

2 The Mathematical Framework of Fuzzy Logic

Bernard Fustier

University of Corsica

2.1 Introduction

In spite of what it may seem, fuzzy logic is not a vague reasoning with indistinct results. On the contrary, it is a rigorous tool that makes it possible for humans to overcome the subtle blend of imprecision and uncertainty of the real world.

It is well-known that fuzzy logic was introduced by Zadeh (1965) in a seminal article entitled “Fuzzy Sets”. This new way of reasoning is based on a very natural principle, the graduality principle, which extends the two-valued classical logic to a more general one where fuzziness is accepted as a matter of science. In particular, we accept that a given proposition is more or less true (or untrue) rather than only true or false. Thus, fuzzy logic can be applied to all these concepts where it is impossible to carry out description in classical mathematical terms because of their natural vagueness.

It seems that the poverty concept falls within the field of fuzzy logic.

However, the majority of applications is still in the industrial world, principally in Japan and Germany (Zimmermann 1993) where fuzzy technology is on the increase with fuzzy tools and fuzzy products such as video cameras, pattern recognition devices etc... Paradoxically, in the area of “soft” sciences, fuzzy logic is of lower penetration. The term of “fuzzy economics” was used for the first time in the summer of 1985 at the First International Fuzzy System Association Congress held at Palma of Mallorca (Ponsard and Fustier 1986). It was the outcome of a long series of research initiated by Ponsard, particularly in the framework of spatial economic analysis (Ponsard 1981a, 1981b, 1982, 1988). Since that time, there has been a certain lack of interest in economic applications of fuzzy subset theory in academic research.

The Chapter is divided into three sections. Sect. 2.1 deals with the graduality principle which applies to “graded” concepts such as fuzzy propositions, fuzzy subsets and fuzzy number concepts. In Sect. 2.2 the basic connectors used in fuzzy logic are illustrated. In Sect. 2.3 the reader can revise the above-mentioned notions referring first to the elaboration of

a decision-making process, and then to the construction of a simple model of evaluation.

2.2 The graduality principle

As its name implies, the graduality principle is a principle of graded concepts, a principle in which everything is a matter of degree. In this section, firstly, the fuzzy proposition concept is examined and, then, the fuzzy subset and fuzzy number concepts are examined.

2.2.1 Fuzzy propositions

Let us consider a property p defined on a set X of elements x . We designate by $p(x)$ the degree of truth of the statement “ x possesses p ” denoted by $P(x)$.

The logic in its classical form recognizes two possibilities (and only two) to express the truth value of any proposition, that is “true” or “false”. According to the custom fixed by Boole the truth value is equal to 0 or 1 when the statement is false or true respectively. In other words, $p(x)$ takes its values in the set $\{0,1\}$ and $P(x)$ is said to be an *ordinary* proposition. This notion supposes that properties p are rigorously defined on the referential sets like, for instance, the masculine gender if we consider a set of persons. In that case, the set $\{0,1\}$ is enough to express truth values (any intervening state between false and true is excluded). Nevertheless, the two-valued (boolean) logic does not hold out against the pervasive imprecision of the real world. In particular, most properties used in natural languages are rather ill-defined. Thus, to estimate the degrees of truth of statements such as “ x is a sympathetic person” or “ x is a beautiful woman”, it is clear that we need a set of values larger than $\{0,1\}$.

Lukasiewicz’s (1928) three-valued logic was a first attempt to make the classical logic suppler (the 0.5 value is used when we have doubts about the true value of a proposition). More general logics (multivalued logics) were worked out afterwards, but it is to Zadeh (1965) that we owe the most general one. Indeed, the interval $[0,1]$ substitutes for the set $\{0,1\}$. When $p(x)$ belongs to $[0,1]$, $P(x)$ is a *fuzzy* proposition. $P(x)$ is true when $p(x) = 1$, untrue when $p(x) = 0$ and “more or less” true (or untrue) for other values of the interval. Notice that $[0,1]$ includes an infinity of values, thus the transition from truth to untruth is gradual rather than abrupt.

The graduality principle applies also to the subset and number notions.

2.2.2 Fuzzy subsets, fuzzy numbers

Let P be a subset of X such that it regroups all the elements x characterized by property p , we can write:

$$P = \{(x / p(x)) \mid x \in X\} \quad (2.1)$$

$p(x)$ is the degree of membership of x to P , that is to say the degree of truth of $P(x)$.

If $p(x) \in \{0,1\}$, then P is an *ordinary* subset of X . If $p(x) \in [0,1]$, then P is a *fuzzy* subset of X . Let us notice that X is an ordinary set, i.e. a non-fuzzy set.

Examples: $X = \{a,b,c,d\}$ represents a set of regions, if a and d are two islands, b and c two mainland regions, then the ordinary subset of “insular” regions can be written as follows: $A = \{(a / 1), (b / 0), (c / 0), (d / 1)\}$. In the classical sets theory it is customary to exclude the elements associated with a zero membership value, one can simply write $A = \{a,d\}$. In the case of fuzzy subsets it is not so easy. Because a fuzzy subset is a collection of objects with unsharp boundaries, we have to review each element of X in order to indicate its membership degree. For instance, the “wealthy” regions fuzzy subset of X can be represented by $B = \{(a / 0.4), (b / 0.8), (c / 0.5), (d / 0.6)\}$. Let us observe that some membership values can be equal to 0 or/and 1, for instance the fuzzy class of regions with “mild weather” can be represented by the following fuzzy subset: $C = \{(a / 1), (b / 0.6), (c / 0), (d / 0.8)\}$. Given P the fuzzy subset defined by (2.1), we give the basic definitions:

– *height* H_P of P :

$$H_P = \vee [p(x) \mid x \in X] \quad (2.2)$$

where \vee represents the max-operator.

– *kernel* K_P of P :

$$K_P = \{x \in X \quad \text{such that} \quad p(x) = 1\} \quad (2.3)$$

– *cardinality* $|P|$ of P :

$$|P| = \sum [p(x) \mid x \in X] \quad (2.4)$$

Furthermore,

$$P \text{ is said } \textit{normalized} \text{ if } H_P = 1 \quad (2.5)$$

and P is *empty*

$$(P = \emptyset) \quad \text{if} \quad \forall x \in X : p(x) = 0 \quad (2.6)$$

Remark: in the particular case where each element of X belongs entirely to P , we have $K_P = X$ and $|P| = |X|$. In other words, P is nothing but the universe X .

Considering C the fuzzy subset of regions with “mild weather”, we have: $H_C = 1$, thus C is normalized. Moreover $K_C = \{a\}$ and $|C| = 2.4$, C is non-empty.

In the specific case where X is the set of real numbers ($p(x)$ is a continuous real mapping), it is possible to introduce the convexity notion. For any pair of real numbers x and x' , and for any value λ of $[0,1]$, P is said to be *convex* if:

$$p[\lambda x + (1 - \lambda)x'] \geq p(x) \wedge p(x') \quad (2.7)$$

where \wedge is the min-operator.

By definition, a *fuzzy number* P is a fuzzy subset of the real line which is normalized and convex such that exactly one real number x_0 exists, called the *mean value* of P , with $p(x_0) = 1$.

When X is a set of discrete values, such as the set of integers, a fuzzy number P can be represented as follows in Figure 2.1.

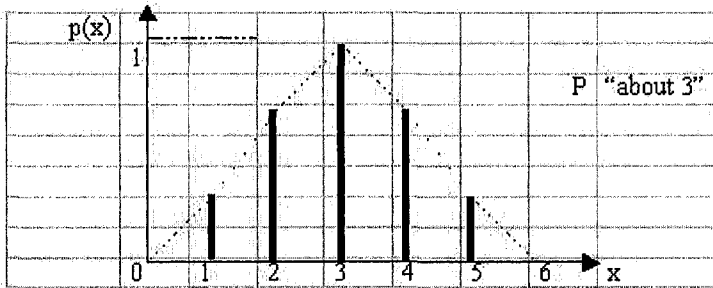


Fig. 2.1. Fuzzy number

On Figure 2.2, the fuzzy subset Q is normalized but not convex: Q is not a fuzzy number, but a *fuzzy quantity*.

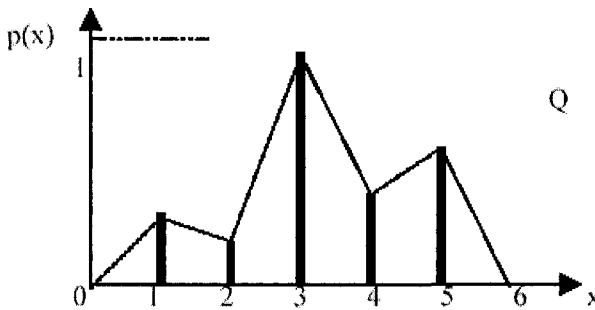


Fig. 2.2. Fuzzy quantity

2.3 The connectors of fuzzy logic

Connectors are operators used to combine fuzzy propositions with the conjunction “and”, the disjunction “or”, or to express the negation of a given statement.

The min, max operators and the complementation (to 1) were first introduced by Zadeh (1965) to express the “and”, the “or” and the “not” respectively.

Other operators have also been suggested. We shall investigate here the basic class of triangular norms and conorms which generalize the use of the min and max operators.

2.3.1 Zadeh’s operators

Considering the degrees of truth $p(x)$ and $q(x)$ of the fuzzy propositions $P(x)$ and $Q(x)$, Zadeh’s operators are given in Table 2.1.

Table 2.1 Zadeh’s operators

proposition:	meaning:	degree of truth:
$P(x)$ and $Q(x)$	“x possesses p and q”	$p(x) \wedge q(x)$
$P(x)$ or $Q(x)$	“x possesses p or q”	$p(x) \vee q(x)$
non- $P(x)$	“x does not possess p”	$1 - p(x)$

\wedge represents the min-operator, \vee represents the max-operator

For example let $P(x)$ be “x is a rich person” with $p(x) = 0.6$. Assuming that “poor” is the opposite of “rich” in such a way that non- $P(x)$ means “x is a poor person”. Under these conditions, the level of truth of:

1. $P(x)$ and non- $P(x)$, i.e. “ x is a rich and poor person”, is $0.6 \wedge (1 - 0.6) = 0.4$
2. $P(x)$ or non- $P(x)$, i.e. “ x is a rich or a poor person”, is $0.6 \vee (1 - 0.6) = 0.6$.

Here we want to emphasize that non-contradiction and excluded middle laws no longer exist in the fuzzy logic context.

In the classical logic, the non-contradiction law means that it is impossible to assert an event and its opposite simultaneously, in other words $P(x)$ and non- $P(x)$ is always false. We see here that $P(x)$ and non- $P(x)$ is not untrue, but a slightly true proposition (0.4). Conversely, in the classical logic a proposition such as $P(x)$ or non- $P(x)$ is always true (excluded middle law). In the case of fuzzy logic, a proposition like “ x is a rich or a poor person” is not totally true (0.6 instead of 1) because, between being “rich as Croesus” and being “poor as Job”, there are still many middle situations that characterize a given person.

Let us note that the \wedge and \vee operators satisfy the following generalization of De Morgan’s laws:

$$1 - [p(x) \wedge q(x)] = [1 - p(x)] \vee [1 - q(x)] \quad (2.8.a)$$

$$1 - [p(x) \vee q(x)] = [1 - p(x)] \wedge [1 - q(x)] \quad (2.8.b)$$

The \wedge and \vee operators are said to be *dual* for the complementation. The duality relations (2.8.a) and (2.8.b) are important because they establish a logical link between the \wedge and \vee operators via the complementation. In the following paragraph, we shall see that the complementation to 1 is also used to show the duality between operators different from \wedge and \vee . Now, let us consider $P = \{(x / p(x)) \mid x \in X\}$ and $Q = \{(x / q(x)) \mid x \in X\}$ two fuzzy subsets of X . The intersection \cap , the union \cup and the complementation $*$ operations correspond to the logical “and”, “or” and “not” respectively, thus we have:

$$P \cap Q = \{(x / p(x) \wedge q(x)) \mid x \in X\} \quad (2.9)$$

$$P \cup Q = \{(x / p(x) \vee q(x)) \mid x \in X\} \quad (2.10)$$

$$P^* = \{(x / 1 - p(x)) \mid x \in X\} \quad (2.11)$$

Let us go back with $B = \{(a / 0.4), (b / 0.8), (c / 0.5), (d / 0.6)\}$ the “wealthy” regions fuzzy subset. We obtain $B^* = \{(a / 0.6), (b / 0.2), (c / 0.5), (d / 0.4)\}$ the “non-wealthy” regions fuzzy subset, $B \cap B^* = \{(a / 0.4), (b / 0.2), (c / 0.5), (d / 0.4)\}$ the fuzzy subset of regions which are simultaneously “wealthy” and “non-wealthy”, and then $B \cup B^* = \{(a / 0.6),$

$(b / 0.8), (c / 0.5), (d / 0.6)\}$ the fuzzy subset of regions which are either “wealthy” or “non-wealthy”. We must observe that:

1. $B \cap B^* \neq \emptyset$ (some regions possess both wealth and poverty features)
2. $|B \cup B^*| < 4$ (the union of wealthy and non-wealthy regions does not give the universe, because there are still many middle regions).

Obviously, these definitions apply to the fuzzy numbers.

Example (Zimmermann 1991, p 18): let us consider two fuzzy real numbers P and Q. The meaning of P is “x is considerably larger than 10” with:

$$p(x) = \begin{cases} 0 & \text{if } x \leq 10 \\ [1 + (x - 10)^{-2}]^{-1} & \text{otherwise} \end{cases}$$

the meaning of Q is “x is approximatively equal to 11” with:

$$q(x) = [1 + (x - 11)^4]^{-1}$$

Then the fuzzy number $P \cap Q$ means “x is considerably larger than 10 and approximatively equal to 11”. Let us write $f(x) = p(x) \wedge q(x)$, we have:

$$f(x) = \begin{cases} 0 & \text{if } x \leq 10 \\ [1 + (x - 10)^{-2}]^{-1} \wedge [1 + (x - 11)^4]^{-1} & \text{otherwise} \end{cases}$$

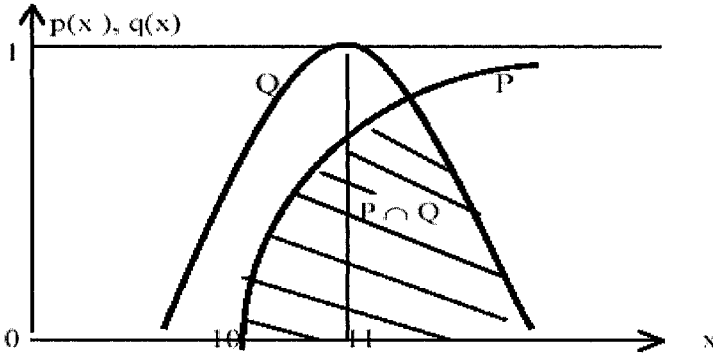


Fig. 2.3. Intersection of fuzzy numbers

The intersection is represented by the curves bordering the hachured part (Figure 2.3).

Algebraic operations with fuzzy numbers have been defined (Dubois and Prade 1979, 1980, 1991; Zimmermann 1991). We consider here the cases of fuzzy addition and fuzzy product.

Let $p(x)$ and $q(y)$ be the membership degrees of the real numbers x and y to fuzzy real numbers P and Q respectively, let $P+Q$ and $P \bullet Q$ be the sum and the product of P and Q respectively. Under these conditions:

1. given the z real values such that $z = x + y$, where $+$ is the ordinary addition, the membership degree of z to $P+Q$, denoted $f(z)$, is defined by:

$$f(z) = \vee [p(x) \wedge q(y) \mid z = x + y] \quad (2.12)$$

2. given the z real values such that $z = x \cdot y$, where \cdot is the ordinary multiplication, the membership degree of z to $P \bullet Q$, denoted $g(z)$, is defined by:

$$g(z) = \vee [p(x) \wedge q(y) \mid z = x \cdot y] \quad (2.13)$$

To simplify matters, let us put ourselves in the context of discrete values, for instance P and Q are fuzzy numbers defined on the set of integers such as:

$P = \{(0 / 0), (1 / 0.5), (2 / 1), (3 / 0.5), (4 / 0)\}$: “ x is approximatively equal to 2”

$Q = \{(1 / 0), (2 / 0.6), (3 / 1), (4 / 0.6), (5 / 0)\}$: “ y is approximatively equal to 3”.

For the addition, the z values are given in the following table:

$x \setminus y$	1	2	3	4	5
0	1	2	3	4	5
1	2	3	4	5	6
2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9

for $z = 1$: $f(1) = 0 \wedge 0 = 0$

for $z = 2$: $f(2) = \vee [(0.5 \wedge 0), (0 \wedge 0.6)] = 0$

for $z = 3$: $f(3) = \vee [(1 \wedge 0), (0.5 \wedge 0.6), (0 \wedge 1)] = 0.5$

for $z = 4$: $f(4) = \vee [(0.5 \wedge 0), (1 \wedge 0.6), (0.5 \wedge 1), [(0 \wedge 0.6)] = 0.6$

for $z = 5$: $f(5) = \vee [(0 \wedge 0), (0.5 \wedge 0.6), (1 \wedge 1), (0.5 \wedge 0.6), (0 \wedge 0)] = 1$

for $z = 6$: $f(6) = \vee [(0 \wedge 0.6), (0.5 \wedge 1), (1 \wedge 0.6), [(0.5 \wedge 0)] = 0.6$

for $z = 7$: $f(7) = \vee [(0 \wedge 1), (0.5 \wedge 0.6), (1 \wedge 0)] = 0.5$

for $z = 8$: $f(8) = \vee [(0 \wedge 0.6), (0.5 \wedge 0)] = 0$

for $z = 9$: $f(9) = 0 \wedge 0 = 0$

Finally, we obtain the representation of $P+Q$ on the figure below:

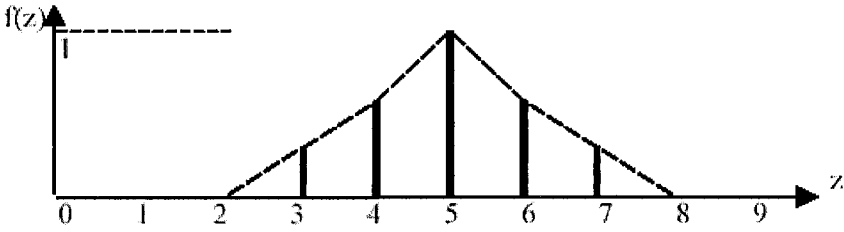


Fig. 2.4. Sum of fuzzy integers

If x_0 and y_0 are the mean values of P and Q respectively, we see that $x_0 + y_0$ is the mean value of the fuzzy number $P+Q$. Presently, this fuzzy number signifies “ z is approximately equal to 5”.

For the multiplication, the z values are given in the following table:

$x \setminus y$	1	2	3	4	5
0	0	0	0	0	0
1	1	2	3	4	5
2	2	4	6	8	10
3	3	6	9	12	15
4	4	8	12	16	20

for $z = 0$: $g(0) = \vee [(0 \wedge 0), (0 \wedge 0.6), (0 \wedge 1), (0 \wedge 0.6), (0 \wedge 0)] = 0$
 for the other z values: $g(1) = 0$, $g(2) = g(3) = 0.5$, $g(4) = 0.6$, $g(5) = 0$,
 $g(6) = 1$, $g(8) = 0.6$, $g(9) = 0.5$, $g(10) = 0$,
 $g(12) = 0.5$, $g(15) = g(16) = g(20) = 0$.

hence the representation of $P \bullet Q$:

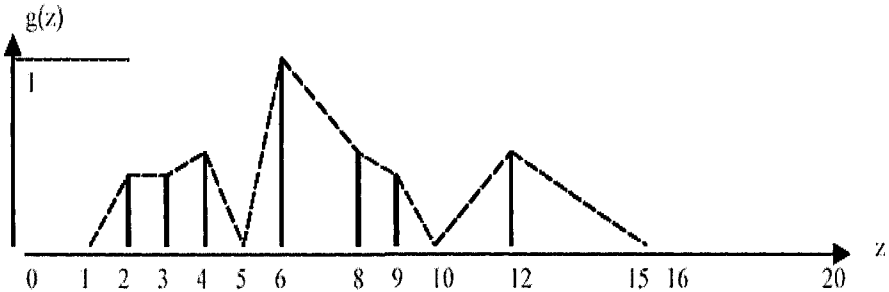


Fig. 2.5. Product of fuzzy integers

We notice that $P \bullet Q$ is normalized but not convex, it is not a fuzzy number, but a fuzzy quantity. The \bullet operator cannot be directly applied to

fuzzy numbers when the universe X is a set of discrete values, the resulting fuzzy subsets may no longer be convex and therefore no longer considered as fuzzy numbers.

2.3.2 Other fuzzy logical connectives

Except for the non-contradiction and excluded middle laws, Zadeh's operators preserve the structure of the classical sets theory. Most general fuzzy logical connectives such as the triangular norms and conorms have been defined even if it means getting off the structure.

A *triangular norm*, sometimes called *t-norm*, is a general operator, denoted by \top , used for indicating the fuzzy logical "and".

Let $p(x)\top q(x) \in [0,1]$ be the degree of truth of $P(x)$ and $Q(x)$, \top must satisfy the following conditions (Bouchon-Meunier 1995, p 39):

1. commutativity: $p(x)\top q(x) = q(x)\top p(x)$
2. associativity: $p(x)\top (q(x)\top r(x)) = (p(x)\top q(x))\top r(x)$
where $r(x)$ is the degree of truth of $R(x)$
3. isotony: $p(x) \leq r(x)$ and $q(x) \leq s(x) \Rightarrow p(x)\top q(x) \leq r(x)\top s(x)$
where $s(x)$ is the degree of truth of $S(x)$
4. neutrality for 1: $p(x)\top 1 = 1\top p(x) = p(x)$

The most frequently used t-norms are (Fodor and Roubens 1994, pp. 7-8):

$$p(x)\top^1 q(x) = p(x) \wedge q(x) \quad (2.14)$$

$$p(x)\top^2 q(x) = p(x) \cdot q(x) \quad (2.15)$$

$$p(x)\top^3 q(x) = [p(x) + q(x) - 1] \vee 0 \quad (2.16)$$

$$p(x)\top^4 q(x) = \begin{cases} p(x) \wedge q(x) & \text{if } p(x) + q(x) > 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.17)$$

$$p(x)\top^5 q(x) = \begin{cases} p(x) \wedge q(x) & \text{if } p(x) \vee q(x) = 1 \\ 0 & \text{otherwise} \end{cases} \quad (2.18)$$

In addition to conditions (1), (2), (3) and (4), any t-norm \top verifies properties $0\top 0 = 0$ and $1\top 1 = 1$.

Corresponding to the t-norms class, a general class of operators for the fuzzy logical "or" is defined analogously; it is the *triangular* or *t-conorms* class.

A t-conorm, denoted by \perp , must satisfy the preceding conditions except for (4). The neutrality condition is defined here in relation to the 0 value, that is to say: $p(x) \perp 0 = 0 \perp p(x) = p(x)$. Moreover, any t-conorm \perp verifies properties $0 \perp 0 = 0$ and $1 \perp 1 = 1$. By considering the duality relation (2.8.b) between the min-max operators, we obtain:

$$1 - ([1 - p(x)] \wedge [1 - q(x)]) = p(x) \vee q(x) \quad (2.19)$$

where \vee is a t-conorm. By substituting the \wedge and \vee operators the most general ones, that is \top and \perp respectively, then (2.19) can be used to transform any t-norm \top into a t-conorm \perp . According to this association, we deduce (Bonissone and Decker 1986) the following t-conorms which correspond to the (2.14)... (2.18) t-norms:

$$p(x) \perp^1 q(x) = p(x) \vee q(x) \quad (2.20)$$

$$p(x) \perp^2 q(x) = p(x) + q(x) - p(x) \cdot q(x) \quad (2.21)$$

$$p(x) \perp^3 q(x) = [p(x) + q(x)] \wedge 1 \quad (2.22)$$

$$p(x) \perp^4 q(x) = \begin{cases} p(x) \vee q(x) & \text{if } p(x) + q(x) < 1 \\ 1 & \text{otherwise} \end{cases} \quad (2.23)$$

$$p(x) \perp^5 q(x) = \begin{cases} p(x) \vee q(x) & \text{if } p(x) \wedge q(x) = 0 \\ 1 & \text{otherwise} \end{cases} \quad (2.24)$$

Let us note that the complement operator used in (2.19) is a particular negation N such that $N[p(x)] = 1 - p(x)$. Although this negation is very commonly used in practice, there are other ones, for instance (Fodor and Roubens 1994, pp 3-4):

$$N[p(x)] = 1 - [p(x)]^2 \quad (2.25)$$

$$N[p(x)] = \begin{cases} 1 & \text{if } p(x) = 0 \\ 0 & \text{if } p(x) > 0 \end{cases} \quad (2.26)$$

$$N[p(x)] = \begin{cases} 1 & \text{if } p(x) < 1 \\ 0 & \text{if } p(x) = 1 \end{cases} \quad (2.27)$$

$$N[p(x)] = [1 - p(x)] / [1 + \lambda p(x)], \quad \lambda > -1 \quad (2.28)$$

More generally, an operator N satisfying the following conditions is a *negation*:

1. $N(0) = 1$ and $N(1) = 0$
2. $p(x) \geq q(x) \Rightarrow N[p(x)] \leq N[q(x)]$

A negation is *strict* if the inequalities in (2) are strict inequalities. Furthermore, if the condition $N\{N[p(x)]\} = p(x)$ is satisfied, then N is said to be *involution*. We see that the complement operator is a strict and involutive negation. For this category of negation, the duality relation (2.19) extended to the \top and \perp operators is written:

$$N\{N[p(x)] \top N[q(x)]\} = p(x) \perp q(x) \quad (2.29.a)$$

Conversely:

$$N\{N[p(x)] \perp N[q(x)]\} = p(x) \top q(x) \quad (2.29.b)$$

These general relations show how the t-norms and t-conorms classes are related in a sense of logical duality. Nevertheless, these connectives are not the only way to express the “and” and the “or”. A certain number of authors have suggested combining the truth values through the medium of aggregating procedures frequently used in statistics such as arithmetic or geometric means (Zimmermann and Zysno 1980, 1983; Dubois and Grabisch 1994). Here we shall only mention one interesting dual pair of these connectives (called *averaging operators*) due to Werners (1988). The first one, denoted Δ , concerns the fuzzy “and”, the second, denoted \sqcup , is the expression of the fuzzy “or”. A distinctive feature of these operators is that they combine the minimum and maximum operators, respectively, with the arithmetic mean. Given $\gamma \in [0,1]$, we have:

$$p(x) \Delta q(x) = \gamma [p(x) \wedge q(x)] + \frac{1}{2} [(1 - \gamma) [p(x) + q(x)]] \quad (2.30.a)$$

$$p(x) \sqcup q(x) = \gamma [p(x) \vee q(x)] + \frac{1}{2} [(1 - \gamma) [p(x) + q(x)]] \quad (2.30.b)$$

If $\gamma = 0$, then $p(x) \Delta q(x) = p(x) \wedge q(x)$ and $p(x) \sqcup q(x) = p(x) \vee q(x)$. Inversely, $\gamma = 1$ implies $p(x) \Delta q(x) = \frac{1}{2} [p(x) + q(x)]$ and $p(x) \sqcup q(x) = \frac{1}{2} [p(x) + q(x)]$. It is clear that the parameter γ indicates the degree of nearness of the Δ and \sqcup operators to the logical meaning of “and” and “or” in the max-min fuzzy logic.

The question arises of how to fix the value of γ within $[0,1]$? In other words, do we have to favour the max-min logic (γ near to 1) or have a high regard for the “aggregating” fuzzy logic (γ near to 0) ?

The question can be broached axiomatically (Bellman and Giertz 1973), but the choice of an operator is essentially a matter of context. It mainly depends upon the real-world situation which is to be represented. As far as the applications are concerned, the estimation process of truth values plays an important part in the choice of operators. If the values are estimated with rather unbiased data, it is possible to use averaging operators without any difficulty. But if the degrees of truth are subjective estimates (to assess the beauty of a landscape for instance), we have to regard these estimates as ordinal values and the max-min operators seem to be suitable for the oc-

casion (truth values are only compared, not aggregated in statistical formula).

2.4 Decision-making and evaluation in a fuzzy context

We consider here two simple models within the framework of the max-min fuzzy logic. The first one is due to Bellman and Zadeh (1970), it concerns the decision-making process. The second model worked out by Fustier (1994, 2000) proposes a fuzzy “aggregation” index and applies to the evaluation field.

2.4.1 Optimal fuzzy decision: the Bellman and Zadeh’s model

In this well-known model, the universe X represents a set of alternatives denoted x and called *actions*. Corresponding to properties p and q respectively, the fuzzy subsets $P = \{(x / p(x)) \mid x \in X\}$ and $Q = \{(x / q(x)) \mid x \in X\}$ are said to be the *fuzzy objective* and *fuzzy constraint*.

Example: $X = \{a, b, c, d, e\}$ is a set of job applicants in a certain company. This one is searching for “a good economist” (property p) provided that the person in question is “capable of working as a team” (property q). Under these conditions, the fuzzy objective is the fuzzy subset of job applicants who are good economists, for instance $P = \{(a / 0.8), (b / 1), (c / 0.5), (d / 0.4), (e / 0.6)\}$. In the same way, the fuzzy constraint is the fuzzy subset of job applicants who are capable of working as a team, for instance $Q = \{(a / 0.6), (b / 0.6), (c / 0.7), (d / 0.8), (e / 0.1)\}$.

The fuzzy subset D such that $D = P \cap Q$ represents the *decision space*. By definition, D regroups the feasible solutions, that is actions which belong both to the fuzzy objective and the fuzzy constraint. Let $d(x)$ be the membership degree of x to D , we know that $d(x) = p(x) \wedge q(x)$. In the Bellman and Zadeh context, a decision is the act of selecting a specific action which is feasible (element of the decision space): the decision is said to be *optimal* if this action corresponds to the maximum of the objective. In other words, an optimal fuzzy decision consists in selecting the action denoted x_0 which has the highest membership degree in the decision set, that is:

$$d(x_0) = \vee [p(x) \wedge q(x) \mid x \in X] \quad (2.31)$$

Remark : x_0 is not always the only solution.

Here we have $D = \{(a / 0.6), (b / 0.6), (c / 0.5), (d / 0.4), (e / 0.1)\}$, thus: $x_0 = a = b$.

The procedure can be extended to any number n of objectives and any number m of constraints, then: $d(x_0) = \vee [p_1(x) \wedge \dots \wedge p_n(x) \wedge q_1(x) \wedge \dots \wedge q_m(x) \mid x \in X]$.

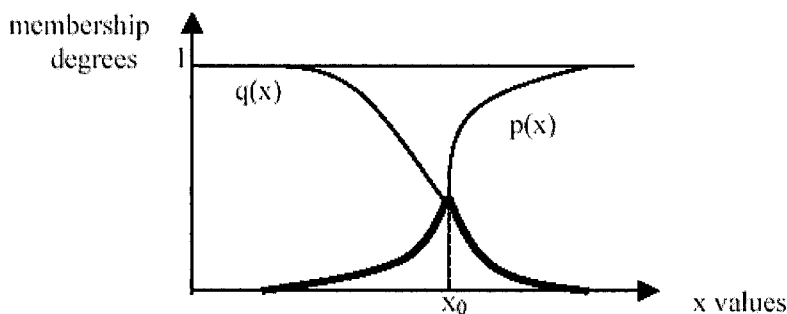


Fig. 2.6. Optimal fuzzy decision in the continuous case

Obviously, the set of actions X can be a set of values. For instance (see Fig. 2.6), the board of directors is trying to find the dividend to be paid to the shareholders. It must be “attractive” for the shareholders (objective, $p(x)$ is increasing). But, the dividend has to be “modest” because of the investment planning of the company (constraint, $q(x)$ is decreasing).

2.4.2 “Fuzzy” aggregation in evaluation problems.

We consider here a set of objects, denoted i , as for example countries that we have to evaluate according to a roughly defined concept like wealth (or its opposite, poverty).

The first step of the evaluating process relies on making the concept of trying to divide the latter into a list of attributes as exhaustive as possible clear. These attributes, denoted j , must be non-redundant and possess different weights denoted $\pi(j)$. If we consider for instance the concept of wealth, we can obtain:

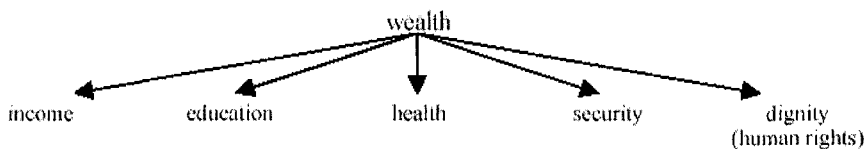


Fig. 2.7. Division of a fuzzy concept into attributes

An attribute is a less vague notion than the initial concept, but it maintains a certain degree of imprecision (from what level of income can we

regard a person as well-to-do ? Is as rich as Croesus ? Is the absence of war enough to assert that the people of a country are safe ?). For this reason, the evaluations of the objects on each attribute and the coefficients of importance of these attributes can be considered as truth degrees of fuzzy propositions:

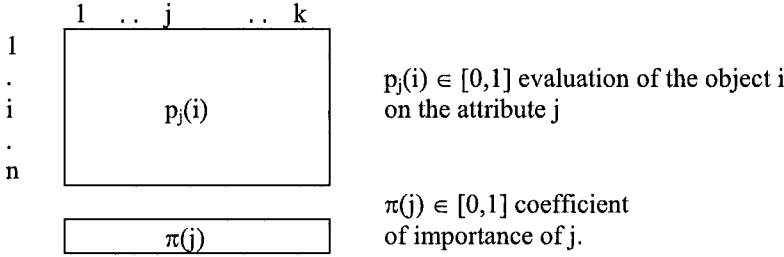


Fig. 2.8. Data

By definition, $p_j(i)$ is the truth degree of the fuzzy proposition: “ i possesses j ” and $\pi(j)$ represents the truth degree of the fuzzy proposition “ j is important”. An attribute with a coefficient of importance equal to 1, is called a *fundamental* attribute; it is assumed that at least one of the attributes is fundamental. Let us note that the vector of coefficients of importance represents the evaluations assigned to an “ideal” object (it possesses j exactly according to the importance of j in the evaluation problem).

If we have to evaluate countries on the first attribute, i.e. according to the monetary wealth (income) and if we can obtain the gross domestic product per capita for each country i , it is possible to consider the formula:

$$p_1(i) = \begin{cases} 0 & \text{if } y(i) < y(-) \\ y(i)/y(+) & \text{otherwise} \end{cases}$$

with $y(i)$ = GDP per capita of i , $y(-)$ = GDP per capita corresponding to the subsistence level, and $y(+)$ = GDP per capita of the richest country in the world.

In case of lack of statistical data concerning purely qualitative attributes (such as “dignity” or the coefficients of importance), we must directly estimate the $p_j(i)$ and $\pi(j)$ in the interval $[0,1]$.

Under these conditions, we wish to define an operator g which assigns a value $g(i) \in [0,1]$ to each object i . Let us observe that $g(i)$ shows how much i fits with the initial concept of evaluation. In the previous example, $g(i)$ is the truth degree of the fuzzy proposition: “the country i is wealthy”. By definition, $1 - g(i)$ is the degree of truth of the proposition: “the country i is not wealthy”. Taking into account a concept like wealth or its opposite (poverty) is equally relevant since the fuzzy complementation enables switching from one concept to the other.

There is a wealth of literature on fuzzy aggregation (Dubois and Prade 1985; Mizumoto 1989a, 1989; Fodor and Roubens 1992, 1994; Dubois and Grabisch 1994). Presently, we have to deal with degrees of truth that are more subjective qualitative estimates than objective numerical data (measures). From this statement of fact it follows that the great majority of the compiled operators (the averaging or *compensatory* operators) must be ignored because their theoretical foundations are in no way different from other traditional statistical operators like means. However, we have to admit that operators which are fully compatible with the max-min fuzzy logic are very rare, the well-known operators of this category are the *weighted maximum*:

$$s(i) = \vee [p_j(i) \wedge \pi(j) \mid j = 1 \dots k] \quad (2.32)$$

and the *weighted minimum* operators (Dubois and Prade 1986):

$$s(i)' = \wedge [p_j(i) \vee (1 - \pi(j)) \mid j = 1 \dots k] \quad (2.33)$$

Let us note that the weighted minimum does not possess concrete meaning in an evaluation problem (because of the non-importance coefficients $1 - \pi(j)$). The weighted maximum formula seems to be appropriate here, but it appears to be too “optimistic”: an evaluation equal to 1 given on a fundamental attribute will suffice to obtain a maximum value of the operator, that is 1. We can see this result in table 2.2 where two objects (a and b) and seven attributes (1, 2...) are considered: we obtain $s(a) = 1$ although we have zero evaluations for all the attributes except for $j = 4$.

To find a solution for that, a differential of *discordance* $r_j(i)$ on each j is calculated between the profile of a given object i and the profile of the ideal object (ie the vector of the coefficients of importance):

$$r_j(i) = \begin{cases} 0 & \text{if } p_j(i) \geq \pi(j) \\ \pi(j) - p_j(i) & \text{otherwise} \end{cases} \quad (2.34)$$

We see that $r_j(i) \in [0,1]$ with $r_j(i) = 1$ if j is fundamental and $p_j(i) = 0$. Following the example of the weighted maximum formula, the max-operator is used to summarize the differentials of discordance. Let $r(i)$ be the index of discordance of i , we have:

$$r(i) = \vee [r_j(i) \mid j = 1 \dots k] \quad (2.35)$$

It is clear that $r(i) \in [0,1]$.

An index of *concordance*, denoted $t(i)$, is obtained by the negation of the discordance notion:

$$t(i) = 1 - r(i) \quad (2.36)$$

Table 2.2 Weighted maximum and discordance calculation

$j \rightarrow$	1	2	3	4	5	6	7	
$p_j(a) \rightarrow$	0	0	0	1	0	0	0	
$p_j(b) \rightarrow$	0.9	0.9	0.9	1	0.9	0.9	0.9	
$\pi(j) \rightarrow$	1	1	1	1	0.8	0.7	0.7	
$p_j(a) \wedge \pi(j) \rightarrow$	0	0	0	1	0	0	0	1 = s(a)
$p_j(b) \wedge \pi(j) \rightarrow$	0.9	0.9	0.9	1	0.8	0.7	0.7	1 = s(b)
$r_j(a) \rightarrow$	1	1	1	0	0.8	0.7	0.7	1 = r(a)
$r_j(b) \rightarrow$	0.1	0.1	0.1	0	0	0	0	0.1 = r(b)

We get $t(a) = 0$ (a is not in concordance with the ideal object) and $t(b) = 0.9$ (b is well in concordance with the ideal object). Finally, a fuzzy “aggregation” operator is given by:

$$g(i) = s(i) \wedge t(i) \quad (2.37)$$

According to the max-min fuzzy logic, the \wedge -operator is used for connecting the two indexes, but in the applications we can stretch the rules and prefer a more “synthetical” operator such as:

$$g(i) = [s(i) + t(i)] / 2 \quad (2.38)$$

With (2.37) we obtain $g(a) = 0$ and $g(b) = 0.90$. With (2.38), we get $g(a) = 0.50$ and $g(b) = 0.95$. Remark: from (2.36) and (2.35) we have $t(i) = 1 - \vee [r_j(i) \mid j = 1 \dots k]$. By using the duality relation (2.8.b), we obtain: $1 - \vee [r_j(i) \mid j = 1 \dots k] = \wedge [1 - r_j(i) \mid j = 1 \dots k]$. Finally, we can also calculate the concordance index according to:

$$t(i) = \wedge [1 - r_j(i) \mid j = 1 \dots k] \quad (2.39)$$

Such a procedure was applied for evaluating the environmental sensibility of tourist zones in the region of Corsica (Fustier and Serra 2001).

From the preceding example, we obtain:

Table 2.3 Using duality relation to calculate the concordance index

$j \rightarrow$	1	2	3	4	5	6	7	
$1 - r_j(a) \rightarrow$	0	0	0	1	0.2	0.3	0.3	0 = t(a)
$1 - r_j(b) \rightarrow$	0.9	0.9	0.9	1	1	1	1	0.9 = t(b)

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