FOUR

Kinematics: An Organized Study of the Geometry of Deformation & Motion

> *"…the various possible types of motion in themselves, leaving out … the causes to which the initiation of motion may be ascribed … constitutes the **Science of Kinematics**."—ET Whittaker*

MetaData

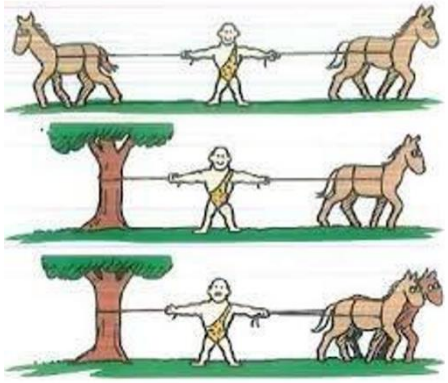
The prose, video, slides and the Q&A in this chapter are directed at scoring the following points:

1. Kinematics is the clincher in the introduction to Continuum Mechanics. The Tensor theory we have learned is designed to make it easy to understand **Kinematics**.
2. It is a mistake to rush to the study of the forces that influence deformation and motion without having the patience to understand what **deformation and motion** themselves are.

3. Kinematics provides the accurate description of the geometry – leading us to precise concepts of deformation terms such as **displacement**, **strain**, and other descriptors of deformation. From there we can know about the rates of strain, stretching rates, vorticity and other descriptors of motion.
4. One of the most important triumphs of Kinematics is the separation of motions and displacements that do not matter to the study of material behavior. This is done in the **multiplicative decomposition** of the fundamental deformation descriptor: The Deformation Gradient.
5. Differentiation of integrals plays key role. **Leibniz Reynold's Transport** Theorem extends the familiar Leibniz theorem from calculus. A generalized version of this is provided for scalars, vectors and tensor valued fields.
6. Basic deformations such as rotation, translation, **extension**, **shear** and as well as constrained flow such as **Irrotational**, **Steady** and **Rigid** flow fields are discussed.

Everything starts from the Geometry

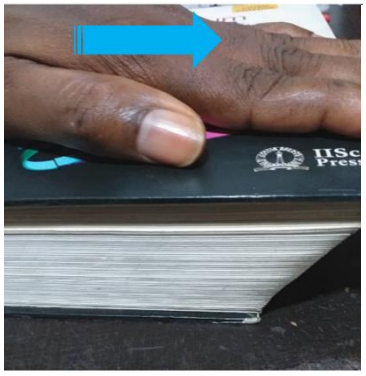
Engineers take great pain and spare no effort to ensure that service components do not undergo stresses or forces beyond their capability. Mechanics of materials provide the science to make the necessary computations to ensure this. It is a mistake, in our journey to understand that science, to quickly run to the study of stresses and forces. Everything we shall learn has a geometric background. Loading concepts such as tension, compression, torsion, bending, etc., have clear geometric implications. The unfortunate fellow in the picture below is obviously in tension. We have three examples of how that tension can be applied.

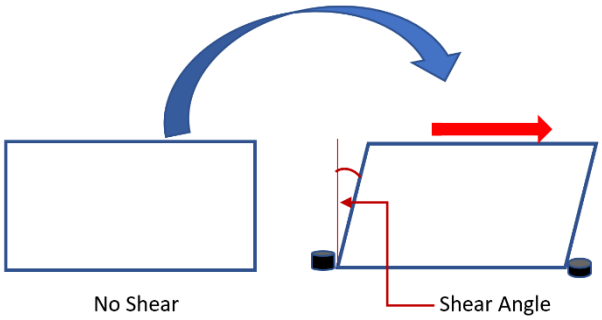


In each case, the geometrical issue is the possible separation of the hands from the body. There are forces involved obviously, but their effects and consequences are about the attempt to change in his geometry: Separation of his hands or the ripping apart of his body! This is tension: It tends to lengthen.

The geometry of compression is the opposite. Try to carry two or more bags of cement on your head, you will get that feeling that your height is being reduced.

In the book shearing deformation diagram shown, the essential geometrical effect is the turning



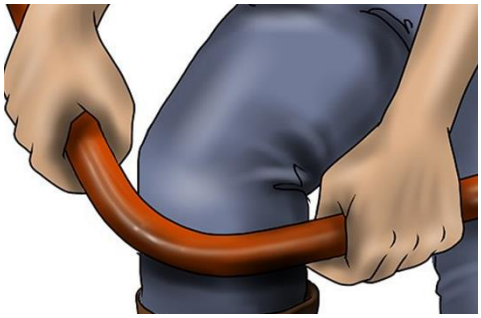


of the roughly rectangular cross section into a parallelogram. Decreasing an angle and increasing another. This kind of deformation and motion occurs all the time when we hoe, shave (sheep

shearing, etc.,) or in a much cataclysmic way, in earthquakes, landslides and other less portentous ways. Shearing creates relative angular displacements and motions in parts of the material body of interest to us.

Torsion also creates relative angular displacements. This is usually caused by moments in the longitudinal direction. In practical cases, it induces, in addition to the relative angular displacements, a warping deformation or motion in the longitudinal direction. Shafts, loaded by longitudinal moments, are often subjected to the shearing and warping caused by torsion.

Bending means, among other things, the alteration, or attempted alteration of the curvature of a body. In a straight bar, the curvature is decreased. Such changes in curvature also creates compression and tension on opposite sides of such bars. In a prismatic body, there is a cease-fire zone that is essentially neutral – free of the tension and compression, in a 3D body, there will usually be accompanying shear stresses. When the latter is NOT the case, it is said to be pure bending – the curvature alteration leading only to tensile and compressive stresses.



While the geometry of motion leads to the definition of purely geometric quantities such as strain, stretch and related quantities, the foregoing show that the loading situations encountered can also be accurately described by the geometry.

The case of thermal stresses is interesting. Consider a bar compelled, as shown in figure ___ to keep its length unchanged. Ordinarily, the heat supplied should lead to elongation. However, the geometrical constraint here creates the forces to prevent it from doing so. This causes thermal stresses.

This bar will be in compression as the effect of geometry is essentially to create the forces decreasing its length from the state the applied heat would have placed it.

Kinematics, the study of the geometry of deformation and motion "is the **machinery for describing** all possible deformations a body can undergo.".

Bodies: Material & Spatial Descriptions

To launch our description of the geometry of deformation, we look at the body we are concerned about in the ambient environment of a 3-Dimensional Euclidean Point Space, $\mathcal{E}$. More specifically, there is a subset of $\mathcal{E}$ between which is in a one to one correspondence with each point in the body.

A deformation is a mapping from this subset to another subset of the same space:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$$

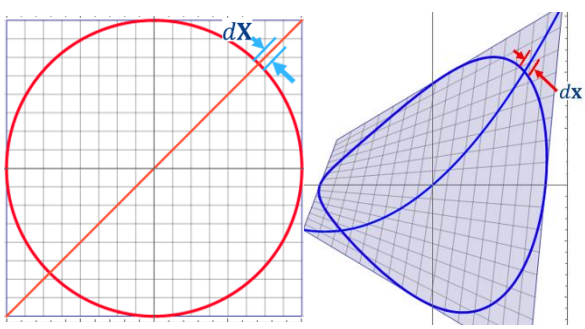
A deformation that changes over time, is a motion. If at specific points $t = 1,2, \dots, n$ in time, we have,

$$\mathbf{x}_1 = \boldsymbol{\chi}_1(\mathbf{X}), \mathbf{x}_2 = \boldsymbol{\chi}_2(\mathbf{X}), \dots, \mathbf{x}_n = \boldsymbol{\chi}_n(\mathbf{X}), \dots$$

Motion can also can be described by the single, continuous, time dependent function,

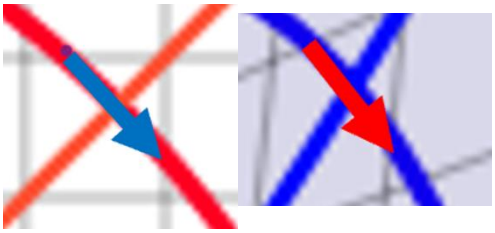
$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) \text{ or } \mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X})$$

so that motion is defined as set of mappings, $\mathbf{x} = \boldsymbol{\chi}(., t), t \in \mathbb{R}$. We assume that our subset of $\mathcal{E}$ is connected. Each member of the set of mappings, that is, each specific deformation in the set, is a known as a configuration or description at a point in time. We can take the configurations as photographs of the body as it undergoes its motion. If we take that view, even though we can have several photographs, at most one of them, represents the current state of the body. This configuration is called the Spatial Configuration. Example in figure shows two configurations of a body. If the second picture is its current state, we may have several pictures of intermediate states from the original. Any of these configurations may be chosen as a reference configuration as distinct from the current, spatial configuration. In several applications, it may be convenient to select the initial configuration as Reference, even though there is no obligation to do so. It is customary, for analytical purposes to have a spatial and a reference configuration in mind. Care must be taken to remember that these two refer to the same actual body at different times. At a particular time, the spatial configuration is always visible. If our reference, for example, were to be the initial



configuration, it means that at the beginning of the process, the initial and spatial configurations coincide as both are visible at that instant.

Our one-to-one mapping of the material points to the subset of $\mathcal{E}$ implies that no two material points occupy the same point at the same time. If we enlarge the infinitesimal segments of the referential and spatial circles marked, we obtain the arrows in figure____:



As a result of the motion, the referential element has deformed from the material (or referential) vector $d\mathbf{X}$ to the spatial vector $d\mathbf{x}$. As we already know, the transformation of one vector to another was caused by a tensor. Here we are looking at the material set of elements in the referential configuration that has been transformed in the motion. These contain, as we have seen, the same elements as in the referential configuration. These differential elements of the arc length represent the tangents and are transformed as,

$$d\mathbf{x} = \mathbf{F}\, d\mathbf{X}$$

where transformation tensor field $\mathbf{F}(\mathbf{X}, t) = \mathrm{Grad}\, \boldsymbol{\chi}(\mathbf{X}, t)$ the material (referential) gradient of the deformation or motion function, $\boldsymbol{\chi}(\mathbf{X}, t)$.

Notice that the transformation stretches (or contracts), rotates and translates the original vector into the new. All elements of that transformation are captured in the transformation tensor called the Deformation Gradient. Our objective is use the deformation gradient and define the geometric concepts of displacement, strain, stretch, rotation, that come from that transformation.

The vector equation here can also be written in terms of components as,

$$x_i = \chi_i(X_1, X_2, X_3), \quad i = 1,2,3$$

for each component because each vector equation is actually three scalar equations

Note also that for something to be dependent on the position vector $\boldsymbol{X}$ means exactly the same thing as to be dependent on its three scalar components.

In Cartesian coordinates, let $\boldsymbol{e}_\alpha,\ \alpha = 1,2,3$ and $\mathbf{E}_k,\ k = 1,2,3$ be the basis vectors in the spatial and reference coordinates respectively

For the differential vectors and the tensor, we can write, in component form,

$$d\mathbf{x} = dx_\alpha \mathbf{e}_\alpha$$

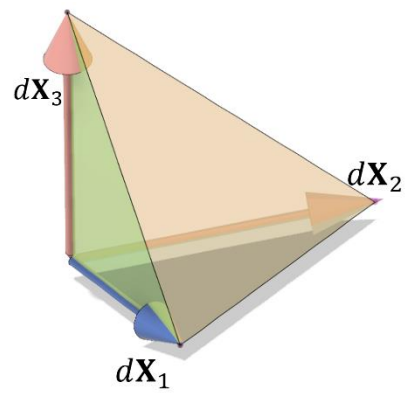
$$\mathbf{F} = \frac{\partial \chi_i}{\partial X_j} \mathbf{e}_i \otimes \mathbf{E}_j \; and$$

$$d\mathbf{X} = dX_k \mathbf{E}_k$$

Deformation gradient, $\mathbf{F}$ is called a two-toe tensor because it belongs to both configurations (one toe on each). It is a mixed tensor, having basis dyads from two configurations. It transforms the infinitesimal tensor from reference to spatial configuration.

We noted above that the function $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ is the deformation.

Volume Ratio



Consider three referential vectors $(d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3)$ forming the tetrahedron in Figure ____. The volume of the tetrahedron,

$$\frac{1}{6}[d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3] \neq 0$$

i.e. the volume will not vanish if the three vectors are neither colinear nor all coplanar. As a result of the motion, the corresponding spatial vectors will form a deformed tetrahedron. Each side will be a transformed referential vector into the spatial: $(d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3)$ will be related to the material vectors in such a way that,

$$d\mathbf{x}_i = \mathbf{F} d\mathbf{X}_{\boldsymbol{i}}$$

The volume ratio between the spatial and material configurations,

$$J = \frac{[d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3]}{[d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3]} = \frac{[\mathbf{F}d\mathbf{X}_1, \mathbf{F}d\mathbf{X}_2, \mathbf{F}d\mathbf{X}_3]}{[d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3]} = \det \mathbf{F}.$$

The linear independence of vectors $(d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3)$ is guaranteed by the non-vanishing of the tetrahedron or we shall have chosen a trivial volume. However, what guarantee do we have for the spatial tetrahedron?

In particular, we examine the situation,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X} = \mathbf{o}$$

the zero vector. What can this mean physically or otherwise? The linear independence of the denominator in the determinant expression guarantees the linear independence and

consequently the non-vanishing of the numerator provided the deformation gradient is an invertible tensor. Mathematically, the Jacobian (determinant of $\mathbf{F}$) of the transformation is zero. We were able to find a non-trivial (not a zero tensor) transformation tensor that transforms a real vector into nothingness! We, by a deformation transformation destroyed matter!

Our physical considerations preclude this possibility. We exclude from consideration such a possibility. And since we cannot have $J = 0$, we can therefore conclude that

$$J > 0$$

Since continuity forces it to pass through zero to negative; if iy cannot be zero, it cannot be negative. The only allowable transformations have a positive determinant.

The Reference Map

The set of mappings that gives each deformation, and consequently, the entire motion is a set of one-to-one mappings. Such mappings are invertible. It follows that, at each time t, we have,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t)$$

From which we can find the reference configuration that resulted in each spatial configuration a time t. The material point that occupied the spatial position $\mathbf{x}$ at time t can be computed by the reference map.

Motion Examples

In the attached $Mathematica^{®}$ Animation Code, it is easy to see that,

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3\mathbf{X} = X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3$$

```
myMap[X1_, X2_, α_] := {X1 + X2 α , X2 (4 X1 α + 1)} // Flatten;

(*distortion proportional to α say between .0 and .5 *)
initialConfig = ParametricPlot[{X1, X2}, {X1, -1, 1}, {X2, -1, 1},
   MeshShading → {{None}}];
circle = ParametricPlot[{Sin[t], Cos[t]}, {t, 0, 2 Pi},
   MeshShading → {{None}}];

line = ParametricPlot[{X1, X1}, {X1, -1, 1}];

Animate[
 deformedConfig = ParametricPlot[myMap[X1, X2, α], {X1, -1, 1}, {X2, -1, 1}];
 deformedCircle = ParametricPlot[myMap[Sin[t], Cos[t], α], {t, 0, 2 Pi}];
 dl1 = ParametricPlot[myMap[X1, X1, α], {X1, -1, 1}];
 Show[deformedConfig, deformedCircle, dl1, PlotRange → All],
 {α, .0, .16, .001}
]
```

We use different base vectors because we are not compelled to refer the referential and spatial to the same base. Let us for simplicity use the same base for now, we can see that the functional relationship is,

$$x_1 = X_1 + X_2 t;\ x_2 = 4X_1X_2t + X_2; x_3 = X_3$$

Which, in this case, is the vector equation or motion,

$$\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X}) \equiv \boldsymbol{\chi}(\mathbf{X},\ t)$$

Now issue the Mathematica command,

$$\text{Grad}[\text{myMap}[X_1, X_2, t], \{X_1, X_2\}]$$

You will easily see that, the deformation gradient in this case is,

$$\{\mathbf{F}(X_1, X_2, X_3, t)\} = \begin{bmatrix} 1 & t & 0 \\ 4tX_2 & 1 + 4tX_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Which is the matrix of the components of the deformation gradient tensor. Consider the more general deformation,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (1 + tX_1 + k_1X_2)\mathbf{e}_1 + (k_2X_1 + tX_2)\mathbf{e}_2 + t\mathbf{e}_3$$

Where k_1 and k_2 are constants, and t is the time variable. To obtain the reference map, we can invert this function and obtain,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) = \frac{tx_1 - k_1x_2}{t^2 - k_1k_2}\mathbf{E}_1 + \frac{tx_2 - k_2x_1}{t^2 - k_1k_2}\mathbf{E}_2 + \frac{x_3}{t}\mathbf{E}_3$$

Mathematica code for this inversion is,

```
Solve[{x1 == t X1 + X2 k1 , x2 == k2 X1 + X2 t, x3 == t X3}, {X1, X2, X3}]
```

$$\left\{\left\{X_1 \to -\frac{-t\,x_1 + k_1\,x_2}{t^2 - k_1\,k_2},\ X_2 \to -\frac{k_2\,x_1 - t\,x_2}{t^2 - k_1\,k_2},\ X_3 \to \frac{x_3}{t}\right\}\right\}$$

This is a simple example of a reference map. For a more specific deformation at a given time, say $t = 1$,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = (X_1 + k_1 X_2)\mathbf{e}_1 + (k_2 X_1 + X_2)\mathbf{e}_2 + t\mathbf{e}_3$$

We can invert this function and obtain,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) = \frac{x_1 - k_1 x_2}{1 - k_1 k_2}\mathbf{E}_1 + \frac{x_2 - k_2 x_1}{1 - k_1 k_2}\mathbf{E}_2 + x_3\mathbf{E}_3$$

Solving this in Mathematica for specific values for $k_1,\ k_2$, we have,

```
Solve[{x1 == t X1 + X2 k1 , x2 == k2 X1 + X2 t, x3 == t X3}, {X1, X2, X3}] /.
 {t → 1, k1 -> 0.15, k2 -> -.2}
```

```
{{X1 → -0.970874 (-x1 + 0.15 x2), X2 → -0.970874 (-0.2 x1 - x2), X3 → x3}}
```

Simple Motions.

The following examples of simple motions have been named:

1. **Pure translation**, $\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{X} + \mathbf{c}(t)$, where $\mathbf{c}$ is a differentiable vector-valued function of time.
2. **Pure rotation**, $\boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{Q}(t)\mathbf{X}$**,** where $\mathbf{Q}$ is a *proper orthogonal* function. (A complicated way of saying that it is a *rotation* function of time).
3. **Simple Shear**. $\boldsymbol{\chi}(\mathbf{X}, t) = (\mathbf{I} + \alpha(t)\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{X}$**,** where α is a differentiable, scalar valued function of time. Q: Transpose the dyad and what do you get? Compare to the original shear motion. The following Mathematica graphic is about Uniform Shear. The Deformation gradient here is easily calculated by hand. Do this to ensure you don't get lost in the mechanical computation and lose the context:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \mathbf{x} = (0.5 + X_1 + 0.5X_2)\mathbf{e}_1 + X_2\mathbf{e}_2 + X_3\mathbf{e}_3$$

for the element occupying $X_1\mathbf{E}_1 + X_2\mathbf{E}_2 + X_3\mathbf{E}_3$ initially.

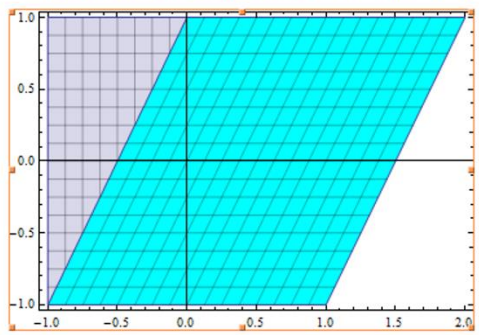
Clearly, $\frac{\partial x_1}{\partial X_1} = 1$, $\frac{\partial x_1}{\partial X_2} = 0.5$, $\frac{\partial x_1}{\partial X_3} = 0$ and $\frac{\partial x_2}{\partial X_2} = \frac{\partial x_3}{\partial X_3}$ with all other components of the deformation gradient vanishing.

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the matrix of the deformation gradient components. Note that the off-diagonal value, 0.5 is the tangent of the angle of shear – the reduction in the original right angle.

Use Mathematica to find **F**, **C**, **E**, **U** and **R**.

```
myMap[X1_, X2_] := {0.5 + X1 + 0.5 X2 , X2} // Flatten

initialConfig = ParametricPlot[{X1, X2}, {X1, -1, 1}, {X2, -1, 1}];

deformedConfig = ParametricPlot[myMap[X1, X2], {X1, -1, 1},
   {X2, -1, 1} , MeshShading → {{Cyan, Cyan}}];

Show[initialConfig, deformedConfig, PlotRange → All]
```



Another Shear Example. Consider a deformation gradient,

$$\mathbf{F} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}$$

Comparing this to the above transformation, we can see that it represents a shearing deformation with gamma being the tangent of the reduction in the originally right angle in the $x_1 - x_2$ plane.

Let $\gamma = 2\tan\beta$. It is easy to show that the above deformation gradient can be broken down to Rotation tensor **R** and Stretch tensors **U** and **V** such that,

$$\mathbf{R} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}$$

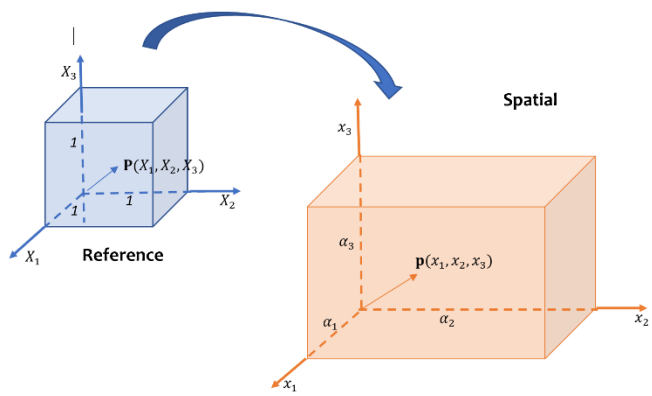
$$\mathbf{U} = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3] \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ \sin\beta & \sec\beta\,(1+\sin^2\beta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{pmatrix}$$

$$\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}} = [\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3] \begin{pmatrix} \sec\beta\,(1+\sin^2\beta) & \sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

4. **Uniform extension.** $\boldsymbol{\chi}(\mathbf{X},t) = (\lambda_1(t)\mathbf{e}_1, \lambda_2(t)\mathbf{e}_2, \lambda_3(t)\mathbf{e}_3\,) \otimes \begin{bmatrix} X_1\mathbf{e}_1 \\ X_2\mathbf{e}_2 \\ X_3\mathbf{e}_3 \end{bmatrix}$, where $\lambda_i, i = 1,2,3$ are differentiable scalar-valued functions of time. In the special case that $\lambda_1 = \lambda_2 = \lambda_3$, the motion is called a pure dilatation.

Consider the unit cube shown below in a triaxial extension so that a typical point $\mathbf{P}$ located at (X_1, X_2, X_3)in the undeformed state, moves to (x_1, x_2, x_3) in such a way that,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + \alpha_2 X_2 \mathbf{e}_2 + \alpha_3 X_3 \mathbf{e}_3$$



Note that uniaxial extension can be obtained by allowing two of the constants to be unity while biaxial will be ensured by one of the constants becoming one as follows:

Uniaxial: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3$

Biaxial: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = X_1 \mathbf{e}_1 + \alpha_2 X_2 \mathbf{e}_2 + \alpha_3 X_3 \mathbf{e}_3$

$$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3) \begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3) \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= \alpha_1 \mathbf{e}_1 \otimes \mathbf{E}_1 + \alpha_2 \mathbf{e}_2 \otimes \mathbf{E}_2 + \alpha_3 \mathbf{e}_3 \otimes \mathbf{E}_3$$

The Green Lagrange strain tensor is,

$$\mathbf{E} = -\frac{1}{2}(1 - \alpha_1^2)\mathbf{E}_1 \otimes \mathbf{E}_1 - \frac{1}{2}(1 - \alpha_2^2)\mathbf{E}_2 \otimes \mathbf{E}_2 - \frac{1}{2}(1 - \alpha_3^2)\mathbf{E}_3 \otimes \mathbf{E}_3$$

```
F := {{α1, 0, 0}, {0, α2, 0}, {0, 0, α3}}
CC = Transpose[F].F
```

$$\{\{\alpha_1^2, 0, 0\}, \{0, \alpha_2^2, 0\}, \{0, 0, \alpha_3^2\}\}$$

```
EE = (1/2) (CC - IdentityMatrix[3]) // MatrixForm
```

atrixForm=

$$\begin{pmatrix} \frac{1}{2}(-1+\alpha_1^2) & 0 & 0 \\ 0 & \frac{1}{2}(-1+\alpha_2^2) & 0 \\ 0 & 0 & \frac{1}{2}(-1+\alpha_3^2) \end{pmatrix}$$

Motions 1 and 2 in the list are called rigid body motions because they do not, by themselves, cause shape changes in the body. Real life motions may be a combination of some of these and more complicated motions like in the earlier examples. It is a goos practice to visualize these motions as we shall do in the examples section later. Mathematica animations of these and other motions are available and tinkering with the motion equations to obtain new motions in combination is a good way to get a clearer view of motions in general. We can then compute the deformation gradients and other useful tensors describing aspects of such motions.

Components of Two-Towed Tensors

The deformation gradient is a special tensor. It is a tensor field that transforms small vectors from referential configuration to current, spatial configuration: $d\mathbf{x} = \mathbf{F}d\mathbf{X}$. This equation can be written in the form,

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$$

In which the inverse deformation gradient tells what vector in the reference state was transformed to $d\mathbf{x}$ in the spatial. Hence $\mathbf{F}^{-1}$, just like $\mathbf{F}$, is two towed: from spatial to referential. From the product in the transformation equation,

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}$$

It makes sense that $d\mathbf{x} = dx_\alpha \mathbf{e}_\alpha,\ and\ d\mathbf{X} = dX_k \mathbf{E}_k$ shows that the deformation gradient should derive its second basis from referential configuration so that,

$$d\mathbf{x} = dx_\alpha \mathbf{e}_\alpha = \left(\frac{\partial \chi_i}{\partial X_j}\mathbf{e}_i \otimes \mathbf{E}_j\right) dX_k \mathbf{E}_k$$

So that scalar products of bases from the same configurations are in scalar product in the dyad operation above. For the same reason, the bases of $\mathbf{F}^{-1}$ are from material to spatial:

$$\mathbf{F}^{-1} = [\mathbf{F}^{-1}]_{ij}\mathbf{E}_i \otimes \mathbf{e}_j$$

One more issue, hidden from us in the ONB system of the Cartesian, is that, since the referential coordinate is in a reciprocal side of the expression, $\frac{\partial \chi_i}{\partial X_j}$, the base vector associated with it MUST

also be a reciprocal base. A fill analysis of this can be seen from reciprocal base systems which is needed for non ONB systems. In Cartesian systems, the natural and the reciprocal bases coincide. In curvilinear coordinates such as Cylindrical and Spherical Polar, this is not so. For example, for spherical polar, the reciprocal basis can be derived from the natural basis (obtained by differentiating the position vector, as follows:

$$\begin{pmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{pmatrix} = \begin{pmatrix} \dfrac{1}{\mathbf{g}_1 \cdot \mathbf{g}_1} & 0 & 0 \\ 0 & \dfrac{1}{\mathbf{g}_2 \cdot \mathbf{g}_2} & 0 \\ 0 & 0 & \dfrac{1}{\mathbf{g}_3 \cdot \mathbf{g}_3} \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \rho \mathbf{e}_\theta \\ \rho \sin\theta\, \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{\rho^2} & 0 \\ 0 & 0 & \dfrac{1}{\rho^2 \sin^2\theta} \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \rho \mathbf{e}_\theta \\ \rho \sin\theta\, \mathbf{e}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{e}_\rho \\ \dfrac{\rho \mathbf{e}_\theta}{\rho^2} \\ \dfrac{\rho \sin\theta\, \mathbf{e}_\phi}{\rho^2 \sin^2\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_\rho \\ \dfrac{\mathbf{e}_\theta}{\rho} \\ \dfrac{\mathbf{e}_\phi}{\rho \sin\theta} \end{pmatrix}$$

The multiplying matrix is made up of the components of the inverse metric tensor. The following table shows the list of natural and reciprocal bases for Cartesian, Cylindrical Polar and Spherical Polar Coordinate systems.

Coordinate System	**Natural Basis Vectors**	**Reciprocal Base Vectors**
Cartesian	$\left\{\dfrac{\partial \mathbf{r}}{\partial x_1} = \mathbf{e}_1;\ \dfrac{\partial \mathbf{r}}{\partial x_2} = \mathbf{e}_2;\ \dfrac{\partial \mathbf{r}}{\partial x_3} = \mathbf{e}_3\right\}$	$\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$
Cylindrical Polar	$\left\{\dfrac{\partial \mathbf{r}}{\partial r} = \mathbf{e}_r;\ \dfrac{\partial \mathbf{r}}{\partial \phi} = r\mathbf{e}_\phi;\ \dfrac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z\right\}$	$\left\{\mathbf{e}_r;\ \dfrac{\mathbf{e}_\phi}{r};\ \mathbf{e}_z\right\}$
Spherical Polar	$\left\{\dfrac{\partial \mathbf{r}}{\partial \rho} = \mathbf{e}_\rho;\ \dfrac{\partial \mathbf{r}}{\partial \theta} = \rho\mathbf{e}_\theta;\ \dfrac{\partial \mathbf{r}}{\partial \phi} = \rho \sin\theta \mathbf{e}_\phi\right\}$	$\left\{\mathbf{e}_\rho;\ \dfrac{\mathbf{e}_\theta}{\rho};\ \dfrac{\mathbf{e}_\phi}{\rho \sin\theta}\right\}$

While the natural basis vectors are computed by simple differentiation, the reciprocal vectors are computed from reciprocity relationships. In the case of orthogonal systems, linear or curvilinear, this relationship becomes simply dividing by the magnitude of the respective natural base vector. The deformation gradient from a material configuration in cylindrical Polar coordinates $\{R, \Theta, Z\}$ to a spatial configuration $\{r, \theta, z\}$ in the same coordinate system is,

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}.$$

We used upper case to depict the Material system. It is the reciprocal system. If both were spherical, $\{\varrho, \Theta, \Phi\} \to \{\rho, \theta, \phi\}$the deformation gradient becomes,

$$\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho \sin\theta\, \mathbf{e}_\phi)\begin{bmatrix} \frac{\partial \rho}{\partial \varrho} & \frac{\partial \rho}{\partial \Theta} & \frac{\partial \rho}{\partial \Phi} \\ \frac{\partial \theta}{\partial \varrho} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial \Phi} \\ \frac{\partial \phi}{\partial \varrho} & \frac{\partial \phi}{\partial \Theta} & \frac{\partial \phi}{\partial \Phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_\varrho \\ \frac{\mathbf{E}_\Theta}{\varrho} \\ \frac{\mathbf{E}_\Phi}{\varrho \sin\Theta} \end{bmatrix}.$$

Polar Decomposition Theorem.

In our discussion of tensors, we saw that two additive decompositions of any second-order tensor can be done. In this section, we are looking at a multiplicative decomposition, motivated by the reality that it is NOT the entire transformation wrought by the deformation gradient that concern us in the study of geometrical changes resulting from the application of loads. The important result called by this name takes its naming roots from complex analysis where a complex variable is represented in two dimensional Polar Coordinates. As we shall see, there is no direct link to this in the proof or application of the theorem. It successfully separates portions of the deformation gradient that do not cause shape changes from the parts that are relevant in geometric modifications resulting from the transformation.

> ***Theorem****. For a given deformation gradient* $\boldsymbol{F}$*, there is a unique rotation tensor* $\boldsymbol{R}$*, and unique, positive definite, symmetric tensors* $\boldsymbol{U}$ *and* $\boldsymbol{V}$ *for which,*
> $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$

This is a fundamental theorem in continuum mechanics called the **Polar Decomposition Theorem**.

Observation. This theorem will be proved shortly. Before embarking on the proof, observe the following:

1. By the results of this theorem, $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$. $\mathbf{R}$ is a rotation tensor while $\mathbf{U}$ and $\mathbf{V}$ are the right (or material) stretch tensor and the left (spatial) stretch tensors respectively. Being a rotation tensor, $\mathbf{R}$ must be proper orthogonal. In addition to its components being an orthogonal matrix, the matrix representation of $\mathbf{R}$ must have a determinant that is positive: $\det \mathbf{R} = +1$.
2. Note that $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{T}}\,\mathbf{R}^{\mathrm{T}}\mathbf{R}\,\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{I}\,\mathbf{U} = \mathbf{U}^2$.

   Definition: **Positive Definite**. A tensor $\mathbf{T}$ is positive definite if for every real vector $\mathbf{u}$, the quadratic form $\mathbf{u} \cdot \mathbf{T}\mathbf{u} > \mathbf{0}$. If $\mathbf{u} \cdot \mathbf{T}\mathbf{u} \geq \mathbf{0}$ Then $\mathbf{T}$ is said to be positive semi-definite.

   Now every positive definite tensor $\mathbf{T}$ has a square root $\mathbf{U}$ such that,

$$\mathbf{U}^2 \equiv \mathbf{U}^{\mathrm{T}}\mathbf{U} = \mathbf{U}\mathbf{U}^{\mathbf{T}} = \mathbf{T}$$

Proof. To prove this theorem, we must first show that $\mathbf{F}^{\mathrm{T}}\mathbf{F}$ is symmetric and positive definite. Take its transpose; symmetry becomes obvious.

To show positive definiteness, for an arbitrary real vector $\mathbf{u}$ consider the expression, $\mathbf{u} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{u}$. Let the vector $\mathbf{b} = \mathbf{F}\mathbf{u}$. Then we can write,

$$\mathbf{u} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}\mathbf{u} = \mathbf{u} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{b} = \mathbf{b} \cdot \mathbf{F}\mathbf{u} = \mathbf{b} \cdot \mathbf{b} = |\mathbf{b}|^2 > 0$$

as the magnitude of any real vector must be positive. Hence $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$ is positive definite.

Since every positive definite tensor has a positive definite square root. Let that square root be $\mathbf{U}$

$$\begin{aligned}\mathbf{F}^{\mathbf{T}}\mathbf{F} &= \mathbf{U}\mathbf{U} = \mathbf{U}^{\mathbf{T}}\mathbf{U} \\ &= \mathbf{U}^{\mathbf{T}}\mathbf{I}\mathbf{U} = \mathbf{U}^{\mathbf{T}}\mathbf{R}^{\mathbf{T}}\mathbf{R}\mathbf{U} \\ &= (\mathbf{R}\mathbf{U})^{\mathbf{T}}\mathbf{R}\mathbf{U}\end{aligned}$$

Which shows that $\mathbf{F} = \mathbf{R}\mathbf{U}$

We can also find a positive definite tensor $\mathbf{V}$ such that $\mathbf{F} = \mathbf{V}\mathbf{R}$

Write $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R} \Rightarrow \mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^{-1}$

The fact that $\mathbf{V}$ is positive definite can also be established from the fact that

$$\begin{aligned}\mathbf{V}^2 &= \mathbf{R}\mathbf{U}\mathbf{R}^{-1}\mathbf{R}\mathbf{U}\mathbf{R}^{-1} \\ &= \mathbf{R}\mathbf{U}\mathbf{U}\mathbf{R}^{-1} = \mathbf{R}\mathbf{U}\mathbf{U}\mathbf{R}^{\mathbf{T}} \\ &= \mathbf{R}\mathbf{U}(\mathbf{R}\mathbf{U})^{\mathbf{T}} \\ &= \mathbf{F}\mathbf{F}^{\mathbf{T}}\end{aligned}$$

which is obviously positive-definite.

To complete the Polar Decomposition Theorem, we now need to show that the $\mathbf{R}$ in

$$\mathbf{F} = \mathbf{RU}$$

is a rotation. Now, from the above equation, we have that,

$$\mathbf{FU}^{-1} = \mathbf{R}$$

so that

$$\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{U}^{-\mathrm{T}}\mathbf{F}^{\mathrm{T}}\mathbf{FU}^{-1} = \mathbf{U}^{-1}\mathbf{U}^{2}\mathbf{U}^{-1} = \mathbf{I}$$

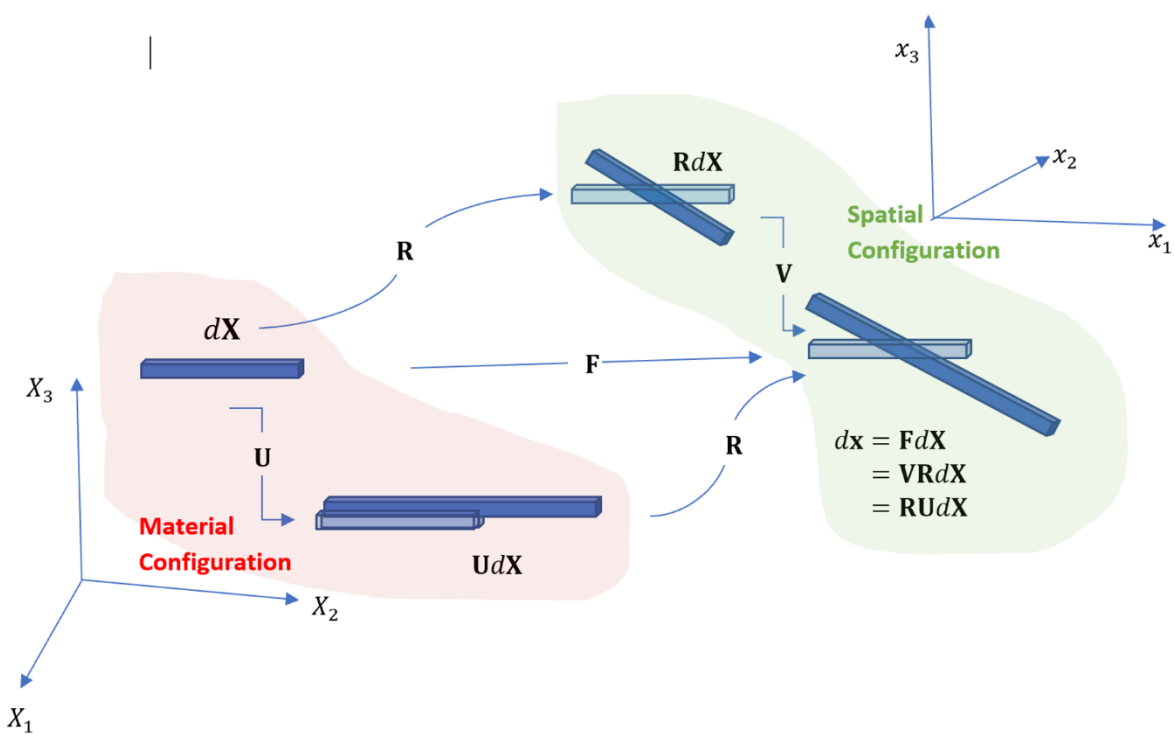
Which shows $\mathbf{R}$ to be an orthogonal tensor. But

$$\det \mathbf{R} = \det(\mathbf{FU}^{-1}) = \det \mathbf{F} \times \det \mathbf{U}^{-1} > 0.$$

From physical considerations, we know that determinant of the deformation gradient is necessarily positive and that of the inverse of $\mathbf{U}$ is positive because $\mathbf{U}^{-1}$ is also positive definite. Hence we can see that, $\det \mathbf{R} = +\mathbf{1}$. Which, when added to the fact that $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ proves thatmeans that $\mathbf{R}$ is a rotation.

End of Theorem

Meaning of the Polar Decomposition Theorem.



The diagram in Figure _____ depicts the polar decomposition theorem.

Beginning from any material configuration, the transformation given by the deformation gradient leads to the spatial configuration. However, this transformation can be achieved in two two-stage processes.

1. A stretch in the material configuration through the Right Stretch Tensor $\mathbf{U}$; followed by a rotation by the rotation tensor $\mathbf{R}$ to the spatial configuration. Note that the rotation tensor is neither a material nor a spatial tensor. It is, like the deformation gradient, a two-toe tensor; operating on a material vector and producing a spatial tensor.
2. A transformation to the spatial configuration by the rotation tensor $\mathbf{R}$, followed by a stretch to the final state in that configuration by the left stretch tensor. The latter is a spatial tensor as it takes a spatial vector (output of the rotation tensor), and returns a spatial vector.

The right stretch tensor is a material tensor. As we can see, the different tensors that come to our attention are classified by what kinds of arguments they can take and what kind of vectors they produce. On the other hand, vectors are classified by where they reside. For example, the material vector is so called because it is made up of elements in the referential (material) configuration. Spatial tensors are similarly defined.

For tensors, when the input as well as output of a tensor are material vectors, such is a material tensor. Examples encountered so far include the Right Stretch Tensor, its inverse, The Right Cauchy-Green Tensor and its inverse. These tensors are also symmetrical and positive definite. Spatial tensors take spatial vectors and produce spatial vectors as do the Left Stretch Tensor, its inverse and the Left Cauchy-Green Tensor and its inverse.

Two-Toe Tensors are neither material not spatial tensors. They take spatial inputs and produce material, or vice versa. Examples include the Rotation Tensor, its transpose (same as its inverse), the Deformation Gradient, its transpose and its inverse.

We the nature of the kind of tensor or the group a tensor belongs can be established by the following reasoning:

Consider a spatial vector $\mathbf{s}$. The dot product $\mathbf{s} \cdot d\mathbf{x}$ has physical significance while $\mathbf{s} \cdot d\mathbf{X}$ does not as the two operands do not exist at the same time so an operation between them makes no physical sense.

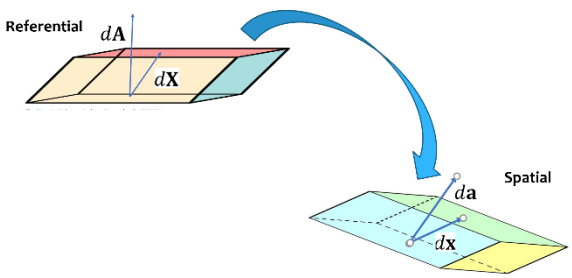
Clearly,

$$\mathbf{s} \cdot d\mathbf{x} = \mathbf{s} \cdot \mathbf{F} d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}} \mathbf{s}$$

meaning that $\mathbf{F}^{\mathrm{T}}\mathbf{s}$ is a material vector so that $\mathbf{F}^{\mathrm{T}}$ transforms spatial vectors to material. Beginning with a material vector $\mathbf{t}$. The physically meaningful product,

$$\mathbf{t} \cdot d\mathbf{X} = \mathbf{t} \cdot \mathbf{F}^{-1} d\mathbf{x} = d\mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}} \mathbf{t}$$

Showing that $\mathbf{F}^{-\mathrm{T}}$ transforms material to spatial while $\mathbf{F}^{-1}$transforms spatial vectors to material. These tensors are two-toed.



Area Transformation

For an element of area $d\mathbf{a}$ in the deformed body with a vector $d\mathbf{x}$ projecting out of its plane (does not have to be normal to it). For the elemental volume, we have the following relationship:

$$d\mathbf{v} = J d\mathbf{V} = d\mathbf{a} \cdot d\mathbf{x} = J d\mathbf{A} \cdot d\mathbf{X}$$

where $d\mathbf{A}$ is the element of area that transformed to $d\mathbf{a}$ and $d\mathbf{X}$ is the image of $d\mathbf{x}$ in the undeformed material. Noting that, $d\mathbf{x} = \mathbf{F} d\mathbf{X}$ we have,

$$d\mathbf{a} \cdot \mathbf{F} d\mathbf{X} - J d\mathbf{A} \cdot d\mathbf{X} = 0$$

$$= (\mathbf{F}^{\mathrm{T}} d\mathbf{a} - J d\mathbf{A}) \cdot d\mathbf{X}$$

For an arbitrary vector $d\mathbf{X}$**,** we have:

$$\mathbf{F}^{\mathrm{T}} d\mathbf{a} - J d\mathbf{A} = \mathbf{o}$$

so that,

$$d\mathbf{a} = J \mathbf{F}^{-\mathrm{T}} d\mathbf{A} = \mathbf{F}^{\mathbf{c}} d\mathbf{A}$$

where $\mathbf{F}^{\mathbf{c}}$ is the cofactor tensor of the deformation gradient. We have used the transformation of volume to obtain an expression for the area transformation. The cofactor tensor is responsible for local area changes while the determinant of the deformation gradient is responsible for local volume changes.

Measuring Shape Changes

Strain

Strain is our attempt to quantify relative displacements and changes in orientations of material line elements as a result of the deformation. Wholesale movements of the entire element itself, by rotation, translation or a combination of both do not qualify as strain. We call such transformations **Rigid Body Motions**. Examples are:

1. **Rotation**: of all material points in the element about an axis
2. **Translation**: of all the material element by the same amount in a given direction.

Strain is a definition. Successful strain functions are so because experience and usage in using them as measuring and prediction tools have been successful. A proper strain function must satisfy two conditions:

- Two deformations, differing only by rigid body motions represent the same strained system in so far as they create the same shape changes in identical materials. A correct strain function will detect this and compute equal quantities for the situations they represent.
- When the deformation gradient becomes $\mathbf{F} = \mathbf{I}$, the identity tensor, the strain function must vanish everywhere. This means that,

Many strain functions can be defined in so far as they satisfy the above conditions. A number have been used successfully in certain situations.

The most successful strain functions are defined from the **Right and Left Cauchy-Green** Tensors. They are defined either as material tensors or spatial tensors. Let us consider first the **Green-Lagrange** Strain Tensor, $\mathbf{E}$ defined as

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$**,** the **right Cauchy Green** Tensor. It will become clear shortly that this strain function is more familiar than it looks. A comparison of what it computes will be made to our elementary conception of strain as the quotient of “increase in length and original length”. It will soon become clearer that this is the strain function we have in mind from that common definition.

Next, we look at the **Euler-Almansi** Strain Tensor, $\mathbf{e}$

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is called the **Left Cauchy-Green** Tensor. ($\mathbf{B}^{-1}$ is named in honor of another great man and referred to as the **Finger** Tensor.). We have shown that $\mathbf{C} = \mathbf{U}^2$ is a material tensor while $\mathbf{B} = \mathbf{V}^2$ is spatial. Consequently, $\mathbf{E}$ is a material strain tensor field while $\mathbf{e}$ is spatial.

Generalized Strain: The Seth-Hill Strain Functions

It has been shown by Seth and Hill that the popular strain functions are special cases of generalized strain functions. These functions, named for the authors, are called the Seth-Hill functions. The referential Seth Hill Strain Function is,

$$\frac{1}{m}(\mathbf{U}^m - \mathbf{I}) \text{ for } m \neq 0,$$

$$\log_{\mathrm{e}} \mathbf{U}\,,\ m = 0$$

It is easy to see that the Euler-Lagrange Strain function is the special case of the Seth-Hill material strain function when $m = 2$.

On the spatial side of things, we have another class of strain function generators. Here is the spatial Seth-Hill Strain function:

$$\frac{1}{m}(\mathbf{V}^m - \mathbf{I}) \text{ for } m \neq 0$$

$$\log_{\mathrm{e}} \mathbf{V}\,,\ m = 0$$

Again, as before, the Euler-Almansi Strain function, $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$ is the special case of the spatial Seth-Hill Strain function when $m = -2$.

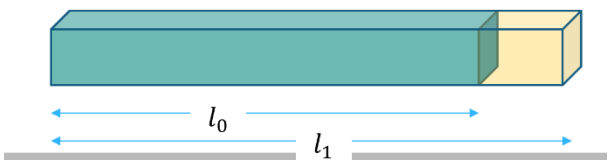
Uniaxial Extension

They told you that strain is Increase in length over original length! Here is what they were talking about:

We noted earlier that Uniaxial extension transformation function is, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3$. Let us write $\alpha_1 = l_1/l_0$ and examine the implications. What is the value of $\alpha_1 X_1$ when $\alpha_1 = l_1/l_0$? Of course, it is zero when $X_1 = 0$, and it is equal to l_1 when $X_1 = l_0$. In one word, it properly defines the spatial configuration for the uniaxial extension we are so used to!

in this case, $\alpha_2 = \alpha_3 = 1$. Consequently, the Lagrangian Strain becomes,

The Green Lagrange strain tensor is,



$$\mathbf{E} = -\frac{1}{2}\left(1 - \left(\frac{l_1}{l_0}\right)^2\right)\mathbf{E}_1 \otimes \mathbf{E}_1$$
$$\approx \frac{l_1 - l_0}{l_0}\mathbf{E}_1 \otimes \mathbf{E}_1$$

To see that this is true, consider that,

$$\frac{l_1^2 - l_0^2}{2l_0^2} = \frac{(l_1 - l_0)(l_1 + l_0)}{2l_0^2}$$

Now, observe that,

$$\lim_{l_0 \to l_1} \frac{(l_1 - l_0)(l_1 + l_0)}{2l_0^2} = \frac{l_1 - l_0}{l_0}$$

When strains are small, in uniaxial extension, it is correct to state that change in length divided by original length is equal to strain! What do the components of the strain tensor mean?

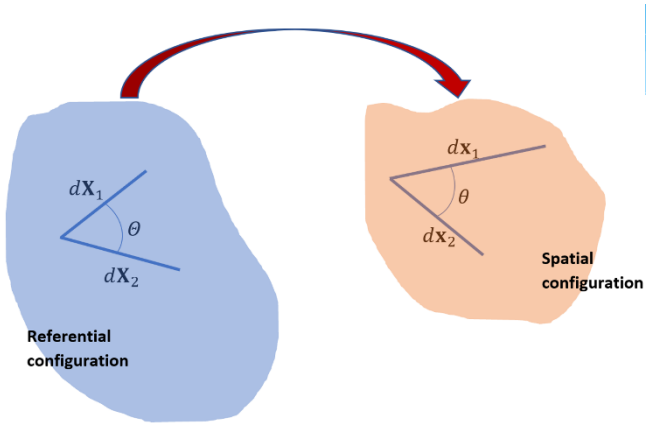
Begin with the meaning of the deformation gradient. The strain tensor components deal with the fibers along the coordinate axes. A look at the strain computations in earlier $Mathematica^{®}$ code reveals the fact that extension (or contraction) supplies the diagonal elements of the strain tensor. Translations and rotations create zero strain as they are rigid-body motions. In the example here, the orientation of the extension along the coordinate axes shows that linear extension or contraction along a coordinate axes creates a strain component element in the diagonal for that coordinate. It can similarly be shown that shear creates off diagonal element in the strain tensor. If it is limited to a coordinate plane, the off-diagonal elements appropriate to that plane will be created. For example, a shear on the $\mathbf{e}_1 - \mathbf{e}_2$ plane creates components $\mathbf{e}_1 \otimes \mathbf{e}_2$ or $\mathbf{e}_2 - \mathbf{e}_1$. It does not matter which is chosen as the strain tensor, by definition, MUST be symmetrical.

Stretch Tensors, Right and Left

The Polar decomposition theorem immediately shows why the deformation gradient cannot be a proper measure of strain. Consider the expression, $\mathbf{F}_1 = \mathbf{R}_1\mathbf{U}$, $\mathbf{F}_2 = \mathbf{R}_2\mathbf{U}$ so that the only difference between the two deformation gradients is the fact that the rotations are different, but the stretch tensors are the same. The strains should be the same but, if we were to use the

deformation gradient as our strain function, we would compute two different values. This disqualifies the deformation gradient as a correct measure of strain.

Consider two infinitesimal material vectors, $d\mathbf{X}_1$and $d\mathbf{X}_2$and subject the material in which they are placed to the deformation gradient $\mathbf{F}$. Clearly, the images of these two elements in the spatial state will be:



$$d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1 \text{and } d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$$

We now proceed to find the magnitude of the image vectors by taking the scalar products as follows:

$$\begin{aligned} d\mathbf{x}_1 \cdot d\mathbf{x}_2 &= \mathbf{F}d\mathbf{X}_1 \cdot \mathbf{F}d\mathbf{X}_2 \\ &= \mathbf{RU}d\mathbf{X}_1 \cdot \mathbf{RU}d\mathbf{X}_2 \\ &= \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{R}^{\mathrm{T}}\mathbf{RU}d\mathbf{X}_2 \\ &= \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2 \end{aligned}$$

Upon recalling that the transpose of a rotation is its inverse. So that, if both vectors are the same, we have that,

$$d\mathbf{x}_1 \cdot d\mathbf{x}_1 = \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_1$$

And after taking square roots, we see that,

$$\|d\mathbf{x}\| = \|\mathbf{U}d\mathbf{X}\|$$

Which tells us that the ***magnitude*** of the spatial vector is governed by a transformation of the material vector, not by the deformation gradient, but by the **right stretch tensor**.

It is left as an exercise for you to make the inverse argument, that, in terms of the spatial lengths, the referential length can be found from,

$$\|d\mathbf{X}\| = \|\mathbf{V}^{-1}d\mathbf{x}\|$$

Shear Strain

The above arguments helps us clarify issues with normal strains on infinitesimal elements. Once we know the Right stretch tensor, we can find the new length of any fibre. In **shear strain**, we are interested, not in elongation or reductions in lengths, but in the changes in the angles between infinitesimal elements.

Using the same diagram, we can take a look at the angle between these two referential elements as they are transformed in the deformation In the referential configuration, the angle between the line elements, $d\mathbf{X}_1$ and $d\mathbf{X}_2$ is,

$$\Theta = \cos^{-1}\left(\frac{d\mathbf{X}_1 \cdot d\mathbf{X}_2}{\|d\mathbf{X}_1\|\|d\mathbf{X}_2\|}\right)$$

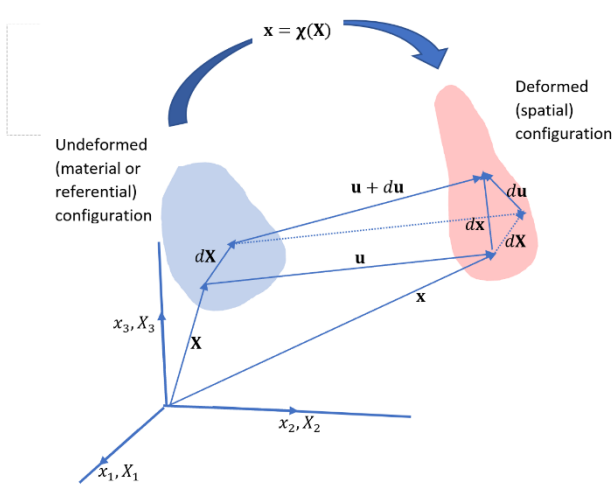
To find the angle between any two elements in the spatial configuration we simply recall that the angle we seek is

$$\theta = \cos^{-1}\left(\frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\|d\mathbf{x}_1\|\|d\mathbf{x}_2\|}\right) = \cos^{-1}\left(\frac{\mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2}{\|\mathbf{U}d\mathbf{X}_1\|\|\mathbf{U}d\mathbf{X}_2\|}\right)$$

To find shear strain, we look at two elements in the referential configuration that are at right angles. Shear strain is DEFINED as the change in the right angle between these two elements: We subtract the new angle θ in radians from $\frac{\pi}{2}$. As it is with elongations or contractions of length, the changes in angles are controlled, not by the deformation gradient or the rotation, but by the right and left stretch tensors. The insight leading to the Seth-Hill generalized strain functions become clearer as they correctly recognized the particular tensor responsible for the shape changes linearly as well, as in relative angular displacements.

Displacement Function

Consider a material that has been subjected to a deformation as shown below. Here, for simplicity, we refer both configurations to the same Cartesian origin and let the two coordinate systems coincide.



Let a point $\mathbf{P}$ be located at the point $\mathbf{X}$ in the material configuration be such that it transforms to the point $\mathbf{p}$ located at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the spatial.

Consider the vector $\mathbf{u} = \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X}$ Let us take the material gradient of this equation and write,

$$\mathbf{H} \equiv \mathbf{Grad}\,\mathbf{u} = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}) - \mathbf{I} = \mathbf{F} - \mathbf{I}$$

Small Strains.

The built environment, using linear elasticity, has at its core the fact tha strains are small: a very reasonable assumption in the days where hard metals such as Iron and its ores or aluminum in its harder varieties were the chief materials for the built environment and manufacturing. Things have changed significantly and those assumptions are no longer always valid. In this section, we will make the assumption of "small strains" and observe its implications on the quantities we have been looking at. In component form, we can write,

$$H_{ij} = \frac{\partial u_i}{\partial X_j}$$

Upon noting that the identity tensor, in Cartesian coordinates has the Kronecker delta as its coefficients, we can therefore write,

$$F_{ij} = \delta_{ij} + H_{ij}$$

Again, in Cartesian, the transpose is simply the reversal of the indices. Hence we can write,

$$\begin{aligned}\mathbf{E} &= \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}) = \frac{1}{2}((\mathbf{H} + \mathbf{I})^{\mathrm{T}}(\mathbf{H} + \mathbf{I}) - \mathbf{I}) \\ &= \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H})\end{aligned}$$

In component form as,

$$\begin{aligned}E_{ij} &= \frac{1}{2}\left(H_{ij} + H_{ji} + H_{ki}H_{kj}\right) \\ &= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_i}{\partial X_j}\right)\end{aligned}$$

If we can neglect second-order terms, and realizing that the spatial is indistinguishable from the material, then we obtain the familiar form for strain-displacement relationships:

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

As we can see, this expression is valid only when the strains are sufficiently small that the exclusion of second-order terms does not affect the results significantly.

Derivatives of Motion

In this section, we take a closer look at the derivatives of motion. The matter deserves a closer look because, while there is a single object under consideration, for the purpose of analysis, we have two different views of it: the referential or material view and the spatial or current view. It is important to get clarity on which we are talking about when we do differential and integral calculus on the body. It is also important to note the relationship between the quantities obtained from the referential perspective and the spatial (current) configuration.

Disambiguation Issues

The terminology of kinematics can be confusing. It can be helpful to be aware of at least four issues as we attempt to compute various derivatives. It may pay us handsomely to have made those decisions early.

1. **Disambiguate "Spatial"**. Sometimes, it refers to the current configuration: the spatial configuration that is visible. At other times it refers to the Euclidean space, as distinct from time – as we talk about variations that are temporal and those that are spatial. We shall use the qualifiers "current" for spatial configurations when it becomes necessary to disambiguate. We may also use the word "field" to qualify the derivatives according to position, to distinguish that from the current configuration.
2. **Resident Configuration of Vectors and Tensors.** In applying the tools of calculus to any quantity, it may be helpful to first identify what configuration it lives in: Spatial or Referential. This helps to avoid several issues of ambiguity in computation and analysis
3. **Material and Spatial Operations**. We can differentiate a tensor or vector or scalar, no matter where it lives, by looking at a material point it occupies or by focusing on the current position it occupies. Talking about a material derivative, for example, we are looking at the derivative based on the material point occupied in the referential state. A spatial derivative is with respect to the current location in the visible configuration.
4. **The Function and its Value**. Matter here is rather pedantic in the sense that the mixing of these two will not likely lead to a confusion or error. Perhaps only a Mathematician will quarrel with you. Yet, in the statement, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, the function itself is on the RHS, the value is on the left. When we write, $\mathbf{x}(\mathbf{X}, t)$, we confuse the two.

Material Derivative

The spatial vector, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, in temporal differentiation, produces the velocity of the particle that was located at $\mathbf{X}$ in the referential configuration. Hence,

$$\mathbf{v} \equiv \left.\frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}}$$

Of course, you will see other textbooks present this as $\mathbf{v} = \frac{\partial \mathbf{x}}{\partial t}$ since the motion function $\boldsymbol{\chi}(\mathbf{X}, t)$ evaluates to $\mathbf{x}$. That is point #4 above. Note that the motion vector as well as velocity, its material derivative, are both spatial vectors. By this we are referring to their "place of residence". In full, spatial vector, velocity, is the material time derivative of the spatial vector valued function called motion. Differentiating the function again, we obtain acceleration:

$$\mathbf{a} \equiv \left.\frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}\right|_{\mathbf{X}}$$

which, as before, could be written with no penalty as,

$$\mathbf{a} = \left.\frac{\partial^2 \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t^2}\right|_{\mathbf{X}} = \frac{\partial \mathbf{v}}{\partial t}$$

We are now in a position to generalize this:

Given differentiable scalar $\phi(\mathbf{X}, t)$, vector $\mathbf{f}(\mathbf{X}, t)$ or tensor $\boldsymbol{\Xi}(\mathbf{X}, t)$ valued function, the following partial derivatives,

$$\frac{D\phi(\mathbf{X}, t)}{Dt} \equiv \left.\frac{\partial \phi(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}}$$

$$\frac{D\mathbf{f}(\mathbf{X}, t)}{Dt} \equiv \left.\frac{\partial \mathbf{f}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}}$$

$$\frac{D\boldsymbol{\Xi}(\mathbf{X}, t)}{Dt} \equiv \left.\frac{\partial \boldsymbol{\Xi}(\mathbf{X}, t)}{\partial t}\right|_{\mathbf{X}}$$

are called **Material Time Derivatives** of the respective functions. Notice that we were silent on whether the functions themselves were spatial. Suffice here is to say that are all expressed in terms of the material vector $\mathbf{X}$ and t.

Each of the above functions could have been expressed in terms of spatial vector $\mathbf{x}$ using the inverse of the relationship, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$, and we could have,

$$\frac{D\phi(\mathbf{X},t)}{Dt} \equiv \left.\frac{\partial\phi(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} = \left.\frac{\partial\phi(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)}$$

At other times, we may choose to write,

$$\hat{\phi}(\mathbf{x},t) = \phi(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)$$

emphasizing the fact that there is an alteration of the functional form once the transformation by the reference map, $\boldsymbol{\chi}^{-1}(\mathbf{x},t)$, is made. When this distinction is not made, it can lead to confusion when actual functions are considered especially when the function is expressed directly in terms of spatial arguments. In either case, the Material time derivative of a spatial function, expressed in terms of the spatial vector can be found using the relationship provided in the next section. Notice that to distinguish between the two functions on either side of the above equation is necessary as they **do not** have the same functional form. At the end of substituting the Reference Map, $\boldsymbol{\chi}^{-1}(\mathbf{x},t)$, you obtain, $\hat{\phi}(\mathbf{x},t)$, which **computes the same value** as $\phi(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)$ but is a **different function from** $\phi(\mathbf{x},t)$. Here is an occasion where *distinguishing a function from the value produced* is not a trivial matter. The example provided here highlights this computational issue clearly. It is important to note that,

$$\begin{aligned}\hat{\phi}(\mathbf{x},t) &= \phi(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)\\ &= \phi(\mathbf{X},t)\\ &\neq \phi(\mathbf{x},t)\end{aligned}$$

Hence the hat! The same argument applied to vector $\mathbf{f}(\mathbf{X},t)$ and tensor $\boldsymbol{\Xi}(\mathbf{X},t)$ valued functions.

Material Derivatives of Spatial Fields

Our focus on the use of the word spatial for representing current configuration, robs us of the proper name of this subsection. Observe that the motion, $\boldsymbol{\chi}(\mathbf{X},t)$, as well as its inverse, the Reference Map, $\boldsymbol{\chi}^{-1}(\mathbf{x},t)$ are also fields. A field is a mapping from a Euclidean Point Space, $\mathcal{E}$. Or, equivalently, each point in $\mathcal{E}$ is associated with a specific value of the function. So are functions defined with respect to the same arguments. We can take gradients of these fields and find their relationships.

For any scalar valued field, $\phi(\mathbf{x},t)$, of the spatial vector $\mathbf{x}$, from multi-variable calculus, we find,

$$d\hat{\phi} = \frac{\partial\hat{\phi}}{\partial\mathbf{x}}\cdot d\mathbf{x} + \frac{\partial\hat{\phi}}{\partial t}dt = \operatorname{grad}\hat{\phi}\cdot d\mathbf{x} + \frac{\partial\hat{\phi}}{\partial t}dt$$

grad in this equation and subsequently, it the spatial field gradient when written with small letter "g". We use the symbolism, $\frac{\partial \hat{\phi}}{\partial \mathbf{x}}$ as a notational convenience here and subsequently. You easily check that this is the same (using Cartesian coordinates) as,

$$\begin{aligned}\frac{\partial \hat{\phi}}{\partial \mathbf{x}} \cdot d\mathbf{x} &= \frac{\partial \hat{\phi}}{\partial x_1} dx_1 + \frac{\partial \hat{\phi}}{\partial x_2} dx_2 + \frac{\partial \hat{\phi}}{\partial x_3} dx_3 \\ &= \left(\frac{\partial \hat{\phi}}{\partial x_1} \mathbf{e}_1 + \frac{\partial \hat{\phi}}{\partial x_2} \mathbf{e}_2 + \frac{\partial \hat{\phi}}{\partial x_3} \mathbf{e}_3 \right) \cdot (\mathbf{e}_1 dx_1 + \mathbf{e}_2 dx_2 + \mathbf{e}_3 dx_3) \\ &= \operatorname{grad} \hat{\phi} \cdot d\mathbf{x} = \left(\hat{\phi}_{,i}\, \mathbf{e}_i \right) \cdot d\mathbf{x}\end{aligned}$$

As will be applicable in general coordinates where the $\mathbf{e}_i$ are no longer Cartesian orthonormal vectors and $\hat{\phi}_{,i}$'s are now covariant derivatives. This symbolism produces, as we have seen, correct results. We shall use it subsequently without necessarily demonstrating validity. Meanwhile, Material Time Derivatives of scalars, vectors and tensors can be found, using the above results, even when their arguments are spatial rather than referential vectors;

$$\begin{aligned}\frac{D\phi(\mathbf{X},t)}{Dt} &\equiv \left.\frac{\partial \phi(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} = \left.\frac{\partial \phi(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)} = \left.\frac{\partial \hat{\phi}(\mathbf{x},t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)} \\ &= \operatorname{grad} \hat{\phi} \cdot \frac{\partial \mathbf{x}}{\partial t} + \frac{\partial \hat{\phi}}{\partial t} = \operatorname{grad} \hat{\phi} \cdot \left.\frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} + \frac{\partial \hat{\phi}}{\partial t} \\ &= \operatorname{grad} \hat{\phi} \cdot \mathbf{v} + \frac{\partial \hat{\phi}}{\partial t}\end{aligned}$$

In the same way, a vector valued function, by the same argument,

$$\begin{aligned}\frac{D\mathbf{f}(\mathbf{X},t)}{Dt} &\equiv \left.\frac{\partial \mathbf{f}(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} = \left.\frac{\partial \mathbf{f}(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)} = \left.\frac{\partial \hat{\mathbf{f}}(\mathbf{x},t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)} \\ &= \left(\operatorname{grad} \hat{\mathbf{f}}\right) \frac{\partial \mathbf{x}}{\partial t} + \frac{\partial \hat{\mathbf{f}}}{\partial t} = \left(\operatorname{grad} \hat{\mathbf{f}}\right) \left(\left.\frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} \right) + \frac{\partial \hat{\mathbf{f}}}{\partial t} \\ &= \left(\operatorname{grad} \hat{\mathbf{f}}\right) \mathbf{v} + \frac{\partial \hat{\mathbf{f}}}{\partial t}\end{aligned}$$

And lastly, for a tensor valued function, by the same argument,

$$\frac{D\boldsymbol{\Xi}(\mathbf{X},t)}{Dt} \equiv \left.\frac{\partial \boldsymbol{\Xi}(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}} = \left.\frac{\partial \boldsymbol{\Xi}(\boldsymbol{\chi}^{-1}(\mathbf{x},t),t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)} = \left.\frac{\partial \hat{\boldsymbol{\Xi}}(\mathbf{x},t)}{\partial t}\right|_{\boldsymbol{\chi}^{-1}(\mathbf{x},t)}$$

$$= \left(\text{grad}\,\widehat{\boldsymbol{\Xi}}\right)\frac{\partial \mathbf{x}}{\partial t} + \frac{\partial \widehat{\boldsymbol{\Xi}}}{\partial t} = \left(\text{grad}\,\widehat{\boldsymbol{\Xi}}\right)\left(\left.\frac{\partial \boldsymbol{\chi}(\mathbf{X},t)}{\partial t}\right|_{\mathbf{X}}\right) + \frac{\partial \widehat{\boldsymbol{\Xi}}}{\partial t}$$

$$= \left(\text{grad}\,\widehat{\boldsymbol{\Xi}}\right)\mathbf{v} + \frac{\partial \widehat{\boldsymbol{\Xi}}}{\partial t}$$

Since velocity is a vector-valued function, its material time derivative, also called the substantial acceleration

$$\frac{D\mathbf{v}}{Dt} = (\text{grad}\,\mathbf{v})\mathbf{v} + \frac{\partial \mathbf{v}}{\partial t}$$

The first term is the convective acceleration, due to change in location while the second term is the acceleration recorded at the location, also called local acceleration.

Example.

Consider a motion defined by

$$\mathbf{x} = (1+t)X_1\mathbf{e}_1 + (1+t)^2X_2\mathbf{e}_2 + (1+t^2)X_3\mathbf{e}_3$$

Let us find the velocity and acceleration. Clearly, in Material terms,

$$\mathbf{V}(\mathbf{X},t) = X_1\mathbf{e}_1 + 2(1+t)X_2\mathbf{e}_2 + 2tX_3\mathbf{e}_3$$

and acceleration,

$$\mathbf{A}(\mathbf{X},t) = 2X_2\mathbf{e}_2 + 2X_3\mathbf{e}_3$$

And if we observe that the reference map here is,

$$\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}) = \frac{x_1}{1+t}\mathbf{e}_1 + \frac{x_2}{(1+t)^2}\mathbf{e}_2 + \frac{x_3}{1+t^2}\mathbf{e}_3$$

We can substitute here and obtain the spatial description of the velocity and acceleration:

$$\mathbf{v}(\mathbf{x},t) = \frac{x_1\mathbf{e}_1}{1+t} + \frac{2x_2\mathbf{e}_2}{1+t} + \frac{2tx_3\mathbf{e}_3}{1+t^2}\,\mathbf{a}(\mathbf{x},t) = \frac{2x_2}{(1+t)^2}\mathbf{e}_2 + \frac{2x_3}{1+t^2}\mathbf{e}_3$$

```
v[x1_, x2_, x3_, t_] := {x1 / (1 + t), 2 x2 / (1 + t), 2 t x3 / (1 + t^2)}
vGrad = Grad[v[x1, x2, x3, t], {x1, x2, x3}]
```

$$\left\{\left\{\frac{1}{1+t}, 0, 0\right\}, \left\{0, \frac{2}{1+t}, 0\right\}, \left\{0, 0, \frac{2t}{1+t^2}\right\}\right\}$$

```
aLocal = D[v[x1, x2, x3, t], t]
```

$$\left\{-\frac{x_1}{(1+t)^2}, -\frac{2x_2}{(1+t)^2}, -\frac{4t^2 x_3}{(1+t^2)^2} + \frac{2x_3}{1+t^2}\right\}$$

```
vGrad.v[x1, x2, x3, t]
```

$$\left\{\frac{x_1}{(1+t)^2}, \frac{4x_2}{(1+t)^2}, \frac{4t^2 x_3}{(1+t^2)^2}\right\}$$

```
aTot = aLocal + vGrad.v[x1, x2, x3, t]
```

$$\left\{0, \frac{2x_2}{(1+t)^2}, \frac{2x_3}{1+t^2}\right\}$$

Now try to evaluate this substantial acceleration from the spatial velocity! Be careful that you are not getting just the local acceleration! The following Mathematica code computes this using the convective and local terms:

The Two Gradients.

When the field argument in a gradient is a spatial vector field variable, we have used grad $(\cdot)$ to express the gradient. On the other hand, if the argument is a referential field variable, we shall use the capitalized form, Grad $(\cdot)$. The relationship between this two can be useful. It is easily established with the symbolism of this chapter as we shall do right away: take the partial derivative of any differentiable scalar, vector or tensor-valued field $(\cdot)$, that can be defined in the spatial as well as the material configuration,

$$\frac{\partial(\cdot)}{\partial \mathbf{X}} = \frac{\partial(\cdot)}{\partial \mathbf{x}} \bullet \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$$

These derivatives with respect to the field variables in the spatial and referential configurations are the gradients. Accordingly, recall that the motion, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$

$$\text{Grad}\,(\cdot) = \big(\text{grad}\,(\cdot)\big)\frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} = \big(\text{grad}\,(\cdot)\big)\mathbf{F}(\mathbf{X}, t)$$

Specifically, for scalar, vector and tensor fields, we have,

$$\text{Grad}\,\phi = (\text{grad}\,\phi)\mathbf{F}(\mathbf{X}, t) = \mathbf{F}^{\mathbf{T}}(\mathbf{X}, t)(\text{grad}\,\phi)$$

$$\text{Grad}\,\mathbf{f} = (\text{grad}\,\mathbf{f})\mathbf{F}(\mathbf{X}, t), \text{and}$$

$$\text{Grad}\,\Xi = (\text{grad}\,\Xi)\mathbf{F}(\mathbf{X}, t).$$

Integrals of Motion

A Review of Leibniz Theorem

Integrals of motion involved the differentiation of integrals of motion. As we shall explore under what conditions such differentiations may take place under the integrals, we make use of the results of Differential and Integral calculus that we need to recall. A quick review of Leibniz theorem from elementary calculus does not hurt at this point.

Leibniz Integral Rule, well-illustrated here states that,

$$\frac{d}{dt}\int_{\phi_0(t)}^{\phi_1(t)} f(x,t)dx = \int_{\phi_0(t)}^{\phi_1(t)} \frac{\partial}{\partial t} f(x,t)dx + f(\phi_1(t),t)\frac{d\phi_1(t)}{dt} - f(\phi_0(t),t)\frac{d\phi_0(t)}{dt}$$

Notice that when ϕ_0 and ϕ_1, the limits of the integration are constants, the remainder terms vanish and it is ok to change the order of integration and the original equation is correct. The following Mathematica code implements a concrete example and can be amended to investigate other examples:

$$\int_{\phi_0(t)}^{\phi_1(t)} f(x,t)dx$$

$$\frac{d}{dt}\int_{\phi_0(t)}^{\phi_1(t)} f(x,t)dx$$

$$\frac{\partial}{\partial t} f(x,t)$$

$$\int_{\phi_0}^{\phi_1} \frac{\partial}{\partial t} f(x,t)dx$$

```
F[α_, b_] := 2 α b + b^2 Sin[α];
ϕ1[α_] := 1 + α; ϕ0[α_] := α^2;
J[α_] = Integrate[F[x, α], {x, ϕ0[α], ϕ1[α]}];
Φ1[t] = D[J[t], t]
1 + 2 t + t^2 - t^4 + t Cos[t^2] - t Cos[1 + t] +
 t (2 + 2 t - 4 t^3 + Cos[t^2] - Cos[1 + t] - 2 t^2 Sin[t^2] + t Sin[1 + t])

Simplify[%]
1 + 4 t + 3 t^2 - 5 t^4 + 2 t Cos[t^2] - 2 t Cos[1 + t] - 2 t^3 Sin[t^2] + t^2 Sin[1 + t]

G[x_, β_] := D[F[x, β], β];
Φ2[t] = Integrate[G[x, t], {x, ϕ0[t], ϕ1[t]}]
1 + 2 t + t^2 - t^4 + 2 t Cos[t^2] - 2 t Cos[1 + t]
1 + 2 t + t^2 - t^4 + 2 t Cos[t^2] - 2 t Cos[1 + t]

Rem[t_] := F[ϕ1[t], t] D[ϕ1[t], t] - F[ϕ0[t], t] D[ϕ0[t], t]
Simplify[Φ1[t] - Rem[t]]
1 + 2 t + t^2 - t^4 + 2 t Cos[t^2] - 2 t Cos[1 + t]
```

Leibniz-Reynolds Transport Theorem

A generalization of the above rule is known as the **Reynold's Transport Theorem** as follows:

The rate of change of an extensive property Φ per unit volume, for the system is equal to the time rate of change of Φ within the volume Ω and the net rate of flux of the property Φ through the surface $\partial\Omega$, or

$$\frac{D}{Dt}\int_{\Omega}\Phi(\mathbf{x},t)dv = \int_{\Omega}\frac{\partial\Phi(\mathbf{x},t)}{\partial t}dv + \int_{\partial\Omega}\Phi(\mathbf{x},t)\mathbf{v}\cdot\mathbf{n}\,ds$$

Our first task is to show that this becomes the Leibniz Rule when restricted to one dimension. That the first integral on the RHS comes from a generalization of Leibniz rule is easy to see. The second integral is for the boundary. Here, we have

$$\int_{x_0(t)}^{x_1(t)}\frac{\partial\Phi(x,t)}{\partial t}dx$$

$\mathbf{v} = \frac{dx(t)}{dt}\mathbf{i}$ the boundary Ω becomes the interval $[x_0(t), x_1(t)]$, at the beginning of the interval, $\mathbf{n}$, the outward drawn normal becomes $-\mathbf{i}$, at the end of the interval, it is $\mathbf{i}$. The line integral occurs only at two points which are now just the beginning points and end points, hence that sum of the evaluations at those two points:

$$\Phi(x_0(t),t)\left(\frac{dx_0(t)}{dt}\mathbf{i}\right)\cdot(-\mathbf{i}) + \Phi(x_1(t),t)\left(\frac{dx_1(t)}{dt}\mathbf{i}\right)\cdot(\mathbf{i})$$

and this recovers the original Leibniz rule.

Proof.

$$\begin{aligned}
\frac{d}{dt}\int_{\Omega}\Phi(\mathbf{x},t)dv &= \frac{d}{dt}\int_{\Omega}\Phi(\mathbf{x},t)\frac{dv}{dV}dV = \frac{D}{Dt}\int_{\Omega_0}\Phi(\mathbf{x},t)JdV \\
&= \int_{\Omega_0}\frac{D}{Dt}[\Phi(\mathbf{x},t)J]dV \\
&= \int_{\Omega_0}\left[\dot{\Phi}(\mathbf{x},t)J + \frac{DJ}{Dt}\Phi(\mathbf{x},t)\right]dV = \int_{\Omega_0}\left[\dot{\Phi}(\mathbf{x},t) + \frac{1}{J}\frac{DJ}{Dt}\Phi(\mathbf{x},t)\right]JdV \\
&= \int_{\Omega}\left[\dot{\Phi}(\mathbf{x},t) + (\operatorname{div}\mathbf{v})\Phi(\mathbf{x},t)\right]dv
\end{aligned}$$

Consequently, if $\Phi(\mathbf{x},t)$ is a scalar function,

$$\dot{I}(t) = \int_{\Omega}\left[\frac{\partial\Phi(\mathbf{x},t)}{\partial t} + \mathbf{v}\cdot\operatorname{grad}\Phi(\mathbf{x},t) + (\operatorname{div}\mathbf{v})\Phi(\mathbf{x},t)\right]dv$$

$$= \int_{\Omega} \left[\frac{\partial \Phi(\mathbf{x}, t)}{\partial t} + \operatorname{div}(\mathbf{v}\Phi) \right] dv$$

which after applying the divergence theorem of Gauss, we find to be,

$$\dot{I}(t) \equiv \int_{\Omega} \left[\frac{\partial \Phi}{\partial t} + \operatorname{div}(\mathbf{v}\Phi) \right] dv = \int_{\Omega} \frac{\partial \Phi}{\partial t} dv + \int_{\partial\Omega} \Phi \mathbf{v} \cdot \mathbf{n} \, ds$$

as required.

If the original integrand was a vector-valued differentiable time dependent field, $\mathbf{f}(\mathbf{x}, t)$. Then we have,

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} \mathbf{f}(\mathbf{x}, t) dv &= \frac{d}{dt} \int_{\Omega} \mathbf{f}(\mathbf{x}, t) \frac{dv}{dV} dV = \frac{d}{dt} \int_{\Omega_0} \mathbf{f}(\mathbf{x}, t) J dV \\
&= \int_{\Omega_0} \frac{D}{Dt} [\mathbf{f}(\mathbf{x}, t) J] dV = \int_{\Omega_0} \left[\frac{D\mathbf{f}}{Dt} J + \mathbf{f} \frac{DJ}{Dt} \right] dV \\
&= \int_{\Omega_0} \left[\frac{\partial \mathbf{f}}{\partial t} J + (\operatorname{grad} \mathbf{f}) J \mathbf{v} + \mathbf{f} \frac{DJ}{Dt} \right] dV \\
&= \int_{\Omega_0} \left[\frac{\partial}{\partial t} \mathbf{f}(\mathbf{x}, t) + (\operatorname{grad} \mathbf{f}) \mathbf{v} + \mathbf{f} \operatorname{div} \mathbf{v} + \right] J dV \\
&= \int_{\Omega} \left[\frac{\partial}{\partial t} \mathbf{f}(\mathbf{x}, t) + \operatorname{div} (\mathbf{f} \otimes \mathbf{v}) \right] dv \\
&= \int_{\Omega} \frac{\partial}{\partial t} \mathbf{f}(\mathbf{x}, t) dv + \int_{\partial\Omega} (\mathbf{f} \otimes \mathbf{v}) \mathbf{n} \, ds.
\end{aligned}$$

We end this section with the material time derivative with a tensor integrand*:* Let $\Xi(\mathbf{x}, t)$ be a tensor valued temporal spatial field. Then, the spatial field material time derivative,

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} \Xi(\mathbf{x}, t) dv &= \frac{d}{dt} \int_{\Omega} \Xi(\mathbf{x}, t) \frac{dv}{dV} dV = \frac{D}{Dt} \int_{\Omega_0} \Xi(\mathbf{x}, t) J dV \\
&= \int_{\Omega_0} \frac{D}{Dt} [\Xi(\mathbf{x}, t) J] dV = \int_{\Omega_0} \left[\frac{D\Xi}{Dt} J + \Xi \frac{DJ}{Dt} \right] dV \\
&= \int_{\Omega_0} \left[\frac{\partial \Xi}{\partial t} J + (\operatorname{grad} \Xi) J \mathbf{v} + \Xi \frac{DJ}{Dt} \right] dV \\
&= \int_{\Omega_0} \left[\frac{\partial}{\partial t} \Xi(\mathbf{x}, t) + (\operatorname{grad} \Xi) \mathbf{v} + \Xi \operatorname{div} \mathbf{v} \right] J dV \\
&= \int_{\Omega} \left[\frac{\partial}{\partial t} \Xi(\mathbf{x}, t) + \operatorname{div} (\Xi \otimes \mathbf{v}) \right] dv
\end{aligned}$$

$$= \int_{\Omega} \frac{\partial}{\partial t} \Xi(\mathbf{x}, t) dv + \int_{\partial\Omega} (\Xi \otimes \mathbf{v}) \mathbf{n} ds$$

After applying Gauss theorem and observing that $\operatorname{div}(\Xi \otimes \mathbf{v}) = \Xi \operatorname{div} \mathbf{v} + (\operatorname{grad} \Xi)\mathbf{v}$. We can now state Reynolds Transport Theorem for scalar, vector or tensor-valued field $\Xi(\mathbf{x}, t)$:

The net rate of change of $\Xi(\mathbf{x}, t)$ in Ω is equal to the rate of change occurring within the boundary Ω and the rate of influx across the boundary $\partial\Omega$:

$$\underbrace{\frac{d}{dt} \int_{\Omega} \Xi(\mathbf{x}, t) dv}_{\text{Net Rate of Change}} = \underbrace{\int_{\Omega} \frac{\partial}{\partial t} \Xi(\mathbf{x}, t) dv}_{\text{Rate of Change Within}} + \underbrace{\int_{\partial\Omega} \Xi(\mathbf{v} \cdot \mathbf{n}) ds}_{\text{influx Across Boundary}}$$

Rates of Shape Changes

Stretching and Strain Rates

Strains, stretches and other tensors associated with deformation and motion may suffice in the characterization of solids in deformation. However, for flowing bodies such as fluids or plastic processes, the rates of shape changes become more important measures. In this section, we delve into some of these aspects.

Based on the earlier pages, we may view the deformation gradient as a Material Gradient of the deformation $\mathbf{x} = \chi(\mathbf{X}, t)$,

$$\mathbf{F} = \operatorname{Grad} \chi(\mathbf{X}, t)$$

The material time derivative (that is, keeping $\mathbf{X}$ fixed) of this is,

$$\begin{aligned} \frac{D\mathbf{F}}{Dt} &= \frac{D}{Dt} \operatorname{Grad} \chi(\mathbf{X}, t) = \operatorname{Grad} \dot{\chi}(\mathbf{X}, t) \\ &= \operatorname{Grad} \mathbf{v} = (\operatorname{grad} \mathbf{v})\mathbf{F} \\ &= \mathbf{L}\mathbf{F} \end{aligned}$$

where $\mathbf{L} \equiv \operatorname{grad} \mathbf{v}$ is called the *velocity gradient*.

From Equation 7b, breaking the deformation gradient using the right stretch tensor, we can write that the velocity gradient,

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \left(\mathbf{R}\dot{\mathbf{U}} + \dot{\mathbf{R}}\mathbf{U}\right)\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}} = \mathbf{R}\dot{\mathbf{U}}\mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}} + \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}$$

Unless otherwise stated, the dot, as in $\dot{\mathbf{F}}$, represents a material time derivative. Before expressing this in terms of skew and symmetric parts, observe first that $\mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$, which, differentiating gives,

$$\begin{aligned}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} + \mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}} &= \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} + \left(\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}} \\ &= 0\end{aligned}$$

showing that $\mathbf{Q}(t) \equiv \dot{\mathbf{R}}(t)\mathbf{R}^{\mathrm{T}}(t)$ is a skew tensor. This is true whenever an orthogonal tensor is a differentiable function of time. And since rotation alters neither symmetry nor skewness, we can write that,

$$\begin{aligned}\mathbf{L} &= \mathbf{R}\,\mathrm{sym}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}\right)\mathbf{R}^{\mathrm{T}} + \mathbf{R}\,\mathrm{skw}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}\right)\mathbf{R}^{\mathrm{T}} + \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \\ &= \mathbf{D} + \mathbf{W}_{\mathrm{str}} + \mathbf{W}_{\mathrm{rot}} \\ &= \mathbf{D} + \mathbf{W}\end{aligned}$$

The symmetric part, $\mathrm{sym}\,\mathbf{L} = \mathbf{D}$, is called the **stretching** or the **rate of strain tensor**, the skew part, $\mathrm{skw}\,\mathbf{L} = \mathbf{W}$ is the **spin tensor**. The axial vector of the skew spin tensor is called the vorticity $\boldsymbol{\omega}$. Furthermore, Again, using the left stretch tensor $\mathbf{V}$, we can write that the velocity gradient,

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1} = \left(\dot{\mathbf{V}}\mathbf{R} + \mathbf{V}\dot{\mathbf{R}}\right)\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} = \dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} = \mathbf{D} + \mathbf{W}$$

the symmetric and skew parts of the velocity gradient. Hence,

$$\begin{aligned}\mathbf{D} &= \mathrm{sym}\left(\dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1}\right) \\ &= \frac{1}{2}\left(\dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}^{-\mathrm{T}}\dot{\mathbf{V}}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} + \mathbf{V}^{-\mathrm{T}}\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\right) \\ &= \frac{1}{2}\left(\dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}^{-1}\dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} - \mathbf{V}^{-1}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}\right)\end{aligned}$$

on account of the symmetry of the left stretch tensor and skewness of $\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$

The skew part of the velocity gradient,

$$\begin{aligned}\mathbf{W} &= \mathrm{skw}\left(\dot{\mathbf{V}}\mathbf{V}^{-1} + \mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1}\right) \\ &= \frac{1}{2}\left(\dot{\mathbf{V}}\mathbf{V}^{-1} - \mathbf{V}^{-\mathrm{T}}\dot{\mathbf{V}}^{\mathrm{T}}\right) + \frac{1}{2}\left(\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} - \mathbf{V}^{-\mathrm{T}}\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}\mathbf{V}^{\mathrm{T}}\right) \\ &= \frac{1}{2}\left(\dot{\mathbf{V}}\mathbf{V}^{-1} - \mathbf{V}^{-1}\dot{\mathbf{V}}\right) + \frac{1}{2}\left(\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} + \mathbf{V}^{-1}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}\right)\end{aligned}$$

again on account of the symmetry of the left stretch tensor and skewness of $\mathbf{R}\dot{\mathbf{R}}^{\mathrm{T}}$

$$\mathbf{D} = \mathbf{R}\,\mathrm{sym}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}\right)\mathbf{R}^{\mathrm{T}} \text{ is the stretching tensor}$$

$$\mathbf{W}_{\mathrm{str}} \equiv \mathbf{R}\,\mathrm{skw}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}\right)\mathbf{R}^{\mathrm{T}} \text{ is stretch spin}$$

$$\mathbf{W}_{\mathrm{rot}} \equiv \dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}} \text{ is rotation spin}$$

Relationship between the stretching tensor and the material derivative of the Lagrange Follows from:

$$\begin{aligned} 2\mathbf{F}^{\mathrm{T}}\mathbf{D}\mathbf{F} &= \mathbf{F}^{\mathrm{T}}(\mathbf{L}+\mathbf{L}^{\mathrm{T}})\mathbf{F} \\ &= \mathbf{F}^{\mathrm{T}}\left(\dot{\mathbf{F}}\mathbf{F}^{-1}+\mathbf{F}^{-\mathrm{T}}\dot{\mathbf{F}}^{\mathrm{T}}\right)\mathbf{F} \\ &= \mathbf{F}^{\mathrm{T}}\dot{\mathbf{F}}+\dot{\mathbf{F}}^{\mathrm{T}}\mathbf{F} = 2\dot{\mathbf{E}} \end{aligned}$$

Instantaneous Rates of Stretch and Spin

Suppose we fix an instant in time and consider the instantaneous values of the stretching and rotation rates. Making that instant our reference configuration, we can write,

$$\mathbf{F} = \mathbf{U} = \mathbf{V} = \mathbf{R} = \mathbf{I}$$

as all these tensors are identity at the instant when displacement coincides with the reference. At this instant, substituting the identity tensor as above, we see that,

$$\begin{aligned} \mathbf{D}|_{inst} &= \mathbf{R}\,\mathrm{sym}\left(\dot{\mathbf{U}}\mathbf{U}^{-1}\right)\mathbf{R}^{\mathrm{T}} \\ &= \mathrm{sym}\left(\dot{\mathbf{U}}\right) \\ &= \dot{\mathbf{U}} = \mathrm{sym}\left(\dot{\mathbf{V}}\mathbf{V}^{-1}+\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1}\right) \\ &= \dot{\mathbf{V}} \\ \mathbf{W}|_{inst} &= \mathrm{skw}\left(\dot{\mathbf{V}}\mathbf{V}^{-1}+\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1}\right) \\ &= \frac{1}{2}\left(\dot{\mathbf{V}}\mathbf{V}^{-1}-\mathbf{V}^{-1}\dot{\mathbf{V}}+\mathbf{V}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1}+\mathbf{V}^{-1}\dot{\mathbf{R}}\mathbf{R}^{\mathrm{T}}\mathbf{V}\right) \\ &= \dot{\mathbf{R}} \end{aligned}$$

Skew Tensors, Vorticity & Circulation

Vorticity is the axial vector of a tensor of motion. We will recall that only skew tensors possess axial vectors. Furthermore, our introduction to Stokes Theorem in the previous chapter shows the relationship between a line integral along a closed path and the curl of a vector. When this integral is the circulation caused by the motion of a fluid body, the vector in question is the axial vector of the skew part of the velocity gradient. These facts make it necessary to take a closer look at some properties of skew tensors.

Given a skew tensor $\mathbf{W}$, its cubic characteristic equation, $\det(\mathbf{W} - \alpha\mathbf{I}) = 0$ necessarily has a solution, $\alpha = 0$. In order to see this, notice that, every skew tensor has a zero eigenvalue. The transpose, $\mathbf{W}^{\mathrm{T}} = -\mathbf{W}$, its negative. Consequently, for any $\mathbf{a}, \mathbf{b} \in \mathbb{E}$,

$$\mathbf{a} \cdot \mathbf{W}\mathbf{b} = -\mathbf{b} \cdot \mathbf{W}\mathbf{a}$$

Now the characteristic equation, $\det(\mathbf{W} - \alpha\mathbf{I}) = 0$, of $\mathbf{W}$, is a cubic polynomial in λ. There is, therefore, at least on real root. Let this be λ and let $\boldsymbol{\xi}_1$ be the associated eigenvector. Clearly,

$$\begin{aligned}\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_1 &= \boldsymbol{\xi}_1 \cdot \mathbf{W}^{\mathrm{T}}\boldsymbol{\xi}_1 = -\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_1 \\ &= \boldsymbol{\xi}_1 \cdot \lambda\boldsymbol{\xi}_1 \\ &= \lambda|\boldsymbol{\xi}_1|^2 = 0\end{aligned}$$

Which can only be true if $\lambda = 0$. Hence one root is zero as the magnitude of the eigenvector of a real eigenvalue can never vanish.

One consequence of this result is that the component representation of any skew tensor becomes especially simple. If we form a positively-oriented orthonormal system (such that, $\boldsymbol{\xi}_i = e_{ijk}\boldsymbol{\xi}_j \times \boldsymbol{\xi}_k$) with the three eigenvectors, $\boldsymbol{\xi}_i, i = 1,2,3,$ (No assumptions that $\boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ are eigenvectors) we can expand $\mathbf{W}$ and write,

$$\begin{aligned}\mathbf{W} &= \left(\boldsymbol{\xi}_i \cdot \mathbf{W}\boldsymbol{\xi}_j\right)\boldsymbol{\xi}_i \otimes \boldsymbol{\xi}_j \\ &= (\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_1 + (\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_2 + \cdots + (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_3)\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_3 \\ &= (\boldsymbol{\xi}_2 \cdot \mathbf{W}\boldsymbol{\xi}_3 - \boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3 \\ &= w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)\end{aligned}$$

where $w = \boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2$. All diagonal elements vanish by the skewness of $\mathbf{W}$ as well as all components containing the first eigenvector since its eigenvalue is zero:

$$\boldsymbol{\xi}_i \cdot \mathbf{W}\boldsymbol{\xi}_1 = -\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_i = 0 \; \forall i = 2, 3.$$

The axial vector of $\mathbf{W}$ is a scaled version of $\boldsymbol{\xi}_1$**.** The scaling factor is tensor coefficient, $\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2$ for is we write, $\mathbf{p} \equiv (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_1$ For any $\mathbf{a} \in \mathbb{E}$**,** we can compute,

$$\begin{aligned}\mathbf{p} \times \mathbf{a} &= (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_1 \times \mathbf{a} \\ &= (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)(\boldsymbol{\xi}_2 \times \boldsymbol{\xi}_3) \times \mathbf{a} \\ &= (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)\big((\boldsymbol{\xi}_2 \cdot \mathbf{a})\boldsymbol{\xi}_3 - (\boldsymbol{\xi}_3 \cdot \mathbf{a})\boldsymbol{\xi}_2\big) \\ &= (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)[\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3]\mathbf{a} \\ &= \mathbf{W}\mathbf{a}\end{aligned}$$

Showing that $\mathbf{p} = (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_1$ is the axial vector of $\mathbf{W}$ as required. This gives an easy method to compute the first eigenvector of a skew tensor: Normalize the axial vector; and obtain the magnitude thereof.

Finally, the remaining two eigenvalues of $\mathbf{W}$ are purely imaginary. We can see that this is so in the following:

First eigenvalue is zero; the first invariant of a skew tensor is zero. This is obvious from the fact that the diagonal components all vanish giving a trace of zero. Let $w = \boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2$, we can also see this without any reference to components remembering that $w\boldsymbol{\xi}_1$ is the axial vector of $\mathbf{W}$:

$$\begin{aligned}
I_1 &= [\mathbf{W}\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] + [\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \boldsymbol{\xi}_3] + [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] \\
&= [\lambda\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] + [\boldsymbol{\xi}_1, w\boldsymbol{\xi}_1, \boldsymbol{\xi}_3] + [\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, 0] \\
&= 0 \\
I_2 &= [\mathbf{W}\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \boldsymbol{\xi}_3] + [\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] + [\mathbf{W}\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] \\
&= [\boldsymbol{\xi}_1, w\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_2, w\boldsymbol{\xi}_1 \times \boldsymbol{\xi}_3] = [\boldsymbol{\xi}_1, w\boldsymbol{\xi}_3, -w\boldsymbol{\xi}_2] \\
&= w^2[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = w^2 \\
I_3 &= [\mathbf{W}\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] = [0, \mathbf{W}\boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] = 0
\end{aligned}$$

So that the characteristic equation is,

$$\lambda^3 + I_2\lambda = \lambda^3 + w^2\lambda = 0$$

yielding roots $\lambda = 0, \pm i|w|$.

Definition. Vorticity. The skew part of the velocity gradient, $\mathbf{L} = \operatorname{grad} \mathbf{v}$,

$$\mathbf{W} \equiv \operatorname{skw} \mathbf{L} = \frac{1}{2}(\operatorname{grad} \mathbf{v} - \operatorname{grad}^{\mathrm{T}} \mathbf{v})$$

As any other skew tensor, has an axial vector. This axial vector, $\boldsymbol{\omega}$ is defined as the vorticity of the motion.

End of Definition

We observe that the curl of velocity, (defined earlier in 3.___). Given the third order alternating tensor, $\mathbf{E} = e_{ijk}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$

$$\begin{aligned}
\operatorname{curl} \mathbf{v} &= \operatorname{div} \mathbf{E}\mathbf{v} \\
&= e_{ijk} v_{k,j}\, \mathbf{e}_i
\end{aligned}$$

in component form. The vector cross of this vector is equal to $\operatorname{grad} \mathbf{v} - \operatorname{grad}^{\mathrm{T}} \mathbf{v}$, for,

$$\begin{aligned}
\big((\operatorname{curl} \mathbf{v}) \times\big) &= e_{ijk} v_{k,j}\, \mathbf{e}_i \times \\
&= e_{\alpha i\beta} e_{ijk} v_{k,j}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta
\end{aligned}$$

$$= \left(\delta_{\beta j}\delta_{\alpha k} - \delta_{\beta k}\delta_{\alpha j}\right) v_{k,j}\, \mathbf{e}_\alpha \otimes \mathbf{e}_\beta$$
$$= v_{k,j}\, \mathbf{e}_k \otimes \mathbf{e}_j - v_{k,j}\, \mathbf{e}_j \otimes \mathbf{e}_k$$
$$= \operatorname{grad} \mathbf{v} - \operatorname{grad}^{\mathrm{T}} \mathbf{v}$$

Showing that

$$\mathbf{W} = \operatorname{skw} \mathbf{L} = \frac{1}{2}\left(\operatorname{grad} \mathbf{v} - \operatorname{grad}^{\mathrm{T}} \mathbf{v}\right)$$
$$= (\boldsymbol{\omega} \times) = \frac{1}{2}\left((\operatorname{curl} \mathbf{v}) \times\right)$$

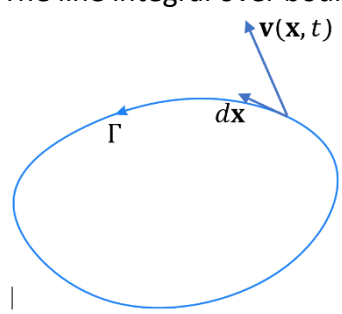
from which, twice the vorticity,

$$2\boldsymbol{\omega} = \operatorname{curl} \mathbf{v}$$

equals the curl of the flow velocity.

Circulation

The line integral over boundary curve Γ of the velocity along the closed path shown is defined as circulation. Given a positively oriented surface $\mathcal{S}$, and bounded as shown by a path Γ, the line integral,



$$C_\Gamma = \int_\Gamma \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x}$$

Clearly, only the resolved component of the velocity along the tangent contributes to this integral. Stokes theorem states that, given a positively oriented surface $\mathcal{S}$, and bounded as shown by a path Γ, the line integral,

$$\int_\Gamma \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = \iint_{\mathcal{S}} (\operatorname{curl} \mathbf{v}) \cdot d\mathbf{s}.$$

That is, the line integral taken over the path shown is equal to the surface integral over the entire surface. The vorticity of the flow, $\boldsymbol{\omega} = \frac{1}{2} \operatorname{curl} \mathbf{v}$, so that the circulation is,

$$\int_{\Gamma} \mathbf{v}(\mathbf{x}, \boldsymbol{t}) \cdot \boldsymbol{d}\mathbf{x} = \mathbf{2} \iint_{\mathcal{S}} \boldsymbol{\omega} \cdot \boldsymbol{d}\mathbf{s}.$$

Constraints on Flow

Irrotational & Circulation-Preserving Flow Fields

A flow field is defined as Irrotational when the vorticity vanishes. In an irrotational field, by Equation___, the circulation also vanishes:

$$C_{\Gamma} = \int_{\Gamma} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = 0$$

Kelvin's Circulation Theorem.

If the material acceleration is the gradient of a potential, then the motion is circulation preserving; that is, its circulation does not change.

To establish this theorem, we need to first show that the rate of change of circulation is the derivative of the integral

$$C_{\Gamma} = \int_{\Gamma} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x}$$

of the velocity taken over the closed path:

Let Γ_0 be the closed path transformed to Γ from the reference configuration by the deformation Gradient field, $\mathbf{F}(\mathbf{X}, t)$. $\mathbf{V}(\mathbf{X}, t)$ is the velocity in referential terms (always evaluates to the same values as $\mathbf{v}(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t)$). Taking a material derivative of both sides,

$$\begin{aligned}
\frac{DC_{\Gamma}}{Dt} &= \frac{D}{Dt} \int_{\Gamma} \mathbf{v}(\mathbf{x}, t) \cdot d\mathbf{x} = \frac{D}{Dt} \int_{\Gamma_0} \mathbf{V}(\mathbf{X}, t) \cdot \mathbf{F} d\mathbf{X} \\
&= \int_{\Gamma_0} \frac{D}{Dt} (\mathbf{V}(\mathbf{X}, t) \cdot \mathbf{F}) d\mathbf{X} \\
&= \int_{\Gamma_0} \left[\frac{D\mathbf{V}}{Dt} \cdot \mathbf{F} d\mathbf{X} + \mathbf{V} \cdot \frac{D\mathbf{F}}{Dt} d\mathbf{X} \right] = \int_{\Gamma_0} \left[\frac{D\mathbf{V}}{Dt} \cdot \mathbf{F} d\mathbf{X} + \mathbf{V} \cdot \mathbf{L}\mathbf{F} d\mathbf{X} \right] \\
&= \int_{\Gamma} \left[\frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \mathbf{v} \cdot \mathbf{L} d\mathbf{x} \right]
\end{aligned}$$

Note that $\mathbf{L}d\mathbf{x} = (\text{grad } \mathbf{v})d\mathbf{x} = d\mathbf{v}$**.** Hence,

$$\frac{DC_{\Gamma}}{Dt} = \int_{\Gamma} \left[\frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \mathbf{v} \cdot d\mathbf{v} \right] = \int_{\Gamma} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} + \frac{1}{2} \int_{\Gamma} d(\mathbf{v} \cdot \mathbf{v})$$

$$= \int_{\Gamma} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x}$$

As the integral of a full differential velocity taken over a closed path vanishes if the function is single valued. If $\exists \phi(\mathbf{X}, t)$ such that acceleration,

$$\frac{D\mathbf{v}}{Dt} = \text{ grad } \phi(\mathbf{X}, t),$$

Then,

$$\begin{aligned} \frac{DC_{\Gamma}}{Dt} &= \int_{\Gamma} \frac{D\mathbf{v}}{Dt} \cdot d\mathbf{x} \\ &= \int_{\Gamma} \big(\text{grad } \phi(\mathbf{X}, t)\big) \cdot d\mathbf{x} \\ &= \int_{\Gamma} d\phi(\mathbf{X}, t) = 0 \end{aligned}$$

Which, again, takes a full differential through a closed path, and hence vanishes.

It follows therefore, that an acceleration, derivable from the gradient of a potential is circulation preserving.

End of Theorem

Rigid Field

Consider two particles $\mathbf{X}$ and $\mathbf{Y}$ in the referential configuration. The distance between these in the spatial configuration, $\phi(t)$, is a function of time:

$$\phi(t) = \|\boldsymbol{\chi}(\mathbf{X}, t) - \boldsymbol{\chi}(\mathbf{Y}, t)\|$$

Differentiating $\frac{1}{2}\big(\phi(t)\big)^2$, we can write,

$$\begin{aligned} \phi(t)\dot{\phi}(t) &= (\mathbf{x} - \mathbf{y}) \cdot \big(\dot{\boldsymbol{\chi}}\,(\mathbf{X}, t) - \dot{\boldsymbol{\chi}}(\mathbf{Y}, t)\big) \\ &= (\mathbf{x} - \mathbf{y}) \cdot \big(\mathbf{v}(\mathbf{X}, t) - \mathbf{v}(\mathbf{Y}, t)\big) \end{aligned}$$

Which must vanish if the distance between the two points never changes. Taking a spatial gradient with respect to $\mathbf{x}$, observing that $\text{grad } (\mathbf{u} \cdot \mathbf{v}) = (\text{grad}^{\mathrm{T}}\mathbf{u})\mathbf{v} + (\text{grad}^{\mathrm{T}}\mathbf{v})\mathbf{u}$

$$\text{grad}^{\mathrm{T}}\, \mathbf{v}(\mathbf{x}, t)(\mathbf{x} - \mathbf{y}) + \mathbf{I}\big(\mathbf{v}(\mathbf{x}, t) - \mathbf{v}(\mathbf{y}, t)\big) = 0$$

or,

$$\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{y}, t) - \text{grad}^{\mathrm{T}}\, \mathbf{v}(\mathbf{x}, t)(\mathbf{x} - \mathbf{y})$$

Taking spatial gradient again with respect to $\mathbf{y}$, we have that,

$$\text{grad}\,\mathbf{v}(\mathbf{y},t) = -\text{grad}^{\mathrm{T}}\,\mathbf{v}(\mathbf{x},t)$$

But $\mathbf{x}$ and $\mathbf{y}$ are two arbitrary points. Setting $\mathbf{x} = \mathbf{y}$, we have that, $\text{grad}\,\mathbf{v}(\mathbf{x},t) = -\text{grad}^{\mathrm{T}}\,\mathbf{v}(\mathbf{x},t)$ showing that the velocity gradient in a rigid velocity field is skew. It follows that $\text{grad}\,\mathbf{v} = \mathbf{W}(t)$, a skew tensor, and $\mathbf{w}(t)$, its axial vector, such that,

$$\begin{aligned}\mathbf{v}(\mathbf{x},t) &= \mathbf{v}(\mathbf{y},t) + \text{grad}\,\mathbf{v}(\mathbf{x},t)(\mathbf{x}-\mathbf{y})\\ &= \mathbf{v}(\mathbf{y},t) + \mathbf{W}(t)(\mathbf{x}-\mathbf{y})\\ &= \mathbf{v}(\mathbf{y},t) + \mathbf{w}(t)\times(\mathbf{x}-\mathbf{y})\end{aligned}$$

Steady Field

The material acceleration,

$$\underbrace{\frac{D\mathbf{v}}{Dt}}_{\substack{\text{Substantial}\\ \text{Acceleration}}} = \underbrace{\frac{\partial\mathbf{v}}{\partial t}}_{\substack{\text{Local}\\ \text{Acceleration}}} + \underbrace{(\text{grad}\,\mathbf{v})\mathbf{v}}_{\substack{\text{Convective}\\ \text{Acceleration}}}$$

The motion in a subset of the flow field is said to be steady if the local derivatives (if we use velocity, it then means local acceleration $\frac{\partial\mathbf{v}}{\partial t} = 0$). In steady motion acceleration only occurs from change of location:

$$\underbrace{\frac{D\mathbf{v}}{Dt}}_{\substack{\text{Substantial}\\ \text{Acceleration}}} = \underbrace{(\text{grad}\,\mathbf{v})\mathbf{v}}_{\substack{\text{Convective}\\ \text{Acceleration}}}$$

In steady motion, the flow field remains unchanged from point to point. Velocities (or other flow properties) may vary from point to point, nevertheless as a result of convective acceleration.

Time Derivatives: Material and Spatial

We end this chapter with a note on the notation for two important partial derivatives. In equation___ we noted a relationship between the time derivatives. It is worth noting that both are partial derivatives with respect to time. The difference in each case is what is kept fixed. It is this difference that necessitates a difference in notations for representing them which causes a significant amount of confusion in the Literature as books and articles are sometimes difficult to follow.

For a given scalar, vector or tensor field $\Xi(\mathbf{X})$,let us write, $\widehat{\Xi}(\mathbf{x},t) \equiv \Xi(\boldsymbol{\chi}^{-1}(\mathbf{X},t))$, its spatial form, the derivative,

$$\frac{\partial \widehat{\Xi}}{\partial t} = \frac{\partial}{\partial t}\widehat{\Xi}(\mathbf{x},t)\Big|_{\mathbf{x}=\boldsymbol{\chi}(\mathbf{X},t)}$$

keeping the spatial variable (current position) constant, is the spatial time derivative. The regular partial differentiation notation is reserved for this description.

Another partial derivative arises from the fact that the motion confers a time value on functions that are defined with respect to the material configuration via the reference map, $\boldsymbol{\chi}^{-1}(\mathbf{X},t)$ which is the inverse of the deformation or motion. If the function $\Xi(\mathbf{X})$ is differentiated with respect to time, keeping the particular particle in reference fixed (following the particle) then, the derivative,

$$\frac{D\Xi}{Dt} = \frac{\partial}{\partial t}\Xi(\mathbf{x},t)\Big|_{\mathbf{X}}$$

We have what is called the substantial derivative. It is also by several other names including advective, convective, hydrodynamic, Lagrangian, particle, substantive, Stokes, total derivative or simply as the derivative following the motion to emphasize the fact that the particle here is kept fixed rather than the location in the current view.

And from equation____ we have this general relationship,

$$\frac{D\Xi}{Dt} = \frac{\partial \widehat{\Xi}}{\partial t} + \left(\operatorname{grad}\widehat{\Xi}\right)\mathbf{v}$$

Between the material and spatial partial derivatives with respect to time which we call material time and spatial time derivatives respectively. As we saw from the definition of steady motion, the fact that a field value is not changing at a point only means that the spatial time derivative vanishes. The material time derivative can be non-zero because of the term including the velocity gradient. In the special case of steady flow, we saw that the convective acceleration remains and is the reason why, despite the fact that the acceleration at a particular point is zero, as seen from its spatial velocity not changing at that point, the particle may still be accelerated to another position through the convective term, resulting in a non-zero value for the substantial acceleration even in steady flow.

Conversely, in isochoric flow, the density as we follow the particle remains unchanged. Here, we have,

$$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + (\operatorname{grad} \rho) \cdot \mathbf{v} = 0$$
$$= \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) - \rho \operatorname{div} \mathbf{v}$$

Examples

1	For the uniform biaxial deformation, given that $x_1 = \lambda_1 X_1, x_1 = \lambda_2 X_2$ and $y^3 = x^3$. Compute the Deformation Gradient tensor, the Lagrangian Strain Tensor as well as the Eulerian Strain Tensor components.
	$$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$ $$= (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$ $$= \alpha_1 \mathbf{e}_1 \otimes \mathbf{E}_1 + \alpha_2 \mathbf{e}_2 \otimes \mathbf{E}_2 + \alpha_3 \mathbf{e}_3 \otimes \mathbf{E}_3$$ The Green Lagrange strain tensor is, $$\mathbf{E} = -\frac{1}{2}(1 - \alpha_1^2)\mathbf{E}_1 \otimes \mathbf{E}_1 - \frac{1}{2}(1 - \alpha_2^2)\mathbf{E}_2 \otimes \mathbf{E}_2$$ Clearly a biaxial state of strain. The rest of the results can be seen from the attached code:

Q4.1 KInematics.nb - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[ ]:= F := {{λ1, 0, 0}, {0, λ2, 0}, {0, 0, 1}}

In[ ]:= CC = Transpose[F].F
        EE = 1 / 2 (CC - IdentityMatrix[3])
        BB = F.Transpose[F]
        ee = 1 / 2 (IdentityMatrix[3] - Inverse[BB])
```

Out[ ]= $\{\{\lambda_1^2, 0, 0\}, \{0, \lambda_2^2, 0\}, \{0, 0, 1\}\}$

Out[ ]= $\left\{\left\{\frac{1}{2}\left(-1+\lambda_1^2\right), 0, 0\right\}, \left\{0, \frac{1}{2}\left(-1+\lambda_2^2\right), 0\right\}, \{0, 0, 0\}\right\}$

Out[ ]= $\{\{\lambda_1^2, 0, 0\}, \{0, \lambda_2^2, 0\}, \{0, 0, 1\}\}$

Out[ ]= $\left\{\left\{\frac{1}{2}\left(1-\frac{1}{\lambda_1^2}\right), 0, 0\right\}, \left\{0, \frac{1}{2}\left(1-\frac{1}{\lambda_2^2}\right), 0\right\}, \{0, 0, 0\}\right\}$

2 If the tensor $\boldsymbol{S}$ is positive definite, Show that $\det\left(\mathbf{S}^{\frac{1}{2}}\right) = [\det(\mathbf{S})]^{\frac{1}{2}}$.

Let the eigenvalues of $\mathbf{S}$ be λ_1, λ_2 and λ_3. Then the determinant of $\mathbf{S}$ is $\lambda_1\lambda_2\lambda_3$ the square root of this is $\sqrt{\lambda_1\lambda_2\lambda_3}$. But since $\mathbf{S}$ is positive definite, The eigenvalues of $\boldsymbol{S}^{\frac{1}{2}}$ are $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$ so that the determinant of $\mathbf{S}^{\frac{1}{2}}$, ie $[\det(\mathbf{S})]^{\frac{1}{2}} = \sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3} = \sqrt{\lambda_1\lambda_2\lambda_3}$

3 Show that rotation alters neither symmetry nor skewness in a tensor.

Consider a symmetric tensor $\mathbf{S}$, and a rotation tensor $\mathbf{R}$. We take a transpose of the rotated tensor $\mathbf{T} = \mathbf{RSR}^{\mathrm{T}}$

$$\begin{aligned}\mathbf{T}^{\mathrm{T}} &= (\mathbf{RSR}^{\mathrm{T}})^{\mathrm{T}} \\ &= (\mathbf{R}^{\mathrm{T}})^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = \mathbf{RSR}^{\mathrm{T}} \\ &= \mathbf{T}\end{aligned}$$

On account of the symmetry of tensor $\mathbf{S}$ and the fact that the transpose of a transpose is the original tensor.

Consider a skew tensor $\mathbf{W}$, and a rotation tensor $\mathbf{R}$. We take a transpose of the rotated tensor $\mathbf{\Omega} = \mathbf{RWR}^{\mathrm{T}}$

$$\begin{aligned}\mathbf{\Omega}^{\mathrm{T}} &= (\mathbf{RWR}^{\mathrm{T}})^{\mathrm{T}} \\ &= (\mathbf{R}^{\mathrm{T}})^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = -\,\mathbf{RWR}^{\mathrm{T}} \\ &= -\mathbf{\Omega}\end{aligned}$$

On account of the skewness of tensor $\mathbf{W}$ and the fact that the transpose of a transpose is the original tensor.

4

Show that the tensor $\mathbf{C} = (\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3)\begin{pmatrix} 163.24 & 34.6 & 4.2 \\ 34.6 & 19. & -30. \\ 4.2 & -30. & 178. \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$ is positive definite. (a) Find the square root of the $\mathbf{C}$ by finding its spectral decomposition from its eigenvalues and eigenvectors. (b) Use the Mathematica function MatrixPower[C, ½] to compare your result.

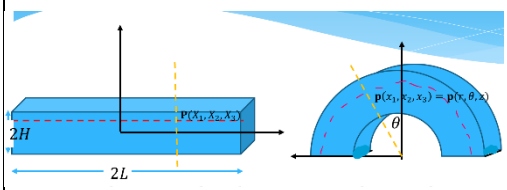
From the code below, it is clear that its eigenvalues are all positive, hence it is positive definite. The tensor $\mathbf{C}$ as well as its square root have the same eigenvectors. It is clear that the eigenvectors of the square root are the square roots of the eigenvalues of $\mathbf{C}$ as the figures show.

Q4.4 KInematics.nb - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[ ]:= CC := {{163.24, 34.6, 4.2}, {34.6, 19, -30},
          {4.2, -30, 178}}

In[ ]:= CEige = Eigenvalues[CC]
        CVec = Eigenvectors[CC]
        BB = MatrixPower[CC, 1 / 2]
        Eigenvalues[BB]
        Eigenvectors[BB]

Out[ ]= {183.793, 170.611, 5.83642}

Out[ ]= {{-0.152601, -0.207928, 0.966167},
         {0.964426, 0.182197, 0.191537},
         {0.215859, -0.961026, -0.172727}}

Out[ ]= {{12.5773, 2.22417, 0.323916},
         {2.22417, 3.25095, -1.86666},
         {0.323916, -1.86666, 13.2065}}

Out[ ]= {13.557, 13.0618, 2.41587}

Out[ ]= {{-0.152601, -0.207928, 0.966167},
         {0.964426, 0.182197, 0.191537},
         {0.215859, -0.961026, -0.172727}}
```

	For a proper orthogonal tensor Q, show that the eigenvalue equation always yields an eigenvalue of +1. This means that $\lambda = 1$ is always a solution for the equation, $\det(\mathbf{Q} - \lambda\mathbf{I})$
	For a proper orthogonal tensor, the cofactor, $$\mathbf{Q}^{\mathrm{c}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$ Showing that it is self-cofactor. The characteristic equation is, $$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0.$$ $I_3 = 1$ for every proper orthogonal tensor; $I_2 = I_1$since it is self cofactor. The second invariant is the trace of the cofactor equaling the first which is the trace of the tensor. Consequently, the characteristic equation becomes, $$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$ Substitute $\lambda = 1$, the equation becomes, $1 - I_1 + I_1 - 1 = 0$, identically. Hence this is an eigenvalue of the tensor.
5	For a vector-valued spatial field, we are given that $\mathrm{Grad}\,\mathbf{f} = (\mathrm{grad}\,\mathbf{f})\mathbf{F}(\mathbf{X}, t)$. Show that, $\mathrm{Div}\,\mathbf{f} = (\mathrm{grad}\,\mathbf{f}) : \mathbf{F}^{\mathrm{T}}$
	We are given, $$\mathrm{Grad}\,\mathbf{f} = (\mathrm{grad}\,\mathbf{f})\mathbf{F}$$ Take the trace of both sides: $$\begin{aligned} \mathrm{tr}\,\mathrm{Grad}\,\mathbf{f} &= \mathrm{tr}\big((\mathrm{grad}\,\mathbf{f})\mathbf{F}\big) \\ &= (\mathrm{grad}^{\mathrm{T}}\,\mathbf{f}) : \mathbf{F} = (\mathrm{grad}\,\mathbf{f}) : \mathbf{F}^{\mathrm{T}} \\ &= \mathrm{Div}\,\mathbf{f} \end{aligned}$$
6	For a vector-valued spatial field, we are given that $\mathrm{Grad}\,\mathbf{f} = (\mathrm{grad}\,\mathbf{f})\mathbf{F}(\mathbf{X}, t)$. Show that, $\mathrm{div}\,\mathbf{f} = (\mathrm{Grad}\,\mathbf{f}) : \mathbf{F}^{-\mathrm{T}}$
	We are given, $$\mathrm{Grad}\,\mathbf{f} = (\mathrm{grad}\,\mathbf{f})\mathbf{F}$$ Post product with $\mathbf{F}^{-1} \Rightarrow \mathrm{grad}\,\mathbf{f} = (\mathrm{Grad}\,\mathbf{f})\mathbf{F}^{-1}$ Take the trace of both sides:

	$= \operatorname{div} \mathbf{f}$
7	Given that $$r = r(X_2),\ \theta = \theta(X_1) \text{ and } z = z(X_3)$$ X_1, X_2, X_3 to r, θ, z are the transformation equations of a straight bar into a semi-circular arc. Find the deformation Gradient and the Strain tensors associated with the deformation.
	 1. If we deform a straight bar into a circular bar as shown below, the transformation function can be found by the following consideration: Note that each horizontal filament in the original bar becomes a circular filament in the spatial configuration. The vertical undeformed sections become radial sections in the spatial state. For the moment, we assume nothing happens in the axial or z direction in each caseLet the centerline be a semicircle at a distance R and let the thickness contract uniformly with a factor α $$\Rightarrow r = R + \alpha X_2, \text{ and}$$ $$\theta = \frac{\pi X_1}{2L}$$ If the bar contracts uniformly in X_3direction, $z = \beta X_3$ $$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$ $$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

	$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$ $$= \left(\frac{\pi r}{2L}\mathbf{e}_\theta \quad \alpha\mathbf{e}_r \quad \beta\mathbf{e}_z\right) \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$ $$= \frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1 + \alpha\mathbf{e}_r \otimes \mathbf{E}_2 + \beta\mathbf{e}_z \otimes \mathbf{E}_3$$ Clearly, $$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \left(\frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{e}_\theta + \alpha\mathbf{E}_2 \otimes \mathbf{e}_r + \beta\mathbf{E}_3 \otimes \mathbf{e}_z\right)\left(\frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1 + \alpha\mathbf{e}_r \otimes \mathbf{E}_2 + \beta\mathbf{e}_z \otimes \mathbf{E}_3\right)$$ $$= \left(\frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{e}_\theta\right)\left(\frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1\right) + \cdots + (\beta\mathbf{E}_3 \otimes \mathbf{e}_z)(\beta\mathbf{e}_z \otimes \mathbf{E}_3)$$ $$= \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2\mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2\mathbf{E}_3 \otimes \mathbf{E}_3$$ since each set of basis vectors is orthonormal, and the Right Stretch Tensor, $$\mathbf{U} = \frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha\mathbf{E}_2 \otimes \mathbf{E}_2 + \beta\mathbf{E}_3 \otimes \mathbf{E}_3$$ Is the square root of the Right Cauchy Green tensor. The positive square roots are taken since both $\mathbf{C}$ as well as $\mathbf{U}$ are necessarily positive definite and can only have positive eigenvalues.
8	In the torsion of the circular bar shown, given that the transformation equations are, $r = R,\ \theta = \Theta + f(Z),\ Z = Z$, find the deformation gradient and the strain function.
	It is convenient to refer the torsion problem to cylindrical coordinates. In consistency with our practice so far, we select R, Θ and Z for the undeformed body and r, θ and z for the typical point in the spatial configuration. For a cylindrical bar, it is reasonable to assume that each there are no changes to the radial and axial components in any element;

Only the angular coordinates are altered by an amount depending on the undeformed value and the axial component Z. Hence,

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta/R \\ \mathbf{E}_Z \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{R} & r\frac{\partial f}{\partial Z} \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

The following Mathematica code computes the appropriate tensors of the deformation:

* From the above computations, we find that the Green Lagrange strains are:

$$\mathbf{E} = \frac{1}{2}\left[\left(\frac{r}{R}\right)^2 - 1\right]\mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + \frac{1}{2}r^2 f^2(Z)\mathbf{E}_Z \otimes \mathbf{E}_Z + \frac{1}{2R}(r^2 f(Z)) \quad (\mathbf{E}_\Theta \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_\Theta)$$

```
F := {{1, 0, 0}, {0, r / R, r f[Z]}, {0, 0, 1}}
CC = Transpose[F].F
```

$$\left\{\{1, 0, 0\}, \left\{0, \frac{r^2}{R^2}, \frac{r^2 f[Z]}{R}\right\}, \left\{0, \frac{r^2 f[Z]}{R}, 1 + r^2 f[Z]^2\right\}\right\}$$

```
EE = (1 / 2) (CC - IdentityMatrix[3]) // MatrixForm
```

trixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(-1 + \frac{r^2}{R^2}\right) & \frac{r^2 f[Z]}{2R} \\ 0 & \frac{r^2 f[Z]}{2R} & \frac{1}{2} r^2 f[Z]^2 \end{pmatrix}$$

And the right Cauchy-Green Tensor for the deformation is:

$$\mathbf{C} = \mathbf{E}_R \otimes \mathbf{E}_R + \left(\frac{r}{R}\right)^2 \mathbf{E}_\Theta \otimes \mathbf{E}_\Theta + [1 + r^2 f^2(Z)]\mathbf{E}_Z \otimes \mathbf{E}_Z + \frac{r^2 f(Z)}{R}(\mathbf{E}_\Theta \otimes \mathbf{E}_Z + \mathbf{E}_Z \otimes \mathbf{E}_\Theta)$$

Explain the meaning of the components

9 In Cartesian Coordinates, the deformation of a rectangular sheet is given by: $\boldsymbol{r} = (\boldsymbol{\lambda_1 x^1} + \boldsymbol{k_1 x^2})\mathbf{g_1} + (\boldsymbol{k_2 x^1} + \boldsymbol{\lambda_2 x^2})\mathbf{g_2} + \boldsymbol{\lambda_3 x_3}\mathbf{g_3}$ Compute the tensors $\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{E}, \boldsymbol{U}$ and $\boldsymbol{R}$. Show that $\boldsymbol{R^T} \cdot \boldsymbol{R} = \boldsymbol{I}$. For $\lambda_1 = 1.1,\ \lambda_2 = 1.25, k_1 = 0.15, k_2 = -0.2$, determine the principal values and directions of $\boldsymbol{E}$. Verify that the principal directions are mutually orthogonal. Compute the strain invariants and show that they are consistent with the characteristic equation.

10	A body undergoes a deformation defined by, $y_1 = \alpha x_1, y_2 = -(\beta x_2 + \gamma x_3), and\ y_3 = \gamma x_2 - \beta x_3$ where $\alpha, \beta\ and\ \gamma$ are constants. Determine $\boldsymbol{F}, \boldsymbol{C}, \boldsymbol{E}, \boldsymbol{U}$ and $\boldsymbol{R}$.

```
xx = {x1, x2, x3};
yy = {α x1, -(β x2 + γ x3), γ x2 - β x3};

F := Table[D[yy[[i]], Evaluate[xx[[j]]]], {i, 1, 3}, {j, 1, 3}]


Evaluate[F] /. {α → 1, β → -Cos[θ], γ → -Sin[θ]}
{{1, 0, 0}, {0, Cos[θ], Sin[θ]}, {0, -Sin[θ], Cos[θ]}}

C1 = Transpose[F] . F
```

$$\{\{\alpha^2, 0, 0\}, \{0, \beta^2 + \gamma^2, 0\}, \{0, 0, \beta^2 + \gamma^2\}\}$$

```
E1 = (1 / 2) (C1 - IdentityMatrix[3])
```

$$\left\{\left\{\frac{1}{2}\left(-1 + \alpha^2\right), 0, 0\right\}, \left\{0, \frac{1}{2}\left(-1 + \beta^2 + \gamma^2\right), 0\right\}, \left\{0, 0, \frac{1}{2}\left(-1 + \beta^2 + \gamma^2\right)\right\}\right\}$$

```
Finger = F . Transpose[F]
```

$$\{\{\alpha^2, 0, 0\}, \{0, \beta^2 + \gamma^2, 0\}, \{0, 0, \beta^2 + \gamma^2\}\}$$

```
Piola = Inverse[Finger]
```

$$\left\{\left\{\frac{\left(\beta^2 + \gamma^2\right)^2}{\alpha^2 \beta^4 + 2 \alpha^2 \beta^2 \gamma^2 + \alpha^2 \gamma^4}, 0, 0\right\},\right.$$
$$\left.\left\{0, \frac{\alpha^2 \beta^2 + \alpha^2 \gamma^2}{\alpha^2 \beta^4 + 2 \alpha^2 \beta^2 \gamma^2 + \alpha^2 \gamma^4}, 0\right\}, \left\{0, 0, \frac{\alpha^2 \beta^2 + \alpha^2 \gamma^2}{\alpha^2 \beta^4 + 2 \alpha^2 \beta^2 \gamma^2 + \alpha^2 \gamma^4}\right\}\right\}$$

11	In the motion, $\mathbf{x} = \left((1+t)X_2 - tX_1\right)\mathbf{e}_1 + ((1+t)^2 X_1 + tX_2)\mathbf{e}_2 + \ (1+t^2)X_3\mathbf{e}_3$, Find the *Reference Map*, *Spatial Velocity* and *Substantial Acceleration*. Show that the latter can be found either by directly differentiating the material velocity or adding the local acceleration to the velocity gradient tensor operation on the spatial velocity.

Here is the Mathematica code:

Local Acceleration Number2.nb - Wolfram Mathematica 11.3

e Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[•]:= (*At this point, we find the reference map for the motion by solving the set of
         motion scalar equations. They are not linear but Mathematica will attempt a
         solution nevertheless:*)
        Solve[{-x1 + (1 + t) X2 - t X1 == 0, -x2 + (1 + t)^2 X1 + t X2 == 0, -x3 + (1 + t^2) X3 == 0},
         {X1, X2, X3}]
```

Out[•]= $\left\{\left\{X1 \to -\frac{t\,x1 - x2 - t\,x2}{1 + 3\,t + 4\,t^2 + t^3},\ X2 \to -\frac{-x1 - 2\,t\,x1 - t^2\,x1 - t\,x2}{1 + 3\,t + 4\,t^2 + t^3},\ X3 \to \frac{x3}{1 + t^2}\right\}\right\}$

```
In[•]:= (* We are now able to compute velocity. It comes directly in Material Configuration. To
         obtain it in Spatial Configuration use the Reference Map. It is a quiet computation
         as the result is NOT shown. You can obtain the result if you want by explicitly
         requesting it: *)
        v[x1_, x2_, x3_, t_] := D[{(1 + t) X2 - t X1, (1 + t)^2 X1 + t X2, (1 + t^2) X3}, t] /.
```

$$\left\{X1 \to -\frac{t\,x1 - x2 - t\,x2}{1 + 3\,t + 4\,t^2 + t^3},\ X2 \to -\frac{-x1 - 2\,t\,x1 - t^2\,x1 - t\,x2}{1 + 3\,t + 4\,t^2 + t^3},\ X3 \to \frac{x3}{1 + t^2}\right\}$$

```
        (* In what follows, aLocal is the local acceleration,
        vGrad function is the velocity gradient,
        matAcce is the Material or Substantial Acceleration which can be obtained by
         either differentiating the material velocity directly or differentiating the
         spatial velocity, obtaining the local acceleration and adding the Velocity
         Gradient operation on the spatial velocity (resulting in aTotal) .*)

In[•]:= aLocal = D[v[x1, x2, x3, t], t];
        vGrad[X1_, X2_, X3_] := Grad[v[X1, X2, X3, t], {X1, X2, X3}];
        matAcce = D[D[{(1 + t) X2 - t X1, (1 + t)^2 X1 + t X2, (1 + t^2) X3}, t], t] /.
```

$$\left\{X1 \to -\frac{t\,x1 - x2 - t\,x2}{1 + 3\,t + 4\,t^2 + t^3},\ X2 \to -\frac{-x1 - 2\,t\,x1 - t^2\,x1 - t\,x2}{1 + 3\,t + 4\,t^2 + t^3},\ X3 \to \frac{x3}{1 + t^2}\right\};$$

```
In[•]:= aTotal = aLocal + vGrad[x1, x2, x3].v[x1, x2, x3, t];

In[•]:= Simplify[%]
```

Out[•]= $\left\{0,\ \frac{-2\,t\,x_1 + 2\,(1 + t)\,x_2}{1 + 3\,t + 4\,t^2 + t^3},\ \frac{2\,x_3}{1 + t^2}\right\}$

```
In[•]:= Simplify[matAcce]
```

Out[•]= $\left\{0,\ \frac{2\,(x2 + t\,(-x1 + x2))}{1 + 3\,t + 4\,t^2 + t^3},\ \frac{2\,x3}{1 + t^2}\right\}$

12	The spatial tensor field, $\mathbf{L} = \text{grad}[\mathbf{v}(\mathbf{x}, t)]$is defined as the velocity gradient. Show that for a motion governed by the deformation gradient, $\mathbf{F}(\mathbf{X}, t) = \text{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t)$the corresponding velocity gradient is $\left(\frac{D\mathbf{F}}{Dt}\right)\mathbf{F}^{-1}$

The material derivative of the deformation gradient is,

$$\frac{D\mathbf{F}(\mathbf{X}, t)}{Dt} = \frac{\partial}{\partial t}\text{Grad}\,\boldsymbol{\chi}(\mathbf{X}, t)\Big|_{X}$$

$$= \text{Grad}\frac{D\boldsymbol{\chi}(\mathbf{X}, t)}{Dt} = \text{grad}\,\mathbf{v}(\mathbf{x}, t)\mathbf{F}$$

So that $\dot{\mathbf{F}} = \mathbf{L}\mathbf{F}$. Therefore,

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$$

13	If the tensor $\mathbf{T}(\alpha)$is a differentiable function of $\alpha \in \mathbb{R}$, and that, $\frac{d}{d\alpha}\det(\mathbf{T}) = \det(\mathbf{T})\,\text{tr}(\dot{\mathbf{T}}\mathbf{T}^{-1})$, if we write, $J \equiv \det\mathbf{F}$, Prove Liouville's theorem that

	$$\frac{DJ}{Dt} = J \operatorname{div} \mathbf{v}$$
	Applying the given equation to the deformation gradient, we have, $$\frac{d}{dt}\det(\mathbf{F}) = \det(\mathbf{F})\operatorname{tr}(\dot{\mathbf{F}}\mathbf{F}^{-1})$$ If we replace the scalar parameter α by t. But from example 26, we find that the velocity gradient, $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. Substituting, we have, $$\frac{d}{dt}\det(\mathbf{F}) = \frac{DJ}{Dt} = \det(\mathbf{F})\operatorname{tr}(\operatorname{grad}\mathbf{v}) = J \operatorname{div}\mathbf{v}.$$
14	Given the Deformation Gradient Tensor $(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 1 & \frac{3}{2} & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$ Find the rotation tensor, the right stretch tensor and the left stretch tensor. Demonstrate that the Rotation tensor is proper orthogonal.
	$F = \{\{1.0, 3/2, 4/3\}, \{0, 1, 0\}, \{0, 0, 1\}\}$ $C_1 = \text{Transpose}[F].F$ $\{\{1., \frac{3}{2}, \frac{4}{3}\}, \{0, 1, 0\}, \{0, 0, 1\}\}$ $\{\{1., 1.5, 1.33333\}, \{1.5, 3.25, 2.\}, \{1.33333, 2., 2.77778\}\}$ $\{\{1.0, \frac{3}{2}, \frac{4}{3}\}, \{0, 1, 0\}, \{0, 0, 1\}\}$ $\{\{1, \frac{3}{2}, \frac{4}{3}\}, \{\frac{3}{2}, \frac{13}{4}, 2\}, \{\frac{4}{3}, 2, \frac{25}{9}\}\}$ $U_1 = \text{MatrixPower}[C_1, 1/2]$ $\{\{0.705882, 0.529412, 0.470588\},$ $\{0.529412, 1.62982, 0.559838\}, \{0.470588, 0.559838, 1.49763\}\}$ $R_1 = F.\text{Inverse}[U_1]$ $\{\{0.705882, 0.529412, 0.470588\},$ $\{-0.529412, 0.8357, -0.146045\}, \{-0.470588, -0.146045, 0.870183\}\}$ $V_1 = F.\text{Inverse}[R_1]$ $\{\{2.12745, 0.529412, 0.470588\},$ $\{0.529412, 0.8357, -0.146045\}, \{0.470588, -0.146045, 0.870183\}\}$
15	In the two dimensional deformation defined by the deformation $$y_1 = 0.1\, x_1(1 + 2\, x_1 + x_2)$$ $$y_2 = 0.2x_2(1 + x_2)$$ where x_i and y_i are Cartesian coordinates of a particle in the reference and deformed configurations respectively. Determine the deformation on a line element $\boldsymbol{a}_0 = \boldsymbol{e}_1 + 2\boldsymbol{e}_2$ passing through the point (2,-2) in the reference configuration.

16	If the element of area $d\boldsymbol{a}$ is the image of the undeformed area element $d\boldsymbol{A}$ that has undergone a deformation given by the deformation gradient $\boldsymbol{F}$, prove Nanson's formula that $d\boldsymbol{a} = Jd\mathbf{A} \cdot \mathbf{F}^{-1}$ where J is the volume ratio for the transformation and is the determinant of the $\boldsymbol{F}$.
17	Given that one eigenvalue of $\mathbf{W}$ is zero. Find $w \in \mathbb{R}$, show that we can express the tensor as, $\mathbf{W} = w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)$ where $\boldsymbol{\xi}_1$ is the eigenvector associated with zero eigenvalue, and $\boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ forms an orthonormal basis with it.
	If we form a positively-oriented orthonormal system (such that, $\boldsymbol{\xi}_i = e_{ijk}\boldsymbol{\xi}_j \times \boldsymbol{\xi}_k$) with the three eigenvectors, $\boldsymbol{\xi}_i, i = 1,2,3$, (No assumptions that $\boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ are eigenvectors) we can expand $\mathbf{W}$ and write, $$\begin{aligned}\mathbf{W} &= (\boldsymbol{\xi}_i \cdot \mathbf{W}\boldsymbol{\xi}_j)\boldsymbol{\xi}_i \otimes \boldsymbol{\xi}_j \\ &= (\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_1)\boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_1 + (\boldsymbol{\xi}_1 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_2 + \cdots + (\boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_3)\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_3 \\ &= (\boldsymbol{\xi}_2 \cdot \mathbf{W}\boldsymbol{\xi}_3 - \boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2)\boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3 \\ &= w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)\end{aligned}$$ where $w = \boldsymbol{\xi}_3 \cdot \mathbf{W}\boldsymbol{\xi}_2$.
18	Given that we can express the skew tensor, $\mathbf{W} = w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)$. Show that the second Invariant is w^2. Note that is has a zero eigenvalue. Show that the other two eigenvalues are purely imaginary with the values $\pm w$.
	The tensor Let the three normalized, positively oriented unit vectors of $\mathbf{W}$ be $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3$ in such a way that ξ_1 corresponds to the zero eigenvalue, recall that $\mathbf{W} = w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)$: $$\begin{aligned} I_2(\mathbf{W}) &= [\mathbf{W}\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \boldsymbol{\xi}_3] + [\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] + [\mathbf{W}\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] \\ &= [\boldsymbol{\xi}_1, \mathbf{W}\boldsymbol{\xi}_2, \mathbf{W}\boldsymbol{\xi}_3] = [\boldsymbol{\xi}_1, w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)\boldsymbol{\xi}_2, w(\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)\boldsymbol{\xi}_3] \\ &= w^2[\boldsymbol{\xi}_1, (\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)\boldsymbol{\xi}_2, (\boldsymbol{\xi}_3 \otimes \boldsymbol{\xi}_2 - \boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_3)\boldsymbol{\xi}_3] \\ &= w^2[\boldsymbol{\xi}_1, \boldsymbol{\xi}_3, -\boldsymbol{\xi}_2] = w^2[\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3] = w^2 \end{aligned}$$ $I_1(\mathbf{W}) = I_3(\mathbf{W}) = 0 \Rightarrow$ Characteristic equation is, $$\lambda^3 - \lambda^2 I_1 + \lambda I_2 - I_3 = \lambda^3 + \lambda I_2 = \lambda(\lambda^2 + w^2) = 0 \Rightarrow$$ $\lambda = 0, \pm w$ are the roots of the tensor's characteristic equation.

19	Define the Lagrangian Stretch Ratio $\Lambda_{\mathbf{N}} = \frac{\lvert d\boldsymbol{r}\rvert}{\lvert d\boldsymbol{R}\rvert}$ where $ds = \lvert d\boldsymbol{r}\rvert$ *and* $dS = \lvert d\boldsymbol{R}\rvert$ the length of elements in the deformed and undeformed configurations respectively, show that $\Lambda_{\mathbf{N}} = \frac{\lvert d\boldsymbol{r}\rvert}{\lvert d\boldsymbol{R}\rvert} = \sqrt{\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}}$ for a material element along the unit vector $\mathbf{N}$ in the reference configuration and that Eulerian stretch Ratio $\lambda_{\mathbf{n}} = \frac{\lvert d\boldsymbol{r}\rvert}{\lvert d\boldsymbol{R}\rvert} = \sqrt{\frac{\mathbf{1}}{\mathbf{N}\cdot\mathbf{B}\cdot\mathbf{N}}}$
	$$dS = \lvert d\boldsymbol{R}\rvert\,\mathbf{N}$$ Hence $$\Lambda_{\mathbf{N}}^2 = \frac{\lvert d\boldsymbol{r}\rvert^2}{\lvert d\boldsymbol{R}\rvert^2} = \frac{d\boldsymbol{r}\cdot d\boldsymbol{r}}{\lvert d\boldsymbol{R}\rvert^2} = \frac{\mathrm{d}\mathbf{R}\cdot\mathbf{F}^{\mathrm{T}}\mathbf{F}\mathrm{d}\mathbf{R}}{\lvert\mathrm{d}\mathbf{R}\rvert^2} = \frac{\lvert d\boldsymbol{R}\rvert^2\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}}{\lvert d\boldsymbol{R}\rvert^2} = \mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}$$ so that $$\Lambda_{\mathbf{N}} = \frac{\lvert d\boldsymbol{r}\rvert}{\lvert d\boldsymbol{R}\rvert} = \sqrt{\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}} = \lvert\mathbf{N}\cdot\mathbf{U}\rvert$$ Similarly, for the direction $\mathbf{n}$, in the spatial configuration, $ds = \lvert d\boldsymbol{r}\rvert\,\mathbf{n}$ $$\left(\frac{1}{\lambda_{\mathbf{n}}}\right)^2 = \left(\frac{\lvert d\boldsymbol{R}\rvert}{\lvert d\boldsymbol{r}\rvert}\right)^2 = \frac{d\boldsymbol{R}\cdot d\boldsymbol{R}}{\lvert d\boldsymbol{r}\rvert^2} = \frac{d\boldsymbol{R}\cdot\boldsymbol{F}^{-T}\cdot\boldsymbol{F}^{-1}\cdot d\boldsymbol{R}}{\lvert d\boldsymbol{r}\rvert^2} = \frac{\lvert d\boldsymbol{r}\rvert^2\mathbf{n}\cdot\mathbf{B}\cdot\mathbf{n}}{\lvert d\boldsymbol{r}\rvert^2} = \mathbf{n}\cdot\mathbf{B}\cdot\mathbf{n} = \mathbf{n}\cdot\mathbf{V}\cdot\mathbf{V}\cdot\mathbf{n}$$ so that $\lambda_{\mathbf{n}} = \sqrt{\frac{\mathbf{1}}{\mathbf{N}\cdot\mathbf{B}\cdot\mathbf{N}}} = \frac{\mathbf{1}}{\lvert\mathbf{V}\cdot\mathbf{n}\rvert}$
20	1. When normal strains are small compared to unity, show that the shears $\Gamma_{\mathbf{e}_1\mathbf{e}_2} = \gamma_{\mathbf{e}_1\mathbf{e}_2}$ approach the lagrangian and Eulerian shear strains respectively.
	Consider first the case $\Gamma_{\mathbf{e}_1\mathbf{e}_2}$ when
21	Find, by direct computation, the physical components of the Deformation gradient if the material and spatial frames are referred to spherical polar coordinates

	$\mathbf{F} = \frac{\partial x^i}{\partial X^j}(\mathbf{g}_i \otimes \mathbf{G}^j)$ $= [\mathbf{e}_\varrho \quad \varrho\mathbf{e}_\vartheta \quad \varrho\sin\vartheta\,\mathbf{e}_\varphi]\begin{bmatrix} \frac{\partial\varrho}{\partial\rho} & \frac{\partial\varrho}{\partial\theta} & \frac{\partial\varrho}{\partial\phi} \\ \frac{\partial\vartheta}{\partial\rho} & \frac{\partial\vartheta}{\partial\theta} & \frac{\partial\vartheta}{\partial\phi} \\ \frac{\partial\varphi}{\partial\rho} & \frac{\partial\varphi}{\partial\theta} & \frac{\partial\varphi}{\partial\phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \frac{\mathbf{e}_\theta}{\rho} \\ \frac{\mathbf{e}_\phi}{\rho\sin\theta} \end{bmatrix}$ $= \frac{\partial\varrho}{\partial\rho}\mathbf{e}_\varrho \otimes \mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\varrho}{\partial\theta}\mathbf{e}_\varrho \otimes \mathbf{e}_\theta + \frac{1}{\rho\sin\theta}\frac{\partial\varrho}{\partial\phi}\mathbf{e}_\varrho \otimes \mathbf{e}_\phi + \varrho\frac{\partial\vartheta}{\partial\rho}\mathbf{e}_\vartheta \otimes \mathbf{e}_\rho + \frac{\varrho}{\rho}\frac{\partial\vartheta}{\partial\theta}\mathbf{e}_\vartheta \otimes \mathbf{e}_\theta$ $+ \frac{\varrho}{\rho\sin\theta}\frac{\partial\vartheta}{\partial\phi}\mathbf{e}_\vartheta \otimes \mathbf{e}_\phi + \varrho\sin\vartheta\frac{\partial\varphi}{\partial\rho}\mathbf{e}_\varphi \otimes \mathbf{e}_\rho + \frac{\varrho\sin\vartheta}{\rho}\frac{\partial\varphi}{\partial\theta}\mathbf{e}_\varphi \otimes \mathbf{e}_\theta$ $+ \frac{\varrho\sin\vartheta}{\rho\sin\theta}\frac{\partial\varphi}{\partial\phi}\mathbf{e}_\varphi \otimes \mathbf{e}_\phi$ $\begin{bmatrix} \frac{\partial\varrho}{\partial\rho} & \frac{1}{\rho}\frac{\partial\varrho}{\partial\theta} & \frac{1}{\rho\sin\theta}\frac{\partial\varrho}{\partial\phi} \\ \varrho\frac{\partial\vartheta}{\partial\rho} & \frac{\varrho}{\rho}\frac{\partial\vartheta}{\partial\theta} & \frac{\varrho}{\rho\sin\theta}\frac{\partial\vartheta}{\partial\phi} \\ \varrho\sin\vartheta\frac{\partial\varphi}{\partial\rho} & \frac{\varrho\sin\vartheta}{\rho}\frac{\partial\varphi}{\partial\theta} & \frac{\varrho\sin\vartheta}{\rho\sin\theta}\frac{\partial\varphi}{\partial\phi} \end{bmatrix}$
22	For vector and tensor fields, $\mathbf{f}(\mathbf{x},t)$ and $\boldsymbol{\Xi}(\mathbf{x},t)$, Given that $\mathrm{Grad}\,\mathbf{f} = (\,\mathrm{grad}\,\mathbf{f})\mathbf{F}(\mathbf{X},t)$, and that $\mathrm{Grad}\,\boldsymbol{\Xi} = (\,\mathrm{grad}\,\boldsymbol{\Xi})\mathbf{F}(\mathbf{X},t)$,show that $\mathrm{div}\,\mathbf{f} = (\mathrm{Grad}\,\mathbf{f}):\mathbf{F}^{-\mathrm{T}}$ and $\mathrm{div}\,\boldsymbol{\Xi} = (\mathrm{Grad}\,\boldsymbol{\Xi}):\mathbf{F}^{-\mathrm{T}}$
	$\mathrm{Grad}\,\mathbf{f} = (\,\mathrm{grad}\,\mathbf{f})\mathbf{F}(\mathbf{X},t) \Rightarrow$ $\mathrm{grad}\,\mathbf{f} = (\mathrm{Grad}\,\mathbf{f})\mathbf{F}^{-1}$ Take the trace of both sides, $\mathrm{div}\,\mathbf{f} = (\mathrm{Grad}\,\mathbf{f}):\mathbf{F}^{-\mathrm{T}}$ The same procedure establishes the second result: $\mathrm{div}\,\boldsymbol{\Xi} = (\mathrm{Grad}\,\boldsymbol{\Xi}):\mathbf{F}^{-\mathrm{T}}$
23	
24	
25	
26	
27	
28	

29	Consider a deformation of the form $\mathbf{x} = \boldsymbol{\omega} \times \mathbf{X}$ Here $\boldsymbol{\omega}$ is a vector with magnitude $\ll 1$, which represents an infinitesimal rotation about an axis parallel to $\boldsymbol{\omega}$ Show that $\mathbf{C} = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{I} - \boldsymbol{\omega} \otimes \boldsymbol{\omega}$.
	Deformation Gradient, $\mathbf{F} = \boldsymbol{\omega} \times$. This is a skew tensor. The transpose, $\mathbf{F}^{\mathrm{T}} = -\boldsymbol{\omega} \times$, its negative. The Right Cauchy-Green Tensor, $$\begin{aligned}\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} &= -(\boldsymbol{\omega} \times)(\boldsymbol{\omega} \times) = -\left(e_{ijk}\omega_j \mathbf{e}_i \otimes \mathbf{e}_k\right)\left(e_{\alpha\beta\gamma}\omega_\beta \mathbf{e}_\alpha \otimes \mathbf{e}_\gamma\right) \\ &= -e_{ijk}\omega_j e_{\alpha\beta\gamma}\omega_\beta \mathbf{e}_i \otimes \mathbf{e}_\gamma \delta_{k\alpha} = -e_{ijk}\omega_j e_{\beta\gamma k}\omega_\beta \mathbf{e}_i \otimes \mathbf{e}_\gamma \\ &= \left(\delta_{i\gamma}\delta_{j\beta} - \delta_{i\beta}\delta_{j\gamma}\right)\omega_\beta \mathbf{e}_i \otimes \mathbf{e}_\gamma \\ &= \omega_j\omega_j \mathbf{e}_i \otimes \mathbf{e}_i - \omega_i\omega_j \mathbf{e}_i \otimes \mathbf{e}_j \\ &= (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{I} - \boldsymbol{\omega} \otimes \boldsymbol{\omega}\end{aligned}$$
30	A body undergoes a deformation defined by, $x_1 = X_1 \cos\alpha - X_2 \sin\alpha$, $X_2 = X_1 \sin\alpha + X_2 \cos\alpha$, and $x_3 = X_3$ where α is a constant. Show that $\mathbf{C} = \mathbf{I}$ and $\mathbf{E} = \mathbf{O}$. Explain the reason for the values of $\mathbf{E}$ components.
	The deformation gradient here is the rotation tensor through angle α around the $\mathbf{e}_3$ axis. Consequently, $$\mathbf{F} = \mathbf{R} = \mathbf{RI} = \mathbf{RU}$$ So that $\mathbf{U} = \mathbf{I}$. $$\mathbf{C} = \mathbf{U}^2 = \mathbf{I}$$ And, $$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \mathbf{O}$$
31	Show that, $\frac{D}{Dt}\int_\Omega \mathbf{u}\boldsymbol{dv} = \int_\Omega \left(\frac{\partial \mathbf{u}}{\partial t} + \boldsymbol{\nabla}(\mathbf{u} \otimes \mathbf{v})\right)\boldsymbol{dv}$ and that $\frac{D}{Dt}\int_\Omega \varrho\mathbf{u}\boldsymbol{dv} = \int_\Omega \varrho\dot{\mathbf{u}}\boldsymbol{dv}$
	The material derivative, $$\frac{D}{Dt}\int_\Omega \mathbf{u}dv = \frac{D}{Dt}\int_{\Omega_0} \mathbf{u}JdV$$ In the last equation, it is permissible to interchange the integral with the derivative since the material volume is invariant and therefore independent of time, so that,

	$$= \int_{\Omega_0} \frac{D}{Dt}(\mathbf{u}J)dV = \int_{\Omega_0} \left(\dot{\mathbf{u}}J + \mathbf{u}\,\dot{J}\right)dV = \int_{\Omega_0} (\dot{\mathbf{u}}J + \mathbf{u}\,J \operatorname{div} \mathbf{v})dV$$ $$= \int_{\Omega_0} (\dot{\mathbf{u}} + \mathbf{u}\, div\, \mathbf{v})JdV = \int_{\Omega} (\dot{\mathbf{u}} + \mathbf{u} \operatorname{div} \mathbf{v})dv$$ $$= \int_{\Omega} \left(\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \mathbf{v}\mathrm{grad}\ \mathbf{u} + \mathbf{u} \operatorname{div} \mathbf{v}\right) dv = \int_{\Omega} \left(\frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \boldsymbol{\nabla}(\mathbf{u} \otimes \mathbf{v})\right) dv$$ Finally, $$\frac{D}{Dt}\int_{\Omega} \varrho \mathbf{u} dv = \frac{D}{Dt}\int_{\Omega} \varrho_0 \mathbf{u} dV = \int_{\Omega_0} \frac{D}{Dt}\varrho_0 \mathbf{u} dV = \int_{\Omega_0} \varrho_0 \dot{\mathbf{u}} dV = \int_{\Omega} \varrho \dot{\mathbf{u}} dv$$ The first equality because $\varrho dv = \varrho_0 dV$, the second because we can interchange differentials and integrals in material coordinates, and the last again because $\varrho dv = \varrho_0 dV$.
32	A body is in the state of plane strain relative to the $x - y$ plane. Assume all the components of the strain are known relative to Cartesian axes (x, y, z). Find the stress components relative to another axes rotated along the z-axis by an angle θ
	In plane strain, $e_{13} = e_{23} = e_{33} = 0$.For a clockwise rotation around the z-axis, the transformation tensor is, $$Q = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ From symmetry, the stress state in plane is, $$E = \begin{pmatrix} e_{11} & e_{12} & 0 \\ e_{12} & e_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ In the rotated axis, the stress state becomes, $$\mathbf{E}_2 = \mathbf{Q}^{\mathrm{T}}\mathbf{E}\mathbf{Q}$$ $$\begin{pmatrix} \cos\alpha(\cos\alpha e_{11} + \sin\alpha e_{12}) + \sin\alpha(\cos\alpha e_{12} + \sin\alpha e_{22}) & \cos\alpha(\cos\alpha e_{12} + \sin\alpha e_{22}) - \sin\alpha(\cos\alpha e_{11} + \sin\alpha e_{12}) & 0 \\ \cos\alpha(\cos\alpha e_{12} - \sin\alpha e_{11}) + \sin\alpha(\cos\alpha e_{22} - \sin\alpha e_{12}) & \cos\alpha(\cos\alpha e_{22} - \sin\alpha e_{12}) - \sin\alpha(\cos\alpha e_{12} - \sin\alpha e_{11}) & 0 \\ 0 & 0 & 0 \end{pmatrix}$$ The state of stress therefore remains in plane strain.

33	A cylindrical tube undergoes the deformation given by $r = R, \phi = \Theta + \vartheta(R), z = Z + w(R)$ where $\{R, \Phi, Z\}$ and $\{r, \phi, z\}$, are polar coordinates of a point in the tube before and after deformation respectively, ϑ and w are scalar functions of R. *(a)* Explain the meaning of the situation where *(i)* $\vartheta = 0$, (ii) $w = 0$. *(b)* Compute $\mathbf{F}, \mathbf{C}$ and $\mathbf{E}$, *(c)* Find the Lagrangian and Eulerian strain components

$$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{R}} = \begin{pmatrix} F_{rR} & F_{r\Phi} & F_{rZ} \\ F_{\phi R} & F_{\phi\Phi} & F_{\phi Z} \\ F_{zR} & F_{z\Phi} & F_{zZ} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Phi} & \frac{\partial r}{\partial Z} \\ \frac{\partial \phi}{\partial R} r & \frac{r}{R}\frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial Z} r \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Phi} & \frac{\partial z}{\partial Z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \vartheta' r & \frac{r}{R} & 0 \\ w' & 0 & 1 \end{pmatrix}$$

(a) When $\vartheta = 0$, The deformation gradient becomes, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ w' & 0 & 1 \end{pmatrix}$. This is a longitudinal elongation as radial and tangential displacements are nil.

When $w = 0$, The deformation gradient becomes, $\begin{pmatrix} 1 & 0 & 0 \\ \vartheta' r & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This is a torsional rotation as there is no other deformation in the material apart from a relative rotation along the longitudinal axis.

(b) The Right Cauchy Green Tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{pmatrix} (w')^2 + r^2(\vartheta')^2 + 1 & \frac{r^2\vartheta'}{R} & w' \\ \frac{r^2\vartheta'}{R} & \frac{r^2}{R^2} & 0 \\ w' & 0 & 1 \end{pmatrix}$$

and the Lagrangian strain,

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \begin{pmatrix} \frac{1}{2}((w')^2 + r^2(\vartheta')^2) & \frac{r^2\vartheta'}{2R} & \frac{w'}{2} \\ \frac{r^2\vartheta'}{2R} & \frac{1}{2}(\frac{r^2}{R^2} - 1) & 0 \\ \frac{w'}{2} & 0 & 0 \end{pmatrix}$$

Finger Tensor, $\mathbf{F}\,\mathbf{F}^{\mathrm{T}}$,

	$$\mathbf{F}\,\mathbf{F}^{\mathrm{T}} = \begin{pmatrix} 1 & r\vartheta' & w' \\ r\vartheta' & (\vartheta')^2 r^2 + \dfrac{r^2}{R^2} & rw'\vartheta' \\ w' & rw'\vartheta' & (w')^2 + 1 \end{pmatrix}$$ The inverse of this also called the Piola Tensor is, $$\mathbf{B} = \mathbf{F}^{-\mathbf{T}}\mathbf{F}^{-\mathbf{1}} = \begin{pmatrix} \dfrac{R^2\left(\dfrac{(w')^2 r^2}{R^2} + (\vartheta')^2 r^2 + \dfrac{r^2}{R^2}\right)}{r^2} & -\dfrac{R^2\vartheta'}{r} & -w' \\ -\dfrac{R^2\vartheta'}{r} & \dfrac{R^2}{r^2} & 0 \\ -w' & 0 & 1 \end{pmatrix}$$ Eulerian strain $$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}) = \begin{pmatrix} \dfrac{1}{2}\left[1 - \dfrac{R^2\left(\dfrac{(w')^2 r^2}{R^2} + (\vartheta')^2 r^2 + \dfrac{r^2}{R^2}\right)}{r^2}\right] & \dfrac{R^2\vartheta'}{2r} & \dfrac{w'}{2} \\ \dfrac{R^2\vartheta'}{2r} & \dfrac{1}{2}\left(1 - \dfrac{R^2}{r^2}\right) & 0 \\ \dfrac{w'}{2} & 0 & 0 \end{pmatrix}$$ To complete the answer to this question, Find the representation of the displacement vector and its gradient in Cylindrical Polar coordinates.
34	When a blood vessel is under pressure, the following deformation transformations were observed, $r = r(R), \phi = \Phi + \psi Z\ , z = \lambda Z$ Compute the deformation gradient, Cauchy-Green Tensor, Lagrangian. and Eulerian strain tensors for this deformation.
	From Q33 above, it is clear that $$\mathbf{F} = \frac{\partial \mathbf{r}}{\partial \mathbf{R}} = \begin{pmatrix} F_{rR} & F_{r\Phi} & F_{rZ} \\ F_{\phi R} & F_{\phi\Phi} & F_{\phi Z} \\ F_{zR} & F_{z\Phi} & F_{zZ} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial r}{\partial R} & \dfrac{1}{R}\dfrac{\partial r}{\partial \Phi} & \dfrac{\partial r}{\partial \mathrm{Z}} \\ \dfrac{\partial \phi}{\partial R} r & \dfrac{r}{R}\dfrac{\partial \phi}{\partial \Phi} & \dfrac{\partial \phi}{\partial \mathrm{Z}} r \\ \dfrac{\partial z}{\partial R} & \dfrac{1}{R}\dfrac{\partial z}{\partial \Phi} & \dfrac{\partial z}{\partial \mathrm{Z}} \end{pmatrix} = \begin{pmatrix} r'(R) & 0 & 0 \\ 0 & \dfrac{r}{R} & \psi r \\ 0 & 0 & \lambda \end{pmatrix}$$ The Right Cauchy Green Tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{pmatrix} (r')^2 & 0 & 0 \\ 0 & \dfrac{r^2}{R^2} & \dfrac{r^2\psi}{R} \\ 0 & \dfrac{r^2\psi}{R} & \lambda^2 + r^2\psi^2 \end{pmatrix}$$

and the Lagrangian strain,

$$\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I}) = \begin{pmatrix} \dfrac{1}{2}\left(\left(\dfrac{dr}{dR}\right)^2 - 1\right) & 0 & 0 \\ 0 & \dfrac{1}{2}\left(\dfrac{r^2}{R^2} - 1\right) & \dfrac{r^2\psi}{2R} \\ 0 & \dfrac{r^2\psi}{2R} & \dfrac{1}{2}(\lambda^2 + r^2\psi^2 - 1) \end{pmatrix}$$

Finger Tensor, $\boldsymbol{F}\,\boldsymbol{F}^T$,

$$\mathbf{F}\,\mathbf{F}^{\mathrm{T}} = \begin{pmatrix} (r')^2 & 0 & 0 \\ 0 & \psi^2 r^2 + \dfrac{r^2}{R^2} & r\lambda\psi \\ 0 & r\lambda\psi & \lambda^2 \end{pmatrix}$$

The inverse of this also called the Piola Tensor is,

$$\mathbf{B} = \mathbf{F}^{-\mathbf{T}}\mathbf{F}^{-\mathbf{1}} = \begin{pmatrix} \dfrac{1}{(r')^2} & 0 & 0 \\ 0 & \dfrac{R^2}{r^2} & -\dfrac{R^2\psi}{r\lambda} \\ 0 & -\dfrac{R^2\psi}{r\lambda} & \dfrac{R^2\left(r^2\psi^2(r')^2 + \dfrac{r^2(r')^2}{R^2}\right)}{r^2\lambda^2(r')^2} \end{pmatrix}$$

Eulerian strain

$$\boldsymbol{e} = \frac{1}{2}(\boldsymbol{I} - \boldsymbol{B}) = \begin{pmatrix} \dfrac{1}{2}\left(1 - \dfrac{1}{(r')^2}\right) & 0 & 0 \\ 0 & \dfrac{1}{2}\left(1 - \dfrac{R^2}{r^2}\right) & \dfrac{R^2\psi}{2r\lambda} \\ 0 & \dfrac{R^2\psi}{2r\lambda} & \dfrac{1}{2}\left[1 - \dfrac{R^2\left(r^2\psi^2(r')^2 + \dfrac{r^2(r')^2}{R^2}\right)}{r^2\lambda^2(r')^2}\right] \end{pmatrix}$$

35	Find the physical components of the Deformation gradient if the material and spatial frames are referred to spherical polar coordinates

Let the material frame be given in the cylindrical coordinates, $\{R, \Phi, Z\}$, the position vector

$$\mathbf{R} = y^i \mathbf{G}_i$$

Find the physical components of the Deformation gradient if the material and spatial frames are referred to spherical polar coordinates= $R\cos\Phi\,\boldsymbol{i} + R\sin\Phi\,\boldsymbol{j} + Z\boldsymbol{k}$

$$\equiv R\boldsymbol{e}_R + Z\boldsymbol{k}$$

If the spatial frame is in cylindrical polar, $\{r, \phi, z\}$, the position vector

$$\begin{aligned}\mathbf{R} &= y^i \mathbf{G}_i \\ &= r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j} + z\boldsymbol{k} \\ &\equiv r\boldsymbol{e}_r + z\boldsymbol{k}\end{aligned}$$

Deformation gradient, upon noting that $\mathbf{G}^j$ is the reciprocal basis while $\mathbf{g}_i$ is the natural basis in their respective frames (see above Question), is

$$\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{R}} = \left(\mathbf{G}^j \frac{\partial}{\partial y^j}\right) \otimes \left(x^i \mathbf{g}_i\right) = \left(\boldsymbol{e}_R \frac{\partial}{\partial R} + \frac{\boldsymbol{e}_\Phi}{R}\frac{\partial}{\partial \Phi} + e_Z \frac{\partial}{\partial Z}\right) \otimes \ (r\boldsymbol{e}_r + z\boldsymbol{k})$$

$$= \boldsymbol{e}_R \frac{\partial}{\partial R} \otimes \ (r\boldsymbol{e}_r + z\boldsymbol{k}) + \left(\frac{\boldsymbol{e}_\Phi}{R}\frac{\partial}{\partial \Phi}\right) \otimes \ (r\boldsymbol{e}_r + z\boldsymbol{k}) + \left(e_Z \frac{\partial}{\partial Z}\right) \otimes \ (r\boldsymbol{e}_r + z\boldsymbol{k})$$

$$= \boldsymbol{e}_R \otimes \frac{\partial}{\partial R}\ (r\boldsymbol{e}_r + ze_Z) + \frac{\boldsymbol{e}_\Phi}{R} \otimes \ \frac{\partial}{\partial \Phi}(r\boldsymbol{e}_r + ze_Z) + e_Z \otimes \frac{\partial}{\partial Z}\ (r\boldsymbol{e}_r + ze_Z)$$

$$\begin{aligned}= \boldsymbol{e}_R \otimes &\left(\frac{\partial}{\partial r}\frac{\partial r}{\partial R} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial R} + \frac{\partial}{\partial z}\frac{\partial z}{\partial R}\right)\ (r\boldsymbol{e}_r + ze_Z) + \frac{\boldsymbol{e}_\Phi}{R} \\ &\otimes \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial \Phi} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial \Phi} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \Phi}\right)\ (r\boldsymbol{e}_r + ze_Z) + e_Z \\ &\otimes \left(\frac{\partial}{\partial r}\frac{\partial r}{\partial \mathrm{Z}} + \frac{\partial}{\partial \phi}\frac{\partial \phi}{\partial \mathrm{Z}} + \frac{\partial}{\partial z}\frac{\partial z}{\partial \mathrm{Z}}\right)\ (r\boldsymbol{e}_r + ze_Z)\end{aligned}$$

$$\begin{aligned}= \boldsymbol{e}_R \otimes &\left[\frac{\partial r}{\partial R}(r\boldsymbol{e}_r + ze_Z) + \frac{\partial \phi}{\partial R}\frac{\partial}{\partial \phi}(r\boldsymbol{e}_r + ze_Z) + \frac{\partial z}{\partial R}\frac{\partial}{\partial z}(r\boldsymbol{e}_r + ze_Z)\right] + \frac{\boldsymbol{e}_\Phi}{R} \\ &\otimes \left[\frac{\partial r}{\partial \Phi}\frac{\partial}{\partial r}(r\boldsymbol{e}_r + ze_Z) + \frac{\partial \phi}{\partial \Phi}\frac{\partial}{\partial \phi}(r\boldsymbol{e}_r + ze_Z) + \frac{\partial z}{\partial \Phi}\frac{\partial}{\partial z}(r\boldsymbol{e}_r + ze_Z)\right] \\ &+ e_Z \otimes \left[\frac{\partial r}{\partial \mathrm{Z}}\frac{\partial}{\partial r}(r\boldsymbol{e}_r + ze_Z) + \frac{\partial \phi}{\partial \mathrm{Z}}\frac{\partial}{\partial \phi}(r\boldsymbol{e}_r + ze_Z) + \frac{\partial z}{\partial \mathrm{Z}}\frac{\partial}{\partial z}(r\boldsymbol{e}_r + ze_Z)\right]\end{aligned}$$

$$= \boldsymbol{e}_R \otimes \left[\frac{\partial r}{\partial R}\boldsymbol{e}_r + \frac{\partial \phi}{\partial R} r\boldsymbol{e}_\phi + \frac{\partial z}{\partial R}\boldsymbol{e}_z\right] + \frac{\boldsymbol{e}_\Phi}{R} \otimes \left[\frac{\partial r}{\partial \Phi}\boldsymbol{e}_r + \frac{\partial \phi}{\partial \Phi} r\boldsymbol{e}_\phi + \frac{\partial z}{\partial \Phi}\boldsymbol{e}_z\right] + e_Z \otimes \left[\frac{\partial r}{\partial \mathrm{Z}}\boldsymbol{e}_r + \frac{\partial \phi}{\partial \mathrm{Z}} r\boldsymbol{e}_\phi + \frac{\partial z}{\partial \mathrm{Z}}\boldsymbol{e}_z\right]$$

Consequently, in matrix notation, we can write,

$$\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{R}} = \begin{pmatrix} F_{rR} & F_{r\Phi} & F_{rZ} \\ F_{\phi R} & F_{\phi\Phi} & F_{\phi Z} \\ F_{zR} & F_{z\Phi} & F_{zZ} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Phi} & \frac{\partial r}{\partial \mathrm{Z}} \\ \frac{\partial \phi}{\partial R} r & \frac{r}{R}\frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial \mathrm{Z}} r \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Phi} & \frac{\partial z}{\partial \mathrm{Z}} \end{pmatrix}$$

36	Find the natural, reciprocal and the physical bases in cylindrical polar coordinates. Also find their derivatives.

The position vector in cylindrical polar coordinates is:

$$\begin{aligned}\mathbf{R} &= y^i \mathbf{G}_i \\ &= r\cos\phi\,\boldsymbol{i} + r\sin\phi\,\boldsymbol{j} + z\boldsymbol{k} \\ &\equiv r\boldsymbol{e}_r + z\boldsymbol{k}\end{aligned}$$

The natural basis set consists of the derivatives of the position vector with respect to the coordinate variables $\{r, \phi, z\}$. Hence natural basis

$$\begin{pmatrix}\mathbf{g}_1\\ \mathbf{g}_2\\ \mathbf{g}_3\end{pmatrix} = \begin{pmatrix}\dfrac{\partial R}{\partial y^1}\\ \dfrac{\partial R}{\partial y^2}\\ \dfrac{\partial R}{\partial y^3}\end{pmatrix} = \begin{pmatrix}\dfrac{\partial R}{\partial r}\\ \dfrac{\partial R}{\partial \phi}\\ \dfrac{\partial R}{\partial z}\end{pmatrix} = \begin{pmatrix}\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}\\ -r\sin\phi\,\boldsymbol{i} + r\cos\phi\,\boldsymbol{j}\\ \boldsymbol{k}\end{pmatrix} \equiv \begin{pmatrix}\boldsymbol{e}_r\\ r\boldsymbol{e}_\phi\\ \boldsymbol{e}_z\end{pmatrix}$$

The reciprocal basis, $\mathbf{g}^i = g^{ij}\mathbf{g}_j$, where g^{ij} is the inverse of the metric tensor.

Accordingly, the reciprocal basis is,

$$\begin{pmatrix}\mathbf{g}^1\\ \mathbf{g}^2\\ \mathbf{g}^3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & \dfrac{1}{r^2} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\boldsymbol{e}_r\\ r\boldsymbol{e}_\phi\\ \boldsymbol{e}_z\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_\rho\\ \dfrac{r\boldsymbol{e}_\phi}{r^2}\\ \boldsymbol{e}_z\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_\rho\\ \dfrac{\boldsymbol{e}_\phi}{r}\\ \boldsymbol{e}_z\end{pmatrix}$$

The natural basis is the normalized natural basis:

$$\begin{pmatrix}\dfrac{\mathbf{g}_1}{|\mathbf{g}_1|}\\ \dfrac{\mathbf{g}_2}{|\mathbf{g}_2|}\\ \dfrac{\mathbf{g}_3}{|\mathbf{g}_3|}\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_\rho\\ r\dfrac{\boldsymbol{e}_\phi}{r}\\ \boldsymbol{e}_z\end{pmatrix} = \begin{pmatrix}\boldsymbol{e}_r\\ \boldsymbol{e}_\phi\\ \boldsymbol{e}_z\end{pmatrix} = \begin{pmatrix}\cos\phi\,\boldsymbol{i} + \sin\phi\,\boldsymbol{j}\\ -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j}\\ \boldsymbol{k}\end{pmatrix}$$

Derivatives:

From the above matrix, a simple application of partial derivatives immediately gives,

$$\begin{pmatrix}\dfrac{\partial \boldsymbol{e}_r}{\partial r} & \dfrac{\partial \boldsymbol{e}_r}{\partial \phi} & \dfrac{\partial \boldsymbol{e}_r}{\partial z}\\ \dfrac{\partial \boldsymbol{e}_\phi}{\partial r} & \dfrac{\partial \boldsymbol{e}_\phi}{\partial \phi} & \dfrac{\partial \boldsymbol{e}_\phi}{\partial z}\\ \dfrac{\partial \boldsymbol{e}_z}{\partial r} & \dfrac{\partial \boldsymbol{e}_z}{\partial \phi} & \dfrac{\partial \boldsymbol{e}_z}{\partial z}\end{pmatrix} = \begin{pmatrix}0 & -\sin\phi\,\boldsymbol{i} + \cos\phi\,\boldsymbol{j} & 0\\ 0 & -\cos\phi\,\boldsymbol{i} - \sin\phi\,\boldsymbol{j} & 0\\ 0 & 0 & 0\end{pmatrix} = \begin{pmatrix}0 & \boldsymbol{e}_\phi & 0\\ 0 & -\boldsymbol{e}_r & 0\\ 0 & 0 & 0\end{pmatrix}$$

37	Find, by direct computation, the physical components of the Deformation gradient if the material and spatial frames are referred to spherical polar coordinates
	Let the material frame be given in the coordinates, $\{\rho, \theta, \phi\}$, the position vector $$\begin{aligned}\mathbf{R} &= y^i \mathbf{G}_i \\ &= \rho \sin\theta \cos\phi\, \boldsymbol{i} + \rho \sin\theta \sin\phi\, \boldsymbol{j} + \rho \cos\theta\, \boldsymbol{k} \\ &\equiv \rho \boldsymbol{e}_\rho\end{aligned}$$ If the spatial frame is in spherical polar, $\{\varrho, \vartheta, \varphi\}$, the position vector $$\begin{aligned}\mathbf{r} &= x^i \mathbf{g}_i \\ &= \varrho \sin\vartheta \cos\varphi\, \boldsymbol{i} + \varrho \sin\vartheta \sin\varphi\, \boldsymbol{j} + \varrho \cos\vartheta\, \boldsymbol{k} \equiv \varrho \boldsymbol{e}_\varrho\end{aligned}$$ Deformation gradient, upon noting that $\mathbf{G}^j$ is the reciprocal basis while $\mathbf{g}_i$ is the natural basis in their respective frames (see above Question), is $$\begin{aligned}\boldsymbol{F} = \frac{\partial \boldsymbol{r}}{\partial \boldsymbol{R}} &= \left(\mathbf{G}^j \frac{\partial}{\partial y^j}\right) \otimes \left(x^i \mathbf{g}_i\right) = \left(\boldsymbol{e}_\rho \frac{\partial}{\partial \rho} + \frac{\boldsymbol{e}_\theta}{\rho}\frac{\partial}{\partial \theta} + \frac{e_\phi}{\rho \sin\theta}\frac{\partial}{\partial \phi}\right) \otimes \left(\varrho \boldsymbol{e}_\varrho\right) \\ &= \left(\boldsymbol{e}_\rho \frac{\partial}{\partial \rho}\right) \otimes \left(\varrho \boldsymbol{e}_\varrho\right) + \left(\frac{\boldsymbol{e}_\theta}{\rho}\frac{\partial}{\partial \theta}\right) \otimes \left(\varrho \boldsymbol{e}_\varrho\right) + \left(\frac{e_\phi}{\rho \sin\theta}\frac{\partial}{\partial \phi}\right) \otimes \left(\varrho \boldsymbol{e}_\varrho\right) \\ &= \boldsymbol{e}_\rho \otimes \frac{\partial}{\partial \rho}\left(\varrho \boldsymbol{e}_\varrho\right) + \frac{\boldsymbol{e}_\theta}{\rho} \otimes \frac{\partial}{\partial \theta}\left(\varrho \boldsymbol{e}_\varrho\right) + \frac{e_\phi}{\rho \sin\theta} \otimes \frac{\partial}{\partial \phi}\left(\varrho \boldsymbol{e}_\varrho\right)\end{aligned}$$

	$$= \boldsymbol{e}_\rho \otimes \left(\frac{\partial}{\partial\varrho}\frac{\partial\varrho}{\partial\rho} + \frac{\partial}{\partial\vartheta}\frac{\partial\vartheta}{\partial\rho} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial\rho}\right)(\varrho\boldsymbol{e}_\varrho) + \frac{\boldsymbol{e}_\theta}{\rho} \otimes \left(\frac{\partial}{\partial\varrho}\frac{\partial\varrho}{\partial\theta} + \frac{\partial}{\partial\vartheta}\frac{\partial\vartheta}{\partial\theta} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial\theta}\right)(\varrho\boldsymbol{e}_\varrho)$$ $$+ \frac{e_\phi}{\rho\sin\theta} \otimes \left(\frac{\partial}{\partial\varrho}\frac{\partial\varrho}{\partial\phi} + \frac{\partial}{\partial\vartheta}\frac{\partial\vartheta}{\partial\phi} + \frac{\partial}{\partial\varphi}\frac{\partial\varphi}{\partial\phi}\right)(\varrho\boldsymbol{e}_\varrho)$$ $$= \boldsymbol{e}_\rho \otimes \left[\frac{\partial\varrho}{\partial\rho}\frac{\partial}{\partial\varrho}(\varrho\boldsymbol{e}_\varrho) + \frac{\partial\vartheta}{\partial\rho}\frac{\partial}{\partial\vartheta}(\varrho\boldsymbol{e}_\varrho) + \frac{\partial\varphi}{\partial\rho}\frac{\partial}{\partial\varphi}(\varrho\boldsymbol{e}_\varrho)\right] + \frac{\boldsymbol{e}_\theta}{\rho}$$ $$\otimes \left[\frac{\partial\varrho}{\partial\theta}\frac{\partial}{\partial\varrho}(\varrho\boldsymbol{e}_\varrho) + \frac{\partial\vartheta}{\partial\theta}\frac{\partial}{\partial\vartheta}(\varrho\boldsymbol{e}_\varrho) + \frac{\partial\varphi}{\partial\theta}\frac{\partial}{\partial\varphi}(\varrho\boldsymbol{e}_\varrho)\right] + \frac{e_\phi}{\rho\sin\theta}$$ $$\otimes \left[\frac{\partial\varrho}{\partial\phi}\frac{\partial}{\partial\varrho}(\varrho\boldsymbol{e}_\varrho) + \frac{\partial\vartheta}{\partial\phi}\frac{\partial}{\partial\vartheta}(\varrho\boldsymbol{e}_\varrho) + \frac{\partial\varphi}{\partial\phi}\frac{\partial}{\partial\varphi}(\varrho\boldsymbol{e}_\varrho)\right]$$ $$= \boldsymbol{e}_\rho \otimes \left[\frac{\partial\varrho}{\partial\rho}\boldsymbol{e}_\varrho + \frac{\partial\vartheta}{\partial\rho}(\varrho\boldsymbol{e}_\vartheta) + \frac{\partial\varphi}{\partial\rho}(\varrho\sin\vartheta\,\boldsymbol{e}_\varphi)\right] + \frac{\boldsymbol{e}_\theta}{\rho}$$ $$\otimes \left[\frac{\partial\varrho}{\partial\theta}\boldsymbol{e}_\varrho + \frac{\partial\vartheta}{\partial\theta}(\varrho\boldsymbol{e}_\vartheta) + \frac{\partial\varphi}{\partial\theta}(\varrho\sin\vartheta\,\boldsymbol{e}_\varphi)\right] + \frac{e_\phi}{\rho\sin\theta}$$ $$\otimes \left[\frac{\partial\varrho}{\partial\phi}\boldsymbol{e}_\varrho + \frac{\partial\vartheta}{\partial\phi}(\varrho\boldsymbol{e}_\vartheta) + \frac{\partial\varphi}{\partial\phi}(\varrho\sin\vartheta\,\boldsymbol{e}_\varphi)\right]$$ Consequently, in matrix notation, we can write, $$\boldsymbol{F} = \frac{\partial\boldsymbol{r}}{\partial\boldsymbol{R}} = \begin{pmatrix} F_{\varrho\rho} & F_{\varrho\theta} & F_{\varrho\phi} \\ F_{\vartheta\rho} & F_{\vartheta\theta} & F_{\vartheta\phi} \\ F_{\varphi\rho} & F_{\varphi\theta} & F_{\varphi\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial\varrho}{\partial\rho} & \frac{1}{\rho}\frac{\partial\varrho}{\partial\theta} & \frac{1}{\rho\sin\theta}\frac{\partial\varrho}{\partial\phi} \\ \varrho\frac{\partial\vartheta}{\partial\rho} & \frac{\varrho}{\rho}\frac{\partial\vartheta}{\partial\theta} & \frac{\varrho}{\rho\sin\theta}\frac{\partial\vartheta}{\partial\phi} \\ \varrho\sin\vartheta\frac{\partial\varphi}{\partial\rho} & \frac{\varrho\sin\vartheta}{\rho}\frac{\partial\varphi}{\partial\theta} & \frac{\varrho\sin\vartheta}{\rho\sin\theta}\frac{\partial\varphi}{\partial\phi} \end{pmatrix}$$
38	Use the formula, $\boldsymbol{F} = F^i_{.j}\mathbf{g}_i \otimes \boldsymbol{G}^j$ to find the tensor as well as physical components of the deformation gradient if the material and spatial frames are referred to spherical polar coordinates
	$$\boldsymbol{F} = F^i_{.j}\mathbf{g}_i \otimes \boldsymbol{G}^j = \begin{pmatrix} F^1_1 & F^1_2 & F^1_3 \\ F^2_1 & F^2_2 & F^2_3 \\ F^3_1 & F^3_2 & F^3_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial\varrho}{\partial\rho} & \frac{\partial\varrho}{\partial\theta} & \frac{\partial\varrho}{\partial\phi} \\ \frac{\partial\vartheta}{\partial\rho} & \frac{\partial\vartheta}{\partial\theta} & \frac{\partial\vartheta}{\partial\phi} \\ \frac{\partial\varphi}{\partial\rho} & \frac{\partial\varphi}{\partial\theta} & \frac{\partial\varphi}{\partial\phi} \end{pmatrix}$$ To obtain physical components we note that the contravariant component is spatial while the covariant is material. If the magnitudes of the material vectors are $\eta_i$and that

of the spatial are $h_i$then, the physical component, $F(ij) = \frac{F_j^i \eta_i}{h_j}$. The vector $\{h_i\} = \{1, \rho, \rho \sin\theta\}$, and $\{\eta_i\} = \{1, \varrho, \varrho \sin\vartheta\}$. Accordingly,

$$[F(ij)] = \left[\frac{F_j^i \eta_i}{h_j}\right] = \begin{pmatrix} F_{\varrho\rho} & F_{\varrho\theta} & F_{\varrho\phi} \\ F_{\vartheta\rho} & F_{\vartheta\theta} & F_{\vartheta\phi} \\ F_{\varphi\rho} & F_{\varphi\theta} & F_{\varphi\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial\varrho}{\partial\rho} & \frac{1}{\rho}\frac{\partial\varrho}{\partial\theta} & \frac{1}{\rho\sin\theta}\frac{\partial\varrho}{\partial\phi} \\ \varrho\frac{\partial\vartheta}{\partial\rho} & \frac{\varrho}{\rho}\frac{\partial\vartheta}{\partial\theta} & \frac{\varrho}{\rho\sin\theta}\frac{\partial\vartheta}{\partial\phi} \\ \varrho\sin\vartheta\frac{\partial\varphi}{\partial\rho} & \frac{\varrho\sin\vartheta}{\rho}\frac{\partial\varphi}{\partial\theta} & \frac{\varrho\sin\vartheta}{\rho\sin\theta}\frac{\partial\varphi}{\partial\phi} \end{pmatrix}$$

39 Find the natural, reciprocal and the physical bases in spherical polar coordinates. Also find their derivatives.

The position vector in spherical polar coordinates is:

$$\begin{aligned} \mathbf{R} &= y^i \mathbf{G}_i \\ &= \rho\sin\theta\cos\phi\,\boldsymbol{i} + \rho\sin\theta\sin\phi\,\boldsymbol{j} + \rho\cos\theta\,\boldsymbol{k} \\ &\equiv \rho\boldsymbol{e}_\rho \end{aligned}$$

The natural basis are the derivatives of the position vector with respect to the coordinate variables $\{\rho, \theta, \phi\}$. Hence natural basis

$$\begin{pmatrix} \mathbf{g}_1 \\ \mathbf{g}_2 \\ \mathbf{g}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial R}{\partial y^1} \\ \frac{\partial R}{\partial y^2} \\ \frac{\partial R}{\partial y^3} \end{pmatrix} = \begin{pmatrix} \frac{\partial R}{\partial\rho} \\ \frac{\partial R}{\partial\theta} \\ \frac{\partial R}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \sin\theta\sin\phi & \cos\theta \\ \rho\cos\theta\cos\phi & \rho\cos\theta\sin\phi & -\rho\sin\theta \\ -\rho\sin\theta\sin\phi & \rho\cos\phi\sin\theta & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \boldsymbol{e}_\rho \\ \rho\boldsymbol{e}_\theta \\ \rho\sin\theta\,\boldsymbol{e}_\phi \end{pmatrix}$$

The reciprocal basis, $\mathbf{g}^i = g^{ij}\mathbf{g}_j$, where g^{ij} is the inverse of the me6tric tensor. Accordingly, the reciprocal basis is,

$$\begin{pmatrix} \mathbf{g}^1 \\ \mathbf{g}^2 \\ \mathbf{g}^3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2\sin^2\theta} \end{pmatrix} \begin{pmatrix} \boldsymbol{e}_\rho \\ \rho\boldsymbol{e}_\theta \\ \rho\sin\theta\,\boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\rho \\ \frac{\rho\boldsymbol{e}_\theta}{\rho^2} \\ \frac{\rho\sin\theta\,\boldsymbol{e}_\phi}{\rho^2\sin^2\theta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\rho \\ \frac{\boldsymbol{e}_\theta}{\rho} \\ \frac{\boldsymbol{e}_\phi}{\rho\sin\theta} \end{pmatrix}$$

The natural basis is the normalized natural basis:

$$\begin{pmatrix} \frac{\mathbf{g}_1}{|\mathbf{g}_1|} \\ \frac{\mathbf{g}_2}{|\mathbf{g}_2|} \\ \frac{\mathbf{g}_3}{|\mathbf{g}_3|} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\rho \\ \frac{\rho \boldsymbol{e}_\theta}{\rho} \\ \frac{\rho \sin\theta\, \boldsymbol{e}_\phi}{\rho \sin\theta} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\rho \\ \boldsymbol{e}_\theta \\ \boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} \cos\phi \sin\theta & \sin\ \theta \sin\ \phi & \cos\ \theta \\ \cos\ \theta \cos\ \phi & \cos\ \theta \sin\ \phi & -\sin\ \theta \\ -\sin\ \phi & \cos\ \phi & 0 \end{pmatrix}$$

Derivatives:

From the above matrix, a simple application of partial derivatives immediately gives,

$$\begin{pmatrix} \frac{\partial \boldsymbol{e}_\rho}{\partial \rho} & \frac{\partial \boldsymbol{e}_\rho}{\partial \theta} & \frac{\partial \boldsymbol{e}_\rho}{\partial \phi} \\ \frac{\partial \boldsymbol{e}_\theta}{\partial \rho} & \frac{\partial \boldsymbol{e}_\theta}{\partial \theta} & \frac{\partial \boldsymbol{e}_\theta}{\partial \phi} \\ \frac{\partial \boldsymbol{e}_\phi}{\partial \rho} & \frac{\partial \boldsymbol{e}_\phi}{\partial \theta} & \frac{\partial \boldsymbol{e}_\phi}{\partial \phi} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & \cos\theta \cos\phi \boldsymbol{i} + \cos\theta \sin\phi\, \boldsymbol{j} - \sin\theta \boldsymbol{k} & -\sin\phi \sin\theta \boldsymbol{i} + \cos\phi \sin\theta \boldsymbol{j} \\ 0 & -\cos\phi \sin\theta \boldsymbol{i} - \sin\theta \sin\phi \boldsymbol{j} - \cos\theta \boldsymbol{k} & -\cos\theta \sin\phi \boldsymbol{i} + \cos\theta \cos\phi \boldsymbol{j} \\ 0 & 0 & -\cos\phi \boldsymbol{i} + \sin\phi \boldsymbol{j} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \boldsymbol{e}_\theta & \boldsymbol{e}_\phi \sin\theta \\ 0 & -\boldsymbol{e}_\rho & \boldsymbol{e}_\phi \cos\theta \\ 0 & 0 & -\cos\phi \boldsymbol{i} + \sin\ \phi \boldsymbol{j} \end{pmatrix}$$

40

41

42

43

44

45

46

47	
48	
49	
50	
51	
52	
53	
54	
55	Relative to Cartesian coordinates, the state of strain at a point in a body is given by, $$\mathbf{E} = [\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3]\begin{pmatrix} 0.5 & 0.3 & 0 \\ 0.3 & 0.4 & -0.2 \\ 0 & -0.2 & 0.2 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$ Determine the change in angle between the two lines emanating from the point in the direction of the lines $2\mathbf{E}_1 + 2\mathbf{E}_2 + \mathbf{E}_3$ and $3\mathbf{E}_1 - 4\mathbf{E}_2$ in the undeformed body.
	The Right Stretch Tensor $\mathbf{U}$ is related to Lagrange Strain $\mathbf{E}$ by: $$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I}), \text{so that}$$ $$\mathbf{U} = \sqrt{2\mathbf{E} + \mathbf{I}}$$ $$\theta = \cos^{-1}\left(\frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\lVert d\mathbf{x}_1 \rVert \lVert d\mathbf{x}_2 \rVert}\right) = \cos^{-1}\left(\frac{\mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2}{\lVert \mathbf{U}d\mathbf{X}_1 \rVert \lVert \mathbf{U}d\mathbf{X}_2 \rVert}\right)$$

Q4.55 Kinematics.nb - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
In[ ]:= EE := {{0.5, 0.3, 0}, {0.3, 0.4, -0.2}, {0, -0.20, 0.2}}
        Line1 := {2, 2, 1}
        Line2 := {3, -4, 0}
        UU := MatrixPower[2 EE + IdentityMatrix[3], 1 / 2]
        FinalAngle :=
         ArcCos[((UU . Line1).(UU.Line2)) / (Norm[UU . Line1] Norm[UU.Line2])]
        InitialAngle := ArcCos[((Line1).(Line2)) / (Norm[Line1] Norm[Line2])]

In[ ]:= AngleChange = 180 * (FinalAngle - InitialAngle) / Pi

Out[ ]= -3.13328
```

56 In the two dimensional deformation defined by the deformation

$$y_1 = 0.1\, x_1(1 + 2\, x_1 + x_2)$$
$$y_2 = 0.2x_2(1 + x_2)$$

where x_i and y_i are Cartesian coordinates of a particle in the reference and deformed configurations respectively. Determine the deformation on a line element $\mathbf{a}_0 = \mathbf{e}_1 + 2\mathbf{e}_2$passing through the point (2,-2) in the reference configuration.

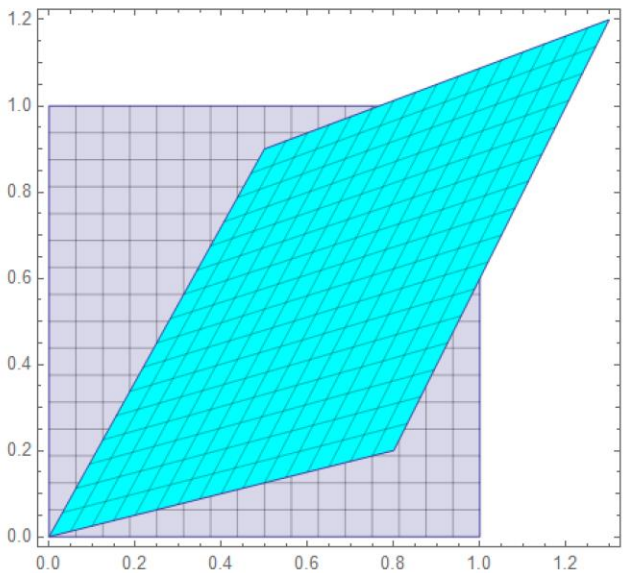
57 Given the transformation of points from $\{X_1, X_2\}$ to $\{x_1, x_2\}$ by the data in the table,

Point	X_1	X_2	x_1	x_2
P_1	0	0	0	0
P_2	1	0	0.8	0.2
P_3	1	1	1.3	1.2
P_4	0	1	0.5	0.9

Q4.57 Kinematics.nb - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

```
Solve[{a0 + 0 a1 + 0 a2 + 0 a3 == 0, b0 + 0 b1 + 0 b2 + 0 b3 == 0, a0 + a1 + 0 a2 + 0 a3 == 0.8,
  b0 + b1 + 0 b2 + 0 b3 == 0.2, a0 + 0 a1 + a2 + 0 a3 == 0.5, b0 + 0 b1 + b2 + 0 b3 == 0.9,
  a0 + a1 + a2 + a3 == 1.3, b0 + b1 + b2 + b3 == 1.2}, {a0, a1, a2, a3, b0, b1, b2, b3}]

{{a0 → 0., a1 → 0.8, a2 → 0.5, a3 → 0., b0 → 0., b1 → 0.2, b2 → 0.9, b3 → 0.1}}

myMap[X1_, X2_] := {0.8 X1 + 0.5 X2 , 0.2 X1 + 0.9 X2 + 0.1 X1 X2} // Flatten

initialConfig = ParametricPlot[{X1, X2}, {X1, 0, 1}, {X2, 0, 1}];

deformedConfig = ParametricPlot[myMap[X1, X2], {X1, 0, 1}, {X2, 0, 1} ,
   MeshShading → {{Cyan, Cyan}}];

Show[initialConfig, deformedConfig, PlotRange → All]
```




```
myFullMap := {0.8 X1 + 0.5 X2 , 0.2 X1 + 0.9 X2 + 0.1 X1 X2, X3}

F[X1_, X2_, X3_] = D[myFullMap, {{X1, X2, X3}}]

{{0.8, 0.5, 0}, {0.2 + 0.1 X2, 0.9 + 0.1 X1, 0}, {0, 0, 1}}

EE = (1 / 2) (Transpose[F[x1, x2, x3]] . F[x1, x2, x3] - IdentityMatrix[3]) // MatrixForm
```

58

The vector function, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \begin{pmatrix} \lambda_1 & k_1 & 0 \\ k_1 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$.

(a) Compute the tensors $\mathbf{F}, \mathbf{C}, \mathbf{E}$ **U and R**

(b) For $\lambda_1 = 1.1, \lambda_2 = 1.25, \lambda_3 = 1, k_1 = 0.15, k_2 = -0.2$, Determine the principal values and directions of $\mathbf{E}$. Verify their mutual orthogonality.

(c) Compute Strain Invariants

(a)	The full solution is in the attached Notebook. Q4.58 Kinematics.nb * - Wolfram Mathematica 11.3 File Edit Insert Format Cell Graphics Evaluation Palettes Window Help $F = \partial_{\{\{x_1,x_2,x_3\}\}} \{k_1 X_2 + \lambda_1 X_1, k_2 X_1 + \lambda_2 X_2, \lambda_3 X_3\}$ $\{\{\lambda_1, k_1, 0\}, \{k_2, \lambda_2, 0\}, \{0, 0, \lambda_3\}\}$ $CC = \text{Simplify}[F^T.F] /. \{\lambda_1 \to 1.1, \lambda_2 \to 1.25, \lambda_3 \to 1., k_1 \to 0.15, k_2 \to -.2\}$ $U = \text{MatrixPower}\left[CC, \frac{1}{2}\right]$ $R = F.\text{Inverse}[U] /. \{\lambda_1 \to 1.1, \lambda_2 \to 1.25, \lambda_3 \to 1.`, k_1 \to 0.15, k_2 \to -.2\}$ $\text{Chop}[\text{Transpose}[R].R]$ $\{\{1., 0, 0\}, \{0, 1., 0\}, \{0, 0, 1.\}\}$ $EE = N\left[\frac{1}{2} (CC - \text{Identity}[3]), 3\right]$
59	
60	
61	
62	
63	
64	
65	

66	
67	
68	
69	

References

Allen, MB, **Continuum Mechanics: The Birthplace of Mathematical Models**, John Wiley, Inc., New Jersey 2016

Bertram, A, **Elasticity and Plasticity of Large Deformations, An Introduction**, Springer, 2008

Dill, EH, **Continuum Mechanics, Elasticity, Plasticity, Viscoelasticity**, CRC Press, 2007

Gurtin, ME, Fried, E & Anand, L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, 2010

Holzapfel, GA, **Nonlinear Solid Mechanics**, Wiley NY, 2007

Jog, CS, **Fluid Mechanics: Foundations and Applications of Mechanics**, Volume II, Cambridge IISc Series, 2015

Negahban, M, **The Mechanical and Thermodynamical Theory of Plasticity**,CRC Press, Taylor and Francis Group, Boca Raton, Fl, 2012

Ogden, RW, **Nonlinear Elastic Deformations**, Dover Publications, Inc. NY, 1997

Romano, A, Lancellotta, R, Marasco, A, **Continuum Mechanics using Mathematica**, Birkhauser, 2006

Taber, LA, **Nonlinear Theory of Elasticity**, World Scientific, 2008

Wegner, JL & Haddow, JB, **Elements of Continuum Mechanics and Thermodynamics**, Cambridge University Press, 2009.