Tutorial Kinematics 1

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For the uniform biaxial deformation, given that $x_1 = \lambda_1 X_1, x_2 = \lambda_2 X_2$ and $x_3 = X_3$. Compute the Deformation Gradient tensor, the Lagrangian Strain Tensor as well as the Eulerian Strain Tensor components.

2

Convenient to use Cartesian Bases$\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for Referential and Spatial respectively. The referential reciprocal bases are the same as natural bases

$$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= \alpha_1 \mathbf{e}_1 \otimes \mathbf{E}_1 + \alpha_2 \mathbf{e}_2 \otimes \mathbf{E}_2 + \alpha_3 \mathbf{e}_3 \otimes \mathbf{E}_3$$

- The Green Lagrange strain tensor is,

$$\mathbf{E} = -\frac{1}{2}(1 - \alpha_1^2)\mathbf{E}_1 \otimes \mathbf{E}_1 - \frac{1}{2}(1 - \alpha_2^2)\mathbf{E}_2 \otimes \mathbf{E}_2$$

- Clearly a biaxial state of strain. The rest of the results can be seen from the attached code:

Q4.1 KInematics.nb - Wolfram Mathematica 11.3

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```
In[ ]:= F := {{λ1, 0, 0}, {0, λ2, 0}, {0, 0, 1}}

In[ ]:= CC = Transpose[F].F
        EE = 1/2 (CC - IdentityMatrix[3])
        BB = F.Transpose[F]
        ee = 1/2 (IdentityMatrix[3] - Inverse[BB])
```

$Out[\,]=$ $\{\{\lambda_1^2, 0, 0\}, \{0, \lambda_2^2, 0\}, \{0, 0, 1\}\}$

$Out[\,]=$ $\{\{\frac{1}{2}(-1+\lambda_1^2), 0, 0\}, \{0, \frac{1}{2}(-1+\lambda_2^2), 0\}, \{0, 0, 0\}\}$

$Out[\,]=$ $\{\{\lambda_1^2, 0, 0\}, \{0, \lambda_2^2, 0\}, \{0, 0, 1\}\}$

$Out[\,]=$ $\{\{\frac{1}{2}\left(1-\frac{1}{\lambda_1^2}\right), 0, 0\}, \{0, \frac{1}{2}\left(1-\frac{1}{\lambda_2^2}\right), 0\}, \{0, 0, 0\}\}$

Use Q2.56 to show that $\mathbf{U}$ and $\mathbf{V}$ in the Polar decomposition, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ are similar tensors. Show that they have the same eigenvalues and Principal Invariants.

- From Q2.56, we see that the two tensors are similar if ∃ an invertible tensor $\mathbf{B}$ such that, $\mathbf{V} = \mathbf{BUB}^{-1}$. But from, the equation, $\mathbf{RU} = \mathbf{VR}$, it follows that
$$\mathbf{V} = \mathbf{RUR}^{-1} = \mathbf{RUR}^{\mathrm{T}}$$
- The rotation tensor $\mathbf{R}$ is always invertible. Hence $\mathbf{U}$ and $\mathbf{V}$ are similar tensors.
- The characteristic equation for $\mathbf{V}$ is,
$$\mathbf{Vv} = \lambda\mathbf{v}$$
- where λ is the eigenvalue of $\mathbf{V}$ and $\mathbf{v}$ its eigenvector. But $\mathbf{V} = \mathbf{RUR}^{-1}$ substituting, we have,
$$\mathbf{RUR}^{-1}\mathbf{v} = \lambda\mathbf{v}$$
- so that $\mathbf{UR}^{-1}\mathbf{v} = \lambda\mathbf{R}^{-1}\mathbf{v}$. If we define
$$\mathbf{v}_1 \equiv \mathbf{R}^{-1}\mathbf{v}$$
- we obtain, $\mathbf{Uv}_1 = \lambda\mathbf{v}_1$ yielding the same characteristic equation as well as eigenvalues and principal invariants as $\mathbf{Vv} = \lambda\mathbf{v}$

Given that $\mathbf{U}$ and $\mathbf{V}$ in the Polar decomposition,

$\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ are similar tensors. Are their eigenvectors the same?.

- From TK4.2, we find that the eigenvalues of $\mathbf{U}$ and $\mathbf{V}$ as well as their Principal Invariants are equal. However, in arriving at that proof, we saw that if λ is the eigenvalue of $\mathbf{V}$ and $\mathbf{v}$ its eigenvector, then $\mathbf{V} = \mathbf{RUR}^{-1}$ substituting, we have,

$$\mathbf{RUR}^{-1}\mathbf{v} = \lambda\mathbf{v}$$

- so that $\mathbf{UR}^{-1}\mathbf{v} = \lambda\mathbf{R}^{-1}\mathbf{v}$.
- Clearly, the eigenvector of $\mathbf{U}$ is $\mathbf{R}^{-1}\mathbf{v} = \mathbf{R}^{\mathrm{T}}\mathbf{v}$. This is the eigenvector of $\mathbf{V}$ rotated in the reverse direction by the same rotation tensor in their Polar Decomposition.

Show that Right Cauchy Green Tensor, CC={{6,5,4},{5,6,4},{4,4,3}} is symmetric and positive definite; and that R={{.888354,-.430577,.159465},{.385919,.888354,.248782},{-.248782,-.159465, .955341}} is a rotation tensor. Use these to find the deformation gradient that produced them as well as the Right and Left Stretch Tensors.

- The positive definiteness of the Cauchy Green Tensor is easily seen by finding its eigenvalues. They are all positive numbers.
- Take the inverse of **R** and compare it to its transpose to see if it is truly a rotation tensor.
- The square root of the Right Cauchy-Green Tensor is the Right Stretch Tensor as shown.
- Observe that the Deformation Gradient is NOT Symmetrical. The Left Stretch Tensor, just like the Right Stretch Tensor is symmetrical. You can check to see the equality of the eigenvalues of the Stretch Tensors.

```
CC = {{6, 5, 4}, {5, 6, 4}, {4, 4, 3}};
R = {{.888354, -.430577, .159465}, {.385919, .888354, .248782},
   {-.248782, -.159465, .955341}};

Eigenvalues[CC]
U = MatrixPower[CC, 0.5];
F = R.U;
V = F.Transpose[R];
```

$\{7+4\sqrt{3}, 1, 7-4\sqrt{3}\}$

```
U // MatrixForm
F // MatrixForm
V // MatrixForm
```

ixForm=

$$\begin{pmatrix} 2. & 1. & 1. \\ 1. & 2. & 1. \\ 1. & 1. & 1. \end{pmatrix}$$

ixForm=

$$\begin{pmatrix} 1.5056 & 0.186665 & 0.617242 \\ 1.90897 & 2.41141 & 1.52306 \\ 0.298312 & 0.387629 & 0.547094 \end{pmatrix}$$

ixForm=

$$\begin{pmatrix} 1.35556 & 0.900421 & 0.185345 \\ 0.900421 & 3.2578 & 0.595583 \\ 0.185345 & 0.595583 & 0.386633 \end{pmatrix}$$

Given that $\mathbf{F} = \{\{1.5056, 0.186665, 0.617242\}, \{1.90897, 2.41141, 1.52306\}, \{0.298312, 0.387629, 0.547094\}\}$ and that $\mathbf{V} = \{\{1.35556, 0.900421, 0.185345\}, \{0.900421, 3.2578, 0.595583\}, \{0.185345, 0.595583, 0.386633\}\}$, Demonstrate that $\mathbf{U}^2 = \mathbf{F}^\mathrm{T}\mathbf{F}$ and that $\mathbf{V}^2 = \mathbf{F}\mathbf{F}^\mathrm{T}$

- We can find the rotation tensor from the relationship, $\mathbf{F} = \mathbf{RU} \Rightarrow \mathbf{R} = \mathbf{FU}^{-1}$and, $\mathbf{V} = \mathbf{RUR}^{-1} = \mathbf{RUR}^\mathrm{T}$.
- Each of the Stretch Tensors can be squared by the MatrixPower[] function shown and the results are compared to the values obtained for the Right and Left Cauchy-Green Tensors.
- Mathematica gave results to five decimal places because they were already approximated at input.

```
MatrixPower[V, 2] // MatrixForm
F.Transpose[F] // MatrixForm
MatrixPower[U, 2] // MatrixForm
Transpose[F].F // MatrixForm

latrixForm=
 2.68265   4.26436   0.859182
 4.26436   11.7788   2.33745
 0.859182  2.33745   0.538557

latrixForm=
 2.68265   4.26436   0.859184
 4.26436   11.7788   2.33746
 0.859184  2.33746   0.538558

latrixForm=
 6.  5.  4.
 5.  6.  4.
 4.  4.  3.

latrixForm=
 5.99999  4.99999  3.99999
 4.99999  5.99999  4.
 3.99999  4.       3.
```

Given the set of orthogonal basis vectors, $\mathbf{E}_i, i = 1, \ldots, 3$ and another set, $\boldsymbol{\xi}_i, i = 1, \ldots, 3$, The latter is called a reciprocal bases if, $\mathbf{E}_i \otimes \boldsymbol{\xi}_i = \mathbf{I}$. Show that the Natural Cartesian basis vectors are self reciprocal

- Given the Cartesian position vector,

$$\mathbf{r} = x_i \mathbf{e}_i$$

- the natural basis vectors come from direct partial differentiation:

$$\mathbf{E}_j = \frac{\partial \mathbf{r}}{\partial x_j} = \delta_{ij} \mathbf{e}_i = \mathbf{e}_j$$

- Writing $\boldsymbol{\xi}_i = \mathbf{e}_i$ and evaluating the sum,

$$\mathbf{e}_i \otimes \boldsymbol{\xi}_i = \mathbf{I}$$

The identity tensor. This shows clearly that Cartesian natural basis vectors are self reciprocal. All orthonormal bases vectors are similarly self reciprocal.

Given the set of orthogonal basis vectors, $\mathbf{E}_i, i = 1, \dots, 3$ and another set, $\boldsymbol{\xi}_i, i = 1, \dots, 3$, The latter is called a reciprocal bases if, $\mathbf{E}_i \otimes \boldsymbol{\xi}_i = \mathbf{I}$. Show that the Natural Cylindrical Polar basis vectors are not self reciprocal. Find their reciprocal bases.

- Given the Cylindrical Polar position vector,

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_3 = r\mathbf{e}_r(\phi) + z\mathbf{e}_z$$

- where $\mathbf{e}_r(\phi) = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$ is a variable unit vector as it depends on ϕ while $\mathbf{e}_z$ is a constant unit vector. The natural basis vectors come from direct partial differentiation:

$$\mathbf{E}_1 = \frac{\partial\mathbf{r}}{\partial r} = \mathbf{e}_r(\phi)$$

$$\mathbf{E}_2 = \frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\phi\,\mathbf{e}_1(\phi) + r\cos\phi\,\mathbf{e}_2 = r\mathbf{e}_\phi(\phi)$$

Where we have defined, $\mathbf{e}_\phi \equiv -\sin\phi\,\mathbf{e}_1(\phi) + \cos\phi\,\mathbf{e}_2$. Differentiating again, we find that,

$$\mathbf{E}_3 = \frac{\partial\mathbf{r}}{\partial z} = \mathbf{e}_z$$

- Writing the sum, $\mathbf{E}_1 \otimes \mathbf{E}_1 + \mathbf{E}_2 \otimes \mathbf{E}_2 + \mathbf{E}_3 \otimes \mathbf{E}_3$ evaluating the sum, we obtain,

$$(\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) \otimes (\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + r^2(-\sin\phi\,\mathbf{e}_1(\phi) + \cos\phi\,\mathbf{e}_2) \otimes(-\sin\phi\,\mathbf{e}_1(\phi) + \cos\phi\,\mathbf{e}_2) + \mathbf{e}_3 \otimes \mathbf{e}_3$$

- The above sum does not yield the desired identity tensor. The problem arises from the fact that the second bases is not a unit vector as the natural bases of curvilinear systems are not orthonormal.
- Reciprocal bases are obtained by dividing the second basis by its magnitude so that, $\boldsymbol{\xi}_2 = \frac{\mathbf{E}_2}{r} = \frac{\mathbf{e}_\phi(\phi)}{r}$. We find therefore that the sum,
- $\mathbf{E}_1 \otimes \boldsymbol{\xi}_1 + \mathbf{E}_2 \otimes \boldsymbol{\xi}_2 + \mathbf{E}_3 \otimes \boldsymbol{\xi}_3$
- $= (\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) \otimes (\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2) + (-\sin\phi\,\mathbf{e}_1(\phi) + \cos\phi\,\mathbf{e}_2) \otimes (-\sin\phi\,\mathbf{e}_1(\phi) + \cos\phi\,\mathbf{e}_2) + \mathbf{e}_3 \otimes \mathbf{e}_3$
- $= (\cos^2\phi + \sin^2\phi)\mathbf{e}_1 \otimes \mathbf{e}_1 + (\cos^2\phi + \sin^2\phi)\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{I}$
- all other terms cancelling out.

Given the set of orthogonal basis vectors, $\mathbf{E}_i, i = 1, \dots, 3$ and another set, $\boldsymbol{\xi}_i, i = 1, \dots, 3$, The latter is called a reciprocal bases if, $\mathbf{E}_i \otimes \boldsymbol{\xi}_i = \mathbf{I}$. Show that the Natural Spherical Polar basis vectors are not self reciprocal. Find their reciprocal bases.

- We can follow the same arguments as in the above slide. Begin with the spherical position vector,

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \rho \sin\theta\, \mathbf{e}_r + \rho\cos\theta$$
$$= \rho\sin\theta\,(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \rho\cos\theta\,\mathbf{k}$$
$$= \rho\sin\theta\cos\phi\,\mathbf{i} + \rho\sin\theta\sin\phi\,\mathbf{j} + \rho\cos\theta\,\mathbf{k} \equiv \rho\mathbf{e}_\rho(\theta,\phi)$$

- Where $\mathbf{e}_\rho(\theta,\phi) \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$, $\mathbf{e}_\theta(\theta,\phi) = \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$, and $\mathbf{e}_\phi(\phi) = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$ in terms of the Cartesian basis vectors. Two things to note: All basis vectors are variables. Only $\mathbf{e}_\rho(\theta,\phi)$ appears in the definition of the Spherical Position Vector. The others come out as we differentiate as follows:

$$\mathbf{r} = \rho\mathbf{e}_\rho(\theta,\phi)$$
$$\mathbf{E}_1 = \frac{\partial\mathbf{r}}{\partial\rho} = \mathbf{e}_\rho(\theta,\phi);$$
$$\mathbf{E}_2 = \frac{\partial\mathbf{r}}{\partial\theta} = \rho\mathbf{e}_\theta(\theta,\phi);$$
$$\mathbf{E}_3 = \frac{\partial\mathbf{r}}{\partial\phi} = \rho\,\sin\theta\,\mathbf{e}_\phi(\phi)$$

- Only the first natural basis is a unit vector.
- We can obtain the reciprocal bases by simply dividing the respective unit vectors by the magnitudes as follows:

$$\boldsymbol{\xi}_1 = \frac{\partial\mathbf{r}}{\partial\rho} = \mathbf{e}_\rho(\theta,\phi);$$
$$\boldsymbol{\xi}_2 = \frac{\partial\mathbf{r}}{\partial\theta} = \frac{\mathbf{e}_\theta(\theta,\phi)}{\rho};$$
$$\boldsymbol{\xi}_3 = \frac{\partial\mathbf{r}}{\partial\phi} = \frac{\mathbf{e}_\phi(\phi)}{\rho\sin\theta}$$

As $\mathbf{e}_\rho(\theta,\phi)$, $\mathbf{e}_\theta(\theta,\phi)$ and $\mathbf{e}_\phi(\phi)$ are always orthonormal, the sum,

$$\mathbf{E}_1 \otimes \boldsymbol{\xi}_1 + \mathbf{E}_2 \otimes \boldsymbol{\xi}_2 + \mathbf{E}_3 \otimes \boldsymbol{\xi}_3 = \mathbf{I}$$

Consider a deformation of the form $\mathbf{x} = \boldsymbol{\omega} \times \mathbf{X}$ Here $\boldsymbol{\omega}$ is a vector with magnitude $\ll 1$, which represents an infinitesimal rotation about an axis parallel to $\boldsymbol{\omega}$ Show that $\mathbf{C} = (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{I} - \boldsymbol{\omega} \otimes \boldsymbol{\omega}$.

- Deformation Gradient, $\mathbf{F} = \boldsymbol{\omega} \times$. This is a skew tensor. The transpose, $\mathbf{F}^{\mathrm{T}} = -\boldsymbol{\omega} \times$, its negative. The Right Cauchy-Green Tensor,

$$\begin{aligned}\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} &= -(\boldsymbol{\omega} \times)(\boldsymbol{\omega} \times) = -\left(e_{ijk}\omega_j \mathbf{e}_i \otimes \mathbf{e}_k\right)\left(e_{\alpha\beta\gamma}\omega_\beta \mathbf{e}_\alpha \otimes \mathbf{e}_\gamma\right) \\ &= -e_{ijk}\omega_j e_{\alpha\beta\gamma}\omega_\beta \mathbf{e}_i \otimes \mathbf{e}_\gamma \delta_{k\alpha} = -e_{ijk}\omega_j e_{\beta\gamma k}\omega_\beta \mathbf{e}_i \otimes \mathbf{e}_\gamma \\ &= \left(\delta_{i\gamma}\delta_{j\beta} - \delta_{i\beta}\delta_{j\gamma}\right)\omega_\beta \mathbf{e}_i \otimes \mathbf{e}_\gamma \\ &= \omega_j\omega_j \mathbf{e}_i \otimes \mathbf{e}_i - \omega_i\omega_j \mathbf{e}_i \otimes \mathbf{e}_j \\ &= (\boldsymbol{\omega} \cdot \boldsymbol{\omega})\mathbf{I} - \boldsymbol{\omega} \otimes \boldsymbol{\omega}\end{aligned}$$

A body undergoes a deformation defined by, $x_1 = X_1 \cos\alpha - X_2 \sin\alpha$, $X_2 = X_1 \sin\alpha + X_2 \cos\alpha$, and $x_3 = X_3$ where α is a constant. Show that $\mathbf{C} = \mathbf{I}$ and $\mathbf{E} = \mathbf{O}$. Explain the reason for the values of $\mathbf{E}$ components.

- The deformation gradient here is the rotation tensor through angle α around the $\mathbf{e}_3$ axis. Consequently,

$$\mathbf{F} = \mathbf{R} = \mathbf{RI} = \mathbf{RU}$$

So that $\mathbf{U} = \mathbf{I}$.

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{I}$$

And,

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \mathbf{O}$$

A cylindrical bar, fixed at one end is twisted at the other as shown. Given that the transformation equations are, $r = R, \theta = \Theta + f(Z), Z = Z$, find the deformation gradient, Right Cauchy Green Tensor and the Euler Lagrange Strain.

- In this case Referential system is Cylindrical Polar, Spatial is also cylindrical Polar.

$$\mathbf{F} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{R} & r\frac{\partial f}{\partial Z} \\ 0 & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

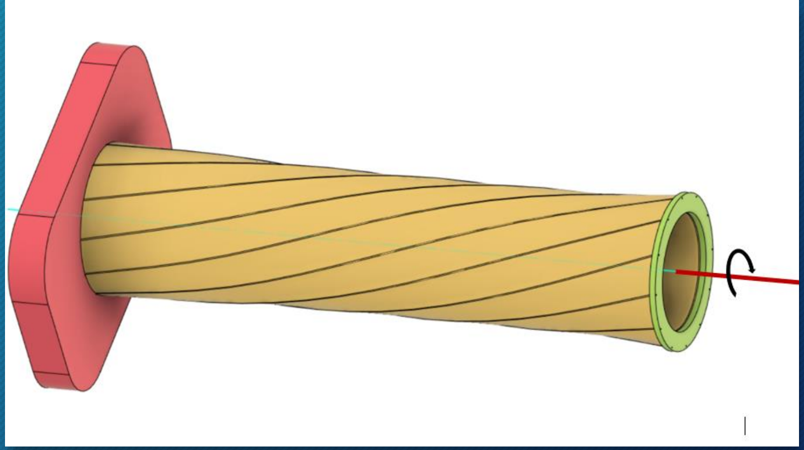
```
F := {{1, 0, 0}, {0, r / R, r f[Z]}, {0, 0, 1}}

CC = Transpose[F].F
```

$$\left\{\{1, 0, 0\}, \left\{0, \frac{r^2}{R^2}, \frac{r^2 f[Z]}{R}\right\}, \left\{0, \frac{r^2 f[Z]}{R}, 1 + r^2 f[Z]^2\right\}\right\}$$

```
EE = (1 / 2) (CC - IdentityMatrix[3]) // MatrixForm
```

trixForm=

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2}\left(-1 + \frac{r^2}{R^2}\right) & \frac{r^2 f[Z]}{2 R} \\ 0 & \frac{r^2 f[Z]}{2 R} & \frac{1}{2} r^2 f[Z]^2 \end{pmatrix}$$



In addition to twisting, if the above cylindrical bar is also subject to elongation and expansion, Find an expression for the deformation and evaluate its Deformation Gradient.

- With the inclusion of extension, it is no longer tenable that $r = R$; Now we have, $r = \chi_r(R), \theta = \chi_\theta(\Theta, \mathrm{Z}) = \Theta + f(Z), Z = \alpha Z$
- This means that the angular deformation function is still adequate and the z-direction is assumed to have a linear extension. More general deformations are possible.
- Same computation code as above can be used.

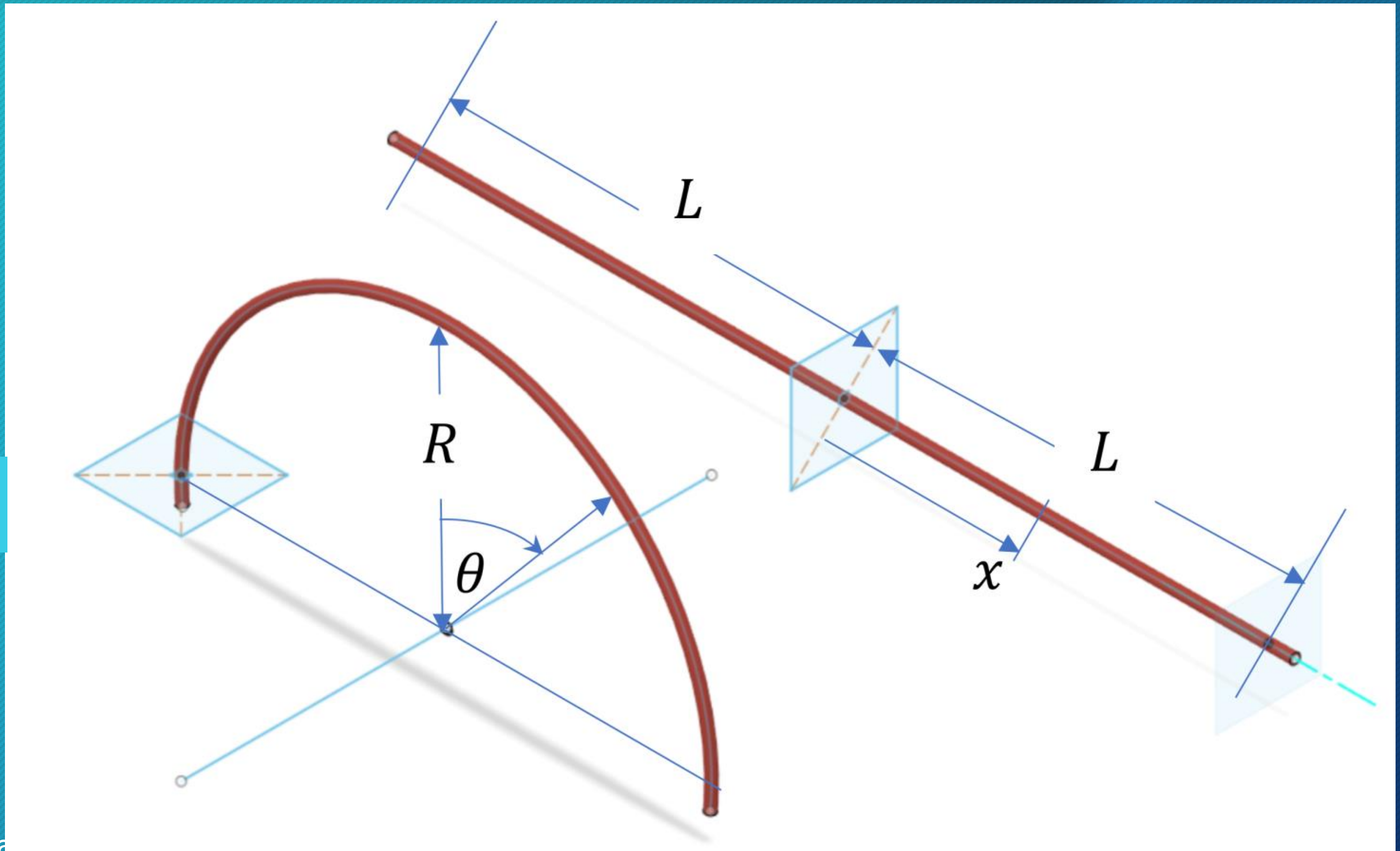
Curved Rod

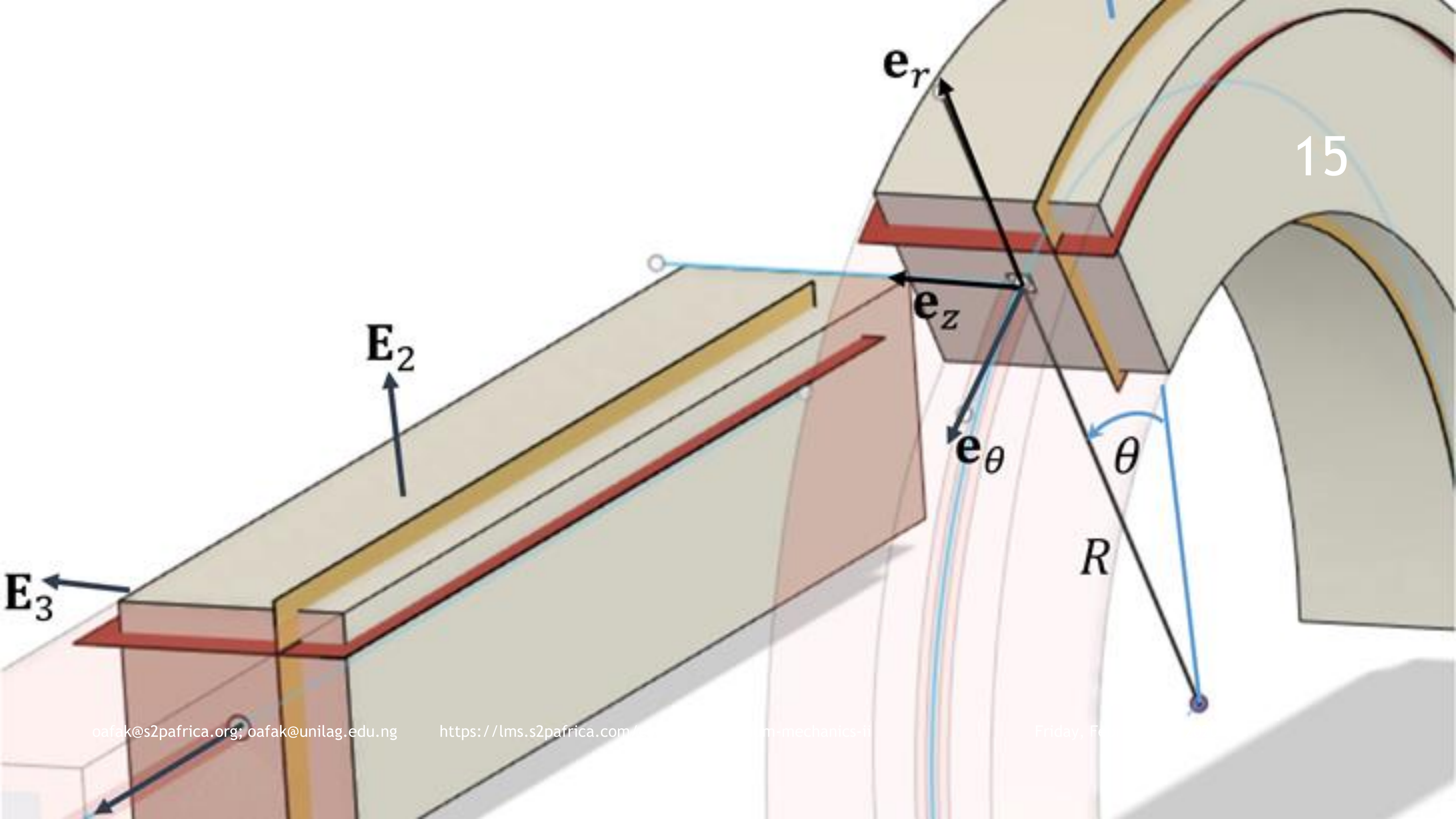
- Consider the length $2L$ of a thin rod uniformly bent into a semicircle as shown.
- Referential configuration is the straight rod, Spatial, after the bending, is the semi-circular rod. If the rod's length does not increase as a result of shape change, then $\pi R = 2L$. Clearly, radius $R = 2L/\pi$
- A point previously located at the distance x from the origin is now at angle θ. The relationship between the two is linear:

$$\frac{x}{2L} = \frac{\theta}{\pi} \Rightarrow \theta = \frac{\pi x}{2L}$$

How else can you obtain this formula?

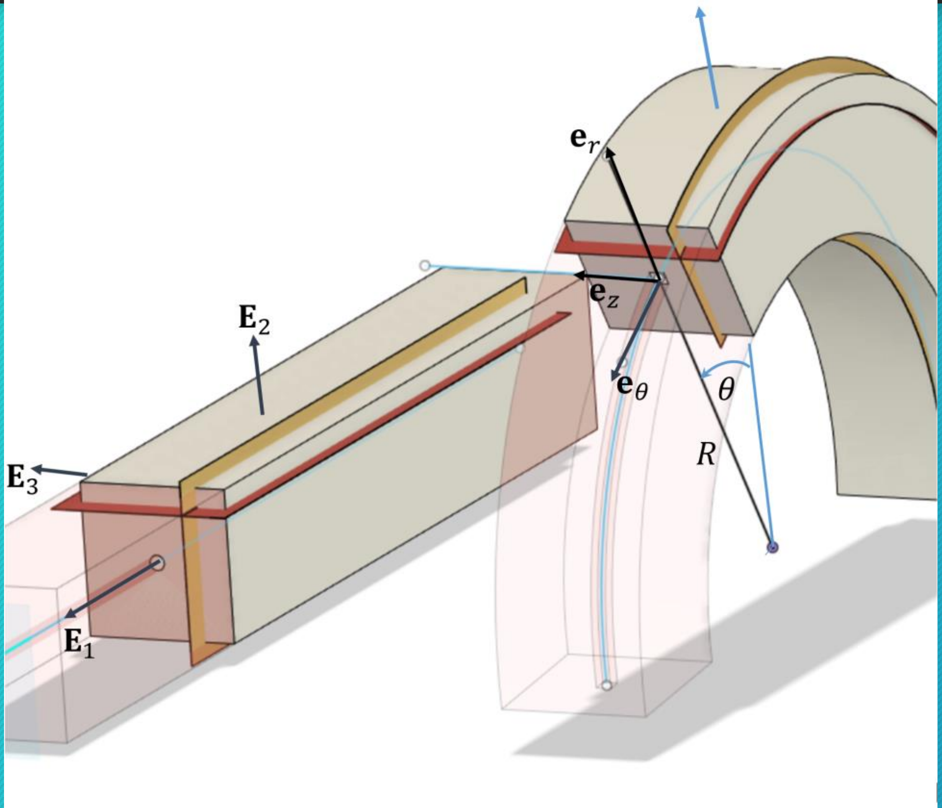
x	0	$\frac{L}{2}$	L	$-L$
θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$-\frac{\pi}{2}$





$\mathbf{e}_r$
$\mathbf{e}_z$
$\mathbf{e}_\theta$
θ
R
$\mathbf{E}_2$
$\mathbf{E}_3$

Bar to Semicircular Region



- Imagine that we bent the bar shown into a semicircular region. Transformation function can be found by the following consideration: Note that each horizontal filament in the original bar becomes a circular filament in the spatial configuration. The vertical undeformed sections become radial sections in the spatial state. Let the centerline be a semicircle at a distance R and let the thickness contract uniformly with a factor α

$$\Rightarrow x_1 = r = \chi_1(X_1, X_2, X_3, t) = R + \alpha X_2, \text{ and}$$

$$x_2 = \theta = \chi_2(X_1, X_2, X_3, t) = \frac{\pi X_1}{2L}$$

- If the bar contracts uniformly in X_3 direction,

$$x_3 = z = \chi_3(X_1, X_2, X_3, t) = \beta X_3$$

Referential & Spatial Configurations

- Clearly, the referential configuration here is the bar; Spatial is the semicircular bar.
- Deformation is such that the spatial is in cylindrical coordinates, the referential is in Cartesian.
- Deformation gradient requires the reciprocal Cartesian bases which are the same as the Cartesian. In the spatial, we use the cylindrical. The full computation given in Q4.7, is repeated here:

Deformation Gradient

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \left(\frac{\pi r}{2L}\mathbf{e}_\theta \quad \alpha\mathbf{e}_r \quad \beta\mathbf{e}_z\right) \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= \frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1 + \alpha\mathbf{e}_r \otimes \mathbf{E}_2 + \beta\mathbf{e}_z \otimes \mathbf{E}_3$$

Right Cauchy-Green/Stretch Tensors

- Clearly,

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \left(\frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{e}_\theta + \alpha \mathbf{E}_2 \otimes \mathbf{e}_r + \beta \mathbf{E}_3 \otimes \mathbf{e}_z\right)\left(\frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1 + \alpha \mathbf{e}_r \otimes \mathbf{E}_2 + \beta \mathbf{e}_z \otimes \mathbf{E}_3\right)$$

$$= \left(\frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{e}_\theta\right)\left(\frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1\right) + \cdots + (\beta \mathbf{E}_3 \otimes \mathbf{e}_z)(\beta \mathbf{e}_z \otimes \mathbf{E}_3)$$

$$= \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2 \mathbf{E}_3 \otimes \mathbf{E}_3$$

- since each set of basis vectors is orthonormal, and the Right Stretch Tensor,

$$\mathbf{U} = \frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta \mathbf{E}_3 \otimes \mathbf{E}_3$$

- Is the square root of the Right Cauchy Green tensor. The positive square roots are taken since both $\mathbf{C}$ as well as $\mathbf{U}$ are necessarily positive definite and can only have positive eigenvalues.

Computing Functions in Cylindrical Systems

$$r = r(R, \Theta, Z) = \chi_r(R, \Theta, Z); \theta = \theta(R, \Theta, Z) = \chi_\theta(R, \Theta, Z); z = z(R, \Theta, Z) = \chi_z(R, \Theta, Z)$$

$$d\mathbf{x} = \frac{d\mathbf{x}}{d\mathbf{X}} d\mathbf{X} = \frac{d\boldsymbol{\chi}}{d\mathbf{X}} d\mathbf{X} = \mathbf{F}\, d\mathbf{x}$$

The spatial position vector, $\mathbf{x} = r\mathbf{e}_r(r, \theta) + z\mathbf{e}_z \Rightarrow$

$$d\mathbf{x} = \frac{\partial \mathbf{x}}{\partial r} dr + \frac{\partial \mathbf{x}}{\partial \theta} d\theta + \frac{\partial \mathbf{x}}{\partial z} dz = \mathbf{e}_r dr + r \frac{\partial \mathbf{e}_r(r, \theta)}{\partial \theta} d\theta + \mathbf{e}_z dz$$

$$= \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta + \mathbf{e}_z dz$$

Similarly, in the Referential,

$$d\mathbf{X} = \frac{\partial \mathbf{X}}{\partial R} dR + \frac{\partial \mathbf{X}}{\partial \Theta} d\Theta + \frac{\partial \mathbf{X}}{\partial Z} dZ = \mathbf{E}_R dR + R\mathbf{E}_\Theta d\Theta + \mathbf{E}_Z dZ$$

Cylindrical Deformation Gradient

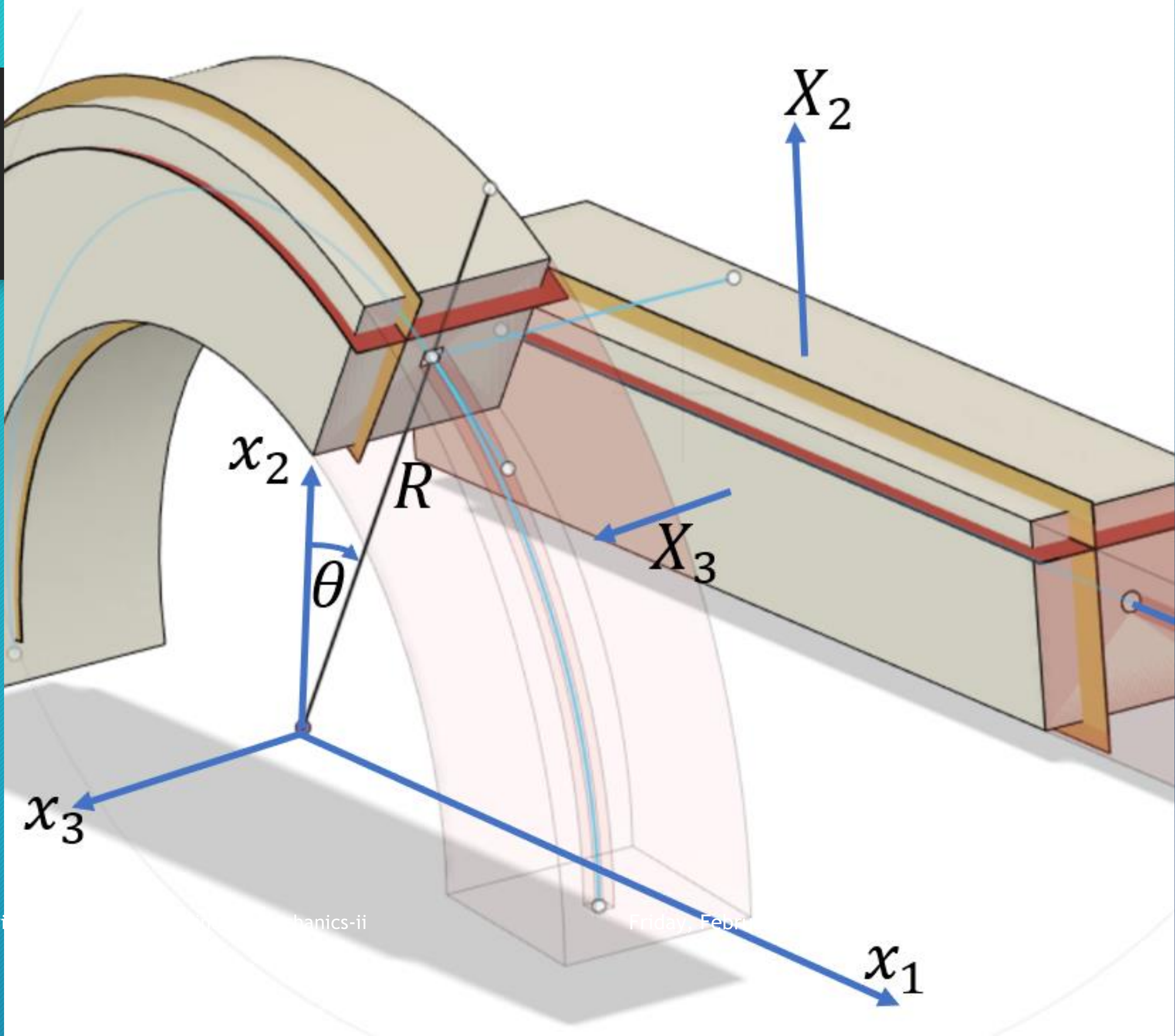
$$d\mathbf{x} = \frac{d\boldsymbol{\chi}}{d\mathbf{X}} d\mathbf{X} = \mathbf{F}\, d\mathbf{X} = \left((\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_z) \begin{pmatrix} \frac{\partial \chi_r}{\partial R} & \frac{\partial \chi_r}{\partial \Theta} & \frac{\partial \chi_r}{\partial Z} \\ \frac{\partial \chi_\theta}{\partial R} & \frac{\partial \chi_\theta}{\partial \Theta} & \frac{\partial \chi_\theta}{\partial Z} \\ \frac{\partial \chi_z}{\partial R} & \frac{\partial \chi_z}{\partial \Theta} & \frac{\partial \chi_z}{\partial Z} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{pmatrix} \right) \begin{pmatrix} \mathbf{E}_R \\ R\mathbf{E}_\Theta \\ \mathbf{E}_Z \end{pmatrix}$$

So that the deformation gradient, in terms of unit vector sets $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$, the matrix of $\mathbf{F}$ can be written as,

$$[\mathbf{F}] = \begin{pmatrix} \frac{\partial \chi_r}{\partial R} & \frac{1}{R}\frac{\partial \chi_r}{\partial \Theta} & \frac{\partial \chi_r}{\partial Z} \\ r\frac{\partial \chi_\theta}{\partial R} & \frac{r}{R}\frac{\partial \chi_\theta}{\partial \Theta} & r\frac{\partial \chi_\theta}{\partial Z} \\ \frac{\partial \chi_z}{\partial R} & \frac{1}{R}\frac{\partial \chi_z}{\partial \Theta} & \frac{\partial \chi_z}{\partial Z} \end{pmatrix}$$

Bar to SemiCircle: Cartesian Solution

- The same Bar to a semicircular region may be solved using Cartesian coordinates. In fact, the two not only give the same results but looking at both brings out the salient issues of the two systems especially the concept of the reciprocal basis.
- For this reason, we present here the Cartesian analysis of the same problem and obtain the Deformation Gradient and other relevant tensors.

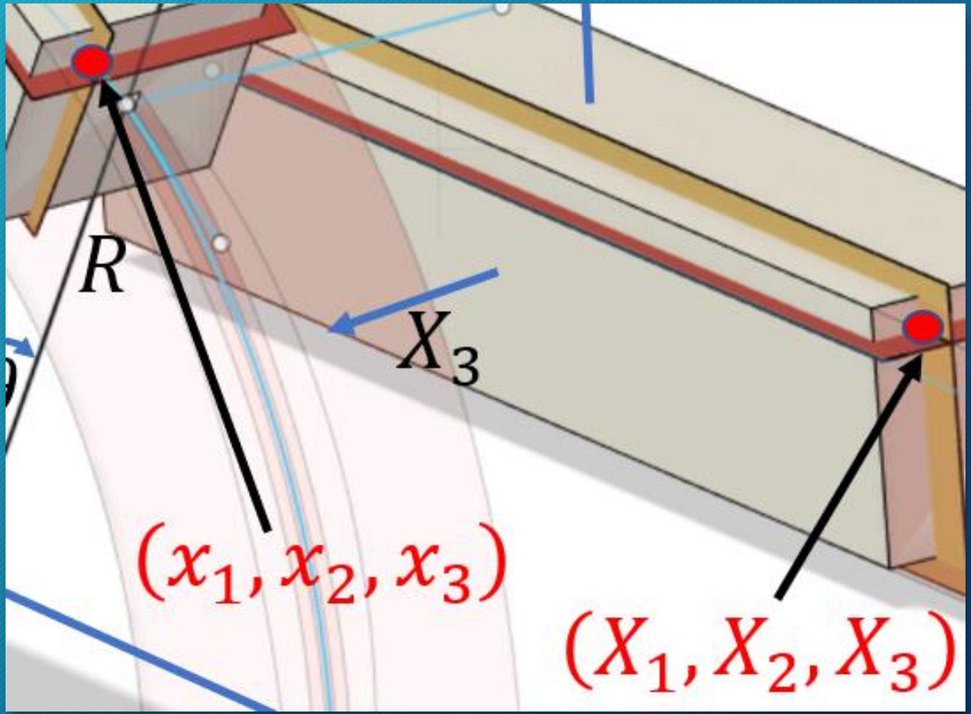


Bar to SemiCircular Bar

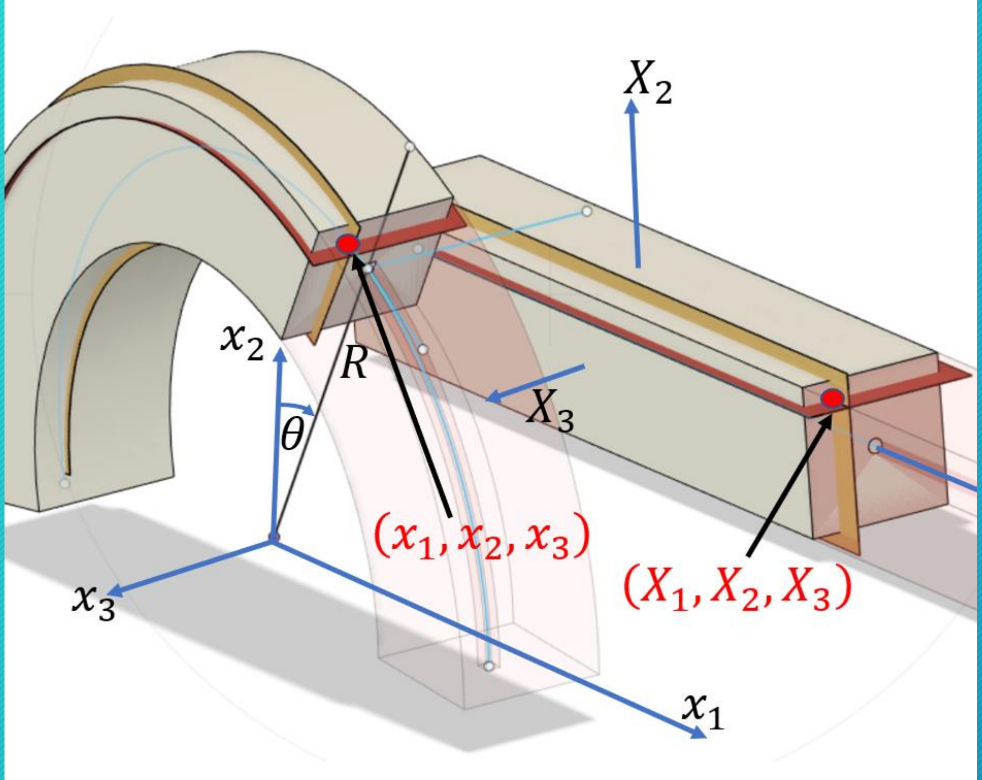
- We are interested in how the referential point, (X_1, X_2, X_3) transformed into the spatial point, (x_1, x_2, x_3).This time, each coordinate system is referred to Cartesian Base vectors. The radial distance of each fiber is still dependent on how far, along the vertical axis, it is from the midplane. It is therefore dependent on X_2 only. We can write that the curved surface at each fibre is located at $r = \chi_r(X_2) = R + \alpha X_2$
- Its angular displacement is dependent on where it is along X_1-axis. Therefore,

$$\theta = \chi_\theta(X_1) = \frac{\pi X_1}{2L}$$

As before, $x_3 = \beta X_3$. We now proceed to express these in Spatial Cartesian coordinates:



Bar to SemiCircular Bar



The Coordinates of the spatial point are:

$$x_1 = \chi_r(X_2)\cos\big(\chi_\theta(X_1)\big) = (R + \alpha X_2)\cos\frac{\pi X_1}{2L}$$

$$x_2 = \chi_r(X_2)\sin\big(\chi_\theta(X_1)\big) = (R + \alpha X_2)\sin\frac{\pi X_1}{2L}$$

$$x_3 = \beta X_3$$

$$\mathbf{F} = \frac{\partial \chi_\alpha}{\partial X_j}\mathbf{e}_\alpha \otimes \mathbf{E}_j$$

$$= (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} \frac{\partial x_1}{\partial X_1} & \frac{\partial x_1}{\partial X_2} & \frac{\partial x_1}{\partial X_3} \\ \frac{\partial x_2}{\partial X_1} & \frac{\partial x_2}{\partial X_2} & \frac{\partial x_2}{\partial X_3} \\ \frac{\partial x_3}{\partial X_1} & \frac{\partial x_3}{\partial X_2} & \frac{\partial x_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

Bar to SemiCircular Bar

$$\mathbf{F} = (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{bmatrix} -\frac{\pi\chi_r}{2L}\sin\frac{\pi X_1}{2L} & \alpha\cos\frac{\pi X_1}{2L} & 0 \\ \frac{\pi\chi_r}{2L}\cos\frac{\pi X_1}{2L} & \alpha\sin\frac{\pi X_1}{2L} & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

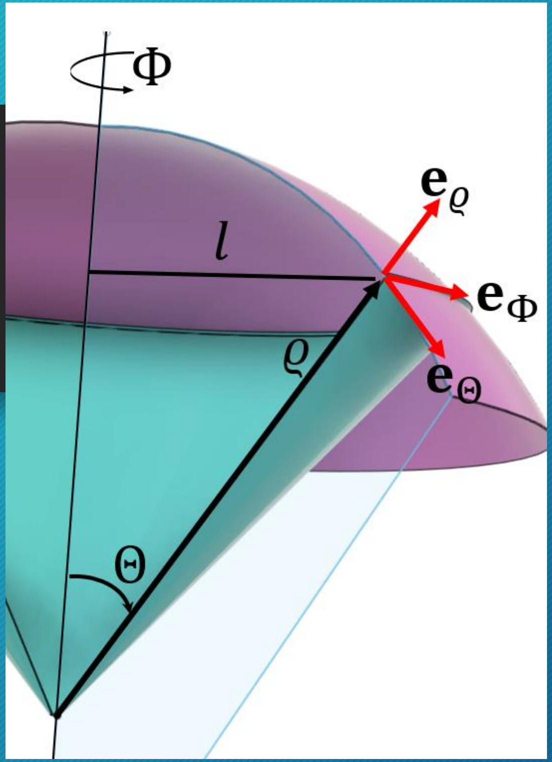
From which we easily find, as before, that

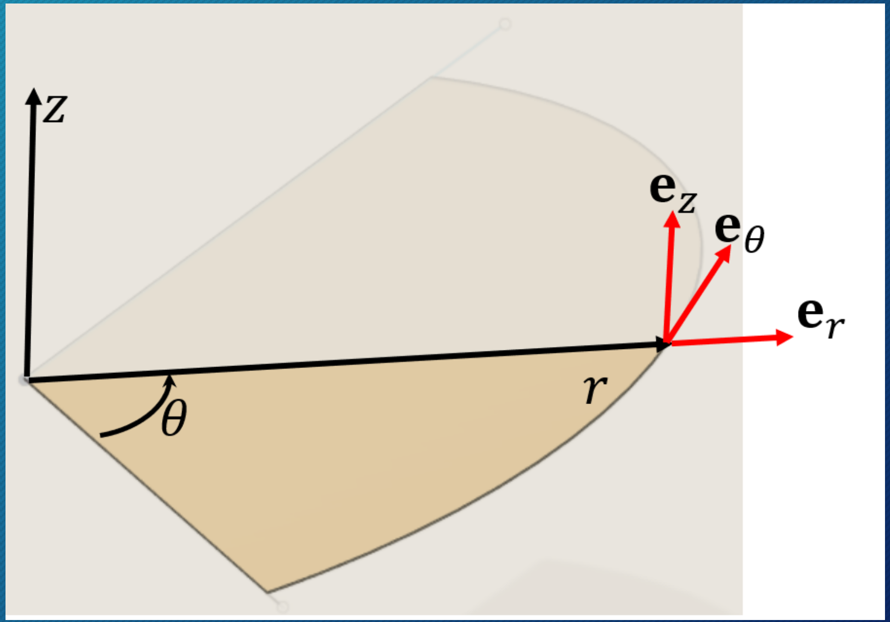
$$\mathbf{C} = \left(\frac{\pi\chi_r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2\mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2\mathbf{E}_3 \otimes \mathbf{E}_3$$

Remember that we had a base system that is self reciprocal here. The deformation gradient looked a bit different but the underlying Right Cauchy-Green Tensor is the same.

Cone to Plane Object

- Consider the cone with half angle Θ as shown. At a slant length of ϱ, clearly, distance l to the foot of the axial line is such that,
$$l = \varrho \sin\Theta$$
- Selecting spherical base vectors $\mathbf{e}_\varrho$, $\mathbf{e}_\Theta$ and $\mathbf{e}_\Phi$ as shown, we can find the transformation equations of the conical lamina to the flat plane shown. Since the total rim length of the cone $2\pi l = 2\pi\varrho\sin\Theta$ must coincide with the curved length of the plane lamina, included angle θ must be such that,
$$0 \le \theta \le 2\pi\sin\Theta$$



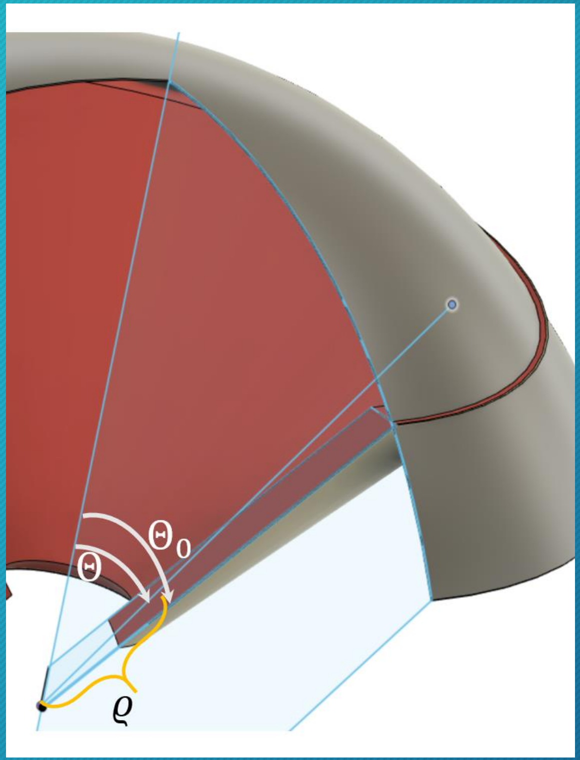


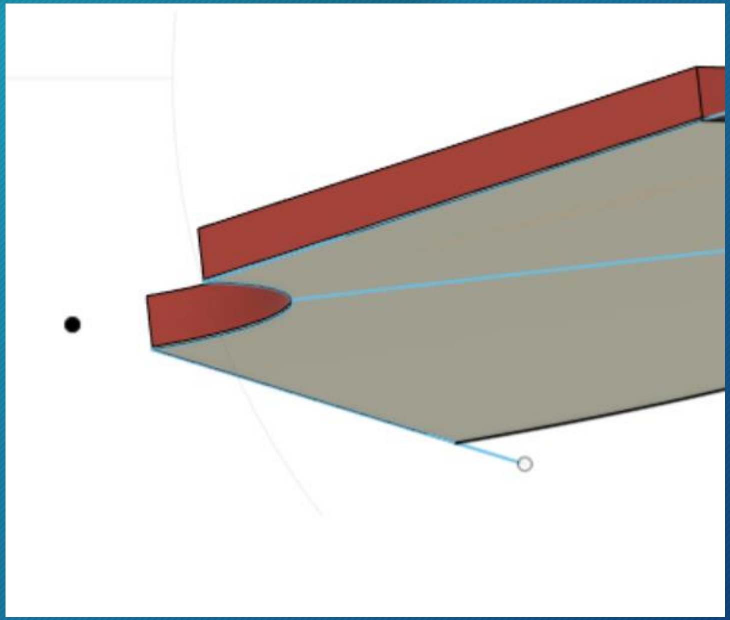
Transformation Equations

$$r = \chi_r(\varrho, \Theta, \Phi) = \varrho$$
$$\theta = \chi_\theta(\varrho, \Theta, \Phi) = \Phi \sin \Theta_0$$
$$z = \chi_z(\varrho, \Theta, \Phi) = \alpha\varrho \tan(\Theta_0 - \Theta)$$

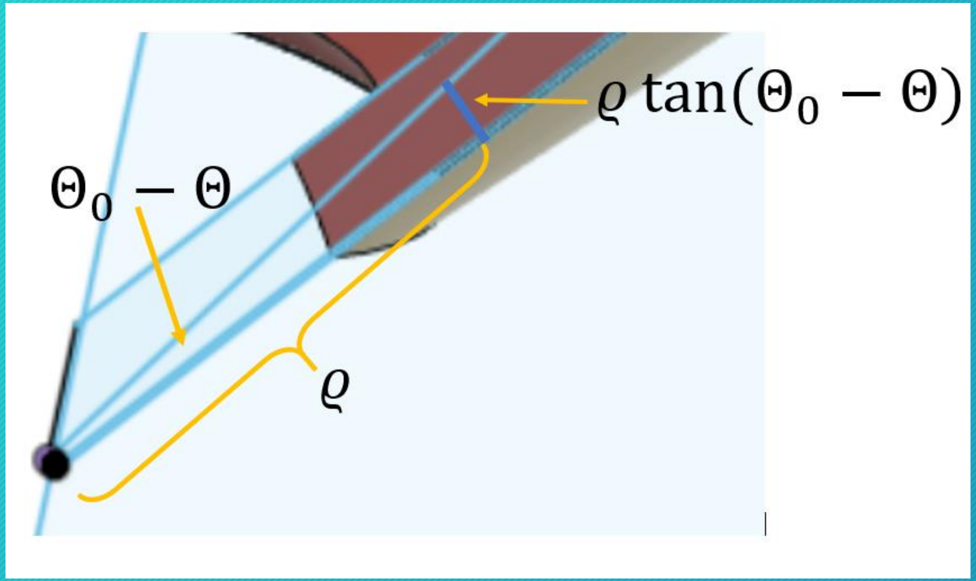
Where α is the shrinkage or expansion factor in the z direction. In finding the deformation gradient, we note that the spherical basis must be reciprocal since it is the referential system. Therefore,

$$\mathbf{F} = (\mathbf{e}_r \quad \rho\mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial \varrho} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial \Phi} \\ \frac{\partial \theta}{\partial \varrho} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial \Phi} \\ \frac{\partial z}{\partial \varrho} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial \Phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\varrho \\ \frac{\mathbf{e}_\Theta}{\varrho} \\ \frac{\mathbf{e}_\Phi}{\varrho \sin \Theta} \end{bmatrix}$$





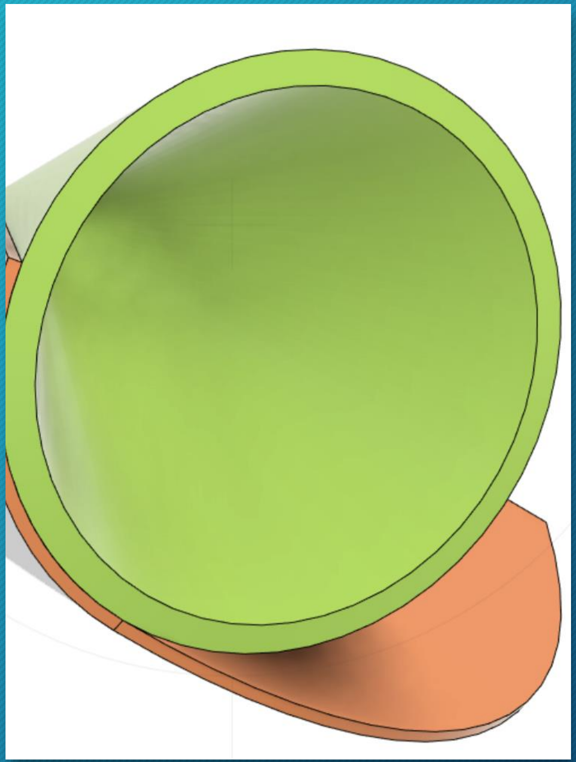
Deformation Gradient

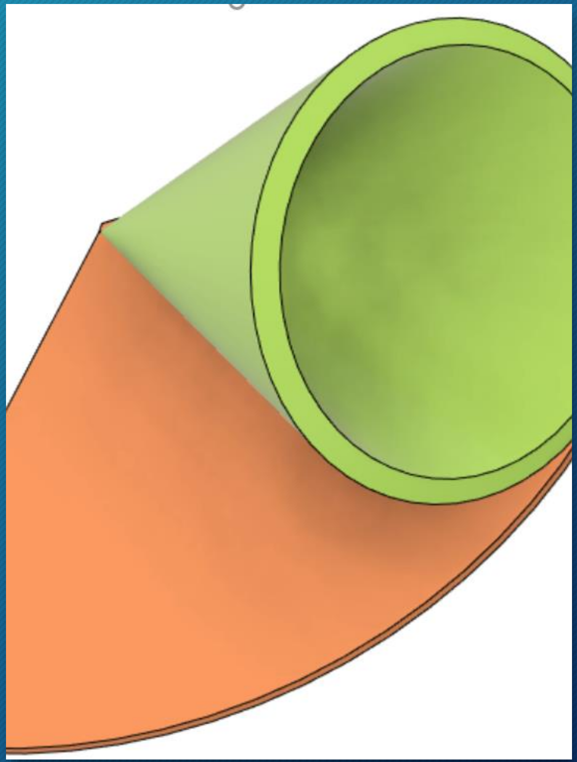


$$= (\mathbf{e}_\rho \quad \mathbf{e}_\theta \quad \mathbf{e}_\phi) \begin{bmatrix} \dfrac{\partial r}{\partial \varrho} & \dfrac{1}{\varrho}\dfrac{\partial r}{\partial \Theta} & \dfrac{1}{\varrho \sin\Theta}\dfrac{\partial r}{\partial \Phi} \\ r\dfrac{\partial \theta}{\partial \varrho} & \dfrac{r}{\varrho}\dfrac{\partial \theta}{\partial \Theta} & \dfrac{r}{\varrho \sin\Theta}\dfrac{\partial \theta}{\partial \Phi} \\ \dfrac{\partial z}{\partial \varrho} & \dfrac{1}{\varrho}\dfrac{\partial z}{\partial \Theta} & \dfrac{1}{\varrho \sin\Theta}\dfrac{\partial z}{\partial \Phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\varrho \\ \mathbf{e}_\Theta \\ \mathbf{e}_\Phi \end{bmatrix}$$

$$= (\mathbf{e}_\rho \quad \mathbf{e}_\theta \quad \mathbf{e}_\phi) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \dfrac{r \sin\Theta_0}{\varrho \sin\Theta} \\ \alpha(\Theta_0 - \Theta) & -\dfrac{\alpha}{\cos^2(\Theta_0 - \Theta)} & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\varrho \\ \mathbf{e}_\Theta \\ \mathbf{e}_\Phi \end{bmatrix}$$

Cone Insulation





Dehin Chemical Company

The metallic hemisphere is shown here. The insulation material is lying under it. It is two tone.

30

Dehin Chemical Company

- They just acquired a large repository of hot liquid mixed liquids that must be kept at high temperature during the manufacture of a product.
- There is an elastomeric double insulating material that will be forced and extended to cover the outer surface of this hemisphere.
- The manufacturer of the insulation material has specifications on the maximum allowable deformation on the material for optimal performance. It is your job to direct the cutting and fitting of this insulation, or they send for an expert in Germany.

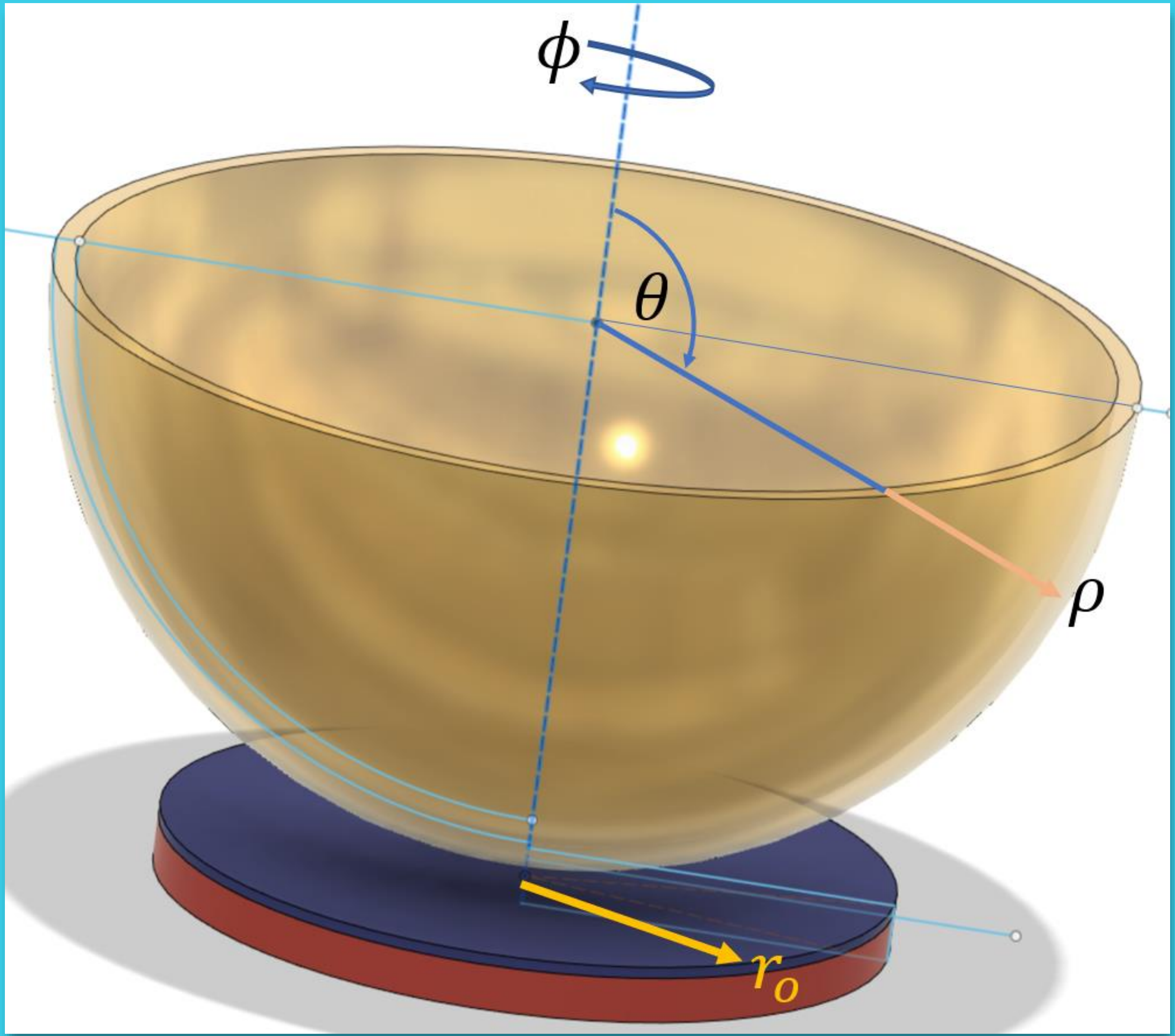
Dehin Chemical Company

- We have a transformation from Cylindrical Polar referential system, R, Θ, Z to the Spherical Polar system, ρ, θ, ϕ. The following bounds on the variables apply:
  $$0 \le R \le r_0; 0 \le \Theta \le 2\pi; 0 \le Z \le -t$$
- At the referential side, and,
  $$\rho_0 \le \rho \le \rho_0 - \alpha Z, \frac{\pi}{2} \le \theta \le \pi, 0 \le \phi \le 2\pi$$

For spatial. The transformation equations are as follows: and here is where the heavy lifting lies:

$$\rho = \rho_0 - \alpha Z$$
$$\theta = -\frac{\pi R}{2r_0} + \pi$$
$$\phi = \Theta$$

- We can now compute the deformation gradient.



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Deformation Gradient

$$\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{\partial\rho}{\partial\theta} & \frac{\partial\rho}{\partial Z} \\ \frac{\partial\theta}{\partial R} & \frac{\partial\theta}{\partial\Theta} & \frac{\partial\theta}{\partial Z} \\ \frac{\partial\phi}{\partial R} & \frac{\partial\phi}{\partial\Theta} & \frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

$$= (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} 0 & 0 & -\alpha \\ -\frac{\pi}{2r_0} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

- Volume Ratio is the determinant of the Deformation Gradient:

$$J = \det\mathbf{F} = \frac{\alpha\pi\rho^2\sin\theta}{2r_0R}$$

- A word on the limiting value of the volume Ratio:

$$J = \det\mathbf{F} = \frac{\alpha\pi\rho^2\sin\theta}{2r_0R}$$

$$\lim_{R\to 0} J = \frac{\alpha\pi\rho^2}{2r_0}$$

- Because as $R \to 0, \theta \to \pi$ so that $\frac{\sin\theta}{R} \to 1$ showing that there is no case of the value becoming indeterminate at the origin.

Deformation Gradient in ONB

$$\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{\partial\rho}{\partial\theta} & \frac{\partial\rho}{\partial Z} \\ \frac{\partial\theta}{\partial R} & \frac{\partial\theta}{\partial\Theta} & \frac{\partial\theta}{\partial Z} \\ \frac{\partial\phi}{\partial R} & \frac{\partial\phi}{\partial\Theta} & \frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

$$= (\mathbf{e}_\rho \quad \mathbf{e}_\theta \quad \mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{1}{R}\frac{\partial\rho}{\partial\theta} & \frac{\partial\rho}{\partial Z} \\ \rho\frac{\partial\theta}{\partial R} & \frac{\rho}{R}\frac{\partial\theta}{\partial\Theta} & \rho\frac{\partial\theta}{\partial Z} \\ \rho\sin\theta\frac{\partial\phi}{\partial R} & \frac{\rho\sin\theta}{R}\frac{\partial\phi}{\partial\Theta} & \rho\sin\theta\frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

$$= (\mathbf{e}_\rho \quad \mathbf{e}_\theta \quad \mathbf{e}_\phi)\begin{bmatrix} 0 & 0 & -\alpha \\ -\frac{\rho\pi}{2r_0} & 0 & 0 \\ 0 & \frac{\rho\sin\theta}{R} & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

Volume Ratio is the determinant of the Deformation Gradient:

$$J = \det\mathbf{F} = \frac{\alpha\pi\rho^2\sin\theta}{2r_0R}$$

Transforming from Cylindrical to Spherical, we have

$$\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{\partial\rho}{\partial\theta} & \frac{\partial\rho}{\partial Z} \\ \frac{\partial\theta}{\partial R} & \frac{\partial\theta}{\partial\Theta} & \frac{\partial\theta}{\partial Z} \\ \frac{\partial\phi}{\partial R} & \frac{\partial\phi}{\partial\Theta} & \frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

. Find its inverse.

- Clearly, in the inverse transformation, we have,

$$R = \chi_R(\rho,\theta,\phi), \qquad \Theta = \chi_\Theta(\rho,\theta,\phi), \qquad Z = \chi_Z(\rho,\theta,\phi)$$

The referential in this transformation is the Spherical system and we use its reciprocal base vectors:

$$\mathbf{F} = (\mathbf{E}_R \quad R\mathbf{E}_\Theta \quad \mathbf{E}_Z)\begin{bmatrix} \frac{\partial R}{\partial\rho} & \frac{\partial R}{\partial\theta} & \frac{\partial R}{\partial\phi} \\ \frac{\partial\Theta}{\partial\rho} & \frac{\partial\Theta}{\partial\theta} & \frac{\partial\Theta}{\partial\phi} \\ \frac{\partial Z}{\partial\rho} & \frac{\partial Z}{\partial\theta} & \frac{\partial Z}{\partial\phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \frac{\mathbf{e}_\theta}{\rho} \\ \frac{\mathbf{e}_\phi}{\rho\sin\theta} \end{bmatrix} = (\mathbf{E}_R \quad \mathbf{E}_\Theta \quad \mathbf{E}_Z)\begin{bmatrix} \frac{\partial R}{\partial\rho} & \frac{1}{\rho}\frac{\partial R}{\partial\theta} & \frac{1}{\rho\sin\theta}\frac{\partial R}{\partial\phi} \\ R\frac{\partial\Theta}{\partial\rho} & \frac{R}{\rho}\frac{\partial\Theta}{\partial\theta} & \frac{R}{\rho\sin\theta}\frac{\partial\Theta}{\partial\phi} \\ \frac{\partial Z}{\partial\rho} & \frac{1}{\rho}\frac{\partial Z}{\partial\theta} & \frac{1}{\rho\sin\theta}\frac{\partial Z}{\partial\phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix}$$

Write the following deformation gradient $\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{\partial\rho}{\partial\theta} & \frac{\partial\rho}{\partial Z} \\ \frac{\partial\theta}{\partial R} & \frac{\partial\theta}{\partial\Theta} & \frac{\partial\theta}{\partial Z} \\ \frac{\partial\phi}{\partial R} & \frac{\partial\phi}{\partial\Theta} & \frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$ in terms or orthonormal basis vectors

- In terms of orthonormal basis vectors, we have,

$$\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{\partial\rho}{\partial\Theta} & \frac{\partial\rho}{\partial Z} \\ \frac{\partial\theta}{\partial R} & \frac{\partial\theta}{\partial\Theta} & \frac{\partial\theta}{\partial Z} \\ \frac{\partial\phi}{\partial R} & \frac{\partial\phi}{\partial\Theta} & \frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix}$$

$$= (\mathbf{e}_\rho \quad \mathbf{e}_\theta \quad \mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial R} & \frac{1}{R}\frac{\partial\rho}{\partial\Theta} & \frac{\partial\rho}{\partial Z} \\ \rho\frac{\partial\theta}{\partial R} & \frac{\rho}{R}\frac{\partial\theta}{\partial\Theta} & \rho\frac{\partial\theta}{\partial Z} \\ \rho\sin\theta\frac{\partial\phi}{\partial R} & \frac{\rho\sin\theta}{R}\frac{\partial\phi}{\partial\Theta} & \rho\sin\theta\frac{\partial\phi}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}$$

Given the deformation, $\rho = \rho_0 - \alpha Z, \theta = -\frac{\pi R}{2r_0} + \pi, \phi = \Theta$. Find the inverse deformation. Find the inverse Deformation Gradient.

- The equations here are easily done by hand. It follows that,

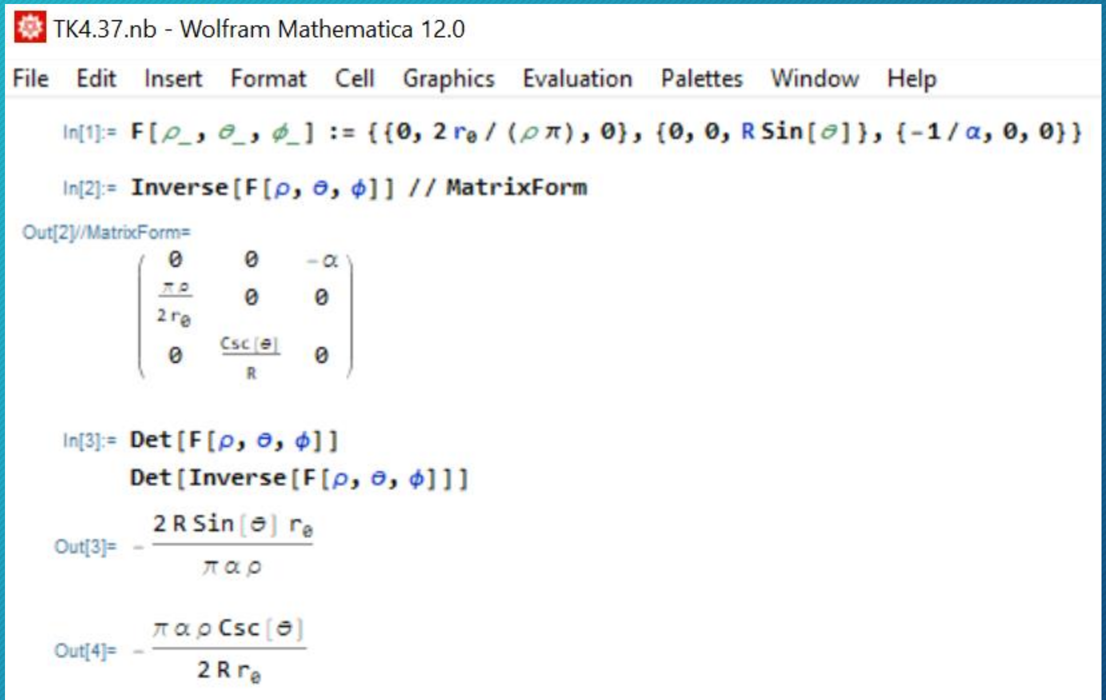
$Z = \chi_Z(\rho,\theta,\phi) = \frac{1}{\alpha}(\rho_0 - \rho)$, $R = \chi_R(\rho,\theta,\phi) = \frac{2r_0}{\pi}(\pi - \theta)$, $\Theta = \chi_\Theta(\rho,\theta,\phi) = \phi$

The deformation gradient,

$$\mathbf{F} = (\mathbf{E}_R \quad \mathbf{E}_\Theta \quad \mathbf{E}_Z)\begin{bmatrix} \frac{\partial R}{\partial \rho} & \frac{1}{\rho}\frac{\partial R}{\partial \theta} & \frac{1}{\rho\sin\theta}\frac{\partial R}{\partial \phi} \\ R\frac{\partial \Theta}{\partial \rho} & \frac{R}{\rho}\frac{\partial \Theta}{\partial \theta} & \frac{R}{\rho\sin\theta}\frac{\partial \Theta}{\partial \phi} \\ \frac{\partial Z}{\partial \rho} & \frac{1}{\rho}\frac{\partial Z}{\partial \theta} & \frac{1}{\rho\sin\theta}\frac{\partial Z}{\partial \phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix} = (\mathbf{E}_R \quad \mathbf{E}_\Theta \quad \mathbf{E}_Z)\begin{bmatrix} 0 & -\frac{2r_0}{\rho\pi} & 0 \\ 0 & 0 & \frac{R}{\rho\sin\theta} \\ -\frac{1}{\alpha} & 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_\rho \\ \mathbf{e}_\theta \\ \mathbf{e}_\phi \end{bmatrix}$$

Use Mathematica to confirm $\mathbf{F}^{-1}$

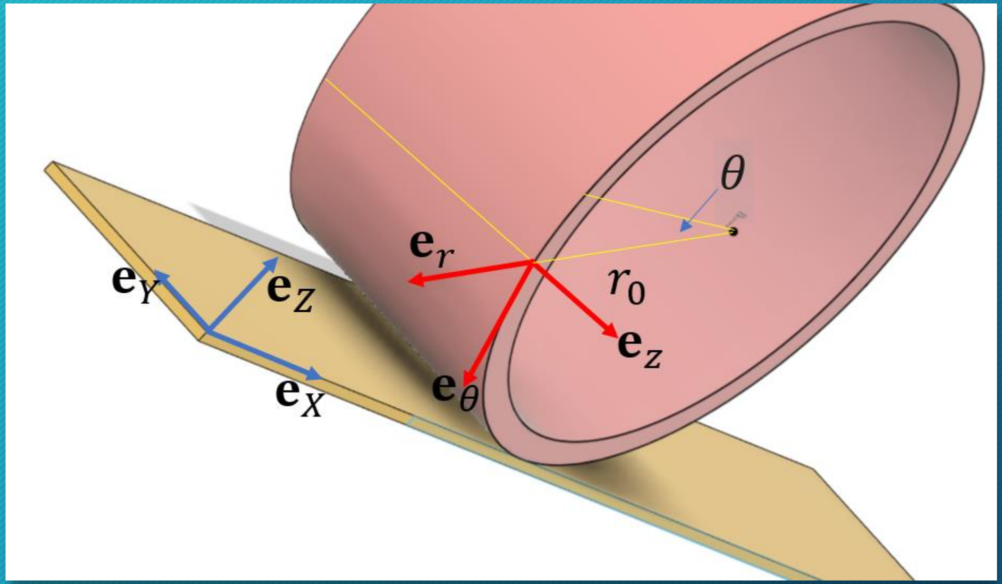
- Here is the Mathematica Code:
- Note that it is only when you refer tensors to orthonormal basis that you can calculate values of physical properties such as Volume or Area ratios.
- The components under such basis are called "Physical Components".

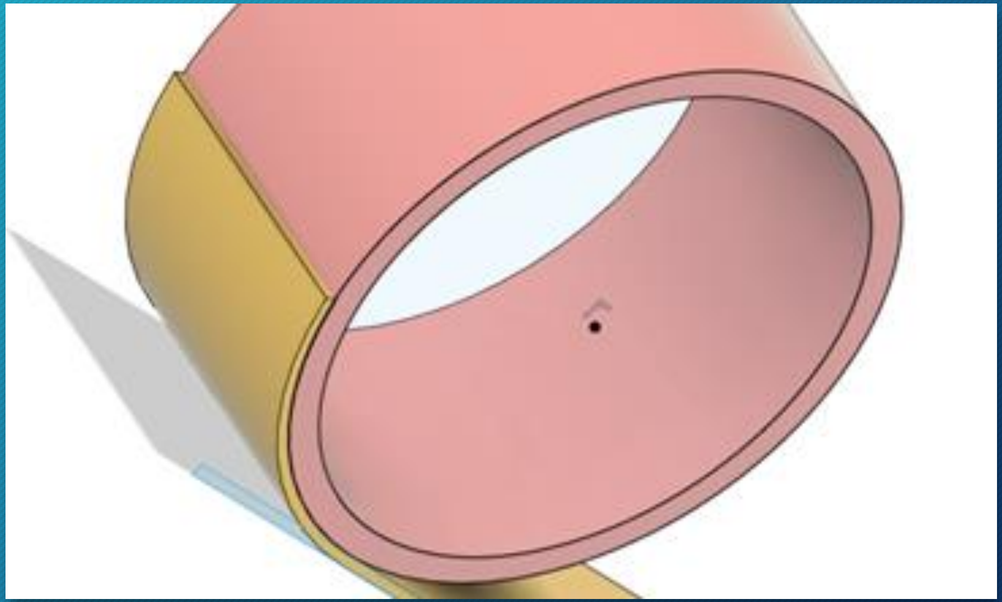


Elastomer Pipe Insulator

- Pipe is insulated by a stretched elastomer. Take Cartesian coordinates as shown on the unstretched insulator. On the pipe, let $\mathbf{e}_r$ be the radial vector with $\mathbf{e}_\theta$ at the rim in the counterclockwise tangential direction. $\mathbf{e}_z$ is the axial unit vector chosen to make the system right-handed.
- Let the insulator be $l \times b$ with a thickness t while pipe width is w. Assuming uniform stretches on the pipe surface the transformations are,

$$r = r_0 - \alpha Z = \chi_r(Z),$$
$$\theta = \frac{\beta}{r_0} X = \chi_\theta(X),$$
$$z = -\gamma Z = \chi_z(Y)$$





Elastomer Pipe Insulator

$2\pi r_0 = l\beta \Rightarrow \beta = \frac{l}{2\pi r_0}$ and $\gamma b = w \Rightarrow \gamma = \frac{w}{b}$.

Deformation gradient

$$\mathbf{F} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial X} & \frac{\partial r}{\partial Y} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial X} & r\frac{\partial \theta}{\partial Y} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_X \\ \mathbf{e}_Y \\ \mathbf{e}_Z \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & 0 & -\alpha \\ \frac{r\beta}{r_0} & 0 & 0 \\ 0 & -\gamma & 0 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_X \\ \mathbf{e}_Y \\ \mathbf{e}_Z \end{bmatrix}$$

Shrinkage under Isochoric Deformation

- When deformation is isochoric, the there is no change in volume. Hence volume ratio is unity.
- Volume ratio is the determinant of the deformation gradient. Note that the deformation gradient MUST be referred to ONB system. Here,

$$\det \mathbf{F} = \frac{\alpha\beta\gamma r}{r_0} = \alpha \frac{l}{2\pi r_0} \frac{w}{b} r = 1$$

- In this case, $\alpha = \frac{2\pi b r_0^2}{wlr}$.

Show that the tensor $\mathbf{C} = (\mathbf{E}_1 \quad \mathbf{E}_2 \quad \mathbf{E}_3)\begin{pmatrix} 163.24 & 34.6 & 4.2 \\ 34.6 & 19. & -30. \\ 4.2 & -30. & 178. \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$ is positive definite.

- Find the square root of the $\mathbf{C}$ by finding its spectral decomposition from its eigenvalues and eigenvectors. (b) Use the Mathematica function MatrixPower[C, ½] to compare your result.
- Eigenvalues are all positive, hence it is positive definite. The tensor $\mathbf{C}$ as well as its square root have the same eigenvectors. Eigenvalues of the square root are the square roots of the eigenvalues of $\mathbf{C}$ as the figures show.

Q4.4 KInematics.nb - Wolfram Mathematica 11.3

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```
In[ ]:= CC := {{163.24, 34.6, 4.2}, {34.6, 19, -30},
          {4.2, -30, 178}}

In[ ]:= CEige = Eigenvalues[CC]
        CVec = Eigenvectors[CC]
        BB = MatrixPower[CC, 1/2]
        Eigenvalues[BB]
        Eigenvectors[BB]

Out[ ]= {183.793, 170.611, 5.83642}

Out[ ]= {{-0.152601, -0.207928, 0.966167},
         {0.964426, 0.182197, 0.191537},
         {0.215859, -0.961026, -0.172727}}

Out[ ]= {{12.5773, 2.22417, 0.323916},
         {2.22417, 3.25095, -1.86666},
         {0.323916, -1.86666, 13.2065}}

Out[ ]= {13.557, 13.0618, 2.41587}

Out[ ]= {{-0.152601, -0.207928, 0.966167},
         {0.964426, 0.182197, 0.191537},
         {0.215859, -0.961026, -0.172727}}
```

Show that rotation alters neither symmetry nor skewness in a tensor.

- Consider a symmetric tensor $\mathbf{S}$, and a rotation tensor $\mathbf{R}$. We take a transpose of the rotated tensor $\mathbf{T} = \mathbf{RSR}^{\mathrm{T}}$

$$\begin{aligned}\mathbf{T}^{\mathrm{T}} &= \left(\mathbf{RSR}^{\mathrm{T}}\right)^{\mathrm{T}} \\ &= \left(\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{S}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = \mathbf{RSR}^{\mathrm{T}} \\ &= \mathbf{T}\end{aligned}$$

- On account of the symmetry of tensor $\mathbf{S}$ and the fact that the transpose of a transpose is the original tensor.
- Consider a skew tensor $\mathbf{W}$, and a rotation tensor $\mathbf{R}$. We take a transpose of the rotated tensor $\boldsymbol{\Omega} = \mathbf{RWR}^{\mathrm{T}}$

$$\begin{aligned}\boldsymbol{\Omega}^{\mathrm{T}} &= \left(\mathbf{RWR}^{\mathrm{T}}\right)^{\mathrm{T}} \\ &= \left(\mathbf{R}^{\mathrm{T}}\right)^{\mathrm{T}}\mathbf{W}^{\mathrm{T}}\mathbf{R}^{\mathrm{T}} = -\,\mathbf{RWR}^{\mathrm{T}} \\ &= -\boldsymbol{\Omega}\end{aligned}$$

- On account of the skewness of tensor $\mathbf{W}$ and the fact that the transpose of a transpose is the original tensor.

For a proper orthogonal tensor Q, show that the eigenvalue equation always yields an eigenvalue of +1. This means that $\lambda = 1$ is always a solution for the equation, $\det(\mathbf{Q} - \lambda\mathbf{I})$

- For a proper orthogonal tensor, the cofactor,
$$\mathbf{Q}^{\mathrm{c}} = (\det \mathbf{Q})\mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}^{-\mathrm{T}} = \mathbf{Q}$$
- Showing that it is self-cofactor. The characteristic equation is,
$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0.$$
- $I_3 = 1$ for every proper orthogonal tensor; $I_2 = I_1$since it is self cofactor. The second invariant is the trace of the cofactor equaling the first which is the trace of the tensor. Consequently, the characteristic equation becomes,
$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0$$
- Substitute $\lambda = 1$, the equation becomes, $1 - I_1 + I_1 - 1 = 0$, identically. Hence this is an eigenvalue of the tensor.

For a vector-valued spatial field, we are given that $\text{Grad}\,\mathbf{f} = (\text{grad}\,\mathbf{f})\mathbf{F}(\mathbf{X}, t)$. Show that, $\text{Div}\,\mathbf{f} = (\text{grad}\,\mathbf{f}):\mathbf{F}^{\text{T}}$

- We are given,

$$\text{Grad}\,\mathbf{f} = (\text{grad}\,\mathbf{f})\mathbf{F}$$

- Take the trace of both sides:

$$\begin{aligned}\text{tr Grad}\,\mathbf{f} &= \text{tr}\big((\text{grad}\,\mathbf{f})\mathbf{F}\big)\\ &= \big(\text{grad}^{\text{T}}\,\mathbf{f}\big):\mathbf{F} = (\text{grad}\,\mathbf{f}):\mathbf{F}^{\text{T}}\\ &= \text{Div}\,\mathbf{f}\end{aligned}$$

If the tensor $\mathbf{S}$ is positive definite, Show that $\det\left(\mathbf{S}^{\frac{1}{2}}\right) = [\det(\mathbf{S})]^{\frac{1}{2}}$. Why is this result important?

- Let the eigenvalues of $\mathbf{S}$ be λ_1, λ_2 and λ_3. Then the determinant of $\mathbf{S}$ is $\lambda_1\lambda_2\lambda_3$ the square root of this is $\sqrt{\lambda_1\lambda_2\lambda_3}$. But since $\mathbf{S}$ is positive definite, The eigenvalues of $\boldsymbol{S}^{\frac{1}{2}}$ are $\sqrt{\lambda_1}, \sqrt{\lambda_2}$ and $\sqrt{\lambda_3}$ so that the determinant of $\mathbf{S}^{\frac{1}{2}}$, ie $[\det(\mathbf{S})]^{\frac{1}{2}} = \sqrt{\lambda_1}\sqrt{\lambda_2}\sqrt{\lambda_3} = \sqrt{\lambda_1\lambda_2\lambda_3}$
- The Right Stretch Tensor, $\mathbf{U}$, is the square root of the Right Cauchy-Green Tensor. The square root of the determinant of the former is therefore the volume ratio since the Determinant of $\mathbf{U}$ is the same as that of the deformation Gradient.

Show that the stretch Tensors have the same determinant as the Deformation Gradient. Why is this important?

- From the Equation, $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ we take the determinant of each side and recall that the determinant of a product of tensors is the product of the determinant of the tensors.

$$\begin{aligned} \det \mathbf{F} = \det \mathbf{RU} &= \det \mathbf{VR} \\ &= (\det \mathbf{R})(\det \mathbf{U}) \\ &= (\det \mathbf{V})(\det \mathbf{R}) \end{aligned}$$

- The result follows upon noting that the determinant of a proper orthogonal tensor is unity. Therefore,

$$\det \mathbf{F} = \det \mathbf{U} = \det \mathbf{V}$$

- We can find volume ratio from the determinant of any stretch tensor.

Given the Deformation Gradient Tensor $(\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3)\begin{pmatrix} 1 & \frac{3}{2} & \frac{4}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$ Find the rotation tensor, the right stretch tensor and the left stretch tensor.


```
F = {{1.0, 3/2, 4/3}, {0, 1, 0}, {0, 0, 1}}
C1 = Transpose[F].F
```

$\{\{1., \frac{3}{2}, \frac{4}{3}\}, \{0, 1, 0\}, \{0, 0, 1\}\}$

```
{{1., 1.5, 1.33333}, {1.5, 3.25, 2.}, {1.33333, 2., 2.77778}}
```

$\{\{1.0, \frac{3}{2}, \frac{4}{3}\}, \{0, 1, 0\}, \{0, 0, 1\}\}$

$\{\{1, \frac{3}{2}, \frac{4}{3}\}, \{\frac{3}{2}, \frac{13}{4}, 2\}, \{\frac{4}{3}, 2, \frac{25}{9}\}\}$

```
U1 = MatrixPower[C1 , 1/2]
{{0.705882, 0.529412, 0.470588},
 {0.529412, 1.62982, 0.559838}, {0.470588, 0.559838, 1.49763}}

R1 = F.Inverse[U1]
{{0.705882, 0.529412, 0.470588},
 {-0.529412, 0.8357, -0.146045}, {-0.470588, -0.146045, 0.870183}}

V1 = F.Inverse[R1]
{{2.12745, 0.529412, 0.470588},
 {0.529412, 0.8357, -0.146045}, {0.470588, -0.146045, 0.870183}}
```

A body undergoes a deformation defined by, $x_1 = X_1 \cos\alpha - X_2 \sin\alpha$, $X_2 = X_1 \sin\alpha + X_2 \cos\alpha$, and $x_3 = X_3$ where α is a constant. Show that $\mathbf{C} = \mathbf{I}$ and $\mathbf{E} = \mathbf{O}$. Explain the reason for the values of $\mathbf{E}$ components.

- The deformation gradient here is the rotation tensor through angle α around the $\mathbf{e}_3$ axis. Consequently,

$$\mathbf{F} = \mathbf{R} = \mathbf{RI} = \mathbf{RU}$$

- So that $\mathbf{U} = \mathbf{I}$.

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{I}$$

- And,

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \mathbf{O}$$

A cylindrical tube undergoes the deformation given by $r = R, \phi = \Theta + \vartheta(R), z = Z + w(R)$ where $\{R, \Phi, Z\}$ and $\{r, \phi, z\}$, are polar coordinates of a point in the tube before and after deformation respectively, ϑ and w are scalar functions of R. Explain the meaning of the situation where *(i)* $\vartheta = 0$, (ii) $w = 0$.

- $[\mathbf{F}] = \left[\frac{\partial \mathbf{r}}{\partial \mathbf{R}}\right] = \begin{pmatrix} F_{rR} & F_{r\Phi} & F_{rZ} \\ F_{\phi R} & F_{\phi\Phi} & F_{\phi Z} \\ F_{zR} & F_{z\Phi} & F_{zZ} \end{pmatrix} = \begin{pmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Phi} & \frac{\partial r}{\partial Z} \\ \frac{\partial \phi}{\partial R} r & \frac{r}{R}\frac{\partial \phi}{\partial \Phi} & \frac{\partial \phi}{\partial Z} r \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Phi} & \frac{\partial z}{\partial Z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ \vartheta' r & \frac{r}{R} & 0 \\ w' & 0 & 1 \end{pmatrix}$. When $\vartheta = 0$, The deformation gradient becomes, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r}{R} & 0 \\ w' & 0 & 1 \end{pmatrix}$. This is a longitudinal elongation as radial and tangential displacements are nil.

  When $w = 0$, The deformation gradient becomes, $\begin{pmatrix} 1 & 0 & 0 \\ \vartheta' r & \frac{r}{R} & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This is a torsional rotation as there is no other deformation in the material apart from a relative rotation along the longitudinal axis.

A cylindrical tube undergoes the deformation given by $r = R, \phi = \Theta + \vartheta(R), z = Z + w(R)$ where $\{R, \Phi, Z\}$ and $\{r, \phi, z\}$, are polar coordinates of a point in the tube before and after deformation respectively, ϑ and w are scalar functions of R. Compute $\mathbf{F}, \mathbf{C}$ and $\mathbf{E}$

- The Right Cauchy Green Tensor

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{pmatrix} (w')^2 + r^2(\vartheta')^2 + 1 & \frac{r^2\vartheta'}{R} & w' \\ \frac{r^2\vartheta'}{R} & \frac{r^2}{R^2} & 0 \\ w' & 0 & 1 \end{pmatrix}$$

- and the Lagrangian strain,

- $$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) = \begin{pmatrix} \frac{1}{2}((w')^2 + r^2(\vartheta')^2) & \frac{r^2\vartheta'}{2R} & \frac{w'}{2} \\ \frac{r^2\vartheta'}{2R} & \frac{1}{2}(\frac{r^2}{R^2} - 1) & 0 \\ \frac{w'}{2} & 0 & 0 \end{pmatrix}$$

A cylindrical tube undergoes the deformation given by $r = R, \phi = \Theta + \vartheta(R), z = Z + w(R)$ where $\{R, \Phi, Z\}$ and $\{r, \phi, z\}$, are polar coordinates of a point in the tube before and after deformation respectively, ϑ and w are scalar functions of R. Find the Lagrangian and Eulerian strain components

- $\mathbf{F}\,\mathbf{F}^{\mathrm{T}} = \begin{pmatrix} 1 & r\vartheta' & w' \\ r\vartheta' & (\vartheta')^2 r^2 + \frac{r^2}{R^2} & rw'\vartheta' \\ w' & rw'\vartheta' & (w')^2+1 \end{pmatrix}$
- The inverse of this also called the Piola Tensor is,
- $\mathbf{B} = \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1} = \begin{pmatrix} \frac{R^2\left(\frac{(w')^2 r^2}{R^2} + (\vartheta')^2 r^2 + \frac{r^2}{R^2}\right)}{r^2} & -\frac{R^2\vartheta'}{r} & -w' \\ -\frac{R^2\vartheta'}{r} & \frac{R^2}{r^2} & 0 \\ -w' & 0 & 1 \end{pmatrix}$
- Eulerian strain

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}) = \begin{pmatrix} \frac{1}{2}\left[1 - \frac{R^2\left(\frac{(w')^2 r^2}{R^2} + (\vartheta')^2 r^2 + \frac{r^2}{R^2}\right)}{r^2}\right] & \frac{R^2\vartheta'}{2r} & \frac{w'}{2} \\ \frac{R^2\vartheta'}{2r} & \frac{1}{2}\left(1 - \frac{R^2}{r^2}\right) & 0 \\ \frac{w'}{2} & 0 & 0 \end{pmatrix}$$

In the deformation, $\mathbf{x} = \chi(\mathbf{X}) = (\mathbf{A} + \mathbf{I})\mathbf{X}$ where $\mathbf{A}$ is independent of $\mathbf{X}$, show that infinitesimal strain, $\boldsymbol{\epsilon} = \text{sym } \mathbf{A}$

- Recall that, $\boldsymbol{\epsilon} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^{\mathrm{T}})$ where $\mathbf{H} = \text{Grad }(\chi(\mathbf{X}) - \mathbf{X})$ the Referential gradient of the difference between the spatial and referential variables. Hence,

$$\begin{aligned}\mathbf{H} &= \text{Grad }(\chi(\mathbf{X}) - \mathbf{X}) \\ &= \text{Grad }((\mathbf{A} + \mathbf{I})\mathbf{X} - \mathbf{I}\mathbf{X}) = \text{Grad } \mathbf{A}\mathbf{X} \\ &= \mathbf{A} \text{ Grad } \mathbf{X} = \mathbf{A}\mathbf{I} = \mathbf{A}\end{aligned}$$

- Consequently $\boldsymbol{\epsilon} = \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) = \frac{1}{2}\left(\mathbf{A} + \mathbf{A}^{\mathrm{T}}\right) = \text{sym } \mathbf{A}$

For the deformation, $x_1 = X_1 + k\,X_3, x_2 = X_2 + k\,X_1, x_3 = X_3 + kX_2$ Find the Deformation Gradient, Lagrangian, Eulerian and Small Strain Tensors when $k = 0.001$.

- Easy to substitute values for $k = 0.001$

```
In[1]:= XX[X1_, X2_, X3_] := {X1 + k X3, X2 + k X1, X3 + k X2};
H = D[XX[X1, X2, X3] - {X1, X2, X3}, {{X1, X2, X3}}];
F = D[XX[X1, X2, X3], {{X1, X2, X3}}];
F // MatrixForm
CG = Transpose[F].F;
Finger = F.Transpose[F];
Piola = Inverse[Finger];
LagrangeStr = (1/2) (CG - Identity[3]);
EulerStrain = 1/2 (Identity[3] - Piola);
SmallStrain = (1/2) (H + Transpose[H]);
LagrangeStr // MatrixForm
EulerStrain // MatrixForm
SmallStrain // MatrixForm
```

4]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & k \\ k & 1 & 0 \\ 0 & k & 1 \end{pmatrix}$$

Out[11]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}(-2+k^2) & \frac{1}{2}(-3+k) & \frac{1}{2}(-3+k) \\ \frac{1}{2}(-3+k) & \frac{1}{2}(-2+k^2) & \frac{1}{2}(-3+k) \\ \frac{1}{2}(-3+k) & \frac{1}{2}(-3+k) & \frac{1}{2}(-2+k^2) \end{pmatrix}$$

Out[12]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2}\left(3-\frac{1+k^2+k^4}{1+2k^3+k^6}\right) & \frac{1}{2}\left(3-\frac{-k+k^2-k^3}{1+2k^3+k^6}\right) & \frac{1}{2}\left(3-\frac{-k+k^2-k^3}{1+2k^3+k^6}\right) \\ \frac{1}{2}\left(3-\frac{-k+k^2-k^3}{1+2k^3+k^6}\right) & \frac{1}{2}\left(3-\frac{1+k^2+k^4}{1+2k^3+k^6}\right) & \frac{1}{2}\left(3-\frac{-k+k^2-k^3}{1+2k^3+k^6}\right) \\ \frac{1}{2}\left(3-\frac{-k+k^2-k^3}{1+2k^3+k^6}\right) & \frac{1}{2}\left(3-\frac{-k+k^2-k^3}{1+2k^3+k^6}\right) & \frac{1}{2}\left(3-\frac{1+k^2+k^4}{1+2k^3+k^6}\right) \end{pmatrix}$$

Out[13]//MatrixForm=

$$\begin{pmatrix} 0 & \frac{k}{2} & \frac{k}{2} \\ \frac{k}{2} & 0 & \frac{k}{2} \\ \frac{k}{2} & \frac{k}{2} & 0 \end{pmatrix}$$

In the deformation, $x_1 = X_1 + k\,X_3, x_2 = X_2 + k\,X_1, x_3 = X_3 + kX_2$ at the point $\{1,1,0\}$ if $k = 0.0001$. Compare the Lagrangian, Eulerian and small strain tensors.

- The attached code computes the Small Strain, Lagrangian and the Eulerian Tensors respectively. The smallness of k shows that strain is very small.
- The differences between the large strain measures and the small strain are predictable small also. It is customary to use the simpler formular for small strain to comnpute these values in such cases.

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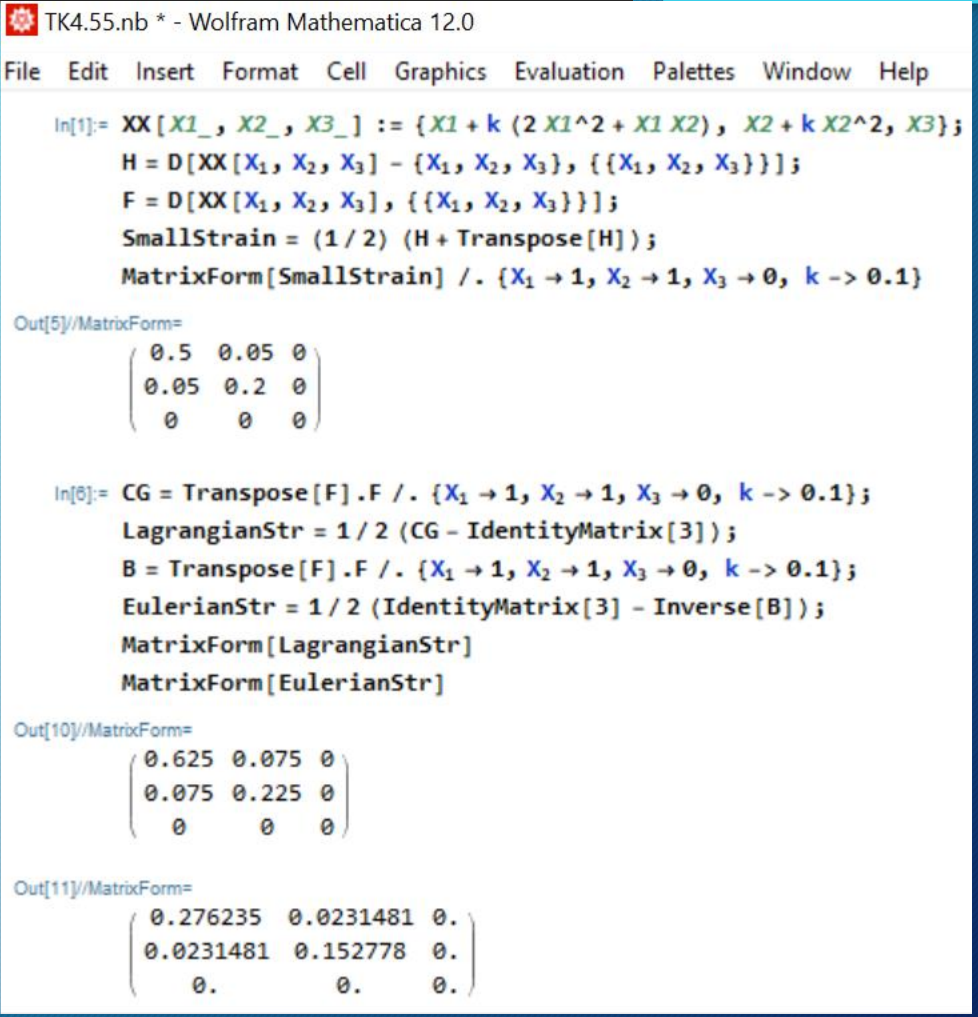
```
In[1]:= XX[X1_, X2_, X3_] := {X1 + k (2 X1^2 + X1 X2), X2 + k X2^2, X3};
        H = D[XX[X1, X2, X3] - {X1, X2, X3}, {{X1, X2, X3}}];
        F = D[XX[X1, X2, X3], {{X1, X2, X3}}];
        SmallStrain = (1/2) (H + Transpose[H]);
        MatrixForm[SmallStrain] /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.0001} /

Out[5]//MatrixForm=
 0.0005    0.00005  0
 0.00005   0.0002   0
 0         0        0

In[6]:= CG = Transpose[F].F /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.0001};
        LagrangianStr = 1/2 (CG - IdentityMatrix[3]);
        B = Transpose[F].F /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.0001};
        EulerianStr = 1/2 (IdentityMatrix[3] - Inverse[B]);
        MatrixForm[LagrangianStr]
        MatrixForm[EulerianStr]

Out[10]//MatrixForm=
 0.000500125  0.000050025  0
 0.000050025  0.000200025  0
 0            0            0

Out[11]//MatrixForm=
 0.00049962   0.000049955  0.
 0.000049955  0.00019994   0.
 0.           0.           0.
```

In the deformation, $x_1 = X_1 + k\,X_3, x_2 = X_2 + k\,X_1, x_3 = X_3 + kX_2$ at the point $\{1,1,0\}$ if $k = 0.1$. Compare the Lagrangian, Eulerian and small strain tensors.

- The attached code computes the Small Strain, Lagrangian and the Eulerian Tensors respectively. $k = 0.1$ creates sufficiently large strains to show a disparity between the different stain measures.
- While the differences between small strains and the Lagrangian values are not too large, the Eulerian strain are vastly different.
- When strains become larger than this, only large strain tensors are usable.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```
In[1]:= XX[X1_, X2_, X3_] := {X1 + k (2 X1^2 + X1 X2), X2 + k X2^2, X3};
H = D[XX[X1, X2, X3] - {X1, X2, X3}, {{X1, X2, X3}}];
F = D[XX[X1, X2, X3], {{X1, X2, X3}}];
SmallStrain = (1/2) (H + Transpose[H]);
MatrixForm[SmallStrain] /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1}
```

Out[5]//MatrixForm=

$$\begin{pmatrix} 0.5 & 0.05 & 0 \\ 0.05 & 0.2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
In[6]:= CG = Transpose[F].F /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1};
LagrangianStr = 1/2 (CG - IdentityMatrix[3]);
B = Transpose[F].F /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1};
EulerianStr = 1/2 (IdentityMatrix[3] - Inverse[B]);
MatrixForm[LagrangianStr]
MatrixForm[EulerianStr]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} 0.625 & 0.075 & 0 \\ 0.075 & 0.225 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[11]//MatrixForm=

$$\begin{pmatrix} 0.276235 & 0.0231481 & 0. \\ 0.0231481 & 0.152778 & 0. \\ 0. & 0. & 0. \end{pmatrix}$$

In the deformation, $x_1 = X_1 + k\, X_3, x_2 = X_2 + k\, X_1, x_3 = X_3 + kX_2$ at the point $\{1,1,0\}$ if $k = 0.1$. Compute the angle between the perpendicular coordinate fibres at this point as a result of the deformation. Discuss

- Remember that the orientation between two fibres is governed by the Right stretch tensor. These fibres are originally at right angles to each other. Using small strain formula, the Tangent of the change in this angle is ϵ_{12}.
- The cosine of the new angular orientation is

$$\theta = \cos^{-1}\left(\frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\|d\mathbf{x}_1\|\|d\mathbf{x}_2\|}\right) = \cos^{-1}\left(\frac{\mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2}{\|\mathbf{U}d\mathbf{X}_1\|\|\mathbf{U}d\mathbf{X}_2\|}\right)$$

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```
In[22]:= XX[X1_, X2_, X3_] := {X1 + k (2 X1^2 + X1 X2), X2 + k X2^2, X3};
         H = D[XX[X1, X2, X3] - {X1, X2, X3}, {{X1, X2, X3}}];
         F = D[XX[X1, X2, X3], {{X1, X2, X3}}];
         SmallStrain = (1/2) (H + Transpose[H]);
         MatrixForm[F] /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1}

Out[26]//MatrixForm=
         1.5  0.1  0
         0    1.2  0
         0    0    1

In[43]:= CG = Transpose[F].F /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1};
         RightStretch = MatrixPower[CG, .5];
         θ = ArcCos[Dot[RightStretch.{1, 0, 0}, RightStretch.{0, 1, 0}] /
             (Norm[RightStretch.{1, 0, 0}] × Norm[RightStretch.{1, 0, 0}])]
         SmallStrain[[1]][[2]] /. {k → .1, X1 → 1, X2 → 1}
         LargeStrain = N[Pi/2] - θ

Out[45]= 1.50408

Out[46]= 0.05

Out[47]= 0.0667161
```

Continued from Previous Slide

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```
In[22]:= XX[X1_, X2_, X3_] := {X1 + k (2 X1^2 + X1 X2), X2 + k X2^2, X3};
         H = D[XX[X1, X2, X3] - {X1, X2, X3}, {{X1, X2, X3}}];
         F = D[XX[X1, X2, X3], {{X1, X2, X3}}];
         SmallStrain = (1/2) (H + Transpose[H]);
         MatrixForm[F] /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1}

Out[26]//MatrixForm=
         ( 1.5  0.1  0 )
         ( 0    1.2  0 )
         ( 0    0    1 )

In[43]:= CG = Transpose[F].F /. {X1 → 1, X2 → 1, X3 → 0, k -> 0.1};
         RightStretch = MatrixPower[CG, .5];
         θ = ArcCos[Dot[RightStretch.{1, 0, 0}, RightStretch.{0, 1, 0}] /
             (Norm[RightStretch.{1, 0, 0}] × Norm[RightStretch.{1, 0, 0}])]
         SmallStrain[[1]][[2]] /. {k → .1, X1 → 1, X2 → 1}
         LargeStrain = N[Pi/2] - θ

Out[45]= 1.50408

Out[46]= 0.05

Out[47]= 0.0667161
```

- From this computation, the spatial fibres are inclined to each other at an angle 1.50408 radians. The original orientation was π radians. The change in the angle is .0667161 radians.
- In contrast, this was computed to .05 radians
- Note that the Right stretch can provide the orientation change between any two fibres.

In the deformation, $x_1 = X_1 + k\, X_3, x_2 = X_2 + k\, X_1, x_3 = X_3 + kX_2$ at the point $\{1,1,0\}$ if $k = 0.1$. Compute the change in length of a line originally oriented at the direction {1,2,1}

- Change in length is governed by the Right Stretch Tensor.

$$d\mathbf{x}_1 \cdot d\mathbf{x}_1 = \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_1$$
$$\|d\mathbf{x}\| = \|\mathbf{U}d\mathbf{X}\|$$

- In the attached code, the unit vector along the direction chosen is obtained from the normalization of the vector.
- Absolute value of the result is provided by taking the norm.
- Change in length is $1.2682 - 1 = 0.2682$.

TK4.58.nb * - Wolfram Mathematica 12.0

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```
In[1]:= XX[X1_, X2_, X3_] := {X1 + k (2 X1^2 + X1 X2), X2 + k X2^2, X3};
        F = D[XX[X1, X2, X3], {{X1, X2, X3}}];
        SmallStrain = (1/2) (H + Transpose[H]);
        CG = Transpose[F].F /. {X1 -> 1, X2 -> 1, X3 -> 0, k -> 0.1};
        RightStretch = MatrixPower[CG, .5];
        Norm[RightStretch.Normalize[{1, 2, 1}]]

Out[5]= 1.2682
```

Show that the strain systems, {{5.,3,0},{3,4,-1},{0,-1,2}} and {{7.61704,0,0},{0,2.48475,0},{0,0,0.898215}} represent the same strain system. Find the rotation tensor that rotates the one to the other.

- One easy way is to compute their eigenvalues. These are equal. They therefore provide the same principal values at the same point.
- Take the eigenvectors of the one; the tensor formed by these normalized eigenvectors can rotate it to its canonical form as shown in the attached code:

```
TK4.59.nb * - Wolfram Mathematica 12.0
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In[13]:= Strain1 = {{5., 3, 0}, {3, 4, -1}, {0, -1, 2}};
         Eigenvalues[Strain1];
         Q = Eigenvectors[Strain1];
         Chop[Q.Strain1.Transpose[Q]]

Out[16]= {{7.61704, 0, 0}, {0, 2.48475, 0}, {0, 0, 0.898215}}
```
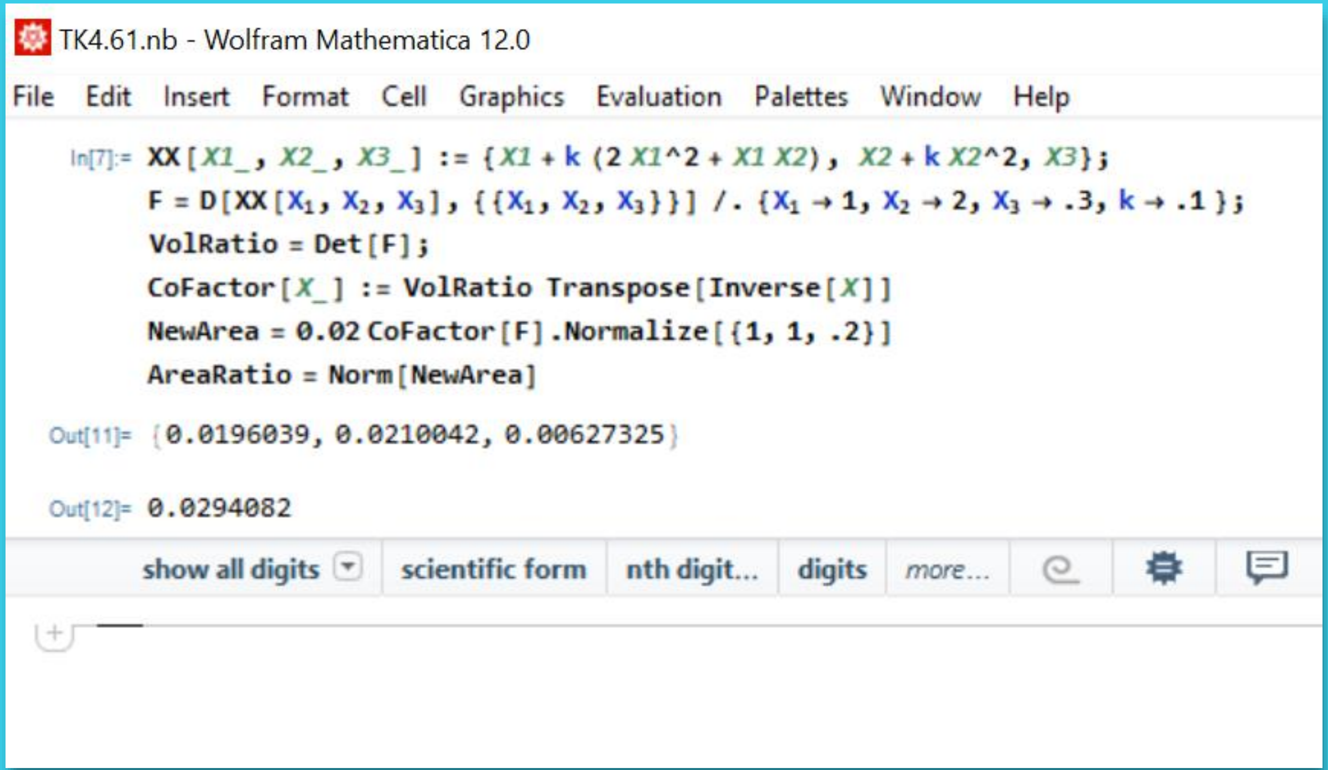

In the deformation, $x_1 = X_1 + k\,X_3, x_2 = X_2 + k\,X_1, x_3 = X_3 + kX_2$ at the point $\{1,2,.3\}$ if $k = 0.1$. Find the volume ratio and the new value of an area $0.02mm^2$ oriented at the direction of $\{1,1,.2\}$.

- Volume ratio is determined by the determinant of the Deformation gradient at the point in question. Area is a vector.
- Mathematica does not define a function for the cofactor tensor. We write the code for that in two statements as shown.
- We can get the value of area by obtaining the Norm of the vector area after transforming the original area by the cofactor. Note that the original area is a vector obtained by the normalized direction vector and the scalar value of area.


TK4.61.nb - Wolfram Mathematica 12.0

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```
In[7]:= XX[X1_, X2_, X3_] := {X1 + k (2 X1^2 + X1 X2), X2 + k X2^2, X3};
F = D[XX[X1, X2, X3], {{X1, X2, X3}}] /. {X1 → 1, X2 → 2, X3 → .3, k → .1};
VolRatio = Det[F];
CoFactor[X_] := VolRatio Transpose[Inverse[X]]
NewArea = 0.02 CoFactor[F].Normalize[{1, 1, .2}]
AreaRatio = Norm[NewArea]

Out[11]= {0.0196039, 0.0210042, 0.00627325}

Out[12]= 0.0294082
```

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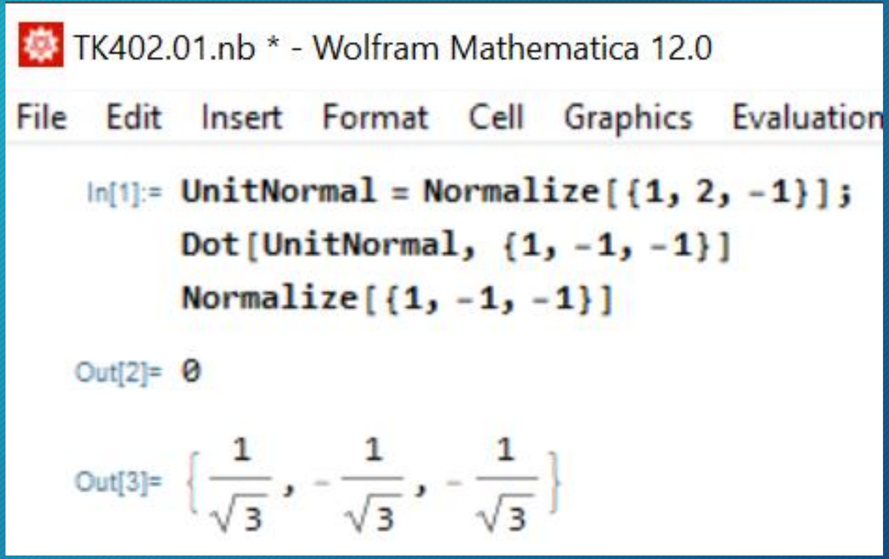
Show that the $\mathbf{E}_1 - \mathbf{E}_2 - \mathbf{E}_3$ is a vector on the plane $X_1 + 2X_2 - X_3 = 10$. Find the unit vector along the same direction.

- The unit normal to the plane can be found by normalizing $\mathbf{E}_1 + 2\mathbf{E}_2 - \mathbf{E}_3$. The vector we are given will lie on the plane if it is at right angles to the normal.
- The attached code shows that this is the case from the dot product of the unit normal and the vector.
- Clearly, $\frac{1}{\sqrt{3}}(\mathbf{E}_1 - \mathbf{E}_2 - \mathbf{E}_3)$ is a unit vector in this direction, lying on the plane.


TK402.01.nb * - Wolfram Mathematica 12.0

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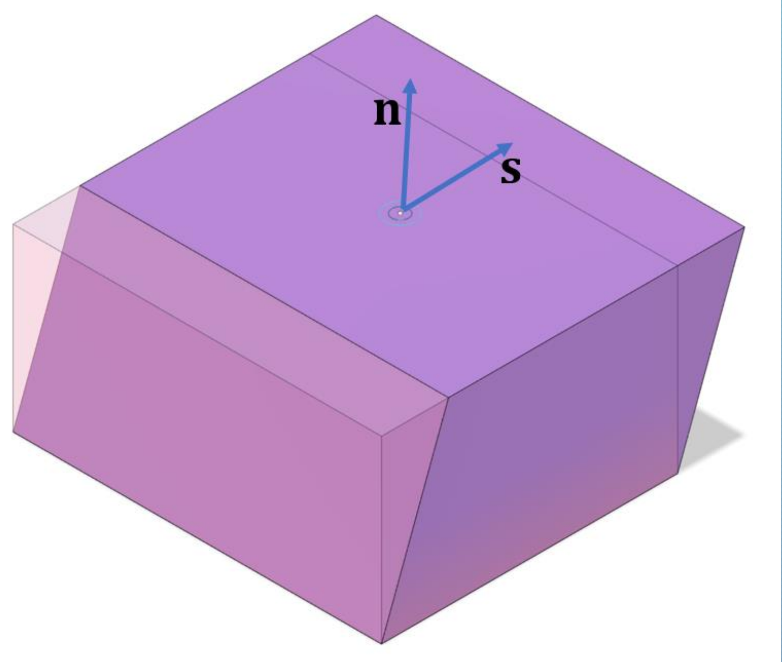
```
In[1]:= UnitNormal = Normalize[{1, 2, -1}];
        Dot[UnitNormal, {1, -1, -1}]
        Normalize[{1, -1, -1}]

Out[2]= 0

Out[3]= {1/√3, -1/√3, -1/√3}
```

The deformation, $\mathbf{x} = (\mathbf{I} + \gamma \mathbf{s} \otimes \mathbf{n})\mathbf{X}$ where $\mathbf{n}$ is the unit normal to a surface and $\mathbf{s}$, a unit vector in shear direction, is a case of simple shear. Find the deformation gradient for in the direction, $\mathbf{E}_1 - \mathbf{E}_2 - \mathbf{E}_3$ with $X_1 + 2X_2 - X_3 = 10$ as the shear plane.

- The figure here depicts simple shear. The attached code computes the general deformation gradient by normalizing $\mathbf{E}_1 - \mathbf{E}_2 - \mathbf{E}_3$ as $\mathbf{s}$ and normalizing the perpendicular vector to the shear plane, $\mathbf{E}_1 + 2\mathbf{E}_2 - \mathbf{E}_3$.



TK402.03.nb * - Wolfram Mathematica 12.0

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```
In[1]:= x[X1_, X2_, X3_] := (IdentityMatrix[3] +
          γ TensorProduct[Normalize[{1, -1, -1}],
           Normalize[{1, 2, -1}]]).{X1, X2, X3};
        DefGrad = D[x[X1, X2, X3], {{X1, X2, X3}}];
        DefGrad // MatrixForm
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 + \frac{\gamma}{3\sqrt{2}} & \frac{\sqrt{2}\gamma}{3} & -\frac{\gamma}{3\sqrt{2}} \\ -\frac{\gamma}{3\sqrt{2}} & 1 - \frac{\sqrt{2}\gamma}{3} & \frac{\gamma}{3\sqrt{2}} \\ -\frac{\gamma}{3\sqrt{2}} & -\frac{\sqrt{2}\gamma}{3} & 1 + \frac{\gamma}{3\sqrt{2}} \end{pmatrix}$$

The deformation, $\mathbf{x} = (\mathbf{I} + \gamma \mathbf{s} \otimes \mathbf{n})\mathbf{X}$ where $\mathbf{n}$ is the unit normal to a surface and $\mathbf{s}$, a unit vector in shear direction, is a case of simple shear. Find the deformation gradient of simple shear with $\mathbf{s}$ as the X_1 −axis on the shear plane being the X_2 −plane.

- In this case, $\mathbf{s} = \mathbf{E}_1$ or $\{1,0,0\}$, and $\mathbf{n} = \mathbf{E}_2$ or $\{0,1,0\}$.

```
x[X1_, X2_, X3_] := (IdentityMatrix[3] +
    γ TensorProduct[Normalize[{1, 0, 0}],
      Normalize[{0, 1, 0}]]).{X1, X2, X3};
DefGrad = D[x[X₁, X₂, X₃], {{X₁, X₂, X₃}}];
DefGrad // MatrixForm

Out[•]//MatrixForm=
```

$$\begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Use a simple example to demonstrate that the deformations, $\mathbf{x} = (\mathbf{I} + \gamma \mathbf{s} \otimes \mathbf{n})\mathbf{X}$ and $\mathbf{x} = (\mathbf{I} + \gamma \mathbf{n} \otimes \mathbf{s})\mathbf{X}$ yield the same infinitesimal strain system.

- Set $\mathbf{s} = \mathbf{E}_1$ or $\{1,0,0\}$, and $\mathbf{n} = \mathbf{E}_2$ or $\{0,1,0\}$.

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1+\gamma^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- Set $\mathbf{s} = \mathbf{E}_2$ or$\{0,1,0\}$, and $\mathbf{n} = \mathbf{E}_2$ or $\{1,0,0\}$.

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \begin{pmatrix} 1 & \gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\gamma^2 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Ignoring second order terms, in both cases, we have,

$$\mathbf{C} = \begin{pmatrix} 1 & \gamma & 0 \\ \gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (1 + t^2)X_1\mathbf{E}_1 + tX_2\mathbf{E}_2 + X_3\mathbf{E}_3$ find the velocity in terms of (i) Referential variables, and (ii) in Spatial variables.

- The deformation function is completely decoupled. It therefore requires no programming to invert the equation. Clearly,

$x_1 = (1 + t^2)X_1 \Rightarrow X_1 = \frac{x_1}{(1+t^2)}$; $x_2 = tX_2 \Rightarrow X_2 = \frac{x_2}{t}$ and $X_3 = x_3$. Velocity is the time derivative of the deformation. Hence,

$$\mathbf{v} = \frac{\partial}{\partial t}\boldsymbol{\chi}(\mathbf{X}, t) = 2tX_1\mathbf{E}_1 + X_2\mathbf{E}_2$$

Referential Variables

$$= 2t\frac{x_1}{(1 + t^2)}\mathbf{E}_1 + \frac{x_2}{t}\mathbf{E}_2$$

$$= \frac{2tx_1}{(1 + t^2)}\mathbf{e}_1 + \frac{x_2}{t}\mathbf{e}_2$$

Spatial Variables

As we have chosen the same ONB as coordinates for spatial and referential configurations.

For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (1 + t^2)X_1\mathbf{E}_1 + tX_2\mathbf{E}_2 + X_3\mathbf{E}_3$ find the substantial acceleration by working only in spatial terms

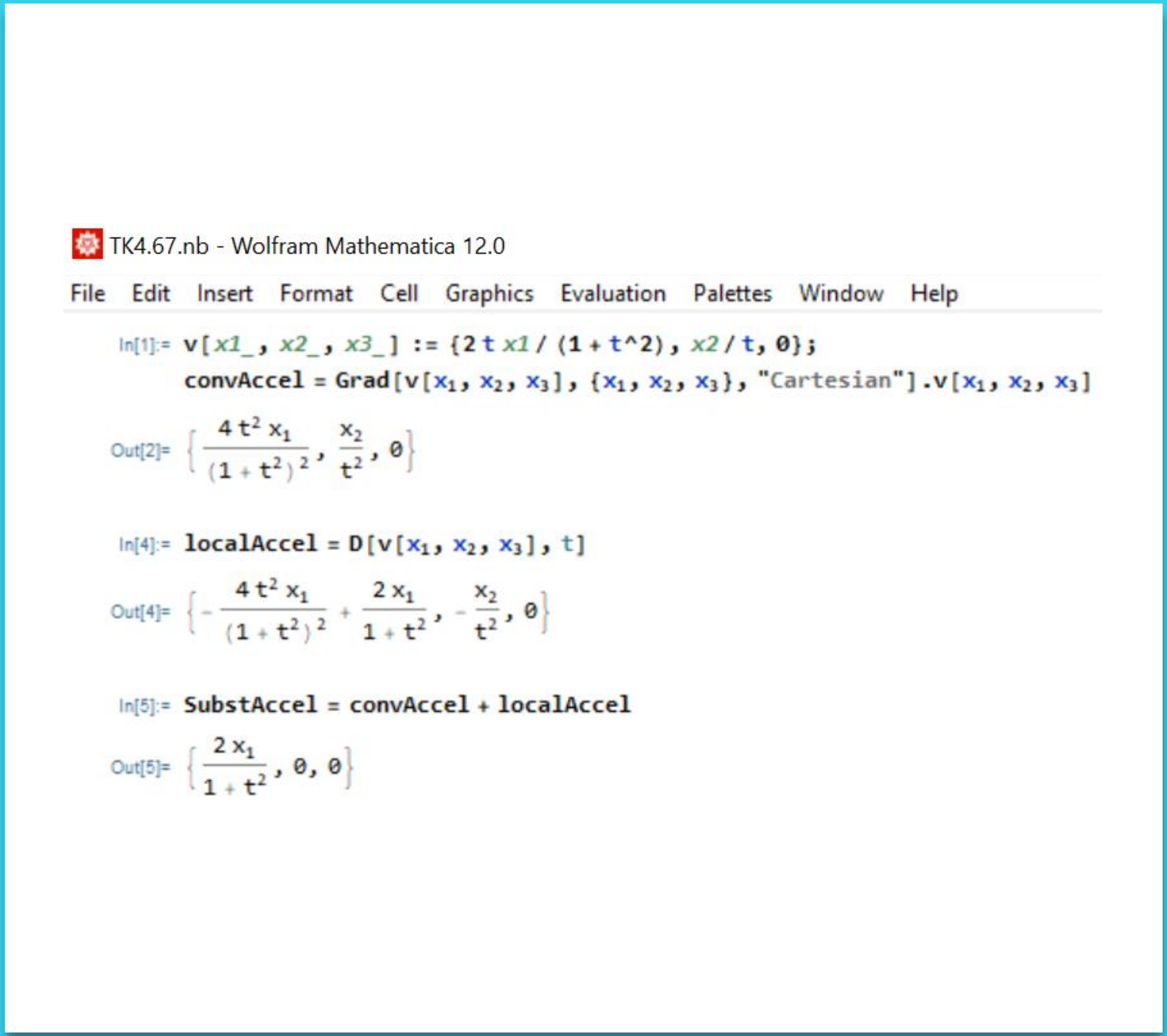
- Substantial acceleration is the material time derivative of velocity. In spatial terms, the velocity for this motion is

$$\mathbf{v}(\mathbf{x}, t) = \frac{2tx_1}{(1+t^2)}\mathbf{e}_1 + \frac{x_2}{t}\mathbf{e}_2$$

- This velocity is the $\hat{\mathbf{f}}$ term as well as the $\mathbf{v}$ term in the equation,

$$(\text{grad }\hat{\mathbf{f}})\mathbf{v} + \frac{\partial \hat{\mathbf{f}}}{\partial t}$$

- The first term being the convective term, and the second, the local acceleration. The computation here is in the attached notebook



For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (1 + t^2)X_1\mathbf{E}_1 + tX_2\mathbf{E}_2 + X_3\mathbf{E}_3$ find the substantial acceleration by working only in referential terms. Show that this result coincides with the computation with spatial variables.

- Substantial acceleration is the material time derivative of velocity. In referential terms, the velocity for this motion is

$$\mathbf{V}(\mathbf{X}, t) = 2tX_1\mathbf{E}_1 + X_2\mathbf{E}_2$$

- Material time derivative of this is simple and does not require programming.

$$\mathbf{A}(\mathbf{X}, t) = \frac{D}{Dt}\mathbf{V}(\mathbf{X}, t) = 2X_1\mathbf{E}_1 = \frac{2x_1}{(1 + t^2)}\mathbf{e}_1$$

The last equality arising from the fact that, $X_1 = \frac{x_1}{(1+t^2)}$. This shows that the substantial acceleration, no matter which configuration is used, returns the same value.

A note on the Mathematica code

- In computing the convective acceleration, we needed to take the spatial gradient of the velocity. You will recall that this produces a tensor term. The expression in the attached code is,

$(\text{grad } \hat{\mathbf{f}})\mathbf{v}$

```
convAccel = Grad[v[x₁, x₂, x₃], {x₁, x₂, x₃}, "Cartesian"].v[x₁, x₂, x₃]
```

- Which we have computed, at this time, in Cartesian coordinates. If the spatial system had been referred to any other coordinate system, the above code computes the same convective acceleration.
- All you need change are the coordinate variable set and the “Cartesian” specification. For example,

```
Grad[v[ρ, θ, ϕ], {ρ, θ, ϕ}, "Spherical"]
```

- will execute the same computation in spherical coordinates!

A note on the Mathematica code, cont'd

- Furthermore, notice also that there could easily be a confusion on the use of capitalization for gradient. In our course, the capitalized gradient, Grad, is a referential gradient while the lower-case gradient, grad, is for spatial. In Mathematica, EVERY function must be capitalized. You distinguish between the two configurations by the arguments you pass to the gradient function. In this example, the gradient we are computing is the spatial gradient even though for compatibility with *Mathematica*, we HAD TO capitalize it.

$(\text{grad}\ \hat{\mathbf{f}})\mathbf{v}$

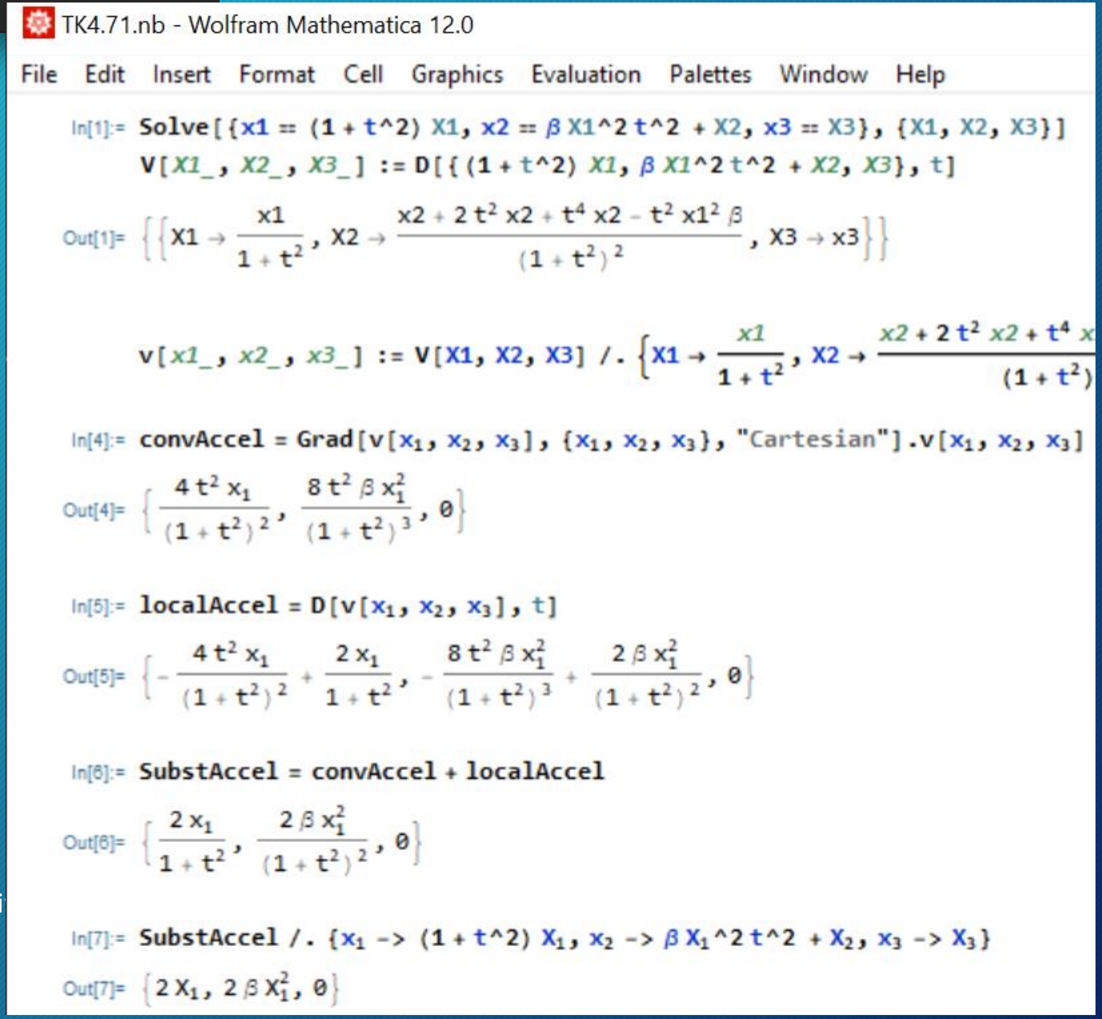
```
convAccel = Grad[v[x1, x2, x3], {x1, x2, x3}, "Cartesian"].v[x1, x2, x3]
```

- Also remember that we will not ordinarily dot a tensor with a vector. However, *Mathematica* uses matrix notation rather than tensor notation. Moving between the conventions and knowing what you are doing is the rite of passage to maturity as a modern engineer. It can only be mastered by practice. I have no consolation to offer to those who memorize everything. There is no alternative to understanding what you are doing!

For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (1 + t^2)X_1\mathbf{E}_1 + (\beta t^2 X_1^2 + X_2)\mathbf{E}_2 + X_3\mathbf{E}_3$ find the velocity and substantial acceleration using both referential and spatial terms. Show equivalency.

- Inverting the motion, using Solve, the reference map here is, $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) =$
$$\frac{x_1}{(1+t^2)}\mathbf{E}_1 + \frac{(1+t^2)^2 x_2 - \beta t^2 x_1^2}{(1+t^2)^2}\mathbf{E}_2 + X_3\mathbf{E}_3$$
- Using this transformation, we obtain the spatial form of velocity and compute the convective term as,
$$\mathbf{v} = \frac{4t^2 x_1^2}{(1+t^2)^2}\mathbf{e}_1 + \frac{8\beta t^2 x_1^2}{(1+t^2)^2}\mathbf{e}_2$$

Local acceleration is obtained by direct differentiation as shown. Addition yields the substantive acceleration which is easily obtained directly by differentiating the motion twice.



For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (1 + t^2)X_1\mathbf{E}_1 + (\beta X_1^2 + X_2)t^2\mathbf{E}_2 + X_3\mathbf{E}_3$ find the reference map, velocity and substantial acceleration using both referential and spatial terms. Show equivalency.

- This is more coupled than the previous problem. The solution is essentially the same and the code we used previously applied here. Inverting the motion, using Solve, the reference map here is, $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) =$

$$\frac{x_1}{(1 + t^2)}\mathbf{E}_1 + \frac{(1 + t^2)^2 x_2 - \beta t^2 x_1^2}{t^2(1 + t^2)^2}\mathbf{E}_2 + X_3\mathbf{E}_3$$

- Using this transformation, we obtain the spatial form of velocity and compute the convective term as,

$$(\text{grad } \mathbf{v})\mathbf{v} = \frac{4t^2 x_1^2}{(1 + t^2)^2}\mathbf{e}_1 + \frac{4x_2}{t^2}\mathbf{e}_2$$

Local acceleration is obtained by direct differentiation as shown. Addition yields the substantive acceleration which is easily obtained directly by differentiating the motion twice.

Coupled Motion

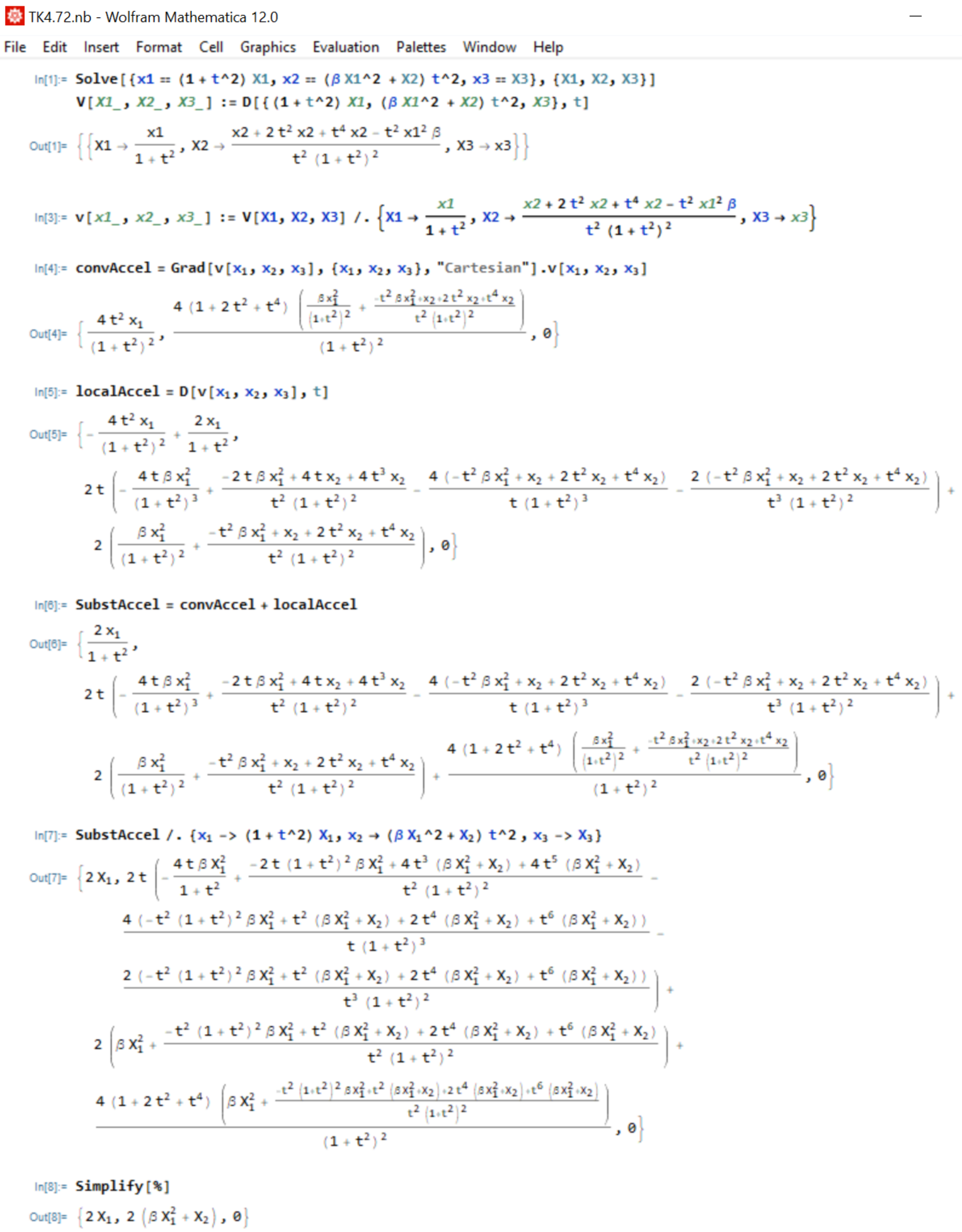
- Note that the local acceleration, after simplification, yields,

$$\frac{\partial \mathbf{v}}{\partial t}=\frac{2(1-t^2)x_1^2}{(1+t^2)^2}\mathbf{e}_1-\frac{2x_2}{t^2}\mathbf{e}_2$$

- Adding the two and transforming back to the referential configuration, we obtain,

$$\frac{D\mathbf{v}}{Dt}=\frac{2x_1}{1+t^2}\mathbf{e}_1+\frac{2x_2}{t^2}\mathbf{e}_2=2X_1\mathbf{E}_1+2(\beta X_1^2+X_2)\mathbf{E}_2$$

- Which, as before, could have been obtained by direct differentiation of the motion in referential form.



For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (\alpha X_2^2 t^2 + X_1)\mathbf{E}_1 + (\beta X_2 t + X_2)\mathbf{E}_2 + X_3\mathbf{E}_3$ find the reference map, velocity and substantial acceleration using both referential and spatial terms. Show equivalency.

- The solution is essentially the same as before, using essentially the same code Inverting the motion, using Solve, the reference map here is, $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) =$

$$\frac{x_1(1+t\beta)^2 - \alpha x_2^2 t^2}{(1+t\beta)^2}\mathbf{E}_1 + \frac{x_2}{1+t\beta}\mathbf{E}_2 + X_3\mathbf{E}_3$$

- Using this transformation, we obtain the spatial form of velocity and compute the convective acceleration term as,

$$(\text{grad } \mathbf{v})\mathbf{v} = \frac{4t\alpha\beta x_2^2}{(1+t\beta)^3}\mathbf{e}_1 + \frac{\beta^2 x_2}{(1+t\beta)^2}\mathbf{e}_2$$

Local acceleration is obtained by direct differentiation as shown. Addition yields the substantive acceleration which is easily obtained directly by differentiating the motion twice.

Cont’d

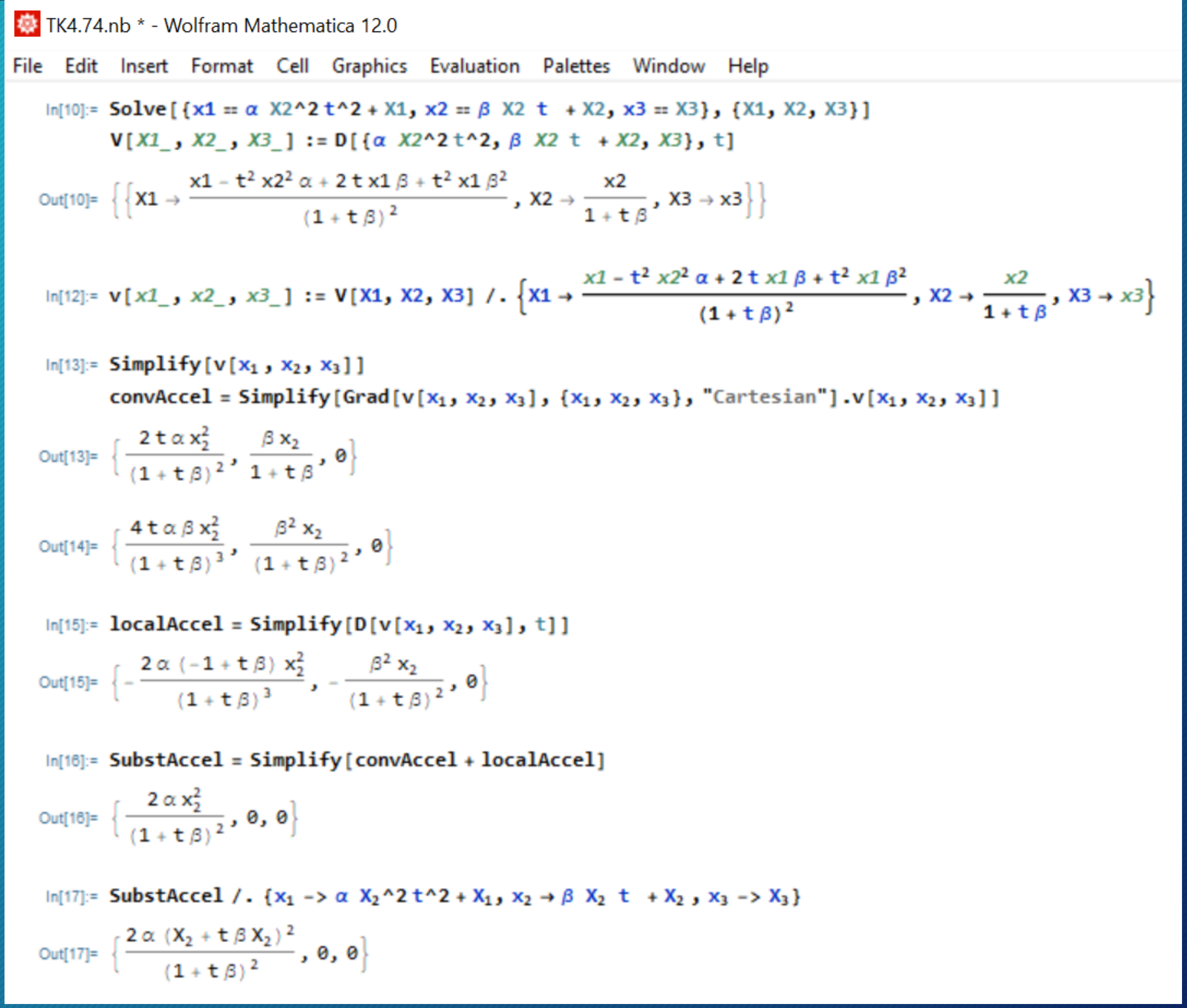
- Note that the local acceleration, after simplification, yields,

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{2\alpha(t\beta - 1)x_2^2}{(1+t\beta)^3}\mathbf{e}_1 - \frac{\beta^2 x_2}{(1+t\beta)^2}\mathbf{e}_2$$

- Adding the two and transforming back to the referential configuration, we obtain,

$$\frac{D\mathbf{v}}{Dt} = \frac{2\alpha x_2^2}{(1+t\beta)^2}\mathbf{e}_1 = \frac{2\alpha(X_2 + t\beta X_2)^2}{(1+t\beta)^2}\mathbf{E}_1$$

- Which, as before, could have been obtained by direct differentiation of the motion in referential form.



For the motion $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = (\alpha X_2^2 t^2 + X_1)\mathbf{E}_1 + (\beta X_2 t + X_2)\mathbf{E}_2 + X_3\mathbf{E}_3$ find the reference map, velocity and substantial acceleration using both referential and spatial terms. Show equivalency.

- The solution is essentially the same as before, using essentially the same code Inverting the motion, using Solve, the reference map here is, $\mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t) =$

$$\frac{x_1(1+t\beta)^2 - \alpha x_2^2 t^2}{(1+t\beta)^2}\mathbf{E}_1 + \frac{x_2}{1+t\beta}\mathbf{E}_2 + X_3\mathbf{E}_3$$

- Using this transformation, we obtain the spatial form of velocity and compute the convective acceleration term as,

$$(\text{grad }\mathbf{v})\mathbf{v} = \frac{4t\alpha\beta x_2^2}{(1+t\beta)^3}\mathbf{e}_1 + \frac{\beta^2 x_2}{(1+t\beta)^2}\mathbf{e}_2$$

Local acceleration is obtained by direct differentiation as shown. Addition yields the substantive acceleration which is easily obtained directly by differentiating the motion twice.

Cont'd

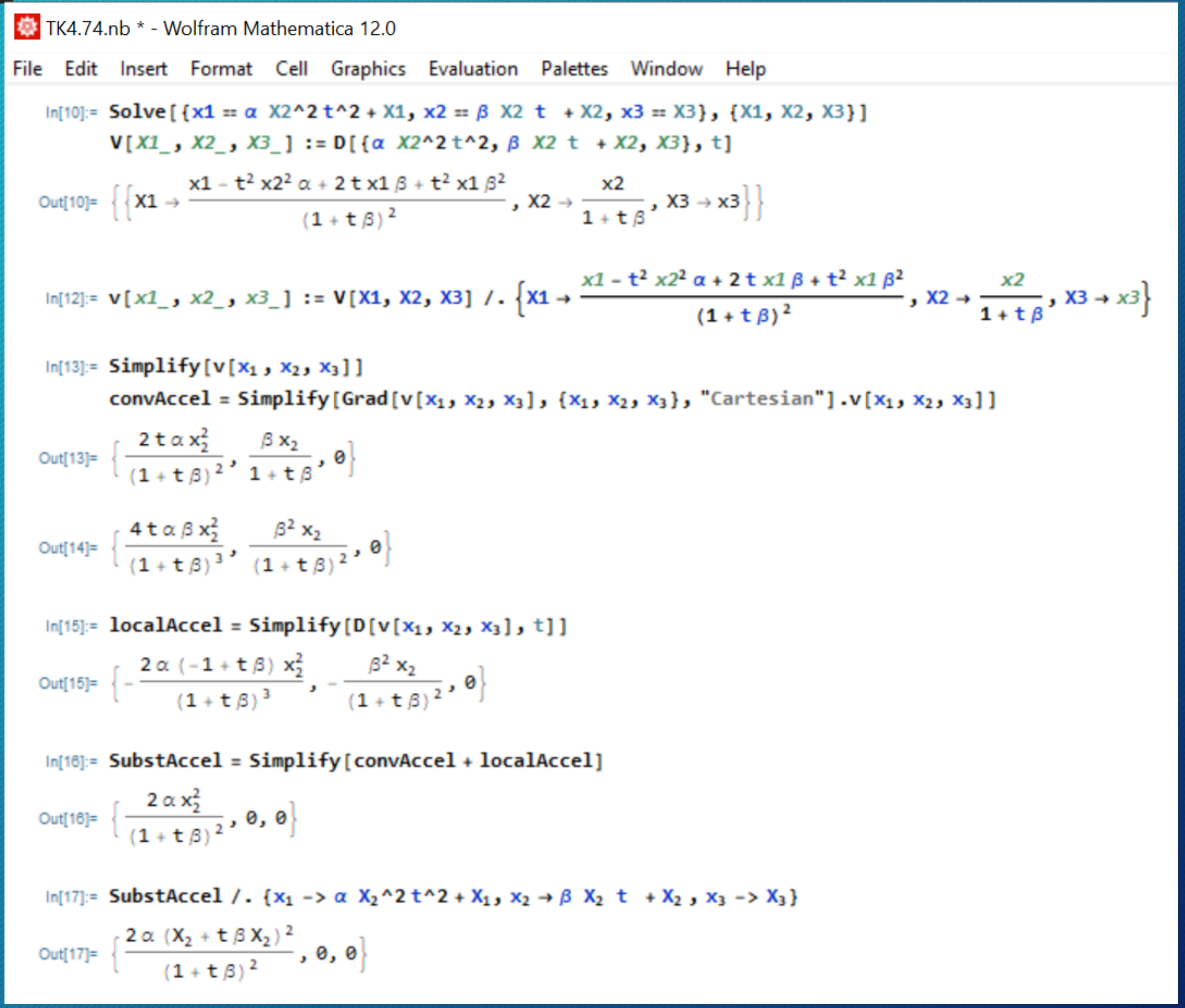
- Note that the local acceleration, after simplification, yields,

$$\frac{\partial \mathbf{v}}{\partial t} = \frac{2\alpha(t\beta - 1)x_2^2}{(1+t\beta)^3}\mathbf{e}_1 - \frac{\beta^2 x_2}{(1+t\beta)^2}\mathbf{e}_2$$

- Adding the two and transforming back to the referential configuration, we obtain,

$$\frac{D\mathbf{v}}{Dt} = \frac{2\alpha x_2^2}{(1+t\beta)^2}\mathbf{e}_1 = \frac{2\alpha(X_2 + t\beta X_2)^2}{(1+t\beta)^2}\mathbf{E}_1$$

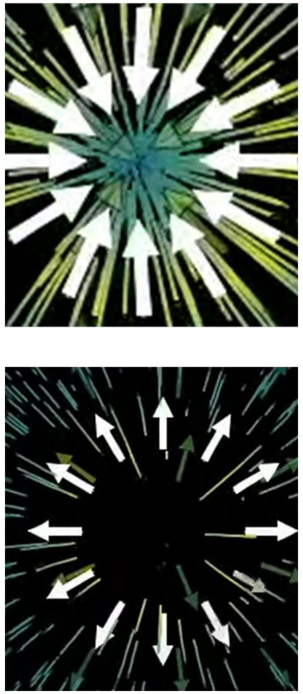
- Which, as before, could have been obtained by direct differentiation of the motion in referential form.

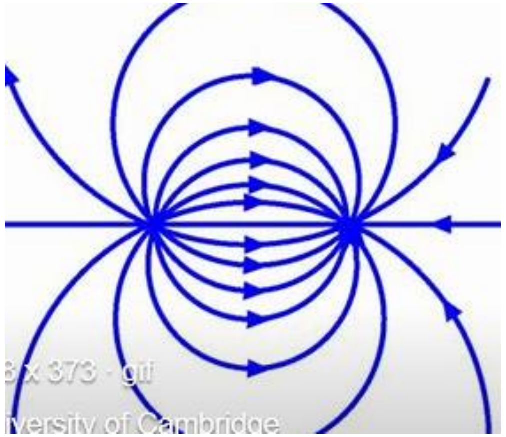


Source & Sink

- Sources and Sinks are useful idealizations in Flow Analyses. The idea that a fluid flows radially out of nothingness or flows inwads and vanishes may be too difficult ti imagine. Look at the magnetic analog of the lines of force near the poles of the magnet. The relevant equations and descriptions are similar.
- We are going to describe a source flow in terms of the velocity field,

$$\mathbf{v}(\mathbf{x}, t) = \frac{\left(\alpha - \frac{\beta}{t}\right)}{r}\mathbf{e}_r$$





For Source Flow, Find Velocity Gradient, Convective Acceleration, Substantial Acceleration, and the Divergence of the Flow. Show that the local acceleration vanishes at steady state. Use Cylindrical Polar Coordinates

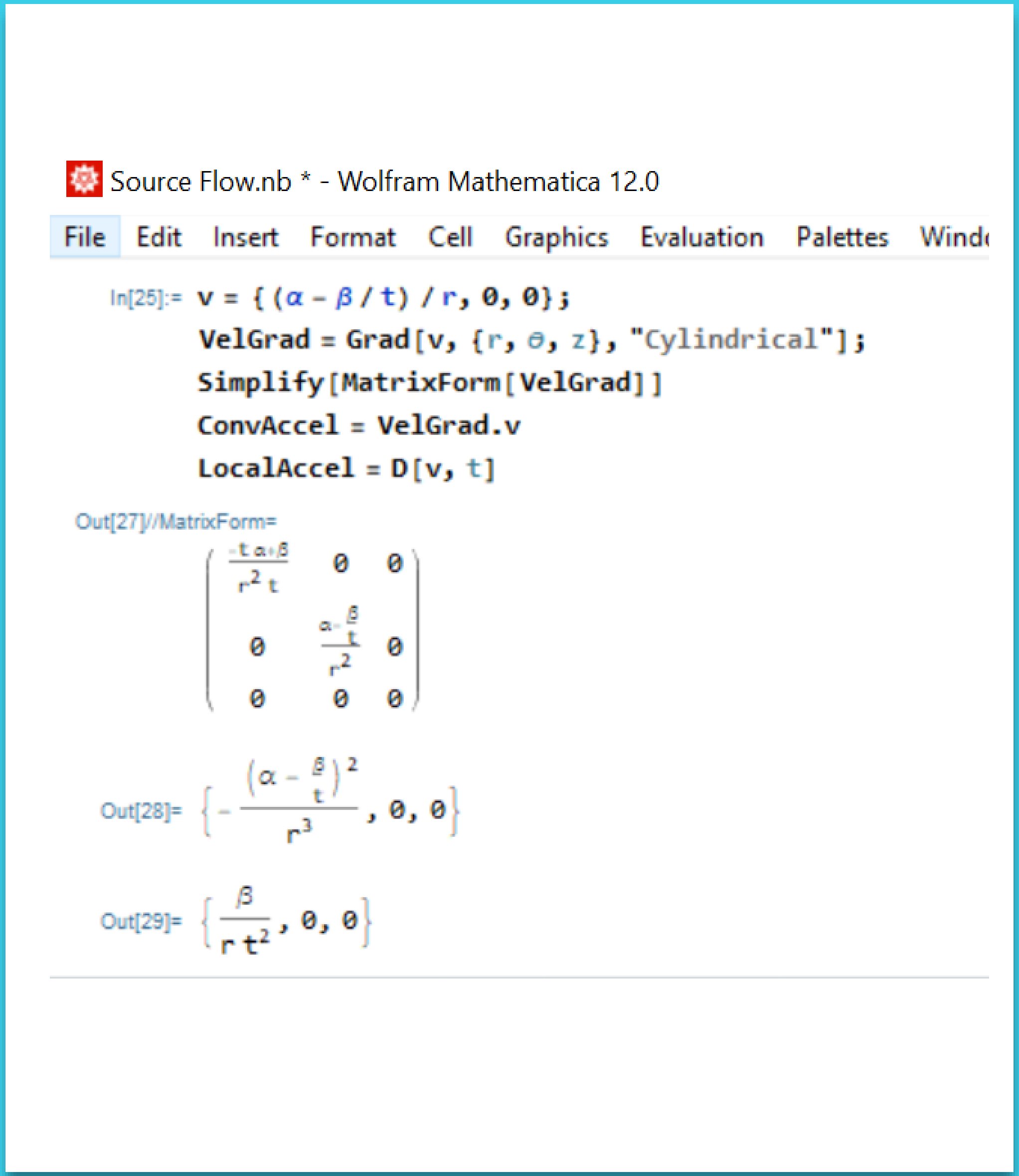
- From the attached code, we can see that both convective as well as local accelerations are in the radial direction.

$$\frac{D\mathbf{v}}{Dt} = -\left(\frac{\left(\alpha - \frac{\beta}{t}\right)^2}{r^3} + \frac{\beta}{rt^2}\right)\mathbf{e}_r$$

At steady state, the term containing t vanishes. We can simply set $\beta = 0$, and find that

$$\frac{D\mathbf{v}}{Dt} = -\frac{\alpha^2}{r^3}\mathbf{e}_r$$

Which is the substantive acceleration in steady flow.



Find the Stretching as well as Spin Tensors for the above flow field. Also find the dimensions of the two parameters in the flow terms.

$$[\mathbf{L}] = \begin{pmatrix} \frac{-t\alpha + \beta}{r^2 t} & 0 & 0 \\ 0 & \frac{t\alpha - \beta}{r^2 t} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It is symmetric and must therefore equal the stretching tensor. The spin tensor is the Annihilator (Zero) tensor here because of the symmetry of the velocity gradient.

For dimensional consistency, α must have the unit of L^2, β the unit of L^2T .

For Source Flow, Find Velocity Gradient, Convective Acceleration, Substantial Acceleration, and the Divergence of the Flow. Use Cartesian Coordinates

- From the attached code, we can see that both convective as well as local accelerations are in the radial direction.

$$\frac{D\mathbf{v}}{Dt} = -\left(\frac{\left(\alpha - \frac{\beta}{t}\right)^2}{r^3} + \frac{\beta}{rt^2}\right)\mathbf{e}_r$$

Source Flow.nb * - Wolfram Mathematica 12.0

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```
In[25]:= v = {(α - β / t) / r, 0, 0};
         VelGrad = Grad[v, {r, θ, z}, "Cylindrical"];
         Simplify[MatrixForm[VelGrad]]
         ConvAccel = VelGrad.v
         LocalAccel = D[v, t]
```

Out[27]//MatrixForm=

$$\begin{pmatrix} \frac{-t\alpha+\beta}{r^2 t} & 0 & 0 \\ 0 & \frac{\alpha-\frac{\beta}{t}}{r^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[28]= $\left\{-\frac{\left(\alpha-\frac{\beta}{t}\right)^2}{r^3}, 0, 0\right\}$

Out[29]= $\left\{\frac{\beta}{r\,t^2}, 0, 0\right\}$

Convert the flow $\mathbf{v}(\mathbf{x},t)=\frac{\left(\alpha-\frac{\beta}{t}\right)}{r}\mathbf{e}_r$ to Cartesian Coordinates and obtain the acceleration in that system of coordinates.

- Beginning with the fact that the radial basis vector,

$$\mathbf{e}_r=\cos\theta\,\mathbf{e}_1+\sin\theta\,\mathbf{e}_2=\frac{x_1\mathbf{e}_1}{\sqrt{x_1^2+x_2^2}}+\frac{x_2\mathbf{e}_2}{\sqrt{x_1^2+x_2^2}}$$

- It follows that the velocity,

$$\mathbf{v}(\mathbf{x},t)=\frac{\left(\alpha-\frac{\beta}{t}\right)}{r}\mathbf{e}_r=\frac{x_1\left(\alpha-\frac{\beta}{t}\right)\mathbf{e}_1}{x_1^2+x_2^2}+\frac{x_2\left(\alpha-\frac{\beta}{t}\right)\mathbf{e}_2}{x_1^2+x_2^2}$$

- With this Cartesian description, we can redo the code as shown. Note that the gradient is now invoked with Cartesian variables.

```
u = {(α - β / t) x1 / (x1^2 + x2^2), (α - β / t) x2 / (x1^2 + x2^2), 0}
ConvAccel = Simplify[Grad[u, {x1, x2, x3}, "Cartesian"].u]
LocalAccel = Simplify[D[u, t]]
```

$$\left\{\frac{x1\left(\alpha-\frac{\beta}{t}\right)}{x1^2+x2^2},\ \frac{x2\left(\alpha-\frac{\beta}{t}\right)}{x1^2+x2^2},\ 0\right\}$$

$$\left\{-\frac{x1\,(-t\,\alpha+\beta)^2}{t^2\,(x1^2+x2^2)^2},\ -\frac{x2\,(-t\,\alpha+\beta)^2}{t^2\,(x1^2+x2^2)^2},\ 0\right\}$$

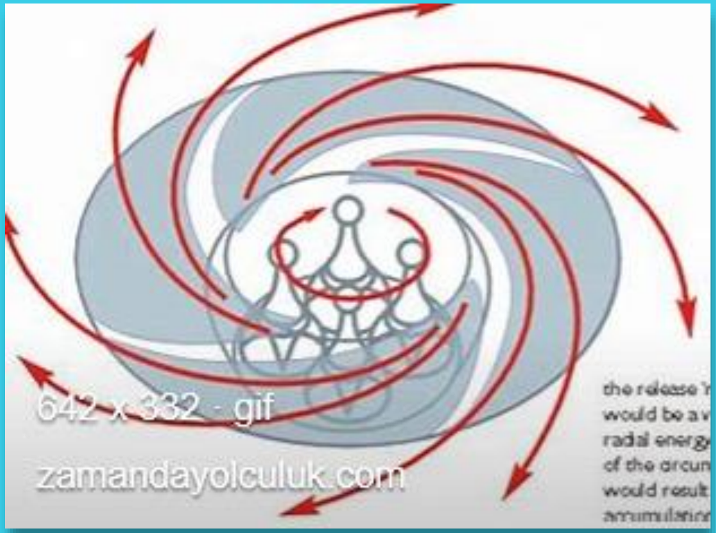
$$\left\{\frac{x1\,\beta}{t^2\,(x1^2+x2^2)},\ \frac{x2\,\beta}{t^2\,(x1^2+x2^2)},\ 0\right\}$$

Planar Vortex Flow

- In the ideal planar vortex, velocity is directed along $\mathbf{e}_\theta$. Here we model a simple vortex with a small temporally variable velocity:

$$\mathbf{v}(\mathbf{x},t) = -\frac{x_2\mathbf{e}_1}{x_1^2+x_2^2} + \frac{x_1\mathbf{e}_2}{x_1^2+x_2^2}$$
$$= -\frac{\sin\theta}{r}\mathbf{e}_1 + \frac{\cos\theta}{r}\mathbf{e}_1$$
$$= \frac{\mathbf{e}_\theta}{r}$$

- We can compute the flow parameters as before.



For Planar Vortex Flow, find the Velocity Gradient, Stretching and Spin Tensors and Convective acceleration in both Cylindrical and Cartesian coordinates

- The attached code calculates the Velocity Gradient and Convective acceleration in both Cylindrical and Cartesian coordinates.
- The equivalency of these is not obvious.
- Note here that the Velocity gradient is a symmetric tensor in both cases and therefore, the spin tensor must vanish.

Planar Vortex.nb * - Wolfram Mathematica 12.0

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```
In[1]:= v = {0, 1 / r, 0};
VelGrad = Grad[v, {r, θ, z}, "Cylindrical"];
Simplify[MatrixForm[VelGrad]]
ConvAccel = VelGrad.v
```

//MatrixForm=

$$\begin{pmatrix} 0 & -\frac{1}{r^2} & 0 \\ -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[4]= $\left\{-\frac{1}{r^3}, 0, 0\right\}$

```
In[5]:= u = {-x2 / (x1^2 + x2^2), x1 / (x1^2 + x2^2), 0};
VelGrad1 = Grad[u, {x1, x2, x3}, "Cartesian"];
Simplify[MatrixForm[VelGrad1]]
ConvAccel = VelGrad1.u
```

Out[7]//MatrixForm=

$$\begin{pmatrix} \frac{2\,x1\,x2}{(x1^2+x2^2)^2} & \frac{-x1^2+x2^2}{(x1^2+x2^2)^2} & 0 \\ \frac{-x1^2+x2^2}{(x1^2+x2^2)^2} & -\frac{2\,x1\,x2}{(x1^2+x2^2)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Out[8]= $\left\{-\frac{2\,x1\,x2^2}{(x1^2+x2^2)^3} + \frac{x1\left(\frac{2\,x2^2}{(x1^2+x2^2)^2} - \frac{1}{x1^2+x2^2}\right)}{x1^2+x2^2}, -\frac{2\,x1^2\,x2}{(x1^2+x2^2)^3} - \frac{x2\left(-\frac{2\,x1^2}{(x1^2+x2^2)^2} + \frac{1}{x1^2+x2^2}\right)}{x1^2+x2^2}, \ldots\right.$

Demonstrate the equivalency of the Velocity gradients computed in the two coordinate systems above.

- In cylindrical and Cartesian coordinates we have,

$$[\mathbf{L}]_{Cyl} = \begin{pmatrix} 0 & -\frac{1}{r^2} & 0 \\ -\frac{1}{r^2} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$[\mathbf{L}]_{Cart} = \begin{pmatrix} \frac{2x_1x_2}{(x_1^2+x_2^2)^2} & \frac{-x_1^2+x_2^2}{(x_1^2+x_2^2)^2} & 0 \\ \frac{-x_1^2+x_2^2}{(x_1^2+x_2^2)^2} & \frac{-2x_1x_2}{(x_1^2+x_2^2)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We proceed to find the eigenvalues of the two Velocity Gradients as shown in the attached code. It is now easy to see the equivalency!

From the symmetry of the Velocity gradient, we see that the spin is zero. The stretching tensor is equal to the velocity gradient.

```
In[9]:= Eigenvalues[VelGrad]
        Eigenvalues[VelGrad1]
```

Out[9]= $\left\{-\frac{1}{r^2}, \frac{1}{r^2}, 0\right\}$

Out[10]= $\left\{0, \frac{-x1^2 - x2^2}{(x1^2 + x2^2)^2}, \frac{1}{x1^2 + x2^2}\right\}$

For Couette flow, the velocity field is $\mathbf{v}(\mathbf{x}, t) = \omega r \mathbf{e}_\theta$. Compute the convective and substantial acceleration. Also compute the stretching and spin rates.

- From the attached code, we can see that both convective as well as local accelerations are in the radial direction.
$$\frac{D\mathbf{v}}{Dt} = -r\omega^2 \mathbf{e}_r$$
- Local acceleration is zero. Note that the velocity, $\omega r \mathbf{e}_\theta$, is tangential, but convective acceleration is radial.
- The stretching rate tensor zero, the spin tensor is the same as the velocity gradient since the latter is antisymmetric. Remember that the former is the symmetric part of the velocity gradient, the latter is the skew part of it.

Couette Flow.nb * - Wolfram Mathematica 12.0

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```
In[1]:= v = {0, ω r, 0};
        VelGrad = Grad[v, {r, θ, z}, "Cylindrical"];
        Simplify[MatrixForm[VelGrad]]
        ConvAccel = VelGrad.v
        LocalAccel = D[v, t]

Out[3]//MatrixForm=
        (0  -ω  0)
        (ω   0  0)
        (0   0  0)

Out[4]= {-r ω^2, 0, 0}

Out[5]= {0, 0, 0}
```

For Couette flow, the velocity field is $\mathbf{v}(\mathbf{x}, t) = \omega r \mathbf{e}_\theta$. Compute the convective and substantial acceleration. Also compute the stretching and spin rates. Work in Cartesian system

- Beginning with the fact that the radial basis vector,

$$\mathbf{e}_\theta = -\sin\theta\,\mathbf{e}_1 + \cos\theta\,\mathbf{e}_2$$
$$= -\frac{x_2\mathbf{e}_1}{\sqrt{x_1^2 + x_2^2}} + \frac{x_1\mathbf{e}_2}{\sqrt{x_1^2 + x_2^2}}$$
$$\mathbf{v}(\mathbf{x}, t) = \omega r \mathbf{e}_\theta = -x_2\mathbf{e}_1 + x_1\mathbf{e}_2$$
$$\frac{D\mathbf{v}}{Dt} = -x_1\omega\mathbf{e}_1 + x_2\omega\mathbf{e}_2$$

- Local acceleration is zero. Note that the velocity, $\omega r \mathbf{e}_\theta$, is tangential, but convective acceleration is radial.
- The stretching rate tensor zero, the spin tensor is the same as the velocity gradient since the latter is antisymmetric. Remember that the former is the symmetric part of the velocity gradient, the latter is the skew part of it.

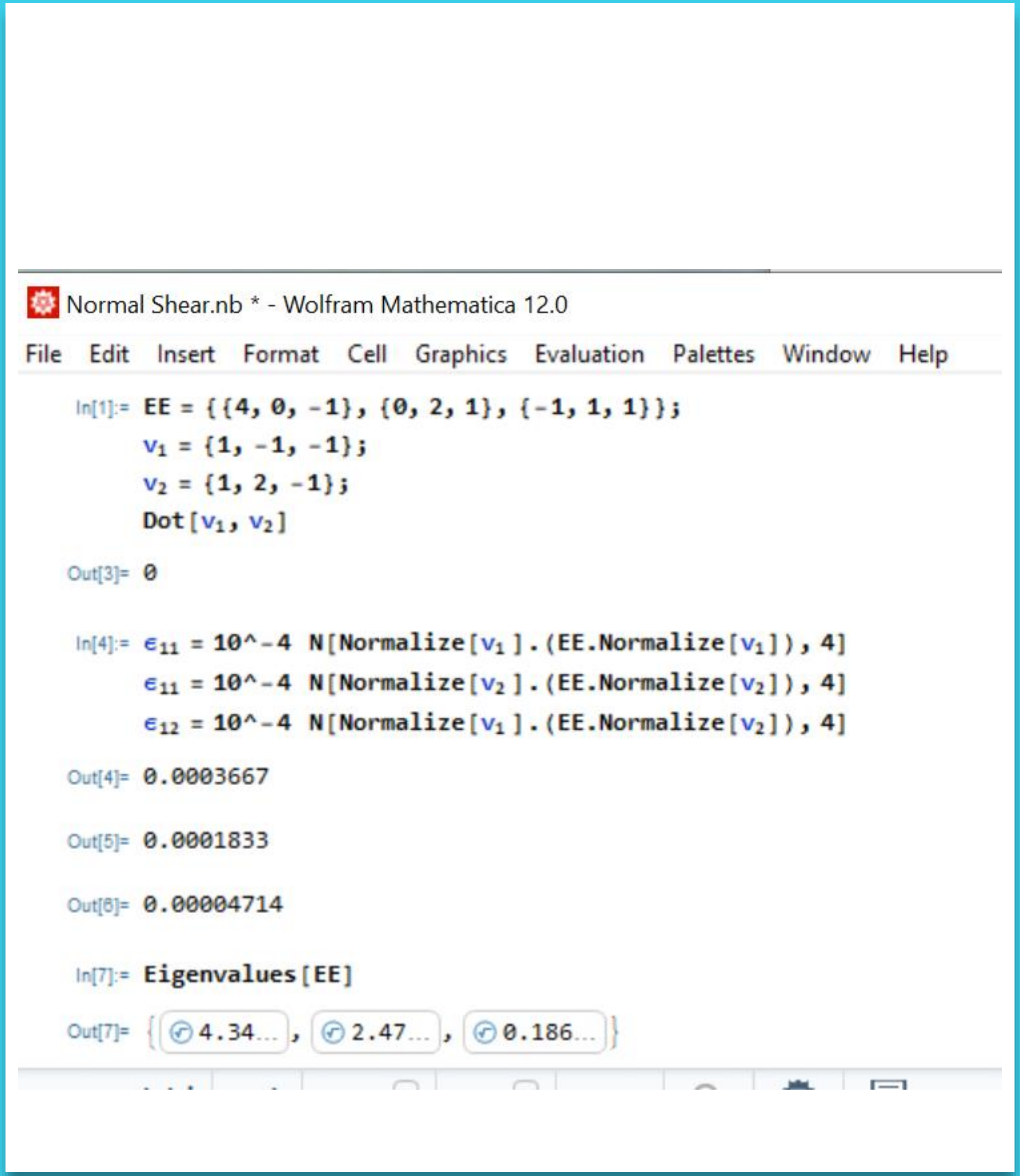
```
= u = {-x1 ω, x2 ω, 0}
  VelGrad1 = Grad[u, {x1, x2, x3}, "Cartesian"];
  Simplify[MatrixForm[VelGrad]]
  ConvAccel = VelGrad1.u
  LocalAccel = D[u, t]

]= {-x1 ω, x2 ω, 0}

MatrixForm=
  ( 0  -ω  0 )
  ( ω   0  0 )
  ( 0   0  0 )

]= {x1 ω^2, x2 ω^2, 0}

]= {0, 0, 0}
```

Show that the directions of vectors $\mathbf{v}_1 = \mathbf{E}_1 - \mathbf{E}_2 - \mathbf{E}_3$ and $\mathbf{v}_2 = \mathbf{E}_1 + 2\mathbf{E}_2 - \mathbf{E}_3$ are orthogonal. If the strain tensor at a point in a material is $\begin{pmatrix} 4 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \times 10^{-4}$. Find the linear strains in the directions of $\mathbf{v}_1$ and $\mathbf{v}_2$ What is the decrease in right angle of elements with unit vectors along $\mathbf{v}_1$and $\mathbf{v}_2$.

- Orthogonality is shown by the vanishing of the dot products. The code here shows that $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$.
- The rest of the computation is as shown in the attached code. Given that $\hat{\mathbf{v}}_1$ and $\hat{\mathbf{v}}_2$ are the normalized vectors in the directions $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively, note that the same answer will be obtained for

$$\epsilon_{12} = \hat{\mathbf{v}}_1 \cdot \mathbf{E}\hat{\mathbf{v}}_2 = \hat{\mathbf{v}}_2 \cdot \mathbf{E}\hat{\mathbf{v}}_1$$

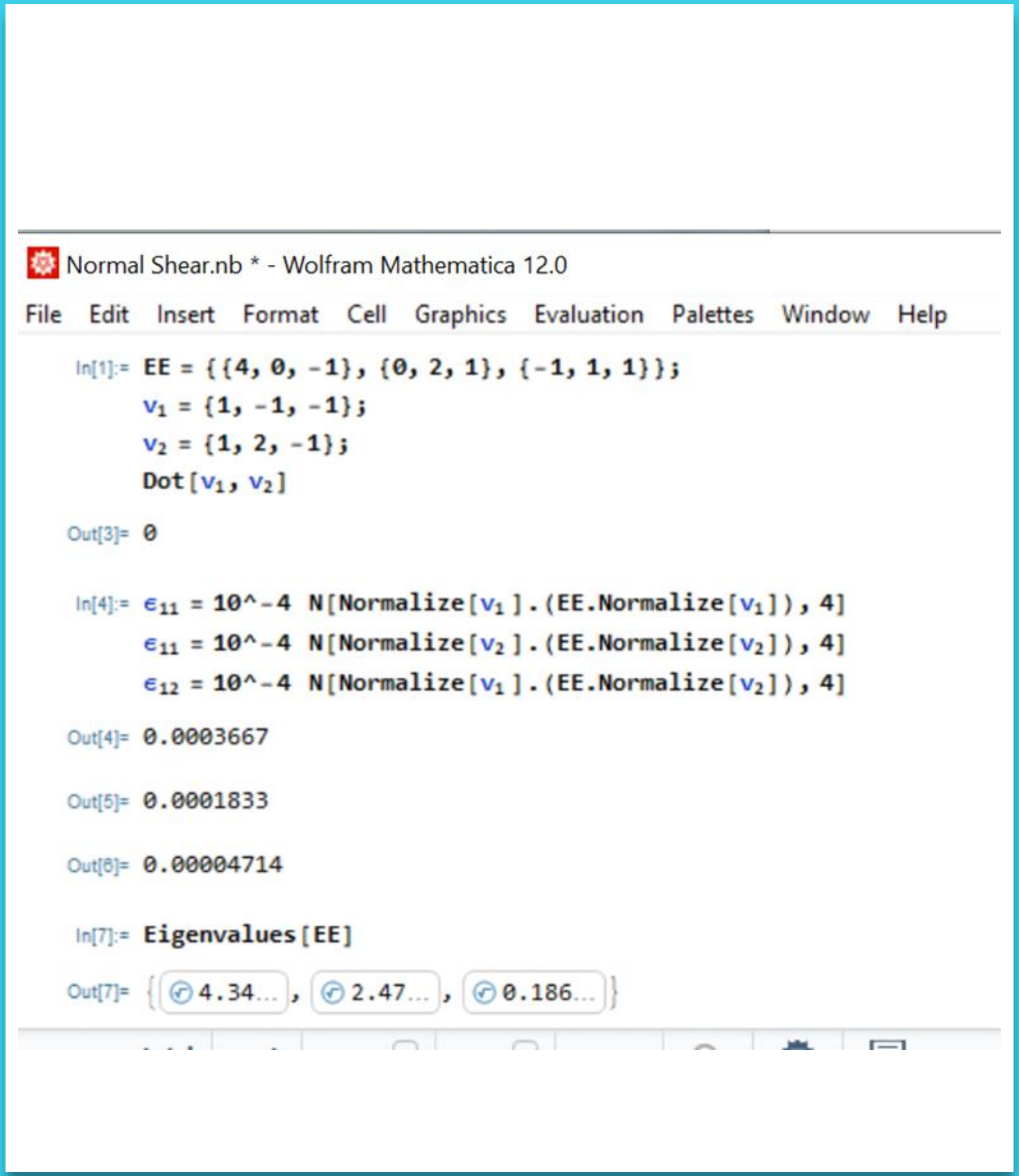
Since $\mathbf{E}$ is symmetrical.



Find the principal strains and dilatation of the strain tensor $[\mathbf{E}] = \begin{pmatrix} 4 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \times 10^{-4}$.

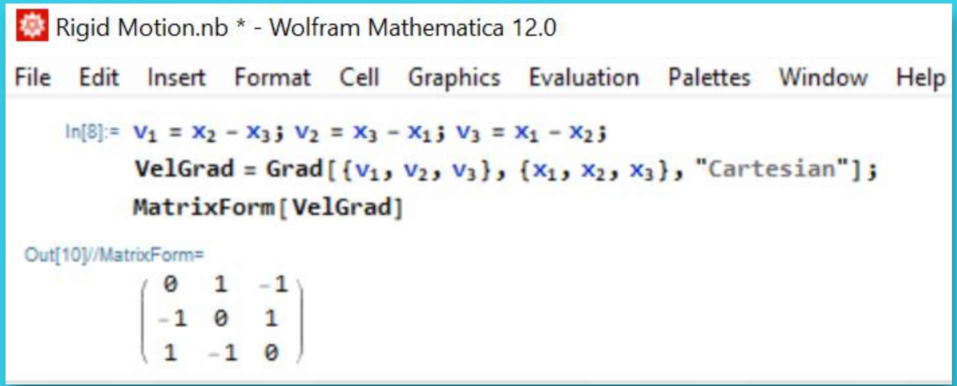
Find the sum of the principal strains and explain why it equals the dilatation.

- The dilatation is the trace of the strain tensor. In this case, dilatation,
  $$\Theta = (4 + 2 + 1) \times 10^{-4} = 7 \times 10^{-4}$$
- The sum of the eigenvalues is
  $$\Theta = (4.34 + 2.47 + 0.186) \times 10^{-4} \approx 7 \times 10^{-4}$$
- The two values must be equal because when a tensor is diagonalized by similarity transformations, only diagonal terms remain. The trace is therefore the same as the sum of its eigenvalues. Similarity transformations do not alter traces nor eigenvalues.

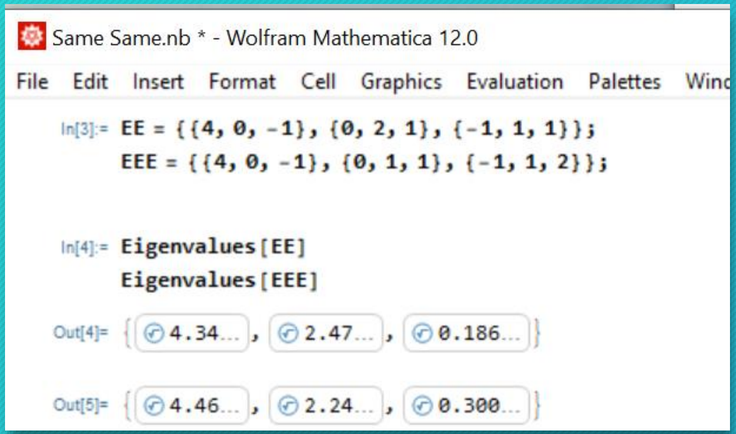


Show that the velocity field, $v_1 = x_2 - x_3, v_2 = x_3 - x_1$, and $v_3 = x_1 - x_2$ constitutes a rigid flow field.

- In a rigid flow field, the stretching tensor will be the annihilator:No stretches. Recall that the stretching tensor is the symmetrical part of the velocity gradient. The attached code computes the velocity gradient.
- Observe that the velocity gradient we obtained is here antisymmetric. Its symmetrical part is the zero tensor. This field therefore is a rigid spinning field.



The strain fields $\begin{pmatrix} 4 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{pmatrix} \times 10^{-4}$ and $\begin{pmatrix} 4 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix} \times 10^{-4}$ look similar. Show what they have in common and show that the strain tensors cannot represent the same strain field



- The two strain tensors given possess the same traces. They also have the same off-diagonal elements. Consequently, at the point in question, they will produce the same dilatation because dilatation is simply the trace of a tensor.
- Next we take the eigenvalues of these tensors as shown in the attached code. The eigenvalues are different.
- Consequently they are for different strain situations. They cannot be converted one to another by similarity transformations.

Components of Grads, Divs & Curls

- Slides 11:29-32 of Kinematics 1 show how to obtain the components of grads in Cartesian coordinates.
- The components of divs and curls follow naturally in such coordinate systems.
- Curvilinear systems in general require the use of general tensor coordinates and the introduction of covariant derivatives. We avoid these complications by resorting to Mathematica coding.

Small Strains

- Again, in Cartesian, the transpose is simply the reversal of the indices. Hence, we can write,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left((\mathbf{H} + \mathbf{I})^{\mathrm{T}}(\mathbf{H} + \mathbf{I}) - \mathbf{I}\right)$$
$$= \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right)$$

- In component form as,

$$E_{ij} = \frac{1}{2}\left(H_{ij} + H_{ji} + H_{ki}H_{kj}\right)$$
$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j}\right)$$

Lagrangian Strain

Can easily be expanded as shown here. YOU ARE EXPECTED to be able to do this manually by simply interpreting the indices

$$E_{xx} = \frac{\partial u_x}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial x}\right)^2 + \left(\frac{\partial u_y}{\partial x}\right)^2 + \left(\frac{\partial u_z}{\partial x}\right)^2\right]$$

$$E_{yy} = \frac{\partial u_y}{\partial y} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial y}\right)^2 + \left(\frac{\partial u_y}{\partial y}\right)^2 + \left(\frac{\partial u_z}{\partial y}\right)^2\right]$$

$$E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_x}{\partial z}\right)^2 + \left(\frac{\partial u_y}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right]$$

$$E_{xy} = E_{yx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) + \frac{1}{2}\left[\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial y}\right]$$

$$E_{xz} = E_{zx} = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) + \frac{1}{2}\left[\frac{\partial u_x}{\partial x}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial x}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial x}\frac{\partial u_z}{\partial z}\right]$$

$$E_{yz} = E_{zy} = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right) + \frac{1}{2}\left[\frac{\partial u_x}{\partial y}\frac{\partial u_x}{\partial z} + \frac{\partial u_y}{\partial y}\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\frac{\partial u_z}{\partial z}\right]$$

Lagrangian Strain in Curvilinear systems

Can easily be expanded as shown here. YOU ARE NOT EXPECTED to be able to do this manually by simply interpreting the indices. Instead, you are expected to use software such as Mathematica to do this.

Cylindrical Coordinates

$$E_{rr} = \frac{\partial u_r}{\partial r} + \frac{1}{2}\left[\left(\frac{\partial u_r}{\partial r}\right)^2 + \left(\frac{\partial u_\theta}{\partial r}\right)^2 + \left(\frac{\partial u_z}{\partial r}\right)^2\right]$$

$$E_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{1}{2r^2}\left[\left(u_\theta - \frac{\partial u_r}{\partial \theta}\right)^2 + \left(u_r + \frac{\partial u_\theta}{\partial \theta}\right)^2 + \left(\frac{\partial u_z}{\partial \theta}\right)^2\right]$$

$$E_{zz} = \frac{\partial u_z}{\partial z} + \frac{1}{2}\left[\left(\frac{\partial u_r}{\partial z}\right)^2 + \left(\frac{\partial u_\theta}{\partial z}\right)^2 + \left(\frac{\partial u_z}{\partial z}\right)^2\right]$$

$$E_{r\theta} = E_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) + \frac{1}{2r}\left[\frac{\partial u_r}{\partial r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial r}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial \theta}\right]$$

$$E_{\theta z} = E_{z\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) + \frac{1}{2r}\left[\frac{\partial u_r}{\partial z}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial z}\left(u_r + \frac{\partial u_\theta}{\partial \theta}\right) + \frac{\partial u_z}{\partial z}\frac{\partial u_z}{\partial \theta}\right]$$

$$E_{zr} = E_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) + \frac{1}{2}\left[\frac{\partial u_r}{\partial r}\frac{\partial u_r}{\partial z} + \frac{\partial u_\theta}{\partial r}\frac{\partial u_\theta}{\partial z} + \frac{\partial u_z}{\partial r}\frac{\partial u_z}{\partial z}\right]$$

Spherical Coordinates

$$E_{\rho\rho} = \frac{\partial u_\rho}{\partial \rho} + \frac{1}{2}\left[\left(\frac{\partial u_\theta}{\partial \rho}\right)^2 + \left(\frac{\partial u_\rho}{\partial \rho}\right)^2 + \left(\frac{\partial u_\phi}{\partial \rho}\right)^2\right]$$

$$E_{\theta\theta} = \frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} + \frac{u_\rho}{\rho} + \frac{1}{2\rho^2}\left[\left(u_\theta - \frac{\partial u_\rho}{\partial \theta}\right)^2 + \left(u_\rho + \frac{\partial u_\theta}{\partial \theta}\right)^2 + \left(\frac{\partial u_\phi}{\partial \theta}\right)^2\right]$$

$$E_{\phi\phi} = \frac{u_\theta \cot\theta}{\rho} + \frac{u_\rho}{\rho} + \frac{1}{\rho\sin\theta}\frac{\partial u_\phi}{\partial \phi} + \frac{1}{2\rho^2\sin^2\theta}\left[\left(\frac{\partial u_\rho}{\partial \phi} - u_\phi \sin\theta\right)^2 + \left(\frac{\partial u_\theta}{\partial \phi} - u_\phi\cos\theta\right)^2 + \left(u_\theta\cos\theta + u_\rho\sin\theta + \frac{\partial u_\phi}{\partial \phi}\right)^2\right]$$

$$E_{\rho\theta} = \frac{1}{2\rho}\left(\rho\frac{\partial u_\theta}{\partial \rho} - u_\theta + \frac{\partial u_\rho}{\partial \theta}\right) + \frac{1}{2\rho}\left[\frac{\partial u_\rho}{\partial \rho}\left(\frac{\partial u_\rho}{\partial \theta} - u_\theta\right) + \frac{\partial u_\theta}{\partial \rho}\left(\frac{\partial u_\theta}{\partial \theta} + u_\rho\right) + \frac{\partial u_\phi}{\partial \theta}\frac{\partial u_\phi}{\partial \rho}\right]$$

$$E_{\rho\phi} = -\frac{u_\phi}{2\rho} + \frac{1}{2\rho}\frac{1}{\sin\theta}\frac{\partial u_\rho}{\partial \phi} + \frac{1}{2}\frac{\partial u_\phi}{\partial \rho} + \frac{1}{2\rho}\left[\frac{\partial u_\rho}{\partial \rho}\left(\frac{1}{\sin\theta}\frac{\partial u_\rho}{\partial \phi} - u_\phi\right) + \frac{\partial u_\theta}{\partial \rho}\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} - u_\phi\cot\theta\right) + \frac{\partial u_\phi}{\partial \rho}\left(u_\theta\cot\theta + u_\rho + \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)\right]$$

$$E_{\theta\phi} = \frac{1}{2\rho\sin\theta}\frac{\partial u_\theta}{\partial \phi} - \frac{u_\phi\cot\theta}{2\rho} + \frac{1}{2\rho}\frac{\partial u_\phi}{\partial \theta} + \frac{1}{2\rho^2}\left|\left(\frac{\partial u_\rho}{\partial \theta} - u_\theta\right)\left(\frac{1}{\sin\theta}\frac{\partial u_\rho}{\partial \phi} - u_\phi\right) + \frac{1}{2\rho^2}\left[\left(\frac{\partial u_\theta}{\partial \theta} + u_\rho\right)\left(\frac{1}{\sin\theta}\frac{\partial u_\theta}{\partial \phi} - u_\phi\cot\theta\right) + \frac{\partial u_\phi}{\partial \theta}\left(u_\theta\cot\theta + u_\rho + \frac{1}{\sin\theta}\frac{\partial u_\phi}{\partial \phi}\right)\right]$$

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https://lms.s2pafrica.com/courses/continuum-mechanics-ii

The coding is straightforward.

- $\mathbf{E} = \frac{1}{2}\left[\text{grad}\,\mathbf{u} + \text{grad}^{\text{T}}\mathbf{u} + \left(\text{grad}^{\text{T}}\mathbf{u}\right)(\text{grad}\,\mathbf{u})\right]$
- Mathematica allows you to compute the grad, div, curl or Laplacian operators of any defined function, scalar, vector or tensor of any order in Cartesian, Cylindrical, Spherical, Oblate or Prolate Spheroida, Conical or any other coordinate system you can invent.
- It is for this reason that we did not need, as many textbooks do, to go into the component forms and derive them. It is sufficient for us to stay in the parsimonious tensor forms.

Familiarity is Important

- While you are going to rely on Finite Element simulations for solutions to most design problems you will encounter. Familiarity with the component forms of the equations you are dealing with is important.
- We will therefore test this. We will allow you to program during your exams. We can therefore ask questions on other coordinate systems apart from the Cartesian that you can find manually directly from the component representations.