Preface

The calculus of variations has a long history of interaction with other branches of mathematics such as geometry and differential equations, and with physics, particularly mechanics. More recently, the calculus of variations has found applications in other fields such as economics and electrical engineering. Much of the mathematics underlying control theory, for instance, can be regarded as part of the calculus of variations.

This book is an introduction to the calculus of variations for mathematicians and scientists. The reader interested primarily in mathematics will find results of interest in geometry and differential equations. I have paused at times to develop the proofs of some of these results, and discuss briefly various topics not normally found in an introductory book on this subject such as the existence and uniqueness of solutions to boundary-value problems, the inverse problem, and Morse theory. I have made "passive use" of functional analysis (in particular normed vector spaces) to place certain results in context and reassure the mathematician that a suitable framework is available for a more rigorous study. For the reader interested mainly in techniques and applications of the calculus of variations, I leavened the book with numerous examples mostly from physics. In addition, topics such as Hamilton's Principle, eigenvalue approximations, conservation laws, and nonholonomic constraints in mechanics are discussed. More importantly, the book is written on two levels. The technical details for many of the results can be skipped on the initial reading. The student can thus learn the main results in each chapter and return as needed to the proofs for a deeper understanding. Several key results in this subject have tractable analogues in finite-dimensional optimization. Where possible, the theory is motivated by first reviewing the theory for finite-dimensional problems.

The book can be used for a one-semester course, a shorter course, or independent study. The final chapter on the second variation has been written with these options in mind, so that the student can proceed directly from Chapter 3 to this topic. Throughout the book, asterisks have been used to flag material that is not central to a first course.

The target audience for this book is advanced undergraduate/ beginning graduate students in mathematics, physics, or engineering. The student is assumed to have some familiarity with linear ordinary differential equations, multivariable calculus, and elementary real analysis. Some of the more theoretical material from these topics that is used throughout the book such as the implicit function theorem and Picard's theorem for differential equations has been collected in Appendix A for the convenience of the reader.

Like many textbooks in mathematics, this book can trace its origins back to a set of lecture notes. The transformation from lecture notes to textbook, however, is nontrivial, and one is faced with myriad choices that, in part, reflect one's own interests and experiences teaching the subject. While writing this book I kept in mind three quotes spanning a few generations of mathematicians. The first is from the introduction to a volume of Spivak's multi-volume treatise on differential geometry [64]:

> I feel somewhat like a man who has tried to cleanse the Augean stables with a Johnny-Mop.

It is tempting, when writing a textbook, to give some modicum of completeness. When faced with the enormity of literature on this subject, however, the task proves daunting, and it soon becomes clear that there is just too much material for a single volume. In the end, I could not face picking up the Johnny-Mop, and my solution to this dilemma was to be savage with my choice of topics. Keeping in mind that the goal is to produce a book that should serve as a text for a one-semester introductory course, there were many painful omissions. Firstly, I have tried to steer a reasonably consistent path by keeping the focus on the simplest type problems that illustrate a particular aspect of the theory. Secondly, I have opted in most cases for the "no frills" version of results if the "full feature" version would take us too far afield, or require a substantially more sophisticated mathematical background. Topics such as piecewise smooth extremals, fields of extremals, and numerical methods arguably belong in any introductory account. Nonetheless, I have omitted these topics in favour of other topics, such as a solution method for the Hamilton-Jacobi equation and Noether's theorem, that are accessible to the general mathematically literate undergraduate student but often postponed to a second course in the subject.

The second quote comes from the introduction to Titchmarsh's book on eigenfunction expansions [70]:

> I believe in the future of 'mathematics for physicists', but it seems desirable that a writer on this subject should understand both physics as well as mathematics.

The words of Titchmarsh remind me that, although I am a mathematician interested in the applications of mathematics, I am not a physicist, and it is best to leave detailed accounts of physical models in the hands of experts. This is not to say that the material presented here lies in some vacuum of pure mathematics, where we merely acknowledge that the material has found some applications. Indeed, the book is written with a definite slant towards "applied mathematics," but it focuses on no particular field of applied mathematics in any depth. Often it is the application not the mathematics that perplexes the student, and a study in depth of any particular field would require either the student to have the necessary prerequisites or the author to develop the subject. The former case restricts the potential audience; the latter case shifts away from the main topic. In any event, I have not tried to write a book on the calculus of variations with a particular emphasis on one of its many fields of applications. There are many splendid books that merge the calculus of variations with particular applications such as classical mechanics or control theory. Such texts can be read with profit in conjunction with this book.

The third quote comes from G.H. Hardy, who made the following comment about A.R. Forsyth's 656-page treatise [27] on the calculus of variations :[1]

> In this enormous volume, the author never succeeds in proving that the shortest distance between two points is a straight line.

Hardy did not mince words when it came to mathematics. The prospective author of any text on the calculus of variations should bear in mind that, although there are many mathematical avenues to explore and endless minutiæ to discuss, certain basic questions that can be answered by the calculus of variations in an elementary text should be answered. There are certain problems such as geodesics in the plane and the catenary that can be solved within our self-imposed regime of elementary theory. I do not hesitate to use these simple problems as examples. At the same time, I also hope to give the reader a glimpse of the power and elegance of a subject that has fascinated mathematicians for centuries.

I wish to acknowlege the help of my former students, whose input shaped the final form of this book. I wish also to thank Fiona Davies for helping me with the figures. Finally, I would like to acknowledge the help of my colleagues at the Institute of Fundamental Sciences, Massey University.

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Fox Glacier, New Zealand *Bruce van Brunt*
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[1] F. Smithies reported this comment in an unpublished talk, "Hardy as I Knew Him," given to the *British Society for the History of Mathematics*, 19 December 1990.

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van Brunt, B.
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