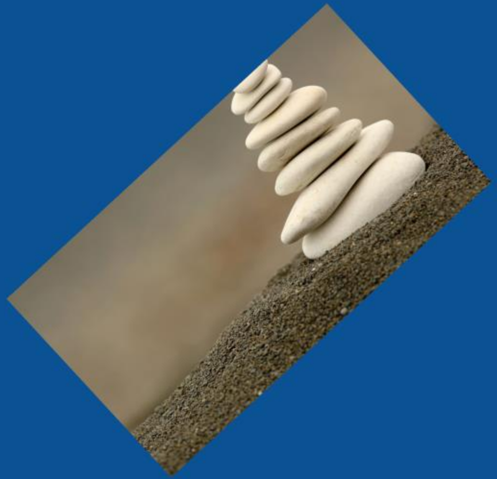

Tensor: A Linear Transformation

SSG 321 Introduction to Continuum Mechanics
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What is a Tensor?

- Clarify Popular Usage
  - The word “tensor” by a historical accident, does not mean Tensor in general!
  - When used without any adjective, you are talking about a Second-Order Tensor.
  - If you mean any other tensor, you MUST be specific
- Consequently,
  - Zero-Order Tensor is a Scalar
  - First Order Tensor is a Vector …
- Assumption:
  - You are familiar with these. We shall be defining the Second-Order Tensor in the following slides. There are higher-order tensors that shall be introduced later.

- A tensor is a Linear Transformation of a vector that produces another vector.
- Mathematically, we write:
  - Given that, $\mathbf{u}, \mathbf{v} \in \mathbb{E}$, the transformation, $\mathbf{T} \in \mathbb{L}$ with operand $\mathbf{u}$, and output $\mathbf{v}$:

$$\mathbf{v} = \mathbf{T}\mathbf{u}$$

is a tensor *if it is linear.* It is a transformation from one Euclidean vector Space to another. Depicting the set of tensors as $\mathbb{L}$, we write, $\mathbf{T} \in \mathbb{L}$ where

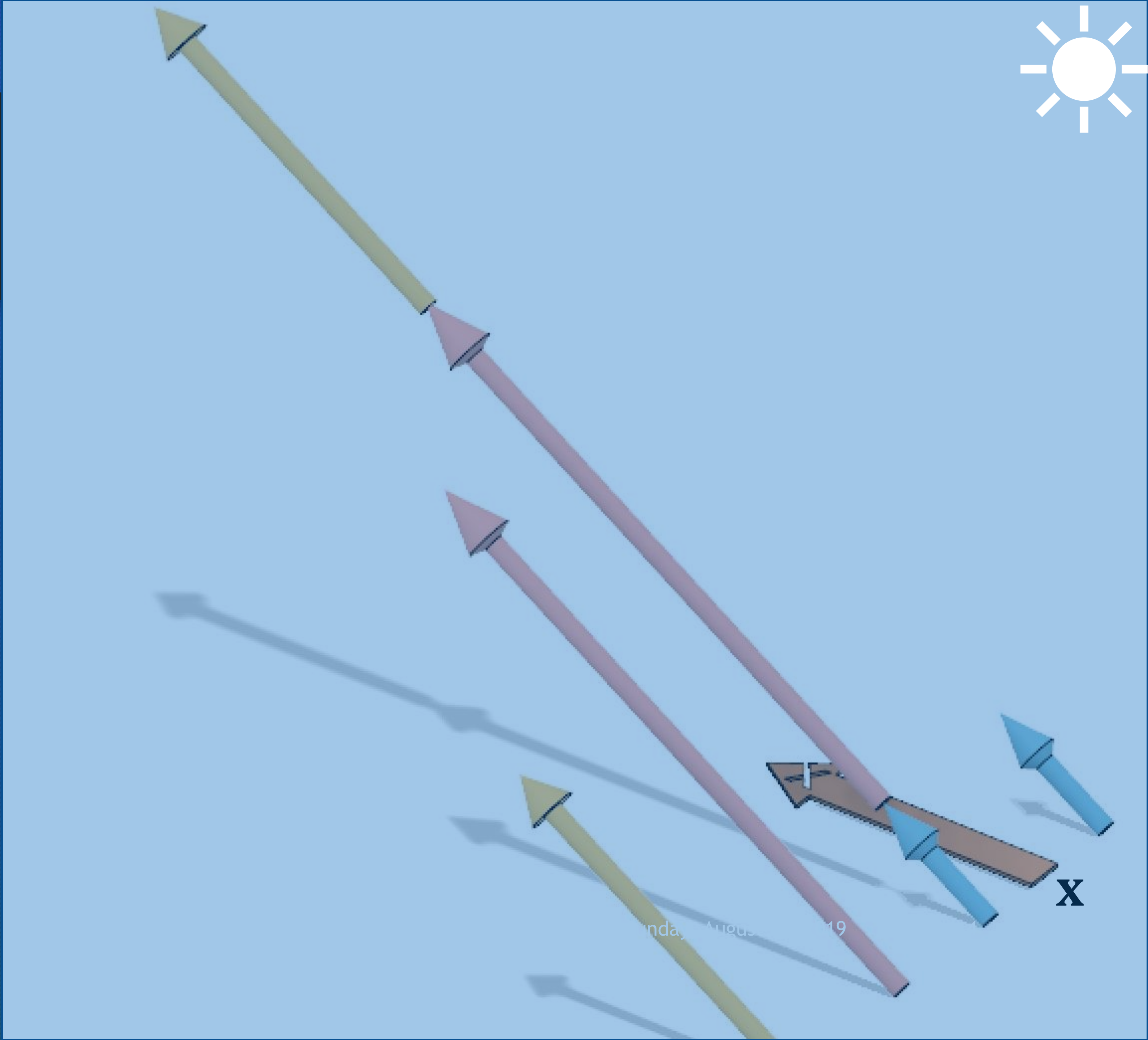
$$\mathbb{L}: \mathbb{E} \to \mathbb{E}$$

is a linear transformation.

Definition

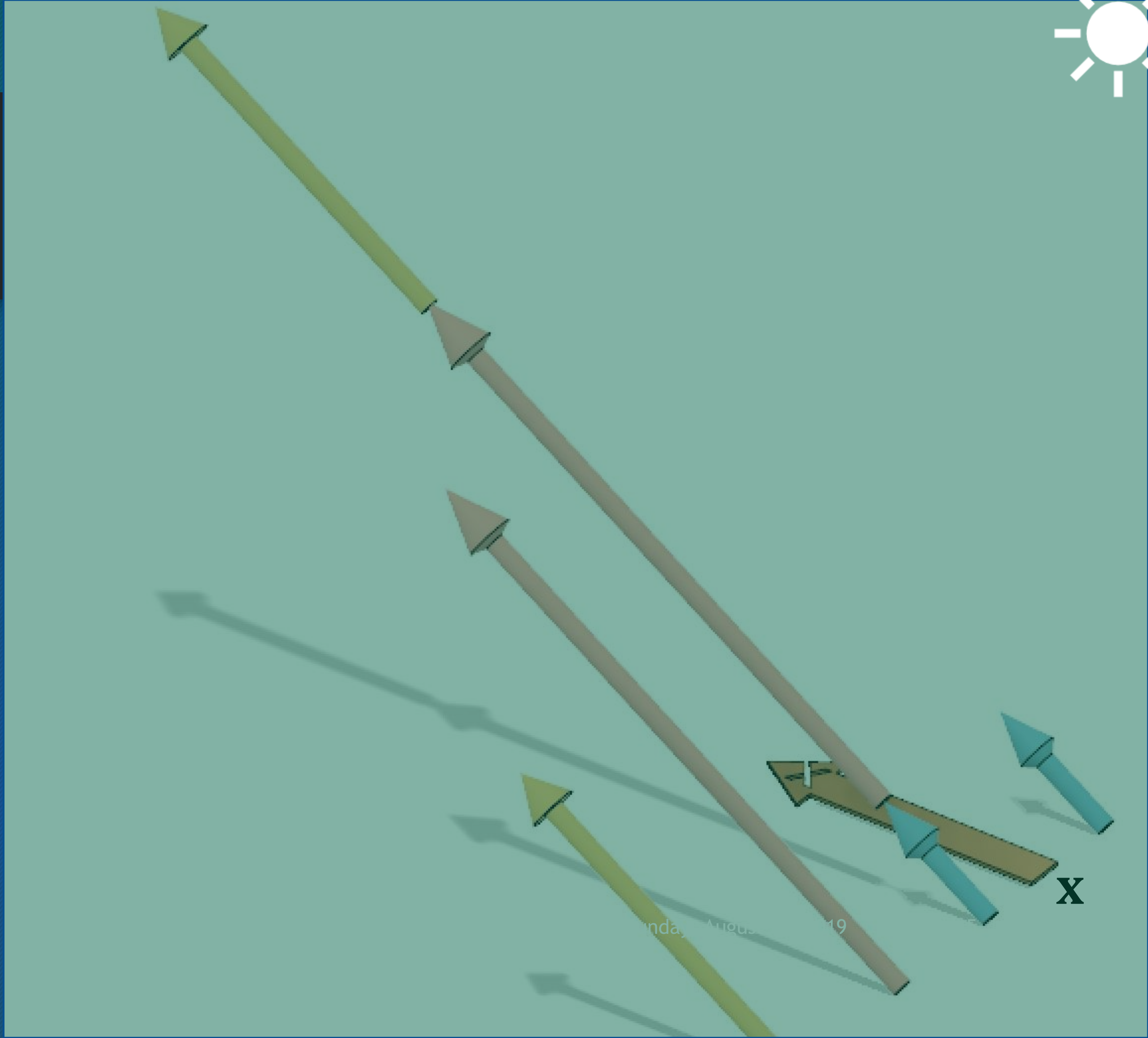
Linear Transformations

- In the figure here, let the brown arrow labelled **x** represent a vector lying on the ground. Assume the other arrows are vectors casting familiar shadows on the ground.
- Examine the relationship between these vectors and their shadows.
- We use the word "transformation" in describing the relationship.



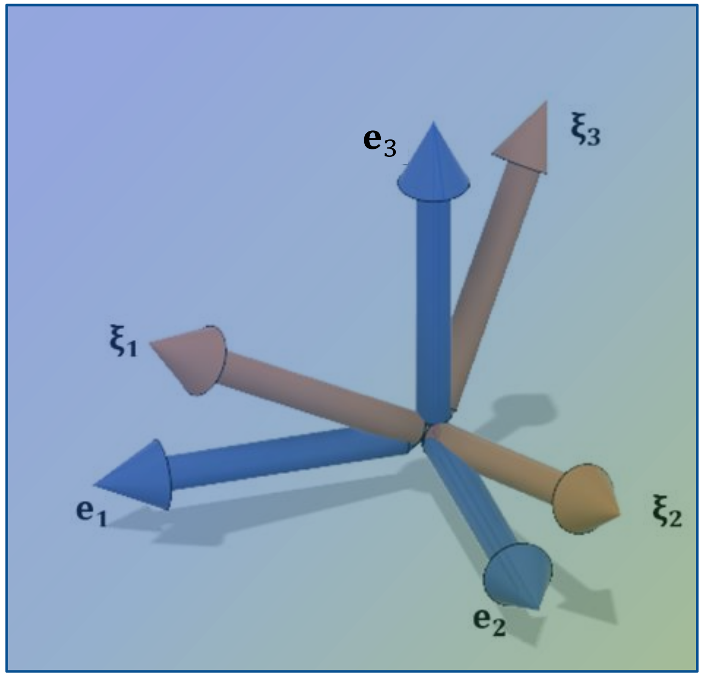
Questions

1. What determines the size or magnitude of each shadow?
2. If two vectors are equal, what can you say about their shadows? If they are opposite? Shadows? If one vector is twice the other?
3. If we add two vectors, what is the relationship between the resultant shadow, their individual shadows, and the vectors that made up the shadow?



Coordinate Transformation

- The figure depicts a coordinate transformation.
  - If a vector $\mathbf{v}$ was fixed to the original orthonormal coordinates in such a way that it moved with it. What will its components be in the new system? In the old system?
- How will the resultant of several vectors transform? What about a weighted sum of vectors?
  - What is the relationship between the image of a weighted sum of vectors, the image of the weighted sum, and the weights as a result of the transformation.



The Dyad

- We encountered the dyad product of two vectors in the last chapter.
  - Recall that a dyad is defined by the way it transforms vectors.
  - We know that the result of that transformation is another vector – satisfying the first requirement to be a tensor.
- Is the dyad a tensor? We will reach than conclusion by the answers provided to the questions we have been raising:
  - If a dyad operates on the (i) sum of two vectors, (ii) the weighted sum of several vectors, (iii) a scalar multiple of two vectors, etc.
  - What is the relationship between the transformed vectors, the originals (input, operand or parameter vectors) and weights?

The Identity Transformation

- We represent the identity transformation by the symbol $\mathbf{I}$
  - Its characteristic may, at first look, appear trivial. It does absolutely NOTHING to the vector it transforms. Mathematically, For any $\mathbf{u} \in \mathbb{E}$,

$$\mathbf{Iu} = \mathbf{u}$$

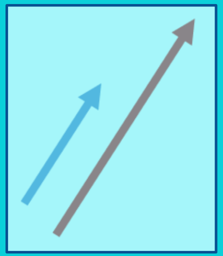
  defines the identity.

- Consider our familiar questions: Given that $\alpha, \beta \in \mathbb{R}$, is

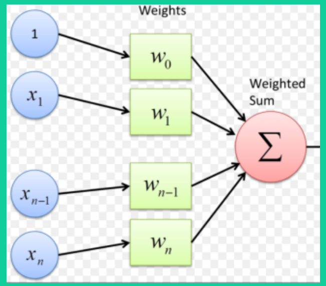
$$\begin{aligned} \mathbf{I}(\mathbf{u} + \mathbf{v}) &= \mathbf{Iu} + \mathbf{Iv}? \\ \mathbf{I}(\alpha\mathbf{u} + \beta\mathbf{v}) &= \mathbf{I}(\alpha\mathbf{u}) + \mathbf{I}(\beta\mathbf{v}) \\ &= \alpha\mathbf{Iu} + \beta\mathbf{Iv}? \end{aligned}$$

The Questions

Is the Identity Transformation of a sum equal to the sum of the identity transformations?



Is the Identity Transformation of a Scalar Multiple equal to the Scalar Multiple of an Identity Transformation?



Is the Identity Transformation of a Weighted Sum equal to the Weighted Sum of the Identity Transformations?

The Annihilator Transformation

- We represent the Annihilator transformation by the symbol $\mathbf{O}$
  - Its characteristic may, at first look, appear trivial. It turns every vector it transforms to the zero vector. Mathematically, for any $\mathbf{u} \in \mathbb{E}$,

$$\mathbf{O}\mathbf{u} = \mathbf{o}$$

defines the annihilator.

- Consider our familiar questions: Given that $\alpha, \beta \in \mathbb{R}$, is

$$\mathbf{O}(\mathbf{u} + \mathbf{v}) = \mathbf{O}\mathbf{u} + \mathbf{O}\mathbf{v}?$$
$$\mathbf{O}(\alpha\mathbf{u} + \beta\mathbf{v}) = \mathbf{O}(\alpha\mathbf{u}) + \mathbf{O}(\beta\mathbf{v}) = \alpha\mathbf{O}\mathbf{u} + \beta\mathbf{O}\mathbf{v}?$$

The Spherical Transformation

- For any $\mathbf{u} \in \mathbb{E}$, the identity transformation, $\mathbf{I}$, and a given $\gamma \in \mathbb{R}$, we represent the spherical transformation as

$$\mathbf{Su} \equiv \gamma \mathbf{Iu}$$

so that $\mathbf{S} = \gamma \mathbf{I}$.

- What answers do we have to our familiar questions when we ask them about the spherical transformation?

The Answer

- If the answer to all the questions asked above is in each case,
  - The transformation of a sum, is the sum of the individual transformations
  - The transformation of a scalar multiple is the scalar multiple of the transformation of the initial vector, and
  - The transformation of a weighted sum is the weighted sum of the transformations of the constituent vectors with same weights.
- Then we can conclude that we have dealing with tensors! The transformations we have discussed are all Linear Transformations!

Simple Tensors

We now proceed to establish the fact that …

- The Projection (or Shadow)
- The Coordinate Transformation
- The Dyad
- The Identity
- The Annihilator
- The Spherical Transformation
- The Vector Cross

As defined, are all tensors

They transform vector → vector, linearly

Identity, Annihilator & Spherical Transformations

- These are simply two special cases of the spherical:
  - $\gamma = 0 \Rightarrow$ Annihilator
  - $\gamma = 1 \Rightarrow$ Identity
- It is easy to see its linearity from that of the Identity. We therefore begin by establishing these.

Annihilator Tensor

$$\mathbf{O}(\mathbf{u}+\mathbf{v})=\mathbf{o}$$

Given that, $\mathbf{u},\mathbf{v}\in\mathbb{E}$, $\mathbf{w}=\mathbf{u}+\mathbf{v}$, by the definition of the vector space, it follows that, $\mathbf{w},\alpha\mathbf{u},\beta\mathbf{v}\in\mathbb{E}$. Consequently, $\mathbf{O}(\mathbf{u}+\mathbf{v})=\mathbf{o}, \mathbf{O}\mathbf{u}=\mathbf{o}, \mathbf{O}(\alpha\mathbf{u})=\mathbf{o}, \mathbf{O}(\beta\mathbf{v})=\mathbf{o}$

$$\mathbf{O}(\mathbf{u}+\mathbf{v})=\mathbf{O}\mathbf{u}+\mathbf{O}\mathbf{v}=\mathbf{o}$$
$$\mathbf{O}(\alpha\mathbf{u}+\beta\mathbf{v})=\mathbf{O}(\alpha\mathbf{u})+\mathbf{O}(\beta\mathbf{v})=\alpha\mathbf{O}\mathbf{u}+\beta\mathbf{O}\mathbf{v}=\mathbf{o}$$

The Annihilator transforms linearly $\Leftrightarrow$ It is a linear Transformation.

Identity Tensor

$$\mathbf{I}(\mathbf{u}+\mathbf{v}) = \mathbf{u}+\mathbf{v}$$

$\mathbf{w}=\mathbf{u}+\mathbf{v}$, is also, by the definition of the vector space, is also a vector. Consequently,

$$\mathbf{Iu}=\mathbf{u}, \mathbf{Iv}=\mathbf{v}, \mathbf{I}(\mathbf{u}+\mathbf{v})=\mathbf{Iu}+\mathbf{Iv}$$

The scalar multiple of a vector is also a vector,

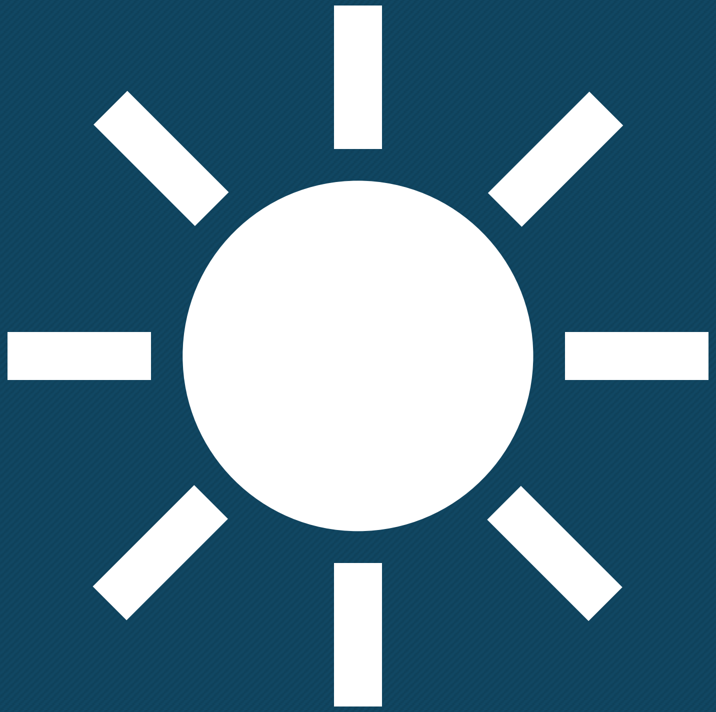
$$\mathbf{I}(\alpha\mathbf{u})=\alpha\mathbf{u}=\alpha\mathbf{Iu}, \mathbf{I}(\beta\mathbf{v})=\beta\mathbf{v}=\beta\mathbf{Iv}$$

And because the linear combination of vectors is also a vector,

$$\begin{aligned}\mathbf{I}(\alpha\mathbf{u}+\beta\mathbf{v}) &= \alpha\mathbf{u}+\beta\mathbf{v}\\ &= \mathbf{I}(\alpha\mathbf{u})+\mathbf{I}(\beta\mathbf{v})\\ &= \alpha\mathbf{Iu}+\beta\mathbf{Iv}\end{aligned}$$

The Identity transforms linearly $\Leftrightarrow$ It is a linear Transformation.

The Projection|Shadow



- We introduced the shadow transformation as if there were parallel rays of light creating the transformation. When this happens, it is called a projection.
- In order to establish its tensor credentials, we shall regenerate this same situation mathematically and introduce the projection transformation:
- Given $\mathbf{x}, \mathbf{v} \in \mathbb{V}$, the transformation, $\mathbf{P}_{\mathbf{x}} \equiv \left(\frac{1}{||\mathbf{x}||}\right)^2 (\mathbf{x} \otimes \mathbf{x})$ transforms any vector $\mathbf{v}$ to a shadow parallel to $\mathbf{x}$.

$$\mathbf{P}_{\mathbf{x}}\mathbf{v} \equiv \left(\frac{1}{||\mathbf{x}||}\right)^2 (\mathbf{x} \otimes \mathbf{x})\mathbf{v} = \left(\frac{1}{||\mathbf{x}||}\right)^2 (\mathbf{x} \cdot \mathbf{v})\mathbf{x}$$

- If $\mathbf{x}$ and $\mathbf{v}$ are perpendicular, the transformation simulates the midday overhead sun casting our shadow on our feet.

The Projection Tensor

- The projection as we have defined it, transforms vectors to vectors. It needs to satisfy one more condition to be a tensor:
  - Is it a linear transformation? We answer as follows:

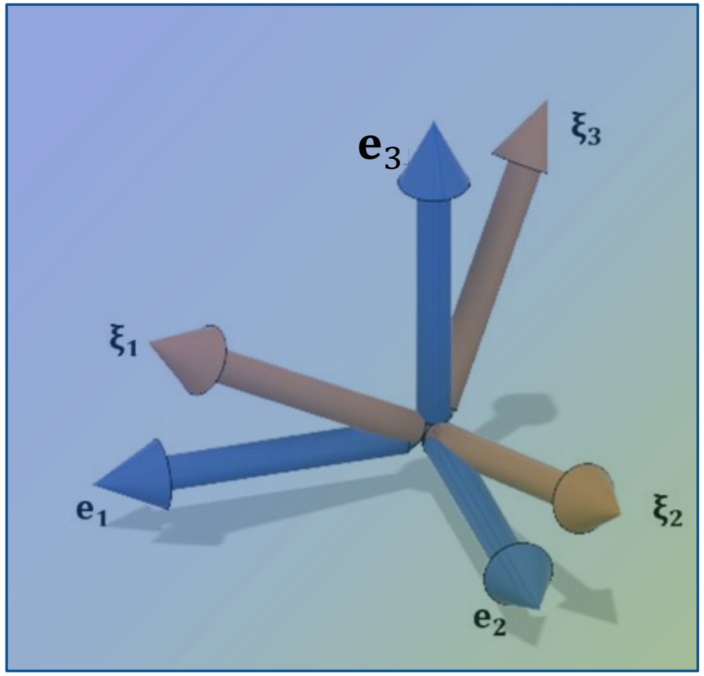
$$\begin{aligned}\mathbf{P_x}(\mathbf{u}+\mathbf{v}) &= \left(\frac{1}{\|\mathbf{x}\|}\right)^2 (\mathbf{x}\otimes\mathbf{x})(\mathbf{u}+\mathbf{v}) = \left(\frac{1}{\|\mathbf{x}\|}\right)^2 \big(\mathbf{x}\cdot(\mathbf{u}+\mathbf{v})\big)\mathbf{x}\\ &= \left(\frac{1}{\|\mathbf{x}\|}\right)^2 (\mathbf{x}\cdot\mathbf{u}+\mathbf{x}\cdot\mathbf{v})\mathbf{x} = \left(\frac{1}{\|\mathbf{x}\|}\right)^2 (\mathbf{x}\cdot\mathbf{u})\mathbf{x} + \left(\frac{1}{\|\mathbf{x}\|}\right)^2 (\mathbf{x}\cdot\mathbf{v})\mathbf{x}\\ &= \mathbf{P_x}\mathbf{u}+\mathbf{P_x}\mathbf{v}\end{aligned}$$

Similarly,

$$\mathbf{P_x}(\alpha\mathbf{u}+\beta\mathbf{v}) = \alpha\mathbf{P_x}\mathbf{u}+\beta\mathbf{P_x}\mathbf{v}$$

Coordinate Transformation

- Given a set of coordinates with basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can transform to the coordinate with basis vectors, $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ with the transformation,
$$\mathbf{C} = (\boldsymbol{\xi}_i \otimes \mathbf{e}_i)$$
- It is easy to prove this: Transform any vector $\mathbf{v}$ referred to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Suppose our operand is
$$\mathbf{v} = 2\mathbf{e}_1 + 3\mathbf{e}_2 - \mathbf{e}_3 = \alpha_j \mathbf{e}_j$$
$$\begin{aligned} \mathbf{C}\mathbf{v} &= (\boldsymbol{\xi}_i \otimes \mathbf{e}_i)\alpha_j \mathbf{e}_j \\ &= \alpha_j \boldsymbol{\xi}_i \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) = \alpha_j \boldsymbol{\xi}_i \delta_{ij} \\ &= \alpha_j \boldsymbol{\xi}_j = 2\boldsymbol{\xi}_1 + 3\boldsymbol{\xi}_2 - \boldsymbol{\xi}_3 \end{aligned}$$



Coordinate Transformation Tensor

- This transformation is a tensor because it satisfies two conditions: The operand is a vector and the output is also a vector: It is a vector to vector transformation. Furthermore,

$$\begin{aligned}\mathbf{C}(\alpha\mathbf{u}+\beta\mathbf{v}) &= (\boldsymbol{\xi}_i\otimes\mathbf{e}_i)(\alpha\mathbf{u}+\beta\mathbf{v}) = \boldsymbol{\xi}_i\big(\mathbf{e}_i\cdot(\alpha\mathbf{u}+\beta\mathbf{v})\big)\\ &= \boldsymbol{\xi}_i(\alpha\mathbf{e}_i\cdot\mathbf{u}+\beta\mathbf{e}_i\cdot\mathbf{v})\\ &= \alpha\boldsymbol{\xi}_i(\mathbf{e}_i\cdot\mathbf{u})+\beta\boldsymbol{\xi}_i(\mathbf{e}_i\cdot\mathbf{v})\\ &= \alpha(\boldsymbol{\xi}_i\otimes\mathbf{e}_i)\mathbf{u}+\beta(\boldsymbol{\xi}_i\otimes\mathbf{e}_i)\mathbf{v}\\ &= \alpha\mathbf{C}\mathbf{u}+\beta\mathbf{C}\mathbf{v}\end{aligned}$$

- Meaning that the transformation is also linear, satisfying the second condition to be a tensor.

The Vector Cross

- We introduce an additional transformation here that we shall show to be a tensor. It is the Vector Cross. You have been able to create a tensor from two vectors We define the vectors Cross here as a tensor created from a single vector using the well-known Levi-Civita symbol.
- Given any vector $\mathbf{v} = v_\alpha \mathbf{e}_\alpha$ the vector cross, $(\mathbf{v}\,\times)$ of $\mathbf{v}$ is defined as,
$$(\mathbf{v}\,\times) \equiv -e_{ijk} v_k \mathbf{e}_i \otimes \mathbf{e}_j$$
- How do we prove that this is a tensor?

1. Prove that it transforms vectors to vectors, and
2. That the transformation is linear.

The first task is straightforward because what we have here is a scalar multiple of a dyad.

The Vector Cross

- The first task is straightforward because what we have here is a scalar multiple of a dyad. Observe that for each term in the sum, we can write,

$$e_{ijk}v_k\mathbf{e}_i\otimes\mathbf{e}_j=(e_{ij1}v_1+e_{ij2}v_2+e_{ij3}v_3)\mathbf{e}_i\otimes\mathbf{e}_j$$

- Which is simply a scalar $=\left(e_{ij1}v_1+e_{ij2}v_2+e_{ij3}v_3\right)$, times the dyad $\mathbf{e}_i\otimes\mathbf{e}_j$ for any pair i,j. Since a dyad transforms vectors to vectors linearly, Vector Cross also transforms vectors to vectors Linearly.
- The Vector Cross is therefore a tensor.

Definition, Again

- A tensor is a Linear Transformation of a vector that produces another vector.
- Mathematically, we write:
  - Given that, $\mathbf{u}, \mathbf{v} \in \mathbb{E}$, the transformation, $\mathbf{T} \in \mathbb{L}$,
$$\mathbf{v} = \mathbf{T}\mathbf{u}$$
  is a tensor *if it is linear*. It is a transformation from one Euclidean vector Space to another. Depicting the set of tensors as $\mathbb{L}$, we write, $\mathbf{T} \in \mathbb{L}$ where
$$\mathbb{L}: \mathbb{E} \to \mathbb{E}$$
  is a linear transformation.

The Linear Operator

The linearity of a transformation is determined by its input and output. A tensor is a linear transformation from a vector to a vector.

Vector to Vector Transformations

- Every transformation, we have seen so far has been shown to be a tensor.
  - This begs the question: Are all vector to vector transformations tensors?
  - The answer is "No, they are not necessarily tensors".
  - Recall, in order for the transformation to be a tensor, in addition to having operand and output as vectors, it MUST also be linear.
- Examples of vector to vector operations that are NOT tensors are easily found. Here are some:
  1. $\mathbf{Tv} = \frac{\mathbf{e}_1}{\|\mathbf{v}\|}$ for a proof of its non tensorial nature see Q2.2
  2. $\mathbf{Tv} = \|\mathbf{v}\|\mathbf{e}_1$ (Q2.3),

Food for Thought

- Is the Inverse of a Tensor a Tensor?

Q&A Range

- At this point in the course, you should go through the following 12 worked examples 1.9-1.12, 1.31, 2.1-2.6, 2.23 in the book.
- Okwesilieze Uwadoka, Mechanical Engineering is the class leader at present. A member of the systems engineering class scored zero. If both continue in the current trend, by the time we are in the final exam, Okwesilieze will still get an “A” grade in this course even if he obtains 40% in the final examination. The other fellow will need to score more than 90% in the final exam just to get the last passing grade.
- “The fault, dear Brutus, is not in our stars,…”. Everyone can still make a good grade in the course. The stakes are much larger. We are building final year course in design on the foundations established here.

Course Grading

- Introduction to Continuum Mechanics 300L
  Systems Engineering
  Mechanical Engineering
  University of Lagos

	Type	Weight	Date Due
Test 1	Supervised Class Test Chapter One	15.0	Week5
Continuous Assessment 1	CBT Self Test	5.0	Week7
Test 2	Supervised Class Test Chapter Two	15.0	Week 8
Continuous Assessment 2	CBT Self Test	5.0	Week 9
Test 3	Supervised Class Test Chapter Three	15.0	Week 11
Online Participation	Various	5.0	Week 13
Examinations	Supervised Exam Three Chapters	40.0	Week 13
Final	Totals	100.0	