FIVE

Stress & Heat Flux: Repetitions from Augustin Louis Cauchy and Joseph Fourier

"In the fields of observation, chance favors only the prepared" – Louis Pasteur

MetaData

The prose, video, slides and the Q&A in this chapter are directed at scoring the following points:

1. The concept of stress attained the current level of precision by AL Cauchy with his stress principle that laid the background for the Cauchy laws of motion. A major cornerstone of mechanics.

2. In establishing the notion of stress, this chapter begins with the concept of the traction vector at a Euclidean point. The surface of interest at this point is identified by its outward drawn normal.
3. Traction gets its direction, not from this surface, but from the resultant load vector. It is important to distinguish between these two directions.
4. Cauchy postulated the existence of a stress tensor from which all other aspects of stress are simple properties. From these we shall establish different kinds of stress states.
5. This chapter continues to the concept of heat flow and the associated concept of heat flux.

Introduction

Until the 17th century the understanding of stress was largely intuitive and empirical. Ancient and medieval architects did develop some geometrical methods and simple formulas to compute the proper sizes of pillars and beams, but the scientific understanding of stress became possible only after the necessary mathematical tools of differential and integral calculus were invented in the 17th and 18th centuries:

We are eight short years away from the bicentennial of the seminal work on the concept of stress. Augustine Louis Cauchy published, in 1827, a generalization of Euler's works on hydrodynamics in order to bequeath to the world, a precise notion of stress. This consummated in the idea that a material body responds to externally applied loads through a tensor-valued field, $\boldsymbol{\sigma}(\mathbf{x}, t)$ that came to be known as the Cauchy Stress Tensor.

What is "Stress"?

Stress is a measure of *force intensity* either within or on the bounding surface of a body subjected to loads. The Continuum Model takes a macroscopic approach: Measurable aggregate behavior rather than the microscopic, atomistic activities that may in fact have led to them, and consequently, the standard results of calculus applicable in the case of limiting values of this quotient as the areas to which the forces are applied become very small. It is necessary to note that the word "stress" can mean a scalar, a vector or a tensor. Cauchy's Stress Principle gives precision to the term. It can mean, depending on the context and usage, any of these three:

- **Tensor** – the *Stress Tensor*. This completely characterizes the stress state at a Euclidean point. That such a tensor exists is proved as Cauchy's theorem – a fundamental law in Continuum Mechanics.
- **Vector** - the *Traction* or intensity of resultant forces on a specific surface. It has a magnitude and a direction; this is, roughly speaking, what we have in mind when we say that stress is "force per unit area."
- **Scalar** – the *scalar magnitude* of the traction vector or some projections of the same in certain directions. Once a direction is given, the scalar stress is the magnitude of the stress intensity, or traction in that direction. Hydrostatic pressure is a pathological case. The

traction has the same magnitude in every direction because the stress tensor at any point in a static fluid is spherical.

Consequences

It is a fact that, even in technical literature, we are often not precise as to which of scalar, vector or tensor we have in mind when we use the word “stress”. This has created a lot of confusion. For example, in public discussion for a, such as Quora, we see such questions as “Is hydrostatic pressure a scalar?”. The number of erroneous notions on terms such as stress, pressure, traction, etc. are common even from PhDs in as STEM areas and in Engineering! It exposes weaknesses in the basic notions of stress.

It becomes more important when one tries to specify loads and stresses in such simulations packages as Fusion 360, ANSYS, etc. The notion of “stress” must be properly understood to be efficient in the use of such tools, first to properly specify boundary conditions, and to properly interpret results at the post processing stage.

This chapter, among other things, aims to give clarity to these notions.

Body Forces, Surface Forces

In order to analyze stress and give it a proper characterization, it is convenient to classify forces into two categories for the purpose of analysis: We follow AEH Love in his:

> *“… a distinction is established between two types of forces which we have called ‘body forces’ and ‘surface tractions’, the former being conceived as due to a direct action at a distance, and the latter to contact action.” (**Ref**)*

It is convenient to examine these forces by categorizing them as follows:

Body forces $\mathbf{b}$ (force per unit volume);

These are forces originating from sources (fields of force usually) outside of the body that act on the volume (or mass) of the body.

Surface forces i.e.: $\mathbf{F}$ (traction on each surface: $\mathbf{t}^{(\mathbf{n})}$ is the force, $\mathbf{t}$ per unit area of surface (whose normal direction is given by the unit vector, $\mathbf{n}$ across they which they act). It cannot be overemphasized that $\mathbf{t}$ itself is a vector whose direction is inherent. The superscript $\mathbf{n}$ refers, not to the direction of the traction but to the normal to the surface on which it acts! This specification is important because a change in the surface orientation without leaving the Euclidean point in question, gives a different value of traction, $\mathbf{t}$.

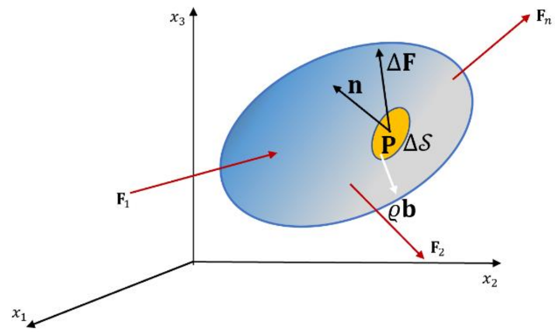
Traction on a Surface

The loads applied directly to the body may themselves be similarly categorized as they can come in form of body or surface forces. As an example, the weight of another body being supported eventually acts as a surface force with a distributed intensity over the coverage of the contact.

Figure 6.1 depicts the forces acting on an element ΔV surrounding a point $\mathbf{P}(x_1, x_2, x_3)$ in a solid body acted upon by the forces $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$. As a result of these actions, the resultant body force per unit volume is $\mathbf{b}(x_1, x_2, x_3)$. Consider an infinitesimal area element oriented in such a way that the unit outward normal to its surface is $\mathbf{n}$. If the resultant force on the surface ΔS is $\Delta\mathbf{F}$, and this results in a traction intensity which will in general vary over ΔS. Let the surface traction vector on this elemental surface be $\mathbf{t}$, it is convenient to label this traction $\mathbf{t}^{(\mathbf{n})}$ in order to emphasise the fact that this traction is the resultant on the *surface whose outward normal* is $\mathbf{n}$. For the avoidance of doubt, it **does not** imply that the direction of $\mathbf{t}^{(\mathbf{n})}$ is $\mathbf{n}$. We can write that,

$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} = \frac{d\mathbf{F}}{dS}$$

The direction of the vector $\mathbf{t}^{(\mathbf{n})}$, as obvious from the above equation, is obtained from that of the resultant force vector itself. The surface label $\mathbf{n}$, its outward drawn normal, is important because, another surface, at the same point will produce a different traction and it will be acting on that other surface.

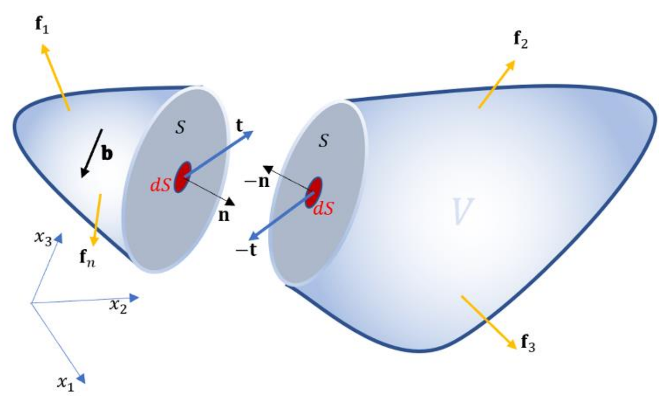


Note that in general, $\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{n})}(x_1, x_2, x_3)$ and $\mathbf{n} = \mathbf{n}(x_1, x_2, x_3)$ as the surface itself is not necessarily a plane. It is only as the limit is approached that $\mathbf{n}$ is a fixed direction for the elemental area and $\Delta\mathbf{F}$ and $\mathbf{t}^{(\mathbf{n})}$ are in the same direction. Furthermore, the body force per unit mass is $\varrho\mathbf{b}$. The density $\varrho = \varrho(x_1, x_2, x_3)$ as it varies over the whole body. If the resultant body force in the volume element ΔV is $\Delta\mathbf{B}$, we can compute the body force per unit volume

$$\mathbf{b} = \frac{1}{\varrho} \lim_{\Delta S \to 0} \frac{\Delta\mathbf{B}}{\Delta V} = \frac{1}{\varrho} \frac{d\mathbf{B}}{dV}.$$

-Euler-Cauchy Stress Principle

In figure 6.2, we have the body earlier depicted in 6.1 now cut in two through the location of the element we are looking at. The outward pointing normal at the same location in the cut body will face the opposite direction as shown.



The Euler–Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equipollent to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field

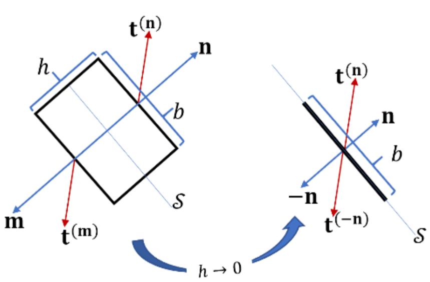
$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}(\mathbf{n}, \mathbf{x}, t)$$

to emphasize that the traction vector is a time dependent field that varies with the surface orientation. The fact that the traction vector, to be fully specified, requires the specification of two directions: The direction of the stress intensity vector, and the orientation of the surface on which the traction acts. This fact, that to be fully specified, stress requires not one, but two directions, has been used by several to define a tensor as possessing magnitude and two directions in contrast with vectors that have magnitude and one direction. (Ref).

Cauchy Reciprocal Theorem

When there is a traction on a surface, what is the relationship of that vector to what happens on the same surface, with the vector drawn in the opposite direction? That question is answered by the Cauchy reciprocal theorem as shown in this section:

Consider the cross section around the small surface element at point $\mathbf{P}$. Let the material element is chosen initially to have a height h around surface $\mathcal{S}$. We look at the equilibrium of this small element as the height h approaches zero so that the element is now at the surface as shown in figure (b).



$$\lim_{h\to 0}\left(\mathbf{t}^{(m)}A + \mathbf{t}^{(n)}A\right) = \mathbf{t}^{(-n)}A + \mathbf{t}^{(n)}A = 0$$

$$\Rightarrow \mathbf{t}^{(-n)} = -\mathbf{t}^{(n)}$$

as it is clear from the picture that $\mathbf{m} \to -\mathbf{n}$; $\mathbf{t}^{(m)} \to \mathbf{t}^{(-n)}$ as $h \to 0$. The stress vector on the surface whose normal is opposite to the surface normal is equal and opposite to the stress vector on the present surface. This is also sometimes called *Cauchy's fundamental lemma* or *Cauchy reciprocal theorem*.

Normal and Shearing stresses

The surface traction is a not a simple quantity. First it is the vector intensity of the vector force on the surface as the surface area approaches a limit in the acceptable process of continuum mechanics. It is defined for a specific surface with an orientation defined by the outward normal $\mathbf{n}$. This implies that the traction at a given point is dependent upon the orientation of the surface. It is a vector that has different values at the same point depending upon the orientation of the surface we are looking at.

Second, it is a function of the coordinate variables. It is a field in the 3D Euclidean Point Space. Therefore proper to write,

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}(\mathbf{n}, \mathbf{x}, t) \equiv \mathbf{t}^{(\mathbf{n})}(x_1, x_2, x_3, t)$$

to make these dependencies explicitly obvious. In general, $\mathbf{t}^{(\mathbf{n})}$ and $\mathbf{n}$ are not in the same direction; that is, there is an angular orientation between the resultant **stress *vector*** and the surface outward normal. Consequently, it is customary to express the stress vector as a vector sum of its projection $\mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)$ along the normal $\mathbf{n}$ and the shearing stress;

$$\mathbf{t}_s^{(\mathbf{n})} \equiv \mathbf{t}^{(\mathbf{n})} - \mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)$$

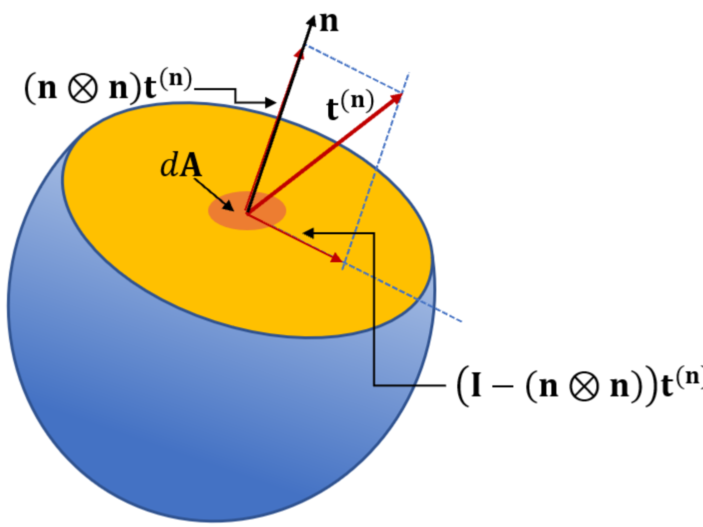
on the surface itself. Since $\mathbf{n}$ is a unit vector, $\|\mathbf{n}\| = 1$,

$$\mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right) = \left(\frac{1}{\|\mathbf{n}\|}\right)^2 (\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})} = (\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})}$$

$$\begin{aligned} \mathbf{t}_s^{(\mathbf{n})} &= \mathbf{t}^{(\mathbf{n})} - \mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right) \\ &= \mathbf{t}^{(\mathbf{n})} - (\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})} \\ &= (\mathbf{I} - \mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})} \end{aligned}$$

$\sigma = \left\|\mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)\right\| = \left\|(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})}\right\|$, and,

$\tau \equiv \left\|\mathbf{t}_s^{(\mathbf{n})}\right\| = \left\|\mathbf{t}^{(\mathbf{n})} - \mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)\right\|$



The normal and shearing components of the stress vector are called the *normal* and *shear tractions* respectively. Since their directions are known, is it customary to refer to their magnitudes alone.

Cauchy's Theorem

According to the *Cauchy Postulate*, the stress vector $\mathbf{t}^{(\mathbf{n})}$ remains unchanged for all surfaces passing through the point $\boldsymbol{P}$ and having the same normal vector $\boldsymbol{n}$ at P, *i.e.* having a common tangent at $\boldsymbol{P}$. This means that the stress vector is a function of the normal vector $\boldsymbol{n}$ only, and it is *not influenced by the curvature of the internal surfaces*.

The state of stress at a point in the body is then defined by all the stress vectors $\mathbf{t}^{(\mathbf{n})}$ associated with all planes (infinite in number) that pass through that point. However, this does not help in determining the stress on another surface oriented at a different normal at that point. In order to find this, we depend on Cauchy's theorem. The principle behind it stems from the fact that knowing the stress vectors on three mutually perpendicular planes, the stress vector on any other plane passing through that point can be found through coordinate transformation equations.

Cauchy's Stress Theorem states that:

> *Provided the stress vector $\boldsymbol{t}^{(\boldsymbol{n})}$ acting on a surface with outwardly drawn unit normal $\boldsymbol{n}$ is a continuous function of the coordinate variables, there exists a second-order tensor valued field $\boldsymbol{\sigma}(\boldsymbol{x})$, independent of $\boldsymbol{n}$, such that $\boldsymbol{t}^{(\boldsymbol{n})}$ is a linear function of $\boldsymbol{n}$ such that: $\boldsymbol{t}^{(\boldsymbol{n})} = \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{n}$*

The tensor $\boldsymbol{\sigma}$ in the above relationship, the tensor of proportionality, is called *Cauchy Stress Tensor*. It is also the "true stress" tensor for reasons that will become clear later. The standard proof of this important theorem examines the balance of forces on an arbitrarily small tetrahedron (Cauchy Tetrahedron) element with the base coinciding with the surface of interest and vertex located at the origin with the three sides coinciding with the Cartesian coordinate planes as shown in figure 6.3.

Proof:

To prove this expression, consider a tetrahedron with three faces oriented in the coordinate planes, and with an infinitesimal base area dA oriented in an arbitrary direction specified by a normal vector $\mathbf{n}$ (Figure 6.3). The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane $\mathbf{n}$. The stress vector on this plane is denoted by $\mathbf{t}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\mathbf{t}^{(\mathbf{e}_1)}, \mathbf{t}^{(\mathbf{e}_2)}$ and $\mathbf{t}^{(\mathbf{e}_3)}$ From equilibrium of forces, Newton's second law of motion, we have

$$\rho\left(\frac{h}{3}dA\right)\mathbf{a} = \mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_1)}dA_1 - \mathbf{t}^{(\mathbf{e}_2)}dA_2 - \mathbf{t}^{(\mathbf{e}_3)}dA_3 = \mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_i)}dA_i$$

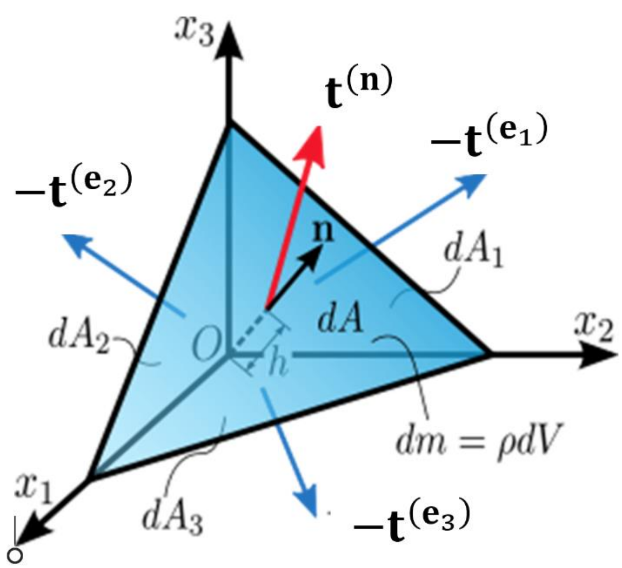


Figure 6.3. Balance of forces in a tetrahedron.(redraw)

where the left-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: ρ is the density, $\mathbf{a}$ is the acceleration, and h is the height of the tetrahedron, considering the plane $\mathbf{n}$ as the base. The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting dA into each face:

$$dA_i = (\mathbf{n} \cdot \mathbf{e}_i)dA = n_i dA$$

and then substituting into the equation to cancel out dA:

$$\mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_i)}n_i dA = \rho\left(\frac{h}{3}dA\right)\mathbf{a}$$

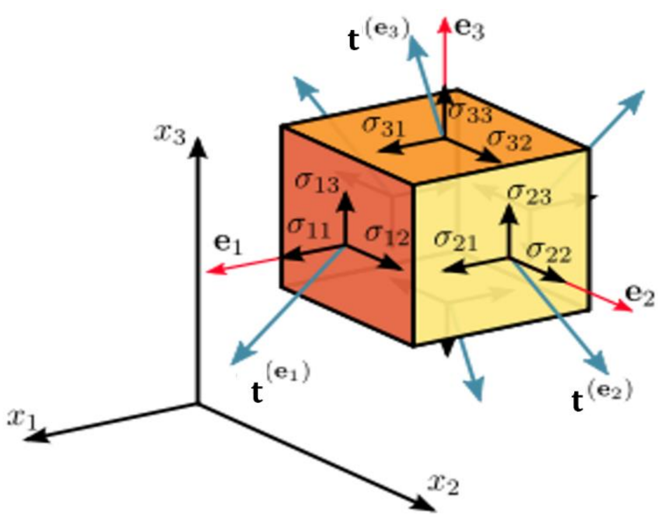
In the limiting case as the tetrahedron shrinks to a point, the height of the tetrahedron approaches zero $(h \to 0)$. As a result, the right-hand-side of the equation approaches 0, so the equation becomes,

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{e}_i)}n_i.$$

We are now to interpret the components $\mathbf{t}^{(\mathbf{e}_i)}$ in this equation. Consider $\mathbf{t}^{(\mathbf{e}_1)}$ the value of the resultant stress traction on the first coordinate plane. Resolving this along the coordinate axes, we have,

$$\begin{aligned}\mathbf{t}^{(\mathbf{e}_1)} &= \left[\mathbf{e}_1 \cdot \mathbf{t}^{(\mathbf{e}_1)}\right]\mathbf{e}_1 + \left[\mathbf{e}_2 \cdot \mathbf{t}^{(\mathbf{e}_1)}\right]\mathbf{e}_2 + \left[\mathbf{e}_3 \cdot \mathbf{t}^{(\mathbf{e}_1)}\right]\mathbf{e}_3 \\ &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ &= \sigma_{1j}\mathbf{e}_j\end{aligned}$$

Where the scalar quantity $\sigma_{1j}$is defined by the above equation as,



$$\sigma_{1j} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_1)}$$

, or in general, we write that,

$$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}, i = 1,2,3$$

. Figure 6.4 is a graphical depiction of this definition where we can see that $\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}$ is the scalar component of the stress vector on the i coordinate plane in the j direction. For any coordinate plane therefore, we may write, $\mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ij}\mathbf{e}_j$, so that the stress or traction vector on an arbitrary plane determined by its orientation in the outward normal $\mathbf{n}$,

$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{e}_i)} n_i = \sigma_{ij} \mathbf{e}_j n_i$

Which is another way of saying that the component of the vector $\mathbf{T}^{(\mathbf{n})}$ along the j coordinate direction is $\sigma_{ij} n_i$ which is the contraction, $\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} = \mathbf{t}^{(\mathbf{n})}$. This proves Cauchy Theorem. Obviously, σ_{ij} are the components of the stress tensor in the coordinate system of computation that we have used so far. The Cauchy law, being a vector equation remains valid in all coordinate systems. We will then have to compute the different values of the stress tensor in the system of choice when for any reason we choose to work in not-Cartesian coordinates.

The normal and shearing stresses can now be written in terms of the Cauchy stress tensor;

Normal stress,

$$\begin{aligned}(\mathbf{n} \otimes \mathbf{n})\mathbf{t}^{(\mathbf{n})} &= (\mathbf{n} \otimes \mathbf{n})\boldsymbol{\sigma}\mathbf{n} \\ &= (\mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n})\mathbf{n},\end{aligned}$$

and the shear stress,

$$\big(\mathbf{I} - (\mathbf{n} \otimes \mathbf{n})\big)\mathbf{t}^{(\mathbf{n})} = \big(\mathbf{I} - (\mathbf{n} \otimes \mathbf{n})\big)\boldsymbol{\sigma}\mathbf{n}.$$

The magnitude of the normal stress, $\sigma = \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n}.$

On the coordinate surfaces, using equation____, in terms of the stress tensor, we can see that,

Stress Component	$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}$
σ_{11}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma}\mathbf{e}_1$
σ_{12}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma}\mathbf{e}_1$
σ_{13}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma}\mathbf{e}_1$
σ_{21}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma}\mathbf{e}_2$
σ_{22}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma}\mathbf{e}_2$
σ_{23}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma}\mathbf{e}_2$
σ_{31}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma}\mathbf{e}_3$
σ_{32}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma}\mathbf{e}_3$
σ_{33}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma}\mathbf{e}_3$

as figure ___ shows,

On the coordinate surfaces, we can see that,

$$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)} = \mathbf{e}_j \cdot \boldsymbol{\sigma}\mathbf{e}_i$$

Here, we have adopted the convention that the first subscript refers to the normal to the coordinate plane, the second to the direction of the stress component. The opposite convention can also be adopted. By the time we examine the law of conservation of angular momentum, it will become clear that the Cauchy stress tensor is necessarily symmetrical, and therefore both conventions give the same values.

The arguments that proved Cauchy theorem could have been based on non-Cartesian coordinate systems. The stress equation must remain unchanged however and the stress tensor characterizing the state of stress at a point remains an invariant. As it is with any second-order tensor, its components in general coordinates will be obtained from,

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \sigma^{i}_{.j}\mathbf{g}_i \otimes \mathbf{g}^j = \sigma_j^{\ i}\mathbf{g}^j \otimes \mathbf{g}_i$$

Because the Cauchy stress is based on areas in the deformed body, it is a spatial field. Whenever it is more convenient to work in Lagrangian coordinates, a stress tensor based on this may become more appropriate. Several such stress tensors are in use. This is the subject of the next section.

Nominal Stress, True Stress

As stated earlier, Cauchy stress tensor is a spatial field. It is the stress tensor in the current configuration. It is the true stress because the traction computed from it is based on the currently measurable area.

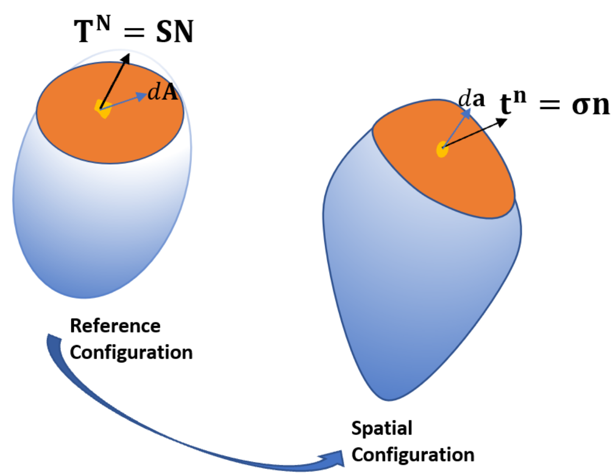
For any given element, a comparison can be made between the current area and the original area that transformed to it. From kinematics, we recall that the vector current area, $d\mathbf{a} = \mathbf{n}da$ in a deformed body

$$d\mathbf{a} = \mathbf{F}^{\mathrm{c}}d\mathbf{A}$$

where $d\mathbf{A} = \mathbf{N}dA$ is its image, in the material coordinates, and $\mathbf{F}^{\mathrm{c}}$ is the cofactor of the deformation gradient. The resultant force acting on an area bounded by ΔS in the deformed coordinates is the sum, $\int_{\Delta S}\mathbf{t}^{\mathbf{n}}da$ of the traction vectors, $\mathbf{t}^{\mathbf{n}}$ over the area; it can be obtained, using Cauchy stress theorem, $\mathbf{t}^{\mathbf{n}} = \boldsymbol{\sigma}\mathbf{n}$, so that,

$$d\mathbf{P} = \int_{\Delta S}\boldsymbol{\sigma}\mathbf{n}da = \int_{\Delta S}\boldsymbol{\sigma}d\mathbf{a}$$

$$= \int_{\Delta S_0}\boldsymbol{\sigma}\mathbf{F}^{\mathrm{c}}d\mathbf{A} = \int_{\Delta S_0}\mathbf{S}\mathbf{N}dA = \int_{\Delta S_0}\mathbf{S}d\mathbf{A}$$



where $\mathbf{S} \equiv \boldsymbol{\sigma}\mathbf{F}^{\mathrm{c}} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$ is called the *First Piola-Kirchhoff Stress* Tensor. Its transpose,

$$\mathbf{S}^{\mathrm{T}} \equiv J\mathbf{F}^{-1}\boldsymbol{\sigma}^{\mathrm{T}}$$

is the stress tensor from the material viewpoint. It is called the nominal stress. Tractions based on this tensor are measured with respect to areas in the material configuration. Symmetry of the Cauchy stress tensor (to be established further on) implies that $\mathbf{S}^{\mathrm{T}} \equiv J\mathbf{F}^{-1}\boldsymbol{\sigma}^{\mathrm{T}} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$

This transformation by the cofactor tensor, applied to $\boldsymbol{\sigma}$ to produce $\mathbf{S}$, when applied as in this or any other case to any tensor is called a *Piola transformation*. Whenever,

$$\mathbf{A} = \mathbf{B}\mathbf{F}^{c} = J\mathbf{B}\mathbf{F}^{-\mathrm{T}}$$

$\mathbf{A}$ is said to be the Piola Transformation of $\mathbf{B}$. In the above expression for nominal stress, we have used the yet-to be proved fact that the Cauchy stress tensor is symmetric. This will be established later. The components of the $\mathbf{S}$ stress are the forces acting on the deformed configuration, per unit undeformed area. They are thought of as acting on the undeformed solid.

Recall that the ratio of elemental volumes in the spatial to material, coordinates

$$\frac{dv}{dV} = J = \det \mathbf{F}$$

where $\mathbf{F}$ is the deformation gradient of the transformation. It therefore follows that the Kirchhoff stress $\boldsymbol{\tau}$, defined by

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}$$

is no different from Cauchy Stress Tensor during isochoric (or volume-preserving) deformations and motions. It is used widely in numerical algorithms in metal plasticity (where there is no change in volume during plastic deformation).

While the Cauchy Stress tensor is symmetric, neither the Piola Kirchoff tensor, nor the norminal stress tensor are. A second tensor, a material stress tensor, is the symmetric Piola-Kirchhoff Stress, $\boldsymbol{\Xi}$ is useful in Conjugate stress analysis. It is defined by,

$$\mathbf{S} \equiv J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \mathbf{F}\boldsymbol{\Xi}$$

In terms of the Cauchy stress $\boldsymbol{\sigma}$ we can write,

$$\boldsymbol{\Xi} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}.$$

This second Piola-Kirchhoff stress is, just like Cauchy stress is symmetric for,

$$\boldsymbol{\Xi}^{\mathrm{T}} = (J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}})^{\mathrm{T}} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \boldsymbol{\Xi}.$$

Stress Invariants & Decompositions

Principal Invariants

As is the case with any second-order tensor, Cauchy Stress Tensor has associated scalar-valued functions called invariants because their values are dependent only on the stress fields alone and independent of coordinate systems chosen to represent them. From equations ____, we easily see that, for a set of arbitrarily selected linearly independent vectors, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$,

$$\begin{aligned}
I_1(\boldsymbol{\sigma}) &= \operatorname{tr} \boldsymbol{\sigma} = \sigma_{ii} \\
&= \frac{[\boldsymbol{\sigma}\mathbf{a}, \mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \mathbf{b}, \boldsymbol{\sigma}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} \\
&= \sigma_{11} + \sigma_{22} + \sigma_{33} \\
I_2(\boldsymbol{\sigma}) &= \operatorname{tr} \boldsymbol{\sigma}^{\mathrm{c}} \\
&= \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}\right) \\
&= \sigma_{11}\sigma_{22} - \sigma_{21}\sigma_{12} + \sigma_{22}\sigma_{33} - \sigma_{32}\sigma_{23} + \sigma_{11}\sigma_{33} - \sigma_{31}\sigma_{13} \\
&= \frac{[\boldsymbol{\sigma}\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \mathbf{c}] + [\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \boldsymbol{\sigma}\mathbf{c}] + [\boldsymbol{\sigma}\mathbf{a}, \mathbf{b}, \boldsymbol{\sigma}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \text{and} \\
I_3(\boldsymbol{\sigma}) &= \det \boldsymbol{\sigma} \\
&= e_{ijk}\sigma_{i1}\sigma_{j2}\sigma_{k3} \\
&= \frac{[\boldsymbol{\sigma}\mathbf{a}, \boldsymbol{\sigma}\mathbf{b}, \boldsymbol{\sigma}\mathbf{c}]}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}
\end{aligned}$$

Additive Decompositions

Again, as is the case with any second order tensor, the stress tensor admits two additive decompositions. It can be broken into its symmetric and skew parts:

$$\begin{aligned}
\boldsymbol{\sigma} &= \frac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{T}}) + \frac{1}{2}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^{\mathrm{T}}) \\
&= \operatorname{sym} \boldsymbol{\sigma} + \operatorname{skw} \boldsymbol{\sigma}
\end{aligned}$$

A more important additive decomposition is the separation into spherical and deviatoric parts. Let

$$s \equiv \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})$$

The spherical and deviatoric parts of $\boldsymbol{\sigma}$ are:

$$\boldsymbol{\sigma} = \operatorname{sph} \boldsymbol{\sigma} + \operatorname{dev} \boldsymbol{\sigma}$$

where $\operatorname{sph} \boldsymbol{\sigma} = s\mathbf{I}$, **and** $\operatorname{dev} \boldsymbol{\sigma} = \boldsymbol{\sigma} - s\mathbf{I}$**.** The second principal invariant of the deviatoric stress tensor,

$$I_2(\operatorname{dev} \boldsymbol{\sigma})$$

Plays a crucial role in the design of metallic elements and is usually computed in the various design software as the Von-Mises or Equivalent Stress.

In a static fluid, shear stresses vanish; and with it, the deviatoric part of the stress. Consequently, for a fluid at rest,

$$\boldsymbol{\sigma} = \operatorname{sph} \boldsymbol{\sigma} = \frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})\mathbf{I} = -p\mathbf{I}$$

so that the pressure is the negative of the third of trace of the stress tensor,

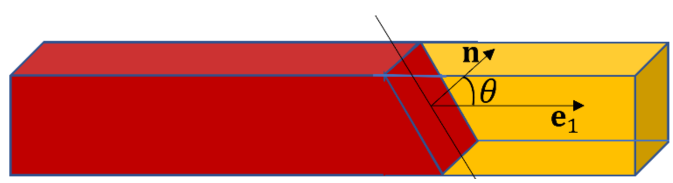
$$p = -\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma}).$$

States of Stress Examples

The law of Cauchy states that the state of stress at any point in a body is completely determined by the Cauchy Stress tensor. We therefore only need to specify the surface of interest once this tensor is known. Equation___ provides the stress at any given surface as the operation of the tensor on the unit normal to the surface. In this section, we see examples of stress tensors for simple states of stress. Uniaxial, biaxial and triaxial stresses, pure shear as well as hydrostatic stress. Each will be computed by the product of Cauchy stress on the surface unit normal.

Uniaxial Stress

Given any vector $\mathbf{v}$ and a scalar stress intensity σ, the Cauchy stress tensor field for uniaxial stress in the same direction as $\mathbf{v}$ is given by:



$$\boldsymbol{\sigma}(\mathbf{x}) = \left(\frac{1}{\|\mathbf{v}\|}\right)^2 \sigma(\mathbf{v} \otimes \mathbf{v})$$

Uniaxial stress of intensity σ in direction of the unit vector, $\mathbf{e}_1$, is therefore given by the stress tensor field,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1).$$

In figure___, a bar is aligned to direction $\mathbf{e}_1$ as shown. Consider a surface with unit outward drawn normal $\mathbf{n}$ at an angle θ as shown. The resultant traction on this surface can be found, using equation ___:

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \boldsymbol{\sigma}\mathbf{n} = \sigma(\mathbf{e}_1 \otimes \mathbf{e}_1)\mathbf{n} \\ &= \sigma\mathbf{e}_1(\mathbf{e}_1 \cdot \mathbf{n}) \\ &= (\sigma\cos\theta)\mathbf{e}_1\end{aligned}$$

From the last expression, uniaxial stress tensor creates traction in only one direction no matter the orientation of the plane on which it acts. When the plane is oriented at right angles, $\theta = \frac{\pi}{2}$, to the direction of uniaxial stress, the surface traction is

$$\mathbf{t}^{(\mathbf{n}_{\theta=\pi/2})} = \left(\sigma\cos\frac{\pi}{2}\right)\mathbf{e}_1 = \mathbf{o}.$$

On a surface with normal in the direction of uniaxial stress, $\theta = 0$, and the surface traction,

$$\mathbf{t}^{(\mathbf{n}_{\theta=0})} = \sigma\mathbf{e}_1.$$

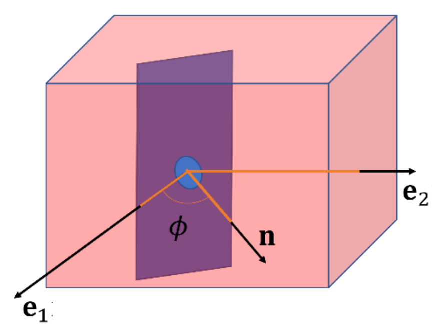
For all other surfaces, it is a non-zero vector of magnitude less than σ and always in the same direction $\mathbf{e}_1$.

Biaxial Stress

Given two perpendicular unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, the stress tensor field,

$\boldsymbol{\sigma}(\mathbf{x}) = \sigma_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3\mathbf{e}_1 \otimes \mathbf{e}_2 + \sigma_4\mathbf{e}_2 \otimes \mathbf{e}_1$

is a bi-axial stress tensor. The traction on an arbitrary plane oriented with a unit normal, outwardly drawn, $\mathbf{n}$



$$\begin{aligned}\mathbf{t}^{(\boldsymbol{n})} &= \boldsymbol{\sigma}(\mathbf{x})\mathbf{n} = (\sigma_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3\mathbf{e}_1 \otimes \mathbf{e}_2 + \sigma_4\mathbf{e}_2 \otimes \mathbf{e}_1)\mathbf{n} \\ &= \sigma_1\mathbf{e}_1(\mathbf{n}\cdot\mathbf{e}_1) + \sigma_2\mathbf{e}_2(\mathbf{n}\cdot\mathbf{e}_2) + \sigma_3\mathbf{e}_1(\mathbf{n}\cdot\mathbf{e}_2) + \sigma_4\mathbf{e}_2(\mathbf{n}\cdot\mathbf{e}_1) \\ &= \sigma_1\mathbf{e}_1\cos\phi + \sigma_2\mathbf{e}_2\sin\phi + \sigma_3\mathbf{e}_1\sin\phi + \sigma_4\mathbf{e}_2\cos\phi\end{aligned}$$

If the eigenvalues (principal stresses) of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{s_1, s_2, 0\}$. $\boldsymbol{\sigma}(\mathbf{x})$ in this case has the spectral form,

$$\boldsymbol{\sigma}(\mathbf{x}) = s_1\mathbf{u}_1 \otimes \mathbf{u}_1 + s_2\mathbf{u}_2 \otimes \mathbf{u}_2$$

where $\mathbf{u}_1$and $\mathbf{u}_2$ are the corresponding eigenvectors (principal planes)

Triaxial Stress

The most general stress tensor in a triaxial field will be in the form,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$$

By the balance of angular momentum, to be shown in the next chapter, only six of them will be independent. In spectral form (using principal directions), the number of independent terms becomes 3. If $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$ are also principal directions, then,

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma_1\mathbf{e}_1 \otimes \mathbf{e}_1 + \sigma_2\mathbf{e}_2 \otimes \mathbf{e}_2 + \sigma_3\mathbf{e}_3 \otimes \mathbf{e}_3$$

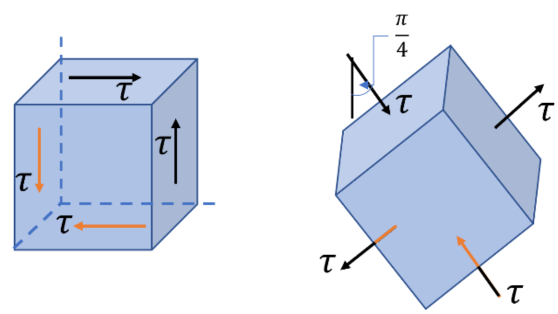
is a triaxial stress field. σ_1, σ_2 and σ_3 are, as we shall soon see, the principal stresses or eigenvalues of the stress tensor.

Pure Shear

The stress tensor field,

$$\boldsymbol{\sigma}(\mathbf{x}) = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)$$

is a state of pure shear of intensity τ.



The traction on an arbitrary plane $\mathbf{n}$

$$\begin{aligned}\mathbf{t}^{(\boldsymbol{n})} &= \boldsymbol{\sigma}(\mathbf{x})\boldsymbol{n} = \tau(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1)\mathbf{n} \\ &= \tau\mathbf{e}_1(\mathbf{n}\cdot\mathbf{e}_2) + \tau\mathbf{e}_2(\mathbf{n}\cdot\mathbf{e}_1) \\ &= \tau\mathbf{e}_1\sin\phi + \tau\mathbf{e}_2\cos\phi\end{aligned}$$

At the $\mathbf{e}_1$ **plane (that is, perpendicular to** $\mathbf{e}_1$) $\phi = 0$.

$$\mathbf{t}^{(\mathbf{n})} = \tau(\mathbf{e}_1\sin 0 + \tau\mathbf{e}_2\cos 0) = \tau\mathbf{e}_2$$

Similarly, at $\phi = \frac{\pi}{2}$,

$$\mathbf{t}^{(\mathbf{n})} = \tau\left(\mathbf{e}_1\sin\frac{\pi}{2} + \tau\mathbf{e}_2\cos\frac{\pi}{2}\right) = \tau\mathbf{e}_1$$

Showing that the resultant stresses on the coordinate planes are shear of the same intensity as τ. This situation is depicted by the diagram in figure___ above.

When $\phi = \frac{\pi}{4}$,

$$\mathbf{t}^{(\boldsymbol{n})} = \tau\left(\frac{\mathbf{e}_1}{\sqrt{2}} + \frac{\mathbf{e}_2}{\sqrt{2}}\right) = \tau\mathbf{n}$$

where n is the surface with normal at 45^o to $\mathbf{e}_1$ axis. When $\phi = \frac{3\pi}{4}$,

$$\mathbf{t}^{(\mathbf{n})} = \tau\left(\frac{\mathbf{e}_1}{\sqrt{2}} - \frac{\mathbf{e}_2}{\sqrt{2}}\right) = -\tau\mathbf{m}$$

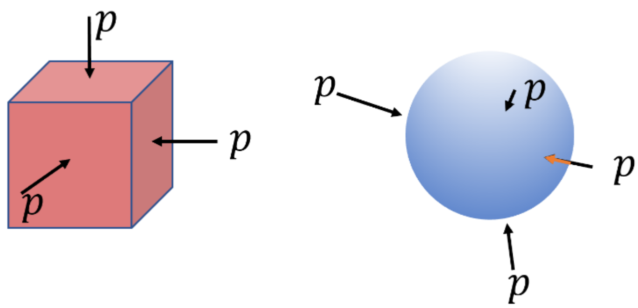
where $\mathbf{n}$ is the surface normal at 135^o to $\mathbf{e}_1$ axis. The tensile and compressive stresses on these planes are shown in figure ___b above.

The eigenvalues of $\boldsymbol{\sigma}(\mathbf{x})$ are $\{\tau, -\tau, 0\}$. Furthermore, $\boldsymbol{\sigma}$ in this case has the spectral form,

$$\boldsymbol{\sigma}(x) = \tau\mathbf{u}_1 \otimes \mathbf{u}_1 - \tau\mathbf{u}_2 \otimes \mathbf{u}_2$$

Hydrostatic Pressure

It is customary to characterize the state of stress called “hydrostatic pressure” in terms of a cuboid with coordinate compressive normal stresses. This way of description only tells part of the story. While it is true that hydrostatic state of stress is usually compressive, it is possible to create a tensile case.



More important is the fact that hydrostatic pressure is a state of stress that creates the same traction at every surface, no matter what the orientation of the surface is, at that point. The spherical stress tensor,

$$\boldsymbol{\sigma}(\mathbf{x}) = -p\mathbf{I}$$

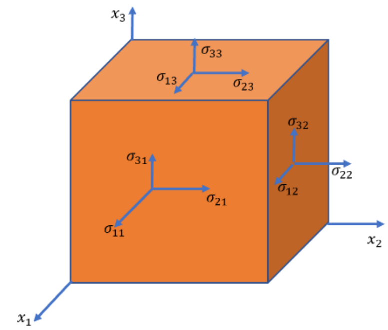
Is hydrostatic pressure of intensity p. In fact, every spherical stress tensor creates hydrostatic pressure. The traction $\mathbf{t}^{(\mathbf{n})}$ on a surface with unit outward normal $\mathbf{n}$ is

$$\begin{aligned}\mathbf{t}^{(\mathbf{n})} &= \boldsymbol{\sigma}(\mathbf{x})\mathbf{n} \\ &= -p\mathbf{I}\mathbf{n} \\ &= -p\mathbf{n}.\end{aligned}$$

This always produces a traction, normal to the surface of the same magnitude, no matter the orientation of the surface. For this reason, the stress intensity of a fluid at rest is independent of the orientation of the surface at that point. While it may vary from point to point, it is constant for every surface orientation at a given point.

Principal Stresses and Principal Planes.

Principal stresses are the eigenvalues of the Cauchy Stress Tensor. The surfaces in which they act, are the eigenvectors. We know from Cauchy's stress law that the components of the stress tensor can be depicted as shown in the figure below.



We also know that by suitable rotations, we are able to transform the components of the stress tensor into other orthonormal systems of coordinates using rotation tensors.

The Stress Eigenvalue Problem

The eigenvalue problem for the stress tensor is simply this:

Can we find suitable rotations such that the only stress components we have to deal with are the normal stresses? If so, what will those normal stresses be? What will those directions be?

Given any tensor $\boldsymbol{\sigma}$ and a vector $\mathbf{n}$, the product $\boldsymbol{\sigma}\mathbf{n}$ is obviously a vector. When can we have the new vector to be such that,

$$\boldsymbol{\sigma}\mathbf{n} = \alpha\mathbf{n}$$

Or, expressed in component form, $\sigma_{ij}n_j = \alpha n_i$.

This will happen only when we can solve the equations,

$$|\sigma_{ij} - \alpha\delta_{ij}| = 0$$

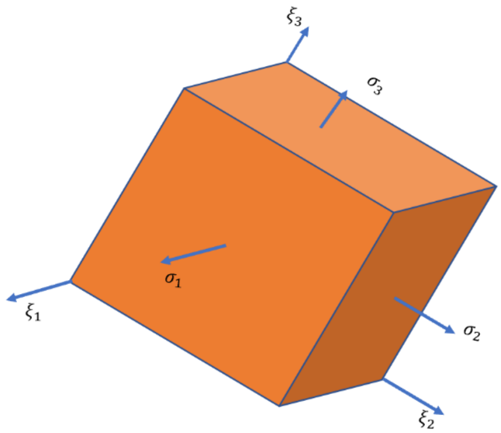
Opening the determinant gives us the equation,

$$-\alpha^3 + I_1\alpha^2 - I_2\alpha + I_3 = 0$$

This is the characteristic equation of the stress tensor. A most important equation in mechanical design as we shall see.

The three solutions to this equation are the principal stresses, σ_1, σ_2 and σ_3. The principal directions can be obtained by substituting each back in the equation, $\sigma_{ij}n_j = \alpha n_i$ and solving for the three vector directions in each case.

I_1, I_2, and I_3 are called the **Principal Invariants** of the stress tensor. They are extremely important scalars in design.



We obtain the stresses and the directions as shown. As usual, we can rotate to this new set of coordinates after the transforming rotation tensor

Given any stress tensor, you can use Mathematica to obtain the eigenvalues and principal directions as the following examples show.

Examples have been movedReword above

Plane Rotations

Let the coordinate system be rotated to point in the direction of the polar coordinates shown so that the normal stresses are now $\sigma_r$and σ_θ, while the shear stresses are $\tau_{r\theta}$. The unit vectors are rotated from $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ such that,

$$\mathbf{e}_r = \cos\theta \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2 \mathbf{e}_\theta = -\sin\theta \mathbf{e}_1 + \cos\theta\, \mathbf{e}_2 \mathbf{e}_z = \mathbf{e}_3$$

For this transformation of coordinates, the rotation tensor is, $\mathbf{e}_1 \otimes \mathbf{e}_r + \mathbf{e}_2 \otimes \mathbf{e}_\theta + \mathbf{e}_3 \otimes \mathbf{e}_z$. Note that this is the transpose of the tensor for vector transformation.

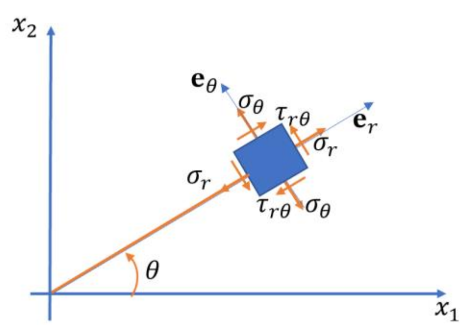
Let the Cartesian unit vectors be $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Let $\sigma_x, \sigma_y$and σ_z be the normal stresses and τ_{xy} the only non-vanishing shear stress.

The tensor components after the rotation to plane polar system shown given below is derived from the following Mathematica Notebook:

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$



```
CauchyStr = {{σx, τxy, 0}, {τxy, σy, 0}, {0, 0, σz}};
e1 = {1, 0, 0};
e2 = {0, 1, 0};
e3 = {0, 0, 1};
er[α_] := Cos[α] e1 + Sin[α] e2;
eθ[α_] := -Sin[α] e1 + Cos[α] e2;
ez[α_] := e3;
Rot[α_] := TensorProduct[e1, er[α]] + TensorProduct[e2, eθ[α]] + TensorProduct[e3, ez[α]]
Rot[θ]

{{Cos[θ], Sin[θ], 0}, {-Sin[θ], Cos[θ], 0}, {0, 0, 1}}

MatrixForm[%]
```

xForm=

$$\begin{pmatrix} \text{Cos}[\theta] & \text{Sin}[\theta] & 0 \\ -\text{Sin}[\theta] & \text{Cos}[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
RotStr[θ] = Rot[θ].CauchyStr.Transpose[Rot[θ]]

{{Sin[θ] (Sin[θ] σy + Cos[θ] τxy) + Cos[θ] (Cos[θ] σx + Sin[θ] τxy),
  Cos[θ] (Sin[θ] σy + Cos[θ] τxy) - Sin[θ] (Cos[θ] σx + Sin[θ] τxy), 0},
 {Cos[θ] (-Sin[θ] σx + Cos[θ] τxy) + Sin[θ] (Cos[θ] σy - Sin[θ] τxy),
  -Sin[θ] (-Sin[θ] σx + Cos[θ] τxy) + Cos[θ] (Cos[θ] σy - Sin[θ] τxy), 0}, {0, 0, σz}}

MatrixForm[Simplify[%]]
```

xForm=

$$\begin{pmatrix} \cos^2(\theta)\,\sigma_x + \sin(2\,\theta)\,\tau_{xy} + \sin^2(\theta)\,\sigma_y & \frac{1}{2}\left(-\sin(2\,\theta)\,\sigma_x + 2\cos(2\,\theta)\,\tau_{xy} + \sin(2\,\theta)\,\sigma_y\right) & 0 \\ \frac{1}{2}\left(-\sin(2\,\theta)\,\sigma_x + 2\cos(2\,\theta)\,\tau_{xy} + \sin(2\,\theta)\,\sigma_y\right) & \sin^2(\theta)\,\sigma_x + \cos(\theta)\left(\cos(\theta)\,\sigma_y - 2\sin(\theta)\,\tau_{xy}\right) & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}$$

Mohr Stress Circle: 2D

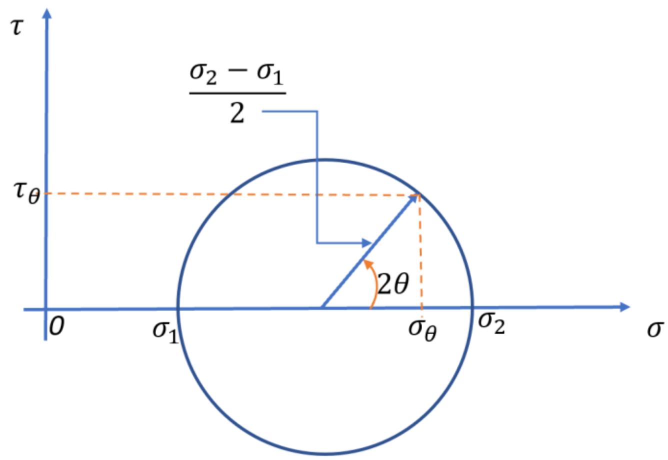
If the coordinates of initial reference were to have been principal coordinates such that $\sigma_1 = \sigma_x < \sigma_2 = \sigma_y$, the rotated equations take an especially simple form,

$$\sigma_r = \frac{\sigma_2 + \sigma_1}{2} - \frac{\sigma_2 - \sigma_1}{2}\cos 2\theta$$

$$\sigma_\theta = \frac{\sigma_2 + \sigma_1}{2} + \frac{\sigma_2 - \sigma_1}{2}\cos 2\theta = a + b\cos\alpha$$

$$\tau_{r\theta} = \frac{\sigma_2 - \sigma_1}{2}\sin 2\theta = b\sin\alpha$$

The parametric representation of a circle radius b, the difference between the principal stresses whose center, a, is at the average of principal stresses. Showing that the normal and shear stresses at an angle θ is a point at angle 2θ on the circle.



To compute the state of stress and obtain the tractions at a given surface or plane, the Mohr stress circle is not so useful in the view of modern computational tools. However, it remains a veritable visualization tool. The 3-D equivalent of this tool can be obtained in the following way:

Mohr Circle: 3D

To find a similar graphical representation for the 3-D situation, given three principal stresses, $\sigma_1 \le \sigma_2 \le \sigma_3$, Observe that, the normal stress at any orientation given by the unit normal, $\mathbf{n}$ is, $\sigma = \mathbf{n} \cdot \boldsymbol{\sigma}\mathbf{n}$, so that,

$$\sigma = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_2 n_3^2$$

The stress vector on the plane, by Cauchy stress law, $\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}\mathbf{n}.$ The square of its magnitude is the square of the normal stress magnitude, σ^2 plus the square of the shear stress, τ^2so that,

$$\sigma^2 + \tau^2 = \sigma_1^2 n_1^2 + \sigma_2^2 n_2^2 + \sigma_2^2 n_3^2$$

And finally observe that, the magnitude of the unit vector $\mathbf{n}$ implies,

$$n_1^2 + n_2^2 + n_3^2 = 1$$

Solving these as three equations for n_1^2, n_2^2 and n_3^2, using the following Mathematica code:

```
y =
 Simplify[
  Solve[{-σ + α1 σ1 + α2 σ2 + α3 σ3 == 0,
    -σ^2 - τ^2 + α1 σ1^2 + α2 σ2^2 + α3 σ3^2 == 0, α1 + α2 + α3 == 1},
   {α1, α2, α3}]]
```

$$\left\{\left\{\alpha_1 \to \frac{\sigma^2 + \tau^2 - \sigma\,\sigma_3 + \sigma_2\,(-\sigma + \sigma_3)}{(\sigma_1 - \sigma_2)\,(\sigma_1 - \sigma_3)},\right.\right.$$
$$\left.\left.\alpha_2 \to \frac{-\sigma^2 - \tau^2 + \sigma_1\,(\sigma - \sigma_3) + \sigma\,\sigma_3}{(\sigma_1 - \sigma_2)\,(\sigma_2 - \sigma_3)},\ \alpha_3 \to -\frac{\sigma^2 + \tau^2 - \sigma\,\sigma_2 + \sigma_1\,(-\sigma + \sigma_2)}{(\sigma_1 - \sigma_3)\,(-\sigma_2 + \sigma_3)}\right\}\right\}$$

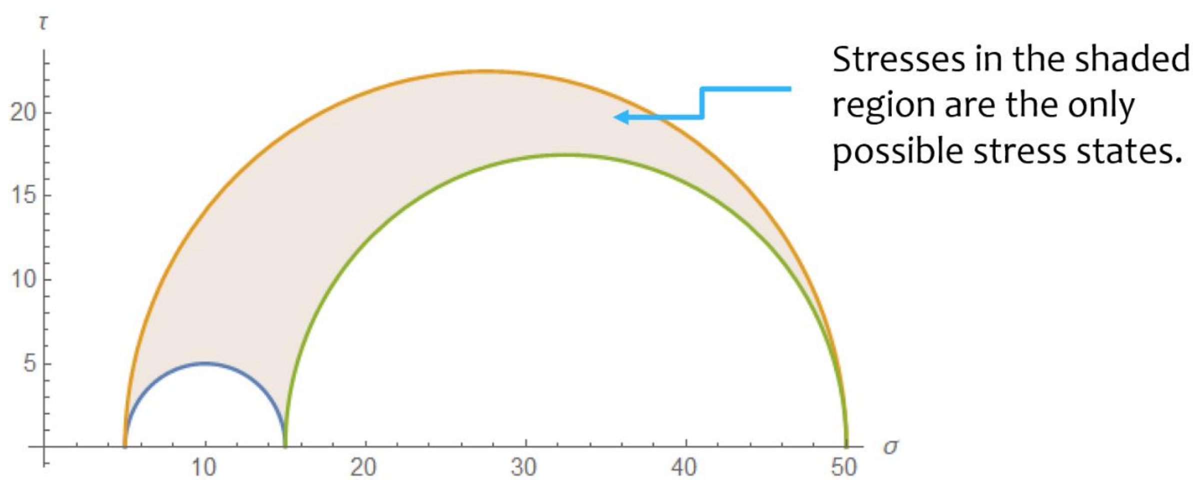
We represented the squares of the magnitudes by $\alpha_1 = n_1^2, \alpha_2 = n_2^2$ and $\alpha_3 = n_3^2$, The numerator of these can be plotted as equations in the τ, σ coordinates by the code below, Where we have used the values of $\sigma_1 = 2, \sigma_2 = 15,$ and $\sigma_3 = 50$

$$\left\{\left\{\alpha_1 \to \frac{\sigma^2 + \tau^2 - \sigma\,\sigma_3 + \sigma_2\,(-\sigma + \sigma_3)}{(\sigma_1 - \sigma_2)\,(\sigma_1 - \sigma_3)},\right.\right.$$
$$\left.\left.\alpha_2 \to \frac{-\sigma^2 - \tau^2 + \sigma_1\,(\sigma - \sigma_3) + \sigma\,\sigma_3}{(\sigma_1 - \sigma_2)\,(\sigma_2 - \sigma_3)},\ \alpha_3 \to -\frac{\sigma^2 + \tau^2 - \sigma\,\sigma_2 + \sigma_1\,(-\sigma + \sigma_2)}{(\sigma_1 - \sigma_3)\,(-\sigma_2 + \sigma_3)}\right\}\right\}$$

```
{{y1, y2, y3}} = y /. {σ1 → 50, σ2 → 15, σ3 → 5}
```

$$\left\{\left\{\alpha_1 \to \frac{15\,(5 - \sigma) - 5\,\sigma + \sigma^2 + \tau^2}{1575},\ \alpha_2 \to \frac{1}{350}\left(50\,(-5 + \sigma) + 5\,\sigma - \sigma^2 - \tau^2\right),\right.\right.$$
$$\left.\left.\alpha_3 \to \frac{1}{450}\left(50\,(15 - \sigma) - 15\,\sigma + \sigma^2 + \tau^2\right)\right\}\right\}$$

```
y1[σ_] := -15 (5 - σ) + 5 σ - σ^2;
y3[σ_] := -50 (15 - σ) + 15 σ - σ^2;
y2[σ_] := 50 (σ - 5) + 5 σ - σ^2;
Plot[{Sqrt[y1[σ]], Sqrt[y2[σ]], Sqrt[y3[σ]]}, {σ, 0, 50},
 AspectRatio → 1 / 2, Filling → {{1 -> {2}}, {3 → {2}}},
 FillingStyle → LightBrown, AxesLabel -> {σ, τ}]
```



Heat Fluxes

The rate at which heat flows through a surface is quantified by Fourier0Stokes law of Heat Fluxes. This is the fundamental law in heat flow. Cauchy's postulated the existence of a stress tensor on the basis of which the load intensity arising from mechanical forces (body and surface forces) can be elegantly quantified in a consistent manner.

The counterpart of this for thermal exchanges with the surroundings is the Fourier-Stokes heat flux theorem. In this section, after stating this law, we shall examine its implications for material (reference) configuration and the transformation of the heat flux from spatial (current_configuration.

Consider a spatial volume $\mathscr{B}_t$ with boundary $\partial\mathscr{B}_t$. Let the outwardly drawn normal to the surface be the unit vector $\mathbf{n}$. Fourier Stokes heat Flux Principle states that $\exists \mathbf{q}(\mathbf{x}, t)$ – vector field such that, the heat flow out of the volume is

$$h(\mathbf{x}, t, \boldsymbol{\partial}\mathscr{B}_t) = h(\mathbf{x}, t, \mathbf{n}) = -\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}$$

$\mathbf{q}(\mathbf{x}, t)$ is called the heat flux through the surface.

Heat flow into the spatial volume $\mathscr{B}_t$ volume is

$$\int_{\partial\mathscr{B}_t} \mathbf{q}(\mathbf{x}, t) \cdot \mathbf{n}\, da = \int_{\partial\mathscr{B}} J\mathbf{q}(\mathbf{x}, t) \cdot \mathbf{F}^{-\mathrm{T}}\mathbf{N}\, dA = \int_{\partial\mathscr{B}} J\mathbf{F}^{-1}\mathbf{q}(\mathbf{x}, t) \cdot d\mathbf{A} = \int_{\partial\mathscr{B}} \mathbf{Q} \cdot d\mathbf{A}$$

$\mathbf{Q}(\mathbf{X}, t)$is a Piola transformation of the spatial heat flux. That is,

$$\mathbf{Q}(\mathbf{X}, t) = J\mathbf{F}^{-1}\mathbf{q}(\mathbf{x}, t)$$

Hence to obtain the material heat flux from the spatial heat flux, we do a Piola transformation on the latter.

Piola Transformation

Recall that, in order to obtain the Piola-Kirchhoff's stress from Cauchy Stress earlier, we also took the latter through the same Piola transformation.

Begin with Cauchy stress law,

$$\boldsymbol{\sigma}\mathbf{n} = \mathbf{t}^{\mathbf{n}}$$

we see that the Cauchy stress tensor is a spatial tensor because it transforms a spatial vector (in this case, the normal to a spatial area) to a spatial vector – the traction in spatial configuration. Recall, in the chapter on Kinematics, we saw the following tensors: Deformation gradient, $\mathbf{F}$, its transpose, $\mathbf{F}^{\mathrm{T}}$, inverse $\mathbf{F}^{-1}$,and cofactor, $\mathbf{F}^{\mathrm{c}}$,the left $\mathbf{V}$,and right $\mathbf{U}$, stretch tensors, as well as the various strain tensors. By the same consideration, we have the following table:

Tensor	Typical Transformation	Type
Deformation gradient, $\mathbf{F}$	$d\mathbf{x} = \mathbf{F}d\mathbf{X}$	Material to Spatial

Transpose of Deformation gradient, $\mathbf{F}^{\mathrm{T}}$		Spatial to material
Inverse of Deformation gradient, $\mathbf{F}^{-1}$	$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$	Spatial to Material
Cofactor of Deformation gradient, $\mathbf{F}^{\mathrm{c}}$	$d\mathbf{a} = \mathbf{F}^{\mathrm{c}}d\mathbf{X}$	Material to Spatial
Rotation Tensor, $\mathbf{R}$		Material to Spatial
Transpose of Rotation Tensor, $\mathbf{R}^{\mathrm{T}}$		Spatial to Material
Left Stretch Tensor, $\mathbf{V}$		Spatial
Right Stretch Tensor, $\mathbf{V}$		Material
Rotation Tensor, $\mathbf{R}$		Material to Spatial
Lagrange Strain Tensor, $\mathbf{E}$	$\mathbf{E} = \frac{1}{2}(\mathbf{U}^2 - \mathbf{I})$	Material
Eulerian Strain Tensor, $\mathbf{e}$	$\mathbf{E} = \frac{1}{2}(\mathbf{I} - \mathbf{V}^{-2})$	Spatial

The list is not exhaustive. From the above we can see that some tensors transform across the configurations while others transform within the configuration. We therefore have material tensors, spatial tensors and two-toe tensors that transform across configurations. In this latter group, we observe that some transform one way (for example material to spatial) while the other may reverse the transformation as can be seen in the pair of the deformation gradient and its inverse; or the rotation tensor and its transpose.

In this light, we see that, while Cauchy stress is a spatial tensor, the first Piola Kirchhoff stress is a two-toe tensor, transforming from material to spatial. To see this, observe that, given an elemental material area, $d\mathbf{A}$,

$$\begin{aligned}\mathbf{S}d\mathbf{A} &= \boldsymbol{\sigma}\mathbf{F}^{\mathrm{c}}d\mathbf{A}\\ &= \boldsymbol{\sigma}d\mathbf{a} = \boldsymbol{\sigma}\mathbf{n}da\\ &= \mathbf{t}^{\mathbf{n}}da\end{aligned}$$

the last expression being the spatial traction on the elemental spatial area. The tensor basis of the first Piola Kirchhoff tensor is therefore from Spatial to Material.

For a spatial tensor $\mathbf{A}$, Piola transformation, $\mathbf{A}\mathbf{F}^{\mathrm{c}}$ creates a two-toe tensor from material to spatial configurations.

For a material vector, $d\mathbf{X}$, Piola Transformation, $\mathbf{F}^{\mathrm{c}\mathrm{T}}d\mathbf{X} = J\mathbf{F}^{-1}d\mathbf{X}$ creates a spatial vector.

Examples

5.1 Consider two tractions $\mathbf{t}^{(\mathbf{n})}$ and $\mathbf{t}^{(\bar{\mathbf{n}})}$ on planes with normal vectors $\mathbf{n}$ and $\bar{\mathbf{n}}$ respectively. Show that

$$\bar{\mathbf{n}} \cdot \mathbf{t}^{(\mathbf{n})} = \mathbf{n} \cdot \mathbf{t}^{(\bar{\mathbf{n}})}$$

if and only if the associated Cauchy stress tensor is symmetric.

Let $\boldsymbol{\sigma}$ be the Cauchy stress tensor. It follows that,

$$\mathbf{n} \cdot \mathbf{t}^{(\bar{\mathbf{n}})} = \mathbf{n} \cdot \boldsymbol{\sigma}\bar{\mathbf{n}} = \bar{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}$$

This is the definition of the transpose of the tensor. But $\bar{\mathbf{n}} \cdot \mathbf{t}^{(\mathbf{n})} = \bar{\mathbf{n}} \cdot \boldsymbol{\sigma}\mathbf{n}$. This can be equal to $\bar{\mathbf{n}} \cdot \boldsymbol{\sigma}^{\mathrm{T}}\mathbf{n}$ only if

$$\boldsymbol{\sigma}^{\mathrm{T}} = \boldsymbol{\sigma}$$

5.2 In the biaxial stress tensor, Equation ___, given that $\sigma_3 = \sigma_4$, find the eigenvalues of the stress tensor. Write the stress tensor in spectral form.

a Mathematica code

```
ChapterFive001.nb - Wolfram Mathematica 11.3
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In[1]:= Eigenvalues[{{σ1, σ3}, {σ3, σ2}}]
```

$$\text{Out[1]=} \left\{\frac{1}{2}\left(\sigma_1+\sigma_2-\sqrt{\sigma_1^2-2\sigma_1\sigma_2+\sigma_2^2+4\sigma_3^2}\right), \frac{1}{2}\left(\sigma_1+\sigma_2+\sqrt{\sigma_1^2-2\sigma_1\sigma_2+\sigma_2^2+4\sigma_3^2}\right)\right\}$$

Shows the eigenvalues. If the eigenvectors are $\mathbf{u}_1$ and $\mathbf{u}_2$ respectively we have,

$$\boldsymbol{\sigma}(\boldsymbol{x}) = s_1\mathbf{u}_1 \otimes \mathbf{u}_1 + s_2\mathbf{u}_2 \otimes \mathbf{u}_2$$

5.3

2. Let the Cauchy stress in Cartesian coordinates be,

$$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 1 & 4 & -3 \\ 4 & 1 & 0 \\ -3 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

(a) Find the component of the traction on the plane $2x_1 + 3x_2 + x_3 = 5$.

(b) Find the stress tensor referred to the orthonormal bases $\boldsymbol{\xi}_2 = \frac{1}{\sqrt{2}}(\mathbf{e}_1 - \mathbf{e}_3)$, $\boldsymbol{\xi}_3 = \frac{1}{3}(2\mathbf{e}_1 - \mathbf{e}_2 + 2\mathbf{e}_3)$

a

The unit normal to the given plane is the vector $\mathbf{n} = \frac{1}{\sqrt{14}}(2\mathbf{e}_1 + 3\mathbf{e}_2 + \mathbf{e}_3)$. The traction vector is $\boldsymbol{\sigma}\,\mathbf{n}$.

In the Mathematica code shown, the Cauchy stress tensor components are entered using the variable Sig. The normal to the surface is entered with the variable NorVec. The tensor operation on this normal as in Cauchy theorem is to take the product $\boldsymbol{\sigma}\,\mathbf{n}$ as shown in symbolic form. Hence the result.

```
Sig = {{1, 4, -3}, {4, 1, 0}, {-3, 0, 0}}
{{1, 4, -3}, {4, 1, 0}, {-3, 0, 0}}

NorVec = {2 / Sqrt[14], 3 / Sqrt[14], 1 / Sqrt[14]}
```

$$\left\{\sqrt{\frac{2}{7}}, \frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right\}$$

```
Trac = Sig . NorVec // MatrixForm
```

trixForm=

$$\begin{pmatrix} -\frac{3}{\sqrt{14}} + \sqrt{14} \\ 4\sqrt{\frac{2}{7}} + \frac{3}{\sqrt{14}} \\ -3\sqrt{\frac{2}{7}} \end{pmatrix}$$

b	We need to refer the same tensor to another orthonormal system. First, we have only two of the three vectors $\boldsymbol{\xi}_2$ and $\boldsymbol{\xi}_3$. We need to find $\boldsymbol{\xi}_1$. $$\boldsymbol{\xi}_1 = \boldsymbol{\xi}_2 \times \boldsymbol{\xi}_3$$ Taking the above cross product, we find that $\boldsymbol{\xi}_1 = \frac{1}{3\sqrt{2}}(-\mathbf{e}_1 - 4\mathbf{e}_2 - \mathbf{e}_3)$. Note that Mathematica computes this directly in Cross($\boldsymbol{\xi}_2, \boldsymbol{\xi}_3$). It is necessary to rotate the Cauchy stress tensor and the appropriate rotation tensor is: $$\mathbf{Q} = \boldsymbol{\xi}_1 \otimes \mathbf{e}_1 + \boldsymbol{\xi}_2 \otimes \mathbf{e}_2 + \boldsymbol{\xi}_3 \otimes \mathbf{e}_3$$ The Mathematica code below computes the rotations $\mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^{\mathbf{T}}$ of the Cauchy tensor σ from $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ to $\boldsymbol{\xi}_\mathbf{1}, \boldsymbol{\xi}_\mathbf{2}, \boldsymbol{\xi}_\mathbf{3}$ **:** Recall from last term that to rotate a vector, you only need to operate the relevant rotation tensor on it. However, in this case, you need to operate the rotation and its inverse – which is its transpose in order to obtain a rotation of the stress tensor to a new coordinate system as we have shown. The decimal floating points are computed to see the simple numbers.

```
e₁ = {1, 0, 0}; e₂ = {0, 1, 0}; e₃ = {0, 0, 1};
ξ₂ = {1 / Sqrt[2], 0, -1 / Sqrt[2]};
ξ₃ = {2 / 3, -1 / 3, 2 / 3};
ξ₁ = Cross[ξ₂, ξ₃]
```

$$\left\{-\frac{1}{3\sqrt{2}},\ -\frac{2\sqrt{2}}{3},\ -\frac{1}{3\sqrt{2}}\right\}$$

```
Q = TensorProduct[ξ₁, e₁] + TensorProduct[ξ₂, e₂] + TensorProduct[ξ₃, e₃]
```

$$\left\{\left\{-\frac{1}{3\sqrt{2}},\ \frac{1}{\sqrt{2}},\ \frac{2}{3}\right\},\ \left\{-\frac{2\sqrt{2}}{3},\ 0,\ -\frac{1}{3}\right\},\ \left\{-\frac{1}{3\sqrt{2}},\ -\frac{1}{\sqrt{2}},\ \frac{2}{3}\right\}\right\}$$

```
NewSig = Q.Sig.Transpose[Q] // MatrixForm
```

ixForm=

$$\begin{pmatrix} \frac{\sqrt{2}}{3}+\frac{\frac{1}{\sqrt{2}}-\frac{2\sqrt{2}}{3}}{\sqrt{2}}-\frac{-2-\frac{1}{3\sqrt{2}}+2\sqrt{2}}{3\sqrt{2}} & -\frac{1}{3\sqrt{2}}-\frac{2}{3}\sqrt{2}\left(-2-\frac{1}{3\sqrt{2}}+2\sqrt{2}\right) & \frac{\sqrt{2}}{3}-\frac{\frac{1}{\sqrt{2}}-\frac{2\sqrt{2}}{3}}{\sqrt{2}}-\frac{-2-\frac{1}{3\sqrt{2}}+2\sqrt{2}}{3\sqrt{2}} \\ -\frac{8}{3}+\frac{4\sqrt{2}}{3}-\frac{1-\frac{2\sqrt{2}}{3}}{3\sqrt{2}} & -\frac{2\sqrt{2}}{3}-\frac{2}{3}\sqrt{2}\left(1-\frac{2\sqrt{2}}{3}\right) & \frac{8}{3}+\frac{4\sqrt{2}}{3}-\frac{1-\frac{2\sqrt{2}}{3}}{3\sqrt{2}} \\ \frac{\sqrt{2}}{3}-\frac{-2-\frac{1}{3\sqrt{2}}-2\sqrt{2}}{3\sqrt{2}}+\frac{-\frac{1}{\sqrt{2}}-\frac{2\sqrt{2}}{3}}{\sqrt{2}} & -\frac{1}{3\sqrt{2}}-\frac{2}{3}\sqrt{2}\left(-2-\frac{1}{3\sqrt{2}}-2\sqrt{2}\right) & \frac{\sqrt{2}}{3}-\frac{-2-\frac{1}{3\sqrt{2}}-2\sqrt{2}}{3\sqrt{2}}-\frac{-\frac{1}{\sqrt{2}}-\frac{2\sqrt{2}}{3}}{\sqrt{2}} \end{pmatrix}$$

```
N[%] // MatrixForm
```

ixForm=

$$\begin{pmatrix} 0.165031 & -0.794529 & 0.498365 \\ -0.794529 & -0.996729 & 4.5388 \\ 0.498365 & 4.5388 & 2.8317 \end{pmatrix}$$

5.4 Let the Cauchy stress in Cartesian coordinates be,

$$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 5x_2x_3 & 3x_2^2 & 0 \\ 3x_2^2 & 0 & -x_1 \\ 0 & -x_1 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Find the component of the traction at the point $\left(\frac{1}{2} \quad \frac{\sqrt{3}}{2} \quad -1\right)$ on the surface $x_1^2 + x_2^2 + x_3 = 0$.

a	We first need to find the normal to the surface. The surface here is not plane. However, there is a tangent to the plane at the point of interest. We take the gradient of its equation to find the unit vector along this normal.

```
ϕ[x1_, x2_, x3_] := x1^2 + x2^2 + x3

NormalVec = Grad[ϕ[x1, x2, x3], {x1, x2, x3}]
{2 x1, 2 x2, 1}

UnitNormalVec = Normalize[NormalVec] /. {x1 → 1 / 2, x2 → Sqrt[3] / 2, x3 → -1}
```

$$\left\{\frac{1}{\sqrt{5}}, \sqrt{\frac{3}{5}}, \frac{1}{\sqrt{5}}\right\}$$

```
VarSigma[x1_, x2_, x3_] := {{5 x2 x3, 3 x2^2, 0}, {3 x2^2, 0, -x1}, {0, -x1, 0}}

TractionVec = VarSigma[1. / 2, Sqrt[3] / 2, -1].UnitNormalVec // MatrixForm
atrixForm=
```

$$\begin{pmatrix} -0.193649 \\ 0.782624 \\ -0.387298 \end{pmatrix}$$

5.5	* At a certain point of a body, Cauchy stress tensor is: $$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 2 & 5 & 3 \\ 5 & 1 & 4 \\ 3 & 4 & 3 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$ (a) Find the components of the traction vector at a point on the plane whose normal has direction ratios 3:1:-2 (b) Find the normal and shear components of this traction.
a	Obtain the unit normal $\mathbf{n}$ and operate the Cauchy stress $\boldsymbol{\sigma}$ on it. To find the normal and the shear stresses, invoke the dyad $(\mathbf{n} \otimes \mathbf{n})$ on slide 18 and the projection tensor $(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$.
5.6	Given that the Cauchy stress in Cartesian coordinates be,

$$\boldsymbol{\sigma} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 3 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Find principal stresses, principal directions and find the traction vector on a plane whose unit normal is $\frac{1}{\sqrt{2}}(0,1,1)$. Also rotate the stress tensor back to its principal directions

```
CauchyStr = {{3, 1, 1}, {1, 0, 2}, {1, 2, 0}};
NorVec = 1 / Sqrt[2] {0, 1, 1};
TracVec = CauchyStr.NorVec
```

$\{\sqrt{2}, \sqrt{2}, \sqrt{2}\}$

```
Eigenvalues[CauchyStr]
```

```
{4, -2, 1}
```

```
vecs = Eigenvectors[CauchyStr]
```

```
{{2, 1, 1}, {0, -1, 1}, {-1, 1, 1}}
```

```
e1 = {1., 0, 0}; e2 = {0, 1., 0}; e3 = {0., 0, 1.};

Q = TensorProduct[e1, Normalize[vecs[[1]]]] + TensorProduct[e2, Normalize[vecs[[2]]]] +
  TensorProduct[e3, Normalize[vecs[[3]]]]
```

```
{{0.816497, 0.408248, 0.408248}, {0., -0.707107, 0.707107}, {-0.57735, 0.57735, 0.57735}}
```

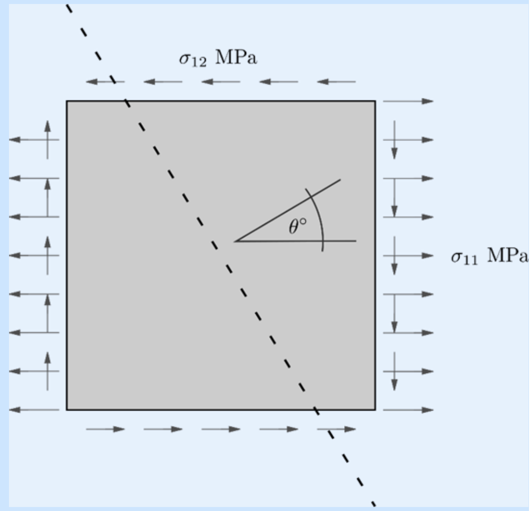
```
NewCauchy = Q. CauchyStr .Transpose[Q] // MatrixForm
```

trixForm=

$$\begin{pmatrix} 4. & 0. & 0. \\ 0. & -2. & 0. \\ -1.66533\times 10^{-16} & 0. & 1. \end{pmatrix}$$

5.7 Find the principal stresses and principal directions in the stress system shown below. Assume $\sigma_{11} = 40\ MPa$, $\sigma_{12} = 100\ MPa$ and $\sigma_{22} = 0\ MPa$

Recall that the principal stresses are the eigenvalues of the stress tensor while the principal directions are the eigenvectors. The principal stresses are:



$$\sigma_2 = \frac{1}{2}\left(\sigma_{11} + \sigma_{22} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2}\right)$$
$$= -81.98$$
$$\sigma_1 = \frac{1}{2}\left(\sigma_{11} + \sigma_{22} + \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2}\right)$$
$$= 121.98$$

The Principal directions are given in the row vectors,

$$\left[-\frac{-\sigma_{11} + \sigma_{22} + \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2}}{2\sigma_{12}}, 1\right] = [-0.82, 1.]$$
$$\left[-\frac{-\sigma_{11} + \sigma_{22} - \sqrt{\sigma_{11}^2 + 4\sigma_{12}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2}}{2\sigma_{12}}, 1\right] = [1.22, 1.]$$

As computed in the following Mathematica code:

Q5.7 Stress.nb - Wolfram Mathematica 11.3

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In[1]:= CauchyStr := {{σ_{11}, σ_{12}}, {σ_{12}, σ_{22}}}

In[2]:= Eigenvalues[CauchyStr]

Out[2]= $\left\{\frac{1}{2}\left(\sigma_{11} + \sigma_{22} - \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2 - 2\,\sigma_{11}\,\sigma_{22} + \sigma_{22}^2}\right), \frac{1}{2}\left(\sigma_{11} + \sigma_{22} + \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2 - 2\,\sigma_{11}\,\sigma_{22} + \sigma_{22}^2}\right)\right\}$

In[3]:= N[%] /. {σ_{11} → 40, σ_{12} → 100, σ_{22} → 0}

Out[3]= {-81.9804, 121.98}

In[4]:= Eigenvectors[CauchyStr]

Out[4]= $\left\{\left\{-\frac{-\sigma_{11} + \sigma_{22} + \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2 - 2\,\sigma_{11}\,\sigma_{22} + \sigma_{22}^2}}{2\,\sigma_{12}}, 1\right\}, \left\{-\frac{-\sigma_{11} + \sigma_{22} - \sqrt{\sigma_{11}^2 + 4\,\sigma_{12}^2 - 2\,\sigma_{11}\,\sigma_{22} + \sigma_{22}^2}}{2\,\sigma_{12}}, 1\right\}\right\}$

In[5]:= N[%] /. {σ_{11} → 40, σ_{12} → 100, σ_{22} → 0}

Out[5]= {{-0.819804, 1.}, {1.2198, 1.}}

5.8 For the given states of stress at a point (x_1, x_2, x_3):

$$C_1 = \begin{pmatrix} 12 & 9 & 0 \\ 9 & -2 & 0 \\ 0 & 0 & 6 \end{pmatrix}, C_2 = \begin{pmatrix} 9 & 0 & 12 \\ 0 & -25 & 0 \\ 12 & 0 & 16 \end{pmatrix}, C_3 = \begin{pmatrix} 1 & -3 & \sqrt{2} \\ -3 & 1 & -\sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 4 \end{pmatrix}$$

Find the following: (a) The stress vectors acting on a plane perpendicular to the vector $2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3$, (b) The magnitude of the traction vector and the normal to the plane, and (c) The magnitude of the normal and tangential components of the stress vector.

In each case, the traction, $\mathbf{t^n} = \boldsymbol{\sigma}\mathbf{n}$. The normal to the plane is the unit vector, $\frac{1}{3}(2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3)$. The scalar product, $\sigma = \mathbf{t^n} \cdot \mathbf{n}$ gives the normal stress while the shear stress can be computed from: $\tau = \sqrt{\|\mathbf{t^n}\|^2 - \sigma^2}$. This is shown in the following code:

```
Q5.8 Stress.nb * - Wolfram Mathematica 11.3
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In[1]:= Cauchy1 = {{12, 9, 0.}, {9, -2, 0}, {0, 0, 6}}
        Cauchy2 = {{9, 0, 12.}, {0, -25, 0}, {12, 0, 16}}
        Cauchy3 = {{1, -3., Sqrt[2]}, {-3, 1, -Sqrt[2]}, {Sqrt[2], -Sqrt[2], 4}}
        VecNor = 1 / 3 {2., -2, 1}
        Trac1 = Cauchy1 . VecNor
        Trac2 = Cauchy2 . VecNor
        Trac3 = Cauchy3 . VecNor

Out[5]= {2., 7.33333, 2.}

Out[6]= {10., 16.6667, 13.3333}

Out[7]= {3.13807, -3.13807, 3.21895}

In[8]:= NorStr1 = Trac1.VecNor
        NorStr2 = Trac2.VecNor
        NorStr3 = Trac3.VecNor
        Shear1 = Sqrt[Norm[Trac1]^2 - NorStr1^2]
        Shear2 = Sqrt[Norm[Trac2]^2 - NorStr2^2]
        Shear3 = Sqrt[Norm[Trac3]^2 - NorStr3^2]

Out[8]= -2.88889

Out[9]= 8.88178×10^-16

Out[10]= 5.25708

Out[11]= 7.30973

Out[12]= 23.5702

Out[13]= 1.55556
```

5.9 A stress field in a region is given by

$$\boldsymbol{\sigma}(x_1,x_2,x_3) = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 12x_1^2 & 9x_1x_2 & 0 \\ 9x_1x_2 & -2x_2x_3^2 & 0 \\ 0 & 0 & 6x_1x_2^2 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

(a)Find the traction vector at the point, $\mathbf{e}_1 + 2\mathbf{e}_2$+3 $\mathbf{e}_3$ acting on the plane $x_1 + x_2 + x_3 = 6$. (b) The normal and shear tractions on the plane, (c) Principal stresses and directions, (d) Maximum shear stress at the point.

a For a plane with equation, $\phi = x_1 + x_2 + x_3 = 6$, the normal vector to the plane is

$$\mathbf{n} = \frac{\operatorname{grad}\phi}{\|\operatorname{grad}\phi\|} = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$$

The point of interest is $(1,2,3)$. Traction vector is: $\mathbf{t}^{\mathbf{n}}(1,2,3) = \boldsymbol{\sigma}(1,2,3)\mathbf{n}$. The following code implements this:

Q5.9 Stress.nb * - Wolfram Mathematica 11.3

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```
In[1]:= f1 = Grad[{x1 + x2 + x3}, {x1, x2, x3}];
        NorVec = f1 / Norm[f1]
        CauchyStr[x1_, x2_, x3_] := {{12 x1^2, 9 x1 x2, 0}, {9 x1 x2, -2 x2 x3^2 , 0 },
           {0, 0, 6 x1 x2^2}};
        tracStr = CauchyStr[1, 2, 3] . NorVec[[1]]
```

Out[2]= $\left\{\left\{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}\right\}$

Out[4]= $\left\{10\sqrt{3}, -6\sqrt{3}, 8\sqrt{3}\right\}$

b Normal traction, $\sigma = \mathbf{t}^{\mathbf{n}} \cdot \mathbf{n}$ and shear is given by: $\tau = \sqrt{\|\mathbf{t}^{\mathbf{n}}\|^2 - \sigma^2}$ is computed here:

```
Sigma = tracStr . NorVec[[1]]
12
Norm[tracStr]
```

$10\sqrt{6}$

c

The principal stresses and directions are the eigenvalues and eigenvectors at this point. They are computed in:

```
In[7]:= Eigenvalues[CauchyStr[1, 2, 3]]
        Eigenvectors[CauchyStr[1, 2, 3]]

Out[7]= {-42, 24, 18}

Out[8]= {{-1, 3, 0}, {0, 0, 1}, {3, 1, 0}}
```

5.10

For the given state of stress at a point (x_1, x_2, x_3):

$$\boldsymbol{\sigma}(x_1, x_2, x_3) = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 12x_1^2 & 9x_1x_2 & 0 \\ 9x_1x_2 & -2x_2x_3^2 & 0 \\ 0 & 0 & 6x_1x_2^2 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$

Find the following: (a) The traction vector acting on elliptic surface $x_2^2 + 2x_3^2 = 6$, at the point, $(5,2,1)$. (b) the traction vector acting at the same point on the plane surface $x_1 + x_2 + 3x_3 = 10$, and (c) Principal stresses and planes in both cases. Comment on the results.

Q5.10 Stress.nb * - Wolfram Mathematica 11.3

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```
CauchyStr[x1_, x2_, x3_] := {{12 x1^2, 9 x1 x2, 0}, {9 x1 x2, -2 x2 x3^2 , 0 },
   {0, 0, 6 x1 x2^2}};
fEllipt = x2^2 2 x3^2
ϕe = Grad[fEllipt, {x1, x2, x3}] /. {x1 → 5, x2 → 2 , x3 → 1}
VecNorEllipt = ϕe / Norm[ϕe]
Trac1e = CauchyStr[5, 2, 1] .VecNorEllipt[[1]]
fPlane = x1 + x2 + 3 x3
ϕp = Grad[fPlane, {x1, x2, x3}]
VecNorPlane = ϕp / Norm[ϕp]
Trac1p = CauchyStr[5, 2, 1] .VecNorPlane[[1]]
```

Function *CauchyStr* in the above code contains the stress tensor at any given point. ϕe and ϕp are the elliptic surface and plane functions respectively. The respective normal vectors are computed as shown by the formula: $\mathbf{n} = \frac{\text{grad}\,\phi}{\|\text{grad}\,\phi\|}$ using the Norm[] function. Tractions are computed from the relation, $\mathbf{t}^{\mathbf{n}} = \boldsymbol{\sigma}\mathbf{n}$ as shown.

```
Eigenvalues[CauchyStr[5, 2, 1]]
Eigenvectors[CauchyStr[5, 2, 1]]
```

Principal stresses and principal planes are computed from the eigenvalues and eigenvectors of the Cauchy stress. Independent of planes of action.

	Results: (a) Traction vector on the elliptical surface, $\{18\sqrt{5}, -\frac{4}{\sqrt{5}}, 48\sqrt{5}\}$ (b) Traction on the plane surface: $\{\frac{390}{\sqrt{11}}, \frac{86}{\sqrt{11}}, \frac{360}{\sqrt{11}}\}$ (c) Eigenvalues $\{2(74+\sqrt{7801}), 120, 2(74-\sqrt{7801})\}$ (d) Eigenvectors are given by the rows of $\begin{pmatrix} \frac{1}{45}(76+\sqrt{7801}) & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{45}(76-\sqrt{7801}) & 1 & 0 \end{pmatrix}$
5.11	The state of stress within a body at the point $(1,1,-2)$is given by the tensor, $$(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 2.0 & 3.5 & 2.5 \\ 3.5 & 0.0 & -1.5 \\ 2.5 & -1.5 & 1.0 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$$ Determine the normal and shear stresses at the point on the surface of an internal sphere whose equation is $x_1^2 + (x_2-2)^2 + x_3^2 = 6$.
	The unit normal here is obtained by normalizing the gradient of the sphere equation. $\mathbf{t}^{\mathbf{n}} = \boldsymbol{\sigma}\mathbf{n}$ is given by taking the dot product of the stress tensor with the unit normal at the point of interest. As before, shear is given by: `CauchyStr5 = {{2.0, 3.5, 2.5}, {3.5, 0.0, -1.5}, {2.5, -1.5, 1.0}};` `ff5[x1_, x2_, x3_] := x1^2 + (x2 - 2)^2 + x3^2;` `UnitNormal5 = Normalize[Grad[ff5[x1, x2, x3], {x1, x2, x3}]] /. {x1 → 1., x2 -> 1, x3 -> 2};` `TracVec5 = CauchyStr5.UnitNormal5 /. {x1 → 1., x2 -> 1, x3 -> 2}` `{1.4288690166235205`, 0.20412414523193156`, 2.4494897427831783`}`   `NorVec5 = TensorProduct[UnitNormal5, UnitNormal5].TracVec5`   `{1.0206207261596578`, -1.0206207261596578`, 2.0412414523193156`}` `ShearStr5 = Sqrt[Norm[TracVec5]^2 - Norm[NorVec5]^2]` `1.3540064007726593`` $$\tau = \sqrt{\lVert \mathbf{t}^{\mathbf{n}} \rVert^2 - \sigma^2}$$
5.12	The state of stress within a body at the point on the plane passing through the origin and the points $(1,2,-5)$ and $(2,0,3)$ is given by the tensor,

$$\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}2.0 & 3.5 & 2.5\\ 3.5 & 0.0 & -1.5\\ 2.5 & -1.5 & 1.0\end{pmatrix}\otimes\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}$$

Determine the normal and shear stresses at this point

a

The first step here is to calculate the unit normal to the plane. To do this, we normalize the cross product of the line joining the points $(1,2,-5)$ and $(2,0,3)$ to the origin.

```
Normalize[Cross[{1, 2., -5}, {2, 0, 3}]]
{0.403604, -0.874475, -0.269069}
```

We can now find the traction vector, $\mathbf{t^n} = \boldsymbol{\sigma}\mathbf{n}$ using this unit normal:

```
CauchyStr = {{2.0, 3.5, 2.5}, {3.5, 0.0, -1.5}, {2.5, -1.5, 1.0}};
TracStr = CauchyStr.NorVec;
NorStr = TracStr.NorVec
ShearStr = Sqrt[Norm[TracStr]^2 - NorStr^2]

-3.32127

2.24484
```

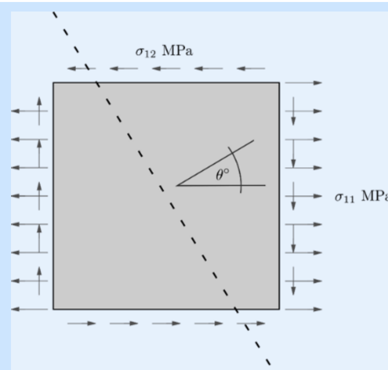
$$\sigma = -3.32, \tau = 2.24$$

5.13

Find the stress and strain on the plane whose normal is oriented at $\theta = 30°$ in the figure below. Assume $\sigma_{11} = 50, \sigma_{12} = 30$.

a

From the formula given in equation ___, using the nomenclature of the question, $\sigma_{22} = 0$



$$\sigma = \frac{\sigma_{11} + \sigma_{22}}{2} + \frac{\sigma_{11} - \sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta$$

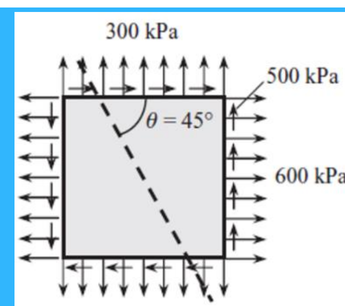
$$\tau = -\frac{\sigma_{11} - \sigma_{22}}{2}\sin 2\theta + \sigma_{12}\cos 2\theta$$

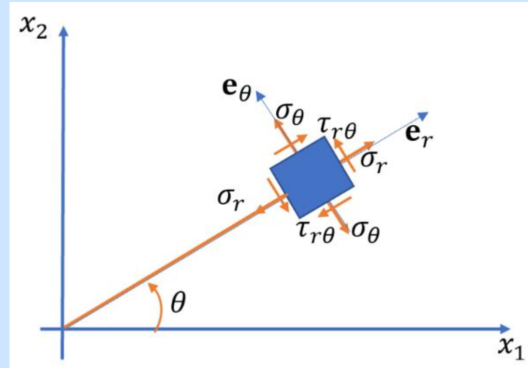
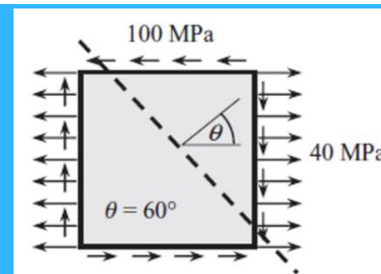
$$\sigma = \frac{50}{2} + \frac{50}{2}\cos\left(\frac{\pi}{3}\right) + 30\sin\left(\frac{\pi}{3}\right) = 63.48$$

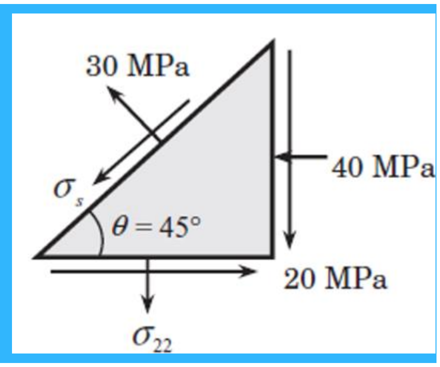
$$\tau = -\frac{50}{2}\sin\left(\frac{\pi}{3}\right) + 30\cos\left(\frac{\pi}{3}\right) = -6.65$$

5.14

Determine the normal and shear stress components on the plane indicated by the dashed lines in the adjacent figure.



a	$$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$ $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$ $$\tau_{r\theta} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$  What we are looking for is the stress after we have rotated the axis by $\theta = 45^o = \pi/4$. Normal stress, $$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$ $$= \frac{300+600}{2} + \frac{300-600}{2}\cos\frac{\pi}{2} + 500\sin\frac{\pi}{2} = 950$$ Shear Stress, $$\tau_{r\theta} = -\frac{300-600}{2}\sin\frac{\pi}{2} + 500\cos\frac{\pi}{2} = 150$$
5.15	Determine the normal and shear stress components on the plane indicated by the dashed lines in the adjacent figure. 
a	$$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$ $$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$ $$\tau_{r\theta} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$ What we are looking for is the stress after we have rotated the axis by $\theta = 60^o = \pi/3$ Normal stress, $$\sigma = \frac{40+0}{2} + \frac{40-0}{2}\cos 2\pi/3 + 100\sin 2\pi/3$$ $$= 20 + 20\cos\frac{2\pi}{3} + 100\sin\frac{2\pi}{2} = 50\sqrt{3} + 10$$ $$\tau = -\frac{40}{2}\sin\frac{2\pi}{3} + 100\cos\frac{2\pi}{3} = -67.32$$

5.16	Determine σ_s and σ_{22} as indicated by the in the adjacent figure. 
a	(see below)

$$\sigma_r = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_\theta = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{r\theta} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

What we given is the stress σ_θ after we have rotated the axis by $\theta = 45^o = \pi/4$

Normal stress,

$$30 = \frac{40 + \sigma_{22}}{2} - \frac{40 - \sigma_{22}}{2}\cos\frac{\pi}{2} - 20\sin\pi/2$$

$$\sigma_{22} = \frac{60 - 40 + 40\cos\frac{\pi}{2} + 20\sin\frac{\pi}{2}}{1 + \cos\frac{\pi}{2}} = 40$$

$$\sigma_s = \tau_{r\theta} = -\frac{40 - 40}{2}\sin\frac{\pi}{2} + 20\cos\frac{\pi}{2} = 0$$

5.17	The matrix of Cartesian stress tensor components at a point is given as

$$\begin{pmatrix} -2x_1^2 & -7 + 4x_1x_2 + x_3 & 1 + x_1 - 3x_2 \\ -7 + 4x_1x_2 + x_3 & 3x_1^2 - 2x_2^2 + 5x_3 & 0 \\ 1 + x_1 - 3x_2 & 0 & x_1 - 5 + 3x_2 + 3x_3 \end{pmatrix}$$

Determine the stress vector at a point on the plane $x_1 + x_2 + x_3 = 5$, (b) The normal components of the traction at point (1,1,3), and (c) Principal stresses and their orientations at the point (1,2,1).

a

```
CauchyTens16[x1_, x2_, x3_] :=
 {{-2 x1^2, -7 + 4 x1 x2 + x3, 1 + x1 - 3 x2},
  {-7 + 4 x1 x2 + x3, 3 x1^2 - 2 x2^2 + 5 x3, 0},
  {1 + x1 - 3 x2, 0, -5 + x1 + 3 x2 + 3 x3}};
NorVec16 = Normalize[{1, 1, 1}];
TracVec16 = CauchyTens16[x1, x2, x3].NorVec16
```

$$\left\{-\frac{2 x_1^2}{\sqrt{3}}+\frac{1+x_1-3 x_2}{\sqrt{3}}+\frac{-7+4 x_1 x_2+x_3}{\sqrt{3}},\ \frac{-7+4 x_1 x_2+x_3}{\sqrt{3}}+\frac{3 x_1^2-2 x_2^2+5 x_3}{\sqrt{3}},\ \frac{1+x_1-3 x_2}{\sqrt{3}}+\frac{-5+x_1+3 x_2+3 x_3}{\sqrt{3}}\right\}$$

```
NorStr16 = TracVec16.NorVec16 /. {x1 → 1., x2 → 1., x3 → 3.}
6.666666666666669`
ShearStr16 =
 Norm[(IdentityMatrix[3] - TensorProduct[NorVec16, NorVec16]).
    TracVec16] /. {x1 → 1., x2 → 1., x3 → 3.}
7.760297817881877`
```

5.18

Given that $\begin{pmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3\end{pmatrix}\begin{pmatrix}12 & 9 & 0\\ 9 & -2 & 0\\ 0 & 0 & 6\end{pmatrix}\otimes\begin{pmatrix}\mathbf{e}_1\\ \mathbf{e}_2\\ \mathbf{e}_3\end{pmatrix}$

Determine the traction at the plane perpendicular to the vector $2\mathbf{e}_1+\mathbf{e}_2+2\mathbf{e}_3$ and (b) its normal and tangential components

a

```
CauchyStr17 = {{12., 9, 0}, {9, -2, 0}, {0, 0, 6}};
NorVec17 = Normalize[{2, 1, 2}];
TracVec17 = CauchyStr17 . NorVec17;
SurfDyad17 = TensorProduct[NorVec17, NorVec17];
NorStr17 = SurfDyad17.TracVec17;
TanStr17 = (IdentityMatrix[3] - SurfDyad17).TracVec17;

TracVec17
{11., 5.33333, 4.}

Norm[NorStr17]
11.7778
```

```
Norm[TanStr17]
5.16995

Sqrt[%^2 + %%^2]
12.8625

Norm[TracVec17]
12.8625
```

5.19	Given that $(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 57 & 9 & 24 \\ 9 & 50 & 0 \\ 24 & 0 & 43 \end{pmatrix} \otimes \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}$ are the stresses at a given point, Determine the principal stresses and directions at this point. (b) What is the maximum shear stress at this point?
a	see code below
5.16	
a	
5.51	A deformation is depicted by the deformation gradient, $$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} x_1^2 & -x_1 x_3 & 2x_2^2 \\ x_1 & 3x_3^2 & x_2 x_3 \\ x_2 x_3 & 5x_2 & x_3 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

```
CauchyStr = {{57., 9, 24}, {9, 50, 0}, {24, 0, 43}};
PrinStr = Eigenvalues[CauchyStr];
PrinDir = Eigenvectors[CauchyStr];
MaxShear = 1 / 2 (PrinStr[[1]] - PrinStr[[3]]);

PrinStr

{76.9635, 49.1961, 23.8404}

PrinDir // MatrixForm

( -0.787925710521062`   -0.2629977767151812`  -0.5567811456422992
  0.084109223748827 9`  -0.9416946420469905`   0.3257864939194993`
 -0.6099989452288006`   0.20986512474173025`   0.7641059587759337` )

MaxShear

26.5615
```

And the Cauchy stress tensor field,

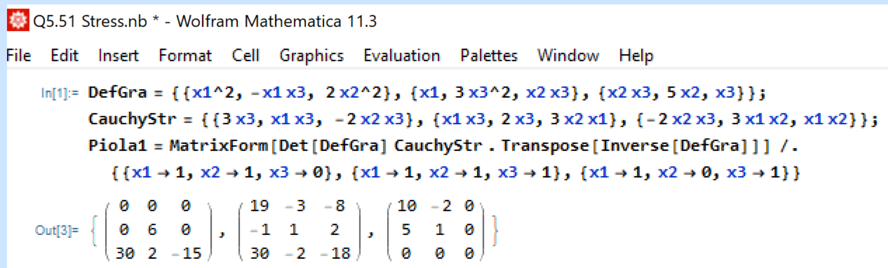
$$\boldsymbol{\sigma}(\mathbf{x}) = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 3x_3 & x_1x_3 & -2x_2x_3 \\ x_1x_3 & 2x_3 & 3x_1x_2 \\ -2x_2x_3 & 3x_1x_2 & x_1x_2 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

Find the first Piola-Kirchhoff stresses at the points (a) (1,1,0), (b) (1,1,1) and (c) (1,0,1)

a

The first Piola Kirchhoff stress, $\mathbf{S}$ is obtained by the Piola transformation of the Cauchy stress $\boldsymbol{\sigma}$:

$$\mathbf{S} = \boldsymbol{\sigma}\mathbf{F}^{\mathrm{c}} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$$



The given code applies Piola transformation using the second formula as Mathematica did not provide for the direct evaluation of the cofactor tensor. As an example, in full tensor form, the solution of (a) is simply,

$$\mathbf{S} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 30 & 2 & -15 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

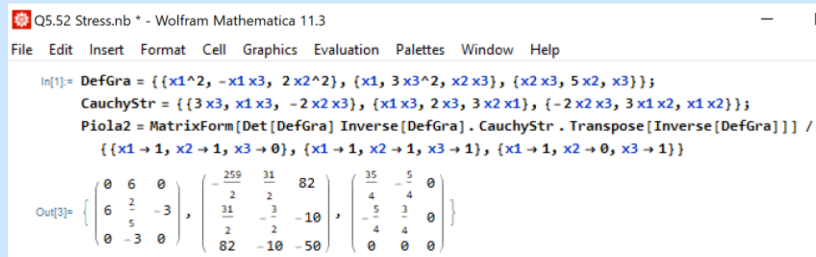
5.52

A deformation is depicted by the deformation gradient,

$$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} x_1^2 & -x_1x_3 & 2x_2^2 \\ x_1 & 3x_3^2 & x_2x_3 \\ x_2x_3 & 5x_2 & x_3 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$, and the Cauchy stress tensor field,

$$\boldsymbol{\sigma}(\mathbf{x}) = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 3x_3 & x_1x_3 & -2x_2x_3 \\ x_1x_3 & 2x_3 & 3x_1x_2 \\ -2x_2x_3 & 3x_1x_2 & x_1x_2 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

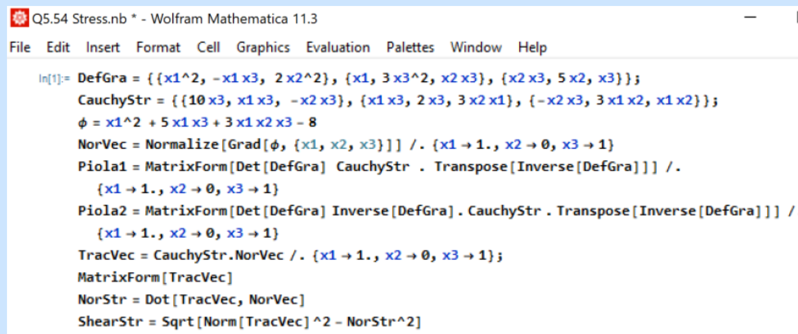
Find the second Piola-Kirchhoff stresses at the points (a) (1,1,0), (b) (1,1,1) and (c) (1,0,1)

a	From equation ____, the second Piola Kirchhoff stress can be found from the first by the formula: $\boldsymbol{\Xi} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$. The second Piola Kirchhoff tensor for case (b) is,  $$\boldsymbol{\Xi} = (\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3)\begin{pmatrix} -\frac{259}{2} & \frac{31}{2} & 82 \\ \frac{31}{2} & -\frac{3}{2} & -10 \\ 82 & -10 & -50 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$
5.53	**Comment on the transformation properties of the First and second Piola Kirchhoff tensors compared to the Cauchy stress tensor.**
a	Cauchy stress tensor $\boldsymbol{\sigma}$ is a spatial tensor. It transforms a spatial vector to a spatial vector. First Piola Kirchhoff is a two-toe tensor. It transforms a material vector to a spatial vector. The second Piola-Kirchhoff tensor transforms from material vector to material vector. It is therefore a material tensor as can be seen from its tensor basis. Note that when a tensor transforms from material to spatial, its first tensor basis is spatial followed by a material basis. The reverse is true for a tensor that transforms from spatial to material. Furthermore, we observe the fact that the second Piola Kirchhoff tensor is symmetric while the first Piola Kirchhoff is not symmetric.
5.54	For a deformation is depicted by the deformation gradient field, $\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} x_1^2 & -x_1x_3 & 2x_2^2 \\ x_1 & 3x_3^2 & x_2x_3 \\ x_2x_3 & 5x_2 & x_3 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$, and the Cauchy stress tensor field, $$\boldsymbol{\sigma}(\mathbf{x}) = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 3x_3 & x_1x_3 & -2x_2x_3 \\ x_1x_3 & 2x_3 & 3x_1x_2 \\ -2x_2x_3 & 3x_1x_2 & x_1x_2 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

Find the traction as well as the shear and normal stresses on the surface given by $x_1^2 + 5x_1x_3 + 3x_1x_2x_3 = 8$ at the point $(1,1,1)$. (b) Find the two Piola stress tensors of the Cauchy stress at the same point.

a

We need to find the unit normal to the surface at the point in question. This is to normalize the gradient of the surface function as shown. At the point $(1,1,1)$, this normal given by NorVec in



```
Q5.54 Stress.nb * - Wolfram Mathematica 11.3
File  Edit  Insert  Format  Cell  Graphics  Evaluation  Palettes  Window  Help
In[1]:= DefGra = {{x1^2, -x1 x3, 2 x2^2}, {x1, 3 x3^2, x2 x3}, {x2 x3, 5 x2, x3}};
        CauchyStr = {{10 x3, x1 x3, -x2 x3}, {x1 x3, 2 x3, 3 x2 x1}, {-x2 x3, 3 x1 x2, x1 x2}};
        ϕ = x1^2 + 5 x1 x3 + 3 x1 x2 x3 - 8
        NorVec = Normalize[Grad[ϕ, {x1, x2, x3}]] /. {x1 → 1., x2 → 0, x3 → 1}
        Piola1 = MatrixForm[Det[DefGra] CauchyStr . Transpose[Inverse[DefGra]]] /.
          {x1 → 1., x2 → 0, x3 → 1}
        Piola2 = MatrixForm[Det[DefGra] Inverse[DefGra] . CauchyStr . Transpose[Inverse[DefGra]]] /.
          {x1 → 1., x2 → 0, x3 → 1}
        TracVec = CauchyStr.NorVec /. {x1 → 1., x2 → 0, x3 → 1};
        MatrixForm[TracVec]
        NorStr = Dot[TracVec, NorVec]
        ShearStr = Sqrt[Norm[TracVec]^2 - NorStr^2]
```

the attached code. Traction stress, $\mathbf{t}^{\mathbf{n}} = \boldsymbol{\sigma}\mathbf{n}$, is obtained by operating Cauchy stress on the normal σ and the shear τ comes from taking the square root of the difference between the norm of the traction and the normal stress: $\mathbf{t}^{\mathbf{n}} = \begin{pmatrix} 8.01 \\ 1.43 \\ 0. \end{pmatrix}$, $\sigma = 6.63, \tau = 4.73$. First and second Piola Kirchhoff stresses are:

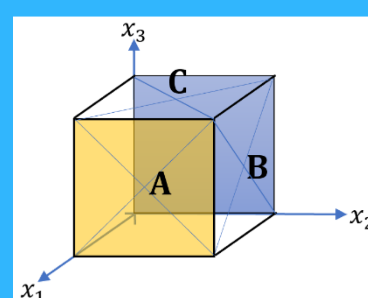
$$\text{Piola1} = \mathbf{S} = \boldsymbol{\sigma}\mathbf{F}^{\mathbf{c}} = \begin{pmatrix} 31. & -9. & 0. \\ 5. & 1. & 0. \\ 0. & 0. & 0. \end{pmatrix}$$

$$\text{Piola2} = \boldsymbol{\Xi} = \mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{\mathbf{c}} = \begin{pmatrix} 24.5 & -6.5 & 0. \\ -6.5 & 2.5 & 0. \\ 0. & 0. & 0. \end{pmatrix}$$

5.55

In the figure, points $\mathbf{A}(1,0.5,0.5)$, $\mathbf{B}(0.5,1.0,0.5)$ and $\mathbf{C}(0.5,0.5,1.0)$ are on the surface of a loaded unit cube with one vertex at the origin of coordinates as shown. The stress field is

$$\boldsymbol{\sigma}(\mathbf{x}) = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3) \begin{pmatrix} x_1 - x_2^2 & x_1 - x_3 & -2x_2x_3 \\ x_1 - x_3 & x_2 - x_1^2 & 0 \\ -2x_2x_3 & 0 & x_1x_2 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

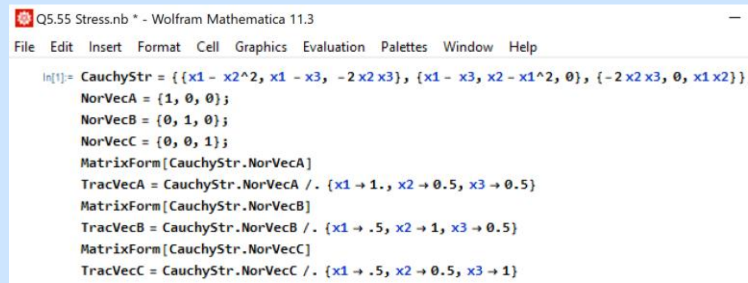


Find the tractions at points $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$.

a	The unit normal at points $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ respectively. The traction vector on the surface containing $\mathbf{A}$ is the vector,

$$\begin{pmatrix} x_1 - x_2^2 & x_1 - x_3 & -2x_2x_3 \\ x_1 - x_3 & x_2 - x_1^2 & 0 \\ -2x_2x_3 & 0 & x_1x_2 \end{pmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 - x_2^2 \\ x_1 - x_3 \\ -2x_2x_3 \end{bmatrix}_{\boldsymbol{A}} = \begin{bmatrix} 0.75 \\ 0.5 \\ -0.5 \end{bmatrix}$$



For the $\mathbf{B}$ and $\mathbf{C}$, we have, $\begin{bmatrix} x_1 - x_3 \\ x_1 - x_1^2 \\ 0 \end{bmatrix}_{\boldsymbol{B}} = \begin{bmatrix} 0.0 \\ 0.75 \\ 0.0 \end{bmatrix}$ and $\begin{bmatrix} -2x_2x_3 \\ 0 \\ x_1x_2 \end{bmatrix}_{\boldsymbol{C}} = \begin{bmatrix} -1.0 \\ 0.0 \\ 0.25 \end{bmatrix}$ respectively.

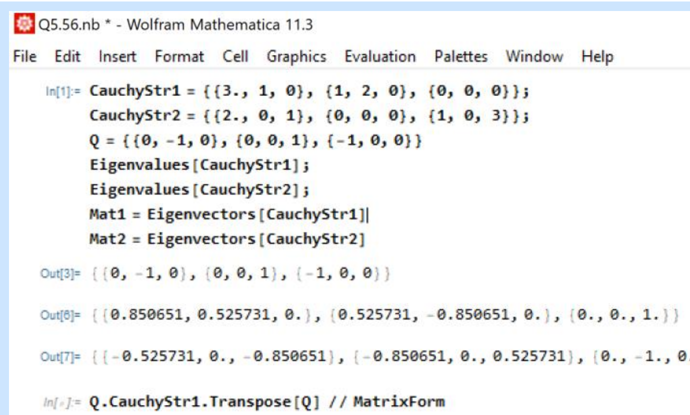
5.56	Show that the stress tensor $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ can be obtained by rotating $\mathbf{B} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ by the orthogonal tensor, $\mathbf{Q} = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}$. Show that they describe the same stress state at a point.

a

It is easy to see that the product

$$\mathbf{QBQ^T} = \mathbf{A}$$

which shows that $\mathbf{A}$ is simply a rotation of $\mathbf{B}$ by the given rotation tensor. The code shown gives the same eigenvalues which means that both stress tensors have the same principal stresses.



However the eigenvectors are not the same. The normalized eigenvectors of $\mathbf{A}$ can be obtained by simply rotating the eigenvectors of $\mathbf{B}$ using the same rotation tensor. Note that, to rotate a vector, you do not post-multiply with the transpose: If vectors $\mathbf{u}$ and $\mathbf{v}$ are eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ respectively, then, $\mathbf{u} = \mathbf{Qv}$ is the required rotation.

5.57 Find the principal invariants of $\mathbf{A}=\begin{pmatrix}2 & 0 & 1\\ 0 & 5 & 0\\ 1 & 0 & 3\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}3 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 5\end{pmatrix}$ and show that they both depict the same state of stress.

a The three principal invariants of these stress tensors are the trace, trace of the cofactor and their determinants. The attached code computes all three. As there is no direct function for the cofactor, we note that for any tensor $\mathbf{T}$, the cofactor is, $\mathbf{T}^{\mathrm{c}}=J\mathbf{T}^{-\mathrm{T}}$ where J is $\det\mathbf{T}$. In the three cases, the answers are the same showing that the states of stress depicted is the same in both cases: $\operatorname{tr}\mathbf{T}=10$, $\operatorname{tr}\mathbf{T}^{\mathrm{c}}=30$, and $\det\mathbf{T}=25$

Q5.57.nb * - Wolfram Mathematica 11.3

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```
CauchyStr1 = {{3., 1, 0}, {1, 2, 0}, {0, 0, 5}};
CauchyStr2 = {{2., 0, 1}, {0, 5, 0}, {1, 0, 3}};
Det[CauchyStr1]
Det[CauchyStr2]
Tr[CauchyStr1]
Tr[CauchyStr2]
Tr[Inverse[Transpose[CauchyStr1]]] Det[CauchyStr1]
Tr[Inverse[Transpose[CauchyStr2]]] Det[CauchyStr2]
```

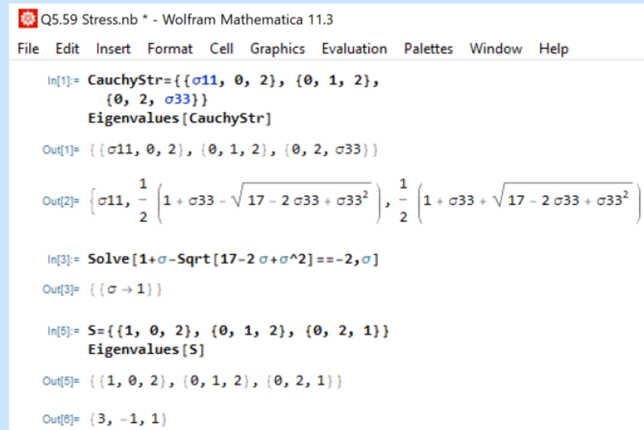
5.58 Conclude that the two matrices $\mathbf{A}=\begin{pmatrix}10 & 20 & 4\\ 20 & 0 & 0\\ 4 & 0 & -5\end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix}4 & 10 & 6\\ 10 & 10 & 0\\ 6 & 0 & 2\end{pmatrix}$ cannot represent the same Cauchy stress.

a We need go no farther than the fact that the trace of these two tensors $\operatorname{tr}\mathbf{A}=5\neq\operatorname{tr}\mathbf{B}=16$ are different. It is not necessary to compute other invariants.

5.59 The principal values of the stress tensor at a point are $3,1,-1$. If the stress tensor at that point has the values, $\begin{pmatrix}\sigma_{11} & 0 & 2\\ 0 & 1 & 2\\ 2 & 2 & \sigma_{33}\end{pmatrix}$. Find the values of σ_{11} and σ_{33}.

a

Substituting the known principal stresses in the eigenvalue expression and testing the outcome as shown in the attached notebook gives the unknown stresses as $\sigma_{11} = 1$ and $\sigma_{33} = 1$. The test also confirms that the eigenvalues of $\begin{pmatrix} \sigma_{11} & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & \sigma_{33} \end{pmatrix}$ are $3, -1, 1$.



Q5.59 Stress.nb * - Wolfram Mathematica 11.3

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```
In[1]:= CauchyStr={{σ11, 0, 2}, {0, 1, 2},
          {0, 2, σ33}}
        Eigenvalues[CauchyStr]
Out[1]= {{σ11, 0, 2}, {0, 1, 2}, {0, 2, σ33}}
Out[2]= {σ11, 1/2 (1 + σ33 - Sqrt[17 - 2 σ33 + σ33^2]), 1/2 (1 + σ33 + Sqrt[17 - 2 σ33 + σ33^2])}
In[3]:= Solve[1+σ-Sqrt[17-2σ+σ^2]==-2,σ]
Out[3]= {{σ → 1}}
In[5]:= S={{1, 0, 2}, {0, 1, 2}, {0, 2, 1}}
        Eigenvalues[S]
Out[5]= {{1, 0, 2}, {0, 1, 2}, {0, 2, 1}}
Out[6]= {3, -1, 1}
```

5.60

The state of stress in a material is partially know at a given point as $\begin{pmatrix} \beta & 0 & 1 \\ 0 & \alpha & 0 \\ 1 & 0 & 3 \end{pmatrix}$. Given that two of the principal stresses at that point are $5, \frac{1}{2}(5 + \sqrt{5})$, Find α and β.

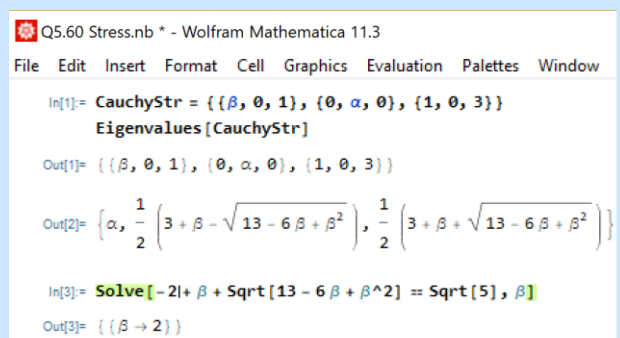
a

In the attached code, the eigenvalues are $\alpha, \frac{1}{2}\left(3 + \beta - \sqrt{13 - 6\beta + \beta^2}\right), \frac{1}{2}\left(3 + \beta + \sqrt{13 - 6\beta + \beta^2}\right)$ which shows that $\alpha = 5$. We solve for β using

$$\left(5 + \sqrt{5}\right) = 3 + \beta + \sqrt{13 - 6\beta + \beta^2}$$

Which yields, $\beta = 2$. The full state of stress is therefore,

$$\begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$



Q5.60 Stress.nb * - Wolfram Mathematica 11.3

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```
In[1]:= CauchyStr = {{β, 0, 1}, {0, α, 0}, {1, 0, 3}}
        Eigenvalues[CauchyStr]
Out[1]= {{β, 0, 1}, {0, α, 0}, {1, 0, 3}}
Out[2]= {α, 1/2 (3 + β - Sqrt[13 - 6 β + β^2]), 1/2 (3 + β + Sqrt[13 - 6 β + β^2])}
In[3]:= Solve[-2|+ β + Sqrt[13 - 6 β + β^2] == Sqrt[5], β]
Out[3]= {{β → 2}}
```

5.61

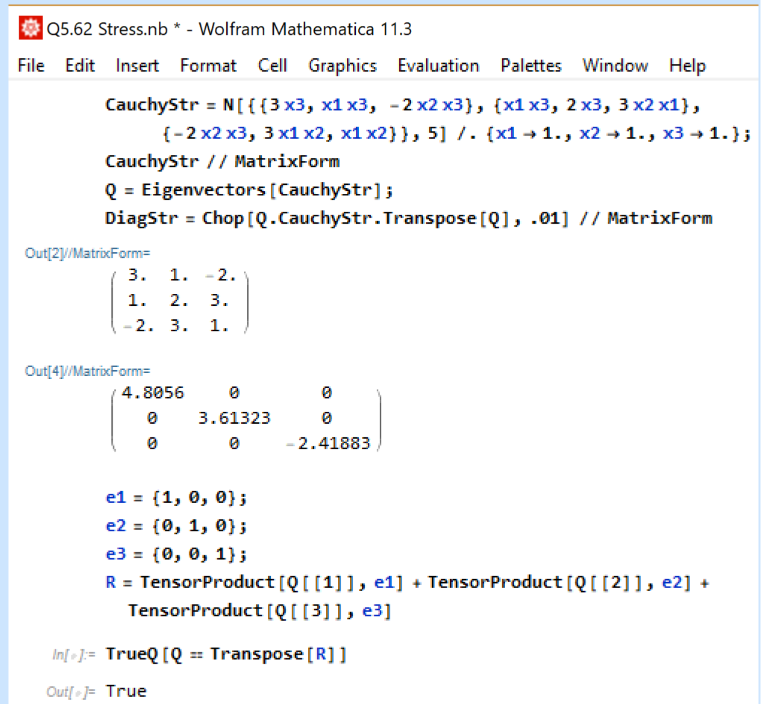
Find the Principal directions of the Cauchy Stress $\begin{pmatrix} 3. & 1. & -2. \\ 1. & 2. & 3. \\ -2. & 3. & 1. \end{pmatrix}$. Rotate the tensor from its present orientation to a coordinate system whose axes are the eigenvectors. Why do you obtain the same answer when you simply use the eigenvector matrix as the rotation tensor?

a

Using the tensor with the matrix of eigenvectors to transform immediately diagonalizes the stress tensor to its diagonal form with its eigenvalues (Principal stresses) on the diagonal.

We can also compose the Rotation tensor of coordinates from $\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3$ to the eigenvectors $\mathbf{q}_1 \quad \mathbf{q}_2 \quad \mathbf{q}_3$. The transpose of the tensor product,

$$\mathbf{q}_1 \otimes \mathbf{e}_1 + \mathbf{q}_2 \otimes \mathbf{e}_2 + \mathbf{q}_3 \otimes \mathbf{e}_3$$



```
Q5.62 Stress.nb * - Wolfram Mathematica 11.3
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CauchyStr = N[{{3 x3, x1 x3, -2 x2 x3}, {x1 x3, 2 x3, 3 x2 x1},
    {-2 x2 x3, 3 x1 x2, x1 x2}}, 5] /. {x1 → 1., x2 → 1., x3 → 1.};
CauchyStr // MatrixForm
Q = Eigenvectors[CauchyStr];
DiagStr = Chop[Q.CauchyStr.Transpose[Q], .01] // MatrixForm

Out[2]//MatrixForm=
 3.   1.  -2.
 1.   2.   3.
-2.   3.   1.

Out[4]//MatrixForm=
 4.8056   0         0
 0        3.61323   0
 0        0        -2.41883

e1 = {1, 0, 0};
e2 = {0, 1, 0};
e3 = {0, 0, 1};
R = TensorProduct[Q[[1]], e1] + TensorProduct[Q[[2]], e2] +
  TensorProduct[Q[[3]], e3]

In[ ]:= TrueQ[Q == Transpose[R]]
Out[ ]= True
```

Because we are rotating, not the tensor, but the coordinate system. We tested if these two tensors are the same in the statement: $\text{TrueQ}[\text{Q} == \text{Transpose}[\text{R}]]$ which returned True. Therefore we could simply use the packed eigenvalue matrix or create a proper rotation tensor. The result is the same because our original tensor is packed into an identity tensor.

5.62 Show that $\operatorname{tr} \mathbf{F}^{-1} = \frac{I_2}{I_3}$

a

Recall that $I_2(\mathbf{F}) = \operatorname{tr} \mathbf{F}^{c} = I_3 \operatorname{tr} \mathbf{F}^{-\mathrm{T}} = I_3 \operatorname{tr} \mathbf{F}^{-1}$ since the trace operation does not change with transposition. The result follows.

$$\operatorname{tr} \mathbf{F}^{-1} = \frac{I_2}{I_3}$$

5.63 Show that $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ depict the same state of stress.

Find a rotation tensor that can transform $\mathbf{A}$ to $\mathbf{B}$.

a The principal invariants of both tensors are the same. in both cases: $\operatorname{tr}\mathbf{A} = \operatorname{tr}\mathbf{B} = 10$, $\operatorname{tr}\mathbf{T}^{c} = \operatorname{tr}\mathbf{B}^{c} = 30$, and $\det\mathbf{A} = \det\mathbf{B} = 25$. They therefore represent the same stress state. The tensor containing the eigenvectors of each tensor will diagonalize each. The transpose of the same tensor with return to the original. The product of the transpose of the eigenvectors of $\mathbf{B}$ with the eigenvectors of $\mathbf{A}$ is the rotation tensor that transforms $\mathbf{A}$ to $\mathbf{B}$ as shown in the attached code.

```
Q5.63 Stress.nb * - Wolfram Mathematica 11.3
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In[*]:= CauchyStr1 = {{2., 0, 1.}, {0, 5., 0}, {1., 0, 3.}};
        Q1 = Eigenvectors[CauchyStr1];
        RotStr1 = Chop[Q1.CauchyStr1.Transpose[Q1], .001];
        CauchyStr2 = {{3., 1., 0}, {1., 2., 0}, {0, 0, 5.}};
        Q2 = Eigenvectors[CauchyStr2];
        RotStr2 = Chop[Q2 . CauchyStr2 . Transpose[Q2], .001];

In[*]:= R1 = Transpose[Q2].Q1;
        R2 = Transpose[Q1].Q2;
        R1.CauchyStr1.Transpose[R1]

Out[*]= {{3., 1., 0.}, {1., 2., 0.}, {0., 0., 5.}}
```

5.64 Given that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ will rotate any vector or tensor through an angle θ around the axial vector $(\mathbf{w}\times) = \mathbf{W}$. For the tensor $(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$. Find the axis of rotation and the rotation angle from current coordinates to principal coordinates.

a

5.65	For any vector $\mathbf{w}$, show that if $\mathbf{W} = (\mathbf{w}\,\times)$, the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ for any choice of θ is a rotation tensor. Demonstrate also that $\mathbf{w}$ is its axis of rotation, and that θ is the angle of rotation.
a	
5.66	Show that $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ depict the same state of stress. Find a rotation axis and angle that can transform $\mathbf{A}$ to $\mathbf{B}$. **Hint:** Remember that the tensor of its eigenvectors will transform each to the same principal coordinate system.
a	

5.67	Given that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ will rotate any vector or tensor through an angle θ around the axial vector $(\mathbf{w}\times) = \mathbf{W}$. Show that $\mathbf{Q}^{\mathrm{T}}(\theta)$ will rotate the same vector in the opposite direction by the same degree.
a	$$\mathbf{Q}^{\mathrm{T}}(\theta) = \mathbf{I} + \mathbf{W}^{\mathrm{T}}\sin\theta + (\mathbf{W}^{\mathrm{T}})^2(1-\cos\theta)$$ Since $\mathbf{W}$ is a skew tensor (all vector crosses are), Its transpose is its negative, Clearly, it becomes obvious that, $$\begin{aligned}\mathbf{Q}^{\mathrm{T}}(\theta) &= \mathbf{I} - \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)\\ &= \mathbf{I} + \mathbf{W}\sin(-\theta) + \mathbf{W}^2(1-\cos(-\theta))\\ &= \mathbf{Q}(-\theta).\end{aligned}$$
5.68	Show that $\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & 6 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 6 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ depict the same state of stress. Find a rotation axis and angle that can transform $\mathbf{A}$ to $\mathbf{B}$. **Hint:** Remember that the tensor of its eigenvectors will transform each to the same principal coordinate system.
a	

5.69	Given that the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ will rotate any vector or tensor through an angle θ around the axial vector $(\mathbf{w}\times) = \mathbf{W}$. For the tensor $(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 2 & 2 & 1 \\ 2 & 6 & -1 \\ 1 & -1 & 3 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$. Find the axis of rotation and the rotation angle from current coordinates to principal coordinates.
a	
5.70	Find the principal directions of the Cauchy Stress $\begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & 6 \end{pmatrix}$. Rotate the tensor from its present orientation to a coordinate system whose axes are the eigenvectors. Why do you obtain the same answer when you simply use the eigenvector matrix as the rotation tensor?
a	

5.71	For any unit vector $\mathbf{w}$, show that if the skew tensor $\mathbf{W} = (\mathbf{w}\,\times)$, the tensor $\mathbf{Q}(\theta) = \mathbf{I} + \mathbf{W}\sin\theta + \mathbf{W}^2(1-\cos\theta)$ for any choice of θ is a rotation tensor. Demonstrate also that $\mathbf{w}$ is its axis of rotation, and that θ is the angle of rotation. Find the rotation axis $\mathbf{w}$, and angle θ that will produce the rotation, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$
a	
5.72	Find the principal invariants of $\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 2 & 6 & -1 \\ 1 & -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 2 & 2 \\ -1 & 2 & 6 \end{pmatrix}$ and show that they both depict the same state of stress. Find the rotation tensor that can transform one into the other. How do these relate to the transformation of the stresses to Principal coordinates?
a	

References

Bertram, A, **Elasticity and Plasticity of Large Deformations, An Introduction**, Springer, 2008

Heinbockel, JH, **Introduction to Tensor Calculus and Continuum Mechanics**, Trafford, 2003

Brannon, RM, **Functional and Structured Tensor Analysis for Engineers**, UNM BOOK DRAFT, 2006, pp 177-184.

Dill, EH, **Continuum Mechanics, Elasticity, Plasticity, Viscoelasticity**, CRC Press, 2007

Gurtin, ME, Fried, E & Anand, L, **The Mechanics and Thermodynamics of Continua**, Cambridge University Press, 2010

Holzapfel, GA, **Nonlinear Solid Mechanics**, Wiley NY, 2007

Kay, DC, **Tensor Calculus**, Schaum Outline Series, McGraw-Hill, 1988

Lai, WM, Rubin, D & Krempl, E, **Introduction to Continuum Mechanics**, Butterworth-Heinemann, Oxford, 2010

Mase,GE, **Theory and Problems of Continuum Mechanics**, Schaum Outline Series, McGraw-Hill, 1970

McConnell, AJ, **Applications of Tensor Analysis**, Dover Publications, NY 1951

Ogden, RW, **Nonlinear Elastic Deformations**, Dover Publications, Inc. NY, 1997

Sokolnikoff, I.S., **Tensor Analysis with Applications to Geometry and the Mechanics of Continua**, John Wiley and Sons, Inc., New York, NY 1964

Spiegel, MR, **Theory and Problems of Vector Analysis**, Schaum Outline Series, McGraw-Hill, 1974

Taber, LA, **Nonlinear Theory of Elasticity**, World Scientific, 2008

Wegner, JL & Haddow, JB, **Elements of Continuum Mechanics and Thermodynamics**, Cambridge University Press, 2009.