# The Deformation Gradient

Course: Continuum Mechanics II; Topic: Kinematics

Instructors: OA Fakinlede, O Adewumi & OA George

# Deformations & Motions



- We are primarily concerned here with deformations and motions in material entities.
- The first important quantity we shall encounter is the deformation gradient. We shall see that is is a tensor. It will also become clear that all the information we need concerning the deformation or motion are contained in this tensor.
- We will try to compute it for important deformations & motions. It will lead us to other geometric quantities of importance. The first lecture will focus on the Deformation Gradient.

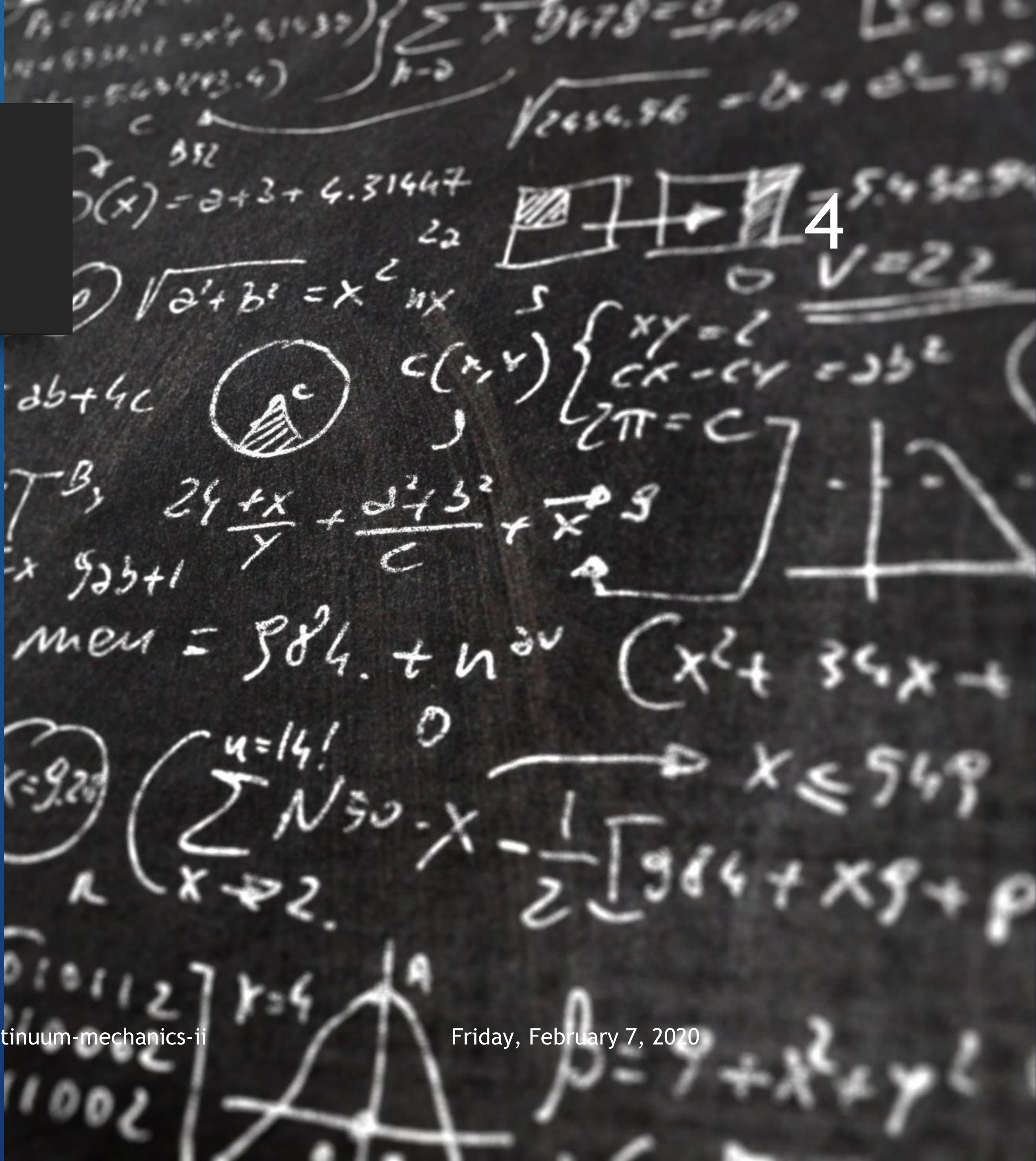
# Topics in this Lecture

- Referential (Material) & Spatial (Current) Configurations
- The Deformation Gradient Tensor
- Polar Decomposition Theorem
- Simple Deformations & Motions
- Displacement, Stretch, Strain & Other Measures of Deformation

# Insight Versus Math & Computation



- This and subsequent chapters call for two achievements: Insight and Computations. Experience shows that you may feel intimidated by the mathematics and computations, thinking, wrongly, that they are difficult. I have news for you, THEY ARE NOT as difficult as they appear to be!
  - Mathematica will be used in many cases to demonstrate the (Symbolic) computations. Simulation software will solve, numerically, the difficult differential equations. Every difficult step will be further explained if you ask for help.
  - Insight is the key. It is possible to be able to work the Math and compute without gaining Insight! This is a trap you should avoid!



# Simulations Input & Output



- Century 21, Engineers do not have to manually solve the differential equations that arise from their analyses. Software is usually available to do that.
  - They need to understand the inputs and results of the software they use in order to supply relevant boundary conditions, obtain correct results and make useful judgments.
  - When, for example, you perform a simulation in Fusion 360, outputs include the symmetrical strain tensor and the displacement vector.
  - Fusion 360 also gives you the Equivalent Strain.
- Weeks Two and Three will supply what you need to …

# Two Week Goals



1. Understand why strain, not deformation gradient, is the choice for quantifying deformation
2. Understand the meaning of each element in the strain tensor; and know why the strain tensor is necessarily symmetrical
3. Know that the elementary definition of strain is only approximate: understand when it cannot properly measure strain
4. See that rotations, translations and other rigid body motions produce zero strain but non-zero deformation gradient.
5. Understand why rigid body motions create ambiguity in strain measurement. How they are removed to make strains unique.
6. Understand various deformations and motions such as Uniaxial, biaxial or triaxial extension, bending, shear, dilatation, torsion, etc.

# Pointwise Transformation



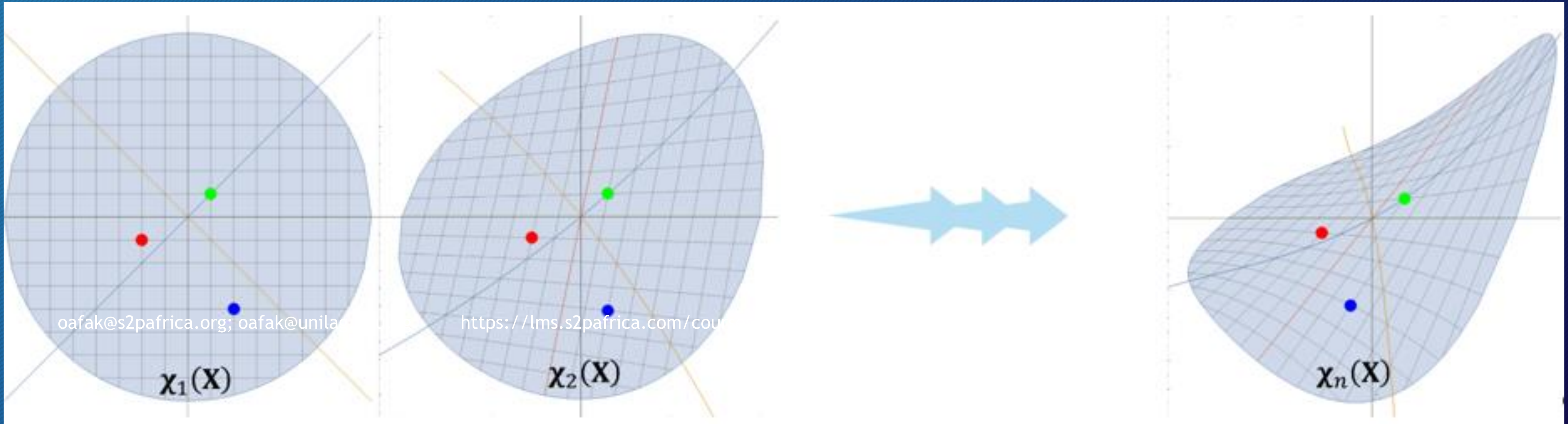
- We look at the body we are concerned about in the ambient environment of a 3-Dimensional Euclidean Point Space, $\mathcal{E}$: there is a subset of $\mathcal{E}$ between which is in a one to one correspondence with each point in the body. A deformation is a mapping from this subset to another subset of the same space:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$$

- Where $\mathbf{X}$ represents an arbitrary point in the material region of concern, and $\mathbf{x}$, its image in the spatial. A deformation that changes over time, is a motion. If at specific $t = 1, 2, \dots, n$ in time, we have the discrete functions,

$$\mathbf{x}_1 = \boldsymbol{\chi}_1(\mathbf{X}), \mathbf{x}_2 = \boldsymbol{\chi}_2(\mathbf{X}), \dots, \mathbf{x}_n = \boldsymbol{\chi}_n(\mathbf{X}), \dots$$

- Motion can also be described by the single, continuous, time dependent function, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ or $\mathbf{x} = \boldsymbol{\chi}_t(\mathbf{X})$



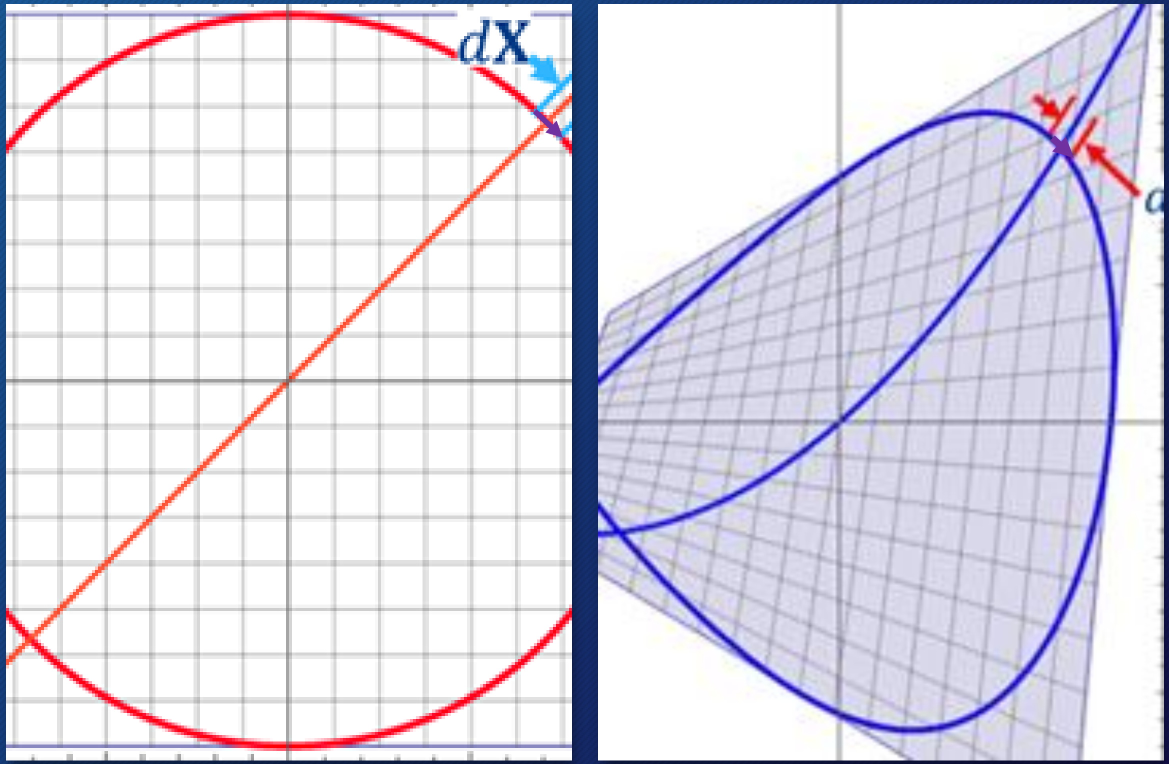
- Motion is defined as set of mappings, $\mathbf{x} = \boldsymbol{\chi}(.,t), t \in \mathbb{R}$.
- We assume that our subset of $\mathcal{E}$ is connected.
  - Each member of the set of mappings, that is, each specific deformation in the set, is a known as a configuration or description at a point in time. We can take the configurations as photographs of the body as it undergoes its motion.
  - If we take that view, even though we can have several photographs, at most one of them, represents the current state of the body. This configuration is called the Spatial Configuration.

Back to elementary calculus: Connected region, simply connected region, multiply connected region.

# Pointwise Transformation

# Referential & Spatial Configurations

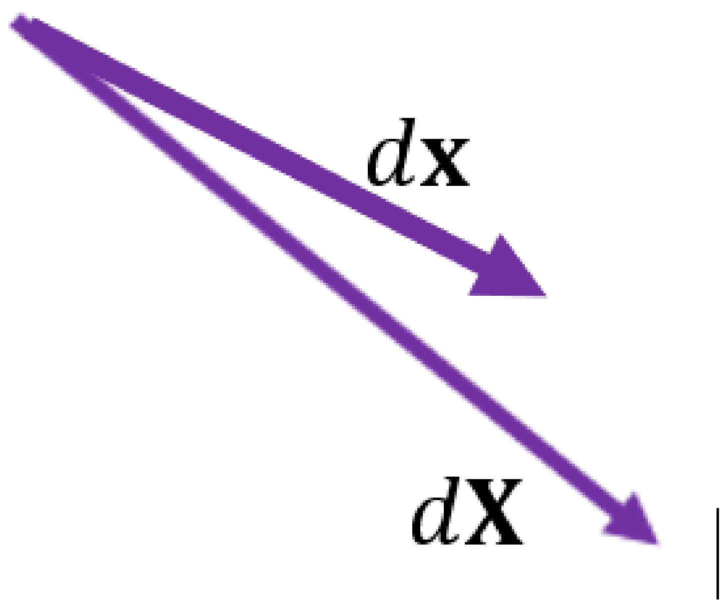
- Consider, for simplicity, the 2-D region undergoing a deformation as shown by what was originally a circle and a straight line.
  - As a result of deforming the region, we have shape changes that we observe.
  - Notice that at the time the second picture is seen, the first no longer exists. However, we keep a photo of it, and refer to it as a "Referential or Material Configuration".
  - To distinguish the one we are presently observing, we call the latter the "Spatial or Current Configuration".
- Mathematically, we can look at this as a vector $d\mathbf{X}$ transformed to vector $d\mathbf{x}$ as shown by the purple arrows.



# Referential & Spatial Configurations

We are looking at the two vectors, $d\mathbf{X}$ in the referential configuration, and $d\mathbf{x}$ in the spatial.

- Note that these do not exist physically together. We bring them up, in the same diagram, artificially, for the purpose of gaining analytical insight.
- Using this artifice, we can see the entire deformation as the transformation of a vector $d\mathbf{X}$ to the vector $d\mathbf{x}$. The former can represent any material vector in the referential (original) configuration, while the latter represents its image in the spatial (or current) configuration.
- We have removed the translation of the vectors, bringing their origins together to show that elongation (or contraction) with the rotation that have been caused by the deformation.



# Spatial and Referential Vectors

- It is obvious that there is a relationship between the infinitesimal spatial and material vector that transformed to it. Each spatial vector, $d\mathbf{x}$ being a vector, consists of three scalars functions, $\{dx_1, dx_2, dx_3\}$, or $dx_\alpha, \alpha = 1,2,3$.
- By the vector equation, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$, we mean $x_\alpha = \chi_\alpha(X_1, X_2, X_3)$, and, from elementary multivariate calculus, we have,

$$dx_\alpha = \frac{\partial \chi_\alpha}{\partial X_j} dX_j$$

Observe that there are nine components here: three for each $\alpha$

- In vector form, assuming the referential system is spanned by $\mathbf{E}_j, j = 1 \dots 3$ and that the spatial system is spanned by $\mathbf{e}_\alpha, \alpha = 1,2,3$. Adding the relevant bases, we have the vector equation,

$$dx_\alpha \mathbf{e}_\alpha = \left( \frac{\partial \chi_\alpha}{\partial X_j} \mathbf{e}_\alpha \otimes \mathbf{E}_j \right) dX_i \mathbf{E}_i$$

Several ways to explain this dyad

# The Deformation Gradient



Textbooks write these equations as equivalent. However, the rightmost is the correct expression because it differentiates the function $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$ while the other differentiates the value.

More compactly, we write,

$$d\mathbf{x} = \mathbf{F} d\mathbf{X}$$

- where transformation tensor field

$$\mathbf{F}(\mathbf{X}, t) = \text{Grad}\, \boldsymbol{\chi}(\mathbf{X}, t) = \frac{\partial x_\alpha}{\partial X_j} \mathbf{e}_\alpha \otimes \mathbf{E}_j = \frac{\partial \chi_\alpha}{\partial X_j} \mathbf{e}_\alpha \otimes \mathbf{E}_j$$

- the material (referential) gradient of the deformation or motion function, $\boldsymbol{\chi}(\mathbf{X}, t)$. This deformation gradient is the tensor that transforms material vectors to spatial vectors in the region of interest.
- It contains ALL information about the deformation or motion.

When is it a deformation? Motion?

# The Deformation Gradient



- Is the fundamental tensor describing deformation and motion. Its second base vector, $\mathbf{E}_j$, is a reciprocal base, as is obvious from the fact that the variable it represents is below.
- Under Cartesian coordinates, there is NO difference between a base vector and its reciprocal base vector.
- Once we are in curvilinear systems such as cylindrical or spherical polar, there will be differences as we shall illustrate.

$$\mathbf{F}(\mathbf{X},t) = \text{Grad}\,\boldsymbol{\chi}(\mathbf{X},t) = \frac{\partial \chi_\alpha}{\partial X_j}\mathbf{e}_\alpha \otimes \mathbf{E}_j$$

# The Deformation Gradient



$$\mathbf{F}(\mathbf{X}, t) = \text{Grad}\, \boldsymbol{\chi}(\mathbf{X}, t) = \frac{\partial \chi_\alpha}{\partial X_j} \mathbf{e}_\alpha \otimes \mathbf{E}_j$$

- It is conventional to use the capital letter Grad for the gradient in the deformation gradient to emphasize the fact that the gradient is taken of the deformation (or motion) function with respect to the Referential system. When a gradient is taken with respect to the Spatial system, we shall write is as grad.
- Observe that

$$\text{Grad}\, \boldsymbol{\chi}(\mathbf{X}, t) \neq \text{grad}\, \boldsymbol{\chi}(\mathbf{X}, t) = \mathbf{I}$$

# The Deformation Gradient



Easily proved because

- The deformation gradient is the material gradient of the deformation function;

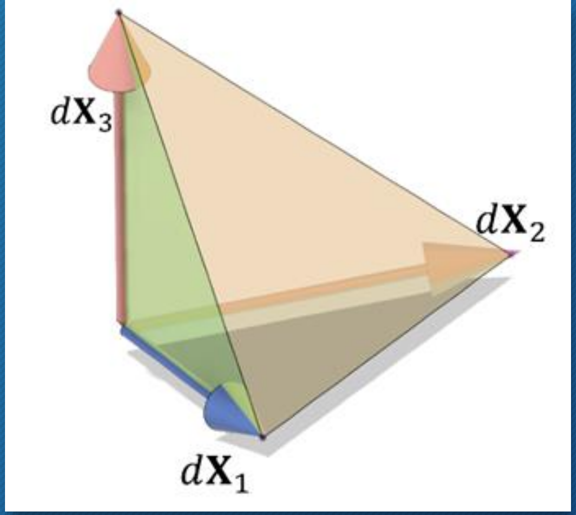
$$\begin{aligned}\text{grad}\,\boldsymbol{\chi}(\mathbf{X},t) &= \frac{\partial x_\alpha}{\partial x_j}\mathbf{e}_\alpha \otimes \mathbf{e}_j \\ &= \delta_{\alpha j}\mathbf{e}_\alpha \otimes \mathbf{e}_j \\ &= \mathbf{I}\end{aligned}$$

- $\mathbf{x} = x_\alpha \mathbf{e}_\alpha$ is, in Cartesian Coordinates, the fundamental spatial variable. It is dependent on the referential vector variable, $\mathbf{X} = X_i \mathbf{E}_i$.
- This relationship is called “the deformation” or “the motion”,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X},t)$$

# Volume Ratio





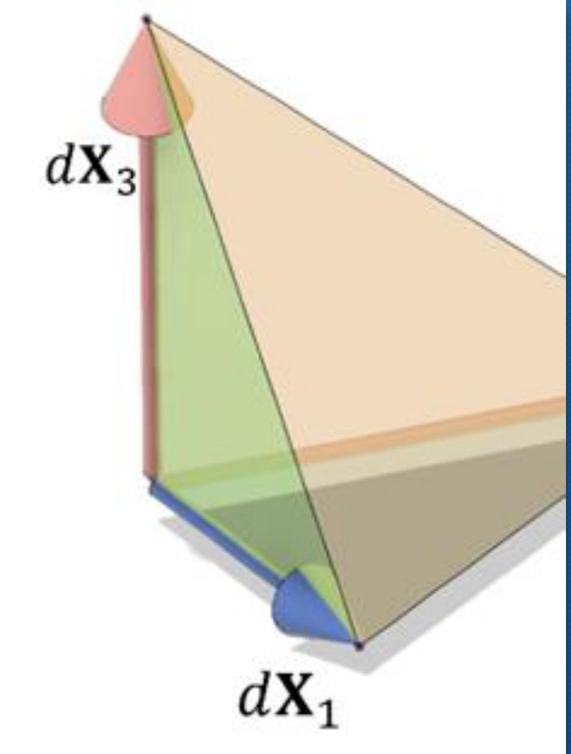

- The change in the material vectors implies there are also changes in areas and volumes. To obtain the change in volume as a result of the deformation, consider an infinitesimal tetrahedron in the referential state.
- The volume of the tetrahedron,

$$\frac{1}{6}[d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3] \neq 0$$

- i.e. the volume will not vanish if the three vectors are neither colinear nor all coplanar. As a result of the motion, the corresponding spatial vectors will form a deformed tetrahedron.

# Volume Ratio






- Each side will be a transformed referential vector into the spatial: $(d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3)$ will be related to the material vectors in such a way that,
$$d\mathbf{x}_i = \mathbf{F}d\mathbf{X}_i$$
- The volume ratio between the spatial and material configurations,
$$J = \frac{[d\mathbf{x}_1, d\mathbf{x}_2, d\mathbf{x}_3]}{[d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3]} = \frac{[\mathbf{F}d\mathbf{X}_1, \mathbf{F}d\mathbf{X}_2, \mathbf{F}d\mathbf{X}_3]}{[d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3]} = \det \mathbf{F}.$$
- The linear independence of vectors $(d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3)$ is guaranteed by the non-vanishing of the tetrahedron or we shall have chosen a trivial volume. However, what guarantee do we have for the spatial tetrahedron?

# Consider the Possibility: $d\mathbf{x} = \mathbf{F}d\mathbf{X} = \mathbf{o}$



Q2.25

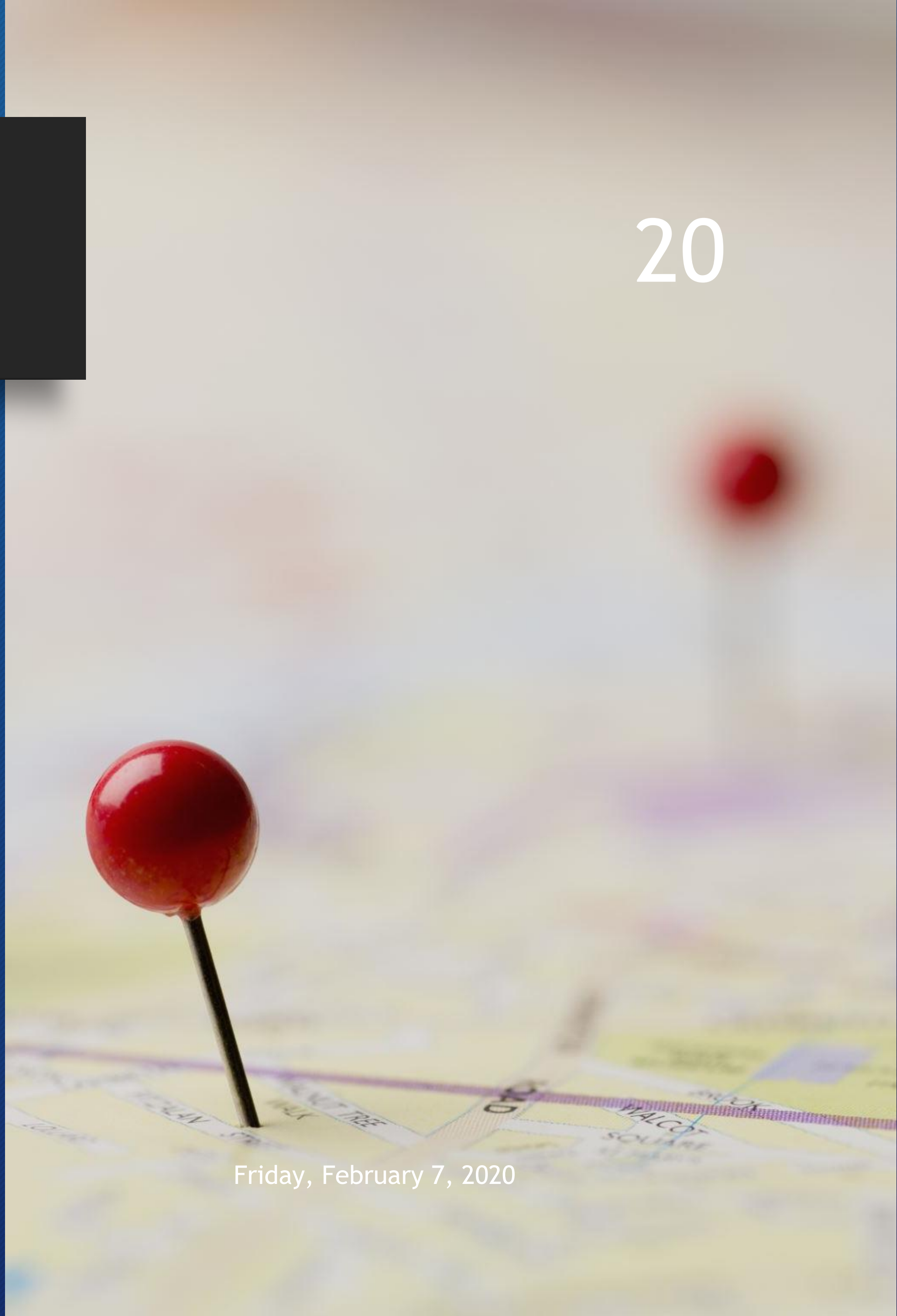
the zero vector. What can this mean physically? The linear independence of the denominator in the determinant expression guarantees non-vanishing of the numerator provided the deformation gradient is an invertible tensor. Mathematically, the Jacobian (determinant of $\mathbf{F}$) of the transformation is zero.

- We were able to find a non-trivial (not a zero tensor) transformation tensor that transforms a real vector into nothingness! We, by a deformation transformation destroyed matter!
- Our physical considerations preclude this possibility. We exclude from consideration such a possibility. And since we cannot have $J = 0$, we can therefore conclude that
$$J > 0$$
- Since continuity forces it to pass through zero to negative; if it cannot be zero, it cannot be negative. The only allowable transformations have a positive determinant.

# The Reference Map



- The set of mappings that gives each deformation, and consequently, the entire motion is a set of one-to-one mappings. Such mappings are invertible. It follows that, at each time $t$, we have,
$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t)$$
- From which we can find the reference configuration that resulted in each spatial configuration at a time $t$. The material point that occupied the spatial position $\mathbf{x}$ at time $t$ can be computed by the reference map.



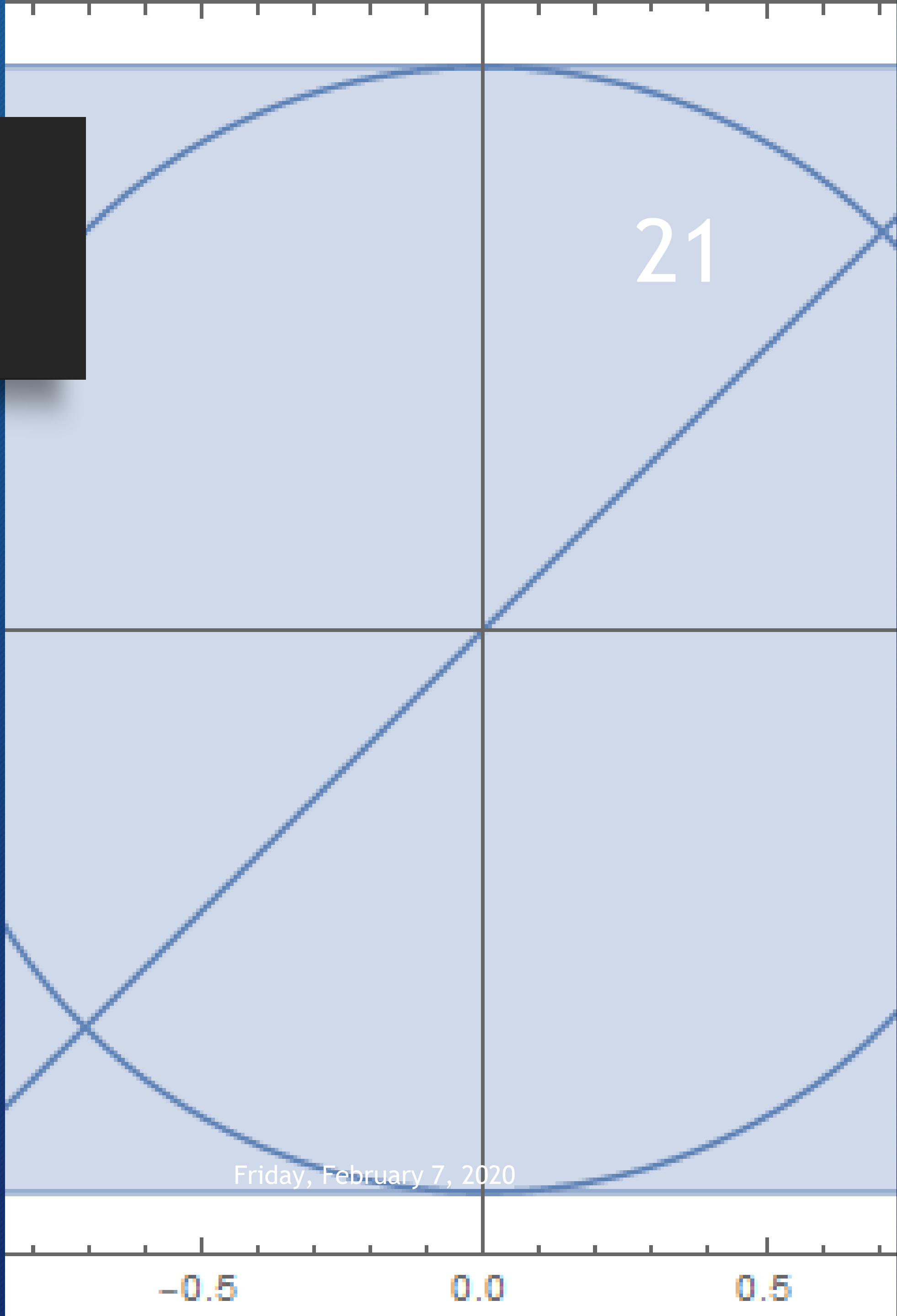
# Simple Deformation



- Consider a rectangular 2-D region and draw a circle of a unit radius and a diagonal line as shown.
- We can plot the line and the circle using parametric expressions for each as shown in the picture.

This is generated by the Mathematica code:

```
initialConfig = ParametricPlot[{X1, X2}, {X1, -1, 1},
   {X2, -1, 1}, MeshShading → {{None}}];
circle = ParametricPlot[{Sin[t], Cos[t]}, {t, 0, 2 Pi},
   MeshShading → {{None}}];
line = ParametricPlot[{X1, X1}, {X1, -1, 1}];
Show[initialConfig, circle, line, PlotRange → All]
```



# Simple Deformation

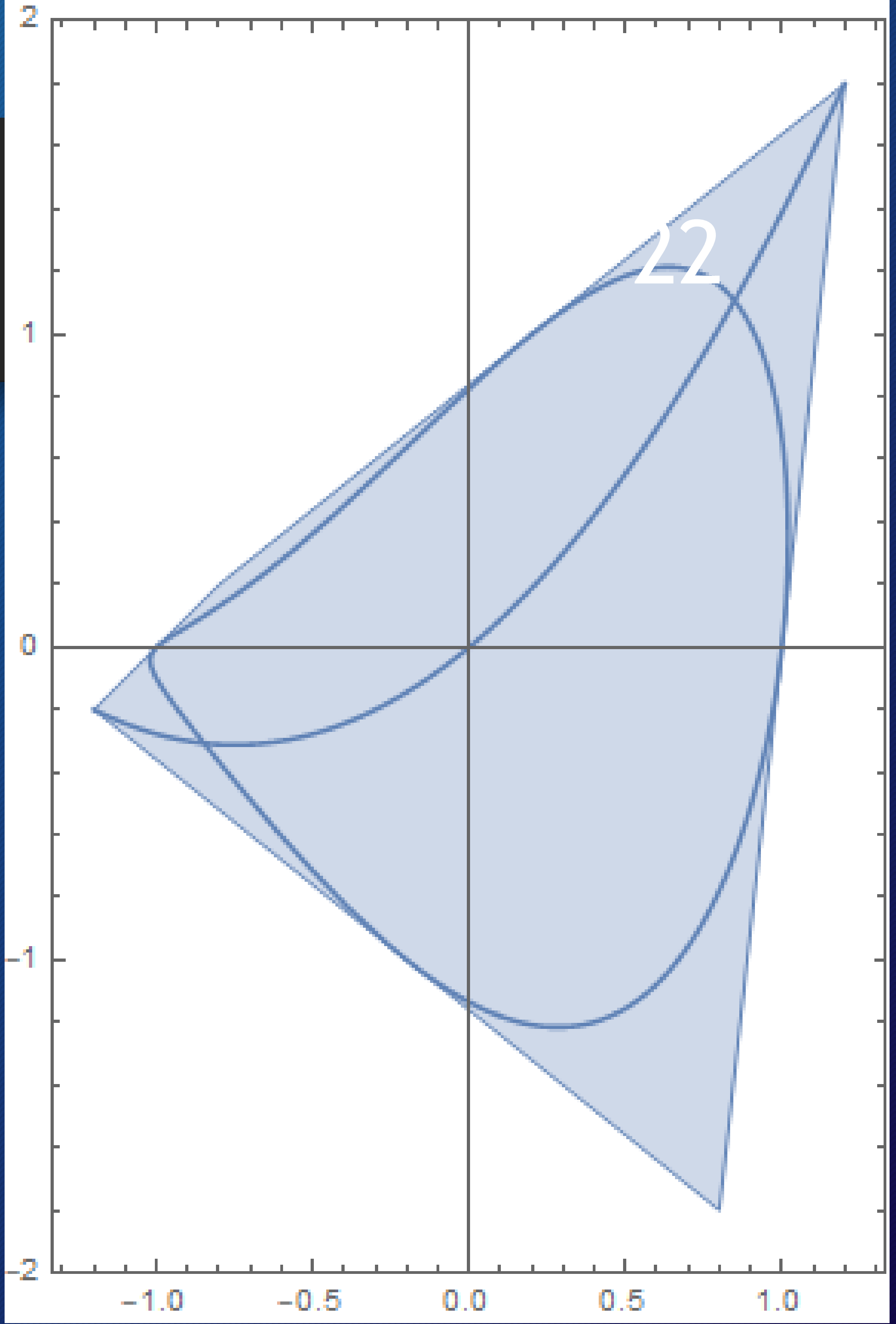


- Consider a simple 2-D deformation function,

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, \alpha) = (X_1 + \alpha X_2)\mathbf{e}_1 + \frac{X_2}{\alpha}\left(4X_1 + \frac{1}{\alpha}\right)\mathbf{e}_2$$

For any $\alpha \neq 0$, the prescribed parameter signifies a specific deformation. Let $\alpha = 0.2$ then, the above deformation can be plotted as shown. The picture here is generated by the Mathematica code:

```
myMap[X1_, X2_, α_] := {X1 + X2 α , X2 (4 X1 + 1 / α) α} // Flatten;
α = .2;
deformedConfig[α_] = ParametricPlot[myMap[X1, X2, α], {X1, -1, 1}, {X2, -1, 1}];
deformedCircle[α_] = ParametricPlot[myMap[Sin[t], Cos[t], α], {t, 0, 2 Pi}];
dl1[α_] = ParametricPlot[myMap[X1, X1, α], {X1, -1, 1}];
Show[deformedConfig[α], deformedCircle[α], dl1[α],  PlotRange → All]
```



# Simple Motion, Reference Map



Consider, for example, the motion,

$$\mathbf{x} = \chi(\mathbf{X}, t) = (tX_1 + k_1X_2)\mathbf{e}_1 + (k_2X_1 + tX_2)\mathbf{e}_2 + t\mathbf{e}_3$$

- Where $k_1$ and $k_2$ are constants, and $t$ is the time variable. To obtain the reference map, we can invert this function and obtain,

$$\mathbf{X} = \chi^{-1}(\mathbf{x}, t) = \frac{tx_1 - k_1x_2}{t^2 - k_1k_2}\mathbf{E}_1 + \frac{tx_2 - k_2x_1}{t^2 - k_1k_2}\mathbf{E}_2 + \frac{x_3}{t}\mathbf{E}_3$$

```
Solve[{x1 == t X1 + X2 k1, x2 == k2 X1 + X2 t, x3 == t X3}, {X1, X2, X3}]
```

$$\left\{\left\{X_1 \to -\frac{-t\,x_1 + k_1\,x_2}{t^2 - k_1\,k_2},\ X_2 \to -\frac{k_2\,x_1 - t\,x_2}{t^2 - k_1\,k_2},\ X_3 \to \frac{x_3}{t}\right\}\right\}$$

# Examples of Simple Motions



The following examples of simple motions have been named:

- **Pure translation**, $\chi(\mathbf{X}, t) = \mathbf{X} + \mathbf{c}(t)$, where $\mathbf{c}$ is a differentiable vector-valued function of time.
- **Pure rotation**, $\chi(\mathbf{X}, t) = \mathbf{Q}(t)\mathbf{X}$, where $\mathbf{Q}$ is a *proper orthogonal* function. (A way of saying that it is a *rotation* function of time).
- **Simple Shear**. $\chi(\mathbf{X}, t) = (\mathbf{I} + \alpha(t)\mathbf{e}_1 \otimes \mathbf{e}_2)\mathbf{X}$, where $\alpha$ is a differentiable, scalar valued function of time. Q: Transpose the dyad and what do you get? Compare to the original shear motion.

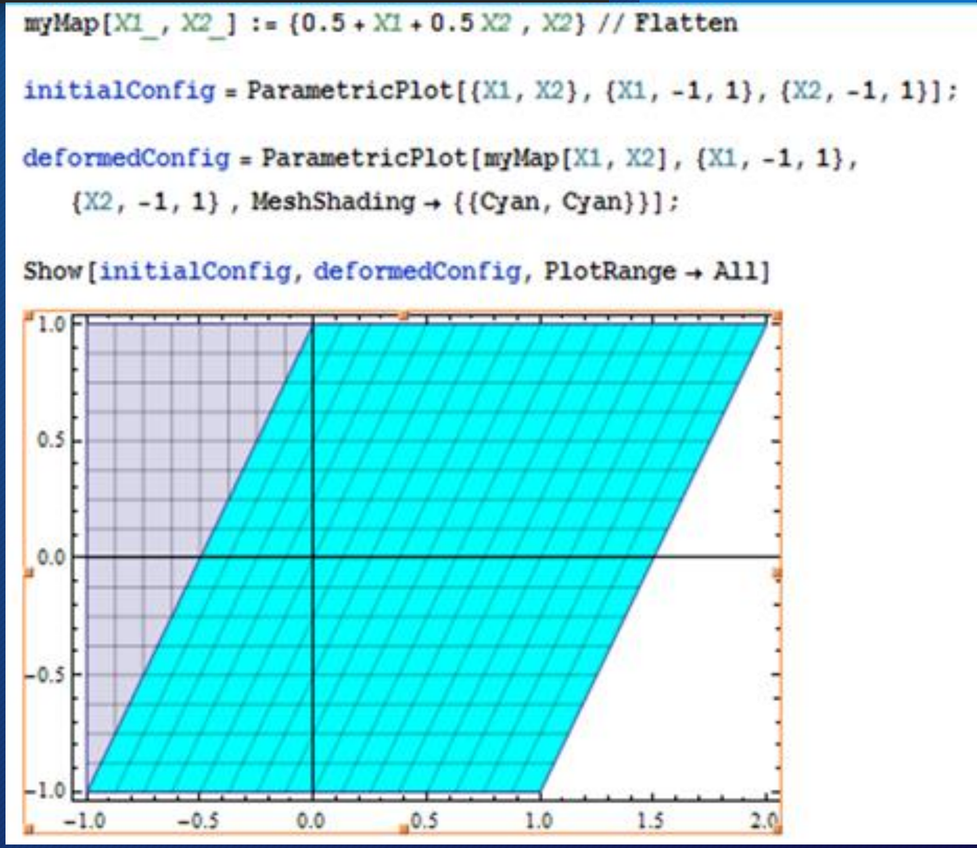
# Simple Shear



- The Mathematica graphic is about Uniform Shear. The Deformation gradient here is easily calculated by hand. Do this to ensure you don't get lost in the mechanical computation and lose the context:

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = (0.5 + X_1 + 0.5X_2)\mathbf{e}_1 + X_2\mathbf{e}_2$$

- for the element occupying $X_1\mathbf{E}_1 + X_2\mathbf{E}_2$ initially.



# Deformation Gradient of Simple Shear



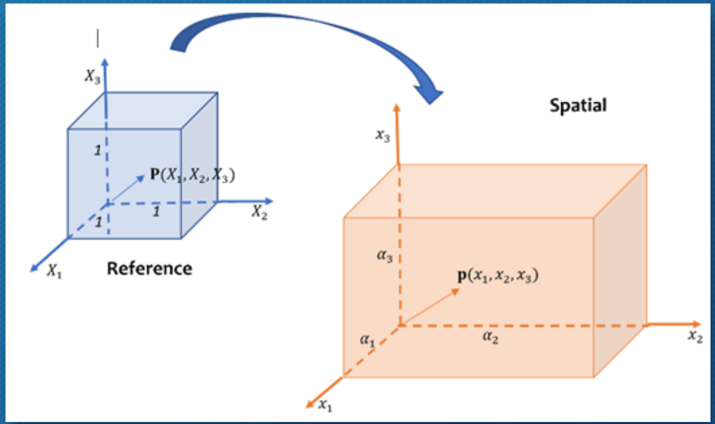
$\frac{\partial x_1}{\partial X_1} = 1, \frac{\partial x_1}{\partial X_2} = 0.5, \frac{\partial x_1}{\partial X_3} = 0$ and $\frac{\partial x_2}{\partial X_2} = \frac{\partial x_3}{\partial X_3}$ with all other components of the deformation gradient vanishing.

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

- is the matrix of the deformation gradient components.

# Uniform Extension





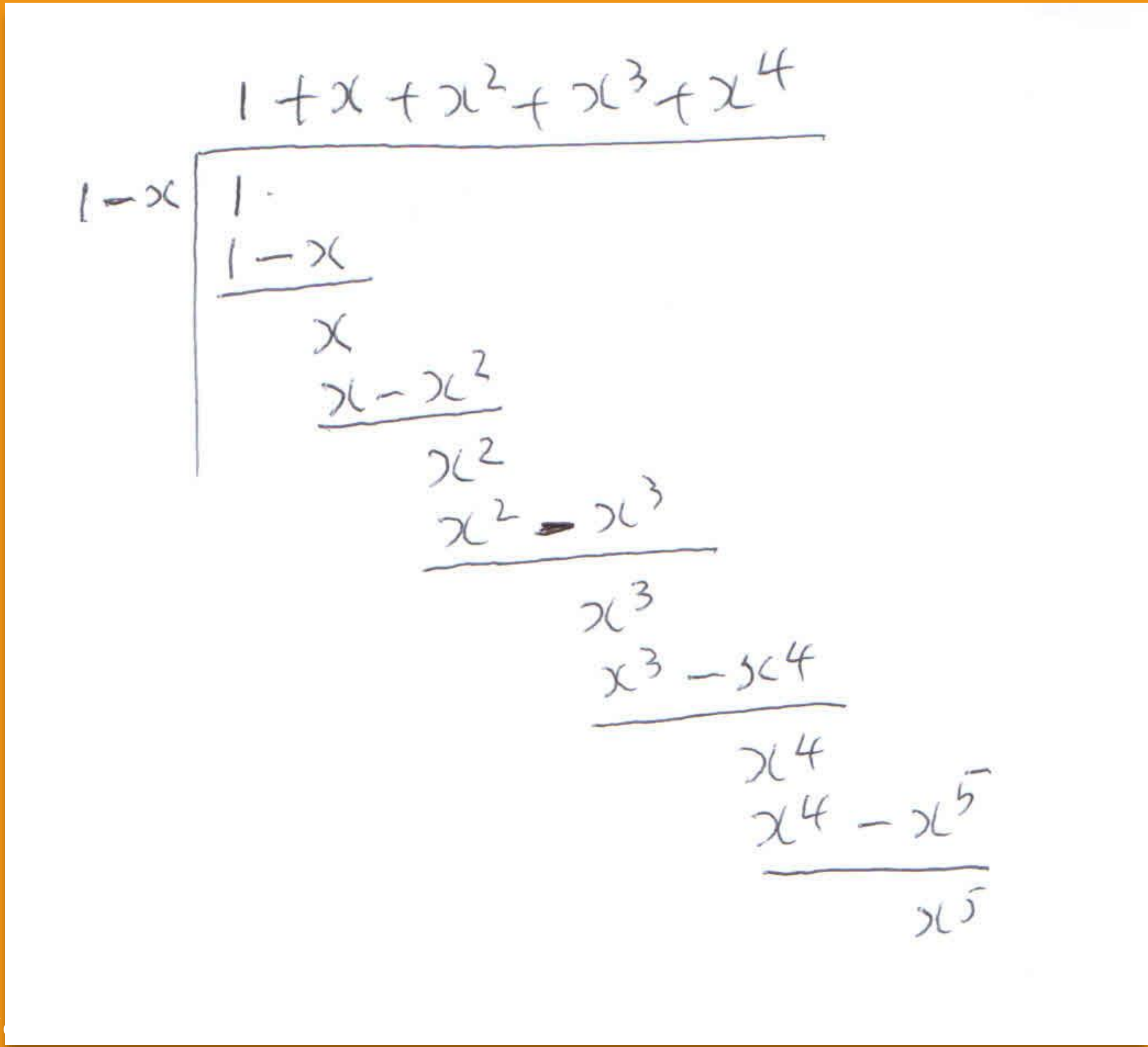

- Consider the unit cube with the transformation vector:
$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + \alpha_2 X_2 \mathbf{e}_2 + \alpha_3 X_3 \mathbf{e}_3$$
- Note that uniaxial extension can be obtained by allowing two of the constants to be unity while biaxial will be ensured by one of the constants becoming one as follows:
- Uniaxial: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha_1 X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3$
- Biaxial: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = X_1 \mathbf{e}_1 + \alpha_2 X_2 \mathbf{e}_2 + \alpha_3 X_3 \mathbf{e}_3$
- Pure Dilatation: $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}) = \alpha(X_1 \mathbf{e}_1 + X_2 \mathbf{e}_2 + X_3 \mathbf{e}_3)$

# The $(1-x)^{-1}$ Riddle

To keep you awake in class!

# Primary 5: Long Division

Long Division




oafak@s2pafrica.org; oafak@unilag.edu.ng 

# Foolish Crammer

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

- No thought, no sense just memorize and give back to the teacher when asked!
- Something a Primary 5 pupil can achieve! Simple Long division!

# Senior Secondary School 3: Binomial Theorem



$$(a+b)^n = a^n b^0 + na^{n-1}b^1 + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

- For example

$$(a+b)^3 = a^3 b^0 + 3a^{3-1}b^1 + \frac{3(3-1)}{2!}a^{3-2}b^2 + \frac{3(3-1)(3-2)}{3!}a^{3-3}b^3 + \cdots$$

$$= a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(1-x)^{-1} = 1^{-1}(-x)^0 + (-1)1^{-2}(-x)^1 + \frac{-1(-2)}{2!}1^{-3}(-x)^2 + \frac{-1(-2)(-3)}{3!}1^{-4}(-x)^3$$

$$+ \frac{-1(-2)(-3)(-4)}{4!}1^{-5}(-x)^4 + \frac{-1(-2)(-3)(-4)(-5)}{5!}1^{-6}(-x)^5 \ldots$$

- Taylor Series near the point $x = a$,

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \cdots + \frac{(x-a)^n}{n!}f^n(a) + \cdots$$

Here we are evaluating the function, $f(x) = \frac{1}{1-x}$ near the point, $x = 0$.

$$f'(x) = \frac{1}{1-x} = -1(1-x)^{-2}(-1) \Rightarrow f'(0) = 1$$

$$f''(x) = \frac{d}{dx}\left(\frac{1}{1-x}\right)^2 = -2(1-x)^{-3}(-1) \Rightarrow f''(0) = 2 \times 1 = 2!$$

It is not difficult to show that $f^n(0) = n!$ so that, as before,

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

# Mathematica Compliant Engineer



```
Series[1/(1 - x), {x, 0, 10}]
```

$$1+x+x^2+x^3+x^4+x^5+x^6+x^7+x^8+x^9+x^{10}+O[x]^{11}$$

# But is the expansion correct?



- For example, let us try $x = 2$ and find the answer given by the expansion:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + x^5 + \cdots$$

- So that,

$$\begin{aligned}(1-2)^{-1} &= 1 + 2 + 2^2 + 2^3 + 2^4 + 2^5 + \cdots \\ &= 1 + 2 + 4 + 8 + 16 + 24 + \cdots\end{aligned}$$

- If at this point, you do not know what is happening and cannot solve this riddle, you are a foolish crammer! You have learned nothing here! Matters not what marks you got!

# Solution to Riddle



Parenthetical Issues:

- Convergence properties of Infinite Series
- Remainder estimate of Taylor expansion
- You did not bother cramming those.

It turns out that they are far more important than cramming the formula. See how easy it is for software to spit out the result. If I need the result in my business, I will rather buy the software than hire you! I need someone that understands how to interpret the results and knows when they are valid! Crammers are more than useless to any serious business!

# The Girl & Boy



- There are several skills that can be picked up this semester:

1. Solid Modeling
2. Animation
3. Simulation
4. Computation
5. Design & Analysis

Continuum Mechanics is to give you insight to the theoretical underpinnings of these. Memorization is OK. Insight and understanding are more important. When you choose cramming over insight, you are self-immolating!

Each of these, on their own, if you were to master them, not only can fetch your daily bread, can also make you an asset to your family, your people and your country. I am afraid not many seem to realize that!


The End

# Polar Decomposition Theorem

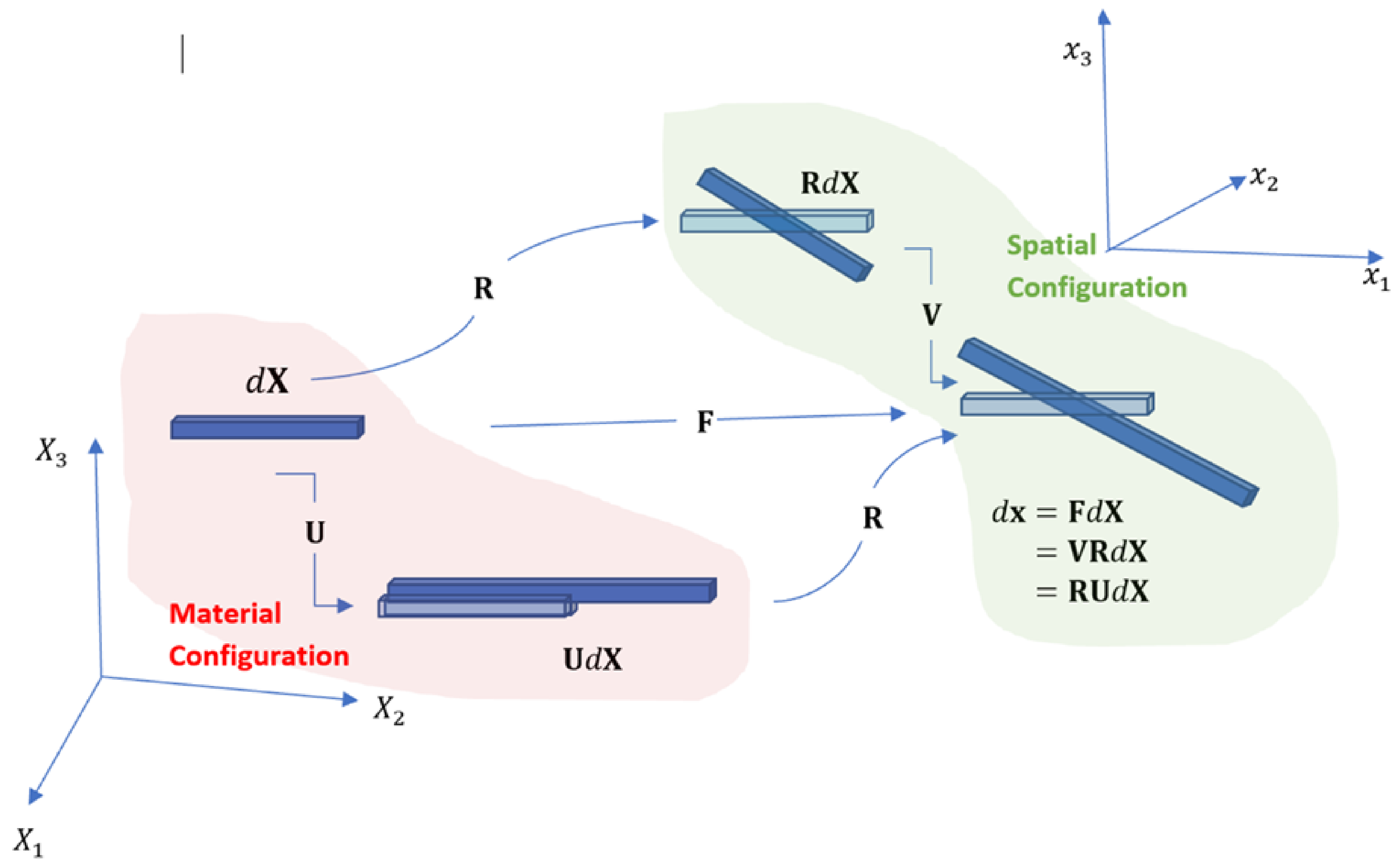
- In this section, we are looking at a multiplicative decomposition, motivated by the reality that it is NOT the entire transformation wrought by the deformation gradient that concerns us in the study of geometrical changes resulting from the application of loads.
- It successfully separates portions of the deformation gradient that do not cause shape changes from the parts that are relevant in geometric modifications resulting from the transformation.

*For a given deformation gradient* $\mathbf{F}$, *there is a unique rotation tensor* $\mathbf{R}$, *and unique, positive definite, symmetric tensors* $\mathbf{U}$ *and* $\mathbf{V}$ *for which,* $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$

- $\mathbf{U}$ *is called the Right Stretch Tensor, and* $\mathbf{V}$ *the Left Stretch Tensor.*

- The proof of this important theorem is given subsequently. More important though is to know the meaning:
- Beginning from any material configuration, the transformation given by the deformation gradient leads to the spatial configuration. However, this transformation can be achieved in two two-stage processes.
  - A stretch in the material configuration through the Right Stretch Tensor $\mathbf{U}$; followed by a rotation by the rotation tensor $\mathbf{R}$ to the spatial configuration. Note that the rotation tensor is neither a material nor a spatial tensor. It is, like the deformation gradient, a two-toe tensor; operating on a material vector and producing a spatial tensor.
  - A transformation to the spatial configuration by the rotation tensor $\mathbf{R}$, followed by a stretch to the final state in that configuration by the left stretch tensor. The latter is a spatial tensor as it takes a spatial vector (output of the rotation tensor), and returns a spatial vector.

# Meaning of Polar Decomposition



$x_3$
$x_2$
$x_1$
$\mathbf{R}d\mathbf{X}$
Spatial
Configuration
$\mathbf{R}$
$\mathbf{V}$
$d\mathbf{X}$
$\mathbf{F}$
$X_3$
$\mathbf{U}$
$\mathbf{R}$
$d\mathbf{x} = \mathbf{F}d\mathbf{X}$
$= \mathbf{VR}d\mathbf{X}$
$= \mathbf{RU}d\mathbf{X}$
Material
Configuration
$\mathbf{U}d\mathbf{X}$
$X_2$
$X_1$

# Polar Decomposition: Proof



There are two stages to establish the Polar Decomposition of the Deformation Gradient:

1. We show that the Right Cauchy-Green Tensor,
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$$
is symmetric and positive definite.
2. Use this fact to find $\mathbf{U}$, such that $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}\mathbf{U}$.
$\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$
3. Then find $\mathbf{V}$ in terms of $\mathbf{U}$.

# Symmetry & Positive Definiteness of $\mathbf{C}$

- It is obvious that $\mathbf{C} = \mathbf{F}^\mathrm{T}\mathbf{F}$ is symmetric because its transpose is
$$\mathbf{C}^\mathrm{T} = \mathbf{F}^\mathrm{T}\mathbf{F} = \mathbf{C}$$
- Now select ANY real non-zero vector $\mathbf{u}$. We can find a vector $\mathbf{b} = \mathbf{Fu}$.

  The quadratic form,
$$\mathbf{u} \cdot \mathbf{F}^\mathrm{T}\mathbf{Fu} = \mathbf{b} \cdot \mathbf{Fu} = ||\mathbf{b}||^2 > 0$$
  Since we selected $\mathbf{u}$ arbitrarily, and we have reached the conclusion that any quadratic form with $\mathbf{C}$ is always greater than zero, we have proved that $\mathbf{C}$ is positive definite.

# Square Root of $\mathbf{C}$



- Every Positive Definite Tensor has a positive definite square root: The right Cauchy-Green Tensor

$$\mathbf{UU} = \mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{T}}\,\mathbf{R}^{\mathrm{T}}\mathbf{R}\,\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{I}\,\mathbf{U} = \mathbf{U}^{2}$$

- Shows that $\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = (\mathbf{RU})^{\mathrm{T}}\mathbf{RU}$ so that $\mathbf{F} = \mathbf{RU}$. To complete the proof, write,

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}$$

and we immediately find that $\mathbf{V} = \mathbf{RUR}^{\mathrm{T}}$ by simply post-multiplying the above by $\mathbf{R}^{\mathrm{T}}$. Since $\mathbf{U}$ is positive definite, so must $\mathbf{V}$. Obvious, if we consider arbitrary real vectors $\mathbf{u}, \mathbf{v}$ such that $\mathbf{v} \equiv \mathbf{R}^{\mathrm{T}}\mathbf{u}$,

$$\mathbf{u} \cdot \mathbf{Vu} = \mathbf{u} \cdot \mathbf{RUR}^{\mathrm{T}}\mathbf{u} = \mathbf{u} \cdot \mathbf{RUv} = \mathbf{v} \cdot \mathbf{UR}^{\mathrm{T}}\mathbf{u} = \mathbf{v} \cdot \mathbf{Uv} > 0$$

# Right Cauchy Green Tensor



- In the proof of Polar Decomposition Theorem, we encounter another important tensor: The Right Cauchy-Green Tensor
$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \mathbf{U}^{\mathrm{T}}\,\mathbf{R}^{\mathrm{T}}\mathbf{R}\,\mathbf{U} = \mathbf{U}^{\mathrm{T}}\mathbf{I}\,\mathbf{U} = \mathbf{U}^{2}$$
- It is said to be “Right” because there is a “Left” Cauchy-Green Tensor: $\mathbf{F}\mathbf{F}^{\mathrm{T}}$ that can be obtained by the product of the deformation gradient and its transpose.
- In the former, the deformation gradient is at the right hand; in the latter, it is at the left side - hence the distinguishing names. It is easily shown that,
$$\mathbf{V}^{2} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$$

- Remember that referential and spatial configurations DO NOT coexist. Bringing vectors from either state together is artificial, not real.
- Tensors that come to our attention are classified by what kinds of arguments they can take and what kind of vectors they produce.
- On the other hand, vectors are classified by where they reside. For example, the material vector is so called because it is made up of elements in the referential (material) configuration. Spatial tensors are similarly defined.

# Referential (Material) & Spatial Vectors, Tensors

# Material, Spatial & Two-Towed Tensors



- A tensor that takes a material argument and produces a material result is defined as a material tensor.
    - Right Cauchy Green Tensor, Right Stretch Tensor
- A tensor that takes a spatial argument and produces a spatial result is defined as a spatial tensor.
    - Left Cauchy-Green Tensor, Left Stretch Tensor
- A Tensor that takes a material argument and produces a spatial result transforms a vector from the referential state to an image in the spatial configuration.
    - Examples: Deformation Gradient, Rotation Tensor

# Determine the Type of Tensor



- Consider a spatial vector $\mathbf{s}$. The dot product $\mathbf{s} \cdot d\mathbf{x}$ has physical significance while $\mathbf{s} \cdot d\mathbf{X}$ does not as the two operands do not exist at the same time so an operation between them makes no physical sense. Clearly,

$$\mathbf{s} \cdot d\mathbf{x} = \mathbf{s} \cdot \mathbf{F}d\mathbf{X} = d\mathbf{X} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{s}$$

- meaning that $\mathbf{F}^{\mathrm{T}}\mathbf{s}$ is a material vector so that $\mathbf{F}^{\mathrm{T}}$ transforms spatial vectors to material. Beginning with a material vector $\mathbf{t}$. The physically meaningful product,

$$\mathbf{t} \cdot d\mathbf{X} = \mathbf{t} \cdot \mathbf{F}^{-1}d\mathbf{x} = d\mathbf{x} \cdot \mathbf{F}^{-\mathrm{T}}\mathbf{t}$$

- Showing that $\mathbf{F}^{-\mathrm{T}}$, just like $\mathbf{F}$, transforms material to spatial while $\mathbf{F}^{-1}$ like $\mathbf{F}^{\mathrm{T}}$ transforms spatial vectors to material. These tensors are two-toed.

# Area Ratio



- For an element of area $d\mathbf{a}$ in the deformed body with a vector $d\mathbf{x}$ projecting out of its plane (does not have to be normal to it). For the elemental volume, we have the following relationship:

$$dv = JdV = d\mathbf{a} \cdot d\mathbf{x} = Jd\mathbf{A} \cdot d\mathbf{X}$$

- where $d\mathbf{A}$ is the element of area that transformed to $d\mathbf{a}$ and $d\mathbf{X}$ is the image of $d\mathbf{x}$ in the undeformed material. Noting that, $d\mathbf{x} = \mathbf{F}d\mathbf{X}$ we have,

$$d\mathbf{a} \cdot \mathbf{F}d\mathbf{X} - Jd\mathbf{A} \cdot d\mathbf{X} = 0$$

$$= \left(\mathbf{F}^{\mathrm{T}}d\mathbf{a} - Jd\mathbf{A}\right) \cdot d\mathbf{X}$$

# Area Ratio

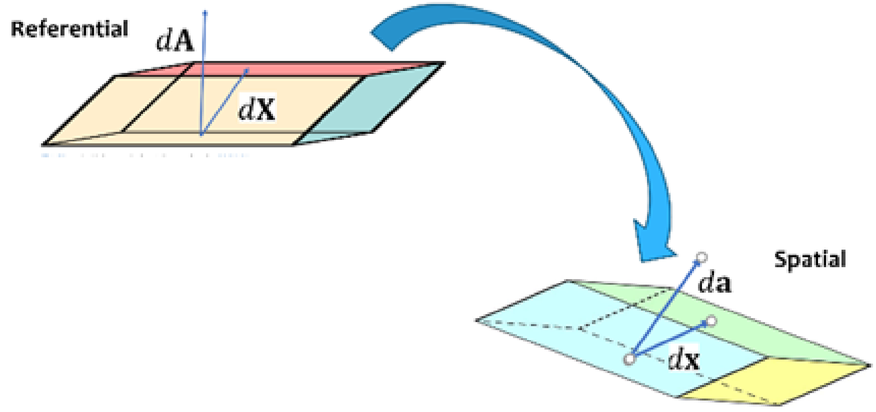
- For an arbitrary vector $d\mathbf{X}$, we have:

$$\mathbf{F}^{\mathrm{T}} d\mathbf{a} - J d\mathbf{A} = \mathbf{o}$$

- so that,

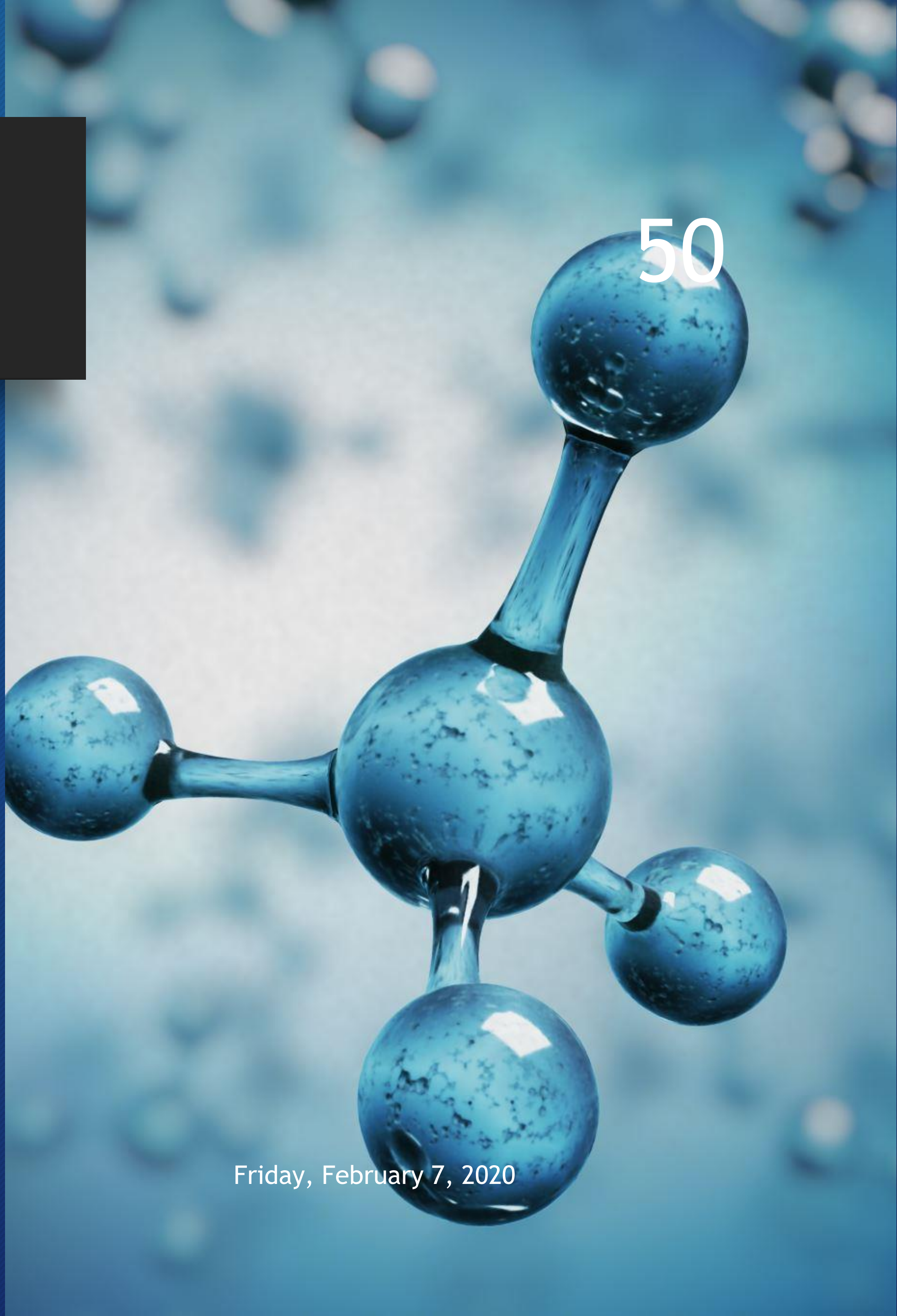
$$d\mathbf{a} = J\mathbf{F}^{-\mathrm{T}} d\mathbf{A} = \mathbf{F}^{\mathrm{c}} d\mathbf{A}$$

- where $\mathbf{F}^{\mathrm{c}}$ is the cofactor tensor of the deformation gradient.



# Deriving Reciprocal Bases

- Reciprocal bases are easily derived. The dyad product of a reciprocal bases with the natural bases yield the identity tensor.
- In Cartesian systems, the natural and the reciprocal bases coincide. In curvilinear coordinates such as Cylindrical and Spherical Polar, this is not so. For example, for spherical polar, the reciprocal basis can be derived from the natural basis (obtained by differentiating the position vector), using superscript to represent the reciprocal bases, as follows:



# Spherical Reciprocal Bases



$$\begin{pmatrix} \mathbf{e}^1 \\ \mathbf{e}^2 \\ \mathbf{e}^3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\mathbf{e}_1 \cdot \mathbf{e}_1} & 0 & 0 \\ 0 & \frac{1}{\mathbf{e}_2 \cdot \mathbf{e}_2} & 0 \\ 0 & 0 & \frac{1}{\mathbf{e}_3 \cdot \mathbf{e}_3} \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \rho \mathbf{e}_\theta \\ \rho \sin\theta \, \mathbf{e}_\phi \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & \frac{1}{\rho^2 \sin^2\theta} \end{pmatrix} \begin{pmatrix} \mathbf{e}_\rho \\ \rho \mathbf{e}_\theta \\ \rho \sin\theta \, \mathbf{e}_\phi \end{pmatrix} = \begin{pmatrix} \mathbf{e}_\rho \\ \frac{\rho \mathbf{e}_\theta}{\rho^2} \\ \frac{\rho \sin\theta \, \mathbf{e}_\phi}{\rho^2 \sin^2\theta} \end{pmatrix} = \begin{pmatrix} \mathbf{e}_\rho \\ \frac{\mathbf{e}_\theta}{\rho} \\ \frac{\mathbf{e}_\phi}{\rho \sin\theta} \end{pmatrix}$$

# Natural & Reciprocal Bases



- This can be repeated for other coordinate systems, e.g. the cylindrical. The results can be summarized as follows:

| Coordinate System | Natural Basis Vectors | Reciprocal Base Vectors |
|---|---|---|
| Cartesian | $\left\{\frac{\partial \mathbf{r}}{\partial x_1} = \mathbf{e}_1;\ \frac{\partial \mathbf{r}}{\partial x_2} = \mathbf{e}_2;\ \frac{\partial \mathbf{r}}{\partial x_3} = \mathbf{e}_3\right\}$ | $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ |
| Cylindrical Polar | $\left\{\frac{\partial \mathbf{r}}{\partial r} = \mathbf{e}_r;\ \frac{\partial \mathbf{r}}{\partial \phi} = r\mathbf{e}_\phi;\ \frac{\partial \mathbf{r}}{\partial z} = \mathbf{e}_z\right\}$ | $\left\{\mathbf{e}_r;\ \frac{\mathbf{e}_\phi}{r};\ \mathbf{e}_z\right\}$ |
| Spherical Polar | $\left\{\frac{\partial \mathbf{r}}{\partial \rho} = \mathbf{e}_\rho;\ \frac{\partial \mathbf{r}}{\partial \theta} = \rho\mathbf{e}_\theta;\ \frac{\partial \mathbf{r}}{\partial \phi} = \rho \sin\theta \mathbf{e}_\phi\right\}$ | $\left\{\mathbf{e}_\rho;\ \frac{\mathbf{e}_\theta}{\rho};\ \frac{\mathbf{e}_\phi}{\rho \sin\theta}\right\}$ |

# Deformation Gradient: Cartesian to Cartesian



- If the Referential State is based on Cartesian unit vectors, $\mathbf{E}_i, i = 1,2,3$ and we have $\mathbf{e}_\alpha, \alpha = 1,2,3$ for the spatial state. Deformation takes the form, $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t) = \boldsymbol{\chi}(X_1, X_2, X_3, t)$ in this case, the Deformation Gradient,

$$\mathbf{F} = \frac{\partial \chi_\alpha}{\partial X_j} \mathbf{e}_\alpha \otimes \mathbf{E}_j$$

$$= (\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3) \begin{bmatrix} \frac{\partial \chi_1}{\partial X_1} & \frac{\partial \chi_1}{\partial X_2} & \frac{\partial \chi_1}{\partial X_3} \\ \frac{\partial \chi_2}{\partial X_1} & \frac{\partial \chi_2}{\partial X_2} & \frac{\partial \chi_2}{\partial X_3} \\ \frac{\partial \chi_3}{\partial X_1} & \frac{\partial \chi_3}{\partial X_2} & \frac{\partial \chi_3}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}.$$

# Deformation Gradient: Cylindrical to Cylindrical



- In the case of orthogonal systems, linear or curvilinear, this relationship becomes simply dividing by the magnitude of the respective natural base vector. The deformation gradient from a material configuration in cylindrical Polar coordinates $\{R, \Theta, Z\}$ to a spatial configuration $\{r, \theta, z\}$ in the same coordinate system is,

$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ \frac{\partial \theta}{\partial R} & \frac{\partial \theta}{\partial \Theta} & \frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial R} & \frac{1}{R}\frac{\partial r}{\partial \Theta} & \frac{\partial r}{\partial Z} \\ r\frac{\partial \theta}{\partial R} & \frac{r}{R}\frac{\partial \theta}{\partial \Theta} & r\frac{\partial \theta}{\partial Z} \\ \frac{\partial z}{\partial R} & \frac{1}{R}\frac{\partial z}{\partial \Theta} & \frac{\partial z}{\partial Z} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_R \\ \mathbf{E}_\Theta \\ \mathbf{E}_Z \end{bmatrix}.$$

- We used upper case to depict the Material system. It is the reciprocal system.

# Deformation Gradient: Spherical to Spherical



- If both were spherical, $\{\varrho, \Theta, \Phi\} \to \{\rho, \theta, \phi\}$ the deformation gradient becomes,
- $$\mathbf{F} = (\mathbf{e}_\rho \quad \rho\mathbf{e}_\theta \quad \rho\sin\theta\,\mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial\varrho} & \frac{\partial\rho}{\partial\Theta} & \frac{\partial\rho}{\partial\Phi} \\ \frac{\partial\theta}{\partial\varrho} & \frac{\partial\theta}{\partial\Theta} & \frac{\partial\theta}{\partial\Phi} \\ \frac{\partial\phi}{\partial\varrho} & \frac{\partial\phi}{\partial\Theta} & \frac{\partial\phi}{\partial\Phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_\varrho \\ \frac{\mathbf{E}_\Theta}{\varrho} \\ \frac{\mathbf{E}_\Phi}{\varrho\sin\Theta} \end{bmatrix} = (\mathbf{e}_\rho \quad \mathbf{e}_\theta \quad \mathbf{e}_\phi)\begin{bmatrix} \frac{\partial\rho}{\partial\varrho} & \frac{1}{\varrho}\frac{\partial\rho}{\partial\Theta} & \frac{1}{\varrho\sin\Theta}\frac{\partial\rho}{\partial\Phi} \\ \rho\frac{\partial\theta}{\partial\varrho} & \frac{\rho}{\varrho}\frac{\partial\theta}{\partial\Theta} & \frac{\rho}{\varrho\sin\Theta}\frac{\partial\theta}{\partial\Phi} \\ \rho\sin\theta\frac{\partial\phi}{\partial\varrho} & \frac{\rho\sin\theta}{\varrho}\frac{\partial\phi}{\partial\Theta} & \frac{\rho\sin\theta}{\varrho\sin\Theta}\frac{\partial\phi}{\partial\Phi} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_\varrho \\ \mathbf{E}_\Theta \\ \mathbf{E}_\Phi \end{bmatrix}.$$

# Coordinate System Pairings



- We are completely free to represent the initial coordinate system any way we like.
  - We can therefore have Cartesian to Spherical Polar or Spherical Polar to Cylindrical Polar transformations.
- It is a matter of which system best describes the deformation transformations as will be shown in examples.

# Illustrative Example

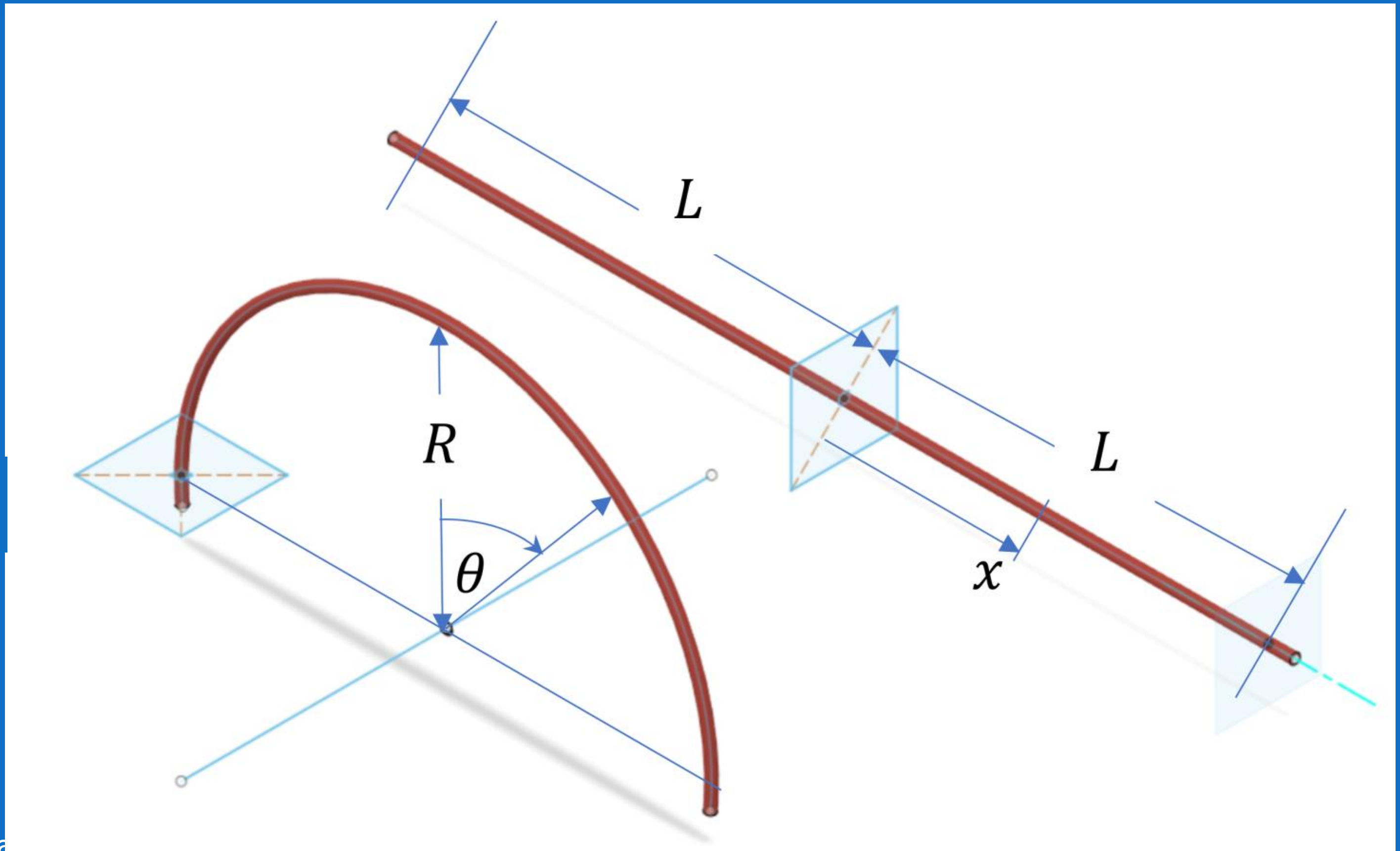


- Consider the length $2L$ of a thin rod uniformly bent into a semicircle as shown.
- Referential configuration is the straight rod, Spatial, after the bending, is the semi-circular rod. If the rod's length does not increase as a result of shape change, then $\pi R = 2L$. Clearly, radius $R = 2L/\pi$
- A point previously located at the distance $x$ from the origin is now at angle $\theta$. The relationship between the two is linear:

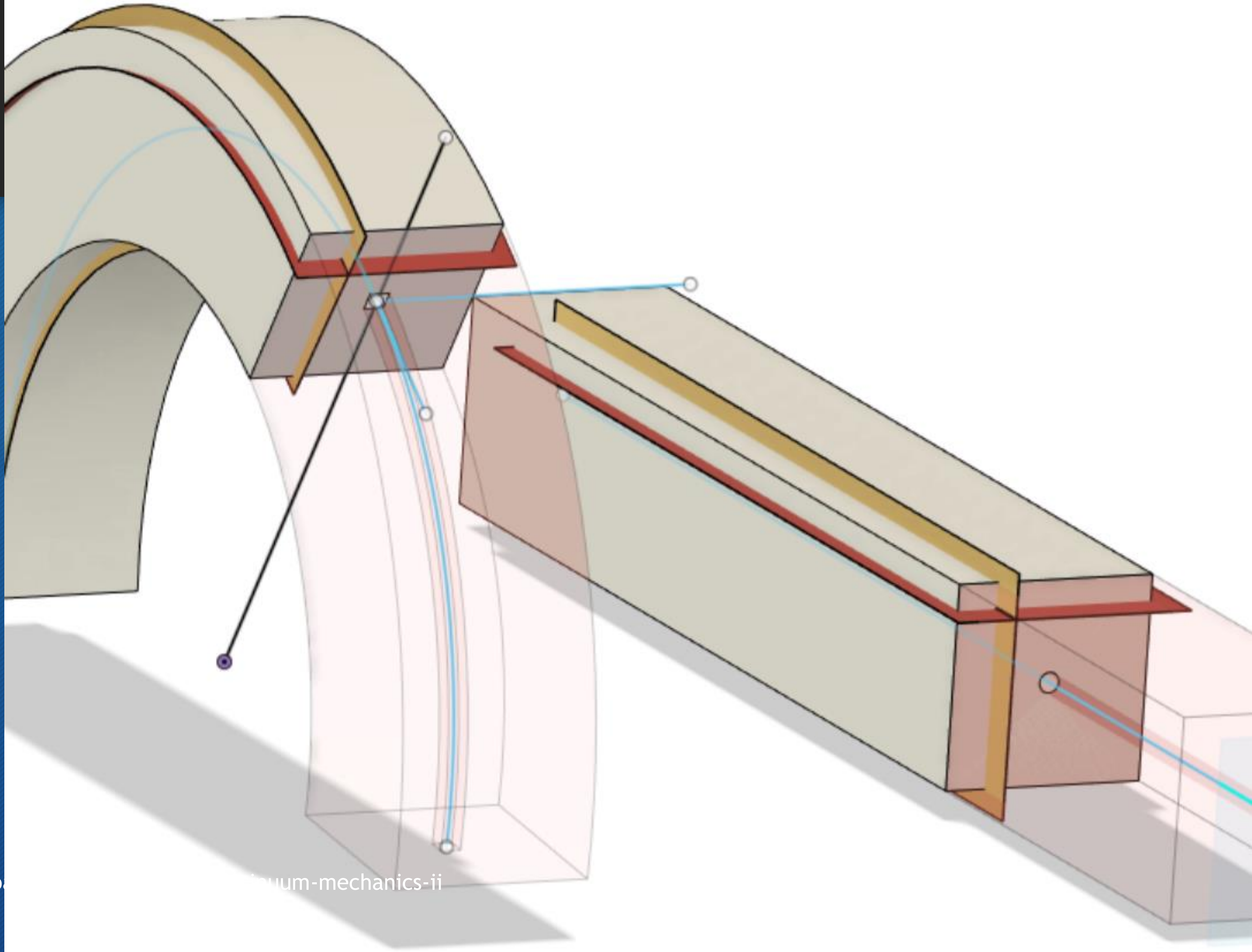
$$\frac{x}{2L} = \frac{\theta}{\pi} \Rightarrow \theta = \frac{\pi x}{2L}$$

How else can you obtain this formula?

| $x$ | $0$ | $\frac{L}{2}$ | $L$ | $-L$ |
|---|---|---|---|---|
| $\theta$ | $0$ | $\frac{\pi}{4}$ | $\frac{\pi}{2}$ | $-\frac{\pi}{2}$ |

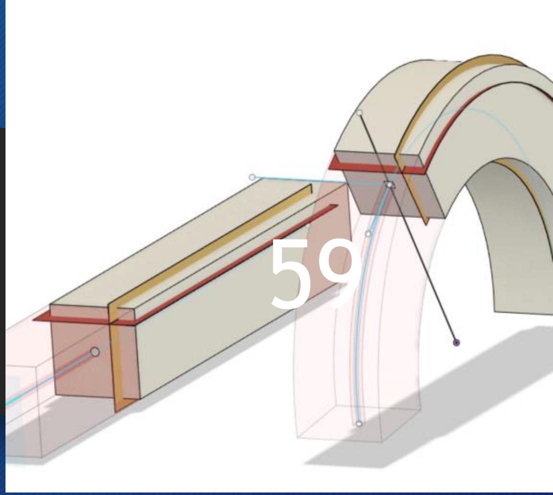


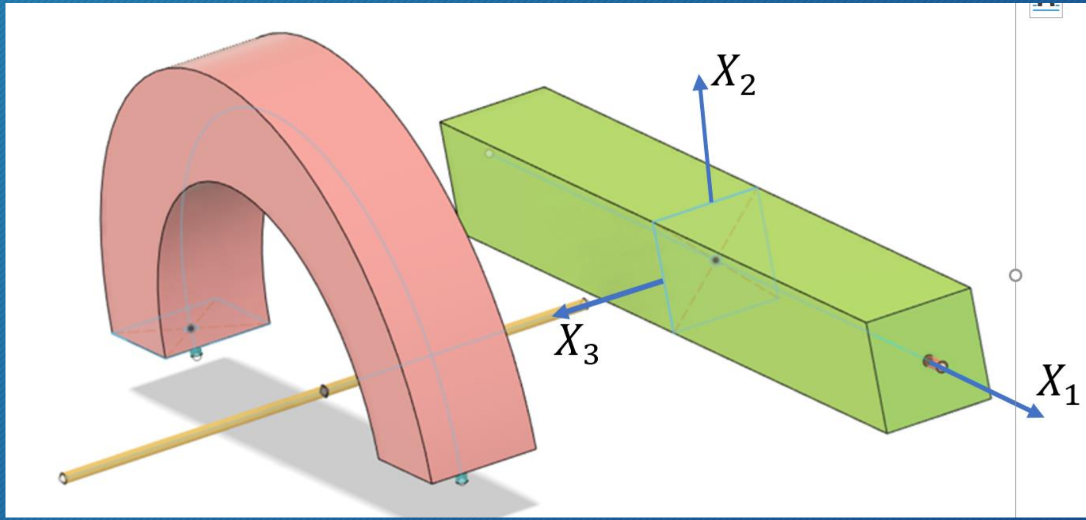
# Bar to Semicircular Region



# Bar to Semicircular Region








Take $X_1$ axis along the bar, $X_2$ vertically, and $X_3$ along the axis of bending.

- Imagine that we bent the bar shown into a semicircular region. Transformation function can be found by the following consideration: Note that each horizontal filament in the original bar becomes a circular filament in the spatial configuration. The vertical undeformed sections become radial sections in the spatial state. Let the centerline be a semicircle at a distance $R$ and let the thickness contract uniformly with a factor $\alpha$

$$\Rightarrow x_1 = r = \chi_1(X_1, X_2, X_3, t) = R + \alpha X_2, \text{and}$$

$$x_2 = \theta = \chi_2(X_1, X_2, X_3, t) = \frac{\pi X_1}{2L}$$

- If the bar contracts uniformly in $X_3$ direction,

$$x_3 = z = \chi_3(X_1, X_2, X_3, t) = \beta X_3$$

# Referential & Spatial Configurations



- Clearly, the referential configuration here is the bar; Spatial is the semicircular bar.
- Deformation is such that the spatial is in cylindrical coordinates, the referential is in Cartesian.
- Deformation gradient requires the reciprocal Cartesian bases which are the same as the Cartesian. In the spatial, we use the cylindrical. The full computation given in Q4.7, is repeated here:

# Deformation Gradient



$$\mathbf{F} = (\mathbf{e}_r \quad r\mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} \frac{\partial r}{\partial X_1} & \frac{\partial r}{\partial X_2} & \frac{\partial r}{\partial X_3} \\ \frac{\partial \theta}{\partial X_1} & \frac{\partial \theta}{\partial X_2} & \frac{\partial \theta}{\partial X_3} \\ \frac{\partial z}{\partial X_1} & \frac{\partial z}{\partial X_2} & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & \frac{\partial r}{\partial X_2} & 0 \\ r\frac{\partial \theta}{\partial X_1} & 0 & 0 \\ 0 & 0 & \frac{\partial z}{\partial X_3} \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= (\mathbf{e}_r \quad \mathbf{e}_\theta \quad \mathbf{e}_z)\begin{bmatrix} 0 & \alpha & 0 \\ \frac{\pi r}{2L} & 0 & 0 \\ 0 & 0 & \beta \end{bmatrix} \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix} = \left(\frac{\pi r}{2L}\mathbf{e}_\theta \quad \alpha\mathbf{e}_r \quad \beta\mathbf{e}_z\right) \otimes \begin{bmatrix} \mathbf{E}_1 \\ \mathbf{E}_2 \\ \mathbf{E}_3 \end{bmatrix}$$

$$= \frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1 + \alpha\mathbf{e}_r \otimes \mathbf{E}_2 + \beta\mathbf{e}_z \otimes \mathbf{E}_3$$

# Right Cauchy-Green/Stretch Tensors



- Clearly,

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}}\mathbf{F} = \left(\frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{e}_\theta + \alpha \mathbf{E}_2 \otimes \mathbf{e}_r + \beta \mathbf{E}_3 \otimes \mathbf{e}_z\right)\left(\frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1 + \alpha \mathbf{e}_r \otimes \mathbf{E}_2 + \beta \mathbf{e}_z \otimes \mathbf{E}_3\right)$$

$$= \left(\frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{e}_\theta\right)\left(\frac{\pi r}{2L}\mathbf{e}_\theta \otimes \mathbf{E}_1\right) + \cdots + (\beta \mathbf{E}_3 \otimes \mathbf{e}_z)(\beta \mathbf{e}_z \otimes \mathbf{E}_3)$$

$$= \left(\frac{\pi r}{2L}\right)^2 \mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha^2 \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta^2 \mathbf{E}_3 \otimes \mathbf{E}_3$$

- since each set of basis vectors is orthonormal, and the Right Stretch Tensor,

$$\mathbf{U} = \frac{\pi r}{2L}\mathbf{E}_1 \otimes \mathbf{E}_1 + \alpha \mathbf{E}_2 \otimes \mathbf{E}_2 + \beta \mathbf{E}_3 \otimes \mathbf{E}_3$$

- Is the square root of the Right Cauchy Green tensor. The positive square roots are taken since both $\mathbf{C}$ as well as $\mathbf{U}$ are necessarily positive definite and can only have positive eigenvalues.

# Computing Functions in Cylindrical Systems



$$r = r(R,\Theta,Z) = \chi_r(R,\Theta,Z); \theta = \theta(R,\Theta,Z) = \chi_\theta(R,\Theta,Z); z = z(R,\Theta,Z) = \chi_z(R,\Theta,Z)$$

$$d\mathbf{x} = \frac{d\mathbf{x}}{d\mathbf{X}} d\mathbf{X} = \frac{d\boldsymbol{\chi}}{d\mathbf{X}} d\mathbf{X} = \mathbf{F}\, d\mathbf{x}$$

The spatial position vector, $\mathbf{x} = r\mathbf{e}_r(r,\theta) + z\mathbf{e}_z \Rightarrow$

$$d\mathbf{x} = \frac{\partial\mathbf{x}}{\partial r} dr + \frac{\partial\mathbf{x}}{\partial\theta} d\theta + \frac{\partial\mathbf{x}}{\partial z} dz = \mathbf{e}_r dr + r\frac{\partial\mathbf{e}_r(r,\theta)}{\partial\theta} d\theta + \mathbf{e}_z dz$$
$$= \mathbf{e}_r dr + r\mathbf{e}_\theta d\theta + \mathbf{e}_z dz$$

Similarly, in the Referential,

$$d\mathbf{X} = \frac{\partial\mathbf{X}}{\partial R} dR + \frac{\partial\mathbf{X}}{\partial\Theta} d\Theta + \frac{\partial\mathbf{X}}{\partial Z} dZ = \mathbf{E}_R dR + R\mathbf{E}_\Theta d\Theta + \mathbf{E}_Z dZ$$

# Cylindrical Deformation Gradient



$$d\mathbf{x} = \frac{d\boldsymbol{\chi}}{d\mathbf{X}} d\mathbf{X} = \mathbf{F}\, d\mathbf{X} = \left( \begin{pmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & \mathbf{e}_z \end{pmatrix} \begin{pmatrix} \frac{\partial \chi_r}{\partial R} & \frac{\partial \chi_r}{\partial \Theta} & \frac{\partial \chi_r}{\partial Z} \\ \frac{\partial \chi_\theta}{\partial R} & \frac{\partial \chi_\theta}{\partial \Theta} & \frac{\partial \chi_\theta}{\partial Z} \\ \frac{\partial \chi_z}{\partial R} & \frac{\partial \chi_z}{\partial \Theta} & \frac{\partial \chi_z}{\partial Z} \end{pmatrix} \otimes \begin{pmatrix} \mathbf{E}_R \\ \frac{\mathbf{E}_\Theta}{R} \\ \mathbf{E}_Z \end{pmatrix} \right) \begin{pmatrix} \mathbf{E}_R \\ R\mathbf{E}_\Theta \\ \mathbf{E}_Z \end{pmatrix}$$

So that the deformation gradient, in terms of unit vector sets $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ and $\{\mathbf{E}_R, \mathbf{E}_\Theta, \mathbf{E}_Z\}$, $\mathbf{F}$ can be written as,

$$\mathbf{F} = \begin{pmatrix} \frac{\partial \chi_r}{\partial R} & \frac{1}{R}\frac{\partial \chi_r}{\partial \Theta} & \frac{\partial \chi_r}{\partial Z} \\ r\frac{\partial \chi_\theta}{\partial R} & \frac{r}{R}\frac{\partial \chi_\theta}{\partial \Theta} & r\frac{\partial \chi_\theta}{\partial Z} \\ \frac{\partial \chi_z}{\partial R} & \frac{1}{R}\frac{\partial \chi_z}{\partial \Theta} & \frac{\partial \chi_z}{\partial Z} \end{pmatrix}$$