt}\otimes\mathbf{v} + \mathbf{u}\otimes\frac{d\mathbf{v}}{dt}.$
3.2	Given that $\alpha(t) \in \mathbb{R}, \mathbf{u}(t), \mathbf{v}(t) \in \mathbb{E}$, and $\mathbf{S}(t), \mathbf{T}(t) \in \mathbb{L}$, are all functions of a scalar variable t, show that $(a) \frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}, (b)\ \frac{d}{dt}\mathbf{TS} = \frac{d\mathbf{T}}{dt}\mathbf{S} + \mathbf{T}\frac{d\mathbf{S}}{dt}, (c) \frac{d}{dt}(\mathbf{Tu}) = \frac{d\mathbf{T}}{dt}\mathbf{u} + \mathbf{T}\frac{d\mathbf{u}}{dt},$ *and* $(d) \frac{d}{dt}\mathbf{S}^{\mathrm{T}} = \left(\frac{d\mathbf{S}}{dt}\right)^{\mathrm{T}}$
b	$\frac{d}{dt}(\mathbf{ST}) = \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{T}(t+h) - \mathbf{S}(t)\mathbf{T}(t)}{t}$

	$= \lim_{h\to 0}\frac{\mathbf{S}(t+h)\mathbf{T}(t+h)-\mathbf{S}(t)\mathbf{T}(t+h)}{t} + \lim_{h\to 0}\frac{\mathbf{S}(t)\mathbf{T}(t+h)-\mathbf{S}(t)\mathbf{T}(t)}{t}$ $= \left(\lim_{h\to 0}\frac{\mathbf{S}(t+h)-\mathbf{S}(t)}{t}\right)\left(\lim_{h\to 0}\mathbf{T}(t+h)\right) + \mathbf{S}(t)\lim_{h\to 0}\frac{\mathbf{T}(t+h)-\mathbf{T}(t)}{t}$ $= \frac{d}{dt}(\mathbf{ST}) = \frac{d\mathbf{S}}{dt}\mathbf{T} + \mathbf{S}\frac{d\mathbf{T}}{dt}.$
3.3	Given that $\mathbf{Q}(t)\mathbf{Q}^{\mathrm{T}}(t) = \mathbf{I}$, the identity tensor, show (a) that $\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}$ is an antisymmetric tensor, and (b) that $\mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}$ is an antisymmetric tensor
	Differentiating $\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I}$, $$\frac{d}{dt}\left(\mathbf{Q}\mathbf{Q}^{\mathrm{T}}\right) = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = \frac{d\mathbf{I}}{dt} = \mathbf{O}$$ Consequently, $$\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} = -\left(\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}\right)^{\mathrm{T}}$$ So we have that the tensor $\mathbf{T} = \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}$ is negative of its own transpose, hence it is skew.
	$$\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{I} = \mathbf{Q}^{\mathrm{T}}\mathbf{Q}$$ Differentiating as before, $$\frac{d}{dt}\left(\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\right) = \frac{d\mathbf{Q}^{\mathrm{T}}}{dt}\mathbf{Q} + \mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt} = \mathbf{O}$$ So that, $$\frac{d\mathbf{Q}^{\mathrm{T}}}{dt}\mathbf{Q} = \left(\mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}\right)^{\mathrm{T}} = -\mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}$$ Tensor $\mathbf{T} = \mathbf{Q}^{\mathrm{T}}\frac{d\mathbf{Q}}{dt}$ is negative of its own transpose, hence skew as required.
3.4	Show that $\frac{d}{d\mathbf{S}}\operatorname{tr}(\mathbf{S}^k) = k(\boldsymbol{S}^{k-1})^T$

	The cases, $k=1, k=2$ are already provided in the text. When $k=3$, $$Df(\mathbf{S}, d\mathbf{S}) = \frac{d}{d\alpha} f(\mathbf{S}+\alpha d\mathbf{S})\Big\vert_{\alpha=0} = \frac{d}{d\alpha} \mathrm{tr}\{(\mathbf{S}+\alpha d\mathbf{S})^3\}\Big\vert_{\alpha=0}$$ $$= \frac{d}{d\alpha} \mathrm{tr}\{(\mathbf{S}+\alpha d\mathbf{S})(\mathbf{S}+\alpha d\mathbf{S})(\mathbf{S}+\alpha d\mathbf{S})\}\Big\vert_{\alpha=0}$$ $$= \mathrm{tr}\left[\frac{d}{d\alpha}(\mathbf{S}+\alpha d\mathbf{S})(\mathbf{S}+\alpha d\mathbf{S})(\mathbf{S}+\alpha d\mathbf{S})\right]\Big\vert_{\alpha=0}$$ $$= \mathrm{tr}[d\mathbf{S}(\mathbf{S}+\alpha d\mathbf{S})(\mathbf{S}+\alpha d\mathbf{S}) + (\mathbf{S}+\alpha d\mathbf{S})d\mathbf{S}(\mathbf{S}+\alpha d\mathbf{S})$$ $$+ (\mathbf{S}+\alpha d\mathbf{S})(\mathbf{S}+\alpha d\mathbf{S})d\mathbf{S}]\vert_{\alpha=0}$$ $$= \mathrm{tr}[d\mathbf{S}\,\mathbf{S}\,\mathbf{S} + \mathbf{S}\,d\mathbf{S}\,\mathbf{S} + \mathbf{S}\,\mathbf{S}\,d\mathbf{S}] = 3(\mathbf{S}^2)^T : d\mathbf{S}$$ It easily follows by induction that, $\frac{d}{d\mathbf{S}} f(\mathbf{S}) = k(\mathbf{S}^{k-1})^{\mathrm{T}}$.
3.5	Define the magnitude of tensor $\mathbf{A}$ as, $\vert\mathbf{A}\vert = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}})}$ Show that $\frac{\partial\vert\mathbf{A}\vert}{\partial\mathbf{A}} = \frac{\mathbf{A}}{\vert\mathbf{A}\vert}$
	Given a scalar α variable, the derivative of a scalar function of a tensor $f(\boldsymbol{A})$ is $$\frac{\partial f(\mathbf{A})}{\partial\mathbf{A}} : \mathbf{B} = \lim_{\alpha\to 0} \frac{\partial}{\partial\alpha} f(\mathbf{A}+\alpha\mathbf{B})$$ for any arbitrary tensor $\mathbf{B}$. In the case of $f(\mathbf{A}) = \vert\mathbf{A}\vert$, $$\frac{\partial\vert\mathbf{A}\vert}{\partial\mathbf{A}} : \mathbf{B} = \lim_{\alpha\to 0} \frac{\partial}{\partial\alpha} \vert\mathbf{A}+\alpha\mathbf{B}\vert$$ $$\vert\mathbf{A}+\alpha\mathbf{B}\vert = \sqrt{\mathrm{tr}(\mathbf{A}+\alpha\mathbf{B})(\mathbf{A}+\alpha\mathbf{B})^{\mathrm{T}}} = \sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{B}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{A}\mathbf{B}^{\mathrm{T}} + \alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}$$ Note that everything under the root sign here is scalar and that the trace operation is linear. Consequently, we can write, $$\lim_{\alpha\to 0} \frac{\partial}{\partial\alpha} \vert\mathbf{A}+\alpha\mathbf{B}\vert = \lim_{\alpha\to 0} \frac{\mathrm{tr}\,(\mathbf{B}\mathbf{A}^{\mathrm{T}}) + \mathrm{tr}\,(\mathbf{A}\mathbf{B}^{\mathrm{T}}) + 2\alpha\,\mathrm{tr}\,(\mathbf{B}\mathbf{B}^{\mathrm{T}})}{2\sqrt{\mathrm{tr}(\mathbf{A}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{B}\mathbf{A}^{\mathrm{T}} + \alpha\mathbf{A}\mathbf{B}^{\mathrm{T}} + \alpha^2\mathbf{B}\mathbf{B}^{\mathrm{T}})}} = \frac{2\mathbf{A}:\mathbf{B}}{2\sqrt{\mathbf{A}:\mathbf{A}}} = \frac{\mathbf{A}}{\vert\mathbf{A}\vert} : \mathbf{B}$$ So that, $$\frac{\partial\vert\mathbf{A}\vert}{\partial\mathbf{A}} : \mathbf{B} = \frac{\mathbf{A}}{\vert\mathbf{A}\vert} : \mathbf{B}$$ or, $$\frac{\partial\vert\mathbf{A}\vert}{\partial\mathbf{A}} = \frac{\mathbf{A}}{\vert\mathbf{A}\vert}$$

	as required since $\mathbf{B}$ is arbitrary.
3.6	Without breaking down into components, use Liouville's theorem, $\frac{\partial}{\partial\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\,\mathrm{tr}\left(\dot{\mathbf{T}}\mathbf{T}^{-1}\right)$, to establish the fact that $\frac{\partial \det(\mathbf{T})}{\partial \mathbf{T}} = \mathbf{T}^{c}$
	Start from Liouville's Theorem, given a scalar parameter such that $\mathbf{T} = \mathbf{T}(\alpha)$, $$\frac{\partial}{\partial\alpha}(\det\mathbf{T}) = \det\mathbf{T}\,\mathrm{tr}\left[\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)\mathbf{T}^{-1}\right] = [(\det\mathbf{T})\mathbf{T}^{-\mathrm{T}}]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)$$ By the chain rule, $$\frac{\partial}{\partial\alpha}(\det\mathbf{T}) = \left[\frac{\partial}{\partial\mathbf{T}}(\det\mathbf{T})\right]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right)$$ It therefore follows that, $$\left[\frac{\partial}{\partial\mathbf{T}}(\det\mathbf{T}) - [(\det\mathbf{T})\mathbf{T}^{-\mathrm{T}}]\right]:\left(\frac{\partial\mathbf{T}}{\partial\alpha}\right) = 0$$ Hence $$\frac{\partial}{\partial\mathbf{T}}(\det\mathbf{T}) = (\det\mathbf{T})\mathbf{T}^{-\mathbf{T}} = \mathbf{T}^{\mathrm{c}}$$
3.7	If $\mathbf{T}$ is invertible, show that $\frac{\partial}{\partial\mathbf{T}}(\log\det(\mathbf{T})) = \mathbf{T}^{-\mathbf{T}}$
	$$\begin{aligned}\frac{\partial}{\partial\mathbf{T}}(\log\det\mathbf{T}) &= \frac{\partial(\log\det\mathbf{T})}{\partial\det\mathbf{T}}\frac{\partial\det\mathbf{T}}{\partial\mathbf{T}}\\ &= \frac{1}{\det\mathbf{T}}\mathbf{T}^{\mathrm{c}} = \frac{1}{\det\mathbf{T}}(\det\ \mathbf{T})\mathbf{T}^{-\mathrm{T}}\\ &= \mathbf{T}^{-\mathrm{T}}\end{aligned}$$
3.8	Use Caley-Hamilton theorem to express the third invariant in terms of traces only (b) Use this result and the fact that $\mathbf{S}^{\mathrm{c}} = (\mathbf{S}^2 - I_1\mathbf{S} + I_2\mathbf{I})^{\mathrm{T}}$ (2.19)to show that the derivative of the determinant is the cofactor.
a	Given a tensor $\mathbf{S}$, by Cayley Hamilton theorem, $$\mathbf{S}^3 - I_1\mathbf{S}^2 + I_2\mathbf{S} - I_3 = 0$$ Note that $I_1 = I_1(\mathbf{S}), I_2 = I_2(\mathbf{S})\ and\ I_3 = I_3(\mathbf{S})$ the first, second and third invariants of $\boldsymbol{S}$ are all scalar functions of the tensor $\mathbf{S}$. Taking the trace of the above equation,

	$$I_1(\mathbf{S}^3) = I_1(\mathbf{S})I_1(\mathbf{S}^2) - I_2(\mathbf{S})I_1(\mathbf{S}) + 3I_3(\mathbf{S})$$ $$= I_1(\mathbf{S})\big(I_1^2(\mathbf{S}) - 2I_2(\mathbf{S})\big) - I_1(\mathbf{S})I_2(\mathbf{S}) + 3I_3(\mathbf{S})$$ $$= I_1^3(\mathbf{S}) - 3I_1(\mathbf{S})I_2(\mathbf{S}) + 3I_3(\mathbf{S})$$ Or, $$I_3(\mathbf{S}) = \frac{1}{3}\big(I_1(\mathbf{S}^3) - I_1^3(\mathbf{S}) + 3I_1(\mathbf{S})I_2(\mathbf{S})\big)$$ $$= \frac{1}{3}\left(I_1(\mathbf{S}^3) - I_1^3(\mathbf{S}) + \frac{3}{2}I_1(\mathbf{S})\big(I_1^2(\mathbf{S}) - I_1(\mathbf{S}^2)\big)\right)$$ $$= \frac{1}{6}\big(2I_1(\mathbf{S}^3) + I_1^3(\mathbf{S}) - 3I_1(\mathbf{S})I_1(\mathbf{S}^2)\big)$$ which show that the third invariant is itself expressible in terms of traces only. It is therefore invariant in value as a result of coordinate transformation.
b	Taking the Fréchet derivative of the third invariant expression above, noting that $\frac{d}{dS}\,\mathrm{tr}\,(\mathbf{S}^k) = k(\mathbf{S}^{k-1})^{\mathrm{T}}$ $$6\frac{\partial}{\partial \mathbf{S}}I_3(\mathbf{S}) = 6\mathbf{S}^{2\mathrm{T}} + 3I_1^2(\mathbf{S})\mathbf{I} - 6I_1(\mathbf{S})\mathbf{S}^{\mathrm{T}} - 3I_1(\mathbf{S}^2)\mathbf{I}$$ so that $$\frac{\partial}{\partial \mathbf{S}}I_3(\mathbf{S}) = \mathbf{S}^{2\mathrm{T}} - I_1^2(\mathbf{S})\mathbf{I} + \mathbf{I}I_2(\mathbf{S}) + 3I_1(\mathbf{S})(I_1(\mathbf{S})\mathbf{I} - \mathbf{S}^{\mathrm{T}})$$ $$= (\mathbf{S}^2 - I_1(\mathbf{S})\mathbf{S} + I_2(\mathbf{S})\mathbf{I})^{\mathrm{T}}$$ $$= \mathbf{S}^{\mathrm{c}}$$
3.9	If $\mathbf{T}$ is invertible, show that $\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = -\mathbf{T}^{-\mathbf{T}}$
	$$\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = \frac{d(\log\det(\mathbf{T}^{-1}))}{d\det(\mathbf{T}^{-1})}\frac{d\det(\mathbf{T}^{-1})}{d\mathbf{T}^{-1}}\frac{d\mathbf{T}^{-1}}{d\mathbf{T}}$$ $$= -\frac{1}{\det(\mathbf{T}^{-1})}\mathbf{T}^{-\mathrm{c}}(\mathbf{T}^{-1}\boxtimes\mathbf{T}^{-\mathrm{T}})$$ $$= -\frac{1}{\det(\mathbf{T}^{-1})}\det(\mathbf{T}^{-1})\,\mathbf{T}^{\mathrm{T}}(\mathbf{T}^{-1}\boxtimes\mathbf{T}^{-\mathrm{T}})$$ $$= -\mathbf{T}^{-\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{T}^{-\mathbf{T}} = -\mathbf{T}^{-\mathrm{T}}$$ An easier method that does not require the evaluation of a fourth-order tensor:

	$$\frac{d}{d\mathbf{T}}(\log\det(\mathbf{T}^{-1})) = \frac{d(\log\det(\mathbf{T}^{-1}))}{d\det(\mathbf{T}^{-1})}\frac{d\det(\mathbf{T}^{-1})}{d\det(\mathbf{T})}\frac{d\det(\mathbf{T})}{d\mathbf{T}}$$ $$= \frac{1}{\det(\mathbf{T}^{-1})}\frac{d\left(1/\det(\mathbf{T})\right)}{d\det(\mathbf{T})}\mathbf{T}^{\mathrm{c}}$$ $$= -\frac{\det(\mathbf{T})}{\det(\mathbf{T}^2)}\mathbf{T}^{\mathbf{c}}$$ $$= -\mathbf{T}^{-\mathrm{T}}$$
3.10	Given that $\mathbf{A}$ and $\mathbf{B}$ are constant tensors, show that $\frac{\partial}{\partial\mathbf{S}}\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = \mathbf{A}^{\mathrm{T}}\mathbf{B}$
	First observe that $\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS})$. If we write, $\mathbf{C} \equiv \mathbf{B}^{\mathrm{T}}\mathbf{A}$, it is obvious from the above that $\frac{\partial}{\partial\mathbf{S}}\mathrm{tr}(\mathbf{CS}) = \mathbf{C}^{\mathrm{T}}$. Therefore, $$\frac{\partial}{\partial\mathbf{S}}\mathrm{tr}(\mathbf{ASB}^{\mathrm{T}}) = (\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathbf{T}} = \mathbf{A}^{\mathrm{T}}\mathbf{B}$$
3.11	Given that $\mathbf{A}$ and $\mathbf{B}$ are constant tensors, show that $\frac{\partial}{\partial\mathbf{S}}\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathbf{B}^{\mathrm{T}}\mathbf{A}$
	Observe that $$\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \mathrm{tr}(\mathbf{B}^{\mathrm{T}}\mathbf{AS}^{\mathrm{T}})$$ $$= \mathrm{tr}[\mathbf{S}(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}]$$ $$= \mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}]$$ [The transposition does not alter trace; neither does a cyclic permutation. Ensure you understand why each equality here is true.] Consequently, $$\frac{\partial}{\partial\mathbf{S}}\mathrm{tr}(\mathbf{AS}^{\mathrm{T}}\mathbf{B}^{\mathrm{T}}) = \frac{\partial}{\partial\mathbf{S}}\mathrm{tr}[(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}\mathbf{S}] = [(\mathbf{B}^{\mathrm{T}}\mathbf{A})^{\mathrm{T}}]^{\mathbf{T}} = \mathbf{B}^{\mathrm{T}}\mathbf{A}$$
3.12	Let $\mathbf{S}$ be a symmetric and positive definite tensor and let $I_1(\mathbf{S}), I_2(\mathbf{S})andI_3(\mathbf{S})$ be the three principal invariants of $\mathbf{S}$ show, using components, that *(a)* $\frac{\partial I_1(\mathbf{S})}{\partial\mathbf{S}} = \mathbf{I}$ the identity tensor, *(b)* $\frac{\partial I_2(\mathbf{S})}{\partial\mathbf{S}} = I_1(\mathbf{S})\mathbf{I} - \mathbf{S}$ and (c) $\frac{\partial I_3(\mathbf{S})}{\partial\mathbf{S}} = I_3(\mathbf{S})\,\mathbf{S}^{-1}$

a	$\frac{\partial I_1(S)}{\partial S}$ can be written in the invariant component form as, $$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j$$ Recall that $I_1(\boldsymbol{S}) = \operatorname{tr} \mathbf{S} = S_{\alpha\alpha}$ hence $$\frac{\partial I_1(\mathbf{S})}{\partial \mathbf{S}} = \frac{\partial I_1(\mathbf{S})}{\partial S_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j = \frac{\partial S_{\alpha\alpha}}{\partial S_{ij}} \mathbf{e}_i \otimes \mathbf{e}_j$$ $$= \delta_{i\alpha}\delta_{\alpha j} \mathbf{e}_i \otimes \mathbf{e}_j = \mathbf{I}$$ which is the identity tensor as expected.
b	$\frac{\partial I_2(S)}{\partial S}$ in a similar way can be written in the invariant component form as, $$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2} \frac{\partial}{\partial S_{ij}} \left[S_{\alpha\alpha} S_{\beta\beta} - S_{\alpha\beta} S_{\beta\alpha} \right] \mathbf{e}_i \otimes \mathbf{e}_j$$ where we have utilized the fact that $I_2(\mathbf{S}) = \frac{1}{2}[\operatorname{tr}^2 \mathbf{S} - \operatorname{tr} \mathbf{S}^2]$. Consequently, $$\frac{\partial I_2(\mathbf{S})}{\partial \mathbf{S}} = \frac{1}{2} \frac{\partial}{\partial S_{ij}} \left[S_{\alpha\alpha} S_{\beta\beta} - S_{\alpha\beta} S_{\beta\alpha} \right] \mathbf{e}_i \otimes \mathbf{e}_j$$ $$= \frac{1}{2} \left[\delta_{i\alpha}\delta_{\alpha j} S_{\beta\beta} + \delta_{i\beta}\delta_{\beta j} S_{\alpha\alpha} - \delta_{\beta i}\delta_{j\alpha} S_{\alpha\beta} - \delta_{\alpha i}\delta_{j\beta} S_{\beta\alpha} \right] \mathbf{e}_i \otimes \mathbf{e}_j$$ $$= \frac{1}{2} \left[\delta_{ij} S_{\beta\beta} + \delta_{ij} S_{\alpha\alpha} - S_{ij} - S_{ji} \right] \mathbf{e}_i \otimes \mathbf{e}_j$$ $$= I_1(\mathbf{S})\mathbf{I} - \mathbf{S}$$
c	$$\det \mathbf{S} = \frac{1}{3!} e_{ijk} e_{rst} S_{ir} S_{js} S_{kt}$$ Differentiating wrt $S_{\alpha\beta}$, we obtain, $$\frac{\partial S}{\partial S_{\alpha\beta}} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \frac{1}{3!} e_{ijk} e_{rst} \left[\frac{\partial S_{ir}}{\partial S_{\alpha\beta}} S_{js} S_{kt} + S_{ir} \frac{\partial S_{js}}{\partial S_{\alpha\beta}} S_{kt} + S_{ir} S_{js} \frac{\partial S_{kt}}{\partial S_{\alpha\beta}} \right] \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$= \frac{1}{3!} e_{ijk} e_{rst} \left[\delta_{i\alpha}\delta_{r\beta} S_{js} S_{kt} + S_{ir} \delta_{j\alpha}\delta_{s\beta} S_{kt} + S_{ir} S_{js} \delta_{k\alpha}\delta_{t\beta} \right] \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$= \frac{1}{3!} e_{ijk} e_{rst} \left[S_{js} S_{kt} + S_{js} S_{kt} + S_{js} S_{kt} \right] \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$= \frac{1}{2!} e_{ijk} e_{rst} S_{js} S_{kt} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \equiv [S^c]^{\alpha\beta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ Which is the cofactor of $\left[S_{\alpha\beta}\right]$ or $\boldsymbol{S}$

3.13	Divergence of a product: Given that φ is a scalar field and $\mathbf{v}$ a vector field, show that $\mathrm{div}(\varphi\mathbf{v}) = (\mathrm{grad}\varphi)\cdot\mathbf{v} + \varphi\,\mathrm{div}\,\mathbf{v}$
	$\mathrm{grad}(\varphi\mathbf{v}) = (\varphi v_i)_{,j}\,\mathbf{e}_i \otimes \mathbf{e}_j$ $= \varphi_{,j}\, v_i \mathbf{e}_i \otimes \mathbf{e}_j + \varphi v_{i,j}\,\mathbf{e}_i \otimes \mathbf{e}_j$ $= \mathbf{v} \otimes (\mathrm{grad}\,\varphi) + \varphi\,\mathrm{grad}\,\mathbf{v}$ Now, $\mathrm{div}(\varphi\mathbf{v}) = \mathrm{tr}\big(\mathrm{grad}(\varphi\mathbf{v})\big)$. Taking the trace of the above, we have: $\mathrm{div}(\varphi\mathbf{v}) = \mathbf{v}\cdot(\mathrm{grad}\,\varphi) + \varphi\,\mathrm{div}\,\mathbf{v}$
3.14	Show that $\mathrm{grad}(\mathbf{u}\cdot\mathbf{v}) = (\mathrm{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$
	$\mathbf{u}\cdot\mathbf{v} = u_i v_i$ is a scalar sum of components. $\mathrm{grad}(\mathbf{u}\cdot\mathbf{v}) = (u_i v_i)_{,j}\,\mathbf{e}_j$ $= u_{i,j}\, v_i \mathbf{e}_j + u_i v_{i,j}\,\mathbf{e}_j$ Now $\mathrm{grad}\,\mathbf{u} = u_{i,j}\,\mathbf{e}_i \otimes \mathbf{e}_j$ swapping the bases, we have that, $(\mathrm{grad}\,\mathbf{u})^{\mathrm{T}} = u_{i,j}\,(\mathbf{e}_j \otimes \mathbf{e}_i)$. Writing $\mathbf{v} = v_k \mathbf{e}_k$, we have that, $(\mathrm{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} = u_{i,j}\, v_k(\mathbf{e}_j \otimes \mathbf{e}_i)\mathbf{e}_k = u_{i,j}\, v_k \mathbf{e}_j \delta_{ik} = u_{i,j}\, v_i \mathbf{e}_j$ It is easy to similarly show that $u_i v_{i,j}\,\mathbf{e}_j = (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}\boldsymbol{u}$. Clearly, $\mathrm{grad}(\mathbf{u}\cdot\mathbf{v}) = (u_i v_i)_{,j}\,\mathbf{e}_j$ $= u_{i,j}\, v_i \mathbf{e}_j + u_i v_{i,j}\,\mathbf{e}_j$ $= (\mathrm{grad}\,\mathbf{u})^{\mathrm{T}}\mathbf{v} + (\mathrm{grad}\,\mathbf{v})^{\mathrm{T}}\mathbf{u}$ As required.
3.16	Show that $\mathrm{grad}(\mathbf{u}\times\mathbf{v}) = (\mathbf{u}\,\times)\mathrm{grad}\,\mathbf{v} - (\mathbf{v}\,\times)\mathrm{grad}\,\mathbf{u}$
	$\mathbf{u}\times\mathbf{v} = e_{ijk} u_j v_k \mathbf{e}_i$ Recall that the gradient of this vector is the tensor, $\mathrm{grad}(\mathbf{u}\times\mathbf{v}) = (e_{ijk} u_j v_k)_{,l}\,\mathbf{e}_i \otimes \mathbf{e}_l$ $= e_{ijk} u_{j,l}\, v_k \mathbf{e}_i \otimes \mathbf{e}_l + e_{ijk} u_j v_{k,l}\,\mathbf{e}_i \otimes \mathbf{e}_l$ $= -e_{ikj} u_{j,l}\, v_k \mathbf{e}_i \otimes \mathbf{e}_l + e_{ijk} u_j v_{k,l}\,\mathbf{e}_i \otimes \mathbf{e}_l$ $= -(\mathbf{v}\,\times)\mathrm{grad}\,\mathbf{u} + (\mathbf{u}\,\times)\mathrm{grad}\,\mathbf{v}$

3.17 Show that $\text{div}\,(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}$

We already have the expression for $\text{grad}(\mathbf{u}\times\mathbf{v})$ above; remember that

$$\begin{aligned}\text{div}\,(\mathbf{u}\times\mathbf{v}) &= \text{tr}[\text{grad}(\mathbf{u}\times\mathbf{v})]\\ &= -e_{ikj}u_{j,l}\,v_k\mathbf{e}_i\cdot\mathbf{e}_l + e_{ijk}u_jv_{k,l}\,\mathbf{e}_i\cdot\mathbf{e}_l\\ &= -e_{ikj}u_{j,l}\,v_k\delta_{il} + e_{ijk}u_jv_{k,l}\,\delta_{il}\\ &= -e_{ikj}u_{j,i}\,v_k + e_{ijk}u_jv_{k,i}\\ &= \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}\end{aligned}$$

3.18 Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that $\text{curl}\,(\phi\mathbf{v}) = \phi\,\text{curl}\,\mathbf{v} + (\text{grad}\,\phi)\times\mathbf{v}$.

$$\begin{aligned}\text{curl}\,(\phi\mathbf{v}) &= e_{ijk}(\phi v_k),_j\,\mathbf{e}_i\\ &= e_{ijk}(\phi,_j\,v_k + \phi v_{k},_j)\mathbf{e}_i\\ &= e_{ijk}\phi,_j\,v_k\mathbf{e}_i + e_{ijk}\phi v_{k},_j\,\mathbf{e}_i\\ &= (\text{grad}\,\phi)\times\mathbf{v} + \phi\,\text{curl}\,\mathbf{v}\end{aligned}$$

3.19 Show that $\text{div}\,(\mathbf{u}\otimes\mathbf{v}) = (\text{div}\,\mathbf{v})\mathbf{u} + (\text{grad}\,\mathbf{u})\mathbf{v}$

$\mathbf{u}\otimes\mathbf{v}$ is the tensor, $u_iv_j\mathbf{e}_i\otimes\mathbf{e}_l$. The gradient of this is the third order tensor,

$$\text{grad}\,(\mathbf{u}\otimes\mathbf{v}) = (u_iv_j),_k\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k$$

And by divergence, we mean the contraction of the last basis vector:

$$\begin{aligned}\text{div}\,(\mathbf{u}\otimes\mathbf{v}) &= (u_iv_j),_k\,(\mathbf{e}_i\otimes\mathbf{e}_j)\mathbf{e}_k\\ &= (u_iv_j),_k\,\mathbf{e}_i\delta_{jk} = (u_iv_j),_j\,\mathbf{e}_i\\ &= u_{i},_j\,v_j\mathbf{e}_i + u_iv_{j},_j\,\mathbf{e}_i\\ &= (\text{grad}\,\mathbf{u})\mathbf{v} + (\text{div}\,\mathbf{v})\mathbf{u}\end{aligned}$$

3.20 For a scalar field ϕ and a tensor field $\mathbf{T}$ show that $\text{grad}\,(\phi\mathbf{T}) = \phi\text{grad}\,\mathbf{T} + \mathbf{T}\otimes\text{grad}\phi$. Also show that $\text{div}\,(\phi\mathbf{T}) = \phi\,\text{div}\,\mathbf{T} + \mathbf{T}\text{grad}\phi$

$$\begin{aligned}\text{grad}(\phi\mathbf{T}) &= (\phi T_{ij}),_k\,\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\\ &= (\phi,_k\,T_{ij} + \phi T_{ij},_k)\mathbf{e}_i\otimes\mathbf{e}_j\otimes\mathbf{e}_k\\ &= \mathbf{T}\otimes\text{grad}\phi + \phi\text{grad}\,\mathbf{T}\end{aligned}$$

Furthermore, we can contract the last two bases and obtain,

$$\begin{aligned}\operatorname{div}(\phi \boldsymbol{T}) &= \left(\phi_{,k} T_{ij} + \phi T_{ij,k}\right)\mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{e}_k \\ &= \left(\phi_{,k} T_{ij} + \phi T_{ij,k}\right)\boldsymbol{g}_i \delta_j^k \\ &= T_{ik}\, \phi_{,k}\, \mathbf{e}_i + \phi T_{ik,k}\, \mathbf{e}_i \\ &= \mathbf{T} \operatorname{grad} \phi + \phi \operatorname{div} \mathbf{T}\end{aligned}$$

3.21 For two arbitrary tensors $\mathbf{S}$ and $\mathbf{T}$, show that $\operatorname{grad}(\mathbf{ST}) = (\operatorname{grad} \mathbf{S}^{\mathrm{T}})^{\mathrm{T}}\mathbf{T} + \mathbf{S} \operatorname{grad} \mathbf{T}$

$$\begin{aligned}\operatorname{grad}(\mathbf{ST}) &= \left(S_{ij}T_{jk}\right)_{,\alpha}\, \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\ &= \left(S_{ij,\alpha}T_{jk} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\ &= \left(T_{kj}S_{ji,\alpha} + S_{ij}T_{jk,\alpha}\right)\mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_\alpha \\ &= (\operatorname{grad} \mathbf{S}^{\mathrm{T}})^{\mathrm{T}}\mathbf{T} + \mathbf{S} \operatorname{grad} \mathbf{T}\end{aligned}$$

3.22 Show that $\operatorname{curl} (\operatorname{grad} \mathbf{v})^{\mathrm{T}} = \operatorname{grad}(\operatorname{curl} \mathbf{v})$

From previous derivation, we can see that,

$\operatorname{curl} \mathbf{T} = e_{ijk} T_{\alpha k,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$. Clearly,

$$\operatorname{curl} \mathbf{T}^{\mathrm{T}} = e_{ijk} T_{k\alpha,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$$

so that $\operatorname{curl} (\operatorname{grad} \mathbf{v})^{\mathrm{T}} = e_{ijk} v_{k,\alpha j}\, \mathbf{e}_i \otimes \mathbf{e}_\alpha$. But $\operatorname{curl} \mathbf{v} = e_{ijk} v_{k,j}\; \mathbf{e}_i$. The gradient of this is,

$$\operatorname{grad}(\operatorname{curl} \mathbf{v}) = \left(e_{ijk} v_{k,j}\right)_{,\alpha} \mathbf{e}_i \otimes \mathbf{e}_\alpha = e_{ijk} v_{k,j\alpha}\, \mathbf{e}_i \otimes \mathbf{e}_\alpha = \operatorname{curl} (\operatorname{grad} \mathbf{v})^{\mathrm{T}}$$

3.23 For a second-order tensor $\mathbf{T} = T_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$, write the explicit expression for $\operatorname{curl} \mathbf{T}$ in Cartesian Coordinates.

$$\operatorname{curl} \mathbf{T} = e_{ijk} T_{k\alpha,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$$

i	α	$e_{ijk} T_{k\alpha,j}\; \mathbf{e}_i \otimes \mathbf{e}_\alpha$	
1	1	$\left(\frac{\partial T_{13}}{\partial x_2} - \frac{\partial T_{12}}{\partial x_3}\right)\mathbf{e}_1 \otimes \mathbf{e}_1$	
i	2	$\left(\frac{\partial T_{23}}{\partial x_2} - \frac{\partial T_{22}}{\partial x_3}\right)\mathbf{e}_1 \otimes \mathbf{e}_3$	
i	3	$\left(\frac{\partial T_{33}}{\partial x_2} - \frac{\partial T_{32}}{\partial x_3}\right)\mathbf{e}_1 \otimes \mathbf{e}_2$	
2	1	$\left(\frac{\partial T_{11}}{\partial x_3} - \frac{\partial T_{13}}{\partial x_1}\right)\mathbf{e}_2 \otimes \mathbf{e}_1$	

	2	2	$\left(\frac{\partial T_{21}}{\partial x_3}-\frac{\partial T_{23}}{\partial x_1}\right)\mathbf{e}_2\otimes\mathbf{e}_2$			
	2	3	$\left(\frac{\partial T_{31}}{\partial x_2}-\frac{\partial T_{33}}{\partial x_3}\right)\mathbf{e}_2\otimes\mathbf{e}_3$			
	3	1	$\left(\frac{\partial T_{12}}{\partial x_1}-\frac{\partial T_{11}}{\partial x_2}\right)\mathbf{e}_3\otimes\mathbf{e}_1$			

3.24	For two arbitrary tensors $\mathbf{S}$ and $\mathbf{T}$, show that $\operatorname{div}(\mathbf{ST})=(\operatorname{grad}\mathbf{S}):\mathbf{T}+\mathbf{T}\operatorname{div}\mathbf{S}$
	$\operatorname{grad}(\mathbf{ST})=(S_{ij}T_{jk})_{,\alpha}\,\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_\alpha$ $=(S_{ij,\alpha}T_{jk}+S_{ij}T_{jk,\alpha})\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_\alpha$ $\operatorname{div}(\mathbf{ST})=(S_{ij,\alpha}T_{jk}+S_{ij}T_{jk,\alpha})\mathbf{e}_i(\mathbf{e}_k\cdot\mathbf{e}_\alpha)$ $=(S_{ij,\alpha}T_{jk}+S_{ij}T_{jk,\alpha})\mathbf{e}_i\delta_{k\alpha}$ $=(S_{ij,k}T_{jk}+S_{ij}T_{jk,k})\mathbf{e}_i$ $=(\operatorname{grad}\mathbf{S}):\mathbf{T}+\mathbf{S}\operatorname{div}\mathbf{T}$
3.25	For two arbitrary vectors, $\mathbf{u}$ and $\mathbf{v}$, show that $\operatorname{grad}(\mathbf{u}\times\mathbf{v})=(\mathbf{u}\times)\operatorname{grad}\mathbf{v}-(\mathbf{v}\times)\operatorname{grad}\mathbf{u}$
	$\operatorname{grad}(\mathbf{u}\times\mathbf{v})=(e_{ijk}u_jv_k)_{,l}\,\mathbf{e}_i\otimes\mathbf{e}_l$ $=(e_{ijk}u_{j,l}v_k+e_{ijk}u_jv_{k,l})\mathbf{e}_i\otimes\mathbf{e}_l$ $=(u_{j,l}e_{ijk}v_k+v_{k,l}e_{ijk}u_j)\mathbf{e}_i\otimes\mathbf{e}_l$ $=-(\mathbf{v}\times)\operatorname{grad}\mathbf{u}+(\mathbf{u}\times)\operatorname{grad}\mathbf{v}$
3.26	For a vector field $\mathbf{u}$, show that $\operatorname{grad}(\mathbf{u}\times)$ is a third ranked tensor. Hence or otherwise show that $\operatorname{div}(\mathbf{u}\times)=-\operatorname{curl}\mathbf{u}$.
	The second–order tensor $(\mathbf{u}\times)$ is defined as $e_{ijk}u_j\mathbf{e}_i\otimes\mathbf{e}_k$. Taking the derivative with an independent base, we have $\operatorname{grad}(\mathbf{u}\times)=e_{ijk}u_{j,l}\,\mathbf{e}_i\otimes\mathbf{e}_k\otimes\mathbf{e}_l$ This gives a third order tensor as we have seen. Contracting on the last two bases,

	$= -\text{curl}\,\mathbf{u}$
3.27	Show that $\text{div}\,(\phi\mathbf{I}) = \text{grad}\,\phi$
	Note that $\phi\mathbf{I} = (\phi\delta_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$. Also note that $\text{grad}\,\phi\mathbf{I} = (\phi\delta_{\alpha\beta})_{,i}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta \otimes \mathbf{e}_i$ The divergence of this third order tensor is the contraction of the last two bases: $\text{div}\,(\phi\mathbf{I}) = \text{tr}(\text{grad}\,\phi\mathbf{I}) = (\phi\delta_{\alpha\beta})_{,i}\,(\mathbf{e}_\alpha \otimes \mathbf{e}_\beta)\mathbf{e}_i = (\phi\delta_{\alpha\beta})_{,i}\,\mathbf{e}_\alpha\delta_{\beta i}$ $= \phi_{,i}\,\delta_{\alpha\beta}\mathbf{e}_\alpha\delta_{\beta i}$ $= \phi_{,i}\,\delta_{\alpha i}\mathbf{e}_\alpha = \phi_{,i}\,\mathbf{e}_i = \text{grad}\,\phi$
3.28	Show that $\text{curl}\,(\phi\mathbf{I}) = (\,\text{grad}\,\phi)\,\times$
	Note that $\phi\mathbf{I} = (\phi\delta_{\alpha\beta})\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, and that $\text{curl}\,\mathbf{T} = e_{ijk}T_{\alpha k,j}\;\mathbf{e}_i \otimes \mathbf{e}_\alpha$ so that, $\text{curl}\,(\phi\mathbf{I}) = e_{ijk}(\phi\delta_{\alpha k})_{,j}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$ $= e_{ijk}(\phi_{,j}\,\delta_{\alpha k})\mathbf{e}_i \otimes \mathbf{e}_\alpha = e_{ijk}\phi_{,j}\,\mathbf{e}_i \otimes \mathbf{e}_k$ $= (\,\text{grad}\,\phi)\,\times$
3.29	Show that the dyad $\mathbf{u} \otimes \mathbf{v}$ is NOT, in general symmetric: $\mathbf{u} \otimes \mathbf{v} = \mathbf{v} \otimes \mathbf{u} - (\mathbf{u} \times \mathbf{v})\,\times$
	$\mathbf{u} \times \mathbf{v} = e_{ijk}u_jv_k\,\mathbf{e}_i$ $((\mathbf{u} \times \mathbf{v})\,\times) = e_{\alpha i\beta}e_{ijk}u_jv_k\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ $= -(\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}\,)u_jv_k\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ $= (-u_\alpha v_\beta + u_\beta v_\alpha)\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ $= \mathbf{v} \otimes \mathbf{u} - \mathbf{u} \otimes \mathbf{v}$
3.30	Show that $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$
	$\text{div}\,(\mathbf{u} \times \mathbf{v}) = (e_{ijk}u_jv_k)_{,i}$ Noting that the tensor e_{ijk} behaves is a constant, we can write, $\text{div}\,(\mathbf{u} \times \mathbf{v}) = (e_{ijk}u_jv_k)_{,i}$ $= e_{ijk}u_{j,i}\,v_k + e_{ijk}u_jv_{k,i}$ $= \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$

3.31	Given a scalar point function ϕ and a vector field $\mathbf{v}$, show that curl $(\phi\mathbf{v}) = \phi$ curl $\mathbf{v} + (\text{grad}\,\phi) \times \mathbf{v}$.
	$$\begin{aligned}\text{curl}\,(\phi\mathbf{v}) &= e_{ijk}(\phi v_k)_{,j}\,\mathbf{e}_i \\ &= e_{ijk}\left(\phi_{,j}\,v_k + \phi v_{k,j}\right)\mathbf{e}_i \\ &= e_{ijk}\phi_{,j}\,v_k\mathbf{e}_i + \epsilon^{ijk}\phi v_{k,j}\,\mathbf{e}_i \\ &= (\text{grad}\,\phi) \times \mathbf{v} + \phi\,\text{curl}\,\mathbf{v}\end{aligned}$$
3.32	Show that curl $(\mathbf{v}\times) = (\text{div}\,\mathbf{v})\mathbf{I} - \text{grad}\,\mathbf{v}$
	$$(\mathbf{v}\times) = e_{\alpha\beta k}v_\beta\,\mathbf{e}_\alpha \otimes \mathbf{e}_k$$ $$\text{curl}\,\mathbf{T} = e_{ijk}T_{\alpha k,j}\;\mathbf{e}_i \otimes \mathbf{e}_\alpha$$ so that $$\begin{aligned}\text{curl}\,(\mathbf{v}\times) &= e_{ijk}e_{\alpha\beta k}v_{\beta,j}\;\mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= \left(\delta_{i\alpha}\delta_{j\beta} - \delta_{j\alpha}\delta_{i\beta}\right)v_{\beta,j}\;\mathbf{e}_i \otimes \mathbf{e}_\alpha \\ &= v_{j,j}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\alpha - v_{i,j}\,\mathbf{e}_i \otimes \mathbf{e}_j \\ &= (\text{div}\,\mathbf{v})\mathbf{I} - \text{grad}\,\mathbf{v}\end{aligned}$$
3.33	Show that curl $(\text{grad}\,\mathbf{v}) = \mathbf{0}$
	For any tensor $\mathbf{T} = T_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ $$\text{curl}\,\mathbf{T} = e_{ijk}T_{\alpha k,j}\;\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ Let $\mathbf{T} = \text{grad}\,\mathbf{v}$. Clearly, in this case, $T_{\alpha\beta} = v_{\alpha,\beta}$ so that $T_{\alpha k,j} = v_{\alpha,kj}$. It therefore follows that, $$\text{curl}\,(\text{grad}\,\mathbf{v}) = e_{ijk}v_{\alpha,kj}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta = \mathbf{0}.$$ The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the vector field is immaterial. This will be true only if the vector field is continuous – a proposition we have assumed in the above.
3.34	Show that curl $(\text{grad}\,\phi) = \mathbf{o}$
	For any tensor $\mathbf{v} = v_\alpha\boldsymbol{e}_\alpha$ $$\text{curl}\,\mathbf{v} = e_{ijk}v_{k,j}\,\boldsymbol{e}_i$$

	Let $\mathbf{v} = \text{grad}\,\phi$. Clearly, in this case, $v_k = \phi_{,k}$ so that $v_{k,j} = \phi_{,kj}$. It therefore follows that, $$\text{curl}\,(\text{grad}\,\phi) = e_{ijk}\phi_{,kj}\,\mathbf{e}_i = \boldsymbol{o}.$$ The contraction of symmetric tensors with anti-symmetric led to this conclusion. Note that this presupposes that the order of differentiation in the scalar field is immaterial. This will be true only if the scalar field is continuous – a proposition we have assumed in the above.
3.35	Show that $\text{div}\,(\text{grad}\,\phi \times \text{grad}\,\theta) = 0$
	$$\text{grad}\,\phi \times \text{grad}\,\theta = e_{ijk}\phi_{,j}\,\theta_{,k}\,\boldsymbol{g}_i$$ The gradient of this vector is the tensor, $$\text{grad}(\text{grad}\,\phi \times \text{grad}\,\theta) = \left(e_{ijk}\phi_{,j}\,\theta_{,k}\right)_{,l}\,\mathbf{e}_i \otimes \mathbf{e}_l$$ $$= e_{ijk}\phi_{,jl}\,\theta_{,k}\,\mathbf{e}_i \otimes \mathbf{e}_l + e_{ijk}\phi_{,j}\,\theta_{,kl}\,\mathbf{e}_i \otimes \mathbf{e}_l$$ The trace of the above result is the divergence we are seeking: $$\text{div}\,(\text{grad}\,\phi \times \text{grad}\,\theta) = \text{tr}[\text{grad}(\text{grad}\,\phi \times \text{grad}\,\theta)]$$ $$= e_{ijk}\phi_{,jl}\,\theta_{,k}\,\mathbf{e}_i \cdot \mathbf{e}_l + e_{ijk}\phi_{,j}\,\theta_{,kl}\,\mathbf{e}_i \cdot \mathbf{e}_l$$ $$= e_{ijk}\phi_{,jl}\,\theta_{,k}\,\delta_{il} + e_{ijk}\phi_{,j}\,\theta_{,kl}\,\delta_{il}$$ $$= e_{ijk}\phi_{,ji}\,\theta_{,k} + e_{ijk}\phi_{,j}\,\theta_{,ki} = 0$$ Each term vanishing on account of the contraction of a symmetric tensor with an antisymmetric.
3.36	Show that $\text{curl curl}\,\mathbf{v} = \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}$
	Let $\boldsymbol{w} = curl\,\boldsymbol{v} \equiv e_{ijk}v_{k,j}\,\mathbf{e}_i$. But curl $\boldsymbol{w} \equiv e_{\alpha\beta\gamma}w_{\gamma,\beta}\,\mathbf{e}_\alpha$. Upon inspection, we find that $w_\gamma = \delta_{\gamma i}e_{ijk}v_{k,j}$ so that $$\text{curl}\,\mathbf{w} \equiv e_{\alpha\beta\gamma}\left(\delta_{\gamma i}e_{ijk}v_{k,j}\right)_{,\beta}\,\mathbf{e}_\alpha = \delta_{\gamma i}e_{\alpha\beta\gamma}e_{ijk}v_{k,j\beta}\,\mathbf{e}_\alpha$$ Now, it can be shown that $\delta_{\gamma i}e_{\alpha\beta\gamma}e_{ijk} = \delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}$ so that, $$\text{curl}\,\mathbf{w} = \left(\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}\right)v_{k,j\beta}\,\mathbf{e}_\alpha$$ $$= v_{\beta,j\beta}\,\mathbf{e}_j - \delta_{\beta j}v_{\alpha,j\beta}\,\mathbf{e}_\alpha$$ $$= \text{grad}(\text{div}\,\mathbf{v}) - \text{grad}^2\mathbf{v}$$ Also recall that the Laplacian (grad^2) of a scalar field ϕ is, $\text{grad}^2\phi = \delta_{ij}\phi_{,ij}$. In Cartesian coordinates, this becomes,

	$$\text{grad}^2\phi = \delta_{ij}\phi,_{ij} = \phi,_{ii}$$ as the unit (metric) tensor now degenerates to the Kronecker delta in this special case. For a vector field, $\text{grad}^2\mathbf{v} = v_\alpha,_{jj}\,\mathbf{e}_\alpha$. Also note that while grad is a vector operator, the Laplacian (grad^2) is a scalar operator.
3.37	For a second-order tensor $\mathbf{T}$ define curl $\mathbf{T} \equiv e_{ijk}T_{\alpha k},_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha$ show that for any constant vector **a,** $(\text{curl }\mathbf{T})\,\mathbf{a} = \text{curl }(\mathbf{T}^{\mathrm{T}}\mathbf{a})$
	Express vector $\mathbf{a}$ in the invariant form with covariant components as $\mathbf{a} = a_\beta\mathbf{e}_\beta$. It follows that $$\begin{aligned}(\text{curl }\mathbf{T})\,\mathbf{a} &= e_{ijk}T_{\alpha k},_j\ (\mathbf{e}_i \otimes \mathbf{e}_\alpha)\mathbf{a}\\ &= e_{ijk}T_{\alpha k},_j\ a_\beta(\mathbf{e}_i \otimes \mathbf{e}_\alpha)\mathbf{e}_\beta\\ &= e_{ijk}T_{\alpha k},_j\ a_\beta\mathbf{e}_i\delta_{\beta\alpha}\\ &= e_{ijk}(T_{\alpha k}),_j\ \mathbf{e}_i a_\alpha\\ &= e_{ijk}(T_{\alpha k}a_\alpha),_j\ \mathbf{e}_i\end{aligned}$$ The last equality resulting from the fact that vector $\boldsymbol{a}$ is a constant vector. Clearly, $$(\text{curl }\mathbf{T})\,\mathbf{a} = \text{curl }(\mathbf{T}^{\mathrm{T}}\mathbf{a})$$
3.38	For any two vectors $\mathbf{u}$ and $\mathbf{v}$, show that $\text{curl }(\mathbf{u}\otimes\mathbf{v}) = [(\text{grad }\mathbf{u})\mathbf{v}\times]^{\mathrm{T}} + (\text{curl }\mathbf{v})\otimes\mathbf{u}$ where $\mathbf{v}\times$ is the skew tensor $e_{ijk}v_k\ \mathbf{e}_i \otimes \mathbf{e}_j$.
	Recall that the curl of a tensor $\mathbf{T}$ is defined by curl $\mathbf{T} \equiv e_{ijk}T_{\alpha k},_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha$. Clearly therefore, $$\begin{aligned}\text{curl }(\mathbf{u}\otimes\mathbf{v}) &= e_{ijk}(u_\alpha v_k),_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha\\ &= e_{ijk}\left(u_\alpha,_j v_k + u_\alpha v_k,_j\right)\mathbf{e}_i \otimes \mathbf{e}_\alpha\\ &= e_{ijk}u_\alpha,_j v_k\ \mathbf{e}_i \otimes \mathbf{e}_\alpha + e_{ijk}u_\alpha v_k,_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha\\ &= \left(e_{ijk}v_k\ \mathbf{e}_i\right)\otimes\left(u_\alpha,_j\,\mathbf{e}_\alpha\right) + \left(e_{ijk}v_k,_j\ \mathbf{e}_i\right)\otimes(u_\alpha\mathbf{e}_\alpha)\\ &= \left(e_{ijk}v_k\mathbf{e}_i\otimes\mathbf{e}_j\right)\left(u_\alpha,_\beta\,\mathbf{e}_\beta\otimes\mathbf{e}_\alpha\right) + \left(e_{ijk}v_k,_j\ \mathbf{e}_i\right)\otimes(u_\alpha\mathbf{e}_\alpha)\\ &= -(\mathbf{v}\times)(\text{grad }\mathbf{u})^{\mathrm{T}} + (\text{curl }\mathbf{v})\otimes\mathbf{u}\\ &= [(\text{grad }\mathbf{u})\mathbf{v}\times]^{\mathrm{T}} + (\text{curl }\mathbf{v})\otimes\mathbf{u}\end{aligned}$$ upon noting that the vector cross is a skew tensor.

3.39	For a second-order tensor field $\mathbf{T}$, show that $\mathrm{div}(\mathrm{curl}\ \mathbf{T}) = \mathrm{curl}(\mathrm{div}\ \mathbf{T}^{\mathrm{T}})$
	Define the second order tensor $\mathbf{S}$ as $$\mathrm{curl}\ \mathbf{T} \equiv e_{ijk}T_{\alpha k},_j\ \mathbf{e}_i \otimes \mathbf{e}_\alpha = S_{i\alpha}\mathbf{e}_i \otimes \mathbf{e}_\alpha$$ The gradient of $\mathbf{S}$ is $S_{i\alpha},_\beta\ \mathbf{e}_i \otimes \mathbf{e}_\alpha \otimes \mathbf{e}_\beta = e_{ijk}T_{\alpha k},_{j\beta}\ \mathbf{e}_i \otimes \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ Clearly, $$\mathrm{div}(\mathrm{curl}\ \mathbf{T}) = e_{ijk}T_{\alpha k},_{j\beta}\ (\mathbf{e}_i \otimes \mathbf{e}_\alpha)\mathbf{e}_\beta$$ $$= e_{ijk}T_{\alpha k},_{j\beta}\ \mathbf{e}_i\ \delta_{\alpha\beta}$$ $$= e_{ijk}T_{\beta k},_{j\beta}\,\mathbf{e}_i = \mathrm{curl}(\mathrm{div}\ \mathbf{T}^{\mathrm{T}})$$
3.40	Show that $\left((\mathrm{curl}\ \mathbf{u})\times\right) = \mathrm{grad}\ \mathbf{u} - \mathrm{grad}^{\mathrm{T}}\mathbf{u}$
	$$\left((\mathrm{curl}\ \mathbf{u})\times\right) = e_{ijk}u_k,_j\ \mathbf{e}_i \times$$ $$= e_{\alpha i\beta}e_{ijk}u_k,_j\ \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$= \left(\delta_{\beta j}\delta_{\alpha k} - \delta_{\beta k}\delta_{\alpha j}\right)u_k,_j\ \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$ $$= u_k,_j\ \mathbf{e}_k \otimes \mathbf{e}_j - u_k,_j\ \mathbf{e}_j \otimes \mathbf{e}_k$$ $$= \mathrm{grad}\ \mathbf{u} - \mathrm{grad}^{\mathrm{T}}\mathbf{u}$$
3.41	Show that $\mathrm{curl}\ (\mathbf{u}\times\mathbf{v}) = \mathrm{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})$
	The vector $\mathbf{w} \equiv \mathbf{u}\times\mathbf{v} = w_k\mathbf{e}_k = e_{k\alpha\beta}u_\alpha v_\beta\mathbf{e}_k$ and $\mathrm{curl}\ \mathbf{w} = e_{ijk}w_k,_j\ \mathbf{e}_i$. Therefore, $$\mathrm{curl}\ (\mathbf{u}\times\mathbf{v}) = e_{ijk}w_k,_j\ \mathbf{e}_i$$ $$= e_{ijk}e_{k\alpha\beta}\left(u^\alpha v^\beta\right),_j\ \mathbf{e}_i$$ $$= \left(\delta_{i\alpha}\delta_{j\beta} - \delta_{j\alpha}\delta_{i\beta}\right)\left(u^\alpha v^\beta\right),_j\ \mathbf{e}_i$$ $$= \left(\delta_{i\alpha}\delta_{j\beta} - \delta_{j\alpha}\delta_{i\beta}\right)\left(u^\alpha,_j\ v^\beta + u^\alpha v^\beta,_j\ \right)\mathbf{e}_i$$ $$= \left[u_i,_j\ v_j + u_i v_j,_j - \left(u_j,_j\ v_i + u_j v_i,_j\ \right)\right]\mathbf{e}_i$$ $$= \left[\left(u_i v_j\right),_j - \left(u_j v_i\right),_j\ \right]\mathbf{e}_i$$ $$= \mathrm{div}(\mathbf{u}\otimes\mathbf{v} - \mathbf{v}\otimes\mathbf{u})$$ since $\mathrm{div}(\mathbf{u}\otimes\mathbf{v}) = \left(u_i v_j\right),_\alpha\left(\mathbf{e}_i \otimes \mathbf{e}_j\right)\mathbf{e}_\alpha = \left(u_i v_i\right),_j\ \mathbf{e}_i$.
3.42	

3.43	Given a scalar point function ϕ and a second-order tensor field $\mathbf{T}$, show that $\text{curl}\,(\phi\mathbf{T}) = \phi\,\text{curl}\,\mathbf{T} + \left((\text{grad}\,\phi)\times\right)\mathbf{T}^{\mathrm{T}}$ where $[(\text{grad}\,\phi)\times]$ is the skew tensor $e_{ijk}\phi_{,j}\,\mathbf{e}_i\otimes\mathbf{e}_k$
	$\text{curl}\,(\phi\mathbf{T}) \equiv e_{ijk}(\phi T_{\alpha k})_{,j}\ \mathbf{e}_i\otimes\mathbf{e}_\alpha$ $= e_{ijk}\left(\phi_{,j} T_{\alpha k} + \phi T_{\alpha k,j}\right)\mathbf{e}_i\otimes\mathbf{e}_\alpha$ $= e_{ijk}\phi_{,j} T_{\alpha k}\,\mathbf{e}_i\otimes\mathbf{e}_\alpha + \phi e_{ijk}T_{\alpha k,j}\ \mathbf{e}_i\otimes\mathbf{e}_\alpha$ $= \left(e_{ijk}\phi_{,j}\,\mathbf{e}_i\otimes\mathbf{e}_k\right)\left(T_{\alpha\beta}\mathbf{e}_\beta\otimes\mathbf{e}_\alpha\right) + \phi e_{ijk}T_{\alpha k,j}\ \mathbf{e}_i\otimes\mathbf{e}_\alpha$ $= \phi\,\text{curl}\,\mathbf{T} + \left((\text{grad}\,\phi)\times\right)\mathbf{T}^{\mathrm{T}}$
3.44	Given a tensor field $\mathbf{T}$, obtain the vector $\mathbf{w}\equiv\mathbf{T}^{\mathrm{T}}\mathbf{v}$ and show that its divergence is $\mathbf{T}:(\text{grad}\,\mathbf{v}) + \mathbf{v}\cdot\text{div}\,\mathbf{T}$
	The gradient of $\mathbf{w}$ is the tensor, $\left(T_{ji}v_j\right)_{,k}\,\mathbf{e}_i\otimes\mathbf{e}_k$. Therefore, divergence of $\boldsymbol{w}$ (the trace of the gradient) is the scalar sum, $T_{ji}v_{j,k}\,\delta_{ik} + T_{ji,k}\,v_j\delta_{ik}$. Expanding, we obtain, $\text{div}\,(\mathbf{T}^{\mathrm{T}}\mathbf{v}) = T_{ji}v_{j,k}\,\delta_{ik} + T_{ji,k}\,v_j\delta_{ik}$ $= T_{ji,i}\,v_j + T_{jk}v_{j,k}$ $= (\text{div}\,\mathbf{T})\cdot\mathbf{v} + \text{tr}(\mathbf{T}^{\mathrm{T}}\text{grad}\,\mathbf{v})$ $= (\text{div}\,\mathbf{T})\cdot\mathbf{v} + \mathbf{T}:(\text{grad}\,\mathbf{v})$ Recall that scalar product of two vectors is commutative so that $\text{div}\,(\mathbf{T}^{\mathrm{T}}\mathbf{v}) = \mathbf{T}:(\text{grad}\,\mathbf{v}) + \mathbf{v}\cdot\text{div}\,\mathbf{T}$
3.45	
3.46	
3.47	
3.48	

3.49	
3.50	
3.51	
3.52	
3.53	Show that if ϕ is a scalar field in the Euclidean space spanned by orthogonal coordinates x_i, with unit basis vectors, $\mathbf{e}_i$, then (a) for the position vector $\mathbf{r}$, $\text{div grad}(\mathbf{r}\phi) = 2 \text{ grad } \phi + \mathbf{r} \text{ div grad } \phi$.(b) Will this expression be valid in the spherical coordinate system?
$(\boldsymbol{a})$	In such a coordinate system, the radius vector will be given by, $$\mathbf{r} = x_i \mathbf{e}_i$$ where $\mathbf{e}_i$ is the unit vector along coordinate x_i. $$\text{grad}(\mathbf{r}\phi) = (x_i\phi),_j \mathbf{e}_i \otimes \mathbf{e}_j$$ $$\text{grad}\big(\text{grad}(\mathbf{r}\phi)\big) = \big((x_i\phi),_j\big),_k \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$$ $$\text{div}\big(\text{grad}(\mathbf{r}\phi)\big) = \text{tr}\left(\text{grad}\big(\text{grad}(\mathbf{r}\phi)\big)\right) = \big((x_i\phi),_j\big),_k \mathbf{e}_i\delta_{jk}$$ $$= \big(\delta_{ij}\phi + x_i\phi,_j\big),_j \mathbf{e}_i$$ $$= \big(\delta_{ij}\phi,_j + x_i,_j \phi,_j + x_i\phi,_{jj}\big)\mathbf{e}_i$$ $$\text{div grad}(\mathbf{r}\phi) = 2 \text{ grad } \phi + \mathbf{r} \text{ div grad } \phi$$
$(\boldsymbol{b})$	The position vector in spherical coordinates are non-linear in the coordinate variables. This form of the expression will not be correct. It works only for Cartesian systems.

3.54	In Cartesian coordinates let x denote the magnitude of the position vector $\mathbf{r} = x_i\mathbf{e}_i$. Show that (a) $x_{,j} = \frac{x_j}{x}$, (b) $x_{,ij} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$, (c) $x_{,ii} = \frac{2}{x}$, (d) If $U = \frac{1}{x}$, then $U_{,ij} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$ $U_{,ii} = 0$ and div $\left(\frac{\mathbf{r}}{x}\right) = \frac{2}{x}$.
(a)	$x = \sqrt{x_i x_i}$ $$x_{,j} = \frac{\partial\sqrt{x_i x_i}}{\partial x_j} = \frac{\partial\sqrt{x_i x_i}}{\partial(x_i x_i)} \times \frac{\partial(x_i x_i)}{\partial x_j} = \frac{1}{2\sqrt{x_i x_i}}\left[x_i\delta_{ij} + x_i\delta_{ij}\right] = \frac{x_j}{x}.$$
(b)	$$x_{,ij} = \frac{\partial}{\partial x_j}\left(\frac{\partial\sqrt{x_i x_i}}{\partial x_i}\right) = \frac{\partial}{\partial x_j}\left(\frac{x_i}{x}\right) = \frac{x\frac{\partial x_i}{\partial x_j} - x_i\frac{\partial x}{\partial x_j}}{(x)^2} = \frac{x\delta_{ij} - \frac{x_i x_j}{x}}{(x)^2} = \frac{1}{x}\delta_{ij} - \frac{x_i x_j}{(x)^3}$$
(c)	$$x_{,ii} = \frac{1}{x}\delta_{ii} - \frac{x_i x_i}{(x)^3} = \frac{3}{x} - \frac{(x)^2}{(x)^3} = \frac{2}{x}.$$
(d)	$U = \frac{1}{x}$ so that $$U_{,j} = \frac{\partial\frac{1}{x}}{\partial x_j} = \frac{\partial\frac{1}{x}}{\partial x} \times \frac{\partial x}{\partial x_j} = -\frac{1}{x^2}\frac{1}{x}x_j = -\frac{x_j}{x^3}$$ Consequently, $$U_{,ij} = \frac{\partial}{\partial x_j}(U_{,i}) = -\frac{\partial}{\partial x_j}\left(\frac{x_i}{x^3}\right) = \frac{x^3\left(\frac{\partial}{\partial x_j}(-x^2)\right) + x_i\frac{\partial}{\partial x_j}(x^3)}{x^6}$$ $$= \frac{x^3(-\delta_{ij}) + x_i\left(\frac{\partial(x^3)}{\partial x}\frac{\partial x}{\partial x_j}\right)}{x^6} = \frac{-x^3\delta_{ij} + x_i\left(3x^2\frac{x_j}{x}\right)}{x^6} = \frac{-\delta_{ij}}{x^3} + \frac{3x_i x_j}{x^5}$$ $$U_{,ii} = \frac{-\delta_{ii}}{x^3} + \frac{3x_i x_i}{x^5} = \frac{-3}{x^3} + \frac{3x^2}{x^5} = 0.$$ $$div\left(\frac{\boldsymbol{r}}{x}\right) = \left(\frac{x_j}{x}\right)_{,j} = \frac{1}{x}x_{j,j} + \left(\frac{1}{x}\right)_{,j} = \frac{3}{x} + x_j\left(\frac{\partial}{\partial x}\left(\frac{1}{x}\right)\frac{dx}{dx_j}\right)$$ $$= \frac{3}{x} + x_j\left[-\left(\frac{1}{x^2}\right)\frac{x_j}{x}\right] = \frac{3}{x} - \frac{x_j x_j}{x^3} = \frac{3}{x} - \frac{1}{x} = \frac{2}{x}$$
3.55	Show that curl $\mathbf{u}\times\mathbf{v} = (\text{grad}\,\mathbf{u})\mathbf{v} - (\text{div}\,\mathbf{u})\mathbf{v} - (\text{grad}\,\mathbf{v})\mathbf{u} + (\text{div}\,\mathbf{v})\mathbf{u}$
	$$\mathbf{u}\times\mathbf{v} = e_{ijk}u_j v_k\mathbf{e}_i$$

$$\text{curl}\,\mathbf{w} = e_{\alpha\beta l} w_{l},_{\beta}\,\mathbf{e}_{\alpha}$$

Clearly,

$$\text{curl}\,\mathbf{u}\times\mathbf{v} = e_{\alpha\beta i}\left(e_{ijk}u_j v_k\right),_{\beta}\,\mathbf{e}_{\alpha}$$

$$= e_{\alpha\beta i}e_{ijk}u_{j,\beta}v_k\mathbf{e}_{\alpha} + e_{\alpha\beta i}e_{ijk}u_j v_{k},_{\beta}\,\mathbf{e}_{\alpha}$$

$$= \left(\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}\right)u_{j,\beta}v_k\mathbf{e}_{\alpha} + \left(\delta_{\alpha j}\delta_{\beta k} - \delta_{\alpha k}\delta_{\beta j}\right)u_j v_{k},_{\beta}\,\mathbf{e}_{\alpha}$$

$$= u_{j},_{k}\,v_k\mathbf{e}_j - u_{j},_{j}\,v_k\mathbf{e}_k + u_j v_{k},_{k}\,\mathbf{e}_j - u_j v_{k},_{j}\,\mathbf{e}_k$$

$$= (\text{grad}\,\mathbf{u})\mathbf{v} - (\text{div}\,\mathbf{u})\mathbf{v} - (\text{grad}\,\mathbf{v})\mathbf{u} + (\text{div}\,\mathbf{v})\mathbf{u}$$

3.56 For a scalar function $\phi$and a vector $\mathbf{v}$ show that the divergence of the vector $\mathbf{v}\phi$ is equal to, $\mathbf{v}\cdot\text{grad}\,\phi + \phi\,\text{div}\,\mathbf{v}$

$$\text{grad}\,(\mathbf{v}\phi) = (v_i\phi),_j\,\mathbf{e}_i\otimes\mathbf{e}_j = \left(\phi v_{i},_{j} + v_i\phi,_j\right)\mathbf{e}_i\otimes\mathbf{e}_j$$

Taking the trace of this equation,

$$\text{div}\,\mathbf{v}\phi = \text{tr}\left(\text{grad}\,(\mathbf{v}\phi)\right) = \left(\phi v_{i},_{j} + v_i\phi,_j\right)\mathbf{e}_i\cdot\mathbf{e}_j$$

$$= \left(\phi v_{i},_{j} + v_i\phi,_j\right)\delta_{ij}$$

$$= \phi v_{i},_{i} + v_i\phi,_i$$

$$= \mathbf{v}\cdot\text{grad}\,\phi + \phi\,\text{div}\,\mathbf{v}$$

Hence the result.

3.57 When $\mathbf{T}$ is symmetric, show that tr(curl $\mathbf{T}$) vanishes.

When $\mathbf{T}$ is symmetric, show that tr(**curl** $\mathbf{T}$) vanishes.

$$\text{curl}\,\mathbf{T} = e_{ijk}T_{\beta k},_{j}\,\mathbf{e}_i\otimes\mathbf{e}_{\beta}$$

$$\text{tr}(\text{curl}\,\mathbf{T}) = e_{ijk}T_{\beta k},_{j}\,\mathbf{e}_i\cdot\mathbf{e}_{\beta}$$

$$= e_{ijk}T_{\beta k},_{j}\,\delta_{i\beta} = e_{ijk}T_{ik},_{j}$$

which obviously vanishes on account of the symmetry and antisymmetry in i and k. In this case,

3.58 For a general tensor field $\mathbf{T}$ show that,

	$+\left(\mathrm{grad}(\mathrm{div}\,\mathbf{T})\right)^{\mathrm{T}} - \mathrm{grad}\left(\mathrm{grad}\,(\mathrm{tr}\,\mathbf{T})\right) - \mathrm{grad}^2\mathbf{T}^{\mathrm{T}}$
	$\mathrm{curl}\,\mathbf{T} = e_{ast}T_{\beta t,s}\,\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ $= S_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ $\mathrm{curl}\,\mathbf{S} = e_{ijk}S_{\alpha k,j}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$ so that $\mathrm{curl}\,\mathbf{S} = \mathrm{curl}(\mathrm{curl}\,\mathbf{T}) = e_{ijk}e_{ast}T_{kt,sj}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$ $= \begin{vmatrix} \delta_{i\alpha} & \delta_{is} & \delta_{it} \\ \delta_{j\alpha} & \delta_{js} & \delta_{jt} \\ \delta_{k\alpha} & \delta_{ks} & \delta_{kt} \end{vmatrix} T_{kt,sj}\,\mathbf{e}_i \otimes \mathbf{e}_\alpha$ $= \begin{bmatrix} \delta_{i\alpha}(\delta_{js}\delta_{kt} - \delta_{jt}\delta_{ks}) + \delta_{is}(\delta_{jt}\delta_{k\alpha} - \delta_{j\alpha}\delta_{kt}) \\ +\delta_{it}(\delta_{j\alpha}\delta_{ks} - \delta_{js}\delta_{k\alpha}) \end{bmatrix} T_{kt,sj}\,\boldsymbol{g}_i \otimes \boldsymbol{g}_\alpha$ $= [\delta_{js}T_{tt,sj} - T_{sj,sj}](\mathbf{e}_\alpha \otimes \mathbf{e}_\alpha) + [T_{\alpha j,sj} - \delta_{j\alpha}T_{tt,sj}](\mathbf{e}_s \otimes \mathbf{e}_\alpha)$ $+ [\delta_{j\alpha}T_{st,sj} - \delta_{js}T_{\alpha t,sj}](\mathbf{e}_t \otimes \mathbf{e}_\alpha)$ $= [\mathrm{grad}^2(\mathrm{tr}\,\mathbf{T}) - \mathrm{div}(\mathrm{div}\,\boldsymbol{T})]\mathbf{I} + \left(\mathrm{grad}(\mathrm{div}\,\mathbf{T})\right)^{\mathrm{T}} - \mathrm{grad}\left(\mathrm{grad}\,(\mathrm{tr}\,\mathbf{T})\right) + \left(\mathrm{grad}(\mathrm{div}\,\mathbf{T})\right)$ $- \mathrm{grad}^2\mathbf{T}^{\mathrm{T}}$
3.59	For any Euclidean coordinate system, show that $\mathrm{div}\,\mathbf{u} \times \mathbf{v} = \mathbf{v} \cdot \mathrm{curl}\,\mathbf{u} - \mathbf{u} \cdot \mathrm{curl}\,\mathbf{v}$
	$\mathbf{u} \times \mathbf{v} = e_{ijk}u_jv_k\mathbf{e}_i$ $\mathrm{grad}\,(\mathbf{u} \times \mathbf{v}) = (e_{ijk}u_jv_k)_{,l}\,\mathbf{e}_i \otimes \mathbf{e}_l$ $\mathrm{div}\,(\mathbf{u} \times \mathbf{v}) = \mathrm{tr}\left(\mathrm{grad}\,(\mathbf{u} \times \mathbf{v})\right) = (e_{ijk}u_jv_k)_{,l}\,\mathbf{e}_i \cdot \mathbf{e}_l$ $= (e_{ijk}u_{j,l}\,v_k + e_{ijk}u_jv_{k,l})\delta_{il}$ $= e_{ijk}u_{j,i}\,v_k + e_{ijk}u_jv_{k,i}$ $= \mathbf{v} \cdot \mathrm{curl}\,\mathbf{u} - \mathbf{u} \cdot \mathrm{curl}\,\mathbf{v}$
3.60	In Cartesian coordinates, If the volume V is enclosed by the surface S, the position vector $\boldsymbol{r} = x^i\mathbf{g}_i$ and $\boldsymbol{n}$ is the external unit normal to each surface element, show that $\frac{1}{6}\int_S \nabla(\boldsymbol{r} \cdot \boldsymbol{r}) \cdot \boldsymbol{n}dS$ equals the volume contained in V.

	$$\mathbf{r}\cdot\mathbf{r} = x^i x^j \mathbf{e}_i \cdot \mathbf{e}_j = x^i x^j \delta_{ij}$$ By the Divergence Theorem, $$\int_S \operatorname{grad}(\boldsymbol{r}\cdot\boldsymbol{r})\cdot\boldsymbol{n}dS = \int_V \nabla\cdot[\nabla(\boldsymbol{r}\cdot\boldsymbol{r})]dV = \int_V \partial_l\left[\partial_k\left(x^i x^j g_{ij}\right)\right]\boldsymbol{g}^l\cdot\ \boldsymbol{g}^k\ dV$$ $$= \int_V \partial_l\left[g_{ij}\left(x^i{}_{,k}\, x^j + x^i x^j{}_{,k}\right)\right]\boldsymbol{g}^l\cdot\ \boldsymbol{g}^k\ dV = \int_V g_{ij}g^{lk}\left(\delta^i_k x^j + x^i\delta^j_k\right)_{,l}\ dV$$ $$= \int_V 2g_{ik}g^{lk}x^i{}_{,l}\ dV = \int_V 2\delta^l_i\delta^i_l\ dV = 6\int_V dV$$??
3.61	
3.62	
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3.64	
3.65	
3.66	
3.67	
3.68	

3.69	
3.70	
3.71	

Fields in Curvilinear Coordinates

Given a vector point function $\mathbf{u}(x^1, x^2, x^3)$ and its covariant components, $u_\alpha(x^1, x^2, x^3)$, $\alpha = 1,2,3$, with $\boldsymbol{g}^\alpha$as the reciprocal basis vectors**,** Let

$$\mathbf{u} = u_\alpha \mathbf{g}^\alpha, \text{ then } d\mathbf{u} = \frac{\partial}{\partial x^k}(u_\alpha \mathbf{g}^\alpha) dx^k$$
$$= \left(\frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}u_\alpha\right) dx^k$$

Clearly,

$$\frac{\partial \mathbf{u}}{\partial x^k} = \frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}u_\alpha$$

And the projection of this quantity on the $\boldsymbol{g}^i$direction is,

$$\frac{\partial \mathbf{u}}{\partial x^k}\cdot \mathbf{g}_i = \left(\frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}u_\alpha\right)\cdot \mathbf{g}_i$$
$$= \frac{\partial u_\alpha}{\partial x^k}\mathbf{g}^\alpha \cdot \mathbf{g}_i + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}\cdot \mathbf{g}_i u_\alpha$$
$$= \frac{\partial u_\alpha}{\partial x^k}\delta_i^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k}\cdot \mathbf{g}_i u_\alpha$$

Now $\mathbf{g}^i \cdot \mathbf{g}_j = \delta_i^j$ so that,

$$\frac{\partial \mathbf{g}^i}{\partial x^k}\cdot \mathbf{g}_j + \mathbf{g}^i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k} = 0.$$

$$\frac{\partial \mathbf{g}^i}{\partial x^k}\cdot \mathbf{g}_j = -\mathbf{g}^i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k} \equiv -\left\{\begin{matrix} i \\ jk \end{matrix}\right\}.$$

This important quantity, necessary to quantify the derivative of a tensor in general coordinates, is called the **Christoffel Symbol** of the second kind.

Using this, we can now write that,

$$\frac{\partial \mathbf{u}}{\partial x^k} \cdot \mathbf{g}_i = \frac{\partial u_\alpha}{\partial x^k} \delta_i^\alpha + \frac{\partial \mathbf{g}^\alpha}{\partial x^k} \cdot \mathbf{g}_i u_\alpha = \frac{\partial u_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} u_\alpha$$

The quantity on the RHS is the component of the derivative of vector $\mathbf{u}$ along the $\boldsymbol{g}_i$ direction using covariant components. It is the covariant derivative of $\mathbf{u}$. Using contravariant components, we could write,

$$\begin{aligned} d\mathbf{u} &= \left(\frac{\partial u^i}{\partial x^k} \mathbf{g}_i + \frac{\partial \mathbf{g}_i}{\partial x^k} u^i \right) dx^k \\ &= \left(\frac{\partial u^i}{\partial x^k} \mathbf{g}_i + \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} \mathbf{g}_\alpha u^i \right) dx^k \end{aligned}$$

So that,

$$\frac{\partial \mathbf{u}}{\partial x^k} = \frac{\partial u^\alpha}{\partial x^k} \mathbf{g}_\alpha + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} u^\alpha$$

The components of this in the direction of $\boldsymbol{g}_i$ can be obtained by taking a dot product as before:

$$\begin{aligned} \frac{\partial}{\partial x^k} \cdot \mathbf{g}^i &= \left(\frac{\partial u^\alpha}{\partial x^k} \mathbf{g}_\alpha + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} u^\alpha \right) \cdot \mathbf{g}^i \\ &= \frac{\partial u^\alpha}{\partial x^k} \delta_\alpha^i + \frac{\partial \mathbf{g}_\alpha}{\partial x^k} \cdot \mathbf{g}^i u^\alpha \\ &= \frac{\partial u^i}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} u^\alpha \end{aligned}$$

The two results above are represented symbolically as,

$$u_{i,k} = \frac{\partial u_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix} u_\alpha$$

for covariant components, and

$$u^i_{,k} = \frac{\partial u^\alpha}{\partial x^k} + \begin{Bmatrix} i \\ \alpha k \end{Bmatrix} u^\alpha$$

for conravariant vector components. Which are the components of the covariant derivatives in terms of the covariant and contravariant components respectively.

It now becomes useful to establish the fact that our definition of the Christoffel symbols here conforms to the definition you find in the books using the transformation rules to define the tensor quantities.

We observe that the derivative of the covariant basis, $\mathbf{g}_i \left(= \frac{\partial r}{\partial x^i}\right)$,

$$\frac{\partial \mathbf{g}_i}{\partial x^j} = \frac{\partial^2 \mathbf{r}}{\partial x^j \partial x^i}$$

$$= \frac{\partial^2 \mathbf{r}}{\partial x^i \partial x^j} = \frac{\partial \mathbf{g}_j}{\partial x^i}$$

Taking the dot product with $\mathbf{g}_k$

$$\frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k = \frac{1}{2}\left(\frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_k + \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}\left[\mathbf{g}_j \cdot \mathbf{g}_k\right] + \frac{\partial}{\partial x^j}\left[\mathbf{g}_i \cdot \mathbf{g}_k\right] - \mathbf{g}_j \cdot \frac{\partial \mathbf{g}_k}{\partial x^i} - \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_k}{\partial x^j}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}\left[\mathbf{g}_j \cdot \mathbf{g}_k\right] + \frac{\partial}{\partial x^j}\left[\mathbf{g}_i \cdot \mathbf{g}_k\right] - \mathbf{g}_j \cdot \frac{\partial \mathbf{g}_i}{\partial x^k} - \mathbf{g}_i \cdot \frac{\partial \mathbf{g}_j}{\partial x^k}\right)$$

$$= \frac{1}{2}\left(\frac{\partial}{\partial x^i}\left[\mathbf{g}_j \cdot \mathbf{g}_k\right] + \frac{\partial}{\partial x^j}\left[\mathbf{g}_i \cdot \mathbf{g}_k\right] - \frac{\partial}{\partial x^k}\left[\mathbf{g}_i \cdot \mathbf{g}_j\right]\right)$$

$$= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

Which is the quantity defined as the ***Christoffel symbols of the first kind*** in the textbooks. It is therefore possible for us to write,

$$[ij,k] \equiv \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_k$$

$$= \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_k$$

$$= \frac{1}{2}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)$$

It should be emphasized that the Christoffel symbols, even though the play a critical role in several tensor relationships, are themselves NOT tensor quantities. (Prove this). However, notice their symmetry in the i and j. The extension of this definition to the Christoffel symbols of the second kind is immediate:

Contract the above equation with the conjugate metric tensor, we have,

$$g^{k\alpha}[ij,\alpha] \equiv g^{k\alpha} \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}_\alpha$$

$$= g^{k\alpha} \frac{\partial \mathbf{g}_j}{\partial x^i} \cdot \mathbf{g}_\alpha = \frac{\partial \mathbf{g}_i}{\partial x^j} \cdot \mathbf{g}^k$$

$$= \begin{Bmatrix} k \\ ij \end{Bmatrix} = -\frac{\partial \mathbf{g}^k}{\partial x^j} \cdot \mathbf{g}_i$$

Which connects the common definition of the second Christoffel symbol with the one defined in the above derivation. The relationship,

$$g^{k\alpha}[ij,\alpha] = \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

apart from defining the relationship between the Christoffel symbols of the first kind and that second kind, also highlights, once more, the index-raising property of the conjugate metric tensor.

We contract the above equation with $g_{k\beta}$and obtain,

$$g_{k\beta}g^{k\alpha}[ij,\alpha] = g_{k\beta} \begin{Bmatrix} k \\ ij \end{Bmatrix} \delta_\beta^\alpha \, [ij,\alpha]$$

$$= [ij,\beta] = g_{k\beta} \begin{Bmatrix} k \\ ij \end{Bmatrix}$$

so that,

$$g_{k\alpha} \begin{Bmatrix} \alpha \\ ij \end{Bmatrix} = [ij,k]$$

Showing that the metric tensor can be used to lower the contravariant index of the Christoffel symbol of the second kind to obtain the Christoffel symbol of the first kind.

We are now in a position to express the derivatives of higher order tensor fields in terms of the Christoffel symbols.

Higher Order Tensors

For a second-order tensor $\mathbf{T}$, we can express the components in dyadic form along the product basis as follows:

$$\begin{aligned} \mathbf{T} &= T_{ij}\mathbf{g}^i \otimes \mathbf{g}^j \\ &= T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j \\ &= T^i_{.j}\mathbf{g}_i \otimes \mathbf{g}^j \\ &= T_j^{.i}\mathbf{g}^j \otimes \mathbf{g}_i \end{aligned}$$

This is perfectly analogous to our expanding vectors in terms of basis and reciprocal bases. Derivatives of the tensor may therefore be expressible in any of these product bases. As an example, take the product covariant bases.

We have:

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial x^k}\otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k}$$

Recall that, $\frac{\partial \mathbf{g}_i}{\partial x^k}\cdot \mathbf{g}^j = \begin{Bmatrix} j \\ ik \end{Bmatrix}$. It follows therefore that**,**

$$\begin{aligned}\frac{\partial \mathbf{g}_i}{\partial x^k}\cdot \mathbf{g}^j - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\delta_\alpha^j &= \frac{\partial \mathbf{g}_i}{\partial x^k}\cdot \mathbf{g}^j - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha \cdot \mathbf{g}^j \\ &= \left(\frac{\partial \mathbf{g}_i}{\partial x^k} - \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha\right)\cdot \mathbf{g}^j = 0.\end{aligned}$$

Clearly, $\frac{\partial \mathbf{g}_i}{\partial x^k} = \begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha$

(Obviously since $\mathbf{g}^j$ is a basis vector it cannot vanish)

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial x^k} &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\frac{\partial \mathbf{g}_i}{\partial x^k}\otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \frac{\partial \mathbf{g}_j}{\partial x^k} \\ &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{ij}\left(\begin{Bmatrix} \alpha \\ ik \end{Bmatrix}\mathbf{g}_\alpha\right)\otimes \mathbf{g}_j + T^{ij}\mathbf{g}_i \otimes \left(\begin{Bmatrix} \alpha \\ jk \end{Bmatrix}\mathbf{g}_\alpha\right) \\ &= \frac{\partial T^{ij}}{\partial x^k}\mathbf{g}_i \otimes \mathbf{g}_j + T^{\alpha j}\left(\begin{Bmatrix} i \\ \alpha k \end{Bmatrix}\mathbf{g}_i\right)\otimes \mathbf{g}_j + T^{i\alpha}\mathbf{g}_i \otimes \left(\begin{Bmatrix} j \\ \alpha k \end{Bmatrix}\mathbf{g}_j\right) \\ &= \left(\frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\begin{Bmatrix} i \\ \alpha k \end{Bmatrix} + T^{i\alpha}\begin{Bmatrix} j \\ \alpha k \end{Bmatrix}\right)\mathbf{g}_i \otimes \mathbf{g}_j = T^{ij}{}_{,k}\mathbf{g}_i \otimes \mathbf{g}_j\end{aligned}$$

Where

$$T^{ij}{}_{,k} = \frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\begin{Bmatrix} i \\ \alpha k \end{Bmatrix} + T^{i\alpha}\begin{Bmatrix} j \\ \alpha k \end{Bmatrix} \text{ or } \frac{\partial T^{ij}}{\partial x^k} + T^{\alpha j}\Gamma^i_{\alpha k} + T^{i\alpha}\Gamma^j_{\alpha k}$$

are the components of the covariant derivative of the tensor $\mathbf{T}$ in terms of contravariant components on the product covariant bases as shown.

In the same way, by taking the tensor expression in the dyadic form of its contravariant product bases, we can write,

$$\begin{aligned}\frac{\partial \mathbf{T}}{\partial x^k} &= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\frac{\partial \mathbf{g}^i}{\partial x^k}\otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k} \\ &= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\Gamma^i_{\alpha k}\otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}\end{aligned}$$

Again, notice from previous derivation above, $\begin{Bmatrix} i \\ jk \end{Bmatrix} = -\frac{\partial \mathbf{g}^i}{\partial x^k}\cdot \mathbf{g}_j$ so that, $\frac{\partial \mathbf{g}^i}{\partial x^k} = -\begin{Bmatrix} i \\ \alpha k \end{Bmatrix}\mathbf{g}^\alpha = -\Gamma^i_{\alpha k}\mathbf{g}^\alpha$. (The Literature has the two representations for the Christoffel Symbols.) Therefore,

$$\frac{\partial \mathbf{T}}{\partial x^k} = \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j + T_{ij}\frac{\partial \mathbf{g}^i}{\partial x^k} \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \frac{\partial \mathbf{g}^j}{\partial x^k}$$

$$= \frac{\partial T_{ij}}{\partial x^k}\mathbf{g}^i \otimes \mathbf{g}^j - T_{ij}\Gamma^i_{\alpha k}\mathbf{g}^\alpha \otimes \mathbf{g}^j + T_{ij}\mathbf{g}^i \otimes \Gamma^j_{\alpha k}\mathbf{g}^\alpha$$

$$= \left(\frac{\partial T_{ij}}{\partial x^k} - T_{\alpha j}\Gamma^\alpha_{ik} - T_{i\alpha}\Gamma^\alpha_{jk}\right)\mathbf{g}^i \otimes \mathbf{g}^j = T_{ij,k}\mathbf{g}^i \otimes \mathbf{g}^j$$

so that

$$T_{ij,k} = \frac{\partial T_{ij}}{\partial x^k} - T_{\alpha j}\Gamma^\alpha_{ik} - T_{i\alpha}\Gamma^\alpha_{jk}$$

Two other expressions can be found for the covariant derivative components in terms of the mixed tensor components using the mixed product bases defined above. It is a good exercise to derive these.

The formula for covariant differentiation of higher order tensors follow the same kind of logic as the above definitions. Each covariant index will produce an additional term similar to that in 3 with a dummy index supplanting the appropriate covariant index. In the same way, each contravariant index produces an additional term like that in 3 with a dummy index supplanting an appropriate contravariant index.

The covariant derivative of the mixed tensor, $A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_n}$ is the most general case for the covariant derivative of an absolute tensor:

$$A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_n \;\; ,j} = \frac{\partial A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_n}}{\partial x^j} - \begin{Bmatrix} \alpha \\ i_1 j \end{Bmatrix} A^{j_1, j_2, \ldots, j_m}_{\alpha, i_2, \ldots, i_n} - \begin{Bmatrix} \alpha \\ i_2 j \end{Bmatrix} A^{j_1, j_2, \ldots, j_m}_{i_1, \alpha, \ldots, i_n} - \cdots - \begin{Bmatrix} \alpha \\ i_n j \end{Bmatrix} A^{j_1, j_2, \ldots, j_m}_{i_1, i_2, \ldots, \alpha}$$
$$+ \begin{Bmatrix} j_1 \\ \beta j \end{Bmatrix} A^{\beta, j_2, \ldots, j_m}_{i_1, i_2, \ldots, i_n} + \begin{Bmatrix} j_2 \\ \beta j \end{Bmatrix} A^{j_1, \beta, \ldots, j_m}_{i_1, i_2, \ldots, i_n} + \cdots +$$

Physical Components in Orthogonal Systems

The components of a vector or tensor in a Cartesian system are projections of the vector on directions that have no dimensions and a value of unity.

These components therefore have the same units as the vectors themselves.

It is natural therefore to expect that the components of a tensor have the same dimensions.

In general, this is not so. In curvilinear coordinates, components of tensors do not necessarily have a direct physical meaning. This comes from the fact that base vectors are not guaranteed to have unit values ($h_i \neq 1$ in general).

They may not be dimensionless. For example, in orthogonal spherical polar, the base vectors are $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$.

These can be expressed in terms of dimensionless unit vectors as, $\rho \boldsymbol{e}_\rho, \rho \sin\phi\, \boldsymbol{e}_\theta,$ and $\boldsymbol{e}_\phi$ since the magnitudes of the basis vectors are $\rho, \rho\sin\phi$, and 1 or $\left(\sqrt{g_{11}}, \sqrt{g_{22}}, \sqrt{g_{33}}\right)$ respectively. As an example consider a force with the contravariant components F^1, F^2 and F^3,

$$\begin{aligned}\mathbf{F} &= F^1\mathbf{g}_1 + F^2\mathbf{g}_2 + F^3\,\mathbf{g}_3 \\ &= \rho F^1\mathbf{e}_\rho + \rho\sin\phi\, F^2\mathbf{e}_\theta\ + F^3\mathbf{e}_\phi\end{aligned}$$

Which may also be expressed in terms of physical components,

$$\mathbf{F} = F_\rho\mathbf{e}_\rho + F_\theta\mathbf{e}_\theta\ + F_\phi\mathbf{e}_\phi.$$

While these physical components $\{F_\rho, F_\theta, F_\phi\}$ have the dimensions of force, for the contravariant components normalized by the in terms of the unit vectors along these axes to be consistent, $\{\rho F^1, \rho\sin\theta\, F^2, F^3\}$ must each be in the units of a force. Hence, F^1, F^2 and F^3 may not themselves be in force units. The consistency requirement implies,

$$F_\rho = \rho F^1,\ \ F_\theta = \rho\sin\theta\, F^2,\ \text{and}\ \ F_\phi = F^3$$

For the same reasons, if we had used covariant components, the relationships,

$$F_\rho = \frac{F_1}{\rho},\ \ F_\theta = \frac{F_2}{\rho\sin\theta},\ \ \text{and}\ \ F_\phi = F_3$$

The magnitudes of the reciprocal base vectors are $\frac{1}{\rho}, \frac{1}{\rho\sin\phi}$, and 1. While the physical components have the dimensions of force, $F_1 = \rho F_\rho$ and $F_2 = \rho\sin\phi\, F_\theta$ have the dimensions of moment, while $F^1 = \frac{F_\rho}{\rho}$ and $F^2 = \frac{F_\theta}{\rho\sin\phi}$ are in dimensions of force per unit length. Only the third components in both cases are given in the dimensions of force.

Physical Component	Contravariant	Covariant	Mixed
$\tau(\rho\rho)$	$\tau^{11}h_1h_1 = \tau^{11}\rho^2$	$\dfrac{\tau_{11}}{h_1\,h_1} = \dfrac{\tau_{11}}{\rho^2}$	$\dfrac{\tau_1^1}{h_1}h_1 = \tau_1^1$

$\tau(\rho\theta)$	$\tau^{12}h_1h_2$ $= \tau^{12}\rho^2 \sin\phi$	$\frac{\tau_{12}}{h_1\, h_1} = \frac{\tau_{12}}{\rho^2 \sin\phi}$	$\frac{\tau_2^1}{h_2}h_1 = \frac{\tau_2^1}{\sin\phi}$
$\tau(\theta\theta)$	$\tau^{22}h_2h_2$ $= \tau^{22}\rho^2 \sin^2\phi$	$\frac{\tau_{22}}{h_2\, h_2} = \frac{\tau_{22}}{\rho^2 \sin^2\phi}$	$\frac{\tau_2^2}{h_2}h_2 = \tau_2^2$
$\tau(\theta\phi)$	$\tau^{23}h_2h_3$ $= \tau^{23}\rho \sin\phi$	$\frac{\tau_{23}}{h_2\, h_3} = \frac{\tau_{23}}{\rho \sin\phi}$	$\frac{\tau_3^2}{h_3}h_2$ $= \tau_3^2\rho \sin\phi$
$\tau(\phi\phi)$	$\tau^{33}h_3h_3 = \tau^{33}$	$\frac{\tau_{33}}{h_3\, h_3} = \tau_{33}$	$\frac{\tau_3^3}{h_3}h_3 = \tau_3^3$
$\tau(\rho\phi)$	$\tau^{31}h_3h_1 = \tau^{31}\rho$	$\frac{\tau_{31}}{h_3\, h_1} = \frac{\tau_{31}}{\rho}$	$\frac{\tau_1^3}{h_1}h_3 = \frac{\tau_1^3}{\rho}$

The transformation equations from the Cartesian to the oblate spheroidal coordinates ξ, η and φ are: $x = f\xi\eta \sin\varphi$, $y = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}$, and $z = f\xi\eta \cos\varphi$, where f is a constant representing the half the distance between the foci of a family of confocal ellipses. Find the components of the metric tensor in this system.

The metric tensor components are:

$$g_{\xi\xi} = \left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2 + \left(\frac{\partial z}{\partial \xi}\right)^2$$

$$= (f\eta \sin\varphi)^2 + f^2\xi^2\left(\frac{1-\eta^2}{\xi^2-1}\right) + (f\eta\cos\varphi)^2 = f^2\left(\frac{\xi^2-\eta^2}{\xi^2-1}\right)$$

$$g_{\eta\eta} = \left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2 = f^2\left(\frac{\xi^2-\eta^2}{1-\xi^2}\right)$$

$$g_{\varphi\varphi} = \left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2 = (f\xi\eta)^2$$

$$g_{\xi\eta} = \left(\frac{\partial x}{\partial \xi}\right)\left(\frac{\partial x}{\partial \eta}\right) + \left(\frac{\partial y}{\partial \xi}\right)\left(\frac{\partial y}{\partial \eta}\right) + \left(\frac{\partial z}{\partial \xi}\right)\left(\frac{\partial z}{\partial \eta}\right)$$

$$= (f\eta \sin\varphi)(f\xi \sin\varphi) - \left(f\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\right)\left(f\xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\right) + (f\eta\cos\varphi)(f\xi\cos\varphi)$$

$$= 0 = g_{\eta\varphi} = g_{\varphi\xi}$$

Find an expression for the divergence of a vector in orthogonal curvilinear coordinates. (***Rework***)

The gradient of a vector $\mathbf{F} = F^i\mathbf{g}_i$ is $\nabla \otimes \boldsymbol{F} = \boldsymbol{g}^j\partial_j \otimes \left(F^i\boldsymbol{g}_i\right) = F^i{}_{,j}\boldsymbol{g}^j \otimes \boldsymbol{g}_i$. The divergence is the contraction of the gradient. While we may use this to evaluate the divergence directly it is often easier to use the computation formula in equation Ex 15:

$$\operatorname{div}\mathbf{F} = F^i{}_{,j}\mathbf{g}^j \cdot \mathbf{g}_i = F^i{}_{,i} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,F^i\right)}{\partial x^i}$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_1h_2h_3F^1) + \frac{\partial}{\partial x^2}(h_1h_2h_3F^2) + \frac{\partial}{\partial x^3}(h_1h_2h_3F^3)\right]$$

Recall that the physical (components having the same units as the tensor in question) components of a contravariant tensor are not equal to the tensor components unless the coordinate system is Cartesian. The physical component $F(i) = F^ih_i$ (no sum on i). In terms of the physical components therefore, the divergence becomes,

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}(h_2h_3F(1)) + \frac{\partial}{\partial x^2}(h_1h_3F(2)) + \frac{\partial}{\partial x^3}(h_1h_2F(3))\right]$$

Find an expression for the Laplacian operator in Orthogonal coordinates.

For the contravariant component of a vector, F^j,

$$F^j{}_{,j} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,F^j\right)}{\partial x^j}.$$

Now the contravariant component of gradient $F^j = g^{ij}\varphi_{,i}$. Using this in place of the vector F^j, we can write,

$$g^{ij}\varphi_{,ji} = \frac{1}{\sqrt{g}}\frac{\partial\left(\sqrt{g}\,g^{ij}\varphi_{,j}\right)}{\partial x^i}$$

given scalar φ, the Laplacian $\nabla^2\varphi$ is defined as, $g^{ij}\varphi_{,ji}$ so that,

$$\nabla^2\varphi = g^{ij}\varphi_{,ji} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^i}\left(\sqrt{g}\, g^{ij}\frac{\partial\varphi}{\partial x^j}\right)$$

When coordinates are orthogonal, $g_{ij} = g^{ij} = 0$ whenever $i \neq j$. Expanding the computation formula therefore, we can write,

$$\nabla^2\varphi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(\frac{h_1h_2h_3}{h_1}\frac{\partial\varphi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(\frac{h_1h_2h_3}{h_2}\frac{\partial\varphi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(\frac{h_1h_2h_3}{h_3}\frac{\partial\varphi}{\partial x^3}\right)\right]$$
$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial x^1}\left(h_2h_3\frac{\partial\varphi}{\partial x^1}\right) + \frac{\partial}{\partial x^2}\left(h_1h_3\frac{\partial\varphi}{\partial x^2}\right) + \frac{\partial}{\partial x^3}\left(h_1h_2\frac{\partial\varphi}{\partial x^3}\right)\right]$$

Show that the oblate spheroidal coordinate systems are orthogonal. Find an expression for the Laplacian of a scalar function in this system.

Example above shows that $g_{\xi\eta} = g_{\eta\varphi} = g_{\varphi\xi} = 0$. This is the required proof of orthogonality. Using the computation formula in example 11, we may write for the oblate spheroidal coordinates that,

$$\nabla^2\Phi = \frac{\sqrt{(\xi^2-1)(1-\eta^2)}}{f^3\xi^2(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(f\xi\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\frac{\partial\Phi}{\partial\xi}\right) + \frac{\partial}{\partial\eta}\left(f\xi\eta\sqrt{\frac{1-\eta^2}{\xi^2-1}}\frac{\partial\Phi}{\partial\eta}\right)\right]$$
$$+ \frac{\partial}{\partial\eta}\left(\frac{f(\xi^2-\eta^2)}{\xi\eta\sqrt{(\xi^2-1)(1-\eta^2)}}\frac{\partial\Phi}{\partial\eta}\right)$$

3.61	Given tensors $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ we define the tensor square product with operator, $\boxtimes$, in the equation, $(\mathbf{A}\boxtimes\mathbf{B})\mathbf{C} = \mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}}$, show that the square product of two tensors $\mathbf{A}$ and $\mathbf{B}$ has the component form, $\mathbf{A}\boxtimes\mathbf{B} = A_{ik}B_{jl}\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^l$
	Let $\mathbf{A} = A_{ij}\mathbf{g}^i\otimes\mathbf{g}^j$, $\boldsymbol{B} = B_{kl}\mathbf{g}^k\otimes\mathbf{g}^l$, $\mathbf{C} = C^{\alpha\beta}\mathbf{g}_\alpha\otimes\mathbf{g}_\beta$. If we assume that $\mathbf{A}\boxtimes\mathbf{B} = A_{ik}B_{jl}\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^l$, Then the product, $$(\mathbf{A}\boxtimes\mathbf{B})\mathbf{C} = A_{ik}B_{jl}(\mathbf{g}^i\otimes\mathbf{g}^j\otimes\mathbf{g}^k\otimes\mathbf{g}^l)(C^{\alpha\beta}\mathbf{g}_\alpha\otimes\mathbf{g}_\beta) = A_{ik}B_{jl}C^{\alpha\beta}\delta^k_\alpha\delta^l_\beta\mathbf{g}^i\otimes\mathbf{g}^l = A_{ik}B_{jl}C^{kl}\mathbf{g}^i\otimes\mathbf{g}^l$$ To complete the proof, we only need to show that the above result equals $\mathbf{A}\mathbf{C}\mathbf{B}^{\mathrm{T}}$. To do so, we change the basis order in $\mathbf{B}$ so that,

	$\mathbf{ACB}^{\mathrm{T}} = (A_{ij}\mathbf{g}^i \otimes \mathbf{g}^j)(C^{\alpha\beta}\mathbf{g}_\alpha \otimes \mathbf{g}_\beta)(B_{kl}\mathbf{g}^l \otimes \mathbf{g}^k)$ $= A_{ij}C^{\alpha\beta}B_{kl}\mathbf{g}^i \otimes \mathbf{g}^k \delta^j_\alpha \delta^l_\beta = A_{ij}C^{jl}B_{kl}\mathbf{g}^i \otimes \mathbf{g}^k$ $= A_{ik}B_{jl}C^{kl}\mathbf{g}^i \otimes \mathbf{g}^j$ $= (\mathbf{A} \boxtimes \mathbf{B})\mathbf{C}$ We can therefore conclude that $\mathbf{A} \boxtimes \mathbf{B} = A_{ik}B_{jl}\mathbf{g}^i \otimes \mathbf{g}^j \otimes \mathbf{g}^k \otimes \mathbf{g}^l$.
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