The Strain Tensors

Topic: Kinematics

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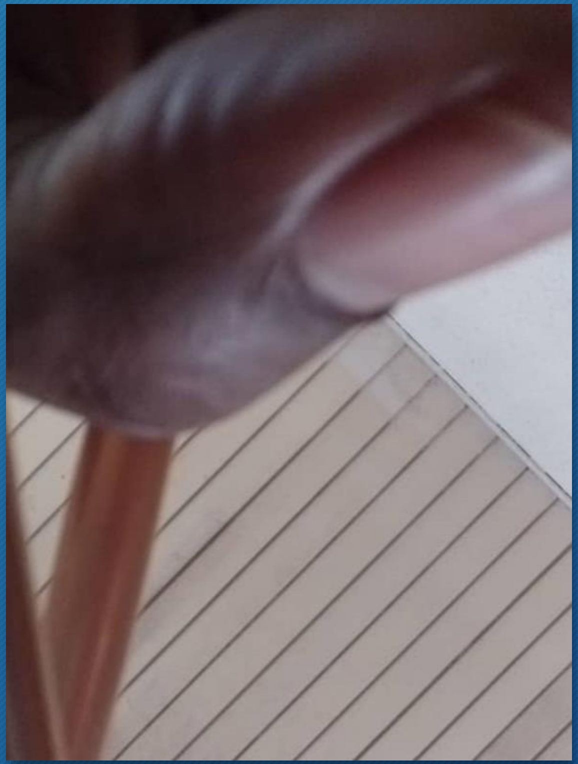
Scope



- Continues development of the Deformation Gradient and defines Strain, Strain Functions
- Convergence of these when strain is small
- Simple examples

What is Strain?

Strain is a well-known word, albeit, much misunderstood. Dictionaries, encyclopedias, etc. attempt to define strain. From our technical perspective, where we are quite specific in what we mean, they are usually wrong!

Strain is not just deformation. Deformation can be perceived in movement of material particles. Not all these movements constitute strain.

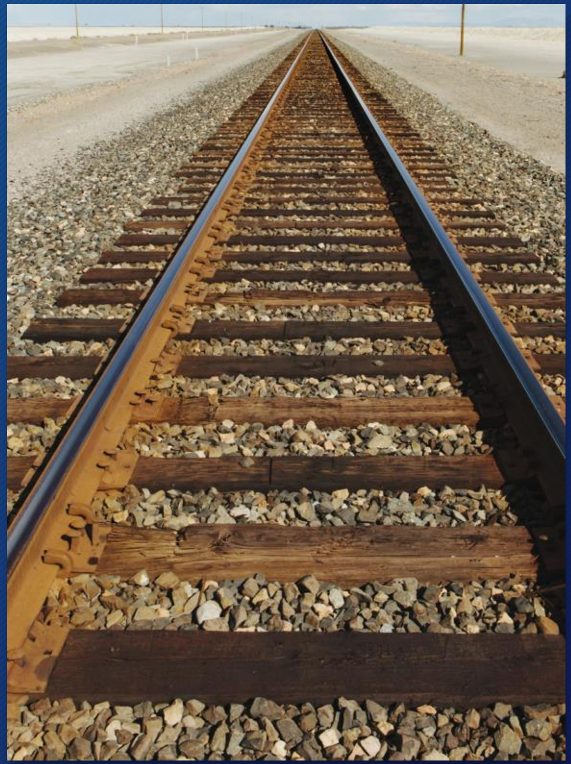
Two fundamental issues come up in properly defining strain: Relativity in the movement, and exclusion of wholesale movements.

What is Strain?

Many, with a technical flair begin to talk about forces, stresses in conjunction with strain. This, it is argued, is because stress is accompanied by strain and vice, versa!

It is possible to create stress without strain. It is also possible to create strain without stress. It is NOT true that both are always complimentary!

Strain is a purely GEOMETRICAL concept, created to uniquely quantify shape changes in a material. Consequently, strain in two different materials can be caused by vastly different force or stress systems due to the material constitution.



Strain: Definition

- If the length of a small, arbitrary line element in the referential state remains unchanged in its spatial image, and if the angle between two line elements in the referential state remains the same in their spatial images, then the body is unstrained.
- Otherwise, the body is strained.
- Strain is a way to measure these changes unambiguously.

The Strain Function: Measuring Shape Changes

Strain is our attempt to quantify relative displacements and changes in orientations of material elements as a result of the deformation. Wholesale movements of the entire element itself, by rotation, translation or a combination of both do not qualify as strain. We call such transformations **Rigid Body Motions**. Examples are:

1. **Rotation**: of all material points in the element about an axis
2. **Translation**: of all the material element by the same amount in a given direction.

Strain is a definition. Successful strain functions are so because experience and usage of them as measuring and prediction tools have been successful. A proper strain function must satisfy two conditions:

Proper Strain Functions

- Two deformations, differing only by rigid body motions represent the same strained system in so far as they create the same shape changes in identical materials.
  - A correct strain function will detect this and compute equal quantities for the situations they represent.
- When the deformation gradient becomes $\mathbf{F} = \mathbf{I}$, the identity tensor, the strain function must vanish everywhere. This means that
  - Many strain functions can be defined in so far as they satisfy the above conditions. A number have been used successfully in certain situations.

- A Material Strain Tensor:

$$\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$$

where $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, the **Right Cauchy Green** Tensor.

- It will become clear shortly that this strain function is more familiar than it looks.
    - A comparison of what it computes will be made to our elementary conception of strain as the quotient of “increase in length and original length”. It will soon become clearer that this is the strain function we have in mind from that common definition.

Green-Lagrange Strain Tensor

Euler-Almansi Strain Tensor

- Next, we look at the Spatial **Euler-Almansi** Strain Tensor, $\mathbf{e}$

$$\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$$

where $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ is called the **Left Cauchy-Green** Tensor. ($\mathbf{B}, \mathbf{B}^{-1}$ are named in honor of two other notable scientists: Finger & Piola Tensors respectively). We have shown that $\mathbf{C} = \mathbf{U}^2$ is a material tensor while $\mathbf{B} = \mathbf{V}^2$ is spatial. Consequently, $\mathbf{E}$ is a material strain tensor field while $\mathbf{e}$ is spatial.

Seth-Hill Strain Tensors

- It has been shown by Seth and Hill that the popular strain functions are special cases of generalized strain functions. These functions, named for the authors, are called the Seth-Hill functions. The referential Seth Hill Strain Function is,

$$\frac{1}{m}(\mathbf{U}^m - \mathbf{I}) \text{ for } m \neq 0,$$
$$\log_e \mathbf{U}, m = 0$$

- It is easy to see that the Green-Lagrange Strain function is the special case of the Seth-Hill material strain function when $m = 2$.

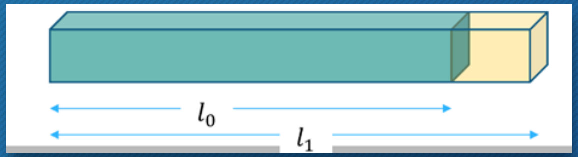
Seth-Hill Strain Tensors

- On the spatial side of things, we have another class of strain function generators. Here is the spatial Seth-Hill Strain function:

$$\frac{1}{m}(\mathbf{V}^m - \mathbf{I}) \text{ for } m \neq 0$$
$$\log_{\mathrm{e}} \mathbf{V}, m = 0$$

- Again, as before, the Euler-Almansi Strain function, $\mathbf{e} = \frac{1}{2}(\mathbf{I} - \mathbf{B}^{-1})$ is the special case of the spatial Seth-Hill Strain function when $m = -2$.

Uniaxial Extension



- They told you (and you believed!) that strain is Increase in length over original length! Here is what they were talking about:
- We noted earlier that Uniaxial extension transformation function is, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3$. Let us write $\alpha_1 = l_1/l_0$ and examine the implications. What is the value of $\alpha_1 X_1$ when $\alpha_1 = l_1/l_0$? Of course, it is zero when $X_1 = 0$, and it is equal to l_1 when $X_1 = l_0$. In one word, it properly defines the spatial configuration for the uniaxial extension we are so used to!
- Consequently, the Lagrangian Strain becomes,

Compute the relevant tensors

$$[\mathbf{F}] = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, [\mathbf{C}] = \begin{pmatrix} \alpha^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, [\mathbf{E}] = \frac{1}{2}\begin{pmatrix} \alpha^2 - 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{E} = -\frac{1}{2}\left(1 - \left(\frac{l_1}{l_0}\right)^2\right)\mathbf{E}_1 \otimes \mathbf{E}_1 \approx \frac{l_1 - l_0}{l_0}\mathbf{E}_1 \otimes \mathbf{E}_1$$

Small Uniaxial Strain

- To see this, consider that,

$$\frac{l_1^2 - l_0^2}{2l_0^2} = \frac{(l_1 - l_0)(l_1 + l_0)}{2l_0^2}$$

- Now, observe that,

$$\lim_{l_0 \to l_1} \frac{(l_1 - l_0)(l_1 + l_0)}{2l_0^2} = \frac{l_1 - l_0}{l_0}$$

- When strains are small, in uniaxial extension, it is correct to state that change in length divided by original length is equal to strain!

l_0	l_1	$\frac{l_1 - l_0}{l_0}$	$\frac{l_1^2 - l_0^2}{2l_0^2}$	% Error
1	1.001	0.001	0.0010005	0.05
1	1.010	0.010	0.0050000	0.50
1	1.100	0.100	0.1050000	4.76
1	2.000	1.000	1.5000000	33.33
1	10.000	9.000	49.5000000	81.82

Steel (l_1=1.002 at yield)

Elastomers (l_1~10)

Computing Strain

- Lagragian strains are computed in the attached Mathematica code.
- The first case is for extension along the X_1 –axis as described earlier. See what it contributes to the strain tensor.
- Last two are shear cases. Observe the relationship with the tangent of the shear angle.

```
myDef[X1_, X2_, X3_] := {1/10 X1, X2, X3};
defGrad = Grad[myDef[X1, X2, X3], {X1, X2, X3}];
LagStrain = 1/2 (Transpose[defGrad].defGrad - IdentityMatrix[3]) // MatrixForm

myDef[X1_, X2_, X3_] := {α1 X1, α2 X2, α3 X3};
defGrad = Grad[myDef[X1, X2, X3], {X1, X2, X3}];
LagStrain = 1/2 (Transpose[defGrad].defGrad - IdentityMatrix[3]) // MatrixForm

In[3]:= myDef[X1_, X2_, X3_] := {α1 X2 + X1, X2, X3};
defGrad = Grad[myDef[X1, X2, X3], {X1, X2, X3}]
LagStrain = 1/2 (Transpose[defGrad].defGrad - IdentityMatrix[3]) // MatrixForm

Out[3]= {{1, α1, 0}, {0, 1, 0}, {0, 0, 1}}

Out[4]//MatrixForm=
```

$$\begin{pmatrix} 0 & \frac{\alpha1}{2} & 0 \\ \frac{\alpha1}{2} & \frac{\alpha1^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

```
In[ ]:= myDef[X1_, X2_, X3_] := {X1, α2 X1 + X2, X3};
defGrad = Grad[myDef[X1, X2, X3], {X1, X2, X3}]
LagStrain = 1/2 (Transpose[defGrad].defGrad - IdentityMatrix[3]) // MatrixForm
```

Observation

From the computations above, we see clearly that extension deformations contribute to the diagonal elements of the strain tensor corresponding to the direction of extension.

Shear deformations contribute to the off-diagonal elements depicting the planes of deformation.

In either case, the tensors are always symmetrical.

When strains are small, the linear strains conform to the elementary definitions. Shear strains are half the angles of shear. When strains are large, extensions become nonlinear, shear are now tangents of the respective angles of shear.

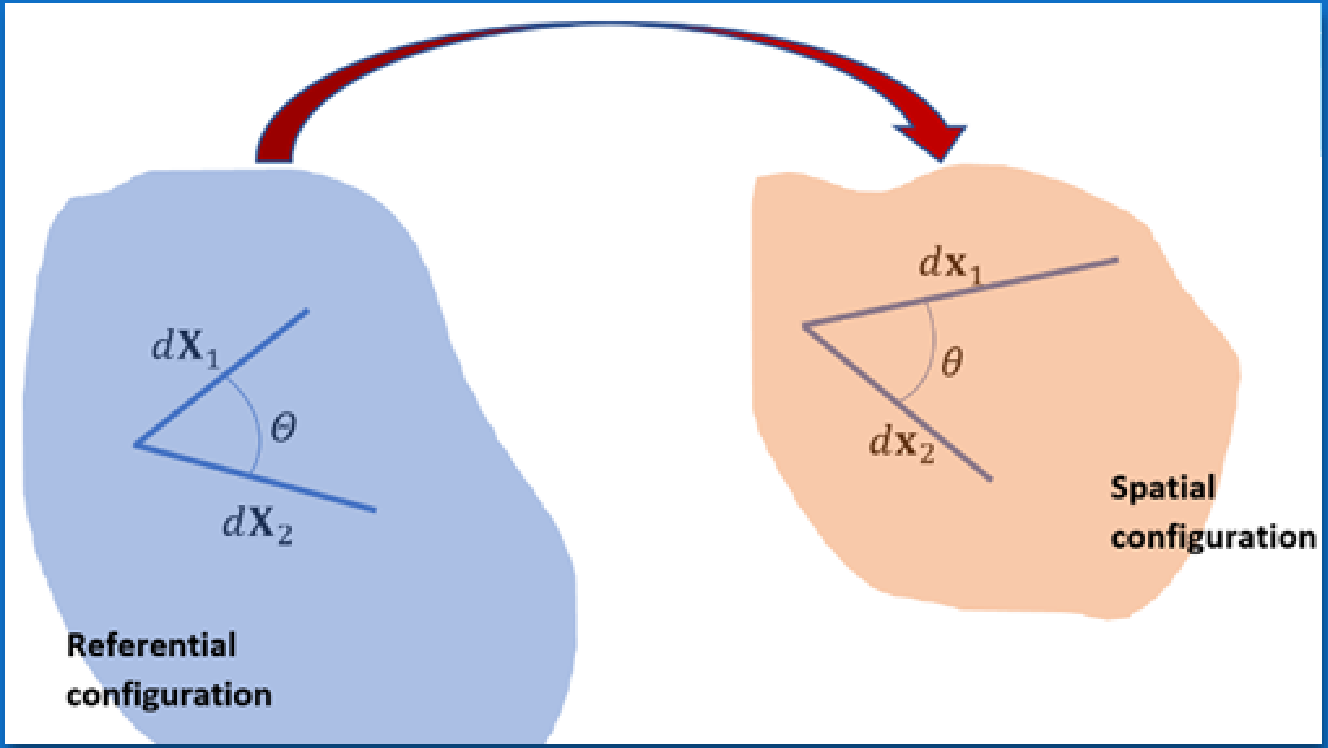
Deformation Gradient as a Strain Function

- The Polar decomposition theorem immediately shows why the deformation gradient cannot be a proper measure of strain. Consider the expression, $\mathbf{F}_1 = \mathbf{R}_1\mathbf{U}, \mathbf{F}_2 = \mathbf{R}_2\mathbf{U}$ so that the only difference between the two deformation gradients is the fact that the rotations are different, but the stretch tensors are the same.
- Deformation gradient creates ambiguity in strain measurement. Therefore the Polar decomposition becomes important to remove rigid body motions and make computed values depend only on deformations and not inconsequential motions.

Stretch Tensors

- Consider two infinitesimal material vectors, $d\mathbf{X}_1$ and $d\mathbf{X}_2$ and subject the material in which they are placed to the deformation gradient $\mathbf{F}$. Clearly, the images of these two elements in the spatial state will be $d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1$ and $d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$
- We now proceed to find the magnitude of the image vectors by taking the scalar products as follows:

$$\begin{aligned} d\mathbf{x}_1 \cdot d\mathbf{x}_2 &= \mathbf{F}d\mathbf{X}_1 \cdot \mathbf{F}d\mathbf{X}_2 \\ &= \mathbf{RU}d\mathbf{X}_1 \cdot \mathbf{RU}d\mathbf{X}_2 \\ &= \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{R}^{\mathrm{T}}\mathbf{RU}d\mathbf{X}_2 \\ &= \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2 \end{aligned}$$



Right Stretch Tensor

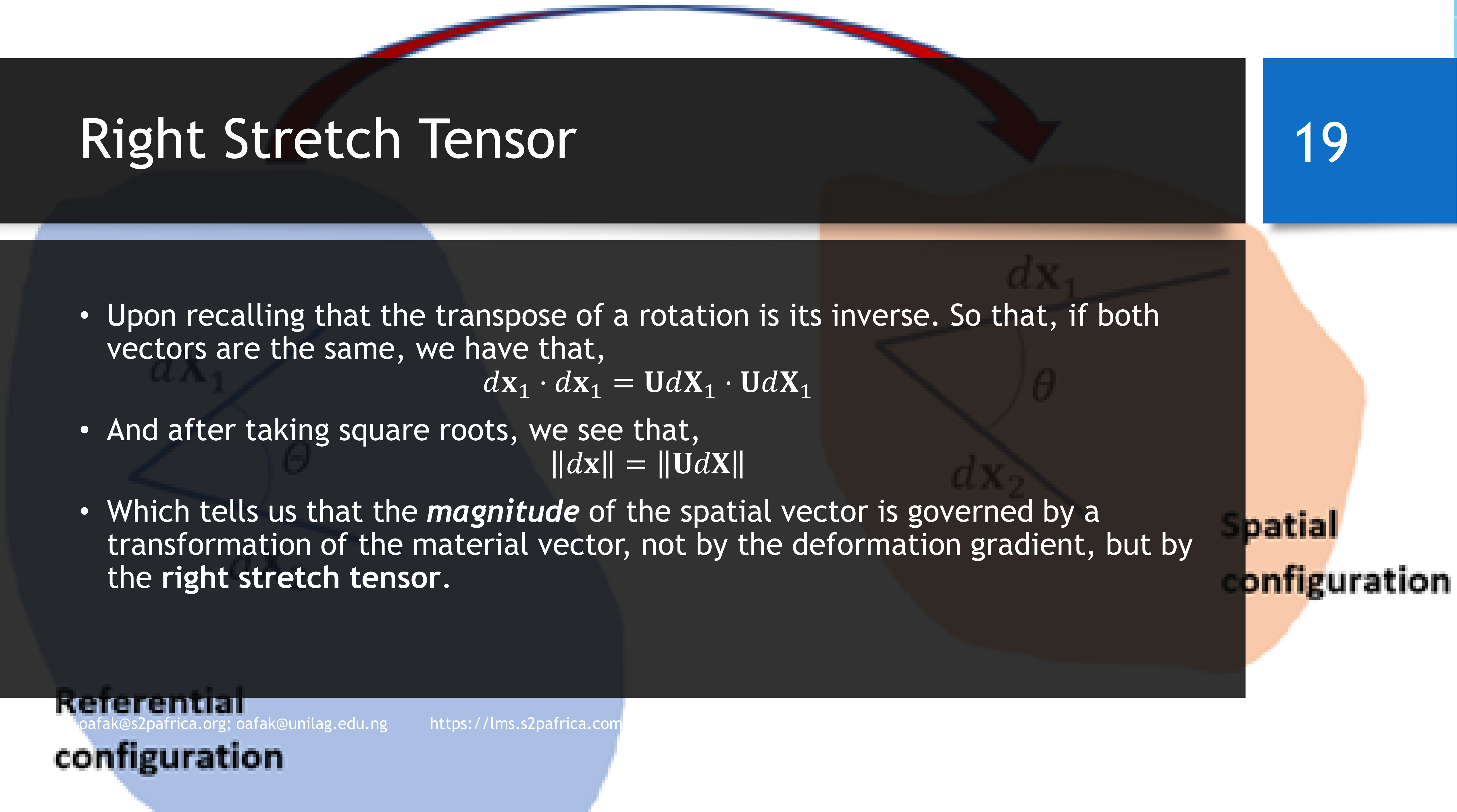
- Upon recalling that the transpose of a rotation is its inverse. So that, if both vectors are the same, we have that,

$$d\mathbf{x}_1 \cdot d\mathbf{x}_1 = \mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_1$$

- And after taking square roots, we see that,

$$\|d\mathbf{x}\| = \|\mathbf{U}d\mathbf{X}\|$$

- Which tells us that the ***magnitude*** of the spatial vector is governed by a transformation of the material vector, not by the deformation gradient, but by the **right stretch tensor**.



Left Stretch Tensor

- The inverse of this argument is immediate: That, in terms of the spatial lengths, the referential length can be found from,
$$\|d\mathbf{X}\| = \|\mathbf{V}^{-1}d\mathbf{x}\|$$
- The inverse of the Left Stretch Tensor computes the magnitude of material (referential) tensors using the spatial image as argument.

Measuring Angles Between Deformed Fibres

- The above arguments helps us clarify issues with normal strains on infinitesimal elements. Once we know the Right stretch tensor, we can find the new length of any fibre. In **shear strain**, we are interested, not in elongation or reductions in lengths, but in the changes in the angles between infinitesimal elements.
- Using the same diagram, we can take a look at the angle between these two referential elements as they are transformed in the deformation In the referential configuration, the angle between the line elements, $d\mathbf{X}_1$ and $d\mathbf{X}_2$ is,

$$\Theta = \cos^{-1}\left(\frac{d\mathbf{X}_1 \cdot d\mathbf{X}_2}{\|d\mathbf{X}_1\|\|d\mathbf{X}_2\|}\right)$$

Measuring Angles Between Deformed Fibres

- To find the angle between any two elements in the spatial configuration we simply recall that the angle we seek is
$$\theta = \cos^{-1}\left(\frac{d\mathbf{x}_1 \cdot d\mathbf{x}_2}{\|d\mathbf{x}_1\|\|d\mathbf{x}_2\|}\right) = \cos^{-1}\left(\frac{\mathbf{U}d\mathbf{X}_1 \cdot \mathbf{U}d\mathbf{X}_2}{\|\mathbf{U}d\mathbf{X}_1\|\|\mathbf{U}d\mathbf{X}_2\|}\right)$$
- To find shear strain, we look at two elements in the referential configuration that are at right angles. Shear strain is DEFINED as the change in the right angle between these two elements: We subtract the new angle θ in radians from $\frac{\pi}{2}$. As it is with elongations or contractions of length, the changes in angles are controlled, not by the deformation gradient or the rotation, but by the right and left stretch tensors. The insight leading to the Seth-Hill generalized strain functions become clearer as they correctly recognized the tensor responsible for the shape changes linearly as well, as in relative angular displacements.

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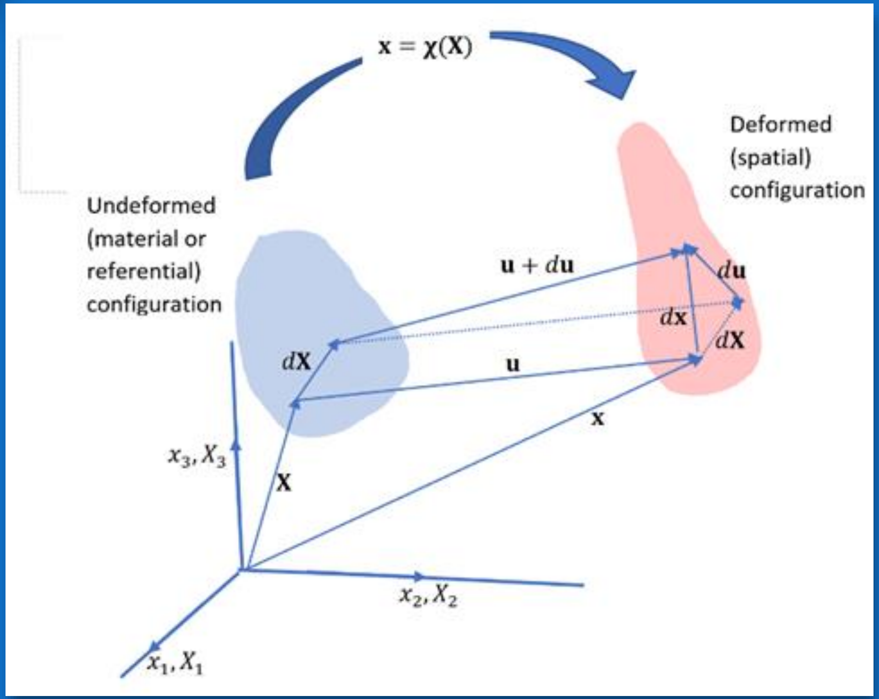
The Displacement Function

- Consider a material that has been subjected to a deformation as shown. Here, for simplicity, we refer both configurations to the same Cartesian origin and let the two coordinate systems coincide.
- Let point $\mathbf{P}$ be located at the point $\mathbf{X}$ in the material configuration be such that it transforms to the point $\mathbf{P}$ located at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$ in the spatial.
- Consider the vector $\mathbf{u} = \boldsymbol{\chi}(\mathbf{X}) - \mathbf{X}$. Let us take the material gradient of this equation and write,

$$\mathbf{H} \equiv \mathbf{Grad\,u} = \mathrm{Grad}\,\boldsymbol{\chi}(\mathbf{X}) - \mathbf{I} = \mathbf{F} - \mathbf{I}$$



Small Strains

- The built environment, using linear elasticity, has at its core the fact the strains are small: a very reasonable assumption in the days where hard metals such as iron and its ores or aluminum in its harder varieties were the chief materials for the built environment and manufacturing. Things have changed significantly, and those assumptions are no longer always valid. In this section, we will assume "small strains" and observe its implications on the quantities we have been looking at. In component form, we can write,

$$H_{ij} = \frac{\partial u_i}{\partial X_j}$$

- Upon noting that the identity tensor, in Cartesian coordinates has the Kronecker delta as its coefficients, we can therefore write,

$$F_{ij} = \delta_{ij} + H_{ij}$$

Small Strains

- Again, in Cartesian, the transpose is simply the reversal of the indices. Hence, we can write,

$$\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{I}\right) = \frac{1}{2}\left((\mathbf{H} + \mathbf{I})^{\mathrm{T}}(\mathbf{H} + \mathbf{I}) - \mathbf{I}\right)$$
$$= \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}} + \mathbf{H}^{\mathrm{T}}\mathbf{H}\right)$$

- In component form as,

$$E_{ij} = \frac{1}{2}\left(H_{ij} + H_{ji} + H_{ki}H_{kj}\right)$$
$$= \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i} + \frac{\partial u_k}{\partial X_i}\frac{\partial u_i}{\partial X_j}\right)$$

Small Strains

- If we can neglect second-order terms, and realizing that the spatial is indistinguishable from the material, then we obtain the familiar form for strain-displacement relationships:

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \approx E_{ij}$$

- As we can see, this expression is valid only when the strains are sufficiently small that the exclusion of second-order terms does not affect the results significantly. Ignoring second-order terms, We can write the small strain tensor $\boldsymbol{\epsilon}$ as,

$$2\boldsymbol{\epsilon} = \mathbf{H} + \mathbf{H}^{\mathrm{T}} \approx 2\mathbf{E}$$

Small Deformation: Euler-Almansi Strain Tensor

- We have based all our computations on Green-Lagrange Strain tensor thus far. Let us consider the Euler strain tensor. Note that,

$$\mathbf{F} = \mathbf{I} + \mathbf{H}$$

- So that, when $\mathbf{H}$ is small, ignoring second order term, $(\mathbf{I} + \mathbf{H})(\mathbf{I} - \mathbf{H}) \approx \mathbf{I}$;

$$\mathbf{F}^{-1} = (\mathbf{I} + \mathbf{H})^{-1} \approx \mathbf{I} - \mathbf{H}$$

- Consequently, under small strains, Euler-Almansi Strain tensor,

$$\mathbf{e} = \frac{1}{2}\left(\mathbf{I} - \left(\mathbf{F}\mathbf{F}^{\mathrm{T}}\right)^{-1}\right) = \frac{1}{2}\left(\mathbf{I} - \mathbf{F}^{-\mathrm{T}}\mathbf{F}^{-1}\right) =$$

$$= \frac{1}{2}\left(\mathbf{I} - (\mathbf{I} - \mathbf{H}^{-\mathrm{T}})(\mathbf{I} - \mathbf{H})\right) \approx \frac{1}{2}\left(\mathbf{H} + \mathbf{H}^{\mathrm{T}}\right) = \boldsymbol{\epsilon}$$

Eulerian & Lagragian Strains Coincide

- Under small strain conditions, the two most used strain tensors give the same result.
- This result also coincides with the elementary definition of strain as change in length divided by original length or final length.
- Under small strains, there is little difference between these.

Small Eulerian & Lagragian Strains

- For uniaxial extension, consider the Mathematica Code:

```
myDef[X1_, X2_, X3_] := {α1 X1, X2, X3};
defGrad = Grad[myDef[X1, X2, X3], {X1, X2, X3}];
defGrad // MatrixForm
LagStrain = 1 / 2 (Transpose[defGrad] . defGrad - IdentityMatrix[3]) // MatrixForm
EulerStrain = 1 / 2 (IdentityMatrix[3] - Inverse[defGrad.Transpose[defGrad]]) // MatrixForm
```

- We have the following results:
- $\mathbf{F} = \begin{pmatrix} \alpha 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\mathbf{E} = \begin{pmatrix} \frac{1}{2}\left(-1+\alpha 1^2\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, and $\mathbf{e} = \begin{pmatrix} \frac{1}{2}\left(1-\frac{1}{\alpha 1^2}\right) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- Let the constant, $\alpha_1 \equiv \frac{l_1}{l_0}$, final length of a line element divided by its original length, From here we see that the deformation gradient has zeroes for all off diagonal elements and unity on the diagonal except $\alpha_1 \equiv \frac{l_1}{l_0}$ at the first element.
- (Green-) Lagragian strain $\frac{1}{2}\left[\left(\frac{l_1}{l_0}\right)^2 - 1\right] = \frac{1}{2l_0^2}(l_1^2 - l_0^2) = \frac{(l_1+l_0)(l_1-l_0)}{2l_0^2} \approx \frac{l_1-l_0}{l_0}$
- (Almansi-) Eulerian Strain $\frac{1}{2}\left[1 - \left(\frac{l_0}{l_1}\right)^2\right] = \frac{1}{2l_1^2}(l_1^2 - l_0^2) = \frac{(l_1+l_0)(l_1-l_0)}{2l_1^2} \approx \frac{l_1-l_0}{l_1}$

So that one is change in length over original length, the other change in length over final length. In small strains, the difference between l_1, l_0 is insignificant.

Small Strain