Coordinate Transformations for Euclidean Spaces

SSG 321 Introduction to Continuum Mechanics

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Last Week Echoes

- Main task last week was the all-important Einstein Summation Convention.
  - Moved from the regular use of the summation symbol, $\sum$ to the use of repeated indices as an indicator of summation.
  - Noted that this method does not alter the meanings of expressions but helps us gain parsimony
  - Observed rules governing its applications – especially how to distinguish between dummy (mutable) indices as distinct from free indices
  - Learned how to know if there are errors in the index notations

More Echoes

- Two major symbols introduced:
    1. The Kronecker delta, δ_{ij}, whose value depends on the relations between its two indices: one when they are explicit and equal, 0 when they are not equal. The summation convention still holding.
    2. The Levi-Civita three index symbol, $e_{\alpha\beta\gamma}$, whose value depends on the uniqueness and arrangements of its indices. One when they are unique arrangements of 1,2,3 in even permutations, Minus one in odd permutations and zero in any other case. Again, the summation convention still valid

Unfinished Business, Last Week

- Time did not allow us to cover the entire scope unfortunately. We were to express, using the newly found power with the summation convention, Kronecker Delta and the Levi-Civita, the component forms of vector operations.
- This includes the scalar, vector and tensor products that we will now define and expand upon.

Dot Product, Component Form

Recall that, $\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$.

Consequently,

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) \\ &= a_i b_j \mathbf{e}_i \cdot \mathbf{e}_j = a_i b_j \delta_{ij} \\ &= a_i b_i \\ &= a_1 b_1 + a_2 b_2 + a_3 b_3\end{aligned}$$

Note on this line that we avoided having four indices of the same type by invoking the fact that a dummy variable is mutable.

Vector Product in Component Form

- Recall that $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$, and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$. The table here shows that

$$\mathbf{e}_i \times \mathbf{e}_j = e_{ijk}\mathbf{e}_k$$

- $\mathbf{a} \times \mathbf{b} = (a_i\mathbf{e}_i) \times (b_j\mathbf{e}_j)$
$= a_i b_j \mathbf{e}_i \times \mathbf{e}_j$
$= e_{ijk} a_i b_j \mathbf{e}_k$

i	j	$\mathbf{e}_i \times \mathbf{e}_j$	$e_{ijk}\mathbf{e}_k,$
1	3	$1 \times 1 \sin 90\,(-\mathbf{e}_2)$	$e_{13k}\mathbf{e}_k = e_{131}\mathbf{e}_1 + e_{132}\mathbf{e}_2 + e_{133}\mathbf{e}_3 = -\mathbf{e}_2$
1	2	$\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$	$e_{12k}\mathbf{e}_k = e_{121}\mathbf{e}_1 + e_{122}\mathbf{e}_2 + e_{123}\mathbf{e}_3 = \mathbf{e}_3$
2	3	$\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$	$e_{23k}\mathbf{e}_k = e_{231}\mathbf{e}_1 + e_{232}\mathbf{e}_2 + e_{233}\mathbf{e}_3 = \mathbf{e}_1$
3	1	$\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$	$e_{31k}\mathbf{e}_k = e_{311}\mathbf{e}_1 + e_{312}\mathbf{e}_2 + e_{313}\mathbf{e}_3 = \mathbf{e}_2$
1	1	$\mathbf{e}_1 \times \mathbf{e}_1 = 0$	$e_{11k}\mathbf{e}_k = e_{111}\mathbf{e}_1 + e_{112}\mathbf{e}_2 + e_{113}\mathbf{e}_3 = 0$
2	2	$\mathbf{e}_2 \times \mathbf{e}_2 = 0$	$e_{22k}\mathbf{e}_k = e_{221}\mathbf{e}_1 + e_{222}\mathbf{e}_2 + e_{223}\mathbf{e}_3 = 0$
2	1	$\mathbf{e}_2 \times \mathbf{e}_1 = -\mathbf{e}_3$	$e_{21k}\mathbf{e}_k = e_{211}\mathbf{e}_1 + e_{212}\mathbf{e}_2 + e_{213}\mathbf{e}_3 = -\mathbf{e}_3$
3	2	$\mathbf{e}_3 \times \mathbf{e}_2 = -\mathbf{e}_1$	$e_{32k}\mathbf{e}_k = e_{321}\mathbf{e}_1 + e_{322}\mathbf{e}_2 + e_{323}\mathbf{e}_3 = -\mathbf{e}_1$
3	3	$\mathbf{e}_3 \times \mathbf{e}_3 = 0$	$e_{33k}\mathbf{e}_k = e_{331}\mathbf{e}_1 + e_{332}\mathbf{e}_2 + e_{333}\mathbf{e}_3 = 0$

The Dyad

- One exceedingly important object that you can also produce from taking a product of two vectors is a **Tensor**. Naturally, we shall call such a product a "Tensor Product"
- The symbol is called a dyad operator, $\otimes$. It combines the product sign and a circle.
- The tensor product is therefore also called a dyad product.
- A dyad is defined by what it does when it acts on another vector:

$$(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

Components of a Dyad

$$\mathbf{a} \otimes \mathbf{b} = (a_i \mathbf{e}_i) \otimes \left(b_j \mathbf{e}_j\right) = a_i b_j \mathbf{e}_i \otimes \mathbf{e}_j$$

- There are nine base dyads for expressing every tensor

$\mathbf{e}_1 \otimes \mathbf{e}_1$	$\mathbf{e}_1 \otimes \mathbf{e}_2$	$\mathbf{e}_1 \otimes \mathbf{e}_3$
$\mathbf{e}_2 \otimes \mathbf{e}_1$	$\mathbf{e}_2 \otimes \mathbf{e}_2$	$\mathbf{e}_2 \otimes \mathbf{e}_3$
$\mathbf{e}_3 \otimes \mathbf{e}_1$	$\mathbf{e}_3 \otimes \mathbf{e}_2$	$\mathbf{e}_3 \otimes \mathbf{e}_3$

Product	Right or wrong	Comments
$\alpha\mathbf{u}$	Correct	Scaling a vector, multiplication of a scalar and a vector; No explicit sign required
$\mathbf{u}\beta\mathbf{v}$	Error	$\mathbf{u}\beta$ is a scaled vector whose product with $\mathbf{v}$ is ambiguous. Possible additional information can make it $(\mathbf{u}\beta)\cdot\mathbf{v}$, $\mathbf{u}\times(\beta\mathbf{v})$, or $\mathbf{u}\otimes(\beta\mathbf{v})$. They have different meanings that cannot be reliable guessed unless you supply the needed information a priori.
$\beta\alpha$	Correct	Product of two scalars; No explicit sign required
$\mathbf{vu}$	Error	Product of two vectors; $\mathbf{v}\cdot\mathbf{u}\neq\mathbf{v}\times\mathbf{u}\neq\mathbf{v}\otimes\mathbf{u}$ Explicit disambiguating sign required. We note here that certain authors imply this simple concatenation as the way they represent the tensor product, $\mathbf{v}\otimes\mathbf{u}$. In most current Literature on the subject, the tensor or dyad sign is the preferred way to represent this product. We retain that more popular convention here and subsequently.
$\beta(\mathbf{u}\times\mathbf{v})$	Correct	Vector product of two vectors gives a vector. Multiplying this result by a scalar does not require another sign. The order of the scaling is NOT important: $\beta(\mathbf{u}\times\mathbf{v})=\beta\mathbf{u}\times\mathbf{v}=\mathbf{u}\times\beta\mathbf{v}=(\mathbf{u}\times\mathbf{v})\beta$ The order of the appearance of the vectors is inviolable: $\beta(\mathbf{u}\times\mathbf{v})\neq\beta\mathbf{v}\times\mathbf{u}=\mathbf{v}\times\beta\mathbf{u}\neq(\mathbf{u}\times\mathbf{v})\beta$

Product	Right or wrong	Comments
$\mathbf{u}\cdot\mathbf{v}\alpha$	Correct	The dot product of a vector with a scaled vector. No ambiguity is created with the location of α; $\mathbf{u}\cdot\mathbf{v}\alpha$, $(\mathbf{u}\alpha)\cdot\mathbf{v}$, or $\alpha\mathbf{u}\cdot\mathbf{v}$ all mean the same thing.
$\beta\mathbf{u}\cdot\mathbf{v}\times\mathbf{w}\alpha$	Correct	Scalar triple product with vector scaling along. Result is the same as $(\beta\alpha)\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = (\beta\alpha)\mathbf{u}\times\mathbf{v}\cdot\mathbf{w}$
$\beta\mathbf{u}\times\mathbf{v}\times\mathbf{w}$	Error	Vector triple product with vector scaling along. Vector product is not associative: $\beta\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \beta(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - \beta(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ $\neq \beta(\mathbf{u}\times\mathbf{v})\times\mathbf{w} = \beta(\mathbf{u}\cdot\mathbf{w})\mathbf{v} - \beta(\mathbf{v}\cdot\mathbf{w})\mathbf{u}$ Parentheses are required to show which product is intended.
$\mathbf{u}\cdot\mathbf{v}\otimes\mathbf{w}$	Error	$(\mathbf{v}\otimes\mathbf{w})\,\mathbf{u} \neq \mathbf{u}(\mathbf{v}\otimes\mathbf{w})$. Error in the book
$\mathbf{u}\times\mathbf{v}\otimes\mathbf{w}$	Correct	Treat the vector cross as a tensor, then obtain the LHS: $(\mathbf{u}\times\mathbf{v})\otimes\mathbf{w} = \mathbf{u}\times(\mathbf{v}\otimes\mathbf{w})$ The two different interpretations evaluate to the same value.

Vectors & Their Matrices

$$\mathbf{a} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3]\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$
$$= a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = a_i\mathbf{e}_i$$

$$\mathbf{b} = [b_1, b_2, b_3]\begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$
$$= b_1\mathbf{e}_1 + b_2\mathbf{e}_2 + b_3\mathbf{e}_3 = b_j\mathbf{e}_j$$

Dyads & Matrices

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \otimes \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

$$\mathbf{a} \otimes \mathbf{b} = [\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3] \otimes \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \begin{bmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{bmatrix}$$

The Trace of a Dyad

Obtain the trace of a dyad by changing the dyad operator into a dot as follows

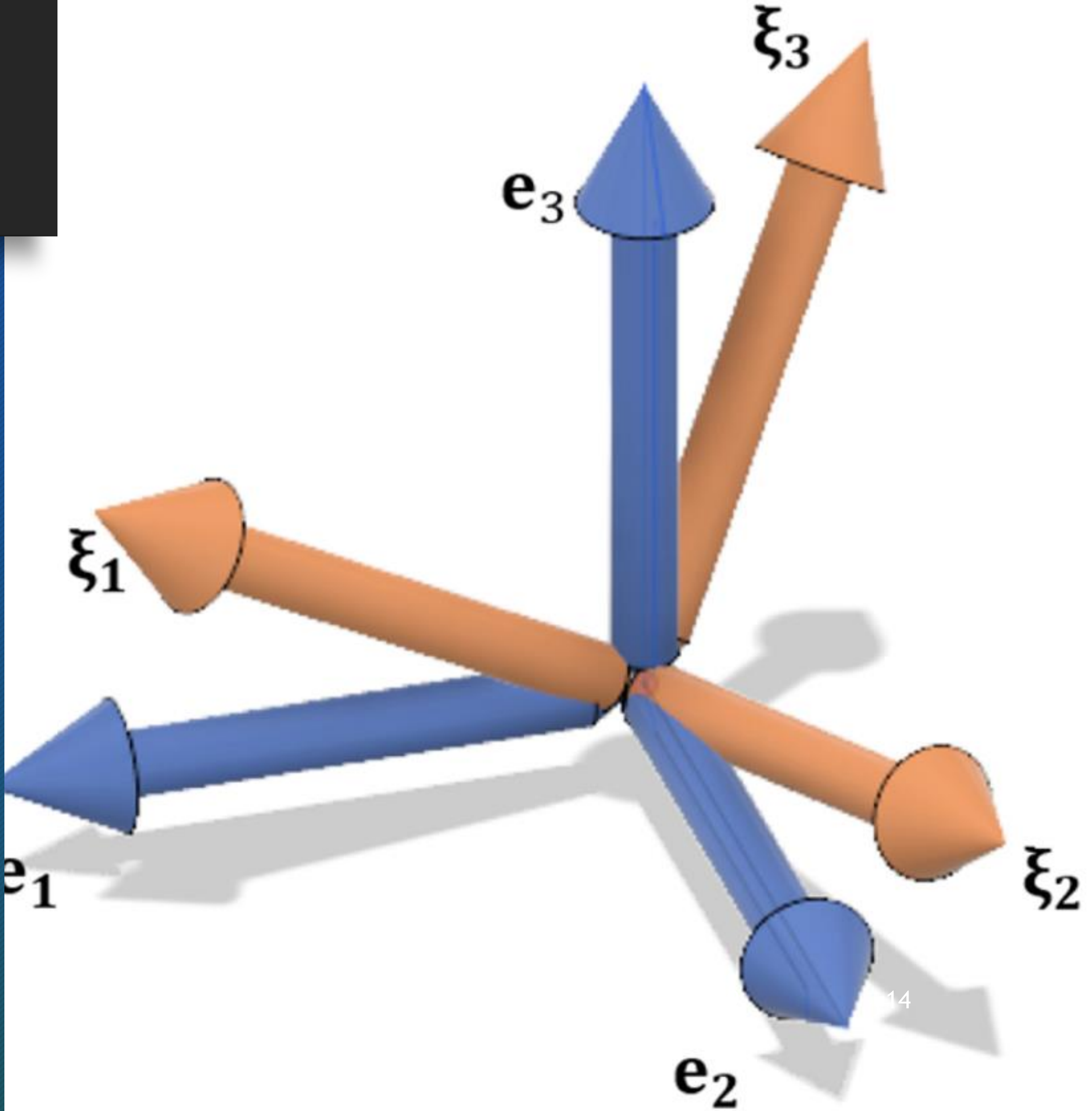
$$\begin{aligned} \mathrm{tr}(\mathbf{a} \otimes \mathbf{b}) &= a_i b_j \,\mathrm{tr}\left(\mathbf{e}_i \otimes \mathbf{e}_j\right) \\ &= a_i b_j \left(\mathbf{e}_i \cdot \mathbf{e}_j\right) \\ &= a_i b_j \,\delta_{ij} \\ &= a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 \end{aligned}$$

It is the inner product as can be seen from the matrix: The scalar product of operands.

$$\begin{bmatrix} \boldsymbol{a_1 b_1} & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & \boldsymbol{a_2 b_2} & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & \boldsymbol{a_3 b_3} \end{bmatrix}$$

Axes Rotation

- Consider a set of Cartesian coordinate orthonormal vectors, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ shown in blue in figure These vectors are position vectors at $\{1,0,0\}, \{0,1,0\}$ and $\{0,0,1\}$ respectively.
- Consider another orthonormal system, shown in pink, whose unit vectors are oriented as shown in the picture. Let these unit vectors be $\{\xi_1, \xi_2, \xi_3\}$.
- All we know about the Coordinate Vectors $\{\xi_1, \xi_2, \xi_3\}$ is that they too are orthonormal, having the same Origin as $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$



New System in Terms of Old

- Set $\{\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}\}$ are orthonormal and are therefore linearly independent; They form a basis. Therefore any vectors in their space can be written in scaled additions (linear) of them: This includes each of the vectors $\boldsymbol{\xi}_\mathbf{1}, \boldsymbol{\xi}_\mathbf{2}$ and $\boldsymbol{\xi}_\mathbf{3}$
- Taking these vectors one by one, we may write,

$$\boldsymbol{\xi}_1 = \alpha_1 \mathbf{e}_1 + \beta_1 \mathbf{e}_2 + \gamma_1 \mathbf{e}_3$$
$$\boldsymbol{\xi}_2 = \alpha_2 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \gamma_2 \mathbf{e}_3$$
$$\boldsymbol{\xi}_3 = \alpha_3 \mathbf{e}_1 + \beta_3 \mathbf{e}_2 + \gamma_3 \mathbf{e}_3$$

where α_i, β_i and γ_i, $i = 1, \dots, 3$ are coefficients to be determined

Small Adjustment

- Suppose instead of $\{\alpha, \beta, \gamma\}$ we used a single indexed symbol $\{a_1, a_2, a_3\}$
- $\{\alpha_j, \beta_j, \gamma_j\} \to \{a_{1j}, a_{2j}, a_{3j}\}, j = 1,2,3.$
- In other words, we can represent all the nine coefficients above as,

$$a_{ij}, i = 1, \dots, 3, j = 1, \dots, 3$$

	α	β	γ
1	a_{11}	a_{21}	a_{31}
2	a_{12}	a_{22}	a_{32}
3	a_{13}	a_{23}	a_{33}

Transformation Coefficients

- Our original transformation equations,

$$\boldsymbol{\xi}_1 = \alpha_1 \mathbf{e}_1 + \beta_1 \mathbf{e}_2 + \gamma_1 \mathbf{e}_3$$
$$\boldsymbol{\xi}_2 = \alpha_2 \mathbf{e}_1 + \beta_2 \mathbf{e}_2 + \gamma_2 \mathbf{e}_3$$
$$\boldsymbol{\xi}_3 = \alpha_3 \mathbf{e}_1 + \beta_3 \mathbf{e}_2 + \gamma_3 \mathbf{e}_3$$

- Now take the form,

$$\boldsymbol{\xi}_i = a_{ij} \mathbf{e}_j.$$

- Scalar multiplication with $\mathbf{e}_\alpha$ results in,

$$\boldsymbol{\xi}_i \cdot \mathbf{e}_\alpha = a_{ij} \mathbf{e}_j \cdot \mathbf{e}_\alpha = a_{ij} \delta_{j\alpha} = a_{i\alpha}$$

from which we conclude that $\boldsymbol{\xi}_i \cdot \mathbf{e}_j = a_{ij}$.

Reverse Transformation

- We now invert the argument this way: Set $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \boldsymbol{\xi}_3\}$ are orthonormal and are therefore linearly independent; They form a basis. Therefore any vectors in their space can be written in scaled additions (linear) of them: This includes each of the vectors $\mathbf{e}_1, \mathbf{e}_2$ and $\mathbf{e}_3$!

$$\mathbf{e}_j = b_{jk}\boldsymbol{\xi}_k = b_{jk}a_{k\alpha}\mathbf{e}_\alpha$$

$$\mathbf{e}_j \cdot \mathbf{e}_\beta = b_{jk}a_{k\alpha}\mathbf{e}_\alpha \cdot \mathbf{e}_\beta$$

$$\delta_{j\beta} = b_{jk}a_{k\alpha}\delta_{\alpha\beta} = b_{jk}a_{k\beta}$$

Showing that the matrix $[b_{ij}]$ is the inverse of $[a_{ij}]$.

Reverse Transformation Again

- Another way to find the b_{jk}s, we multiply scalarly by $\boldsymbol{\xi}_i$ again and obtain,

$$\begin{aligned}\boldsymbol{\xi}_i \cdot \mathbf{e}_j &= \boldsymbol{\xi}_i \cdot \left(b_{jk}\boldsymbol{\xi}_k\right)\\ &= b_{jk}\boldsymbol{\xi}_i \cdot \boldsymbol{\xi}_k\\ &= b_{jk}\delta_{ik}b_{ji}\\ &= b_{ji}\end{aligned}$$

- Showing that the reverse transformation matrix is also a transpose of the original transformation.

$$\mathbf{A} = \mathbf{B}^{-1} = \mathbf{B}^{\mathrm{T}}$$

Road Sign: Geometry of Transformation

In the next five slides, we present a two-dimensional view of the geometry of the transformation of coordinates.

This is tedious even though not rigorous. You need to be patient. What are trying to achieve here is for you to believe the easy formulas we will later use.

When we shall have established the connection, we will not need to repeat this tedious process again. However, those who do not give sufficient attention will find it hard to apply the easy formulas later!

- $\mathbf{O}\ x_1\ x_2\ \rightarrow \mathbf{O}\ y_1\ y_2$, find coordinates of vector $\mathbf{OP}$ in the new system
- Let $\mathbf{OP} = \mathbf{v} = a_i\mathbf{e}_i$ where $\mathbf{e}_1$and $\mathbf{e}_2$ are unit vectors along $\mathbf{O}\ x_1\ x_2$. If the coordinates are rotated to $\mathbf{O}\ y_1\ y_2$ such that the same vector now becomes $\mathbf{v} = b_i\boldsymbol{\xi}_i$ where $\boldsymbol{\xi}_1$and $\boldsymbol{\xi}_2$ are unit vectors along the $\mathbf{O}\ y_1\ y_2$ system.

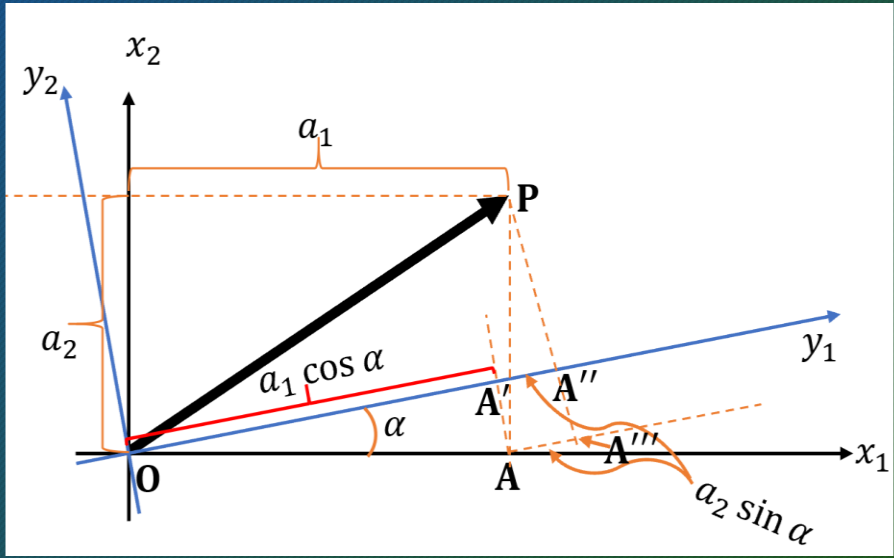
$$\mathbf{OA} = a_1; \mathbf{OB} = a_2; \mathbf{OA}'' = b_1? \mathbf{OB}'' = b_2?$$

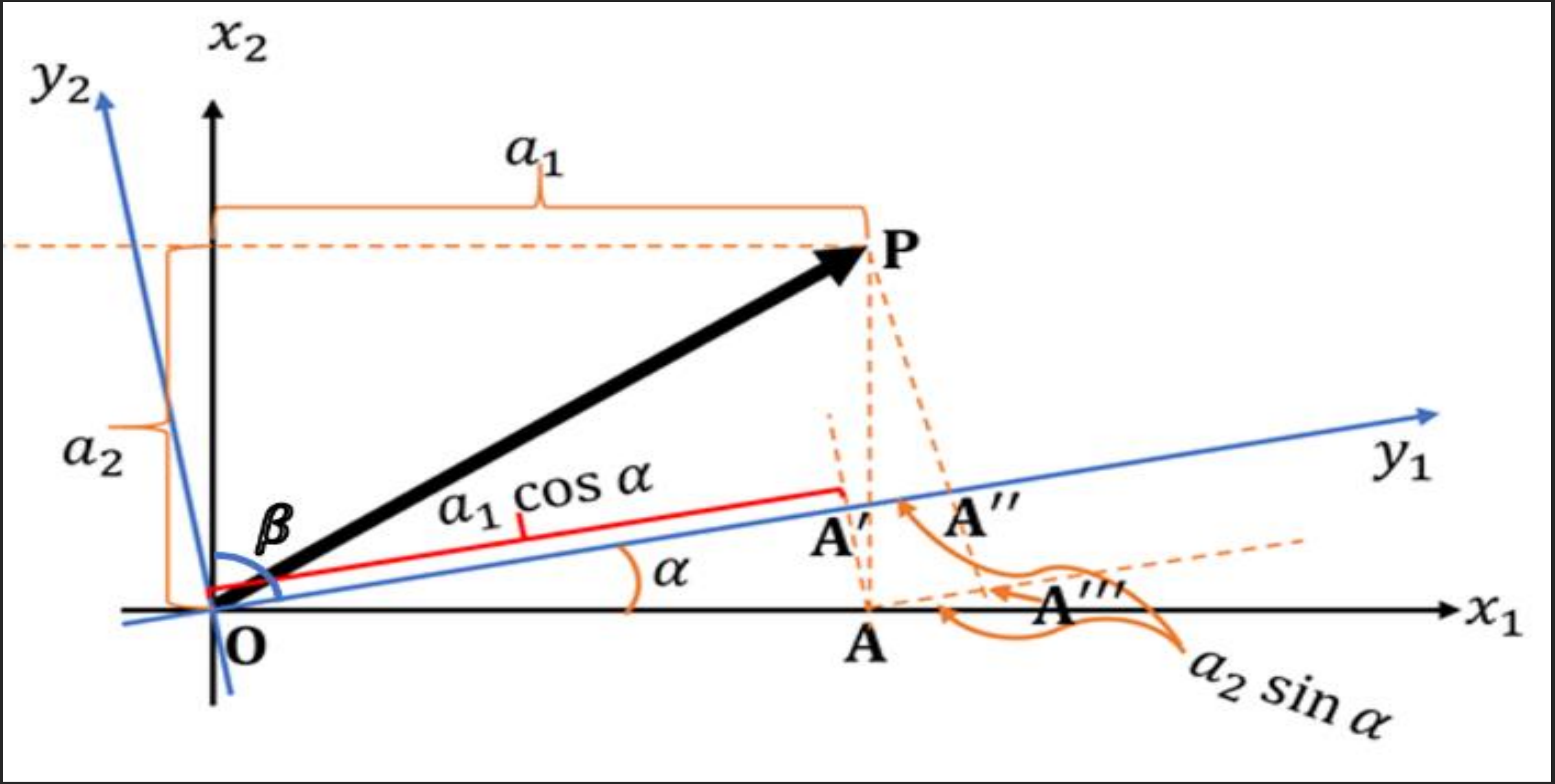
- We drop perpendicular lines to the lines $\mathbf{O}\ y_1$ and $\mathbf{O}\ y_2$ meeting them at $\mathbf{A}''$and $\mathbf{B}''$respectively.

$$\mathbf{OA}' = a_1 \cos\alpha \ \ \mathbf{AA}'' = a_2 \sin\alpha$$

- because $\mathbf{PA}$ is the hypotenuse of a right-angled triangle $\mathbf{APA}'''$with angle α at $\mathbf{APA}'''$ And it is easy to see that $\mathbf{AA}'\mathbf{A}''\mathbf{A}'''$is a rectangle. Its opposite sides are equal, consequently, the length

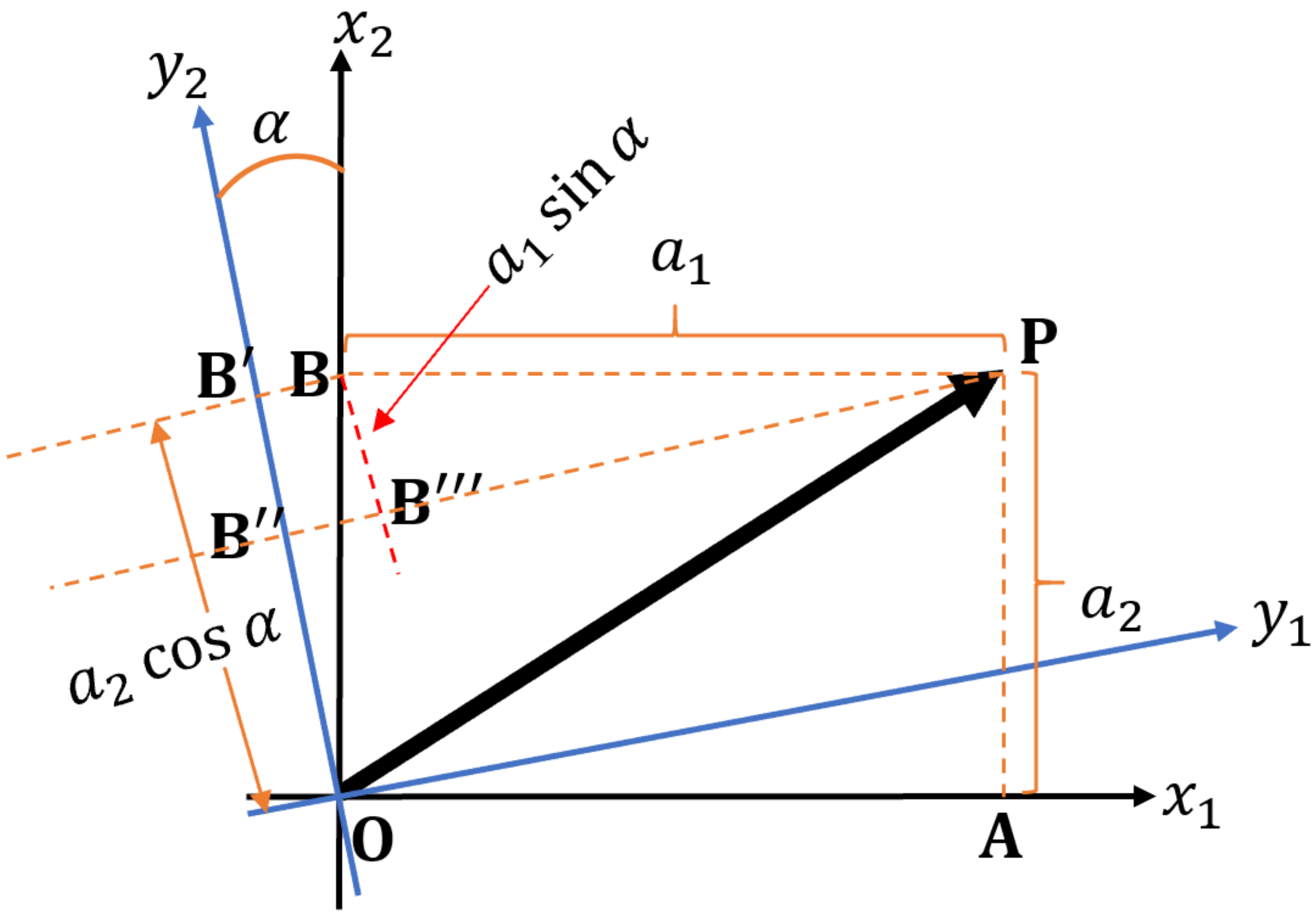
$$\mathbf{OA}'' = b_1 = a_1\cos\alpha + a_2 \sin\alpha.$$
$$= a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)$$




x_2
y_2
a_1
P
a_2
β
$a_1 \cos \alpha$
α
A′
A″
A‴
y_1
x_1
O
A
$a_2 \sin \alpha$

The Other Coordinate

- $\mathbf{B}''$ is the foot of the perpendicular from point $\mathbf{P}$ to the $\mathbf{O}\, y_2$-axis. $\mathbf{BB}'$is parallel to $\mathbf{PB}''$. $\mathbf{B}'''$ is the foot of the perpendicular from $\mathbf{B}$ to $\mathbf{PB}''$. By the same arguments as before, $\mathbf{BB}'\mathbf{B}''\mathbf{B}'''$ is also a rectangle. Clearly,
$$\begin{aligned}\mathbf{OB}'' &= b_2 = -a_1 \sin\alpha + a_2 \cos\alpha.\\ &= a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2)\end{aligned}$$
- The rotation tensor is: $\mathbf{R}^{\mathrm{T}} = \mathbf{e}_j \otimes \boldsymbol{\xi}_j$. Hence, we have



x_2
y_2
α
$a_1 \sin\alpha$
a_1
P
B′
B
B‴
B″
a_2
y_1
$a_2 \cos\alpha$
x_1
O
A

The Rotation

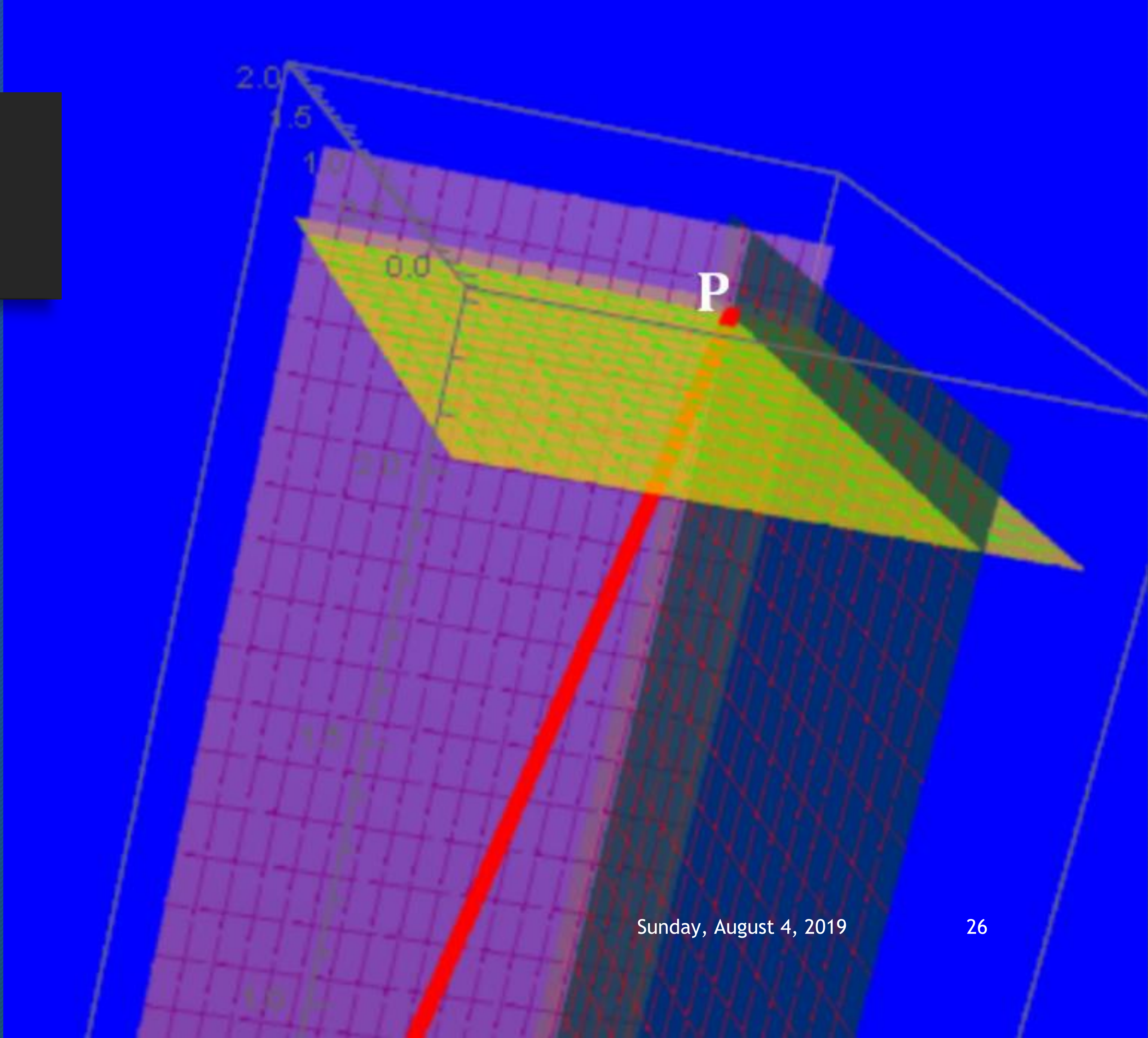
- Consider the Dyad Sum:

$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = \left(\mathbf{e}_j \otimes \boldsymbol{\xi}_j\right) a_i \mathbf{e}_i$$
$$= a_i \mathbf{e}_j \left(\boldsymbol{\xi}_j \cdot \mathbf{e}_i\right)$$
$$\mathbf{R}^{\mathrm{T}}\mathbf{v} = \mathbf{e}_1\left(a_1(\boldsymbol{\xi}_1 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_1 \cdot \mathbf{e}_2)\right) + \mathbf{e}_2\left(a_1(\boldsymbol{\xi}_2 \cdot \mathbf{e}_1) + a_2(\boldsymbol{\xi}_2 \cdot \mathbf{e}_2)\right)$$

- Exactly the same expression we found geometrically!
- Look again at the dyad sum! It is simply the dyads formed by the coordinate basis vectors!
- To compute any rotation, we only need to form the dyads from the set of coordinates before and after!

The Cartesian System of Coordinates

- Three orthonormal unit vectors as basis set.
- The intersection of two planes creates a coordinate line.
- Intersection of three planes create three coordinate lines all meeting at the same point at which the three planes intersect to give the location of the point.



Mathematica Code

- The graphics you have just seen were mathematically generated. The code is something like we show here:

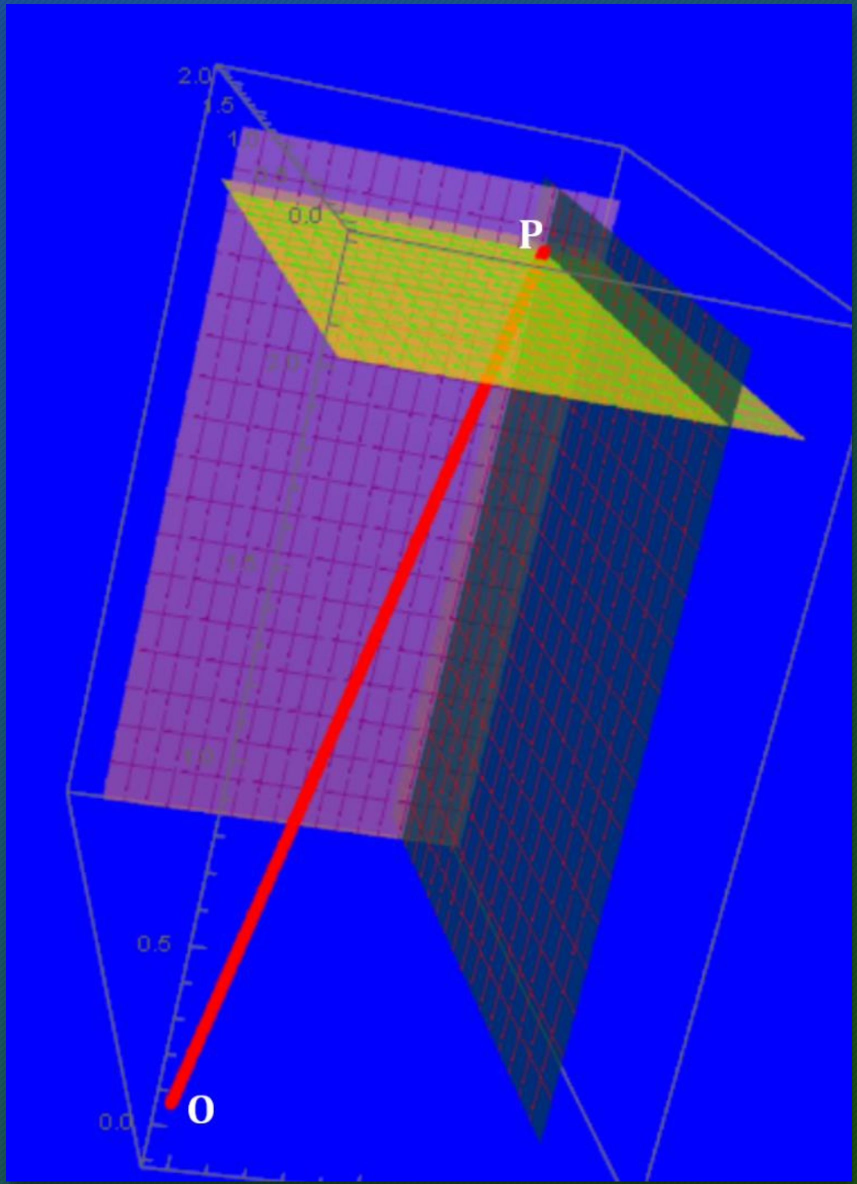
```
Cart1 = ParametricPlot3D[{1, y, z}, {y, 0, 1.4}, {z, 0, 1.4}, PlotStyle → Red];
Cart2 = ParametricPlot3D[{x, 1, z}, {x, 0, 1.4}, {z, 0, 1.4}, PlotStyle → Green];
Cart3 = ParametricPlot3D[{x, y, 1}, {x, 0, 1.4}, {y, 0, 1.4}, PlotStyle → Yellow];
Show[Cart1, Cart2, Cart3, PlotRange → {{0, 1.5}, {0, 1.5}, {0, 1.5}}, Ticks → None]
```

The Euclidean Point Space

- All engineering objects of interest reside. This space contains point locations that can be occupied by a location in an object at a particular time. It is often of interest to be able to do several things:
  - Locate the point in an unambiguous way,
  - Relate the point to one or more other points in its vicinity, and
  - Define quantities that take up values of interest at that point: Fields.
- Examples of Fields
  - Temperature map of this classroom (one thousand thermometers)
  - Temperature distribution, Temperature field.
  - Tensor Fields

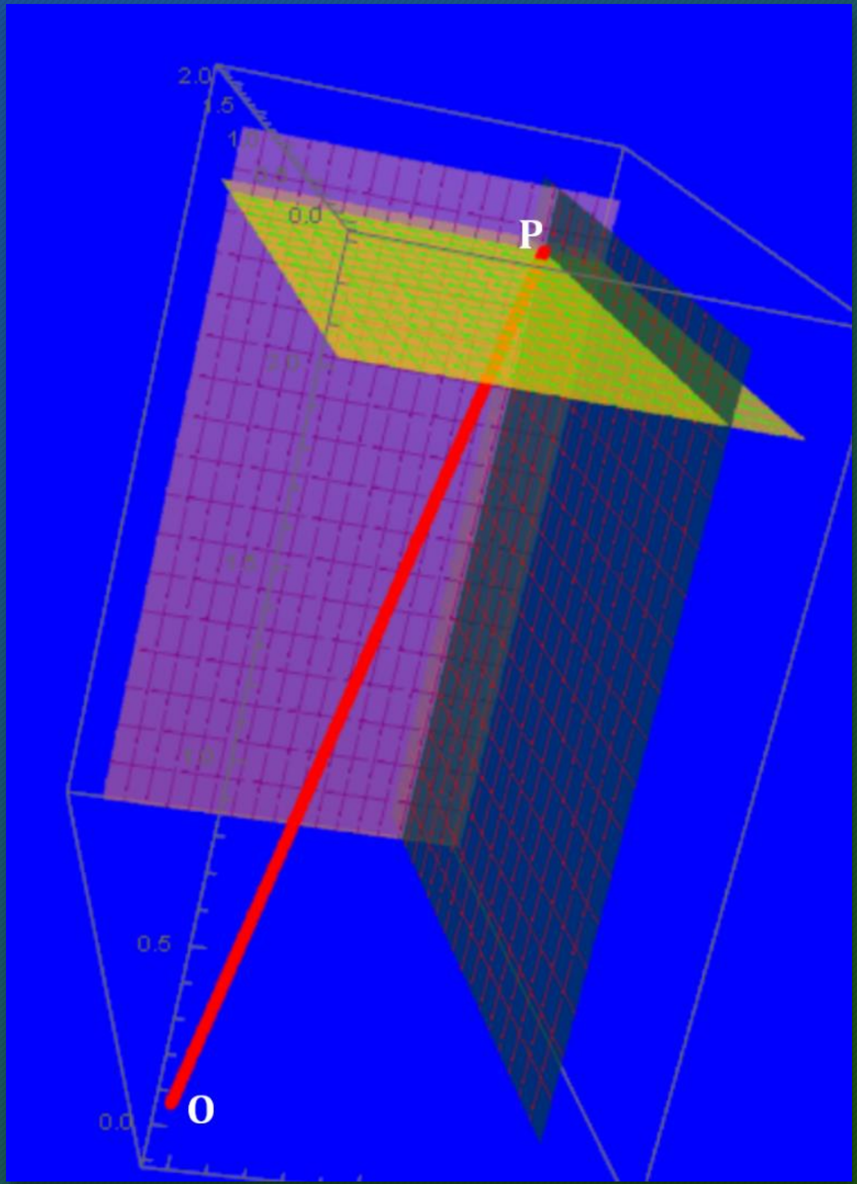
Points & Vectors

- The Euclidean Point Space that we have used our Cartesian Coordinate System to describe contains points, NOT vectors.
- It is critically important for you to be able to do graphics, to note the distinction.
- It is even more serious to note the difference in order to do mechanical analysis such as Solid Mechanics, Fluid Mechanics, Thermodynamics, etc., that our knowledge of Continuum Mechanics lead us to.



Points & Vectors

- $\mathbf{O}$ and $\mathbf{P}$ are points.
- Drawing a line between $\mathbf{O}$ and $\mathbf{P}$ creates the vector $\mathbf{OP}$ possessing all the attributes of vectors as we have previously defined:
  - Magnitude defined by the length of the line $\mathbf{OP}$,
  - Direction defined by the direction of the line $\mathbf{OP}$, and
  - Sense as from $\mathbf{O}$ to $\mathbf{P}$
- The vector we have just created is no ordinary vector. It was brought to life by joining the point $\mathbf{P}$ to the origin.



Position Vectors

- Note that we could create a vector by joining any two points.
- It is a no brainer that every point in the space can be treated as we have just treated $\mathbf{P}$: Create a vector by simply joining the point to the origin.
- The vectors created this way have a name: Position Vectors.
- We are emphasizing the fact that we have created a vector from a point by simply using a line to join the point to the origin.
- They are special vectors. Not all vectors are created this way.

Cartesian System: Special Attributes

* Each coordinate surface is a plane. The three defined at a particular point are respectively parallel to the three you can define at any other point.
* Each coordinate line: the intersection of these planes that are parallel to the axes are similarly parallel straight lines at all points in the system.
* The basis vectors – usually defined as unit vectors along the axes, are always the same at any point in the Cartesian system. It does not matter where the point P is located, the basis vectors are the same unit vectors we define as $(\mathbf{i}, \mathbf{j}$ and $\mathbf{k})$ or $(\mathbf{e}_1, \mathbf{e}_2,$ and $\mathbf{e}_3)$ along the coordinate lines at the origin.

Consequences of Attributes

- Position Vector is a linear function of the coordinates:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

- We can easily write the vector field in terms of three scalar fields that we call its components;

$$\mathbf{v}(x_1, x_2, x_3) = v_1(x_1, x_2, x_3)\mathbf{e}_1 + v_2(x_1, x_2, x_3)\mathbf{e}_2 + v_3(x_1, x_2, x_3)\mathbf{e}_3$$

  - In order, say to find acceleration, we may need to differentiate this function spatially or temporally; We need to worry only about the components as their vector bases are all constants.
- These nice features occurs only in Cartesian System of coordinates.

Relate Position Vectors to Basis Set

- A partial differentiation of the position vector with respect to the coordinate variables yield the basis vectors for the coordinate system as shown here:

$$\mathbf{r} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = x_i\mathbf{e}_i$$

$$\frac{\partial \mathbf{r}}{\partial x_i} = \mathbf{e}_i, i = 1,2,3.$$

- The partial derivative of the position vector to the three coordinate variables constitute a set of linearly independent vectors that can form the basis set.
- In the Cartesian System, they are the same as our usual basis set

Nonlinear Coordinate Systems

To form a coordinate system,

- Select three variables (ξ_1, ξ_2, ξ_3). When each takes a value, say, $\xi_i = \alpha_i$ where each α_i is a real number, then we have the point $(\alpha_1, \alpha_2, \alpha_3)$. We can write this point in at least two other ways: $\xi_i = \alpha_i,\ i = 1, \dots, 3$ or as $(\xi_1 = \alpha_1, \xi_2 = \alpha_2, \xi_3 = \alpha_3)$.
- For each, $\xi_i = \alpha_i$, we have defined a coordinate surface. In the case of Cartesian coordinates, given any three $\alpha_i \in \mathbb{R}, i = 1,2,3,$ we have $x_1 = \alpha_1$, defining a plane with normal along the $\mathbf{e}_1$axis, $x_2 = \alpha_2$, defining a plane with normal along the $\mathbf{e}_2$axis and $x_3 = \alpha_3$, which is a plane with normal along the $\mathbf{e}_3$ axis.
- A systematic choice leads to specific systems: Cylindrical, Spherical Polar

Cylindrical Polar Coordinate System

- We now introduce a transformation (called a polar transformation) of $\{x_1, x_2\} \to \{r, \phi\}$ such that, $x_1 = r\cos\phi,$ and $x_2 = r\sin\phi$ .Note also that this transformation is invertible: $r = \sqrt{x_1^2 + x_2^2}$,and $\phi = \tan^{-1}\frac{x_2}{x_1}$
- With such a transformation, we can locate any point in the 3-D space with three scalars $\{\xi_1, \xi_2, \xi_3\} \to \{r, \phi, z\}$ instead of our previous set $\{x_1, x_2, x_3\}$

Cylindrical Position Vector

- What does the position vector look like?

$$\mathbf{r} = r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z = r\mathbf{e}_r + z\mathbf{e}_z$$

Where we have defined,

$$\mathbf{e}_r = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$$
$$\mathbf{e}_z = \mathbf{e}_3$$

There are several methods to obtain the basis set of vectors. One instructive way is to do a partial differentiation of the position vector:

$$\frac{\partial\mathbf{r}}{\partial r} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2 = \mathbf{e}_r,\ \frac{\partial\mathbf{r}}{\partial\phi} = -r\sin\phi\,\mathbf{e}_1 + r\cos\phi\,\mathbf{e}_2 \equiv r\mathbf{e}_\phi,\ \frac{\partial\mathbf{r}}{\partial z} = \mathbf{e}_z$$

Cylindrical Basis Vector Set

By differentiating the position vector,

$$\begin{aligned}\mathbf{r} &= r\cos\phi\,\mathbf{e}_1 + r\sin\phi\,\mathbf{e}_2 + z\mathbf{e}_z \\ &= r\mathbf{e}_r + z\mathbf{e}_z\end{aligned}$$

with respect to the coordinate variables ξ_1, ξ_2, ξ_3 which now are r, ϕ, z, the basis vectors are shown in the table shown:

Derivative	Explicit Form	
$\frac{\partial \mathbf{r}}{\partial r}$	$\cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_2$	$\mathbf{e}_r$
$\frac{\partial \mathbf{r}}{\partial \phi}$	$-r\sin\phi\,\mathbf{e}_1 + r\cos\phi\,\mathbf{e}_2$	$r\mathbf{e}_\phi$
$\frac{\partial \mathbf{r}}{\partial \mathbf{z}}$	$\mathbf{e}_z$	

$$\|\mathbf{e}_r\|^2 = \cos^2\phi + \sin^2\phi = 1$$
$$\|\mathbf{e}_\phi\|^2 = \sin^2\phi + \cos^2\phi = 1$$
$$\|\mathbf{e}_z\|^2 = 1$$

They are individually normalized with each having a norm or magnitude of 1. Now let's take them in pairs:

$$\mathbf{e}_r \cdot \mathbf{e}_\phi = -\cos\phi\sin\phi + \cos\phi\sin\phi = 0$$
$$\mathbf{e}_\phi \cdot \mathbf{e}_z = -\sin\phi \times 0 + \cos\phi \times 0 + 1 \times 0 = 0$$
$$\mathbf{e}_z \cdot \mathbf{e}_r = \cos\phi \times 0 + \sin\phi \times 0 + 1 \times 0 = 0$$

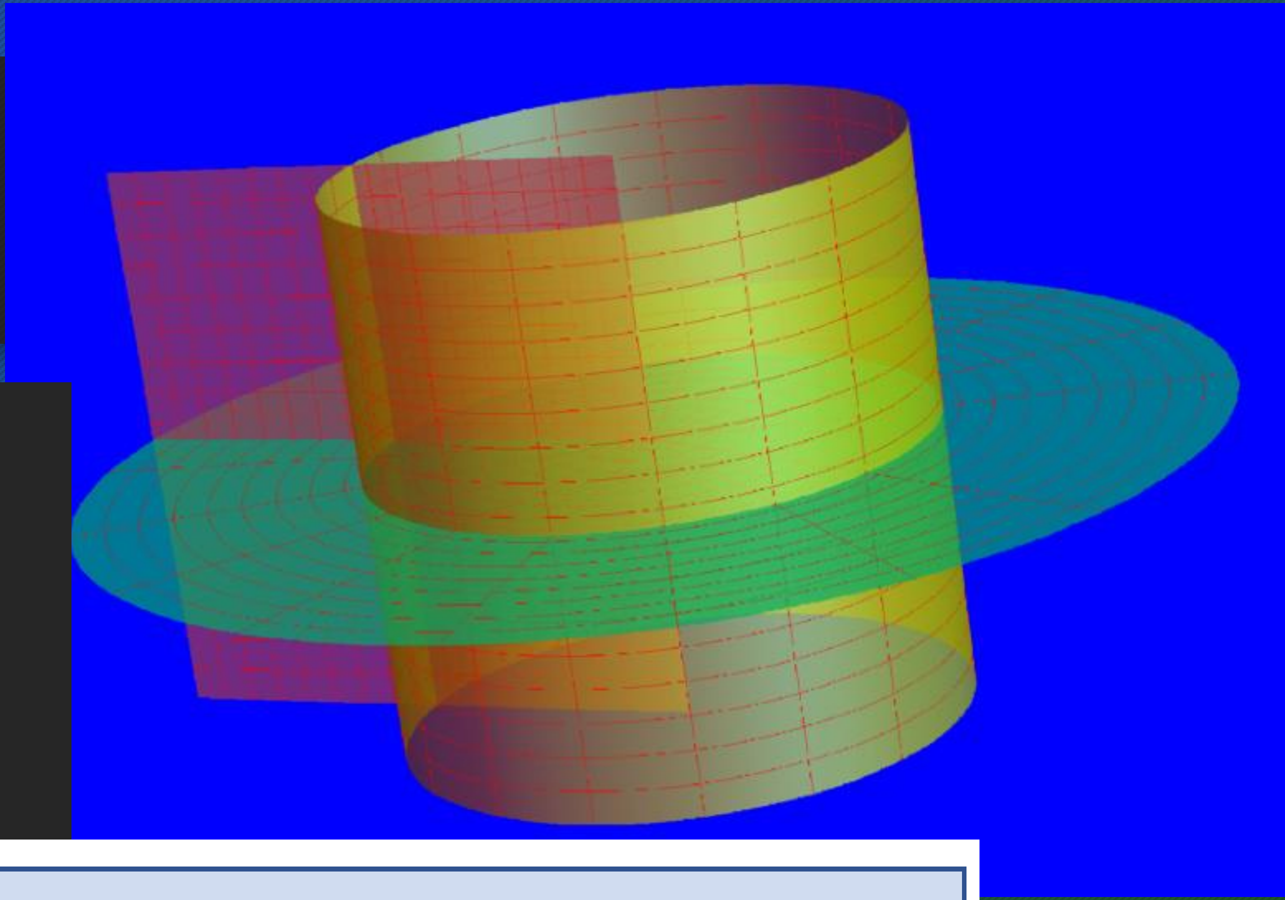
So that they are pairwise orthogonal.

Cylindrical Polar basis vectors constitute an orthonormal system

The code to plot the coordinate surfaces are given here

Type it into the Mathematica Notebook …

Coordinate Surfaces in Cylindrical System



```
c1 = ParametricPlot3D[{Sin[ϕ], Cos[ϕ], z}, {ϕ, 0, π}, {z, 1.5, 3.5}, PlotStyle → Red];
c2 = ParametricPlot3D[{r Sin[π / 3], r Cos[π / 3], z}, {r, 0, 2}, {z, 1.5, 3.5}, PlotStyle → Green];
c3 = ParametricPlot3D[{r Sin[ϕ], r Cos[ϕ], 2}, {ϕ, 0, 2 π}, {r, 0.5, 2.5}, PlotStyle → Yellow];
Show[c1, c2, c3, PlotRange -> {{0, 1.4}, {0, 1.5}, {1, 2.5}}, Ticks → None]
```

Mistakes to Avoid

- **That the Cylindrical position vector is $r\mathbf{e}_r(\phi) + \phi\mathbf{e}_\phi(\phi) + z\mathbf{e}_z$**
  - A simplistic copy of the Cartesian formula. This is wrong in at least two ways. For one thing, it is dimensionally incorrect because the unit of the middle basis component is an angle while the other components are measuring lengths. Secondly, we cannot obtain the Cartesian result from this via a coordinate transformation.
- **That the basis vectors are constants.**
  - They are NOT all constants. $\mathbf{e}_r(\phi)$ and $\mathbf{e}_\phi(\phi)$ are both functions of ϕ unlike in the Cartesian case, but $\mathbf{e}_z$ is a constant like the Cartesian case.

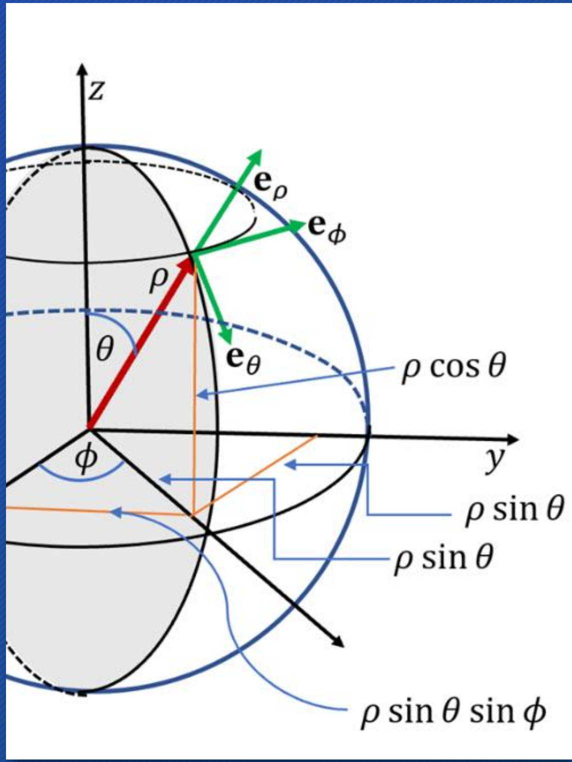
Spherical Coordinates

- The spherical Polar coordinate system selects its three ordered triplets with yet another strategy. This can be explained by the same transformation route we started. Continuing further with our transformation, we may again introduce two new scalars such that $\{r, z\} \to \{\rho, \theta\}$ in such a way that the position vector,

$$\mathbf{r} = r\mathbf{e}_r + z\mathbf{e}_z = \rho \sin\theta \, \mathbf{e}_r + \rho \cos\theta \, \mathbf{e}_z \equiv \rho \mathbf{e}_\rho$$

- Here, $r = \rho \sin\theta$, $z = \rho \cos\theta$. As before, we can use three scalars, $\{\rho, \theta, \phi\}$ instead of $\{r, \phi, z\}$. In comparison to the original Cartesian system we began with, we have that

Spherical Polar Coordinates



$$\begin{aligned}\mathbf{r} &= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \\ &= \rho\sin\theta\,\mathbf{e}_r + \rho\cos\theta\,\mathbf{e}_z \\ &= \rho\sin\theta\,(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \rho\cos\theta\,\mathbf{k} \\ &= \rho\sin\theta\cos\phi\,\mathbf{i} + \rho\sin\theta\sin\phi\,\mathbf{j} + \rho\cos\theta\,\mathbf{k} \\ &\equiv \rho\mathbf{e}_\rho(\theta,\phi)\end{aligned}$$

where

$$\mathbf{e}_\rho \equiv \sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k}$$

a nonlinear function of the coordinate variables ρ, θ and ϕ

Spherical Coordinate Surfaces

- Again, we introduce the unit vector,

$$\mathbf{e}_\theta \equiv \cos\theta\cos\phi\,\mathbf{i} + \cos\theta\sin\phi\,\mathbf{j} - \sin\theta\,\mathbf{k}$$

- and retain

$$\mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}$$

- as before. It is easy to demonstrate the fact that these vectors constitute another orthonormal set. Combining the two transformations, we can move from $\{x, y, z\}$ system of coordinates to $\{\rho, \phi, \theta\}$ directly by the transformation equations, $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$ and $z = \rho\cos\theta$ .The orthonormal set of basis for the $\{\rho, \theta, \phi\}$ system is $\{\mathbf{e}_\rho(\theta,\phi), \mathbf{e}_\theta\,(\theta,\phi), \mathbf{e}_\phi\,(\phi)\}$