Stress & Heat Flux I

1

Course Codes: MEG 416, SSG 431

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Topics to be Covered

DEFINE STRESS TENSOR, TRACTION VECTOR & SCALAR COMPONENTS

BODY & SURFACE FORCES

EULER-CAUCHY PRINCIPLE OF EQUIPOLLENCY; CAUCHY’S RECIPROCITY LAW

CAUCHY STRESS THEOREM

NOMINAL, KIRCHHOFF, PIOLA-KIRCHHOFF & OTHER STRESS TENSORS

“In the fields of observation, chance favors only the prepared” – Louis Pasteur

- Until the 17th century the understanding of stress largely intuitive and empirical.
  - Ancient and medieval architects: geometrical methods, simple formulas to compute the proper sizes of pillars and beams
  - Scientific understanding of stress: Sequel to mathematical tools of differential and integral calculus $\Rightarrow$17th and 18th centuries:
- Augustine Louis Cauchy published, in 1827, a generalization of Euler’s works on hydrodynamics in order to bequeath to the world, a precise notion of stress.
  - This consummated in the idea that a material body responds to externally applied loads through a tensor-valued field, $\boldsymbol{\sigma}(\mathbf{x}, t)$

What is Stress? Scalar, Vector, Tensor?

- Stress is a measure of *force intensity* either within or on the bounding surface of a body subjected to loads.
  - The Continuum Model takes a macroscopic approach: Measurable aggregate behavior rather than the microscopic, atomistic activities that may in fact have led to them, and consequently, the standard results of calculus applicable in the case of limiting values of this quotient as the areas to which the forces are applied become very small applies.
  - It is necessary to note that the word “stress” can mean a scalar, a vector or a tensor. Cauchy’s Stress Principle gives precision to the term. It can mean, depending on the context and usage, any of these three:

Stress Tensor

- **Tensor** – the *Stress Tensor*. This completely characterizes the stress state at a Euclidean point.
  - We begin with Euler-Cauchy's Principle of Equipollency; Cauchy's Reciprocity Theorem and we shall establish for a fact that such a tensor exists
  - This is Cauchy's theorem – a fundamental law in Continuum Mechanics.
  - You can be forgiven if you thought that Hooke's law is the fundamental law of this science: It is NOT! Cauchy's Stress Law is!

The Traction: or Stress Vector

- **Next time you hear that Stress** is Force per unit area; **This is it!**
- **Vector** - the *Traction* or Intensity of resultant forces on a specific surface. It has a magnitude and a direction; this is, roughly speaking, what we have in mind when we say that stress is “force per unit area.”
- It is the Stress Vector.

Several Meanings: Stress Scalars

- **Scalar** – the *scalar magnitude* of the traction vector or some projections of the same in certain directions. Once a direction is given, the scalar stress is the magnitude of the stress intensity, stress vector or traction in that direction.
  - Components of the stress tensor is another thing we may be thinking of when we say that stress is a scalar.
  - Hydrostatic pressure is a pathological case. The traction has the same magnitude in every direction because the stress tensor at any point in a static fluid is spherical.
  - Stress invariants such as the Von-Mises Stress is also a scalar stress.

Body Forces, Surface Forces

- "… a distinction is established between two types of forces which we have called 'body forces' and 'surface tractions', the former being conceived as due to a direct action at a distance, and the latter to contact action."

Body Forces: Act without contact – Field action

- Force per unit volume
- These are forces originating from sources (fields of force usually) outside of the body that act on the volume (or mass) of the body.
- Examples:
    - Electro-Magnetic Forces
    - Inertial Forces
    - Force Fields such as gravity,

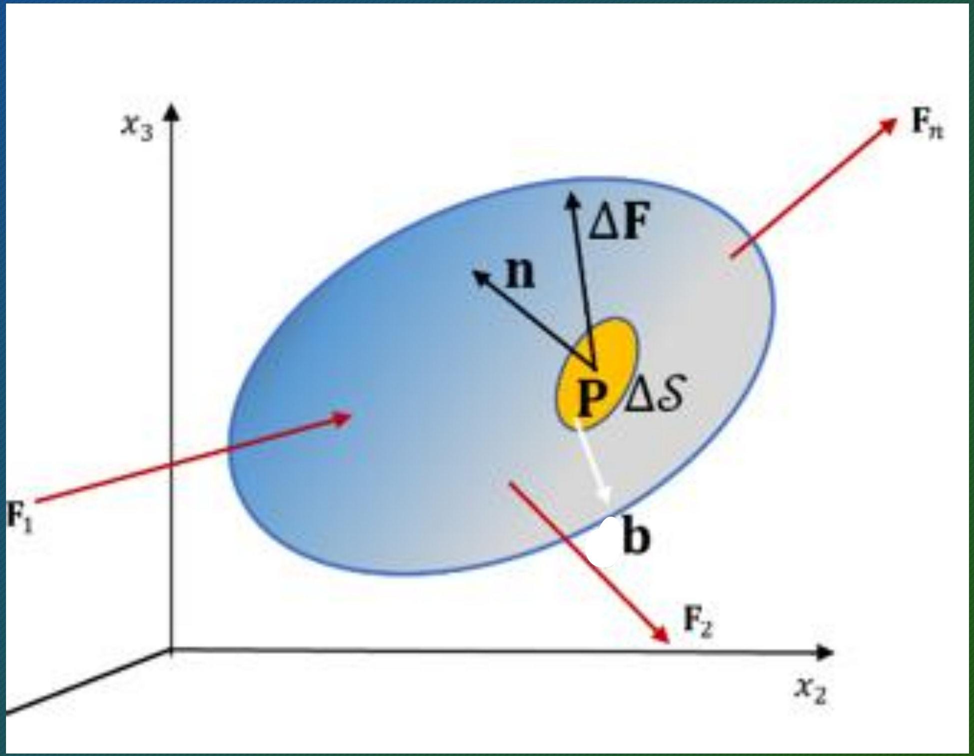
Surface Forces: They make contact to act

- Traction on each surface: $\mathbf{t}^{(\mathbf{n})}$ is the force, $\mathbf{t}$ per unit area of surface (whose normal direction is given by the unit vector, $\mathbf{n}$ across they which they act.
- It cannot be overemphasized that $\mathbf{t}$ itself is a vector whose direction is inherent.
  - The superscript $\mathbf{n}$ refers, not to the direction of the traction but to the normal to the surface on which it acts!
  - This specification is important because a change in the surface orientation without leaving the Euclidean point in question, gives a different value of traction, $\mathbf{t}$.

Traction on a surface

- The figure depicts the forces acting on an element ΔV surrounding a point $\mathbf{P}(x_1, x_2, x_3)$ in a solid body acted upon by the forces $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$.
- Consider an infinitesimal area element oriented in such a way that the unit outward normal to its surface is $\mathbf{n}$. If the resultant force on the surface ΔS is $\Delta\mathbf{F}$, and this results in a traction intensity which will in general vary over ΔS.
- Let the surface traction vector on this elemental surface be $\mathbf{t}$, it is convenient to label this traction $\mathbf{t}^{(\mathbf{n})}$ in order to emphasize the fact that this traction is the resultant on the *surface whose outward normal* is $\mathbf{n}$. For the avoidance of doubt, it **does not** imply that the direction of $\mathbf{t}^{(\mathbf{n})}$ is $\mathbf{n}$. We can write that,

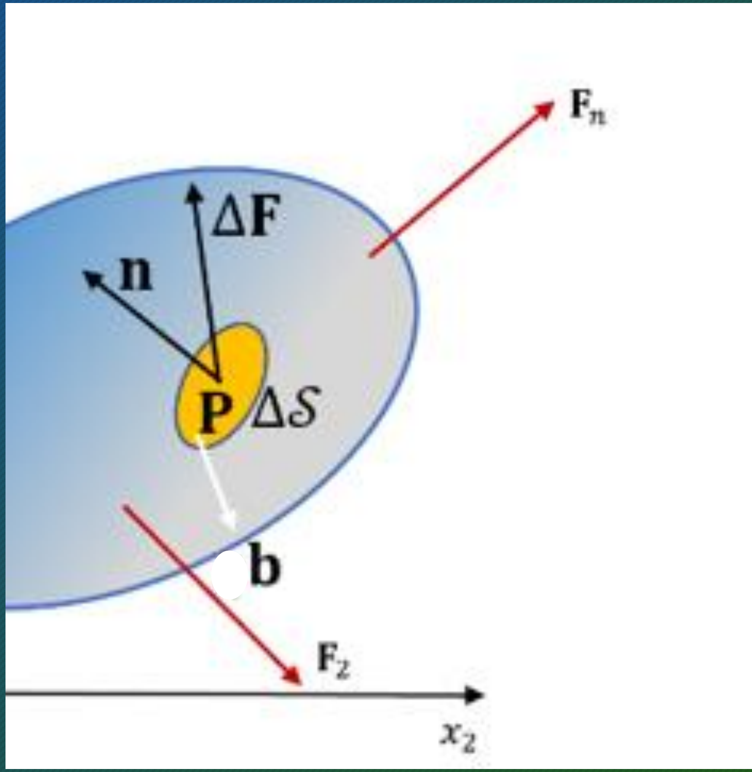
$$\mathbf{t}^{(\mathbf{n})} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{F}}{\Delta S} = \frac{d\mathbf{F}}{dS}$$



Traction on a surface

- Note that in general, $\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{n})}(x_1, x_2, x_3)$ and $\mathbf{n} = \mathbf{n}(x_1, x_2, x_3)$ as the surface itself is not necessarily a plane. It is only as the limit is approached that $\mathbf{n}$ is a fixed direction for the elemental area and $\Delta\mathbf{F}$ and $\mathbf{t}^{(\mathbf{n})}$ are in the same direction.
- If the resultant body force in the volume element ΔV is $\Delta\mathbf{B}$, we can compute the body force per unit volume

$$\mathbf{b} = \lim_{\Delta S \to 0} \frac{\Delta\mathbf{B}}{\Delta V} = \frac{d\mathbf{B}}{dV}.$$

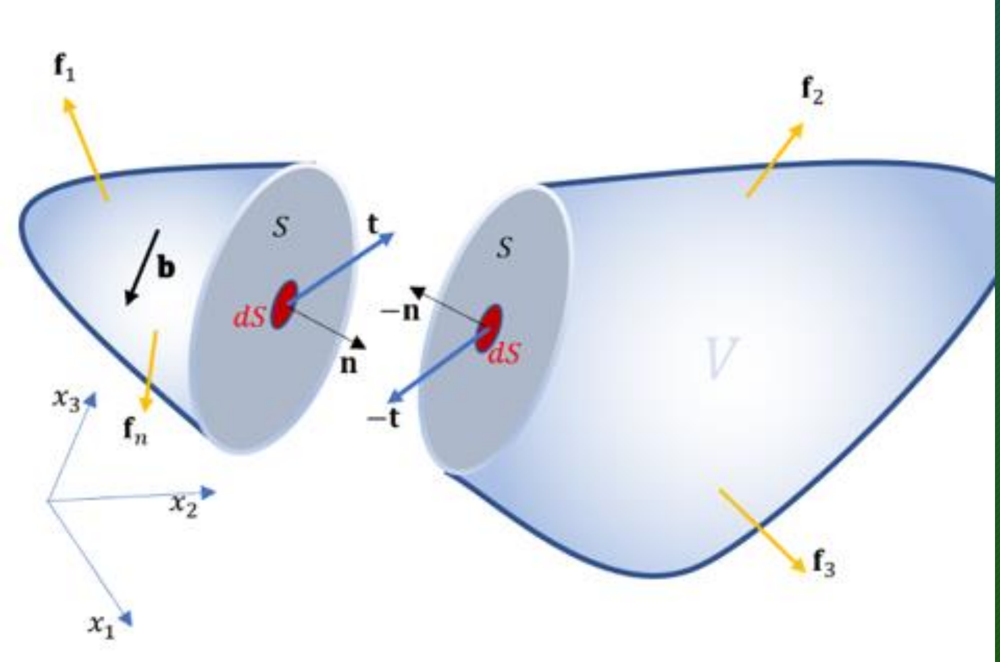


Euler-Cauchy Stress Principle

- In the figure, we have the body earlier depicted in 6.1 now cut in two through the location of the element we are looking at. The outward pointing normal at the same location in the cut body will face the opposite direction as shown.
- The Euler-Cauchy stress principle states that *upon any surface (real or imaginary) that divides the body, the action of one part of the body on the other is equipollent to the system of distributed forces and couples on the surface dividing the body*, and it is represented by a vector field

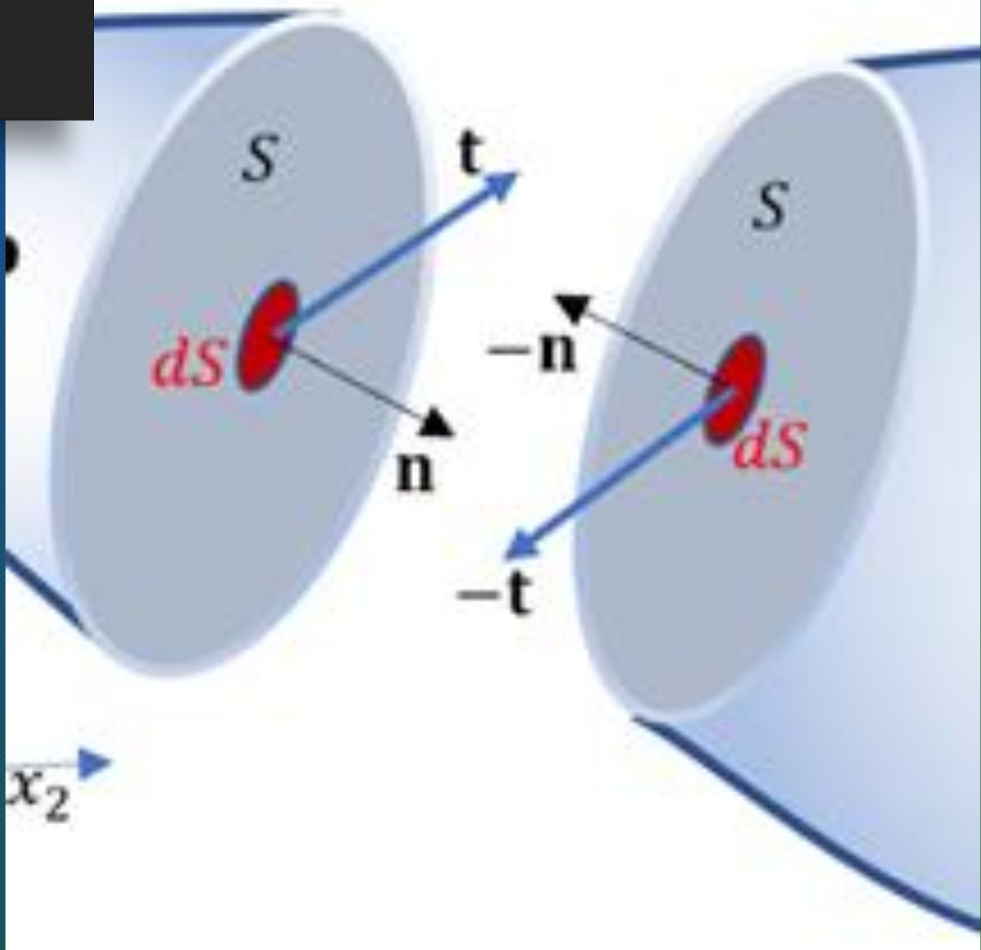
$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}(\mathbf{n}, \mathbf{x}, t)$$

- to emphasize that the traction vector is a time dependent field that varies with the surface orientation.



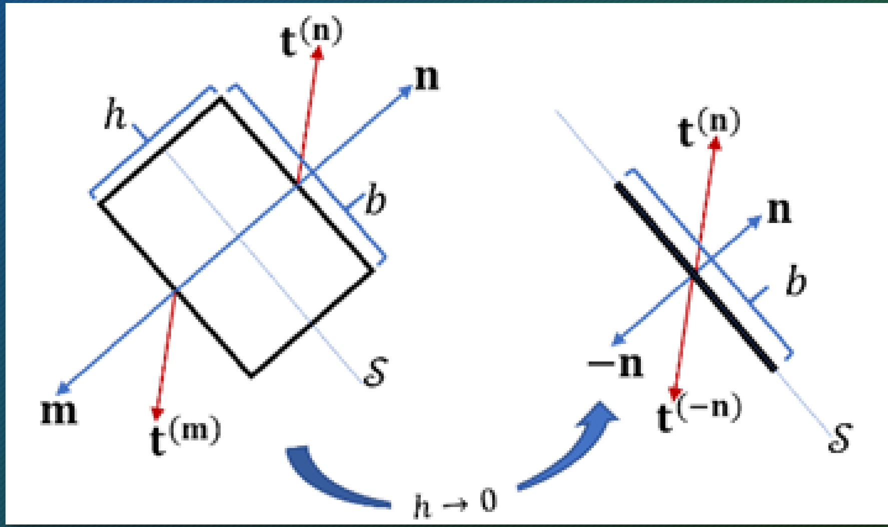
Euler-Cauchy Stress Principle

- The fact that the traction vector, to be fully specified, requires the specification of two directions:
  - The direction of the stress intensity vector, and the orientation of the surface on which the traction acts.
  - This fact, that to be fully specified, stress requires not one, but two directions, has been used by several to define a tensor as possessing magnitude and two directions in contrast with vectors that have magnitude and one direction.



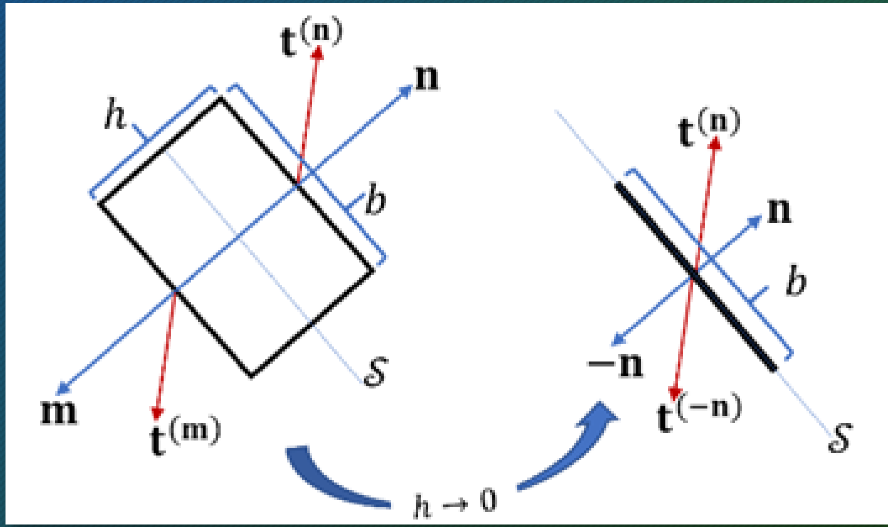
Cauchy Reciprocal Theorem

- When there is a traction on a surface, what is the relationship of that vector to what happens on the same surface, with the vector drawn in the opposite direction? That question is answered by the Cauchy reciprocal theorem as shown in this section:
- Consider the cross section around the small surface element at point $\mathbf{P}$. Let the material element is chosen initially to have a height h around surface $\mathcal{S}$. We look at the equilibrium of this small element as the height h approaches zero so that the element is now at the surface as shown in



Cauchy Reciprocal Theorem

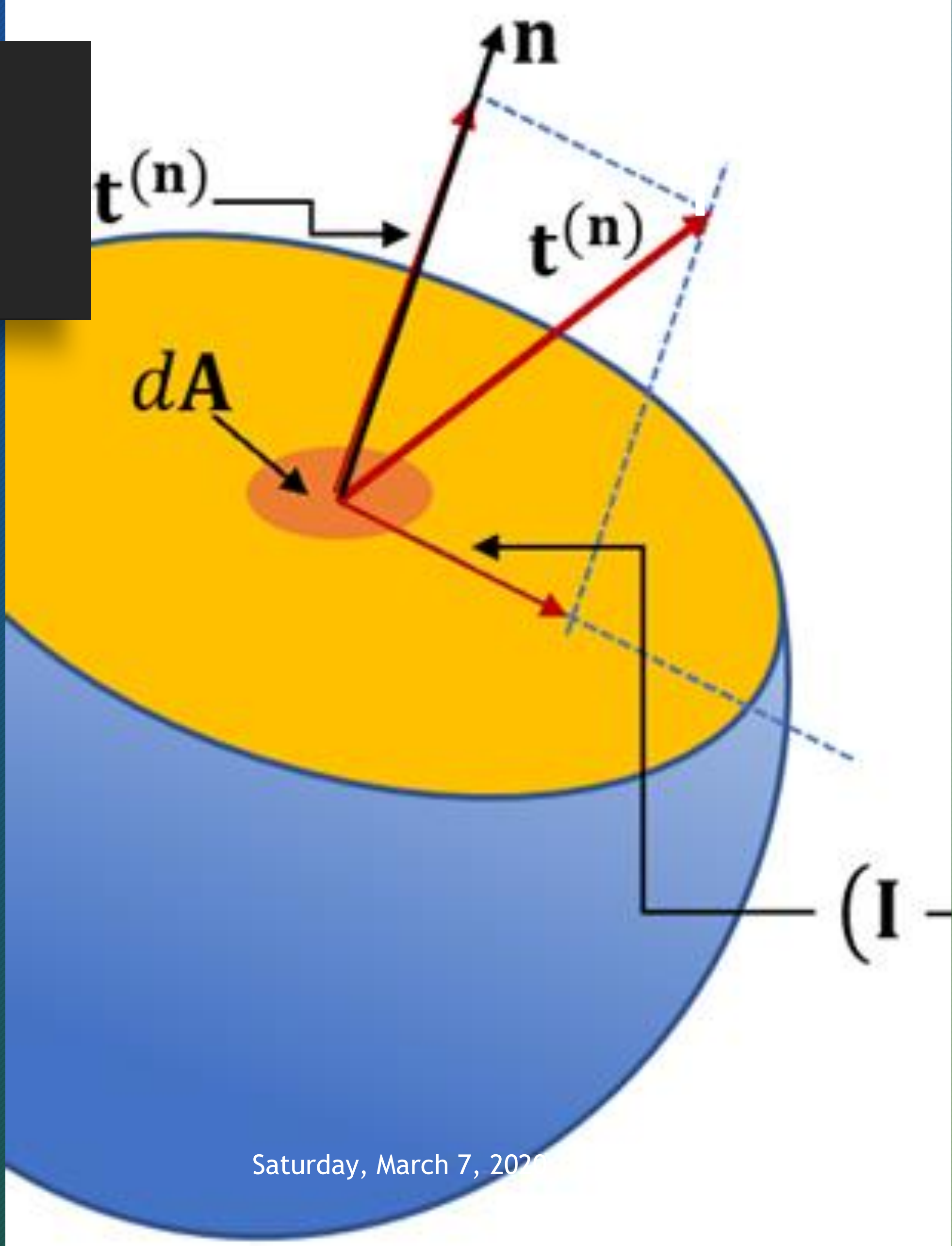
$$\lim_{h\to 0}\left(\mathbf{t}^{(\mathbf{m})}A + \mathbf{t}^{(\mathbf{n})}A\right) = \mathbf{t}^{(-\mathbf{n})}A + \mathbf{t}^{(\mathbf{n})}A$$
$$= 0$$
$$\Rightarrow \mathbf{t}^{(-\mathbf{n})} = -\mathbf{t}^{(\mathbf{n})}$$



- as it is clear from the picture that
$$\mathbf{m} \to -\mathbf{n};\ \mathbf{t}^{(\mathbf{m})} \to \mathbf{t}^{(-\mathbf{n})}$$
- as $h \to 0$. The stress vector on the surface whose normal is opposite to the surface normal is equal and opposite to the stress vector on the present surface. This is also sometimes called *Cauchy's fundamental lemma* or *Cauchy reciprocal theorem*

Normal & Shearing Stresses

- The surface traction is a not a simple quantity.
  - First it is the vector intensity of the vector force on the surface as the surface area approaches a limit in the acceptable process of continuum mechanics.
  - It is defined for a specific surface with an orientation defined by the outward normal $\mathbf{n}$. This implies that the traction at a given point is dependent upon the orientation of the surface.
  - It is a vector that has different values at the same point depending upon the orientation of the surface we are looking at.

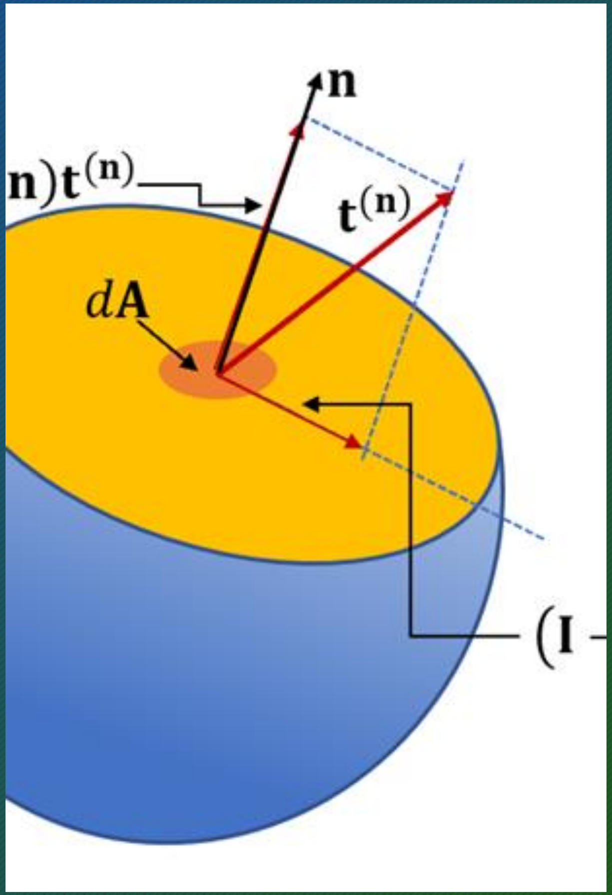


Normal & Shearing Stresses

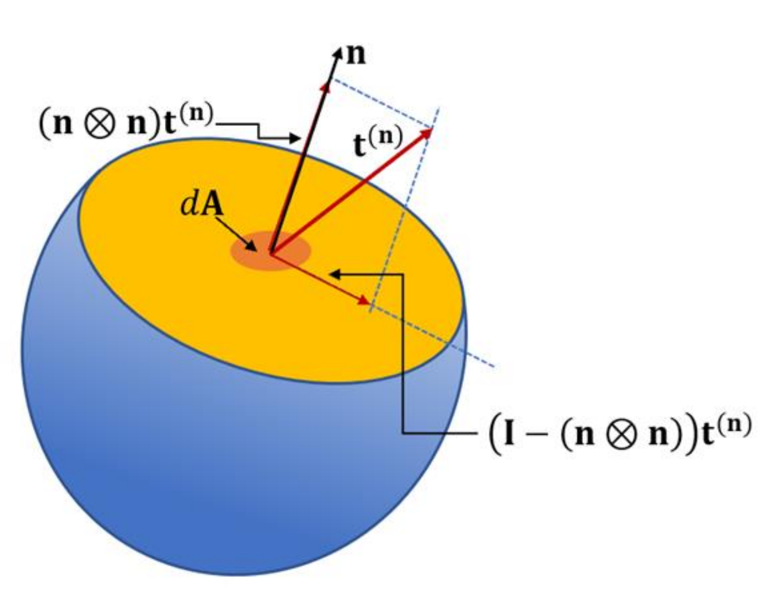
- Second, it is a function of the coordinate variables. It is a field in the 3D Euclidean Point Space. It is therefore proper to write,

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}(\mathbf{n}, \mathbf{x}, t) \equiv \mathbf{t}^{(\mathbf{n})}(x_1, x_2, x_3, t)$$

- to make these dependencies explicitly obvious. In general, $\mathbf{t}^{(\mathbf{n})}$ and $\mathbf{n}$ are not in the same direction; That is, there is an angular orientation between the resultant **stress *vector*** and the surface outward normal. Consequently, it is customary to express the stress vector as a vector sum of its projection $\mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)$ along the normal $\mathbf{n}$ and the shearing stress $\mathbf{t}_s^{(\mathbf{n})} \equiv \mathbf{t}^{(\mathbf{n})} - \mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)$



Normal & Shearing Stresses



- on the surface itself. Since $\mathbf{n}$ is a unit vector, $\|\mathbf{n}\| = 1$,

$$\mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right) = \left(\frac{1}{\|\mathbf{n}\|}\right)^2 (\mathbf{n}\otimes\mathbf{n})\mathbf{t}^{(\mathbf{n})} = (\mathbf{n}\otimes\mathbf{n})\mathbf{t}^{(\mathbf{n})}$$

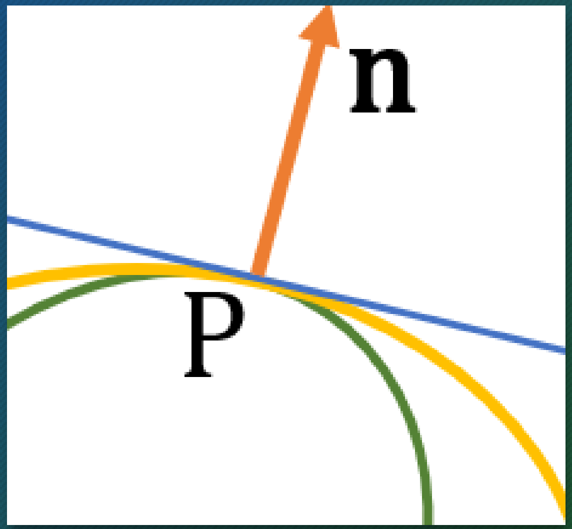
$$\begin{aligned}\mathbf{t}_s^{(\mathbf{n})} &= \mathbf{t}^{(\mathbf{n})} - \mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)\\ &= \mathbf{t}^{(\mathbf{n})} - (\mathbf{n}\otimes\mathbf{n})\mathbf{t}^{(\mathbf{n})}\\ &= (\mathbf{I} - \mathbf{n}\otimes\mathbf{n})\mathbf{t}^{(\mathbf{n})}\end{aligned}$$

- $\sigma = \left\|\mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)\right\| = \left\|(\mathbf{n}\otimes\mathbf{n})\mathbf{t}^{(\mathbf{n})}\right\|$, and, $\tau \equiv \left\|\mathbf{t}_s^{(\mathbf{n})}\right\| = \left\|\mathbf{t}^{(\mathbf{n})} - \mathrm{prj}_{\mathbf{n}}\left(\mathbf{t}^{(\mathbf{n})}\right)\right\|$

- normal and shearing components of the stress vector are called the *normal* and *shear tractions* respectively. Since their directions are known, is it customary to refer to their magnitudes alone.

Cauchy’s Theorem

- According to the *Cauchy Postulate*, the stress vector $\mathbf{t}^{(\mathbf{n})}$ remains unchanged for all surfaces passing through the point P and having the same normal vector $\mathbf{n}$ at $\mathbf{P}$, *i.e.* having a common tangent at $\mathbf{P}$. This means that the stress vector is a function of the normal vector $\mathbf{n}$ only, and it is *not influenced by the curvature of the internal surfaces*.
- *The state of stress at a point* in the body is then defined by all the stress vectors $\mathbf{t}^{(\mathbf{n})}$ associated with all planes (infinite in number) that pass through that point.
- To find the traction from orientation to orientation, we cannot examine this infinite number! In order to find this, we depend on Cauchy’s theorem.



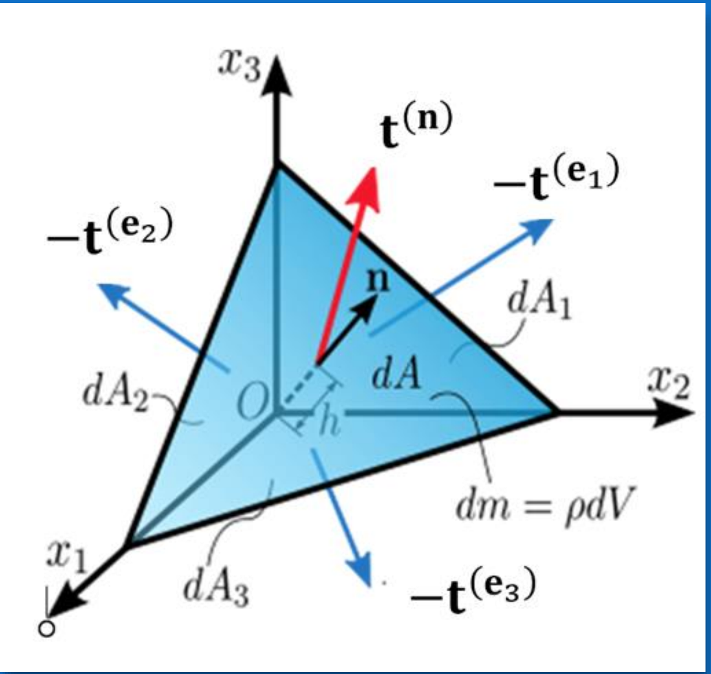
Cauchy's Stress Theorem

- **Cauchy's Stress Theorem** states that:
  - *Provided the stress vector* $\mathbf{t}^{(\mathbf{n})}$ *acting on a surface with outwardly drawn unit normal* $\mathbf{n}$ *is a continuous function of the coordinate variables, there exists a second-order tensor valued field* $\boldsymbol{\sigma}(\mathbf{x})$, *independent of* $\mathbf{n}$, *such that* $\mathbf{t}^{(\mathbf{n})}$ *is a linear function of* $\mathbf{n}$ *such that:* $\mathbf{t}^{(\mathbf{n})} = \boldsymbol{\sigma}(\mathbf{x})\mathbf{n}$
- The tensor $\boldsymbol{\sigma}$ in the above relationship, the tensor of proportionality, is called *Cauchy Stress Tensor*. It is also the "true stress" tensor for reasons that will become clear later.
- The standard proof of this important theorem examines the balance of forces on an arbitrarily small tetrahedron (Cauchy Tetrahedron) element with the base coinciding with the surface of interest and vertex located at the origin with the three sides coinciding with the Cartesian coordinate planes as shown

The Cauchy Tetrahedron

- To sketch a proof of Cauchy's stress theorem, we are greatly aided by the special tetrahedron called the Cauchy Tetrahedron.
- It has three faces oriented in the coordinate planes, and with an infinitesimal base area dA oriented in an arbitrary direction specified by a normal vector $\mathbf{n}$. The tetrahedron is formed by slicing the infinitesimal element along an arbitrary plane $\mathbf{n}$. The stress vector on this plane is denoted by $\mathbf{t}^{(\mathbf{n})}$. The stress vectors acting on the faces of the tetrahedron are denoted as $\mathbf{t}^{(\mathbf{e}_1)}, \mathbf{t}^{(\mathbf{e}_2)}$ and $\mathbf{t}^{(\mathbf{e}_3)}$.
- From equilibrium of forces, Newton's second law of motion, we have

$$\rho\left(\frac{h}{3}dA\right)\mathbf{a} = \mathbf{t}^{(\mathbf{n})}A - \mathbf{t}^{(\mathbf{e}_1)}dA_1 - \mathbf{t}^{(\mathbf{e}_2)}dA_2 - \mathbf{t}^{(\mathbf{e}_3)}dA_3$$
$$= \mathbf{t}^{(\mathbf{n})}dA - \mathbf{t}^{(\mathbf{e}_i)}dA_i$$



Cauchy Stress Theorem: Proof

- where the left-hand-side of the equation represents the product of the mass enclosed by the tetrahedron and its acceleration: ρ is the density, $\mathbf{a}$ is the acceleration, and h is the height of the tetrahedron, considering the plane $\mathbf{n}$ as the base.
- The area of the faces of the tetrahedron perpendicular to the axes can be found by projecting dA into each face:

$$dA_i = (\mathbf{n} \cdot \mathbf{e}_i)dA = n_i dA$$

- and then substituting into the equation to cancel out dA:

$$\mathbf{t}^{(\mathbf{n})} dA - \mathbf{t}^{(\mathbf{e}_i)} n_i dA = \rho \left(\frac{h}{3} dA \right) \mathbf{a}$$

Cauchy Stress Theorem: Proof

- In the limiting case as the tetrahedron shrinks to a point, the height of the tetrahedron approaches zero ($h \to 0$).
- As a result, the right-hand-side of the equation approaches 0, so the equation becomes,

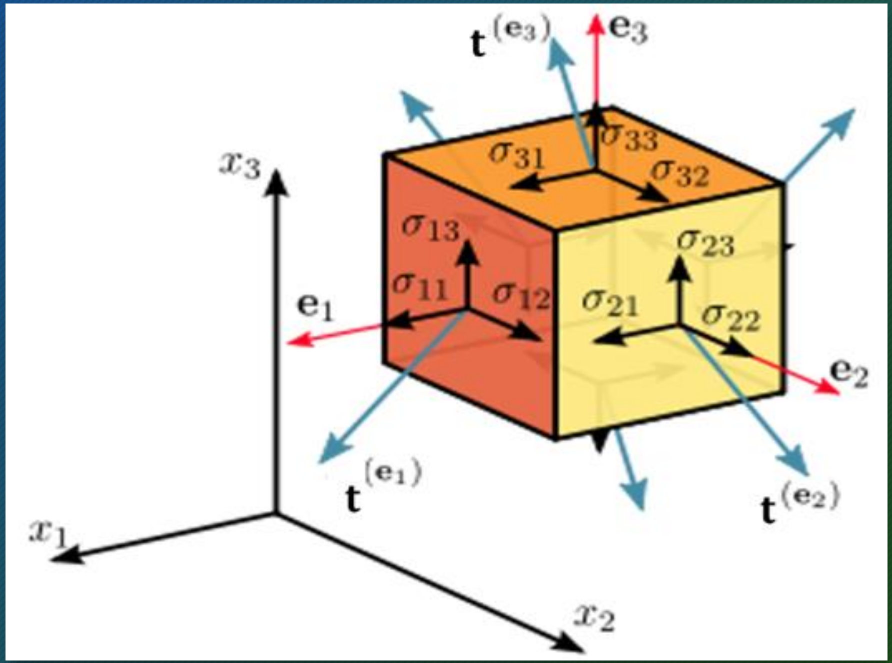
$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{e}_i)} n_i$$

Tractions on Coordinate Surfaces

- We are now to interpret the components $\mathbf{t}^{(\mathbf{e}_i)}$ in this equation. Consider $\mathbf{t}^{(\mathbf{e}_1)}$ the value of the resultant stress traction on the first coordinate plane. Resolving this along the coordinate axes, we have,

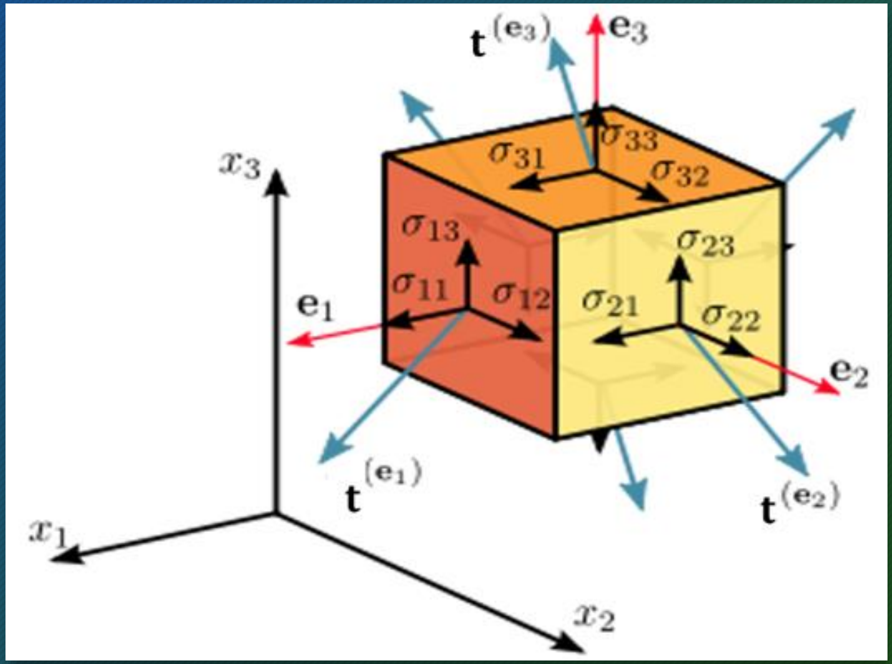
$$\begin{aligned}\mathbf{t}^{(\mathbf{e}_1)} &= \left[\mathbf{e}_1 \cdot \mathbf{t}^{(\mathbf{e}_1)}\right]\mathbf{e}_1 + \left[\mathbf{e}_2 \cdot \mathbf{t}^{(\mathbf{e}_1)}\right]\mathbf{e}_2 + \left[\mathbf{e}_3 \cdot \mathbf{t}^{(\mathbf{e}_1)}\right]\mathbf{e}_3 \\ &= \sigma_{11}\mathbf{e}_1 + \sigma_{12}\mathbf{e}_2 + \sigma_{13}\mathbf{e}_3 \\ &= \sigma_{1j}\mathbf{e}_j\end{aligned}$$

- Where the scalar quantity σ_{1j} is defined by the above equation as,



Tractions on Coordinate Surfaces

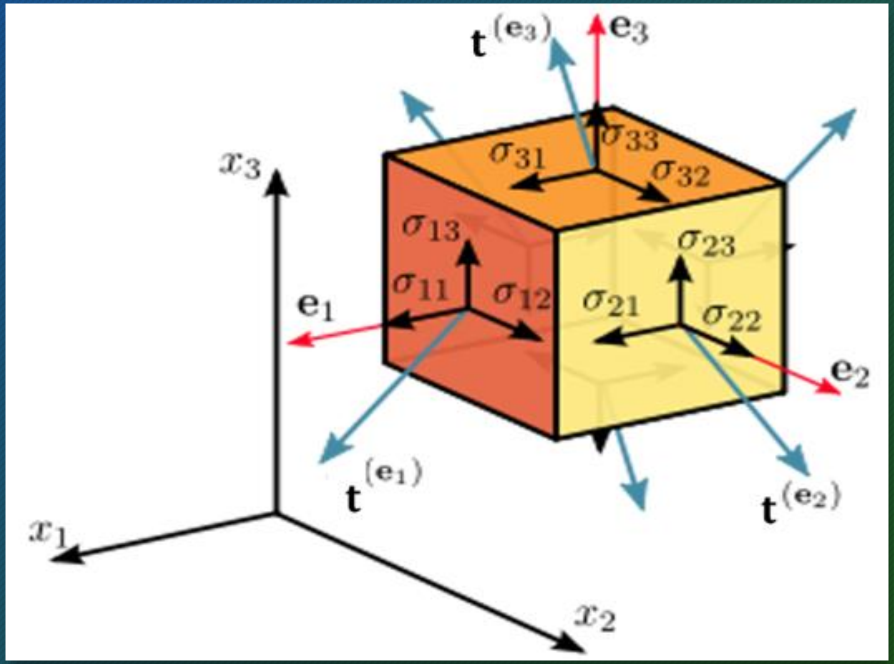
- $\sigma_{1j} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_1)}$, or in general, we write that,
  $$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}, i = 1,2,3$$
- This figure is a graphical depiction of this definition where we can see that $\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}$ is the scalar component of the stress vector on the i coordinate plane in the j direction.
- For any coordinate plane therefore, we may write, $\mathbf{t}^{(\mathbf{e}_i)} = \sigma_{ij}\mathbf{e}_j$, so that the stress or traction vector on an arbitrary plane determined by its orientation in the outward normal $\mathbf{n}$,



QED

$$\mathbf{t}^{(\mathbf{n})} = \mathbf{t}^{(\mathbf{e}_i)} n_i = \sigma_{ij} \mathbf{e}_j n_i$$

- Which is another way of saying that the component of the vector $\mathbf{t}^{(\mathbf{n})}$ along the j coordinate direction is $\sigma_{ij} n_i$ which is the contraction, $\boldsymbol{\sigma}(\mathbf{x}, t)\mathbf{n} = \mathbf{t}^{(\mathbf{n})}$.
- This proves Cauchy Theorem.
- Obviously, σ_{ij} are the components of the stress tensor in the coordinate system of computation that we have used so far. The Cauchy law, being a vector equation remains valid in all coordinate systems.



Normal & shear components

- On the coordinate surfaces, using S25 in terms of the stress tensor, we can see that, as figure S24 shows,
- On the coordinate surfaces, we can see that,
$$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)} = \mathbf{e}_j \cdot \boldsymbol{\sigma} \mathbf{e}_i$$
- Here, we have adopted the convention that the first subscript refers to the normal to the coordinate plane, the second to the direction of the stress component. The opposite convention can also be adopted. By the time we examine the law of conservation of angular momentum, it will become clear that the Cauchy stress tensor is necessarily symmetrical, and therefore both conventions give the same values.

Component	$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}$
σ_{11}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma} \mathbf{e}_1$
σ_{12}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma} \mathbf{e}_1$
σ_{13}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma} \mathbf{e}_1$
σ_{21}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma} \mathbf{e}_2$
σ_{22}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma} \mathbf{e}_2$
σ_{23}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma} \mathbf{e}_2$
σ_{31}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma} \mathbf{e}_3$
σ_{32}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma} \mathbf{e}_3$
σ_{33}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma} \mathbf{e}_3$

Normal & shear components

- The arguments that proved Cauchy theorem could have been based on non-Cartesian coordinate systems. The stress equation must remain unchanged however and the stress tensor characterizing the state of stress at a point remains an invariant. As it is with any second-order tensor, its components in general coordinates will be obtained from,

$$\boldsymbol{\sigma} = \sigma_{ij}\mathbf{g}^i \otimes \mathbf{g}^j = \sigma^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = \sigma^i_{.j}\mathbf{g}_i \otimes \mathbf{g}^j = \sigma_j^{\,i}\mathbf{g}^j \otimes \mathbf{g}_i$$

- Because the Cauchy stress is based on areas in the deformed body, it is a spatial field. Whenever it is more convenient to work in Lagrangian coordinates, a stress tensor based on this may become more appropriate. Several such stress tensors are in use. This is the subject of the next section.

Component	$\sigma_{ij} = \mathbf{e}_j \cdot \mathbf{t}^{(\mathbf{e}_i)}$
σ_{11}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma}\mathbf{e}_1$
σ_{12}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma}\mathbf{e}_1$
σ_{13}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma}\mathbf{e}_1$
σ_{21}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma}\mathbf{e}_2$
σ_{22}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma}\mathbf{e}_2$
σ_{23}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma}\mathbf{e}_2$
σ_{31}	$\mathbf{e}_1 \cdot \boldsymbol{\sigma}\mathbf{e}_3$
σ_{32}	$\mathbf{e}_2 \cdot \boldsymbol{\sigma}\mathbf{e}_3$
σ_{33}	$\mathbf{e}_3 \cdot \boldsymbol{\sigma}\mathbf{e}_3$

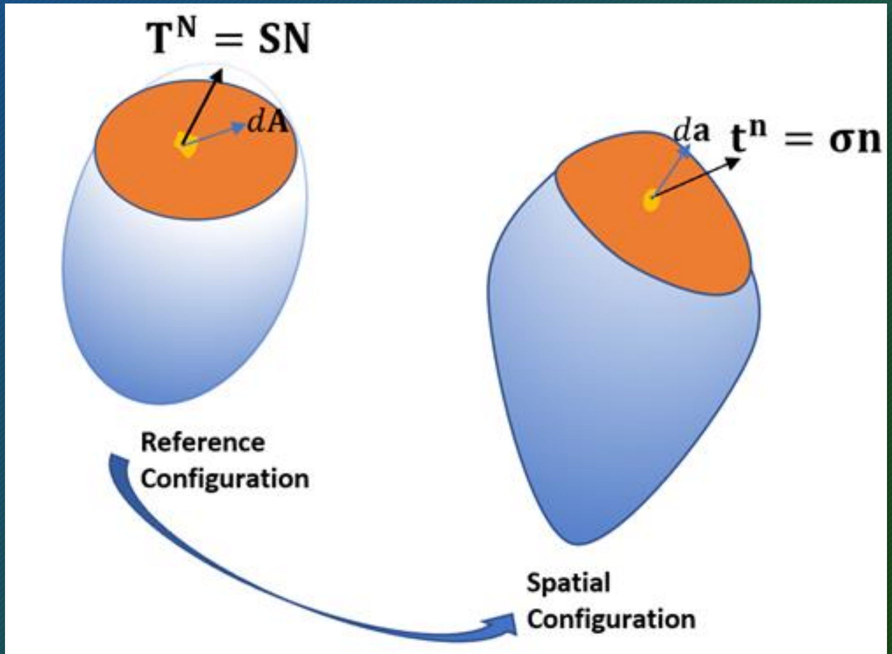
Nominal Stress Tensor, True Stress

- As stated earlier, Cauchy stress tensor is a spatial field. It is the stress tensor in the current configuration. It is the true stress because the traction computed from it is based on the currently measurable area.
- For any given element, a comparison can be made between the current area and the original area that transformed to it. Recall that the vector current area, $d\mathbf{a} = \mathbf{n}da$ in a deformed body

$$d\mathbf{a} = \mathbf{F}^{\mathrm{c}} d\mathbf{A}$$

where $d\mathbf{A} = \mathbf{N}dA$ is its image, in the material coordinates, and $\mathbf{F}^{\mathrm{c}}$ is the cofactor of the deformation gradient.

- The resultant force acting on an area bounded by ΔS in the deformed coordinates is the sum, $\int_{\Delta S} \mathbf{t}^{\mathbf{n}} da$ of the traction vectors, $\mathbf{t}^{\mathbf{n}}$ over the area; it can be obtained, using Cauchy stress theorem, $\mathbf{t}^{\mathbf{n}} = \boldsymbol{\sigma}\mathbf{n}$, so



The First Piola-Kirchhoff Stress Tensor

- that,

$$d\mathbf{P} = \int_{\Delta S} \boldsymbol{\sigma}\mathbf{n}da = \int_{\Delta S} \boldsymbol{\sigma}d\mathbf{a}$$
$$= \int_{\Delta S_0} \boldsymbol{\sigma}\mathbf{F}^{c}d\mathbf{A} = \int_{\Delta S_0} \mathbf{S}\mathbf{N}dA = \int_{\Delta S_0} \mathbf{S}d\mathbf{A}$$

- where $\mathbf{S} \equiv \boldsymbol{\sigma}\mathbf{F}^{c} = J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}$ is called the *First Piola-Kirchhoff Stress* Tensor. Its transpose,

$$\mathbf{S}^{\mathrm{T}} \equiv J\mathbf{F}^{-1}\boldsymbol{\sigma}^{\mathrm{T}}$$

- is the stress tensor from the material viewpoint. It is called the nominal stress. Tractions based on this tensor are measured with respect to areas in the material configuration.

Piola Transformation

- Symmetry of the Cauchy stress tensor (to be established further on) implies that $\mathbf{S}^{\mathrm{T}} \equiv J\mathbf{F}^{-1}\boldsymbol{\sigma}^{\mathrm{T}} = J\mathbf{F}^{-1}\boldsymbol{\sigma}$
- This transformation by the cofactor tensor, applied to $\boldsymbol{\sigma}$ to produce $\mathbf{S}$, when applied as in this or any other case to any tensor is called a *Piola transformation*. Whenever,

$$\mathbf{A} = \mathbf{B}\mathbf{F}^{c} = J\mathbf{B}\mathbf{F}^{-\mathrm{T}}$$

- $\mathbf{A}$ is said to be the Piola Transformation of $\mathbf{B}$. In the above expression for nominal stress, we have used the yet-to be proved fact that the Cauchy stress tensor is symmetric. This will be established later. The components of the $\mathbf{S}$ stress are the forces acting on the deformed configuration, per unit undeformed area. They are thought of as acting on the undeformed solid.

Kirchhoff Stress Tensor

- Recall that the ratio of elemental volumes in the spatial to material, coordinates

$$\frac{dv}{dV} = J = \det \mathbf{F}$$

- where $\mathbf{F}$ is the deformation gradient of the transformation. It therefore follows that the Kirchhoff stress $\boldsymbol{\tau}$, defined by

$$\boldsymbol{\tau} = J\boldsymbol{\sigma}$$

- is no different from Cauchy Stress Tensor during isochoric (or volume-preserving) deformations and motions. It is used widely in numerical algorithms in metal plasticity (where there is no change in volume during plastic deformation).

Second Piola-Kirchhoff Stress Tensor

- While the Cauchy Stress tensor is symmetric, neither the Piola Kirchhoff tensor, nor the nominal stress tensor are. A second tensor, a material stress tensor, is the symmetric Piola-Kirchhoff Stress, $\boldsymbol{\Xi}$ is useful in Conjugate stress analysis. It is defined by,

$$\mathbf{S} \equiv J\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \mathbf{F}\boldsymbol{\Xi}$$

- In terms of the Cauchy stress $\boldsymbol{\sigma}$ we can write,

$$\boldsymbol{\Xi} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}.$$

- This *Second Piola-Kirchhoff* stress is, just like Cauchy stress is symmetric for,

$$\boldsymbol{\Xi}^{\mathrm{T}} = \left(J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}}\right)^{\mathrm{T}} = J\mathbf{F}^{-1}\boldsymbol{\sigma}\mathbf{F}^{-\mathrm{T}} = \boldsymbol{\Xi}.$$