2 Equisingular Deformations of Plane Curve Singularities

In this section, we study deformations of plane curve singularities leaving certain invariants fixed, in particular, the multiplicity, the δ-invariant and the Milnor number. We define these notions also for non-reduced base spaces, especially for fat points, and we develop the theory of the corresponding equimultiple, equinormalizable and equisingular deformations.

We again focus on the issue of versality in our study, and we approach it from two points of view: as deformations of the equation, and as deformations of the parameterization. The second approach culminates in a new proof of the smoothness of the base of a *versal equisingular* deformation. The *equisingularity ideal* plays a central role in the theory. It represents the space of first order equisingular deformations and, geometrically, its quotient by the Tjurina ideal represents the tangent space to the base of the semiuniversal equisingular deformation inside the base of a semiuniversal deformation.

2.1 Equisingular Deformations of the Equation

We study now special deformations of plane curve singularities, requiring that the topological type is preserved. Recall that the topological type of a reduced plane curve singularity $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ is the equivalence class of $(C,\mathbf{0})$ under local, embedded homeomorphisms (Definition I.3.30), and that the topological type is equivalently characterized by numerical data such as the system of multiplicity sequences (Theorem I.3.42).[8]

To study deformations which do not change the topological type in the nearby germs we must, first of all, specify the point of the nearby fibre where we take the germ. More precisely, we have to introduce the notion of a *deformation with section.*

However, in order to apply the full power of deformation theory, we need deformations over *non-reduced* base spaces. In particular, we have to define first order equisingular deformations, that is, equisingular deformations over the fat point T_ε. Since "constant multiplicity" can be generalized to "equimultiplicity" (along a section) over a non-reduced base, the system of multiplicity sequences is an appropriate invariant for defining equisingular deformations over arbitrary base spaces. This approach was chosen and developed by J. Wahl in his thesis. Based on Zariski's studies in equisingularity [Zar1], he created the infinitesimal theory of equisingular deformations and gave several applications (cf. [Wah, Wah1]).

Throughout the following, let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity, and let $f \in \mathfrak{m}^2 \subset \mathbb{C}\{x,y\}$ be a defining power series. We call $f=0$, or just f the *(local) equation* of $(C,\mathbf{0})$. Deformations of $(C,\mathbf{0})$ (respectively embedded deformations of $(C,\mathbf{0})$) will also be called *deformations of the equation* in contrast to *deformations of the parametrization*, as considered in Section 2.3.

Definition 2.1. A *deformation with section* of $(C,\mathbf{0})$ over a complex germ (T,t_0) consists of a deformation $(i,\phi):(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$ of $(C,\mathbf{0})$ over (T,t_0) and a section of ϕ, that is, a morphism $\sigma:(T,t_0)\to(\mathscr{C},x_0)$ satisfying $\phi\circ\sigma = \mathrm{id}_{(T,t_0)}$. It is denoted by (i,ϕ,σ) or just by (ϕ,σ).

The category of deformations with section of $(C,\mathbf{0})$ is denoted by $\mathcal{D}ef^{\,sec}_{(C,\mathbf{0})}$, where morphisms are morphisms of deformations which commute with the

[8] It is a general fact from topology (proved by Timourian [Tim] and King [Kin1]) that, if the embedded type of the fibres of a family of hypersurfaces is constant, then the family is even topologically trivial.

sections. Isomorphism classes of deformations with sections over (T, t_0) are denoted by $\underline{\mathcal{D}ef}^{sec}_{(C,\mathbf{0})}(T, t_0)$.

It follows from the definition that the section σ is a closed embedding, mapping (T, t_0) isomorphically to $\sigma(T, t_0)$. Moreover, by Corollary 1.6, we may assume the deformation to be embedded, that is, any deformation with section is given by a commutative diagram

$$\begin{tikzcd}
(C,\mathbf{0}) \arrow[r, hook, "i"] \arrow[d] & (\mathscr{C}, x_0) \arrow[r, hook] \arrow[d, shift left, "\phi"] & (\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \arrow[dl, "\mathrm{pr}"] \\
\{t_0\} \arrow[r, hook] & (T, t_0) \arrow[u, bend left, shift left, "\sigma"] &
\end{tikzcd} \tag{2.1.1}$$

where $(\mathscr{C}, x_0)$ is a hypersurface germ in $(\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ and pr the natural projection. $(\mathscr{C}, x_0)$ is defined by an unfolding $F \in \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ satisfying $F \circ \sigma = 0$. Hence, F is an element of $\mathrm{Ker}\big(\sigma^\sharp \colon \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \to \mathcal{O}_{T,t_0}\big) =: I_\sigma$, the ideal of $\sigma(T, t_0)$. After fixing local coordinates x, y for $(\mathbb{C}^2, \mathbf{0})$, we get

$$I_\sigma = \langle x - \sigma_1, y - \sigma_2 \rangle, \qquad \sigma_1 := \sigma^\sharp(x),\ \sigma_2 := \sigma^\sharp(y) \in \mathcal{O}_{T,t_0}\,.$$

Hence, I_σ determines the section σ.

The section σ is called the *trivial section* if $\sigma(T, t_0) = (\{\mathbf{0}\} \times T, t_0)$, that is, $I_\sigma = \langle x, y \rangle$. It is called a *singular section* if we have an inclusion of germs $\sigma(T, t_0) \subset (\mathrm{Sing}(\phi), p)$.

Next, we show that the section can be *trivialized*, that is, each embedded deformation with section is isomorphic to an embedded deformation with trivial section, that is, given by a diagram (2.1.1) with σ the trivial section (see Proposition 2.2, below). The proof is based on the relative lifting Lemma I.1.27. In geometric terms, this lemma says that any commutative diagram of morphisms of complex germs (with solid arrows)

$$\begin{tikzcd}
(\mathbb{C}^n \times T, (\mathbf{0}, t_0)) \arrow[rr, dashed] & & (\mathbb{C}^m \times T, (\mathbf{0}, t_0)) \\
(\mathscr{X}, x_0) \arrow[u, hook] \arrow[rr] \arrow[dr] & & (\mathscr{Y}, y_0) \arrow[u, hook] \arrow[dl] \\
& (T, t_0) &
\end{tikzcd}$$

where $(\mathscr{X}, x_0) \to (T, t_0)$ and $(\mathscr{Y}, y_0) \to (T, t_0)$ are induced by the projection, can be completed to a commutative diagram by a dashed arrow. The dashed arrow can be chosen as an isomorphism if $n = m$ and $(\mathscr{X}, x_0) \to (\mathscr{Y}, y_0)$ is an isomorphism (respectively as a closed embedding if $n \le m$ and $(\mathscr{X}, x_0) \to (\mathscr{Y}, y_0)$ is a closed embedding).

Proposition 2.2. *Let* $i : (\mathscr{X}, x_0) \hookrightarrow (\mathbb{C}^n, \mathbf{0}) \times (T, t_0)$ *be a closed embedding, and let* $\mathrm{pr} : (\mathbb{C}^n, \mathbf{0}) \times (T, t_0) \to (T, t_0)$ *be the projection to the second factor. Then each section* $\sigma : (T, t_0) \to (\mathscr{X}, x_0)$ *of* $\mathrm{pr} \circ i$ *can be trivialized. That is, there is an isomorphism*

$$\psi : (\mathbb{C}^n, \mathbf{0}) \times (T, t_0) \xrightarrow{\cong} (\mathbb{C}^n, \mathbf{0}) \times (T, t_0)$$

commuting with pr *such that* $\psi \circ \sigma$ *is the canonical inclusion*

$$\psi \circ \sigma : (T, t_0) \to \{\mathbf{0}\} \times (T, t_0) \subset (\mathbb{C}^n, \mathbf{0}) \times (T, t_0)\,.$$

Proof. Since $(\sigma(T), x_0) \xrightarrow{\mathrm{pr}} (T, t_0) \hookrightarrow \{\mathbf{0}\} \times (T, t_0)$ is an isomorphism over (T, t_0), the statement follows by applying the relative lifting lemma to the isomorphism of $\mathscr{O}_{T,t_0}$-algebras $\mathscr{O}_{\sigma(T),x_0} \xrightarrow{\cong} \mathscr{O}_{\{\mathbf{0}\}\times(T,t_0)}$. □

Corollary 2.3. *With the above notations, we have*

$$T^{1,sec}_{(C,\mathbf{0})} := \underline{\mathcal{D}ef}^{\,sec}_{(C,\mathbf{0})}(T_\varepsilon) \cong \mathfrak{m}/\langle f, \mathfrak{m}j(f)\rangle\,,$$

where $j(f) \subset \mathbb{C}\{x, y\}$ *denotes the Jacobian ideal and* $\mathfrak{m} \subset \mathbb{C}\{x, y\}$ *the maximal ideal.*

Proof. Since each section can be trivialized, each deformation with section of $(C, \mathbf{0})$ over T_ε is represented by $f + \varepsilon g$ with $g \in \mathfrak{m}$. Such a deformation is trivial iff $g \in \langle f, \mathfrak{m}j(f)\rangle$ as shown in the proof of Proposition 1.25 and Remark 1.25.1. □

Definition 2.4. Let (i, ϕ, σ), $\phi : (\mathscr{C}, x_0) \hookrightarrow (\mathbb{C}^2 \times T, (\mathbf{0}, t_0)) \to (T, t_0)$, be an embedded deformation with section $\sigma : (T, t_0) \to (\mathscr{C}, x_0)$ of $(C, \mathbf{0})$, and let f be an equation for $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ of multiplicity $\mathrm{mt}(f)$. Moreover, let $F \in \mathscr{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ be a defining power series for $(\mathscr{C}, x_0) \subset (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$, and let $I_\sigma \subset \mathscr{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ denote the ideal of $\sigma(T, t_0) \subset (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$. Then (i, ϕ, σ) is called *equimultiple* (or, the deformation (i, ϕ) is called *equimultiple along* σ) iff

$$F \in I_\sigma^{\mathrm{mt}(f)}\,.$$

Note that this definition is independent of the chosen embedding and local equation.

Definition 2.5. Let T be a complex space, $U \subset \mathbb{C}^2 \times T$ be open and $\sigma : T \to U$, $t \mapsto (\sigma_1(t), \sigma_2(t), t)$, a section of the second projection. We define the *blowing up* of U *along* σ (or the *blowing up of the section* σ) as the complex space

$$\begin{aligned} B\ell_\sigma(U) := B\ell_{\sigma(T)}(U) &:= \big\{(z; L) \in U \times \mathbb{P}^1 \,\big|\, z - \sigma(t) \in L \times \{t\}\big\} \\ &:= \big\{(x, y, t;\, a_1 : a_2) \in U \times \mathbb{P}^1 \,\big|\, a_2(x - \sigma_1(t)) = a_1(y - \sigma_2(t))\big\}\,, \end{aligned}$$

together with the projection $\pi : B\ell_\sigma(U) \to U$. In particular, if σ is the trivial section with $\sigma_1(t) = \sigma_2(t) = 0$ for all $t \in T$, then $B\ell_\sigma(\mathbb{C}^2 \times T) = B\ell_{\mathbf{0}}(\mathbb{C}^2) \times T$.

As previously (when blowing up points), we can cover $U \times \mathbb{P}^1$ by two charts $U \times V_i := \{a_i \neq 0\} \subset U \times \mathbb{P}^1$, $i = 1, 2$. For the first chart we obtain (with $v := a_2/a_1$)

$$(U \times V_1) \cap B\ell_\sigma(U) = \big\{(x, y, t, v) \,\big|\, v(x - \sigma_1(t)) = y - \sigma_2(t)\big\}$$

with ideal sheaf $\langle v(x - \sigma_1) - y + \sigma_2\rangle \mathcal{O}_{U \times V_1}$. Setting $u := x - \sigma_1$ and eliminating y, we see that $(U \times V_1) \cap B\ell_\sigma(U)$ is isomorphic to an open subset of $\mathbb{C}^2 \times T$ with coordinates u, v, t. That is, if $U = U_1 \times U_2 \times T$, $U_i \subset \mathbb{C}$ open, then

$$(U \times V_1) \cap B\ell_\sigma(U) = \big\{(u, v, t) \in U_1 \times \mathbb{C} \times T \,\big|\, uv + \sigma_2(t) \in U_2\big\}$$

is an open neighbourhood of $\{0\} \times \mathbb{C} \times T$, and v is an affine coordinate of $\mathbb{C}$, not just a coordinate of the germ $(\mathbb{C}, 0)$. In these coordinates π is given as

$$\pi : (U \times V_1) \cap B\ell_\sigma(U) \to U \subset \mathbb{C}^2 \times T\,,\ (u, v, t) \mapsto \big(u + \sigma_1(t),\, uv + \sigma_2(t),\, t\big)\,.$$

Similarly, we have coordinates $\overline{u}, \overline{v}, t$ in the second chart (with $\overline{u}$ affine) and

$$\pi : (U \times V_2) \cap B\ell_\sigma(U) \to U\,, \quad (\overline{u}, \overline{v}, t) \mapsto \big(\overline{uv} + \sigma_1(t),\, \overline{v} + \sigma_2(t),\, t\big)\,.$$

As $B\ell_\sigma(U)$ can be covered by these two charts, both being isomorphic over T to open subsets in $\mathbb{C}^2 \times T$, we can blow up sections of the composition $B\ell_\sigma(U) \to U \to T$ by choosing coordinates of the charts and proceeding as above. Different coordinates give results which are isomorphic over T.

Furthermore, the construction is local along the sections. Hence, we can blow up finitely many pairwise disjoint sections in an arbitrary order, or simultaneously, and get a blown up complex space, which is unique up to isomorphism over T. By passing to small representatives, we can also blow up sections of morphisms of germs of complex spaces.

For each point $\sigma(t) \in \sigma(T)$ we get $\pi^{-1}(\sigma(t)) = \mathbb{P}^1$ with local equation $u = 0$ in the first chart and with $\overline{v} = 0$ in the second chart. Hence,

$$\mathscr{E} := \pi^{-1}\big(\sigma(T)\big) = \sigma(T) \times \mathbb{P}^1$$

is a divisor in $B\ell_\sigma(U)$, called the *exceptional divisor* of the blowing up (which we describe below in local coordinates).

Now, let (T, t_0) be a germ, let $\sigma : (T, t_0) \to (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ be a section of the projection to (T, t_0), and let $(\mathscr{C}, x_0)$ be the hypersurface germ of $(\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$ defined by $F \in \mathcal{O}_{\mathbb{C}^2 \times T, (\mathbf{0}, t_0)}$. Fixing local coordinates, we can write F as

$$F(x, y, \boldsymbol{t}) = \sum_{i,j} a_{ij}(\boldsymbol{t}) \cdot \big(x - \sigma_1(\boldsymbol{t})\big)^i \big(y - \sigma_2(\boldsymbol{t})\big)^j\,, \qquad a_{ij} \in \mathcal{O}_{T, t_0}\,,$$

and $F(x, y, \mathbf{0}) = f(x, y)$. Then F defines an embedded deformation of $(C, \mathbf{0}) = (V(f), \mathbf{0})$ which is equimultiple along σ iff

$$\min\{i+j \mid a_{ij} \neq 0\} = \mathrm{mt}(f)\,.$$

Let $\pi : B\ell_\sigma(\mathbb{C}^2{\times}T,(\mathbf{0},t_0)) \to (\mathbb{C}^2{\times}T,(\mathbf{0},t_0))$ be the blowing up along the section σ, which is a germ along the exceptional divisor $\sigma(T)\times\mathbb{P}^1 \subset B\ell_\sigma(U)$ in the blowing up of a small representative $\sigma : T \to U \subset \mathbb{C}^2\times T$. Assume that F is equimultiple along σ. Then, in the first chart, we have

$$\widehat{F}(u,v,\boldsymbol{t}) := (F\circ\pi)(u,v,\boldsymbol{t}) = \sum_{i,j} a_{ij}(\boldsymbol{t})u^i(uv)^j = u^{\mathrm{mt}(f)}\cdot\widetilde{F}(u,v,\boldsymbol{t})\,,$$

and, in the second chart,

$$\widehat{F}(\overline{u},\overline{v},\boldsymbol{t}) := (F\circ\pi)(\overline{u},\overline{v},\boldsymbol{t}) = \overline{v}^{\mathrm{mt}(f)}\cdot\widetilde{F}(\overline{u},\overline{v},\boldsymbol{t})\,.$$

The functions $\widetilde{F}(u,v,\boldsymbol{t})$ and $\widetilde{F}(\overline{u},\overline{v},\boldsymbol{t})$ (which are defined by these relations) are holomorphic in the respective charts, and they define a unique zero-set in the intersection of these charts.

We define the following (Cartier-)divisors in $B\ell_\sigma(\mathbb{C}^2{\times}T,(\mathbf{0},t_0))$:

- $\widehat{\mathscr{C}}$, the divisor given by $\widehat{F}=0$, called the *total transform of* $(\mathscr{C},x_0)$.
- $\widetilde{\mathscr{C}}$, the divisor given by $\widetilde{F}=0$, called the *strict transform of* $(\mathscr{C},x_0)$.

As a divisor, we have

$$\widehat{\mathscr{C}} = \widetilde{\mathscr{C}} + \mathrm{mt}(f)\cdot\mathscr{E}\,,$$

and $\widetilde{\mathscr{C}}$ and $\mathscr{E}$ have no common component. The divisor $\widetilde{\mathscr{C}}+\mathscr{E}$ is called the *reduced total transform of* $(\mathscr{C},x_0)$. In the first chart, it is given by $u\cdot\widetilde{F}(u,v,\boldsymbol{t})=0$, in the second by $\overline{v}\cdot\widetilde{F}(\overline{u},\overline{v},\boldsymbol{t})=0$.

We shall call a family of plane curve singularities equisingular if it is equimultiple and if the reduced total transform in all successive blowing ups (until the special fibre is resolved) are again equimultiple along the singular sections. This is Wahl's [Wah] definition (if the base space is a fat point), and it implies that all fibres are equisingular in the sense of Zariski [Zar1].

Definition 2.6. Let $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve germ, and let (i,ϕ,σ) be an embedded deformation with section of $(C,\mathbf{0})$ over (T,t_0). If $(C,\mathbf{0})$ is singular, then (i,ϕ,σ) is called an *equisingular deformation of* $(C,\mathbf{0})$ or an *equisingular deformation of the equation of* $(C,\mathbf{0})$ if the following holds: There exist small representatives for (i,ϕ,σ) and a commutative diagram of complex spaces and morphisms

$$\begin{tikzcd}
\mathscr{C}^{(N)} \arrow[r] \arrow[d, hook] & \mathscr{C}^{(N-1)} \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & \mathscr{C}^{(0)} \arrow[d, hook] \arrow[dr, "\phi"] & \\
\mathscr{M}^{(N)} \arrow[r, "\pi_N"] \arrow[dr, phantom, "\square"] & \mathscr{M}^{(N-1)} \arrow[r, "\pi_{N-1}"] \arrow[dr, phantom, "\square"] & \cdots \arrow[r, "\pi_1"] \arrow[dr, phantom, "\square"] & \mathscr{M}^{(0)} \arrow[r] \arrow[dr, phantom, "\square"] & T \\
M^{(N)} \arrow[r] \arrow[u, hook] & M^{(N-1)} \arrow[r] \arrow[u, hook] & \cdots \arrow[r] & M^{(0)} \arrow[r] \arrow[u, hook] & \{t_0\} \arrow[u, hook]
\end{tikzcd} \qquad (2.1.2)$$

together with pairwise disjoint sections

$$\sigma_1^{(\ell)}, \ldots, \sigma_{k_\ell}^{(\ell)} : T \to \mathscr{C}^{(\ell)} \subset \mathscr{M}^{(\ell)}, \quad \ell = 0, \ldots, N-1,$$

of the composition $\mathscr{M}^{(\ell)} \xrightarrow{\pi_\ell} \mathscr{M}^{(\ell-1)} \xrightarrow{\pi_{\ell-1}} \ldots \xrightarrow{\pi_1} \mathscr{M}^{(0)} \to T$ with the following properties:

(1) The lower row of (2.1.2) induces a minimal embedded resolution of the plane curve germ $(C, \mathbf{0}) \subset (M^{(0)}, \mathbf{0}) = (\mathbb{C}^2, \mathbf{0})$.
(2) For $\ell = 0$, we have $(\mathscr{M}^{(0)}, x_0) = (\mathbb{C}^2 \times T, (\mathbf{0}, t_0))$, $(\mathscr{C}^{(0)}, x_0) = (\mathscr{C}, x_0)$, $k_0 = 1$. Moreover, $\sigma_1^{(0)} : T \to \mathscr{M}^{(0)}$ is the section (induced by) σ, and $(\mathscr{C}^{(0)}, x_0) \hookrightarrow (\mathscr{M}^{(0)}, x_0) \to (T, t_0)$ defines an equimultiple (embedded) deformation of $(C, \mathbf{0})$ along $\sigma_1^{(0)}$.
(3) For $\ell = 1$, we have that $\pi_1 : \mathscr{M}^{(1)} \to \mathscr{M}^{(0)}$ is the blowing up of $\mathscr{M}^{(0)}$ along the section $\sigma_1^{(0)}$, $\mathscr{C}^{(1)}$ is the strict transform of $\mathscr{C}^{(0)} \subset \mathscr{M}^{(0)}$, and $\mathscr{E}^{(1)}$ is the exceptional divisor of π_1.
(4) For $\ell \geq 1$, we require inductively that
  - $\sigma_1^{(\ell)}(t_0), \ldots, \sigma_{k_\ell}^{(\ell)}(t_0)$ are precisely the non-nodal singular points of the reduced total transform of $(C, \mathbf{0}) \subset (M^{(0)}, \mathbf{0}) = (\mathbb{C}^2, \mathbf{0})$.
  - $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)} \hookrightarrow \mathscr{M}^{(\ell)} \to T$ induces (embedded) equimultiple deformations along $\sigma_1^{(\ell)}, \ldots, \sigma_{k_\ell}^{(\ell)}$, of the respective germs of the reduced total transform $C^{(\ell)} \cup E^{(\ell)}$ of $(C, \mathbf{0})$ in $M^{(\ell)}$.
  - The sections are *compatible*, that is, for each $j = 1, \ldots, k_\ell$ there is some $1 \leq i \leq k_{\ell-1}$ such that $\pi_{\ell+1} \circ \sigma_j^{(\ell)} = \sigma_i^{(\ell-1)}$.
  - $\pi_{\ell+1} : \mathscr{M}^{(\ell+1)} \to \mathscr{M}^{(\ell)}$ is the blowing up of $\mathscr{M}^{(\ell)}$ along $\sigma_1^{(\ell)}, \ldots, \sigma_{k_\ell}^{(\ell)}$, $\mathscr{C}^{(\ell+1)}$ is the strict transform of $\mathscr{C}^{(\ell)} \subset \mathscr{M}^{(\ell)}$, and $\mathscr{E}^{(\ell+1)}$ is the exceptional divisor of the composition $\pi_1 \circ \ldots \circ \pi_{\ell+1}$.

If $(C, \mathbf{0})$ is smooth, each deformation with section is called *equisingular*.

We call a diagram (2.1.2) together with the sections $\sigma_j^{(\ell)}$ such that (1)–(4) hold an *equisingular deformation of the resolution of* $(C, \mathbf{0})$ associated to the embedded deformation with section (i, ϕ, σ).

Remark 2.6.1. (1) The sections $\sigma_i^{(\ell)}$ are also called *equimultiple sections* for the equisingular deformation. By Proposition 2.2, p. 269, all sections can be locally trivialized, that is, for each $p = \sigma_j^{(\ell)}(t_0)$, there are isomorphisms of germs $(\mathscr{M}^{(\ell)}, p) \cong (\mathbb{C}^2, \mathbf{0}) \times (T, t_0)$ over (T, t_0) trivializing the section $\sigma_j^{(\ell)}$.

(2) Considering the restriction of the strict transforms $\mathscr{C}^{(\ell)}$ to the special fibre over t_0, we get a minimal embedded resolution of $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$,

$$\begin{array}{ccccccccc}
M^{(N)} & \xrightarrow{\pi_N} & M^{(N-1)} & \xrightarrow{\pi_{N-1}} & \cdots & \xrightarrow{\pi_2} & M^{(1)} & \xrightarrow{\pi_1} & (\mathbb{C}^2, \mathbf{0}) \\
\cup & & \cup & & & & \cup & & \cup \\
C^{(N)} & \longrightarrow & C^{(N-1)} & \longrightarrow & \cdots & \longrightarrow & C^{(1)} & \longrightarrow & (C, \mathbf{0}) .
\end{array}$$

π_1 is the blowing up of the origin, and $\pi_{\ell+1}$, $\ell = 1, \ldots, N-1$, is the simultaneous blowing up of all non-nodal singularities $p_j = \sigma_j^{(\ell)}(t_0)$, $j = 1, \ldots, k_\ell$ of the respective reduced total transforms of $(C, \mathbf{0})$. However, it is not important that we blow up the points simultaneously. As the construction is local, we can blow the points up successively in any order, the result is always isomorphic. In the same way, $\pi_{\ell+1} : \mathscr{M}^{(\ell+1)} \to \mathscr{M}^{(\ell)}$ can either blow up $\mathscr{M}^{(\ell)}$ simultaneously along the sections $\sigma_j^{(\ell)}$ or successively in an arbitrary order.

(3) By semicontinuity of the multiplicity[9], equimultiplicity of the reduced total transform $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)}$ along $\sigma_i^{(\ell)}$ is equivalent to equimultiplicity of the strict transform $\mathscr{C}^{(\ell)}$ *and* of the reduced exceptional divisor $\mathscr{E}^{(\ell)}$ along $\sigma_i^{(\ell)}$. Indeed, if we want to preserve the topological type of the singularities along σ in the nearby fibres, it is not sufficient to require only equimultiplicity of the strict transforms as is shown in Example 2.6.2, below.

(4) If the germ $(C, \mathbf{0})$ is smooth, then each (embedded) deformation of $(C, \mathbf{0})$, $(C, \mathbf{0}) \hookrightarrow (\mathscr{C}, x_0) \to (T, t_0)$, with section $\sigma : (T, t_0) \to (\mathscr{C}, x_0)$ is equimultiple along σ.

If the reduced total transform in the special fibre $C^{(\ell)} \cup E^{(\ell)}$, $\ell \geq 1$, has a node at $q \in C^{(\ell)} \cap E^{(\ell)}$, that is, if $C^{(\ell)}$, $E^{(\ell)}$ are smooth and intersect transversally at q, then there exists a unique section σ_q such that $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)}$ is equimultiple along σ_q.

This implies that the definition of equisingularity remains unchanged if, in Definition 2.6, we start with any (not necessarily minimal) embedded resolution as special fibre (in the bottom row of diagram (2.1.2)).

(5) It follows also that, for $\ell \geq 1$ and $q \in C^{(\ell)} \cap E^{(\ell)}$,

$$(\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)}, q) \hookrightarrow (\mathscr{M}^{(\ell)}, q) \to (T, t_0)$$

is an equisingular embedded deformation of the germ $(C^{(\ell)} \cup E^{(\ell)}, q)$.

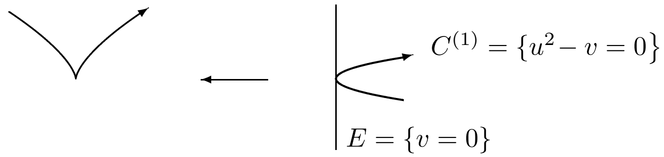
(6) By Proposition 2.8 on page 275, the sections $\sigma_j^{(\ell)}$ are uniquely determined. Since the minimal resolution is unique (Exercise I.3.3.1), it follows that the associated equisingular deformation of the resolution is uniquely determined (up to isomorphism) by (i, ϕ, σ). By (4), the same holds if the lower row of (2.1.2) is any (not necessarily minimal) embedded resolution of $(C, \mathbf{0})$.

Example 2.6.2. Consider the one-parameter deformation of the cusp given by $F := x^2 - y^3 - t^2 y^k$, $k \geq 0$. For $k \geq 2$, the deformation given by F is equimultiple along the trivial section $\sigma : t \mapsto (0, 0, t)$ (and σ is the unique equimultiple section), while, for $k \leq 1$ there is no equimultiple section.

After blowing up σ, we obtain (in the second chart) the reduced total transform $\{v(u^2 - v - t^2 v^{k-2}) = 0\}$. In the special fibre we get the reduced total transform of the cusp, which is the union of the smooth germ

[9] For hypersurfaces, this is easy: if $F_{\boldsymbol{t}}(\boldsymbol{x}) = F(\boldsymbol{x}, \boldsymbol{t}) = f(\boldsymbol{x}) + g_{\boldsymbol{t}}(\boldsymbol{x})$, $g_0(\boldsymbol{x}) = 0$, is an unfolding of f then, for $\boldsymbol{t}$ sufficiently close to $\mathbf{0}$, the terms of lowest order of f cannot be cancelled by terms of $g_{\boldsymbol{t}}$. Hence, $\mathrm{mt}(f) \geq \mathrm{mt}\, F_{\boldsymbol{t}}$ for $\boldsymbol{t}$ close to $\mathbf{0}$.

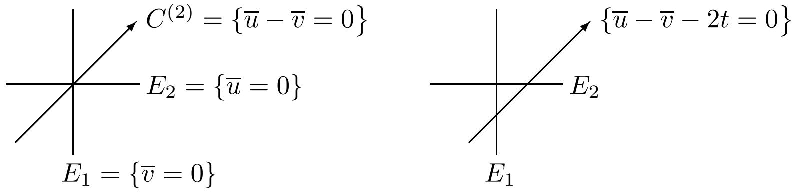
$C^{(1)} = \{u^2 - v = 0\}$ and the exceptional divisor $E = \{v = 0\}$, intersecting with multiplicity 2 at the origin:



For $k \geq 3$, the blown-up deformation has the trivial section $\sigma^{(1)}$ as unique equimultiple section.

For $k = 2$ there are two different equimultiple sections through the origin being compatible with σ. Indeed, a section $\sigma^{(1)}$ is compatible with the trivial section σ iff its image lies in the exceptional divisor $\mathscr{E}^{(1)} = \{v = 0\}$. In other words, a section $\sigma^{(1)}$ through the origin is compatible with σ iff it is given by an ideal $\langle u - t\alpha, v\rangle$, $\alpha \in \mathbb{C}\{t\}$. Since the ideal of the reduced total transform $v(u^2 - v - t^2)$ is contained in $\langle u - t, v\rangle^2$ and in $\langle u + t, v\rangle^2$, we get two equimultiple sections $\sigma_\pm^{(1)}$ given by the ideals $\langle u \pm t, v\rangle$. Geometrically, the reduced total transform of the special fibre (which is an A_3-singularity) is deformed into the union of a line and a parabola, meeting transversally in two points, and the equimultiple sections are the singular sections through the nodes. [10] After blowing up $\sigma^{(1)}$ (respectively one of the sections $\sigma_\pm^{(1)}$), the reduced total transform in the special fibre is the union of three concurrent lines.

Hence, for $k = 2$, we find no equimultiple section through the origin of the respective reduced total transform $\{\overline{u}\overline{v}(\overline{u} \mp 2t - \overline{v}) = 0\}$ $(\overline{u} = u \pm t)$. Geometrically, this is caused by the fact that the D_4-singularity (of multiplicity 3) in the special fibre is deformed into three nodes (each of multiplicity 2) in the nearby fibres:



If $k \geq 3$, the reduced total transform of F, $\overline{u}\overline{v}(\overline{u} - \overline{v} - t^2\overline{u}^{k-3}\overline{v}^{k-2})$, is contained in $\langle \overline{u}, \overline{v}\rangle^3$. Hence, it defines an equimultiple deformation along the trivial section $\sigma^{(2)}$.

[10] Note that replacing t^2 by t in the definition of F, there is no equimultiple section of the strict transform in case $k = 2$. At first glance, this might seem strange, since fibrewise the A_3-singularity is still deformed into 2 nodes. But there is a monodromy phenomenon which cannot be observed in the real pictures: a loop around the origin in the base of the deformation interchanges the nodes of the nearby fibres. Algebraically, this corresponds to the fact that there is no square root of t in $\mathbb{C}\{t\}$. See also Figure 2.7.

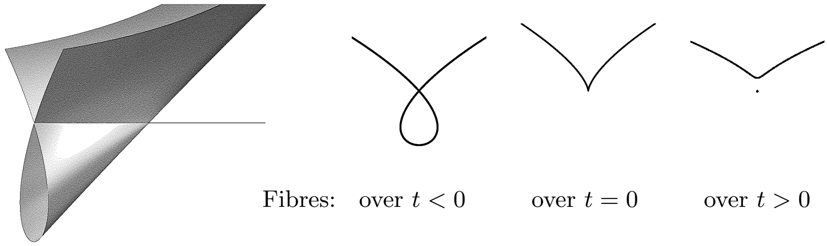


Fig. 2.7. The deformation of the cusp given by $x^2 - y^3 + ty^2$ is equimultiple along the trivial section but not equisingular. Note that the real pictures are misleading: the complex fibres are always connected.

We conclude that F defines an equisingular deformation iff $k \geq 3$: it is even a trivial deformation, since $F = x^2 - y^3(1 - t^2 y^{k-3})$.

Finally, the case $k = 2$ shows that it is *not sufficient to require equimultiplicity of the strict transforms* $\mathscr{C}^{(\ell)}$, $\ell \geq 0$. Indeed, the strict transforms $\mathscr{C}^{(2)}_{\pm}$, given by $(\overline{u} \mp 2t - \overline{v})$, are equimultiple along the section $\sigma^{(2)}_{\pm}$ with ideal $\langle \overline{u}, \overline{v} \pm 2t \rangle$, and the latter is compatible with $\sigma^{(1)}_{\pm}$ (since its image lies in the exceptional divisor $\mathscr{E}^{(2)}_{\pm} = \{\overline{u} = 0\}$).

Definition 2.7. A deformation (i, ϕ) of $(C, \mathbf{0})$ over (T, t_0),

$$(C, \mathbf{0}) \xhookrightarrow{i} (\mathscr{C}, x_0) \xrightarrow{\phi} (T, t_0)\,,$$

is called *equisingular* (or an *ES-deformation*) if there exists an embedded deformation with section (i, ϕ, σ) inducing (i, ϕ) such that (i, ϕ, σ) is equisingular in the sense of Definition 2.6. Two equisingular deformations of $(C, \mathbf{0})$ over (T, t_0) are isomorphic if they are isomorphic as deformations over (T, t_0). The set of isomorphism classes of equisingular deformations of $(C, \mathbf{0})$ over (T, t_0) is denoted by $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}(T, t_0)$, and

$$\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})} \colon (\textit{complex germs}) \longrightarrow \mathcal{S}ets\,, \quad (T, t_0) \longmapsto \underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}(T, t_0)$$

is called the *equisingular deformation functor.*

Proposition 2.8. *Let (i, ϕ) be an equisingular deformation of $(C, \mathbf{0})$ over (T, t_0). Then the system of equimultiple sections $\sigma^{(\ell)}_i$, $\ell \geq 0$, for the diagram* (2.1.2) *is uniquely determined.*

Proof.[11] This result is basically due to Wahl, who proved it if (T, t_0) is a fat point, and we refer to his proof [Wah, Thm. 3.2]. In general, let $\sigma^{(\ell)}_i$ and

[11] Another proof of Proposition 2.8 is given in [CGL2], where it is shown that uniqueness of the sections fails in positive characteristic.

$\widetilde{\sigma}_i^{(\ell)}$ be equimultiple sections with $\sigma_i^{(\ell)}(t_0) = \widetilde{\sigma}_i^{(\ell)}(t_0) =: p_i$. Then, by Wahl's result, we may assume that $\sigma_i^{(\ell)\,\sharp}(x_\nu) - \widetilde{\sigma}_i^{(\ell)\,\sharp}(x_\nu) \in \mathcal{O}_{T,t_0}$ vanishes modulo an arbitrary power of $\mathfrak{m}_{T,t_0}$, where x_ν denote generators of the maximal ideal of $\mathcal{O}_{\mathscr{M}^{(\ell)},p_i}$. Hence, $\sigma_i^{(\ell)\,\sharp} = \widetilde{\sigma}_i^{(\ell)\,\sharp}$, by Krull's intersection theorem. $\square$

The approach of Wahl to equisingular deformations is slightly different. He considers diagrams as in Definition 2.6, *together with a system of (equimultiple) sections* satisfying all the required properties. Morphisms in this category (denoted by $\mathcal{D}ef^N_{(C,\mathbf{0})}$) are morphisms of diagrams commuting with the given sections. This approach is necessary to show that the corresponding functor of isomorphism classes $\underline{\mathcal{D}ef}^N_{(C,\mathbf{0})}$ satisfies Schlessinger's conditions and, hence, has a formal semiuniversal deformation. By Proposition 2.8, the natural forgetful functor $\underline{\mathcal{D}ef}^N_{(C,\mathbf{0})} \to \underline{\mathcal{D}ef}_{(C,\mathbf{0})}$ is injective, and we denote the image by $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$.

Next, we want to show that equisingular deformations of reducible plane curve singularities induce equisingular deformations of the respective branches. For the proof we need the following statement which is interesting in its own:

Proposition 2.9. *Let $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity and let $\big(\widetilde{C}_i,\widetilde{0}_i\big)$, $i=1,\dots,r$, be reduced (not necessarily plane) curve singularities. Let $\big(\widetilde{C},\widetilde{0}\big) := \coprod_{i=1}^r \big(\widetilde{C}_i,\widetilde{0}_i\big)$ be the (multigerm of the) disjoint union and let $\pi : \big(\widetilde{C},\widetilde{0}\big) \to (C,\mathbf{0})$ be a finite morphism such that, for sufficiently small representatives, π induces an isomorphism*

$$\pi : \widetilde{C} \setminus \{\widetilde{0}\} \xrightarrow{\cong} C \setminus \{\mathbf{0}\}\,.$$

Moreover, let (T,t_0) be an arbitrary complex germ and consider a Cartesian diagram

$$\begin{tikzcd}
\big(\widetilde{C},\widetilde{0}\big) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \big(\widetilde{\mathscr{C}},\widetilde{x}_0\big) \arrow[d, "\widetilde{\pi}"'] \arrow[dd, bend left=60, "\phi"] \\
(\mathbb{C}^2,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \big(\mathbb{C}^2\times T,(\mathbf{0},t_0)\big) \arrow[d, "p"'] \\
\{t_0\} \arrow[r, hook] & (T,t_0)
\end{tikzcd}$$

with ϕ a flat morphism. Let $(\mathscr{C},x_0) := \widetilde{\pi}\big(\widetilde{\mathscr{C}},\widetilde{x}_0\big)$ be the image of $\widetilde{\pi}$, endowed with its Fitting structure (see Definition I.1.45). Then the Fitting ideal $\mathrm{Fitt}\big(\widetilde{\pi}_*\big(\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}\big)$ *is a principal ideal in $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$, the induced map $(\mathscr{C},x_0) \to (T,t_0)$ is flat, and $(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$ is an (embedded) deformation of $(C,\mathbf{0})$.*

Furthermore, the Fitting structure is the unique analytic structure on $\widetilde{\pi}\big(\widetilde{\mathscr{C}},\widetilde{x}_0\big)$ such that the projection to (T,t_0) defines a deformation of $(C,\mathbf{0})$. It coincides with the annihilator structure, that is, the ideal in $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ defining $(\mathscr{C},x_0)$ is the kernel of $\widetilde{\pi}^\sharp : \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \to \mathcal{O}_{\widetilde{\mathscr{C}},\widetilde{x}_0}$.

Proof. We work with representatives of the above germs which we always assume to be sufficiently small.

By Proposition I.1.70, π and $\widetilde{\pi}$ are finite morphisms. By the finite coherence theorem I.1.67, we may assume that $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}$ has a free resolution $\mathcal{F}_\bullet$ by $\mathcal{O}_{U\times T}$-modules of finite rank ($U \subset \mathbb{C}^2$ a neighbourhood of $\mathbf{0}$). Moreover, we can assume that the matrices in the free resolution $\mathcal{F}_\bullet$ have only entries in $\mathcal{J}(t_0)$, the ideal sheaf of $\{t_0\}$ in $\mathcal{O}_T$.

Step 1. We show that $\mathrm{Fitt}\big(\widetilde{\pi}_*\big(\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}\big)$ is a principal ideal in $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$:

Since the above diagram is Cartesian, tensoring with $\mathbb{C} = \mathcal{O}_T/\mathcal{J}(t_0)$ gives $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}} \otimes_{\mathcal{O}_T} \mathbb{C} = \pi_*\mathcal{O}_{\widetilde{C}}$, and its stalk at $\mathbf{0}$ is a finitely generated $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module of depth 1 (since $(\widetilde{C},\widetilde{0})$ is a reduced curve germ, hence Cohen-Macaulay). The Auslander-Buchsbaum formula (in the form of Corollary B.9.4) implies that each minimal free resolution of $\big(\pi_*\mathcal{O}_{\widetilde{C}}\big)_\mathbf{0}$ has length 1.

Since $\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}$ is a flat $\mathcal{O}_T$-module (via p_*), tensoring the exact sequence (of $\mathcal{O}_{U\times T}$-modules)

$$\ldots \longrightarrow \mathcal{F}_2 \xrightarrow{M_2} \mathcal{F}_1 \xrightarrow{M_1} \mathcal{F}_0 \longrightarrow \widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}} \longrightarrow 0$$

with $\mathbb{C}$ over $\mathcal{O}_T$ leads to an exact sequence of $\mathcal{O}_U$-modules

$$\ldots \longrightarrow \mathcal{F}_2 \otimes_{\mathcal{O}_T} \mathbb{C} \xrightarrow{\overline{M}_2} \mathcal{F}_1 \otimes_{\mathcal{O}_T} \mathbb{C} \xrightarrow{\overline{M}_1} \mathcal{F}_0 \otimes_{\mathcal{O}_T} \mathbb{C} \longrightarrow \pi_*\mathcal{O}_{\widetilde{C}} \longrightarrow 0\,.$$

By the choice of $\mathcal{F}_\bullet$, all $\mathcal{O}_U$-entries of the matrices $\overline{M}_i$ vanish at $\mathbf{0}$. Hence, $\mathcal{F}_\bullet \otimes_{\mathcal{O}_T} \mathbb{C}$ induces a minimal free resolution of the stalk $\big(\pi_*\mathcal{O}_{\widetilde{C}}\big)_\mathbf{0}$, which has length 1 by the above. It follows that the germ at $\mathbf{0}$ of $\overline{M}_1$ is injective, that is, we have a short exact sequence of $\mathcal{O}_U$-modules

$$0 \longrightarrow \mathcal{F}_1 \otimes_{\mathcal{O}_T} \mathbb{C} \xrightarrow{\overline{M}_1} \mathcal{F}_0 \otimes_{\mathcal{O}_T} \mathbb{C} \longrightarrow \pi_*\mathcal{O}_{\widetilde{C}} \longrightarrow 0\,.$$

Since the support of $\pi_*\mathcal{O}_{\widetilde{C}}$ is C (hence, of codimension 1 in U), the free modules $\mathcal{F}_1$ and $\mathcal{F}_0$ must have the same rank. Moreover, Proposition B.5.3 implies that we may also assume M_1 to be injective. In particular, $\mathcal{F}itt_{\mathcal{O}_{U\times T}}\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)$ is a principal ideal in $\mathcal{O}_{U\times T}$, generated by the determinant of M_1.

Step 2. $(C,\mathbf{0}) \hookrightarrow (\mathscr{C},x_0) \to (T,t_0)$ is an (embedded) deformation of $(C,\mathbf{0})$:

Since $\pi_*\mathcal{O}_{\widetilde{C}}|_{U\cap C\setminus\{\mathbf{0}\}} \cong \mathcal{O}_C|_{U\cap C\setminus\{\mathbf{0}\}}$ by assumption, all germs outside $\mathbf{0}$ of $\det(\overline{M}_1)$ are reduced. Hence, $\det(\overline{M}_1)$ is reduced, and $\det(M_1) \otimes_{\mathcal{O}_T} \mathbb{C} = \det(\overline{M}_1)$ generates the ideal of $C \subset U$. It follows that $(\mathscr{C},x_0)$ with the Fitting structure is flat over (T,t_0) and defines a deformation of $(C,\mathbf{0})$.

Step 3. The Fitting and annihilator structure on $\widetilde{\pi}\big(\widetilde{\mathscr{C}},\widetilde{0}\big)$ coincide:

In general, $\mathcal{F}itt := \mathcal{F}itt\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big) \subset \mathcal{A}nn\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big) =: \mathcal{A}nn$. If we tensor the cokernel by $\mathbb{C}$ over $\mathcal{O}_T$, the result is a $\mathcal{O}_U$-sheaf with support at $\mathbf{0} \in C$, since the sheaves $\pi_*\mathcal{O}_{\widetilde{C}}$ and $\mathcal{O}_C$ are isomorphic outside $\mathbf{0}$.

However, we know already that $\mathcal{F}itt \otimes_{\mathcal{O}_T} \mathbb{C} = \mathcal{F}itt\big(\pi_*\mathcal{O}_{\widetilde{C}}\big)$ is a radical ideal. Since $\mathcal{F}itt \otimes_{\mathcal{O}_T} \mathbb{C} \subset \mathcal{A}nn \otimes_{\mathcal{O}_T} \mathbb{C}$ and both have C as zero-set, Hilbert's Nullstellensatz implies that they must coincide. Hence, we have $\mathcal{A}nn / \mathcal{F}itt \otimes_{\mathcal{O}_T} \mathbb{C} = 0$. On the other hand, Proposition B.5.3 gives that the stalk $(\mathcal{A}nn / \mathcal{F}itt)_{x_0}$ is $\mathcal{O}_{T,t_0}$-flat, hence, faithfully flat as $\mathcal{O}_{T,t_0}$ is local. It follows that $\mathcal{A}nn / \mathcal{F}itt = 0$.

Step 4. To see the uniqueness of the analytic structure of $(\mathscr{C}, x_0)$, let $(\mathscr{C}', x_0)$ denote $\widetilde{\pi}\big(\widetilde{\mathscr{C}}, \widetilde{0}\big)$ with any analytic structure such that

$$(C, \mathbf{0}) \hookrightarrow (\mathscr{C}', x_0) \to (T, t_0)$$

is a deformation of $(C, \mathbf{0})$. Then $\mathcal{O}_{\mathscr{C}',x_0} \to \big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}$ is injective by Proposition B.5.3, since this is so after tensoring with $\mathbb{C}$ over $\mathcal{O}_{T,t_0}$ and since $\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}$ is $\mathcal{O}_{T,t_0}$-flat. It follows that the ideal of $(\mathscr{C}', x_0)$ is the kernel of $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \twoheadrightarrow \mathcal{O}_{\mathscr{C}',x_0} \hookrightarrow \big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}$. Since $\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}$ is a ring with 1, the kernel is just the annihilator of $\big(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}}\big)_{(\mathbf{0},t_0)}$ which coincides with the ideal of $(\mathscr{C}, x_0)$ as shown in Step 3 of the proof. □

Corollary 2.10. *With the assumptions of Proposition 2.9, we have:*

(1) Let F be a generator of $\mathrm{Fitt}(\widetilde{\pi}_*\mathcal{O}_{\widetilde{\mathscr{C}}})_{(\mathbf{0},t_0)}$*. Then F is a non-zerodivisor of $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$.*
(2) If (T, t_0) is reduced (respectively Cohen-Macaulay), then also $(\mathscr{C}, x_0)$ is reduced (respectively Cohen-Macaulay). If (T, t_0) and $(\widetilde{C}, \widetilde{x}_0)$ are normal, then $(\widetilde{\mathscr{C}}, \widetilde{x}_0)$ is also normal, and $(\widetilde{\mathscr{C}}, \widetilde{x}_0) \to (\mathscr{C}, x_0)$ is the normalization of $(\mathscr{C}, x_0)$.

Proof. (1) Tensoring $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \xrightarrow{\cdot F} \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)} \to \mathcal{O}_{\mathscr{C},x_0} \to 0$ by $\mathbb{C}$ over $\mathcal{O}_{T,t_0}$, we can argue as in Step 1 of the proof of Proposition 2.9 to see that multiplication by F is injective.

(2) If (T, t_0) is reduced, $\mathcal{O}_{\widetilde{\mathscr{C}},\widetilde{x}_0}$ and, hence, $\mathcal{O}_{\mathscr{C},x_0}$ (which is a subring by the second part of Proposition 2.9), has no nilpotent elements. If (T, t_0) is Cohen-Macaulay, also $\mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ and, since F is a non-zerodivisor, $\mathcal{O}_{\mathscr{C},x_0}$ are Cohen-Macaulay rings (Corollary B.8.3).

If $\big(\widetilde{C}, \widetilde{0}\big)$ is normal, it is smooth and each deformation of $\big(\widetilde{C}, \widetilde{0}\big)$ is trivial. Hence, $\big(\widetilde{\mathscr{C}}, \widetilde{x}_0\big) \cong \big(\widetilde{C}, \widetilde{0}\big) \times (T, t_0)$ which is normal if (T, t_0) is normal. The singular locus $\mathrm{Sing}(\mathscr{C})$ is everywhere of codimension one (since the fibres of $\mathscr{C} \to T$ have isolated singularities). Thus, Sard's theorem, applied to $\widetilde{\pi} : \widetilde{\mathscr{C}} \setminus \widetilde{\pi}^{-1}(\mathrm{Sing}(\mathscr{C})) \to \mathscr{C} \setminus \mathrm{Sing}(\mathscr{C})$, shows that $\widetilde{\pi}$ is generically an isomorphism. The result follows now from the universal property of normalization (see Theorem I.1.95). □

Proposition 2.11. *Let $f = f_1 f_2$, with $f_1, f_2 \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$ non-units, define a reduced plane curve singularity $(C, \mathbf{0})$, and let $F \in \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ define an equisingular deformation of $(C, \mathbf{0})$ over an arbitrary complex space germ (T, t_0).*

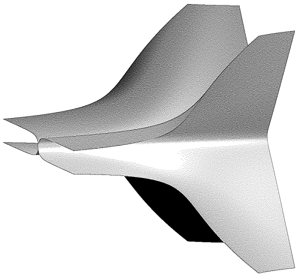


Fig. 2.8. The deformation of an A_3-singularity given by $x^2 - y^4 + tx^2y^2$ is equisingular along the trivial section. It splits into the equisingular deformations of the smooth branches given by $x\sqrt{1+ty^2} - y^2$ and $x\sqrt{1+ty^2} + y^2$ (real picture).

Then F decomposes as $F = F_1F_2$, where $F_1, F_2 \in \mathcal{O}_{\mathbb{C}^2\times T,(\mathbf{0},t_0)}$ define equisingular deformations of the plane curve germs at $\mathbf{0}$ defined by f_1 and f_2, respectively. Moreover, F_1 and F_2 are unique up to multiplication by units.

Proof. Since F defines an (embedded) equisingular deformation of $(C,\mathbf{0})$, Definition 2.6 gives rise to a Cartesian diagram

$$\begin{tikzcd}
\big(C^{(N)},0^{(N)}\big) \arrow[r, hook] \arrow[d, "\pi"] \arrow[dr, phantom, "\square"] & \big(\mathscr{C}^{(N)},0^{(N)}\big) \arrow[d, "\widetilde{\pi}"'] \\
(\mathbb{C}^2,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathbb{C}^2\times T,(\mathbf{0},t_0)) \arrow[d, "p"'] \\
\{t_0\} \arrow[r, hook] & (T,t_0)\,,
\end{tikzcd} \tag{2.1.3}$$

where $(C^{(N)},0^{(N)})$ is the multigerm of the strict transform of $(C,\mathbf{0}) = V(f)$ at the intersection points with the exceptional divisor. Moreover,

$$\widetilde{\pi}\big(\mathscr{C}^{(N)},0^{(N)}\big) = V(F) \subset (\mathbb{C}^2\times T,(\mathbf{0},t_0))\,,$$

and $\pi : \big(C^{(N)},0^{(N)}\big) \to (C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ is a resolution of the singularity of $(C,\mathbf{0})$. In particular, outside the special fibre π is an isomorphism onto $(C,\mathbf{0})\setminus\{\mathbf{0}\}$, the multigerm $\big(C^{(N)},0^{(N)}\big)$ is smooth, and it can be written as the disjoint union of (multi)germs $\big(C^{(N)},0^{(N)}\big) = \big(C_1^{(N)},0_1^{(N)}\big) \amalg \big(C_2^{(N)},0_2^{(N)}\big)$ such that $\pi\big(C_i^{(N)},0_i^{(N)}\big) = (C_i,\mathbf{0}) := V(f_i)$ for $i = 1,2$.

Hence, $\big(\mathscr{C}^{(N)},0^{(N)}\big) = \big(\mathscr{C}_1^{(N)},0_1^{(N)}\big) \amalg \big(\mathscr{C}_2^{(N)},0_2^{(N)}\big)$ and the composition

$$\big(\mathscr{C}_i^{(N)},0_i^{(N)}\big) \xrightarrow{\widetilde{\pi}} (\mathbb{C}^2\times T,(\mathbf{0},t_0)) \longrightarrow (T,t_0)$$

is flat and specializes to $\big(C_i^{(N)}, 0_i^{(N)}\big) \xrightarrow{\pi} (\mathbb{C}^2, \mathbf{0}) \to \{t_0\}$. We get diagrams analogous to (2.1.3) for $\big(C_i^{(N)}, 0_i^{(N)}\big) \hookrightarrow \big(\mathscr{C}_i^{(N)}, 0_i^{(N)}\big)$, $i = 1, 2$, and all these diagrams satisfy the assumptions of Proposition 2.9. Applying the latter yields $F_1, F_2 \in \mathcal{O}_{\mathbb{C}^2 \times T, 0}$ such that $V(F_i) = \widetilde{\pi}\big(\mathscr{C}_i^{(N)}, 0_i^{(N)}\big) =: (\mathscr{C}_i, x_0)$.

Since, as a set, $(\mathscr{C}, x_0) = (\mathscr{C}_1, x_0) \cup (\mathscr{C}_2, x_0)$, and since the structures defined by F, respectively by F_1F_2, define both deformations of $(C, \mathbf{0})$, the uniqueness statement of Proposition 2.9 implies $\langle F \rangle = \langle F_1 F_2 \rangle$. That is, $F = F_1F_2$ up to multiplication by units.

It is clear that the diagram (2.1.2) for $(C, \mathbf{0}) \hookrightarrow (\mathscr{C}, x_0) \to (T, t_0)$ induces diagrams for $(C_i, \mathbf{0}) \hookrightarrow (\mathscr{C}_i, x_0) \to (T, t_0)$, $i = 1, 2$. Since the strict transforms of $(\mathscr{C}, x_0)$ are equimultiple along the sections in the diagram, and since the multiplicities of the strict transforms of $(\mathscr{C}_1, x_0)$ and $(\mathscr{C}_2, x_0)$ add up to the multiplicity of the respective strict transform of $(\mathscr{C}, x_0)$, it follows from semicontinuity of multiplicities that the strict transforms of $(\mathscr{C}_1, x_0), (\mathscr{C}_2, x_0)$ are equimultiple along the sections, too. As the reduced exceptional divisors are equimultiple along the sections in the diagram, the reduced total transforms of $(\mathscr{C}_1, x_0), (\mathscr{C}_2, x_0)$ are equimultiple along the sections, that is, the deformations defined by F_1, F_2 are equisingular. □

Remark 2.11.1. Conversely, in general, not every product of equisingular deformations of the branches defines an equisingular deformation of $(C, \mathbf{0})$ (even if the singular sections coincide). However, if $f = f_1 \cdot \ldots \cdot f_s$ and if the germs defined by the factors f_i have pairwise no common tangent direction, then every product of equisingular deformations along a (unique) singular section σ defines an equisingular deformation of $(C, \mathbf{0})$.

To show this, we may assume that $s = 2$ and that σ is the trivial section. Let $F_1 = f_1 + h_1$, $F_2 = f_2 + h_2$ define equisingular deformations of $V(F_1)$, $V(F_2)$ along σ. Then the product F_1F_2 obviously defines an equimultiple deformation along σ. Since no branch of $V(f_1)$ has the same tangent direction as a branch of $V(f_2)$, the equimultiple sections $\sigma_j^{(\ell)}$, $\ell \geq 1$, for the equisingular deformation defined by F_1 are disjoint to the equimultiple sections for the equisingular deformation defined by F_2. As the strict transform of f_2 at $\sigma_j^{(\ell)}(t_0)$ is a unit, the multiplicity of the strict transform of F_1F_2 along such a section $\sigma_j^{(\ell)}$ equals the multiplicity of the strict transform of $f_1 + h_1$ along $\sigma_j^{(\ell)}$. Thus, F_1F_2 defines an equimultiple deformation of the strict transform of $V(f_1f_2)$ along $\sigma_j^{(\ell)}$. As the deformation along $\sigma_j^{(\ell)}$ of the reduced exceptional divisor induced by F_1F_2 coincides with the one induced by F_1, and as the analogous statements hold for F_2, Remark 2.6.1 (3) implies that F_1F_2 defines an equisingular deformation of $V(f_1f_2)$.

Proposition 2.12. *Let $\phi : (\mathscr{C}, x_0) \hookrightarrow (\mathscr{M}, x_0) \to (T, t_0)$ be an embedded equisingular deformation of $(C, \mathbf{0})$ along the section $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, \mathbf{0})$ with $(T, \mathbf{0})$ reduced. Assume further that $(C, \mathbf{0}) = (C_1, \mathbf{0}) \cup (C_2, \mathbf{0})$ where $(C_1, \mathbf{0})$ and $(C_2, \mathbf{0})$ are reduced plane curve singularities without common components, and* *let* $\phi_i : (\mathscr{C}_i, x_0) \hookrightarrow (\mathscr{M}, x_0) \to (T, t_0)$ *be the induced deformations of* $(C_i, \mathbf{0})$ *along* σ, $i = 1, 2$ *(Proposition 2.11). Then, for a sufficiently small representative* $\mathscr{C} \to \mathscr{M} \to T$*, the following holds:*

(1) The number of branches of $\mathscr{C}$ *is constant along* σ*, that is,*

$$r(\mathscr{C}_t, \sigma(t)) = r(C, \mathbf{0}), \quad t \in T,$$

where $\mathscr{C}_t = \phi^{-1}(t)$*.*

(2) The intersection multiplicity of $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *is constant along* σ*, that is,*

$$i_{\sigma(t)}(\mathscr{C}_{1,t}, \mathscr{C}_{2,t}) = i_{\mathbf{0}}(C_1, C_2), \quad t \in T,$$

where $\mathscr{C}_{i,t} = \phi_i^{-1}(t)$*.*

We call families $\mathscr{C}_1 \to T$ and $\mathscr{C}_2 \to T$ satisfying property (2) *equiintersectional along* σ.[12]

Proof. (1) We use the notations of Definition 2.6 and consider the induced sequence over $t \in T$, $\mathscr{C}_t^{(N)} \to \mathscr{C}_t^{(N-1)} \to \ldots \to \mathscr{C}_t^{(0)} = \mathscr{C}_t$. Since the space $\mathscr{C}_t^{(N)}$ has $r = r(C, \mathbf{0})$ connected components, $r(\mathscr{C}_t, \sigma(t)) \geq r$. If we would have $r(\mathscr{C}_t, \sigma(t)) > r$, then the map $\mathscr{C}_t^{(N)} \to \mathscr{C}_t$ cannot be surjective on all branches and, hence, there exists some ℓ such that the number of points in $\mathscr{C}_t^{(\ell)} \cap \mathscr{E}_t^{(\ell)}$ exceeds the number of points in $\mathscr{C}_{\mathbf{0}}^{(\ell)} \cap \mathscr{E}_{\mathbf{0}}^{(\ell)}$. Then there is some $1 \leq j \leq k_\ell$ such that $\operatorname{mt}\bigl(\mathscr{C}_t^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_j^{(\ell)}(t)\bigr) < \operatorname{mt}\bigl(\mathscr{C}_{\mathbf{0}}^{(\ell)} \cup \mathscr{E}_{\mathbf{0}}^{(\ell)}, \sigma_j^{(\ell)}(\mathbf{0})\bigr)$ contradicting equisingularity.

(2) It follows from Proposition I.3.21, p. 190, that

$$i_{\mathbf{0}}(C_1, C_2) = \sum_q \operatorname{mt}(C_1^{(\ell)}, q) \operatorname{mt}(C_2^{(\ell)}, q), \tag{2.1.4}$$

where q runs through all infinitely near points belonging to $(C, \mathbf{0})$. Note that $\operatorname{mt}(C_i^{(\ell)}, q) = 0$ if $q \notin C_i^{(\ell)}$.

Since $\operatorname{mt}(C_{i,t}^{(\ell)}, \sigma(t))$ and $r(C_{i,t}^{(\ell)}, \sigma(t))$ (by (1)) are constant, the induced sequence

$$\mathscr{C}_t^{(N)} \to \mathscr{C}_t^{(N-1)} \to \ldots \to \mathscr{C}_t^{(0)} = \mathscr{C}_t$$

is an embedded resolution of $\mathscr{C}_t$. Since, by definition of equisingularity, $\operatorname{mt}\bigl(\mathscr{C}_{i,t}^{(\ell)}, \sigma_j^{(\ell)}(t)\bigr)$ is constant in t (for $i = 1, 2$ and all ℓ and j), we get the equality $i_{\sigma(t)}(\mathscr{C}_{1,t}, \mathscr{C}_{2,t}) = i_{\mathbf{0}}(C_1, C_2)$ by applying (2.1.4) to $\mathscr{C}_{1,t}$ and $\mathscr{C}_{2,t}$. $\square$

[12] This notion is generalized to non-reduced base spaces (T, t_0) in Definition 2.65, p. 364.

Proposition 2.13. *Let $(C_i, \mathbf{0})$, $i = 1, \ldots, r$, be the branches of the reduced plane curve singularity $(C, \mathbf{0})$. Let $(\mathscr{C}_i, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be embedded deformations of $(C_i, \mathbf{0})$ given by $F_i \in \mathcal{O}_{\mathscr{M},\mathbf{0}}$, and let $(\mathscr{C}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be the deformation of $(C, \mathbf{0})$ given by $F = F_1 \cdot \ldots \cdot F_r$. Let $(T, \mathbf{0})$ be reduced. Then $(\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along a section $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, \mathbf{0})$ iff for a sufficiently small representative T of $(T, \mathbf{0})$ the following holds:*

(1) the number of branches of $\mathscr{C}$ is constant along σ, that is, $r(\mathscr{C}_t, \sigma(t)) = r(C, \mathbf{0})$ for $t \in T$,
(2) the pairwise intersection multiplicity of $\mathscr{C}_i$ and $\mathscr{C}_j$ is constant along σ, that is, $i_{\sigma(t)}(\mathscr{C}_{i,t}, \mathscr{C}_{j,t}) = i_{\mathbf{0}}(C_i, C_j)$ for $i \neq j$ and $t \in T$, and
(3) $(\mathscr{C}_i, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along σ for $i = 1, \ldots, r$.

Proof. If $(\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along σ, then (1)–(3) follow from Propositions 2.11 and 2.12. For the converse, we use the notation as in Proposition 2.12. If $r(\mathscr{C}_t, \sigma(t))$ is constant then $\mathscr{C}_t^{(N)} \to \mathscr{C}_t$ is an embedded resolution of $\mathscr{C}_t$, since $\mathrm{mt}(\mathscr{C}_t, \sigma(t))$ is constant by (3). Since $(\mathscr{C}_i, \mathbf{0}) \to (T, \mathbf{0})$ is equisingular along σ, the multiplicity $\mathrm{mt}\big(\mathscr{C}_{i,t}^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_k^{(\ell)}(t)\big)$ is constant for all ℓ, k such that $\sigma_k^{(\ell)}(t)$ belongs to $\mathscr{C}_{i,t}^{(\ell)}$. Since the intersection multiplicity $i_{\sigma(t)}(\mathscr{C}_{i,t}, \mathscr{C}_{j,t})$ is constant, we have that $\mathrm{mt}\big(\mathscr{C}_{i,t}^{(\ell)} \cup \mathscr{C}_{j,t}^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_k^{(\ell)}(t)\big)$ is constant if $\sigma_k^{(\ell)}(t)$ belongs to $\mathscr{C}_{i,t}^{(\ell)} \cap \mathscr{C}_{j,t}^{(\ell)}$ by (2.1.4). It follows that $\mathrm{mt}\big(\mathscr{C}_t^{(\ell)} \cup \mathscr{E}_t^{(\ell)}, \sigma_k^{(\ell)}(t)\big)$ is constant for all $\ell = 0, \ldots, N-1$ and $1 \leq k \leq k_\ell$. $\square$

2.2 The Equisingularity Ideal

In this section, we study first order equisingular deformations, that is, equisingular deformations over the fat point $T_\varepsilon = \big(\{0\}, \mathbb{C}[\varepsilon]/\varepsilon^2\big)$. The main result is the following proposition:

Proposition 2.14. *Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity with local equation $f \in \mathbb{C}\{x, y\}$. Then the following holds:*

(1) The set

$$I^{es}(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} \text{there exists a section } \sigma \text{ such that } f + \varepsilon g \\ \text{defines an equisingular deformation of} \\ (C, \mathbf{0}) \text{ over } T_\varepsilon \text{ along } \sigma \end{array} \right\}$$

is an ideal containing the Tjurina ideal $\langle f, j(f) \rangle$, where $j(f) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$.

(2) The subset

$$I^{es}_{\mathit{fix}}(f) := \left\{ g \in I^{es}(f) \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation} \\ \text{of } (C, \mathbf{0}) \text{ along the trivial section over } T_\varepsilon \end{array} \right\}.$$

of $I^{es}(f)$ is an ideal in $\mathbb{C}\{x, y\}$ containing $\langle f, \mathfrak{m} j(f) \rangle$. Moreover, as complex vector subspace of $\mathbb{C}\{x, y\}$, $I^{es}(f)$ is spanned by $I^{es}_{\mathit{fix}}(f)$ and the transversal 2-plane spanned by the partials $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. Furthermore, we have

$$\mathrm{mt}(g) \geq \begin{cases} \mathrm{mt}(f) - 1 & \text{if } g \in I^{es}(f)\,, \\ \mathrm{mt}(f) & \text{if } g \in I^{es}_{fix}(f)\,. \end{cases}$$

Definition 2.15. The ideal $I^{es}(f) \subset \mathbb{C}\{x, y\}$ is called the *equisingularity ideal* of $f \in \mathbb{C}\{x, y\}$. $I^{es}_{fix}(f)$ is called the *fixed equisingularity ideal* of f.

We prove Proposition 2.14 by induction on the Milnor number, making blowing-ups as induction steps.

Recall that just requiring the multiplicities of the strict transforms in the blown up family to stay constant is not sufficient to get equisingularity of the original deformation. Indeed, the equisingularity condition in the induction step translates to an equisingularity condition for the strict transform *plus extra conditions on the intersection with fixed smooth germs*, namely the components of the exceptional divisor. This corresponds to the requirement that the multiplicities of the reduced *total* transforms are constant in the definition of equisingularity. Therefore, we have to consider a slightly more general situation in the induction step. This is the reason for introducing the ideals $I^{es}_L(f)$ and $I^{es}_{fix,L_1\dots L_k}(f)$ below.

The following example of the cusp $f = x^2 - y^3$ might be helpful for understanding the general situation: A first order (equisingular) deformation of the strict transform $C^{(1)} = \{u^2 - v = 0\}$ corresponds to an equisingular deformation $f + \varepsilon g$ of the cusp along the trivial section exactly if its equation is of the form $u^2 - v + \varepsilon g(uv, v)/v^2$ and if there is a section $\sigma_\alpha : T_\varepsilon \to E$ such that the intersection multiplicity with $E = \{v = 0\}$ along σ_α is constant. In other words, if $I_{\sigma_\alpha} = \langle v, u - \varepsilon\alpha\rangle$ with $\alpha \in \mathbb{C}$, then we require

$$\mathrm{ord}_t\Big((t + \varepsilon\alpha)^2 + \varepsilon g^{(1)}(t{+}\varepsilon\alpha, 0)\Big) = 2\,, \tag{2.2.5}$$

for $g^{(1)}(u, v) := g(uv, v)/v^2$. Now, we must continue blowing up. Note that

$$(t + \varepsilon\alpha)^2 + \varepsilon g^{(1)}(t{+}\varepsilon\alpha, 0) = t^2 + 2\varepsilon\alpha t + \varepsilon g^{(1)}(t, 0)\,.$$

Hence, replacing $g^{(1)}$ by $g^{(1)} - 2\alpha u = g^{(1)} - \alpha\frac{\partial(u^2-v)}{\partial u}$, we may assume that $\alpha = 0$, that is, σ_α is the trivial section. Then the above condition on the intersection multiplicity is equivalent to $i_{\mathbf{0}}(g^{(1)} - 2\alpha u, E) \geq 2 = i_{\mathbf{0}}(u^2 - v, E)$.

Similarly, a first order (equisingular) deformation of $C^{(2)}$ corresponds to an (equisingular) deformation of $C^{(1)}$ satisfying (2.2.5) for $\alpha = 0$ iff its equation has the form $\overline{u} - \overline{v} + \varepsilon g^{(1)}(\overline{u}, \overline{uv})/\overline{u}$ and the intersection point with the components of the exceptional divisor does not move. The latter means that

$$\begin{aligned} \mathrm{ord}_t\big(t + \varepsilon g^{(2)}(t, 0)\big) &\geq 1 = i_{\mathbf{0}}(\overline{u} - \overline{v}, E_1)\,, \\ \mathrm{ord}_t\big({-t} + \varepsilon g^{(2)}(0, t)\big) &\geq 1 = i_{\mathbf{0}}(\overline{u} - \overline{v}, E_2)\,, \end{aligned}$$

where $g^{(2)}(\overline{u}, \overline{v}) := g^{(1)}(\overline{u}, \overline{uv})/\overline{u}$.

This example suggests that in the inductive proof we should not only consider $I^{es}(f)$ but also the following auxiliary objects: let $L, L_1, \dots, L_k \subset (\mathbb{C}^2, \mathbf{0})$ denote smooth germs (respectively their local equations) through the origin with different tangent directions. Consider the sections $\sigma_\alpha : T_\varepsilon \to L$ given by the ideal $I_{\sigma_\alpha} := \langle x - \varepsilon\alpha\ell_1, y - \varepsilon\alpha\ell_2\rangle$, $\alpha \in \mathbb{C}$, where $\boldsymbol{\ell} = (\ell_1, \ell_2) \in \mathbb{C}^2$ is a fixed tangent vector to L, and let $\sigma_0 : T_\varepsilon \to (\mathbb{C}^2, \mathbf{0})$ be the trivial section. Define

$$I_L^{es}(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation} \\ \text{of } (C, \mathbf{0}) \text{ with singular section } \sigma_\alpha \text{ in } L \text{ and} \\ i_{\sigma_\alpha}(f + \varepsilon g, L) = i_{\mathbf{0}}(f, L) \end{array} \right\},$$

$$I^{es}_{\mathit{fix}, L_1..L_k}(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation} \\ \text{with trivial singular section } \sigma_0 \text{ and} \\ i_{\sigma_0}(f + \varepsilon g, L_j) = i_{\mathbf{0}}(f, L_j) \text{ for } j = 1, \dots, k \end{array} \right\}.$$

Here i_{σ_α} denotes the intersection multiplicity along σ_α, that is,

$$i_{\sigma_\alpha}(f + \varepsilon g, L) := \operatorname{ord}_t\big(f(t\boldsymbol{\ell} - \varepsilon\alpha\boldsymbol{\ell}) + \varepsilon g(t\boldsymbol{\ell} - \varepsilon\alpha\boldsymbol{\ell})\big),$$

and we assume that the intersection multiplicities $i_{\mathbf{0}}(f, L)$ and $i_{\mathbf{0}}(f, L_j)$, $j = 1, \dots, k$, are finite.

Proof of Proposition 2.14. We show that, for any smooth germs $L, L_1, \dots, L_k$ ($k \geq 0$) as above, $I^{es}_{\mathit{fix}, L_1 \dots L_k}(f)$, $I^{es}_L(f)$ and $I^{es}(f)$ are ideals in $\mathbb{C}\{x,y\}$.

Step 1. We show that it suffices to prove the claim for $I^{es}_{\mathit{fix}, L_1 \dots L_k}(f)$, $k \geq 0$.

Step 1a. Assume that $I^{es}_{\mathit{fix}, L}(f)$ is an ideal. Then $I^{es}_L(f)$ is an ideal, spanned as a linear space by $I^{es}_{\mathit{fix}, L}(f)$ and f'_L, the derivative of f in the direction of L. Furthermore, f'_L does not belong to $I^{es}_{\mathit{fix}, L}(f)$.

Indeed, $f + \varepsilon g$ defines an equisingular deformation with singular section σ_α, $\alpha \in \mathbb{C}$, iff the deformation induced by

$$\begin{aligned} &f(x - \varepsilon\alpha\ell_1, y - \varepsilon\alpha\ell_2) + \varepsilon g(x - \varepsilon\alpha\ell_1, y - \varepsilon\alpha\ell_2) \\ &\qquad \equiv f(x,y) - \varepsilon \cdot \Big(\alpha \cdot \underbrace{\big(\ell_1 \tfrac{\partial f}{\partial x}(x,y) + \ell_2 \tfrac{\partial f}{\partial y}(x,y)\big)}_{=:\, f'_L(x,y)} - g(x,y)\Big), \end{aligned}$$

is equisingular along the trivial section. We conclude that $I^{es}_L(f)$ is spanned as a linear space by $I^{es}_{\mathit{fix}, L}(f)$ and by f'_L.

To show that $I^{es}_L(f)$ is an ideal, we show that $\mathfrak{m} \cdot f'_L \subset I^{es}_{\mathit{fix}, L}(f)$. Indeed,

$$\mathfrak{m} \cdot f'_L \subset j_{\mathit{fix}, L}(f) := \big\{ g \in \mathbb{C}\{x,y\} \;\big|\; i_{\mathbf{0}}(g, L) \geq i_{\mathbf{0}}(f, L) \big\} \cap \mathfrak{m} \cdot j(f),$$

since $i_{\mathbf{0}}(f'_L, L) = i_{\mathbf{0}}(f, L) - 1$. On the other hand, $j_{\mathit{fix}, L}(f) \subset I^{es}_{\mathit{fix}, L}(f)$, since the ideal $j_{\mathit{fix}, L}(f)$ just describes the infinitesimal locally trivial deformations of first order with trivial singular section and fixed intersection multiplicity with L. Since $i_{\sigma_0}(f + \varepsilon f'_L, L) = i_{\mathbf{0}}(f, L) - 1$, we have $f'_L \notin I^{es}_{\mathit{fix}, L}(f)$.

Step 1b. Assuming that $I^{es}_{\mathit{fix}}(f)$ is an ideal, we see in the same way that $I^{es}(f)$ is spanned as a linear space by $I^{es}_{\mathit{fix}}(f)$ and the transverse 2-plane spanned by $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$. We deduce that $I^{es}(f)$ is an ideal, the sum of the ideals $I^{es}_{\mathit{fix}}(f)$ and $j(f)$, since we have the linear decomposition $j(f) = \mathbb{C}\frac{\partial f}{\partial x} + \mathbb{C}\frac{\partial f}{\partial y} + \mathfrak{m}j(f)$. The inclusion $\mathfrak{m}j(f) \subset I^{es}_{\mathit{fix}}(f)$ results from the fact that $\mathfrak{m}j(f)$ describes the infinitesimal locally trivial deformations of first order with trivial singular section.

Observe that $\mathrm{mt}(g) \geq \mathrm{mt}(f)$ for all elements $g \in I^{es}_{\mathit{fix}}(f)$, since all germs equisingular to f have the same multiplicity at the singular point. In view of the preceding result, this yields, in particular, that $\mathrm{mt}(g) \geq \mathrm{mt}(f) - 1$ for all elements $g \in I^{es}(f)$, since $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ satisfy the latter inequality.

Step 2. We prove that $I^{es}_{\mathit{fix},L_1\dots L_k}(f)$, $k \geq 0$, is an ideal. To do so, we proceed by induction on the number of blowing ups needed to resolve f.

Step 2a. As base of induction, we consider the case of a non-singular germ $f \in \mathbb{C}\{x,y\}$. Then

$$I^{es}_{\mathit{fix},L_1\dots L_k}(f) = \bigl\{g \in \mathbb{C}\{x,y\} \bigm| i_{\mathbf{0}}(g, L_j) \geq i_{\mathbf{0}}(f, L_j)\,,\ j = 1, \dots, k\bigr\}\,,$$

which obviously defines an ideal in $\mathbb{C}\{x,y\}$.

Step 2b. Assume that f is singular. Let $\pi : M \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of the origin, and let $E \subset M$ be the exceptional divisor. Denote by $\widetilde{L}_1, \dots, \widetilde{L}_k \subset M$ the strict transforms of $L_1, \dots, L_k$, by $\widetilde{C}$ the strict transform of the germ $(C, \mathbf{0})$, and by $q_1, \dots, q_s$ the intersection points of $\widetilde{C}$ with E. Let $m := \mathrm{mt}(C, \mathbf{0})$, and let $\widetilde{f}_i \in \mathcal{O}_{M,q_i}$ be a local equation for the germ $\bigl(\widetilde{C}, q_i\bigr)$, $i = 1, \dots, s$.

From Definition 2.6, we see that $f + \varepsilon g$ is the defining equation of an equisingular deformation of $(C, \mathbf{0})$ with trivial singular section and fixed intersection multiplicities with $L_1, \dots, L_k$ iff it is mapped under the injective morphism

$$(\pi \times \mathrm{id}_{T_\varepsilon})^\sharp : \mathcal{O}_{\mathbb{C}^2 \times T_\varepsilon, (\mathbf{0},0)} \hookrightarrow \mathcal{O}_{M \times T_\varepsilon, (q_i, 0)}$$

to the product of the m-th power of the equation of the exceptional divisor E and the equation of an equisingular deformation of the germ $\bigl(\widetilde{C}, q_i\bigr)$ satisfying the following conditions:

- if one of $\widetilde{L}_1, \dots, \widetilde{L}_k$ passes through q_i, then the equisingular deformation of $\bigl(\widetilde{C}, q_i\bigr)$ has trivial singular section and fixed intersection multiplicities with $\widetilde{L}_1, \dots, \widetilde{L}_k$ and E (cf. Proposition I.3.21),
- if none of $\widetilde{L}_1, \dots, \widetilde{L}_k$ passes through q_i, then the equisingular deformation of $\bigl(\widetilde{C}, q_i\bigr)$ has singular section with values in E and it has fixed intersection multiplicity with E.

Correspondingly, $I^{es}_{\mathit{fix},L_1\dots L_k}(f)$ is the preimage of $\bigoplus_{i=1}^{s} E^m \cdot I_i$ under

$$\pi^\sharp : \mathbb{C}\{x,y\} \to \bigoplus_{i=1}^{s} \mathcal{O}_{M,q_i}\,,$$

where

$$I_i := \begin{cases} I^{es}_{fix,\widetilde{L}_1\ldots\widetilde{L}_k E}\big(\widetilde{f}_i\big), & \text{if } q_i \in \widetilde{L}_1 \cup \ldots \cup \widetilde{L}_k,\\ I^{es}_E\big(\widetilde{f}_i\big), & \text{if } q_i \notin \widetilde{L}_1 \cup \ldots \cup \widetilde{L}_k\,. \end{cases}$$

Finally, since resolving $\widetilde{f}_i$, $i = 1, \ldots, s$, needs less blowing ups than resolving f, the induction hypothesis and the result of Step 1a assure that $I_i \subset \mathcal{O}_{M,q_i}$, $i = 1, \ldots, s$, are ideals. Hence, $I^{es}_{fix,L_1\ldots L_k}(f)$ is an ideal, too. □

Example 2.15.1. We reconsider the proof of Proposition 2.14 to compute the equisingularity ideal for A_μ- and D_μ-singularities.

(1) Let $f_\mu := x^2 - y^{\mu+1} \in \mathbb{C}\{x,y\}$, $\mu \geq 1$. Then

$$I^{es}(f_\mu) = \langle f_\mu, j(f_\mu)\rangle = \langle x, y^\mu\rangle = \big\{g \in \mathbb{C}\{x,y\} \,\big|\, i_{\mathbf{0}}(x,g) \geq \mu\big\}\,, \tag{2.2.6}$$

and, for $L := \{y = 0\}$, we get

$$I^{es}_L(f_\mu) = \langle x, y^{\mu+1}\rangle\,, \quad I^{es}_{fix,L}(f_\mu) = I^{es}_{fix}(f_\mu) = \langle f_\mu, \mathfrak{m}\cdot j(f_\mu)\rangle\,.$$

Indeed, as L is transversal to $\{f_\mu = 0\}$, each equimultiple deformation along the trivial section preserves the intersection multiplicity with L (which is 2), hence, $I^{es}_{fix,L}(f_\mu) = I^{es}_{fix}(f_\mu)$. The proof of Proposition 2.14 then shows that, as a linear space, $I^{es}_L(f_\mu)$ is spanned by $I^{es}_{fix}(f_\mu)$ and the derivative $\frac{\partial f}{\partial x} = 2x$. Now, we proceed by induction on μ: For $\mu = 1$, equisingularity is equivalent to equimultiplicity. Hence, $I^{es}_{fix}(f_1) = \mathfrak{m}^2 = \langle f_1, \mathfrak{m}\cdot j(f_1)\rangle$ and $I^{es}(f_1) = \mathfrak{m} = \langle f_1, j(f_1)\rangle$.

For $\mu = 2$, the considerations right before the proof of Proposition 2.14 show that $g \in \mathbb{C}\{x,y\}$ defines an equisingular deformation of the cusp along the trivial section iff $g^{(2)} \in \langle\overline{u}, \overline{v}\rangle$, which is equivalent to $g^{(1)} - 2\alpha u \in \langle u^2, v\rangle$, and thus to $g \in \langle x^2, xy, y^3\rangle = \langle f_\mu, \mathfrak{m}\cdot j(f_2)\rangle$. As $I^{es}(f_2)$ is spanned by $I^{es}_{fix}(f_2)$ and the two partials of f_2, we get $I^{es}(f_2) = \langle f_2, j(f_2)\rangle$.

For $\mu \geq 3$, let $\pi : M \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of the origin. Then there is a unique intersection point of the strict transform of $\{f_\mu = 0\}$ with the exceptional divisor $E = \pi^{-1}(\mathbf{0})$. Locally at this point, the exceptional divisor is given by $\{v = 0\}$, and the strict transform is given by $\{u^2 - v^{\mu-1} = 0\}$. Together with the induction hypothesis, the proof of Proposition 2.14 shows that $I^{es}_{fix}(f_\mu)$ is the preimage of $v^2\cdot\langle u, v^{\mu-1}\rangle$ under $\pi^\sharp : (x,y) \mapsto (uv, v)$. Thus, $I^{es}_{fix}(f_\mu) = \langle x^2, xy, y^{\mu+1}\rangle$. Finally, $I^{es}(f_\mu)$ is spanned by $I^{es}_{fix}(f_\mu)$ and the two partials of f_μ. Thus, $I^{es}(f_\mu) = \langle x, y^\mu\rangle$.

(2) Let $g_\mu := y(x^2 - y^{\mu-2}) \in \mathbb{C}\{x,y\}$, $\mu \geq 4$. Then

$$I^{es}(g_\mu) = \langle g_\mu, j(g_\mu)\rangle\,, \quad I^{es}_{fix}(g_\mu) = \langle g_\mu, \mathfrak{m}\cdot j(g_\mu)\rangle = \langle x^3, x^2y, xy^2, y^{\mu-1}\rangle\,.$$

For $\mu = 4$, equisingularity is again equivalent to equimultiplicity. Hence, we get $I^{es}_{\mathit{fix}}(g_4) = \mathfrak{m}^3 = \langle g_4, \mathfrak{m} \cdot j(g_4)\rangle$ and $I^{es}(g_4) = \langle g_4, j(g_4)\rangle$.

Now, let $\mu \geq 5$, and let $\pi : M \to (\mathbb{C}^2, \mathbf{0})$ be the blowing up of the origin. Then there is a unique non-nodal singular point of the reduced total transform of $\{g_\mu = 0\}$ on the exceptional divisor $E = \pi^{-1}(\mathbf{0})$. Locally at this point, the exceptional divisor is given by $\{v = 0\}$, and the strict transform is given by $\{u^2 - v^{\mu-4} = 0\}$. For $\mu \geq 6$, the proof of Proposition 2.14, together with Case (1), gives that $I^{es}_{\mathit{fix}}(g_\mu)$ is the preimage of $v^3 \cdot \langle u, v^{\mu-4}\rangle$ under $\pi^\sharp : (x, y) \mapsto (uv, v)$. Thus, $I^{es}_{\mathit{fix}}(g_\mu) = \langle x^3, x^2y, xy^2, y^{\mu-1}\rangle = \langle g_\mu, \mathfrak{m}j(g_\mu)\rangle$, and $I^{es}(g_\mu) = \langle g_\mu, j(g_\mu)\rangle$.

It remains the case $\mu = 5$. Here, $I^{es}_{\mathit{fix}}(g_5)$ is the preimage under $\pi^\sharp$ of the ideal $v^3 \cdot I^{es}_{\mathit{fix},E}(u^2 - v)$ (with $E = \{v = 0\}$). As $I^{es}_{\mathit{fix},E}(u^2 - v) = \langle u^2, v\rangle$, this gives $I^{es}_{\mathit{fix}}(g_5) = \langle x^3, x^2y, xy^2, y^4\rangle$. Hence, $I^{es}(g_5) = \langle g_5, j(g_5)\rangle$.

This example shows that, for f defining an A_k- or a D_k-singularity, the equisingularity and the Tjurina ideal coincide. The same holds for f defining a singularity of type E_6, E_7, E_8, which we leave as an exercise:

Lemma 2.16. *If $f \in \mathbb{C}\{x, y\}$ defines an ADE-singularity, then*

$$I^{es}(f) = \langle f, j(f)\rangle\,, \quad I^{es}_{\mathit{fix}}(f) = \langle f, \mathfrak{m}j(f)\rangle\,.$$

The next proposition gives a general description of the equisingularity ideal in the case of reduced semiquasihomogeneous, respectively Newton non-degenerate (NND, see Definition I.2.15), plane curve singularities. Note that, for the NND polynomial $f = (x^2 - y^3)(y^2 - x^3)$, we get $I^{es}(f) = \langle f, j(f)\rangle$ and $I^{es}_{\mathit{fix}}(f) = \langle f, \mathfrak{m} \cdot j(f)\rangle$, but f does not define an ADE-singularity. Hence, the inverse implication in Lemma 2.16 does not hold.

Proposition 2.17. *Let $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity with local equation $f \in \mathbb{C}\{x, y\}$.*

(1) If $f = f_0 + f'$ is semiquasihomogeneous with principal part f_0 being quasihomogeneous of type $(w_1, w_2; d)$, then

$$I^{es}(f) = \langle j(f),\, x^\alpha y^\beta \mid w_1\alpha + w_2\beta \geq d\rangle\,.$$

(2) If f is Newton non-degenerate with Newton diagram $\Gamma(f, \mathbf{0})$ at the origin, then the equisingularity ideal is generated by $j(f)$ and the monomials corresponding to points $(\alpha, \beta) \in \mathbb{N}^n$ on or above $\Gamma(f, \mathbf{0})$.[13]

[13] These monomials are just the monomials of Newton order ≥ 1, where we say that a monomial has *Newton order* $\delta \in \mathbb{R}$ (w.r.t. f) iff it corresponds to a point on the hypersurface $\delta \cdot \Gamma(f, \mathbf{0}) \subset \mathbb{R}^2$.

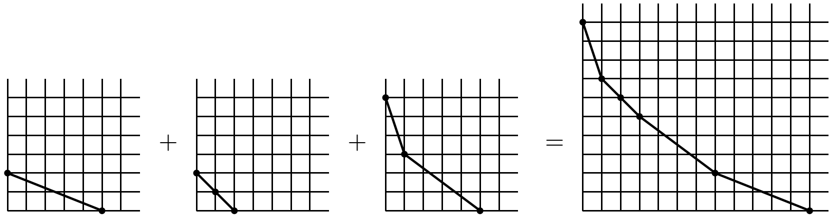


Fig. 2.9. Newton diagram of $f = (x^5 - y^2)(x^2 + xy - y^2)(y^6 - y^3x + x^5)$.

Remark 2.17.1. In fact, for a reduced Newton non-degenerate power series f, we prove the following: let $J, I^s(f) \subset \mathbb{C}\{x,y\}$ be the ideals

$$J = \langle x^\alpha y^\beta \mid x^\alpha y^\beta \text{ has Newton order } \geq 1 \rangle$$

and

$$I^s(f) := \left\{ g \in \mathbb{C}\{x,y\} \;\middle|\; \begin{array}{c} f + \varepsilon g \text{ defines an equisingular deformation of} \\ (C, \mathbf{0}) \text{ where the equimultiple sections through} \\ \text{all the infinitely near non-nodes of the reduced} \\ \text{total transform of } (C, \mathbf{0}) \text{ are trivial sections} \end{array} \right\}.$$

Then

$$I^{es}(f) = \langle j(f), I^{es}_{fix}(f) \rangle = \langle j(f), J \rangle = \langle j(f), I^s(f) \rangle .$$

The proof uses the following general facts for the Newton diagram at the origin of a power series $f \in \mathbb{C}\{x,y\}$ (see, e.g., [BrK, §8.4], [DJP, §5.1]):

- if f is irreducible then $\Gamma(f, \mathbf{0})$ has at most one facet;
- if $f = f_1 \cdot \ldots \cdot f_s$ then $\Gamma(f, \mathbf{0})$ is obtained by gluing the facets of $\Gamma(f_i, \mathbf{0})$ (suitably displaced, such that the resulting diagram looks like the graph of a convex piece-wise linear function, see Figure 2.9);
- the blowing up map $\pi^\sharp$ maps monomials of Newton order 1 (resp. ≥ 1, resp. ≤ 1) with respect to f to monomials of Newton order 1 (resp. ≥ 1, resp. ≤ 1) with respect to the total transform of f.

Proof of Proposition 2.17. As in the proof of Proposition 2.14 we proceed by induction, making blowing-ups as induction steps. We simultaneously treat the case of semiquasihomogeneous and Newton non-degenerate singularities.

Actually, what we suppose is that f is reduced and either Newton non-degenerate, or a product of type $f = xf'$ (respectively $f = yf'$), or $f = xyf'$, with f' Newton non-degenerate. In the latter ("non-convenient") cases, we consider a modified Newton diagram $\Gamma(f, \mathbf{0})$ to define the Newton order. For this, we omit vertical and horizontal facets (if they exist) and extend (if necessary) the facets of maximal and minimal slope such that they touch the x- and y-axis, respectively.

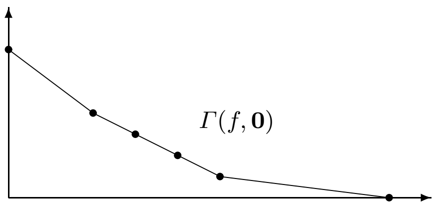


Fig. 2.10. Newton diagram $\Gamma(f,\mathbf{0})$ of a unitangential NND singularity.

Proposition 2.14 shows that we always have $I^{es}(f) = \langle j(f), I^{es}_{fix}(f)\rangle$ and $\langle j(f), I^s(f)\rangle \subset I^{es}(f)$. Thus, it suffices to prove that, under the given assumptions, $\langle j(f), I^{es}_{fix}(f)\rangle$ is contained in the ideal generated by $j(f)$ and the monomials of Newton order ≥ 1 w.r.t. f, and that the latter ideal is contained in $\langle j(f), I^s(f)\rangle$.

Step 1. We show that $I^s(f)$ contains the ideal generated by all monomials of Newton order ≥ 1 w.r.t. f.

Case A. f defines an ordinary singularity (including f smooth).

Then, $I^s(f) = I^{es}_{fix}(f) = \langle x, y\rangle^{\mathrm{mt}(f)}$, hence the statement.

Case B. f is singular and unitangential.

Since f is either SQH or NND, the tangent can only be x or y. Let us assume that it is y, that is, the Newton diagram has no facet of slope ≤ -1 (see also Figure 2.10). In particular, when blowing-up the origin, it suffices to consider the chart $x = u$, $y = uv$.

Now, let $g = x^\alpha y^\beta$ be a monomial of Newton order ≥ 1. Then, in particular, $f + \varepsilon g$ is equimultiple along the trivial section and its reduced total transform is given by

$$u \cdot \left(\frac{f(u,uv)}{u^{\mathrm{mt}(f)}} + \varepsilon \cdot \frac{g(u,uv)}{u^{\mathrm{mt}(f)}} \right) = u\widetilde{f}(u,v) + \varepsilon \cdot \frac{g(u,uv)}{u^{\mathrm{mt}(f)-1}}\,. \tag{2.2.7}$$

Since $g(u,uv)$ is a monomial of Newton order ≥ 1 w.r.t. the total transform $u^{\mathrm{mt}(f)}\widetilde{f}$, the induction hypothesis gives that (2.2.7) defines an equisingular deformation with all equimultiple sections through non-nodes of the reduced total transform of $u\widetilde{f}$ being trivial sections. Hence, $g \in I^s(f)$.

Case C. f has at least two tangential components.

It might happen that the Newton diagram has facets of slope < -1 (all corresponding to branches with tangent x), > -1 (tangent y), and $= -1$ (tangent $\alpha x + \beta y$, $\alpha, \beta \neq 0$). Since f is NND, the last-mentioned branches define an ordinary singularity, hence, impose no equisingularity condition to the respective strict transforms.

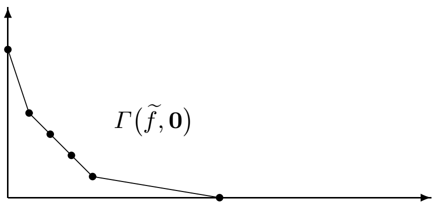


Fig. 2.11. Newton diagram at the origin of the strict transform.

Thus, we can conclude as in Case B, considering both charts of the blowing-up map and noting that the Newton diagram at the origin of the respective strict transform equals the Newton diagram of the strict transform of the respective tangential component.

Step 2. We prove that $I^{es}_{\mathit{fix}}(f)$ coincides with the ideal J generated by xf_y, yf_x and the monomials of Newton order ≥ 1 w.r.t. f.

What we actually claim is that $I^{es}_{\mathit{fix}}(f) \subset J$ (the other inclusion is given by Step 1 and Proposition 2.14). To see this, note that the inclusion $I^{es}_{\mathit{fix}}(f) \subset J$ holds true for ordinary singularities (see Case A, above). It remains to consider Cases B,C.

Case B. f is singular and unitangential.

Let us assume that $f + \varepsilon g$, $g \in \mathbb{C}\{x, y\}$, defines an equisingular deformation with trivial singular section. In particular, this implies that the reduced total transform (2.2.7) is equisingular (with singular section in the exceptional divisor $\{u = 0\}$). Proposition 2.11 implies that

$$u \cdot \left(\widetilde{f}(u,v) + \varepsilon \cdot \frac{g(u,uv)}{u^{\mathrm{mt}(f)}} \right) = \Big(u + \varepsilon g_1(u,v) \Big) \cdot \Big(\widetilde{f}(u,v) + \varepsilon g_2(u,v) \Big) \ ,$$

such that both factors define equisingular deformations with singular section in $\{u = 0\}$. Hence, $g_1 \in u \cdot \mathbb{C}\{u, v\}$, and Proposition 2.14 (respectively its proof) and the induction hypothesis give

$$\frac{g(u,uv)}{u^{\mathrm{mt}(f)}} \in \left\langle v\widetilde{f}_u, \widetilde{f}_v, \text{terms of Newton order} \geq 1 \text{ w.r.t. } \widetilde{f} \right\rangle . \tag{2.2.8}$$

Those monomials in g leading to terms of Newton order ≥ 1 w.r.t. $\widetilde{f}$ have Newton order ≥ 1 w.r.t. f, hence, are contained in $I^{es}_{\mathit{fix}}(f)$ by Step 1. Moreover, we compute that

$$\widetilde{f}_v = \frac{\partial}{\partial v} \left(\frac{f(u,uv)}{u^{\mathrm{mt}(f)}} \right) = \frac{u \cdot f_y(u,uv)}{u^{\mathrm{mt}(f)}}$$

is the image for $g = xf_y$, which is in $I^{es}_{\mathit{fix}}(f)$, too.

The latter proves the claim as long as $\widetilde{f}$ has no component with tangent u (then $v\widetilde{f}_u$ has Newton order ≥ 1). On the other hand, if some components have tangent u, then some of the higher equimultiple sections of (2.2.7) have to be in the strict transform of $\{u = 0\}$.

More precisely, let $-\rho$ be the slope of the steepest facet in $\Gamma\big(\widetilde{f}, \mathbf{0}\big)$ (see also Figure 2.11), then we have the above restriction for $N := \lceil \rho \rceil$ successive equimultiple sections (including the present one).

Of course, the latter imposes $N-1$ independent conditions to $g(u, uv)$. Since $u\widetilde{f}_u$ has Newton order ≥ 1 w.r.t. $\widetilde{f}$, this means that we have to exclude $v\widetilde{f}_u, \ldots, v^{N-1}\widetilde{f}_u$ on the right-hand side of (2.2.8). But, due to the choice of N, $v^N\widetilde{f}_u$ has Newton order ≥ 1. Hence, we conclude that

$$g \in \big\langle x f_y, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f \big\rangle .$$

Case C. f has at least two tangential components.

We can suppose that $f = f_1 f_2 f_3$, where f_1 has tangent x, f_2 defines an ordinary singularity, and f_3 has tangent y. Then, again by Proposition 2.11, any equisingular deformation $f + \varepsilon g$ with trivial singular section splits as

$$f + \varepsilon g = (f_1 + \varepsilon g_1) \cdot (f_2 + \varepsilon g_2) \cdot (f_3 + \varepsilon g_3) ,$$

where (in view of the above)

$$\begin{aligned} g_1 &\in \big\langle y f_{1,x}, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f_1 \big\rangle , \\ g_2 &\in \langle x, y \rangle^{\mathrm{mt}(f_2)} , \\ g_3 &\in \big\langle x f_{3,y}, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f_3 \big\rangle . \end{aligned}$$

In particular, since products of terms of Newton order ≥ 1 w.r.t. f_1, f_2, f_3, respectively, have Newton order ≥ 1 w.r.t. f, it is not difficult to see that the latter implies that

$$\begin{aligned} g \in \big\langle & y^{1+\mathrm{mt}(f_2)+\mathrm{mt}(f_3)} f_{1,x},\ x^{\mathrm{mt}(f_1)+\mathrm{mt}(f_2)+1} f_{3,y}, \\ & \qquad \text{terms of Newton order} \geq 1 \text{ w.r.t. } f \big\rangle . \\ = \big\langle & y f_x, x f_y, \text{ terms of Newton order} \geq 1 \text{ w.r.t. } f \big\rangle . \end{aligned}$$

Step 3. Combining Steps 1, 2 and Proposition 2.14, we conclude that

$$I^{es}(f) = \langle j(f), I^{es}_{\mathit{fix}}(f) \rangle \subset \langle j(f), J \rangle \subset \langle j(f), I^s(f) \rangle \subset I^{es}(f) .$$

Hence all inclusions are equalities, which implies the statement of the proposition. □

The proof of Proposition 2.17 shows that for f Newton non-degenerate, the equisingularity ideal is generated by the Tjurina ideal and the ideal $I^s(f)$.

This is caused by the fact that the equimultiple sections $\sigma_j^{(\ell)}$ through all infinitely near non-nodes can be simultaneously trivialized in this case. That is, each equisingular deformation of a reduced Newton non-degenerate plane curve singularity is isomorphic to an equisingular deformation where all the equimultiple sections $\sigma_j^{(i)}$ through the infinitely near non-nodes of the reduced total transform are globally trivial sections (see Proposition 2.69).

If f is Newton degenerate, however, this is not necessarily the case as the following example shows:

Example 2.17.2. Consider the Newton degenerate polynomial

$$f = (x-2y)^2(x-y)^2x^2y^2 + x^9 + y^9,$$

defining a germ consisting of four transversal cusps. Blowing up the origin, we get four intersection points $q_{1,1}, \dots, q_{1,4}$ of the strict transform with the exceptional divisor E_1, the germ of the strict transform being smooth (and tangential to E_1) at each $q_{1,j}$, see Figure 2.12. Thus, each deformation of the (multigerm of the) strict transform is equisingular along some sections $\sigma_1^{(1)}, \dots, \sigma_4^{(1)}$. However, since the cross-ratio of the four intersection points $q_{1,1}, \dots, q_{1,4}$ is preserved under isomorphisms, usually the sections can not be simultaneously trivialized. Consider, for instance, the 1-parameter deformation given by

$$F(x,y,t) = (x-2y)^2(x-y)^2(x+ty)^2y^2 + x^9 + y^9, \quad t \in \mathbb{C} \text{ close to } 0\,.$$

After blowing up the trivial section, the strict transform of F is a locally trivial deformation of the strict transform of f along sections $\sigma_1^{(1)}, \dots, \sigma_4^{(1)}$ in the exceptional divisor, where the cross-ratio of $\sigma_1^{(1)}(t), \dots, \sigma_4^{(1)}(t)$ varies in t. Although the family defined by F is topologically trivial, it cannot be isomorphic (not even C^1-diffeomorphic) to an equisingular deformation with trivial equimultiple sections. The induced deformation over T_ε is defined by $f + \varepsilon g$, with

$$g = 2(x^5y^3 - 6x^4y^4 + 13x^3y^5 - 12x^2y^6 + 4xy^7)\,.$$

We will see below, that, indeed, $g \in I^{es}(f) \setminus \langle f, j(f), I^s(f)\rangle$ (see Example 2.63.1).

We use the previous discussion of the equisingularity ideal to show that a generic element $g \in I^{es}(f)$ intersects f with the same multiplicity $\kappa(f)$ as a generic polar of f, $\alpha\frac{\partial f}{\partial x} + \beta\frac{\partial f}{\partial y}$, $(\alpha:\beta) \in \mathbb{P}^1$ generic, does. Indeed, this is an immediate consequence of the following lemma (since $j(f) \subset I^{es}(f)$):

Lemma 2.18. *Let $f \in \mathbb{C}\{x,y\}$ be reduced, and let $g \in I^{es}(f)$. Then*

$$i(f,g) \geq \kappa(f)\,.$$

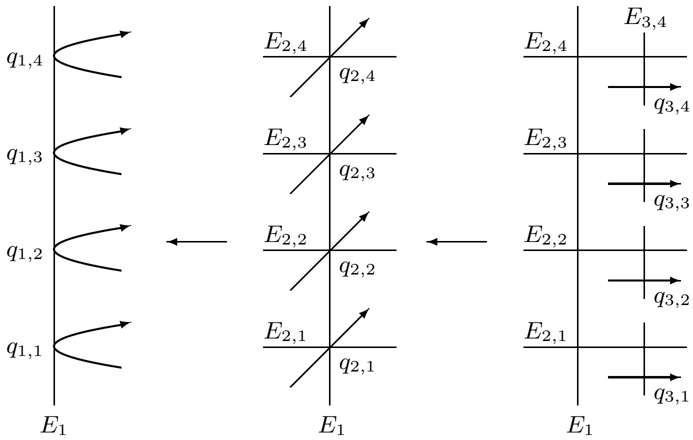


Fig. 2.12. Resolution of $\big\{(x-2y)^2(x-y)^2x^2y^2+x^9+y^9=0\big\}$.

Moreover, with the notations introduced in the proof of Proposition 2.14, we have

$$i(f,g) \geq \begin{cases} \kappa(f)+\mathrm{mt}(f), & \textit{if } g \in I^{es}_{\mathit{fix}}(f)\,, \\ \kappa(f)+i(f,L)-\mathrm{mt}(f)\,, & \textit{if } g \in I^{es}_{L}(f)\,, \\ \kappa(f)+\mathrm{mt}(f)+\sum\limits_{j=1}^{k}\big(i(f,L_j)-\mathrm{mt}(f)\big)\,, & \textit{if } g \in I^{es}_{\mathit{fix},L_1\dots L_k}(f)\,. \end{cases}$$

Proof. We proceed by induction on the number of blowing ups needed to resolve the plane curve singularity $\{f=0\}$ *and* to make $\{f=0\}$ transversal to the (strict transforms of) $L, L_1, \dots, L_k$.

Step 1. As base of induction, we have to show for a non-singular f and transverse smooth germs $L, L_1, \dots, L_k$ that $i(f,g) \geq 0$ if $g \in I^{es}(f)$ or if $g \in I^{es}_L(f)$, and that $i(f,g) \geq 1$ if $g \in I^{es}_{\mathit{fix}}(f)$ or if $g \in I^{es}_{\mathit{fix},L_1\dots L_k}(f)$. But this is obvious.

Step 2. We show that it suffices to prove the statement for $g \in I^{es}_{\mathit{fix},L_1\dots L_k}(f)$. Thus, let us assume that, for each $g \in I^{es}_{\mathit{fix},L_1\dots L_k}(f)$,

$$i(f,g) \geq \kappa(f)+\mathrm{mt}(f)+\sum_{j=1}^{k}\big(i(f,L_j)-\mathrm{mt}(f)\big)\,, \tag{2.2.9}$$

The case $k=0$ implies that $i(f,g) \geq \kappa(f)+\mathrm{mt}(f)$ for each $g \in I^{es}_{\mathit{fix}}(f)$. Moreover, due to Proposition 2.14, the equisingularity ideal $I^{es}(f)$ is generated as a linear space by $I^{es}_{\mathit{fix}}(f)$ and the partial derivatives of f. As $i(f,g) \geq \kappa(f)+\mathrm{mt}(f)$ for $g \in I^{es}_{\mathit{fix}}(f)$, and as each element of $j(f)$ intersects f with multiplicity at least $\kappa(f)$, we get $i(f,g) \geq \kappa(f)$ for each $g \in I^{es}(f)$.

Finally, we have seen in Step 1a of the proof of Proposition 2.14 that $\mathfrak{m}I^{es}_L(f) \subset I^{es}_{\mathit{fix},L}(f)$. Thus, for each $g \in I^{es}_L(f)$, and for each $h \in \mathfrak{m}$, we get by (2.2.9)

$$\begin{aligned} i(f,g)+i(f,h)=i(f,hg) &\geq \kappa(f)+\mathrm{mt}(f)+i(f,L)-\mathrm{mt}(f) \\ &= \kappa(f)+i(f,L)\,. \end{aligned}$$

If we choose $h \in \mathfrak{m}$ generically, then $i(f,h)=\mathrm{mt}(f)$ (Exercise I.3.2.1), and we obtain the wanted inequality $i(f,g) \geq \kappa(f)+i(f,L)-\mathrm{mt}(f)$ for $g \in I_L^{es}(f)$.

Step 3. Let $f \in \mathbb{C}\{x,y\}$ be an arbitrary reduced element defining a curve germ $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$, and let $g \in I^{es}_{fix,L_1\dots L_k}(f)$. Further, let $\pi : M \to (\mathbb{C}^2,\mathbf{0})$ be the blowing up of the origin, and let $E \subset M$ be the exceptional divisor. Denote by $\widetilde{L}_1,\dots,\widetilde{L}_k \subset M$ the strict transforms of $L_1,\dots,L_k$, by $\widetilde{C}$ the strict transform of the germ $(C,\mathbf{0})$, and by $q_1,\dots,q_s$ the intersection points of $\widetilde{C}$ with E. Since $L_1,\dots,L_k$ are supposed to be *transversal* smooth germs, each q_i is contained in at most one of the $\widetilde{L}_j$, and each $\widetilde{L}_j$ contains at most one of the q_i. Thus, we may assume that for some $0 \leq \ell \leq \min\{k,s\}$, we have

$$q_i \in \widetilde{L}_j \iff i=j\leq \ell\,. \tag{2.2.10}$$

Now, let $\widetilde{f}_i \in \mathcal{O}_{M,q_i}$, respectively $e_i \in \mathcal{O}_{M,q_i}$, be local equations for the germ of $\widetilde{C}$, respectively of E, at q_i, and denote by $\widehat{g}_i$ the total transform of g at q_i $i=1,\dots,s$. Then, due to Proposition 2.14, $\mathrm{mt}(g) \geq \mathrm{mt}(f)$, and

$$\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}} \in I_i := \begin{cases} I^{es}_{fix,\widetilde{L}_1\dots\widetilde{L}_k E}\big(\widetilde{f}_i\big)\,, & \text{if } i\leq \ell\,,\\ I^{es}_E\big(\widetilde{f}_i\big)\,, & \text{if } i>\ell\,. \end{cases} \tag{2.2.11}$$

Moreover, due to Proposition I.3.21, and since $\sum_{i=1}^s i\big(\widetilde{f}_i,e_i\big)=\mathrm{mt}(f)$, we have

$$\begin{aligned} i(f,g) &= \mathrm{mt}(f)\cdot\mathrm{mt}(g)+\sum_{i=1}^s i\big(\widetilde{f}_i,\widetilde{g}_i\big)\\ &= \mathrm{mt}(f)^2+\sum_{i=1}^s\left(i\left(\widetilde{f}_i,\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}}\right)+\big(\mathrm{mt}(g)-\mathrm{mt}(f)\big)\cdot i\big(\widetilde{f}_i,e_i\big)\right)\\ &\geq \mathrm{mt}(f)^2+\sum_{i=1}^s i\left(\widetilde{f}_i,\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}}\right). \end{aligned}$$

Here, by (2.2.11) and the induction hypothesis,

$$i\left(\widetilde{f}_i,\frac{\widehat{g}_i}{e_i^{\mathrm{mt}(f)}}\right) \geq \begin{cases} \kappa\big(\widetilde{f}_i\big)+i\big(\widetilde{f}_i,e_i\big)+i\big(\widetilde{f}_i,\widetilde{L}_i\big)-\mathrm{mt}\big(\widetilde{f}_i\big), & \text{if } i\leq\ell\,,\\ \kappa\big(\widetilde{f}_i\big)+i\big(\widetilde{f}_i,e_i\big)-\mathrm{mt}\big(\widetilde{f}_i\big)\,, & \text{if } i>\ell\,. \end{cases}$$

Thus,

$$i(f,g)\geq \mathrm{mt}(f)^2+\sum_{i=1}^s\Big(\kappa\big(\widetilde{f}_i\big)-\mathrm{mt}\big(\widetilde{f}_i\big)\Big)+\sum_{i=1}^s i\big(\widetilde{f}_i,e_i\big)+\sum_{i=1}^{\ell} i\big(\widetilde{f}_i,\widetilde{L}_i\big)\,.$$

The statement follows, since $\sum_{i=1}^{s} i\big(\widetilde{f}_i, e_i\big) = \mathrm{mt}(f)$, since

$$\begin{aligned}\sum_{i=1}^{s}\Big(\kappa\big(\widetilde{f}_i\big) - \mathrm{mt}\big(\widetilde{f}_i\big)\Big) &= \sum_{i=1}^{s}\Big(2\delta\big(\widetilde{f}_i\big) - r\big(\widetilde{f}_i\big)\Big)\\ &= 2\delta(f) - \mathrm{mt}(f)\big(\mathrm{mt}(f)-1\big) - r(f)\\ &= \kappa(f) - \mathrm{mt}(f)^2\end{aligned}$$

(due to Propositions I.3.38, I.3.35 and I.3.34), and since

$$\sum_{i=1}^{\ell} i\big(\widetilde{f}_i, \widetilde{L}_i\big) = \sum_{j=1}^{k}\sum_{i=1}^{s} i\big(\widetilde{f}_i, \widetilde{L}_j\big) = \sum_{j=1}^{k}\Big(i(f, L_j) - \mathrm{mt}(f)\Big)\,,$$

due to the assumption (2.2.10). □

Proposition 2.19. *Let (i, ϕ, σ) be an equisingular deformation over $(\mathbb{C}, 0)$ of the reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$. Moreover, let*

$$\begin{tikzcd} C \arrow[r, hook, "i"] \arrow[d] & \mathscr{C} \arrow[r, hook] \arrow[d, shift left, "\phi"] & B \times T \arrow[dl, "\mathrm{pr}_T"] \\ \{0\} \arrow[r, hook] & T \arrow[u, bend left, "\sigma"] & \end{tikzcd}$$

be a representative for (i, ϕ, σ) with $T \subset \mathbb{C}$, $B \subset \mathbb{C}^2$ neighbourhoods of the origin. Then, for all sufficiently small open neighbourhoods $U \subset B$ of the origin, we can choose an open neighbourhood $W = W(0) \subset T$ such that, for all $t \in W$, we have:

(i)

$$i_U(C, C_t) := \sum_{z \in U} i_z(C, C_t) \geq \kappa(C, \mathbf{0})\,,$$

where $C_t \times \{t\} = \mathscr{C}_t \subset U \times \{t\}$ is the fibre of ϕ over t.

(ii) If σ is the trivial section, we have even

$$i_U(C, C_t) \geq \kappa(C, \mathbf{0}) + \mathrm{mt}(C, \mathbf{0})\,.$$

Proof. The hypersurface germ $(\mathscr{C}, \mathbf{0}) \subset (\mathbb{C}^2 \times \mathbb{C}, \mathbf{0})$ is defined by an unfolding $F \in \mathbb{C}\{x, y, t\}$ with $f := F_0 \in \mathbb{C}\{x, y\}$ being a local equation for $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$. We may assume that $F \neq f$, otherwise the left-hand side in (i) and (ii) are infinite. We write

$$F(x, y, t) = f(x, y) + t^m f_m(x, y) + t^{m+1} g(x, y, t)\,, \quad m \geq 1\,.$$

Then $\overline{F} := f + t^m f_m$ defines an equisingular deformation of $(C, \mathbf{0})$ over the fat point $\big(\{0\}, \mathbb{C}\{t\}/\langle t^{m+1}\rangle\big)$, with the (uniquely determined) singular section

$\overline{\sigma} = \sigma \bmod \langle t^{m+1}\rangle$ given by the ideal $I_{\overline{\sigma}} = \langle x - t^m\alpha, y - t^m\beta\rangle$, $\alpha,\beta \in \mathbb{C}$. Substituting t^m by ε, we get that $f + \varepsilon f_m$ defines an equisingular deformation over T_ε with the singular section being defined by the ideal $\langle x - \varepsilon\alpha, y - \varepsilon\beta\rangle$. If σ is the trivial section, then $\alpha = \beta = 0$. We obtain

$$f_m \in \begin{cases} I^{es}_{\mathit{fix}}(f) & \text{if } \sigma \text{ is the trivial section}, \\ I^{es}(f) & \text{otherwise}, \end{cases}$$

and Lemma 2.18 gives

$$i(f, f_m) \geq \begin{cases} \kappa(f) + \mathrm{mt}(f) & \text{if } \sigma \text{ is the trivial section}, \\ \kappa(f) & \text{otherwise}. \end{cases}$$

It remains to show that for a sufficiently small neighbourhood $U \subset \mathbb{C}^2$ of the origin, we find some $\rho > 0$ such that

$$i(f, f_m) = i_U(C, C_t) = \sum_{\boldsymbol{z}\in U} i_{\boldsymbol{z}}(C, C_t) \qquad \text{for } 0 < |t| < \rho\,.$$

We consider the unfolding of f_m given by $G := f_m + tg$. By Proposition I.3.14, for $U \subset \mathbb{C}^2$ a sufficiently small neighbourhood of the origin, we find some open neighbourhood $W \subset \mathbb{C}$ of 0 such that G converges on $U \times W$ and such that, for each $t \in W$,

$$i(f, f_m) = i_U(C, D_t) = \sum_{\boldsymbol{z}\in U} i_{\boldsymbol{z}}(C, D_t)\,, \qquad D_t := V(G_t)\,.$$

Now, the statement follows, since from the definition of the intersection multiplicity we get

$$\begin{aligned} i_{\boldsymbol{z}}(C, C_t) &= i(f\circ\phi_{\boldsymbol{z}},\, f\circ\phi_{\boldsymbol{z}} + t^m f_m\circ\phi_{\boldsymbol{z}} + t^{m+1} g\circ\phi_{\boldsymbol{z}}) \\ &= i(f\circ\phi_{\boldsymbol{z}},\, f_m\circ\phi_{\boldsymbol{z}} + tg\circ\phi_{\boldsymbol{z}}) \;=\; i_{\boldsymbol{z}}(C, D_t) \end{aligned}$$

with $\phi_{\boldsymbol{z}}$ the linear coordinate change $x \mapsto x + z_1$, $y \mapsto y + z_2$, $\boldsymbol{z} = (z_1, z_2)$. □

2.3 Deformations of the Parametrization

We describe now a different approach to equisingular deformations of a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ by considering deformations of the parametrization.

To define deformations of the parametrization (with section) we need deformations of a sequence of morphisms.

Definition 2.20. Let $(X_n, x_n) \xrightarrow{f_n} (X_{n-1}, x_{n-1}) \xrightarrow{f_{n-1}} \ldots \xrightarrow{f_1} (X_0, x_0)$ be a sequence of morphisms of complex (multi-) germs.

(1) A *deformation of the sequence of morphisms* over a complex germ (T, t_0) is a Cartesian diagram

$$
\begin{tikzcd}
(X_n, x_n) \arrow[r, hook] \arrow[d, "f_n"'] \arrow[dr, phantom, "\square"] & (\mathscr{X}_n, x_n) \arrow[d, "F_n"] \\
(X_{n-1}, x_{n-1}) \arrow[r, hook] \arrow[d, "f_{n-1}"'] \arrow[dr, phantom, "\square"] & (\mathscr{X}_{n-1}, x_{n-1}) \arrow[d, "F_{n-1}"] \\
\vdots \arrow[d, "f_1"'] \arrow[dr, phantom, "\square"] & \vdots \arrow[d, "F_1"] \\
(X_0, x_0) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{X}_0, x_0) \arrow[d, "F_0"] \\
\{t_0\} \arrow[r, hook] & (T, t_0)
\end{tikzcd}
$$

such that the composition $F_0 \circ \ldots \circ F_i : (\mathscr{X}_i, x_i) \to (T, t_0)$, $i = 0, \ldots, n$, is flat (hence a deformation of (X_i, x_i)).

(2) If $(\mathscr{X}_n', x_n') \to \ldots \to (\mathscr{X}_0', x_0') \to (T', t_0')$ is another deformation of the sequence over a complex germ (T', t_0'), then a *morphism* from the second deformation to the first one is given by a morphism $\varphi : (T', t_0') \to (T, t_0)$ and liftings $(\mathscr{X}_i', x_i') \to (\mathscr{X}_i, x_i)$, $i = 0, \ldots, n$, such that the obvious diagram commutes.

(3) The category of deformations of the sequence $(X_n, x_n) \to \ldots \to (X_0, x_0)$ is denoted by $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}$.

If we consider only deformations over a fixed germ (T, t_0), then we get the (non-full) subcategory $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T, t_0)$ with morphisms being the identity on (T, t_0). $\underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T, t_0)$ denotes the set of isomorphism classes of deformations in $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T, t_0)$.

(4) We call $T^1_{(X_n,x_n)\to\ldots\to(X_0,x_0)} := \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T_\varepsilon)$ the set of isomorphism classes of *(first order) infinitesimal deformations* of the sequence $(X_n, x_n) \to \ldots \to (X_0, x_0)$.

(5) Deformations of the sequence $(X_n, x_n) \to \ldots \to (X_0, x_0)$ over (T, t_0) satisfying $(\mathscr{X}_0, x_0) = (X_0, x_0) \times (T, t_0)$, together with morphisms of deformations satisfying that the first lifting

$$(\mathscr{X}_0', x_0') = (X_0, x_0) \times (T', t_0') \to (X_0, x_0) \times (T, t_0) = (\mathscr{X}_0, x_0)$$

is of type $\mathrm{id}_{(X_0,x_0)} \times \varphi$, form a subcategory of $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_0,x_0)}$ denoted by $\mathcal{D}ef_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}$. The set of isomorphism classes of first order deformations of $(X_n, x_n) \to \ldots \to (X_1, x_1)/(X_0, x_0)$ is

$$T^1_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)} := \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}(T_\varepsilon)\,.$$

(6) The functor

$$\underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)} : (\textit{complex germs}) \longrightarrow \mathcal{S}ets\,,$$
$$(T,t_0) \longmapsto \underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_0,x_0)}(T,t_0)$$

is called the *deformation functor of the sequence* $(X_n,x_n)\to\ldots\to(X_0,x_0)$. In the same way, we define the functor $\underline{\mathcal{D}ef}_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}$.

Since this functor satisfies Schlessinger's conditions (H_0), (H_1) and (H_2), it follows that $T^1_{(X_n,x_n)\to\ldots\to(X_0,x_0)}$ and $T^1_{(X_n,x_n)\to\ldots\to(X_1,x_1)/(X_0,x_0)}$ are complex vector spaces (see Appendix C).

Remark 2.20.1. Let $\{\mathrm{pt}\}$ denote the reduced complex germ consisting of one point. Then deformations of (X,x) can be identified with deformations of the morphism $(X,x)\to\{\mathrm{pt}\}$, and deformations of (X,x) with section can be identified with deformations of the sequence $\{x\}\to(X,x)\to\{\mathrm{pt}\}$. In other words, the category $\mathcal{D}ef_{(X,x)}$ (respectively $\mathcal{D}ef^{\,sec}_{(X,x)}$) is naturally equivalent to $\mathcal{D}ef_{(X,x)\to\{\mathrm{pt}\}}$ (respectively $\mathcal{D}ef_{\{x\}\to(X,x)\to\{\mathrm{pt}\}}$).

These definitions can obviously be generalized to deformations of diagrams instead of deformations of sequences of morphisms and to multigerms instead of germs and the corresponding deformation functors again satisfy Schlessinger's conditions (H_0), (H_1) and (H_2). We formulate this only for a special case, which is needed below.

Definition 2.21. Let $\big(\overline{X},\overline{x}\big)=\coprod_{j=1}^r\big(\overline{X}_j,\overline{x}_j\big)$ and $(X,x)=\coprod_{i=1}^s(X_i,x_i)$ be multigerms, and let $f:\big(\overline{X},\overline{x}\big)\to(X,x)$ be a morphism mapping the set $\big\{\overline{x}\big\}=\big\{\overline{x}_1,\ldots,\overline{x}_r\big\}$ onto $\{x\}=\{x_1,\ldots,x_s\}$. Then a *deformation of the diagram*

$$\begin{tikzcd}
\{\overline{x}\} \arrow[r, hook] \arrow[d] & \big(\overline{X},\overline{x}\big) \arrow[d, "f"] \\
\{x\} \arrow[r, hook] & (X,x)
\end{tikzcd}$$

over (T,t_0) consists of deformations over (T,t_0) of $\big(\overline{X},\overline{x}\big)$ and of $\{\overline{x}\}\to\{x\}$, which fit into an obvious commutative diagram. As a deformation of a finite set of reduced points is trivially isomorphic to the disjoint union of the same number of copies of (T,t_0), such a deformation is equivalently given by a commutative diagram

$$\begin{tikzcd}
\big(\overline{X},\overline{x}\big) \arrow[r, hook] \arrow[d, "f"'] \arrow[dr, phantom, "\Box"] & \big(\overline{\mathscr{X}},\overline{x}\big) \arrow[d, "F"] \\
(X,x) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\Box"] & (\mathscr{X},x) \arrow[d] \\
\{t_0\} \arrow[r, hook] & (T,t_0) \arrow[u, bend right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"']
\end{tikzcd}$$

where $\sigma = \{\sigma_1, \dots, \sigma_s\}$ and $\overline{\sigma} = \{\overline{\sigma}_1, \dots, \overline{\sigma}_r\}$ are (multi-)sections satisfying $\sigma_i(t_0) = x_i$, $\overline{\sigma}_j(t_0) = \overline{x}_j$ for all i, j. Moreover, for each $i \in \{1, \dots, s\}$ and $j \in \{1, \dots, r\}$ such that $x_i = f(\overline{x}_j)$ we have $\sigma_i = F \circ \overline{\sigma}_j$. In this situation, we say that the (multi-)sections σ and $\overline{\sigma}$ are *compatible*.

We call such deformations *deformations of* $\big(\overline{X}, \overline{x}\big) \to (X, x)$ *with compatible sections* (or just *deformations with section*), and denote the corresponding category by $\mathcal{D}ef^{sec}_{(\overline{X},\overline{x})\to(X,x)}$. Recall that the multisections σ and $\overline{\sigma}$ can be trivialized (Proposition 2.2).

We wish to apply all this to deformations of the parametrization of a plane curve singularity. To keep the notations shorter and to avoid overlaps in the notations, from now on we denote the base points of the complex germs appearing by $\mathbf{0}$, $\overline{0}$ or $\overline{0}_i$ (without necessarily referring to an embedding in some $(\mathbb{C}^n, \mathbf{0})$).

Consider the commutative diagram of complex (multi-) germs

$$\begin{tikzcd}
(\overline{C}, \overline{0}) \arrow[d, two heads, "n"'] \arrow[dr, "\varphi"] & \\
(C, \mathbf{0}) \arrow[r, hook, "j"'] & (\mathbb{C}^2, \mathbf{0})
\end{tikzcd}$$

where $(C, \mathbf{0})$ is a reduced plane curve singularity, j the given embedding, n the *normalization*, and $\varphi = j \circ n$ the *parametrization* of $(C, \mathbf{0})$.

If $(C, \mathbf{0}) = (C_1, \mathbf{0}) \cup \ldots \cup (C_r, \mathbf{0})$ is the decomposition of $(C, \mathbf{0})$ into irreducible components, then $(\overline{C}, \overline{0}) = (\overline{C}_1, \overline{0}_1) \amalg \ldots \amalg (\overline{C}_r, \overline{0}_r)$ is a multigerm with $(\overline{C}_i, \overline{0}_i) \cong (\mathbb{C}, 0)$ mapped onto $(C_i, \mathbf{0})$, inducing the normalization of the component $(C_i, \mathbf{0})$. On the level of (semi-) local rings we have

$$\begin{tikzcd}
\mathcal{O}_{\overline{C},\overline{0}} = \bigoplus\limits_{i=1}^{r} \mathcal{O}_{\overline{C}_i,\overline{0}_i} \cong \bigoplus\limits_{i=1}^{r} \mathbb{C}\{t_i\} & \\
\mathcal{O}_{C,\mathbf{0}} \arrow[u, hook, "n^\sharp"] & \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \cong \mathbb{C}\{x, y\}\,. \arrow[l, two heads] \arrow[ul, "\varphi^\sharp = (\varphi_1^\sharp, \dots, \varphi_r^\sharp)"']
\end{tikzcd}$$

We fix coordinates x, y for $(\mathbb{C}^2, \mathbf{0})$ and, for each $i = 1, \dots, r$, a local coordinate t_i of $(\overline{C}_i, \overline{0}_i)$, identifying this germ with $(\mathbb{C}, 0)$. Then the parametrization $\varphi = \{\varphi_i \mid i = 1, \dots, r\}$ is given by r holomorphic map germs

$$\varphi_i = \varphi|_{(\overline{C}_i,\overline{0}_i)} : (\mathbb{C}, 0) \longrightarrow (\mathbb{C}^2, \mathbf{0})\,, \quad t_i \longmapsto \big(x_i(t_i), y_i(t_i)\big)\,.$$

If $f \in \mathbb{C}\{x, y\}$ defines $(C, \mathbf{0})$, f decomposes in r irreducible factors $f_1, \dots, f_r$ with $(C_i, \mathbf{0}) = \big(V(f_i), \mathbf{0}\big)$. With the identification $\mathcal{O}_{\overline{C},\overline{0}} = \bigoplus_{i=1}^r \mathbb{C}\{t_i\}$, we have

$$\varphi^\sharp = (\varphi_i^\sharp)_{i=1}^r : \mathbb{C}\{x,y\} \to \bigoplus_{i=1}^r \mathbb{C}\{t_i\}\,,$$

with $\varphi_i^\sharp(x) = x_i(t_i)$, $\varphi_i^\sharp(y) = y_i(t_i)$, and $\operatorname{Ker}(\varphi_i^\sharp) = \langle f_i \rangle$, $\operatorname{Ker}(\varphi^\sharp) = \langle f \rangle$.

Remark 2.21.1. Since $(\overline{C},\overline{0})$ and $(\mathbb{C}^2,\mathbf{0})$ are smooth (multi-)germs, any deformation of these germs is trivial (Exercise 1.3.1). Hence, any deformation of the parametrization $\varphi : (\overline{C},\overline{0}) \to (\mathbb{C}^2,\mathbf{0})$ over a germ $(T,\mathbf{0})$ is given by a Cartesian diagram and isomorphisms

$$\begin{tikzcd}
(\overline{C},\overline{0}) \arrow[r, hook, "i"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & \big(\overline{\mathscr{C}},\overline{0}\big) \arrow[r, "\cong"] \arrow[d, "\phi"] & \big(\overline{C}\times T,\overline{0}\big) \arrow[d, "\phi"] \\
(\mathbb{C}^2,\mathbf{0}) \arrow[r, hook, "j"] \arrow[d] \arrow[dr, phantom, "\square"] & \big(\mathscr{M},\mathbf{0}\big) \arrow[r, "\cong"] \arrow[d, "\phi_0"] & \big(\mathbb{C}^2\times T,\mathbf{0}\big) \arrow[d, shift right, "\mathrm{pr}"'] \\
\{\mathbf{0}\} \arrow[r, hook] & (T,\mathbf{0}) \arrow[r, equal] & (T,\mathbf{0}) \arrow[u, shift right, bend right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"']
\end{tikzcd}$$

with pr the projection, $\big(\overline{\mathscr{C}},\overline{0}\big) = \coprod_{i=1}^r \big(\overline{\mathscr{C}}_i,\overline{0}_i\big)$, and $\big(\overline{\mathscr{C}}_i,\overline{0}_i\big) \cong \big(\overline{C}_i \times T,\overline{0}_i\big)$. Compatible sections $\overline{\sigma}$ and σ consist of disjoint sections $\overline{\sigma}_i : (T,\mathbf{0}) \to (\overline{\mathscr{C}}_i,\overline{0}_i)$ of $\mathrm{pr}\circ\phi_i$, where $\phi_i : \big(\overline{\mathscr{C}}_i,\overline{0}_i\big) \to (\mathscr{M},\mathbf{0})$ denotes the restriction of ϕ, and a section σ of pr such that $\phi\circ\overline{\sigma}_i = \sigma$, $i = 1,\dots,r$. Note that pr and $\mathrm{pr}\circ\phi$ are automatically flat by Corollary I.1.88 and there is no further requirement on ϕ.

Let $(\mathscr{C},\mathbf{0}) := \phi\big(\overline{\mathscr{C}},\overline{0}\big)$ with Fitting structure. Then, by Proposition 2.9, the restriction $\phi_0 : (\mathscr{C},\mathbf{0}) \to (T,\mathbf{0})$ is a deformation of $(C,\mathbf{0})$. Having fixed local coordinates x, y for $(\mathbb{C}^2,\mathbf{0})$ and t_i for $(\overline{C}_i,\overline{0}_i)$, the morphism

$$\phi = \{\phi_i \mid i = 1,\dots,r\} : (\overline{C}\times T,\overline{0}) \to (\mathbb{C}^2\times T,\mathbf{0})$$

is given by r holomorphic map germs

$$\phi_i : (\mathbb{C}\times T,\mathbf{0}) \to (\mathbb{C}^2\times T,\mathbf{0})\,,\quad (t_i,s) \mapsto \big(\phi_{i,1}(t_i,s),s\big)\,,$$

with $\phi_{i,1}(t_i,s) = \big(X_i(t_i,s),Y_i(t_i,s)\big)$, $X_i(t_i,\mathbf{0}) = x_i(t_i)$, $Y_i(t_i,\mathbf{0}) = y_i(t_i)$.

A section $\overline{\sigma} : (T,\mathbf{0}) \to (\overline{C}\times T,\overline{0})$, $s \mapsto \coprod_{i=1}^r \overline{\sigma}_i(s)$, compatible with the trivial section σ, $\sigma(s) = (\mathbf{0},s)$, is then given by r holomorphic germs

$$\overline{\sigma}_i : (T,\mathbf{0}) \to (\overline{C}_i\times T,\overline{0}_i)\,,\quad \overline{\sigma}_i(s) = \big(\overline{\sigma}_{i,1}(s),s\big)$$

such that $\big(X_i(\overline{\sigma}_i(s)),Y_i(\overline{\sigma}_i(s))\big) = (0,0) \in \mathbb{C}^2$.

Definition 2.22. Let $n : (\overline{C},\overline{0}) \to (C,\mathbf{0})$ be the normalization of the reduced plane curve germ $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$, let $\varphi : (\overline{C},\overline{0}) \to (\mathbb{C}^2,\mathbf{0})$ be its parametrization, and let $(T,\mathbf{0})$ be a complex space germ.

(1) Objects in the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}(T,\mathbf{0})$, respectively in the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$, are called *deformations of the normalization*, respectively *deformations of the parametrization* of $(C,\mathbf{0})$ over $(T,\mathbf{0})$. They are denoted by (i,j,ϕ,ϕ_0) or just by ϕ.

(2) The corresponding deformations of the normalization $\big(\overline{C},\overline{0}\big)\to(C,\mathbf{0})$, resp. of the parametrization $\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$, with compatible sections are objects in the category $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}(T,\mathbf{0})$, resp. in $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$. Objects in these categories are called *deformations with section of the normalization*, resp. *of the parametrization*, of $(C,\mathbf{0})$. They are denoted by $(\phi,\overline{\sigma},\sigma)$.

(3) $T^{1,sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$, resp. $T^{1,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$, denotes the corresponding vector space of *(first order) infinitesimal deformations* of the normalization, resp. parametrization, with section.

We show now that isomorphism classes of deformations of the normalization and of the parametrization are essentially the same thing.

Proposition 2.23. *If* $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ *is a reduced plane curve singularity, then there is a surjective functor from* $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ *to* $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$, *inducing an isomorphism between the deformation functors* $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ *and* $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$. *The same holds for* $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$, *resp.* $\mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ *and the corresponding deformation functors.*

Proof. We consider the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}$ and show that the natural forgetful functors from this category to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ and to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ induce isomorphisms for the corresponding deformation functors.

By Proposition 2.9, we have a functor from the category $\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}$ and, by forgetting $(\mathbb{C}^2,\mathbf{0})$, to $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$. The relative lifting lemma 1.27 says that, for a given germ $(T,\mathbf{0})$, the functor $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\to\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})}(T,\mathbf{0})$ is surjective (full) and injective on the set of isomorphism classes. Hence, the deformation functors are isomorphic.

To see that the two functors $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})}$ and $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ are isomorphic, note that Proposition 2.9 easily implies that the forgetful map $\mathcal{D}ef_{(\overline{C},\overline{0})\to(C,\mathbf{0})/(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\to\mathcal{D}ef_{(\overline{C},\overline{0})/(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ is an isomorphism of categories. Since $(\mathbb{C}^2,\mathbf{0})$ is a smooth germ, each deformation of $(\mathbb{C}^2,\mathbf{0})$ is trivial (Exercise 1.3.1), hence, $\mathcal{D}ef_{(\overline{C},\overline{0})/(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})\to\mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ induces a bijection on the set of isomorphism classes, and similar for $(\overline{C},\overline{0})\to(C,\mathbf{0})\to(\mathbb{C}^2,\mathbf{0})$. Together this implies the required isomorphism.

As sections from the base space to the total space of deformations are not affected by the previous arguments, it follows that $\underline{\mathcal{D}ef}^{sec}_{(\overline{C},\overline{0})\to(C,\mathbf{0})}$ and $\underline{\mathcal{D}ef}^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ are isomorphic, too. □

As an immediate consequence of Proposition 2.23, we obtain the following corollary:

Corollary 2.24. *Each deformation of the parametrization (with compatible sections) induces an embedded deformation (with section) of the curve germ* $(C, \mathbf{0})$.

Considering deformations over T_ε, this yields vector space homomorphisms

$$T^1_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} \xrightarrow{\alpha'} T^1_{(C,\mathbf{0})}\,, \qquad T^{1,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} \xrightarrow{\beta'} T^{1,sec}_{(C,\mathbf{0})}\,.$$

In Section 2.4 below, we describe these maps α' and β' in explicit terms.

Example 2.24.1. Consider the cusp, parametrized by $\varphi : t \mapsto (t^3, t^2)$, and the deformation of the parametrization $\phi : (t, s) \mapsto (t^3 - s^2 t, t^2 - s^2)$ over $(\mathbb{C}, 0)$. According to Proposition 2.9, the induced embedded deformation of $(C, \mathbf{0})$ is given by $\operatorname{Ker}(\phi^\sharp : \mathbb{C}\{x, y\} \to \mathbb{C}\{t, s\})$. Hence, the deformation of the equation is given by $\bigl(V(x^2 - y^3 - s^2y^2), \mathbf{0}\bigr) \to (\mathbb{C}, 0)$, $(x, y, s) \mapsto s$, which is the deformation of the cusp into an ordinary double point along the trivial (singular) section $s \mapsto (\mathbf{0}, s)$ with image $\{\mathbf{0}\} \times (\mathbb{C}, 0)$. The preimage in $(\overline{\mathscr{C}}, \overline{0}) = (\mathbb{C} \times \mathbb{C}, \mathbf{0})$ of this image is $\{(s, t) \mid t^2 - s^2 = 0\}$.

It follows that the deformation $(\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{C}, \mathbf{0}) \to (\mathbb{C}, 0)$ admits two sections $s \mapsto \{(s, t) \mid t = \pm s\}$ which both map to the unique singular section of $(\mathscr{C}, \mathbf{0}) \to (\mathbb{C}, 0)$.

Equimultiple Deformations

We are now going to define equimultiple deformations of the parametrization.

The multiplicity of $(C, \mathbf{0})$ satisfies $\operatorname{mt}(C, \mathbf{0}) = \sum_{i=1}^r \operatorname{mt}(C_i, \mathbf{0})$, and the multiplicity of the i-th branch satisfies

$$\operatorname{mt}(C_i, \mathbf{0}) = \min\bigl\{\operatorname{ord}_{t_i} x_i(t_i), \operatorname{ord}_{t_i} y_i(t_i)\bigr\} =: \operatorname{ord}(\varphi_i, \overline{0}_i) =: \operatorname{ord}\varphi_i\,,$$

$\operatorname{ord}\varphi_i$ being the *order of the parametrization* of the i-th branch. This follows from Proposition I.3.12 (see also Exercise I.3.2.1 and Proposition I.3.21). We call

$$\mathbf{mt}(C, \mathbf{0}) := \bigl(\operatorname{mt}(C_1, \mathbf{0}), \dots, \operatorname{mt}(C_r, \mathbf{0})\bigr)$$

the *multiplicity vector* of $(C, \mathbf{0})$ which, therefore, equals

$$\mathbf{ord}\,\varphi := \mathbf{ord}(\varphi, \overline{0}) := \bigl(\operatorname{ord}\varphi_1, \dots, \operatorname{ord}\varphi_r\bigr)\,,$$

the *order of the parametrization* of $(C, \mathbf{0})$.

Note that $\operatorname{ord}\varphi_i = \max\bigl\{m \bigm| \varphi_i^\sharp(\mathfrak{m}_{(\overline{C}_i,\overline{0}_i)}) \subset \mathfrak{m}^m_{\mathbb{C}^2,\mathbf{0}}\bigr\}$, where the right-hand side does not involve any choice of coordinates.

Let $(\phi, \overline{\sigma}, \sigma)$ be a deformation with section of the parametrization of $(C, \mathbf{0})$ over $(T, \mathbf{0})$. We set

$$I_{\overline{\sigma}_i} := \operatorname{Ker}\bigl(\overline{\sigma}_i^\sharp : \mathcal{O}_{\overline{\mathscr{C}}_i,\overline{0}_i} \to \mathcal{O}_{T,\mathbf{0}}\bigr)\,, \quad i = 1,\dots,r\,,$$
$$I_\sigma := \operatorname{Ker}\bigl(\sigma^\sharp : \mathcal{O}_{\mathscr{C},\mathbf{0}} \to \mathcal{O}_{T,\mathbf{0}}\bigr)\,,$$

which are the ideals of the respective sections. We have $\phi_i^\sharp(I_\sigma) \subset I_{\overline{\sigma}_i}$ and define the order of the deformation of the parametrization of the i-th branch (along $\overline{\sigma}_i$) as

$$\operatorname{ord}(\phi_i, \overline{\sigma}_i, \sigma) := \max\bigl\{m \bigm| \phi_i^\sharp(I_\sigma) \subset I_{\overline{\sigma}_i}^m\bigr\}\,.$$

The r-tuple

$$\mathbf{ord}\,\phi := \mathbf{ord}(\phi, \overline{\sigma}, \sigma) := \bigl(\operatorname{ord}(\phi_1, \overline{\sigma}_1), \dots, \operatorname{ord}(\phi_r, \overline{\sigma}_r)\bigr)$$

is called the *order (vector) of the deformation of the parametrization* of $(C, \mathbf{0})$ (along $\overline{\sigma}, \sigma$).

Definition 2.25. (1) A deformation of the parametrization with section $(\phi, \overline{\sigma}, \sigma) \in \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$ is called *equimultiple* if $\mathbf{ord}\,\phi = \mathbf{ord}\,\varphi$.

We denote by $\mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0}) \subset \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$ the full subcategory of *equimultiple deformations of the parametrization.* Moreover, the corresponding set of isomorphism classes is denoted by $\underline{\mathcal{D}ef}^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$, and we set

$$T^{1,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} := \underline{\mathcal{D}ef}^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T_\varepsilon)\,.$$

(2) More generally, let $\boldsymbol{m} = (m_1, \dots, m_r)$, $1 \le m_i \le \operatorname{ord}\varphi_i$, be an integer vector. Then we say that $(\phi, \overline{\sigma})$ is $\boldsymbol{m}$*-multiple* if $\phi_i^*(I_\sigma) \subset I_{\overline{\sigma}_i}^{m_i}$ for $i = 1, \dots, r$.

$\mathcal{D}ef^{\,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$, $\underline{\mathcal{D}ef}^{\,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$, and $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ have the obvious meaning.

Note that $\mathcal{D}ef^{\,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ coincides with $\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ for $\boldsymbol{m} = (1, \dots, 1)$, and with $\mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ for $\boldsymbol{m} = (\operatorname{ord}\varphi_1, \dots, \operatorname{ord}\varphi_r)$.

If σ and all the $\overline{\sigma}_i$ are trivial sections (which we always may assume by Proposition 2.2), then $\operatorname{ord}(\phi_i, \overline{\sigma}_i)$ is the minimum of the t_i-orders of $X_i(t_i, s)$ and $Y_i(t_i, s)$. If this minimum is attained by, say, X_i, then equimultiple implies that the leading term of (the power series expansion in t_i of) X_i is a unit in $\mathcal{O}_{T,\mathbf{0}}$. Moreover, the deformation is $\boldsymbol{m}$-multiple iff $\operatorname{ord}_{t_i} X_i \ge m_i$ and $\operatorname{ord}_{t_i} Y_i \ge m_i$ for all i. Furthermore, an equimultiple deformation of the parametrization of $(C, \mathbf{0})$ induces an equimultiple deformation (of the equation) of $(C, \mathbf{0})$:

Lemma 2.26. *Let $(\phi, \overline{\sigma}, \sigma)$ be an equimultiple deformation of the parametrization of $(C, \mathbf{0})$. Then the induced embedded deformation of each branch of $(C, \mathbf{0})$ and, hence, of $(C, \mathbf{0})$ itself, is equimultiple along σ, too.*

Proof. First, assume that the base $(T,\mathbf{0})$ of the deformation is reduced. For each $t \in T$ near $\mathbf{0}$, ϕ induces a parametrization $\phi_t : \big(\overline{\mathscr{C}}_t, \overline{\sigma}(t)\big) \to \big(\mathbb{C}^2, \sigma(t)\big)$ of the fibre $\big(\mathscr{C}_t, \sigma(t)\big)$ of $(\mathscr{C},\mathbf{0}) \to (T,\mathbf{0})$ over t. Since ϕ is equimultiple,

$$\mathbf{mt}(C,\mathbf{0}) = \mathbf{ord}(\varphi,\overline{0}) = \mathbf{ord}\big(\phi_t, \overline{\sigma}(t)\big) = \mathbf{mt}(\phi_t, \sigma(t))$$

by Exercise I.3.2.1.

For an arbitrary base $(T,\mathbf{0})$, we may assume that $(T,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$, that $\phi : \big(\overline{C} \times \overline{T}, \overline{0}\big) \to \big(\mathbb{C}^2 \times T, \mathbf{0}\big)$, and that the sections are trivial. Then it is clear that there is an extension $\widetilde{\phi} : \big(\overline{C} \times \mathbb{C}^n, \overline{0}\big) \to \big(\mathbb{C}^2 \times \mathbb{C}^n, \mathbf{0}\big)$ of ϕ which is equimultiple along trivial sections and the result follows as before. □

However, the converse of Lemma 2.26 is not true as the following example shows.

Example 2.26.1. (Continuation of Example 2.24.1) The deformation

$$(\mathscr{C},\mathbf{0}) = \big(V(x^2 - y^3 - s^2y^2), \mathbf{0}\big) \longrightarrow (\mathbb{C}, 0)\,, \quad (x,y,s) \longmapsto s\,,$$

of the cusp to a node is equimultiple along the trivial section σ. It is induced by the deformation of the parametrization

$$\phi : (\mathbb{C} \times \mathbb{C}, \mathbf{0}) \longrightarrow (\mathbb{C}^2 \times \mathbb{C}, \mathbf{0})\,, \quad (t,s) \longmapsto (t^3 - s^2t, t^2 - s^2, s)$$

either along the section $\overline{\sigma} : s \mapsto (s,s)$, or along the section $\overline{\sigma} : s \mapsto (-s,s)$. However, $(\phi, \overline{\sigma}, \sigma)$ is not equimultiple: $I_\sigma = \langle x, y\rangle$, $I_{\overline{\sigma}} = \langle t - s\rangle$ (or $\langle t+s\rangle$), $\mathbf{ord}\,\varphi = \mathrm{mt}(C,\mathbf{0}) = 2$, while $\phi^\sharp(I_\sigma) = \langle t^3 - s^2t, (t-s)(t+s)\rangle$ is contained in $I_{\overline{\sigma}}$, but not in $I_{\overline{\sigma}}^2$.

Next, we give an explicit description for the vector space $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ of first order $\boldsymbol{m}$-multiple deformations of the parametrization.

Let $\varphi : \big(\overline{C}, \overline{0}\big) \to (\mathbb{C}^2, \mathbf{0})$ be the parametrization of $\phi(\overline{C},\overline{0}) = (C,\mathbf{0}) = \bigcup_{i=1}^r (C_i, \mathbf{0})$, given by the system of parametrizations for the branches $t_i \mapsto \varphi_i(t_i) = \big(x_i(t_i), y_i(t_i)\big)$, $i = 1,\dots,r$. In the following, we identify $\mathcal{O}_{C,\mathbf{0}}$ with $n^\sharp\mathcal{O}_{C,\mathbf{0}} = \varphi^\sharp\mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \subset \mathcal{O}_{\overline{C},\overline{0}}$, and also any ideal of $\mathcal{O}_{C,\mathbf{0}}$ with its image in $\mathcal{O}_{\overline{C},\overline{0}}$. Then the subalgebra

$$\mathcal{O}_{C,\mathbf{0}} = \mathbb{C}\left\{\begin{pmatrix} x_1 \\ \vdots \\ x_r\end{pmatrix}, \begin{pmatrix} y_1 \\ \vdots \\ y_r\end{pmatrix}\right\} \subset \bigoplus_{i=1}^r \mathbb{C}\{t_i\} = \mathcal{O}_{\overline{C},\overline{0}}$$

has $\mathbb{C}$-codimension $\delta = \delta(C,\mathbf{0})$. We set

$$\dot{\varphi} := \dot{\boldsymbol{x}} \cdot \frac{\partial}{\partial x} + \dot{\boldsymbol{y}} \cdot \frac{\partial}{\partial y} \in \mathcal{O}_{\overline{C},\overline{0}} \cdot \frac{\partial}{\partial x} \oplus \mathcal{O}_{\overline{C},\overline{0}} \cdot \frac{\partial}{\partial y}\,,$$

with

$$\dot{\boldsymbol{x}} := \dot{\varphi^\sharp(x)} := \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_r \end{pmatrix}, \quad \dot{\boldsymbol{y}} := \dot{\varphi^\sharp(y)} := \begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_r \end{pmatrix},$$

and with $\dot{x}_i, \dot{y}_i$ denoting the derivatives of x_i, y_i with respect to t_i. Let

$$\mathfrak{m}_{\overline{C},\overline{0}} := \bigoplus_{i=1}^r \mathfrak{m}_{\overline{C}_i,\overline{0}_i} = \bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}\,.$$

be the Jacobson radical of $\mathcal{O}_{\overline{C},\overline{0}}$, and set, for any r-tuple $\boldsymbol{m} = (m_1,\dots,m_r)$ of integers,

$$\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}} := \bigoplus_{i=1}^r \mathfrak{m}^{m_i}_{\overline{C}_i,\overline{0}_i} = \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}\,.$$

If $1 \le m_i \le \operatorname{ord}\varphi_i$ for all $i = 1,\dots,r$, we introduce the complex vector space

$$\begin{aligned} M^{\boldsymbol{m}}_\varphi &:= \left(\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}\right) \Big/ \left(\dot{\varphi}\cdot \mathfrak{m}_{\overline{C},\overline{0}} + \mathfrak{m}_{C,\boldsymbol{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{C,\boldsymbol{0}}\frac{\partial}{\partial y}\right) \\ &= \left((\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}/\mathfrak{m}_{C,\boldsymbol{0}})\frac{\partial}{\partial x} \oplus (\mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}/\mathfrak{m}_{C,\boldsymbol{0}})\frac{\partial}{\partial y}\right) \Big/ \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}/\mathfrak{m}_{C,\boldsymbol{0}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right). \end{aligned}$$

For $\boldsymbol{0} = (0,\dots,0)$, we set

$$\begin{aligned} M^{\boldsymbol{0}}_\varphi &:= \left(\mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}\right) \Big/ \left(\dot{\varphi}\cdot \mathcal{O}_{\overline{C},\overline{0}} + \mathcal{O}_{C,\boldsymbol{0}}\frac{\partial}{\partial x} \oplus \mathcal{O}_{C,\boldsymbol{0}}\frac{\partial}{\partial y}\right). \\ &= \left((\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\boldsymbol{0}})\frac{\partial}{\partial x} \oplus (\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\boldsymbol{0}})\frac{\partial}{\partial y}\right) \Big/ \mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\boldsymbol{0}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right). \end{aligned}$$

Proposition 2.27. *Using the above notations, the following holds:*

(1) $T^1_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{\boldsymbol{0}}_\varphi$ *and* $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{\boldsymbol{m}}_\varphi$ *if* $1 \le m_i \le \operatorname{ord}\varphi_i$ *for all* $i = 1,\dots,r$. *In particular,*

$$T^{1,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{(1,\dots,1)}_\varphi\,, \quad T^{1,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})} \cong M^{(\operatorname{ord}\varphi_1,\dots,\operatorname{ord}\varphi_r)}_\varphi\,.$$

(2) Let $(T,\boldsymbol{0}) = (\mathbb{C}^k,\boldsymbol{0})$ *with local coordinates* $\boldsymbol{s} = (s_1,\dots,s_k)$. *Moreover, let* $\phi : (\overline{C}\times\mathbb{C}^k,\overline{0}) \to (\mathbb{C}^2\times\mathbb{C}^k,\boldsymbol{0})$ *define an* $\boldsymbol{m}$*-multiple deformation of the parametrization along the trivial sections* $\overline{\sigma}$ *and* σ*, given by* r *holomorphic germs*

$$\phi_i : (\overline{C}_i\times\mathbb{C}^k,\overline{0}_i) \to (\mathbb{C}^2\times\mathbb{C}^k,\boldsymbol{0})\,, \quad (t_i,\boldsymbol{s}) \mapsto \big(X_i(t_i,\boldsymbol{s}), Y_i(t_i,\boldsymbol{s}),\boldsymbol{s}\big)\,.$$

Then $(\phi,\overline{\sigma},\sigma)$ *is a versal (respectively semiuniversal)* $\boldsymbol{m}$*-multiple deformation iff the column vectors*

$$\left(\frac{\partial X_i}{\partial s_j}(t_i,\boldsymbol{0})\frac{\partial}{\partial x} + \frac{\partial Y_i}{\partial s_j}(t_i,\boldsymbol{0})\frac{\partial}{\partial y}\right)_{i=1}^r \in \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}\,,$$

$j = 1,\dots,k$, *represent a system of generators (respectively a basis) for the vector space* $M^{\boldsymbol{m}}_\varphi$.

(3) Let $\boldsymbol{a}^j, \boldsymbol{b}^j \in \mathfrak{m}^{\boldsymbol{m}}_{\overline{C},\overline{0}} = \bigoplus_{i=1}^r t_i^{m_i}\mathbb{C}\{t_i\}$ *be such that*

$$\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial x} = \begin{pmatrix} a_1^j \\ \vdots \\ a_r^j \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} b_1^j \\ \vdots \\ b_r^j \end{pmatrix} \frac{\partial}{\partial y}, \quad j = 1, \ldots, k,$$

represent a basis for $M^{\boldsymbol{m}}_{\varphi}$*. Then the deformation of the parametrization* $\phi : (\overline{C}, \overline{0}) \times (\mathbb{C}^k, \mathbf{0}) \to (\mathbb{C}^2, \mathbf{0}) \times (\mathbb{C}^k, \mathbf{0})$ *given by* $\phi_i = (X_i, Y_i, \boldsymbol{s})$ *with*

$$X_i(t_i, \boldsymbol{s}) = x_i(t_i) + \sum_{j=1}^k a_i^j(t_i) s_j\,,$$

$$Y_i(t_i, \boldsymbol{s}) = y_i(t_i) + \sum_{j=1}^k b_i^j(t_i) s_j\,,$$

$i = 1, \ldots, r$*, is a semiuniversal* $\boldsymbol{m}$*-multiple deformation of the parametrization* φ *over* $(\mathbb{C}^k, \mathbf{0})$*.*

In particular, $\boldsymbol{m}$*-multiple deformations of the parametrization are unobstructed and have a smooth semiuniversal base space of dimension* $\dim_{\mathbb{C}}(M^{\boldsymbol{m}}_{\varphi})$*.*

Proof. Let $\phi \in \mathcal{D}ef_{(\overline{C},\overline{0}) \to (\mathbb{C}^2,\mathbf{0})}(T_\varepsilon)$ be as in Remark 2.21.1, that is, ϕ is given by

$$X_i(t_i, \varepsilon) = x_i(t_i) + \varepsilon a_i(t_i)\,, \quad Y_i(t_i, \varepsilon) = y_i(t_i) + \varepsilon b_i(t_i)\,,$$

with $a_i, b_i \in \mathbb{C}\{t_i\}$, $i = 1, \ldots, r$, $\varepsilon^2 = 0$.

ϕ is trivial iff there exist isomorphisms $\big(\overline{C} \times T_\varepsilon, \overline{0}\big) \xrightarrow{\cong} \big(\overline{C} \times T_\varepsilon, \overline{0}\big)$ and $\big(\mathbb{C}^2 \times T_\varepsilon, \mathbf{0}\big) \xrightarrow{\cong} \big(\mathbb{C}^2 \times T_\varepsilon, \mathbf{0}\big)$ over T_ε, being the identity modulo ε, such that via these isomorphisms ϕ is mapped to the product deformation (that is, the deformation as above with $a_i, b_i = 0$). On the ring level, these isomorphisms are given as

$$x \longmapsto x + \varepsilon\psi_1(x, y)\,, \quad y \longmapsto y + \varepsilon\psi_2(x, y)\,,$$

$\psi_1, \psi_2 \in \mathbb{C}\{x, y\}$ arbitrary, and as

$$t_i \longmapsto \widetilde{t}_i := t_i + \varepsilon h_i(t_i)\,, \quad i = 1, \ldots, r\,,$$

$h_i \in \mathbb{C}\{t_i\}$ arbitrary, such that

$$x_i(t_i) + \varepsilon a_i(t_i) = x_i(\widetilde{t}_i) + \varepsilon\psi_1\big(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i)\big)\,,$$
$$y_i(t_i) + \varepsilon a_i(t_i) = y_i(\widetilde{t}_i) + \varepsilon\psi_2\big(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i)\big)\,.$$

Using Taylor's formula and $\varepsilon^2 = 0$, we get $x_i(\widetilde{t}_i) = x_i(t_i) + \varepsilon\dot{x}_i(t_i)h_i(t_i)$ and $\varepsilon\psi_1\big(x_i(\widetilde{t}_i), y_i(\widetilde{t}_i)\big) = \varepsilon\psi_1\big(x_i(t_i), y_i(t_i)\big)$, and the analogous equations for $y_i(\widetilde{t}_i)$ and $\varepsilon\psi_2$.

Hence, the necessary and sufficient condition for ϕ to be trivial reads

$$a_i = \dot{x}_i h_i + \psi_1(x_i, y_i)\,, \quad b_i = \dot{y}_i h_i + \psi_2(x_i, y_i)\,,$$

that is,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} b_1 \\ \vdots \\ b_r \end{pmatrix} \frac{\partial}{\partial y} \in \dot{\varphi} \cdot \mathcal{O}_{\overline{C},\overline{0}} + \mathcal{O}_{C,\mathbf{0}} \cdot \frac{\partial}{\partial x} \oplus \mathcal{O}_{C,\mathbf{0}} \cdot \frac{\partial}{\partial y} \,.$$

Moreover, ϕ is $\boldsymbol{m}$-multiple along the trivial sections iff $a_i, b_i \in t^{m_i}\mathbb{C}\{t_i\}$. ϕ is trivial along the trivial sections iff the above isomorphisms respect the trivial sections, that is, $\psi_1, \psi_2 \in \mathfrak{m}_{\mathbb{C}^2,\mathbf{0}}$ and $h_i \in t_i\mathbb{C}\{t_i\}$. This proves statement (1).

As the proofs of (2) and (3) are similar to (but simpler than) the proofs of the respective statements for equisingular deformations, we omit them here. $\square$

Example 2.27.1. (1) Consider the irreducible plane curve singularity $(C, \mathbf{0})$ parametrized by $\varphi : t \mapsto (t^2, t^7)$. Then

$$M_\varphi^m \cong (t^m\mathbb{C}\{t\})^2 \Big/ (2t, 7t^6) \cdot t^\delta \cdot \mathbb{C}\{t\} + \langle t^2, t^7\rangle^\delta \mathbb{C}\{t^2, t^7\}^2 \,,$$

with $\delta = 0$ if $m = 0$, and $\delta = 1$ if $m > 0$. As a $\mathbb{C}$-vector space, M_φ^0 has the basis $\{(0, t), (0, t^3), (0, t^5)\}$. Hence,

$$X(t, \boldsymbol{s}) = t^2 \,, \quad Y(t, \boldsymbol{s}) = t^7 + s_1 t + s_2 t^3 + s_3 t^5 \,,$$

defines a semiuniversal deformation of the parametrization of $(C, \mathbf{0})$. Similarly, for $m = 1$,

$$X(t, \boldsymbol{s}) = t^2 + s_1 t \,, \quad Y(t, \boldsymbol{s}) = t^7 + s_2 t + s_3 t^3 + s_4 t^5$$

defines a semiuniversal deformation of the parametrization with section, and

$$X(t, \boldsymbol{s}) = t^2 \,, \quad Y(t, \boldsymbol{s}) = t^7 + s_1 t^3 + s_2 t^5$$

a semiuniversal equimultiple deformation of the parametrization.

(2) Consider the reducible plane curve singularity $(C, \mathbf{0})$ given by the local equation $x(x^3 - y^5)$, and let $(x_i(t), y_i(t))$, $i = 1, 2$, be parametrizations for the branches of $(C, \mathbf{0})$. To save indices, we write $\mathcal{O}_{\overline{C},\overline{0}} = \mathbb{C}\{t\} \oplus \mathbb{C}\{t\}$ instead of $\mathbb{C}\{t_1\} \oplus \mathbb{C}\{t_2\}$ and

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 \end{pmatrix} , \quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}$$

as column vectors in $\mathbb{C}\{t\} \oplus \mathbb{C}\{t\}$. Then $\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\mathbf{0}}$ has dimension $\delta = 9$ and has the $\mathbb{C}$-basis

$$\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ t \end{pmatrix}, \begin{pmatrix} 0 \\ t^2 \end{pmatrix}, \begin{pmatrix} 0 \\ t^3 \end{pmatrix}, \begin{pmatrix} 0 \\ t^4 \end{pmatrix}, \begin{pmatrix} 0 \\ t^6 \end{pmatrix}, \begin{pmatrix} 0 \\ t^7 \end{pmatrix}, \begin{pmatrix} 0 \\ t^9 \end{pmatrix}, \begin{pmatrix} 0 \\ t^{12} \end{pmatrix} \right\} .$$

Now, $M_\varphi^{(0,0)}$ is $\left(\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\mathbf{0}}\right)^2$ modulo

$$\left(\begin{pmatrix}\dot{x}_1\\ \dot{x}_2\end{pmatrix},\begin{pmatrix}\dot{y}_1\\ \dot{y}_2\end{pmatrix}\right)\cdot\mathcal{O}_{\overline{C},\overline{0}}=\left(\begin{pmatrix}0\\ 5t^4\end{pmatrix},\begin{pmatrix}1\\ 3t^2\end{pmatrix}\right)\cdot\big(\mathbb{C}\{t\}\oplus\mathbb{C}\{t\}\big)\,.$$

We compute a $\mathbb{C}$-basis of $M_\varphi^{(0,0)}$ as

$$\left\{\left(\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^2\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^3\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\right.$$
$$\left.\left(\begin{pmatrix}0\\t^4\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^6\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\t^9\end{pmatrix},\begin{pmatrix}0\\0\end{pmatrix}\right),\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}0\\t\end{pmatrix}\right)\right\}.$$

Hence, a semiuniversal deformation of the parametrization of $(C,\mathbf{0})$ is given by:

$$\begin{aligned}\begin{pmatrix}X_1(t,\boldsymbol{s})\\ X_2(t,\boldsymbol{s})\end{pmatrix}&=\begin{pmatrix}s_1\\ t^5+s_2t+s_3t^2+s_4t^3+s_5t^4+s_6t^6+s_7t^9\end{pmatrix},\\ \begin{pmatrix}Y_1(t,\boldsymbol{s})\\ Y_2(t,\boldsymbol{s})\end{pmatrix}&=\begin{pmatrix}t\\ t^3+s_8t\end{pmatrix}.\end{aligned}$$

2.4 Computation of T^1 and T^2

In the previous section, we gave an explicit description of the semiuniversal deformation of the parametrization of a reduced plane curve singularity $j:(C,\mathbf{0})\hookrightarrow(\mathbb{C}^2,\mathbf{0})$. In this section, we consider infinitesimal deformations and obstructions for deformations of the parametrization and for related deformations. We are interested in explicit formulas for T^1 and T^2 in terms of basic invariants of $(C,\mathbf{0})$, because these modules contain important information on the deformation functors. For example, if $T^2=0$, then the semiuniversal deformation has a smooth base space of dimensiom $\dim_{\mathbb{C}} T^1$.

The main tool is the cotangent braid of the normalization of $(C,\mathbf{0})$, $n:\big(\overline{C},\overline{0}\big)\to(C,\mathbf{0})$, and of the parametrization $\varphi:=j\circ n:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$. This can be found in Appendix C.5, as well as the notations to be used and the formula

$$T^i_{X\backslash X\to Y/Y}\cong T^{i-1}_Y(F_*\mathcal{O}_X)\,,\quad i\geq 0\,, \tag{2.4.12}$$

where $F:X\to Y$ is any morphism of complex spaces, respectively of germs of complex spaces.

To simplify notations, throughout this section we usually omit the base points. That is, we write $\overline{C}$ instead of $\big(\overline{C},\overline{0}\big)$, C instead of $(C,\mathbf{0})$, and $\mathbb{C}^2$ instead of $(\mathbb{C}^2,\mathbf{0})$. Furthermore, we set

$$\mathcal{O}=\mathcal{O}_{C,\mathbf{0}}=\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}/\langle f\rangle\,,\quad \overline{\mathcal{O}}=\mathcal{O}_{\overline{C},\overline{0}}=\bigoplus_{i=1}^r\mathbb{C}\{t_i\}\,.$$

The maps $n^\sharp:\mathcal{O}\to\overline{\mathcal{O}}$, resp. $\varphi^\sharp:\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\to\overline{\mathcal{O}}$, are the $\mathbb{C}$-algebra maps of n and φ, sending x to $(x_1,\dots,x_r)$ and y to $(y_1,\dots,y_r)$ in $\overline{\mathcal{O}}$. We set

$$\dot{x}_i := \frac{\partial x_i}{\partial t_i}\,, \quad \dot{y}_i := \frac{\partial y_i}{\partial t_i}\,.$$

For computing T^1 and T^2, we also need T^0, which we describe first:

Lemma 2.28. *With the above notations, we have*

(1) $T^0_{\overline{C}\to\mathbb{C}^2} = \big\{(\xi,\eta) \in \mathrm{Der}_{\mathbb{C}}\big(\overline{\mathcal{O}},\overline{\mathcal{O}}\big) \times \mathrm{Der}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big) \;\big|\; \xi \circ \varphi^\sharp = \varphi^\sharp \circ \eta\big\}\,,$
$T^0_{\overline{C}\to C} \xrightarrow{\cong} T^0_C\,.$

(2) $T^0_{\overline{C}/C} = T^0_{\overline{C}/\mathbb{C}^2} = 0\,.$

(3) $T^0_{C\backslash\overline{C}\to C/C} = T^0_{C\backslash\overline{C}\to\mathbb{C}^2/\mathbb{C}^2} = 0\,.$

(4) $T^0_{C\backslash\mathbb{C}^2} = \big\{\eta \in \mathrm{Der}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big) \;\big|\; \varphi^\sharp \circ \eta = 0\big\} = \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \cdot \Big(f\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\Big)\,,$
$T^0_{\overline{C}\backslash C} = 0\,.$

(5) $T^0_C = \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathcal{O}) = \mathrm{Hom}_{\mathbb{C}}(\Omega^1_{C,\mathbf{0}},\mathcal{O})\,.$

(6) For each $\overline{\mathcal{O}}$-module N, respectively each $\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$-module M, we have

$$T^0_{\overline{C}}(N) = \bigoplus_{i=1}^{r} N\frac{\partial}{\partial t_i}\,, \quad T^0_{\mathbb{C}^2}(M) = M\frac{\partial}{\partial x} \oplus M\frac{\partial}{\partial y}\,.$$

Moreover, $T^0_{\overline{C}} = T^0_{\overline{C}}\big(\overline{\mathcal{O}}\big)\,, \quad T^0_{\mathbb{C}^2} = T^0_{\mathbb{C}^2}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big)\,.$

Proof. (1) The first statement is just the definition of $T^0_{\overline{C}\to\mathbb{C}^2}$. The definition of $T^0_{\overline{C}\to C}$ is analogous. From $T^0_{\overline{C}/C} = 0$ (shown in (2)) and from the exact sequence $\dashrightarrow$ in the braid of $\overline{C} \to C$ (see Figure 2.14), it follows that the map $T^0_{\overline{C}\to C} \to T^0_C$ is injective. However, in characteristic 0, every derivation of $\mathcal{O}$ lifts to $\overline{\mathcal{O}}$ (cf. [Del1]), hence, we have an isomorphism.

(2) By definition, we have $T^0_{\overline{C}/C} = \big\{\xi \in \mathrm{Der}_{\mathbb{C}}\big(\overline{\mathcal{O}},\overline{\mathcal{O}}\big) \;\big|\; \xi \circ n^\sharp = 0\big\}$. Each derivation $\xi \in \mathrm{Der}_{\mathbb{C}}\big(\overline{\mathcal{O}},\overline{\mathcal{O}}\big)$ is of the form $\xi = \sum_{i=1}^r h_i \frac{\partial}{\partial t_i}$ for some $h_i \in \mathbb{C}\{t_i\}$. Now, the equality $\xi \circ n^\sharp = 0$ implies that

$$0 = \xi \circ n^\sharp(x) = \xi\big(x_1(t_1),\ldots,x_r(t_r)\big) = \big(h_1\dot{x}_1,\ldots,h_r\dot{x}_r\big)\,,$$

and, in an analogous manner, $\big(h_1\dot{y}_1,\ldots,h_r\dot{y}_r\big) = 0$. Hence, for all $i = 1,\ldots,r$, $h_i(\dot{x}_i,\dot{y}_i) = 0$, which implies $h_i = 0$ as $(\dot{x}_i,\dot{y}_i) \neq (0,0)$. The same argument applies to $T^0_{\overline{C}/\mathbb{C}^2}$.

(3) follows from the definition, respectively from the isomorphism (2.4.12).

(4) $T^0_{\overline{C}\backslash C} = \{\xi \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathcal{O}) \mid n^\sharp \circ \xi = 0\} = 0$, since $n^\sharp$ is injective. The result for $T^0_{C\backslash\mathbb{C}^2}$ follows in the same way, since $\mathrm{Ker}\,\varphi^\sharp = \mathcal{O}f$.

(5),(6) are just the definitions. $\square$

In the following, we use that, for (X,x) a smooth germ (respectively a complete intersection germ), we have $T^i_{X,x}(M)=0$ for each finitely generated $\mathcal{O}_{X,x}$-module and $i\geq 1$ (respectively $i\geq 2$). In particular, as plane curve singularities are complete intersections, $T^i_C(M)=0$ for all $i\geq 2$.

The non-zero terms of the braid for the parametrization are shown in Figure 2.13, with $T^1_{\overline{C}\backslash\overline{C}\to\mathbb{C}^2/\mathbb{C}^2}$ being replaced by $T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)$ according to (2.4.12).

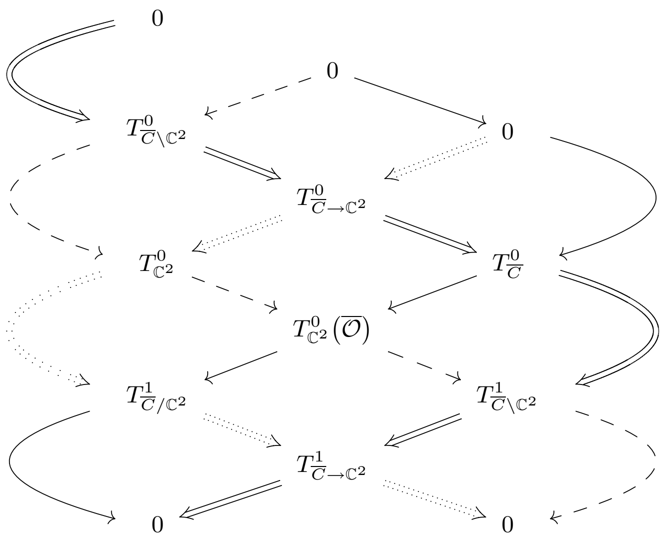


Fig. 2.13. The cotangent braid for the parametrization $\varphi:\overline{C}\to\mathbb{C}^2$.

The maps $\varphi^*:T^0_{\mathbb{C}^2}\dashrightarrow T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)$ and $\varphi':T^0_{\overline{C}}\longrightarrow T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)$ in the braid can be made explicit by using the isomorphisms in Lemma 2.28. Namely,

$$\varphi^*:\mathbb{C}\{x,y\}\frac{\partial}{\partial x}\oplus\mathbb{C}\{x,y\}\frac{\partial}{\partial y}\longrightarrow\bigoplus_{i=1}^r\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\oplus\bigoplus_{i=1}^r\mathbb{C}\{t_i\}\frac{\partial}{\partial y}$$

is componentwise the structure map

$$x\mapsto\big(x_1(t_1),\dots,x_r(t_r)\big)\,,\quad y\mapsto\big(y_1(t_1),\dots,y_r(t_r)\big)\,,$$

while $\varphi'=(\varphi'_1,\dots,\varphi'_r)$ is the tangent map

$$\varphi'_i:\mathbb{C}\{t_i\}\frac{\partial}{\partial t_i}\to\mathbb{C}\{t_i\}\frac{\partial}{\partial x}\oplus\mathbb{C}\{t_i\}\frac{\partial}{\partial y}\,,\qquad\frac{\partial}{\partial t_i}\mapsto\dot{x}_i(t_i)\frac{\partial}{\partial x}+\dot{y}_i(t_i)\frac{\partial}{\partial y}\,.$$

In particular, we have

$$\begin{aligned} \varphi^*(T^0_{\mathbb{C}^2}) &= \mathcal{O}\cdot\frac{\partial}{\partial x}\oplus\mathcal{O}\cdot\frac{\partial}{\partial y} \\ \varphi'(T^0_{\overline{C}}) &= \overline{\mathcal{O}}\cdot\dot{\varphi} = \overline{\mathcal{O}}\cdot\dot{\boldsymbol{x}}\,\frac{\partial}{\partial x}\oplus\overline{\mathcal{O}}\cdot\dot{\boldsymbol{y}}\,\frac{\partial}{\partial y} \end{aligned} \tag{2.4.13}$$

with

$$\dot{\varphi} = \varphi'\left(\frac{\partial}{\partial t_1},\dots,\frac{\partial}{\partial t_r}\right) = \dot{\boldsymbol{x}}\cdot\frac{\partial}{\partial x}+\dot{\boldsymbol{y}}\cdot\frac{\partial}{\partial y} = \begin{pmatrix}\dot{x}_1\\ \vdots\\ \dot{x}_r\end{pmatrix}\cdot\frac{\partial}{\partial x}+\begin{pmatrix}\dot{y}_1\\ \vdots\\ \dot{y}_r\end{pmatrix}\cdot\frac{\partial}{\partial y}\,.$$

Using the results of Lemma 2.28 and the isomorphism (2.4.12), the braid for the normalization looks as displayed in Figure 2.14.

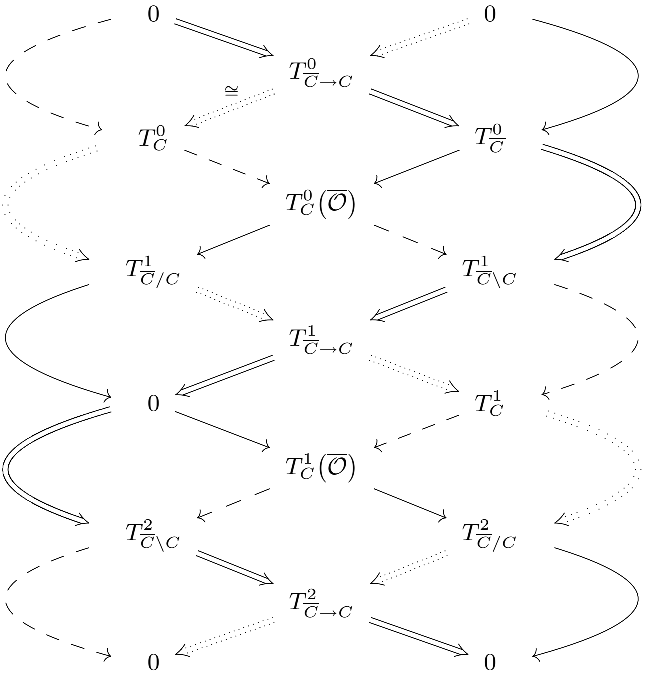


Fig. 2.14. The cotangent braid for the normalization $n:\overline{C}\to C$.

Since we have $T^0_C(M)\subset T^0_{\mathbb{C}^2}(M)$ for each $\mathcal{O}$-module M, we can give the following description of $n^*: T^0_C \dashrightarrow T^0_C(\overline{\mathcal{O}})$ and $n': T^0_{\overline{C}}\longrightarrow T^0_C(\overline{\mathcal{O}})$:

$$n^*:\mathcal{O}\frac{\partial}{\partial x}\oplus\mathcal{O}\frac{\partial}{\partial y}\supset T^0_C\longrightarrow T^1_{\overline{C}\backslash\overline{C}\to C/C}\cong T^0(\overline{\mathcal{O}})\subset\overline{\mathcal{O}}\frac{\partial}{\partial x}\oplus\overline{\mathcal{O}}\frac{\partial}{\partial y}$$

is given by $x \mapsto \big(x_1(t_1), \dots, x_r(t_r)\big)$, $y \mapsto \big(y_1(t_1), \dots, y_r(t_r)\big)$, and

$$n' : \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\} \frac{\partial}{\partial t_i} \cong T^0_{\overline{C}} \longrightarrow T^1_{\overline{C}\backslash \overline{C} \to C/C} \subset \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\} \frac{\partial}{\partial x} \oplus \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\} \frac{\partial}{\partial y}\,,$$

is given by $\frac{\partial}{\partial t_i} \mapsto \dot{x}_i(t_i)\frac{\partial}{\partial x} + \dot{y}_i(t_i)\frac{\partial}{\partial y}$.

Lemma 2.29. *With the notations introduced above, we have*

$$T^i_{\overline{C}\backslash C} \cong T^{i-1}_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,, \quad i \geq 0\,.$$

Proof. $T^i_{\overline{C}\backslash C}$ appears in the exact sequence of complex vector spaces

$$0 \dashrightarrow T^0_C \dashrightarrow T^0_C\big(\overline{\mathcal{O}}\big) \dashrightarrow T^1_{\overline{C}\backslash C} \dashrightarrow T^1_C \dashrightarrow T^1_C\big(\overline{\mathcal{O}}\big) \dashrightarrow T^2_{\overline{C}\backslash C} \dashrightarrow \cdots$$

of the cotangent braid for the normalization (see Figure 2.14). Moreover, by Appendix C.4, we have the long T^i_C-sequence induced by the exact sequence $0 \to \mathcal{O} \to \overline{\mathcal{O}} \to \overline{\mathcal{O}}/\mathcal{O} \to 0$ of $\mathcal{O}$-modules. Since $T^i_C(\mathcal{O}) = T^i_C$, we can replace $T^i_{\overline{C}\backslash C}$ by $T^{i-1}_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)$ in the above exact sequence, whence the result. $\square$

The following proposition is the main result of this section. As usually, τ denotes the Tjurina number, δ the δ-invariant, mt the multiplicity, and r the number of branches of $(C, \mathbf{0})$.

Proposition 2.30. *Let $(C, \mathbf{0}) \overset{j}{\hookrightarrow} (\mathbb{C}^2, \mathbf{0})$ be a reduced plane curve singularity, defined by $f \in \mathbb{C}\{x, y\}$. Let $n : \big(\overline{C}, \overline{0}\big) \to (C, \mathbf{0})$ be the normalization, and let $\varphi := j \circ n$ be the parametrization of $(C, \mathbf{0})$. Then the following holds:*

(1) (i) $T^1_{\overline{C}\backslash \mathbb{C}^2} \cong \big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial x} \oplus \big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial y}$ is a complex vector space of dimension 2δ.
(ii) $T^2_{\overline{C}\backslash \mathbb{C}^2} = 0$.

(2) (i) $T^1_{\overline{C}/\mathbb{C}^2} \cong \Big(\overline{\mathcal{O}}\frac{\partial}{\partial x} \oplus \overline{\mathcal{O}}\frac{\partial}{\partial y}\Big) \Big/ \overline{\mathcal{O}}\Big(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\Big)$ is an $\overline{\mathcal{O}}$-module of rank one.
(ii) $T^2_{\overline{C}/\mathbb{C}^2} = 0$.

(3) (i) $T^1_{\overline{C}\to\mathbb{C}^2} \cong \Big(\big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial x} \oplus \big(\overline{\mathcal{O}}/\mathcal{O}\big)\frac{\partial}{\partial y}\Big) \Big/ \big(\overline{\mathcal{O}}/\mathcal{O}\big)\Big(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\Big)$ is a $\mathbb{C}$-vector space of dimension $2\delta - \dim_{\mathbb{C}}(T^0_{\overline{C}}/T^0_C) = \tau - \delta$.
(ii) $T^2_{\overline{C}\to\mathbb{C}^2} = 0$.

(4) (i) $T^1_C \cong \mathcal{O} \Big/ \Big(\mathcal{O}\frac{\partial f}{\partial x} + \mathcal{O}\frac{\partial f}{\partial y}\Big)$ is a $\mathbb{C}$-vector space of dimension τ.
(ii) $T^2_C = 0$.

(5) (i) $T^1_{\overline{C}\to C} \cong T^1_{\overline{C}\to\mathbb{C}^2}$ has $\mathbb{C}$-dimension $\tau - \delta$.
(ii) $T^2_{\overline{C}\to C} \cong \overline{\mathcal{O}}/\mathcal{O}$ has $\mathbb{C}$-dimension δ.

(6) (i) $T^1_{\overline{C}\backslash C} \cong T^1_{\overline{C}\backslash\mathbb{C}^2}$ *has* $\mathbb{C}$*-dimension* 2δ.

(ii) $T^2_{\overline{C}\backslash C} \cong \overline{\mathcal{O}}/\mathcal{O}$ *has* $\mathbb{C}$*-dimension* δ.

(7) (i) $T^0_C(\overline{\mathcal{O}}) \cong \overline{\mathcal{O}}\left(\dot{\bar{\boldsymbol{x}}}\frac{\partial}{\partial x} + \dot{\bar{\boldsymbol{y}}}\frac{\partial}{\partial y}\right)$ *is a free* $\overline{\mathcal{O}}$*-module of rank* 1*. Here,* $\dot{\bar{\boldsymbol{x}}} = (\dot{\bar{x}}_1, \dots, \dot{\bar{x}}_r)$, $\dot{\bar{\boldsymbol{y}}} = (\dot{\bar{y}}_1, \dots, \dot{\bar{y}}_r)$, *where*

$$\dot{\bar{x}}_i := \frac{\dot{x}_i}{\gcd(\dot{x}_i, \dot{y}_i)} = t_i^{-m_i+1}\dot{x}_i(t_i),\ \ \dot{\bar{y}}_i := \frac{\dot{y}_i}{\gcd(\dot{x}_i, \dot{y}_i)} = t_i^{-m_i+1}\dot{y}_i(t_i),$$

where $m_i = \min\{\mathrm{ord}_{t_i} x_i(t_i), \mathrm{ord}_{t_i} y_i(t_i)\}$.

(ii) $T^1_C(\overline{\mathcal{O}}) \cong T^2_{\overline{C}\backslash C}$ *is of* $\mathbb{C}$*-dimension* δ.

(8) (i) $T^1_{\overline{C}/C} \cong \overline{\mathcal{O}}\left(\dot{\bar{\boldsymbol{x}}}\frac{\partial}{\partial x} + \dot{\bar{\boldsymbol{y}}}\frac{\partial}{\partial y}\right) \Big/ \overline{\mathcal{O}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right)$ *is a* $\mathbb{C}$*-vector space of dimension* $\mathrm{mt} - r$.

(ii) $T^2_{\overline{C}/C}$ *has* $\mathbb{C}$*-dimension* $2\delta + \mathrm{mt} - r$.

Proof. (1) (i) From the exact sequence $\dashrightarrow$ in the cotangent braid for the parametrization, we get $T^1_{\overline{C}\backslash\mathbb{C}^2} = \mathrm{Coker}\big(\varphi^* : T^0_{\mathbb{C}^2} \to T^0_{\mathbb{C}^2}(\overline{\mathcal{O}})\big)$, and then the formula follows from the explicit description of φ^*. (ii) is also a consequence of the same exact sequence, noting that $T^1_{\mathbb{C}^2}(\varphi_*\mathcal{O}_{\overline{C}}) = 0 = T^2_{\mathbb{C}^2}$, since $(\mathbb{C}^2, \mathbf{0})$ is smooth.

(2) (i) and (ii) follow in the same way from the exact sequence $\longrightarrow$ in the cotangent braid for the parametrization and the explicit description of φ'.

For the next statements, consider the exact sequences $\Longrightarrow$ in the braids for the normalization and for the parametrization. From these we obtain the rows in the following commutative diagram with exact rows and columns (with $I = \mathcal{O}f$)

$$\begin{tikzcd}
0 \arrow[r] & T^0_{\overline{C}\to C} \arrow[r] & T^0_{\overline{C}} \arrow[r] & T^1_{\overline{C}\backslash C} \arrow[r] \arrow[d, hook] & T^1_{\overline{C}\to C} \arrow[r] \arrow[d, hook] & 0 \\
& T^0_{\overline{C}\to\mathbb{C}^2} \arrow[r] \arrow[u, two heads, "\alpha"] & T^0_{\overline{C}} \arrow[r] \arrow[u, equal] & T^1_{\overline{C}\backslash\mathbb{C}^2} \arrow[r] \arrow[d, "d^*"] & T^1_{\overline{C}\to\mathbb{C}^2} \arrow[r] & 0 \\
& & & \mathrm{Hom}_{\mathcal{O}}\big(I/I^2, \overline{\mathcal{O}}/\mathcal{O}\big) \arrow[d, two heads] & & \\
& & & T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,. & &
\end{tikzcd} \tag{2.4.14}$$

To define the map α, note that $(\xi, \rho) \in T^0_{\overline{C}\to\mathbb{C}^2}$ satisfies $\xi \circ \varphi^* = \varphi^* \circ \rho$. Since $\ker\varphi^\sharp = I$, we get $\rho(I) \subset I$. Hence, ρ induces a derivation η of $\mathcal{O}_C$, and we define $\alpha(\xi, \rho) = (\xi, \eta)$. As $\varphi^* = n^* \circ j^\sharp$, we have $\xi \circ n^\sharp = n^\sharp \circ \eta$, and α is well-defined.

To see that the map α is surjective, apply the functor $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\_\_,\mathcal{O})$ to the surjection $\Omega^1_{\mathbb{C}^2,\mathbf{0}} \twoheadrightarrow \Omega^1_{C,\mathbf{0}}$, and deduce that $\mathrm{Hom}_{\mathcal{O}}(\Omega^1_{C,\mathbf{0}},\mathcal{O})$ injects into $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\mathcal{O})$. On the other hand, applying $\mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\_\_)$ to the exact sequence $0 \to I \to \mathcal{O}_{\mathbb{C}^2,\mathbf{0}} \to \mathcal{O} \to 0$ gives rise to the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},I) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}) \to \mathrm{Hom}_{\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}}(\Omega^1_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}) \to 0$$

since $\Omega^1_{\mathbb{C}^2,\mathbf{0}}$ is free. Thus, each $\eta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O},\mathcal{O}) \cong \mathrm{Hom}_{\mathcal{O}}(\Omega^1_{C,\mathbf{0}},\mathcal{O})$ lifts to an element $\rho \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}})$, which shows that α is surjective.

The third column results from applying $\mathrm{Hom}_{\mathcal{O}}(\_\_,\overline{\mathcal{O}}/\mathcal{O})$ to the defining exact sequence of $\Omega^1_{C,\mathbf{0}}$,

$$0 \longrightarrow I/I^2 \xrightarrow{\;d\;} \Omega^1_{\mathbb{C}^2,\mathbf{0}} \otimes \mathcal{O} \longrightarrow \Omega^1_{C,\mathbf{0}} \longrightarrow 0\,, \tag{2.4.15}$$

with d induced by the exterior derivation. Note that (see Lemma 2.29)

$$\begin{aligned} T^1_{\overline{C}\backslash C} &\cong T^0_C\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{C,\mathbf{0}},\overline{\mathcal{O}}/\mathcal{O}\big)\,,\\ T^1_{\overline{C}\backslash \mathbb{C}^2} &\cong T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2,\mathbf{0}}\otimes\mathcal{O},\overline{\mathcal{O}}/\mathcal{O}\big)\,, \end{aligned}$$

and (see Proposition 1.25 and Generalization 1.27)

$$\begin{aligned} &T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \mathrm{Coker}\,\big(d^*\colon \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2,\mathbf{0}}\otimes\mathcal{O},\overline{\mathcal{O}}/\mathcal{O}\big) \to \mathrm{Hom}_{\mathcal{O}}\big(I/I^2,\overline{\mathcal{O}}/\mathcal{O}\big)\big)\,,\\ &T^1_{C/\mathbb{C}^2}\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \mathrm{Hom}_{\mathcal{O}}\big(I/I^2,\overline{\mathcal{O}}/\mathcal{O}\big)\,. \end{aligned}$$

The last column in (2.4.14) is induced by the previous one. The commutativity is obvious.

(3) Consider the cotangent braid for the parametrization to conclude that

$$T^1_{\overline{C}\to\mathbb{C}^2} = \mathrm{Coker}\big(T^0_{\overline{C}} \to T^1_{\overline{C}\backslash\mathbb{C}^2}\big) = \mathrm{Coker}\big(T^0_{\overline{C}} \xrightarrow{\varphi'} T^0_{\mathbb{C}^2}\big(\overline{\mathcal{O}}\big)/\varphi^*(T^0_{\mathbb{C}^2})\big)\,,$$

and then use (2.4.13) to get the first formula for $T^1_{\overline{C}\to\mathbb{C}^2}$.

To compute its dimension, we use the diagram (2.4.14), statement (1) (i), and that $T^0_{\overline{C}\to\mathbb{C}^2} \cong T^0_{\overline{C}}$ by Lemma 2.28:

$$\begin{aligned} \dim_{\mathbb{C}} T^1_{\overline{C}\to\mathbb{C}^2} &= \dim_{\mathbb{C}} T^1_{\overline{C}\backslash\mathbb{C}^2} - \dim_{\mathbb{C}} \mathrm{Im}(T^0_{\overline{C}} \to T^1_{\overline{C}\backslash\mathbb{C}^2})\\ &= 2\delta - \dim_{\mathbb{C}} \mathrm{Im}(T^0_{\overline{C}} \to T^1_{\overline{C}\backslash\mathbb{C}^2}) \;=\; 2\delta - \dim_{\mathbb{C}}(T^0_{\overline{C}}/T^0_C)\,. \end{aligned}$$

A formula of Deligne (for the dimension of smoothing components for not necessarily plane curve singularities, see [Del1, GrL]) gives, in our situation,

$$\dim_{\mathbb{C}}(T^0_{\overline{C}}/T^0_C) = 3\delta - \tau\,.$$

(For an independent proof, see Lemma 2.32.) This proves (3) (i). The vanishing of $T^2_{\overline{C}\to\mathbb{C}^2}$ follows from the cotangent braid for the parametrization.

(4) follows from Propositions 1.25 and 1.29 (see also Corollary 1.17).

(5) By Proposition 2.23, we know that $\underline{\mathcal{D}ef}_{\overline{C}\to C}(T) \cong \underline{\mathcal{D}ef}_{\overline{C}\to\mathbb{C}^2}(T)$ for each complex germ T, in particular, for $T = T_\varepsilon$. Hence, $T^1_{\overline{C}\to C} \cong T^1_{\overline{C}\to\mathbb{C}^2}$, which proves (i).

To show (ii), we notice that the same argument proves $T^1_{\overline{C}\backslash C} \cong T^1_{\overline{C}\backslash\mathbb{C}^2}$. From the commutative diagram (2.4.14), it follows that d^* is the zero map and

$$\mathrm{Hom}_{\mathcal{O}}\big(I/I^2, \overline{\mathcal{O}}/\mathcal{O}\big) \xrightarrow{\cong} T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,,$$

and, since $I/I^2 \cong \mathcal{O}f$, we get

$$T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big) \cong \overline{\mathcal{O}}/\mathcal{O}\,,$$

which has $\mathbb{C}$-dimension δ. Furthermore, by Lemma 2.29, and by the braid for the normalization, we get

$$T^2_{\overline{C}\to C} \cong T^2_{\overline{C}\backslash C} \cong T^1_C\big(\overline{\mathcal{O}}/\mathcal{O}\big)\,,$$

whence (ii). [14]

(6) We proved in (5) that $T^1_{\overline{C}\backslash C} \cong T^1_{\overline{C}\backslash\mathbb{C}^2}$, the latter being isomorphic to $\mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2,\mathbf{0}} \otimes \mathcal{O}, \overline{\mathcal{O}}/\mathcal{O}\big) \cong \overline{\mathcal{O}}/\mathcal{O} \oplus \overline{\mathcal{O}}/\mathcal{O}$, which shows (i). (ii) was already proved in (5).

(7) Applying $\mathrm{Hom}_{\mathcal{O}}\big(\_\_, \overline{\mathcal{O}}\big)$ to the sequence (2.4.15), we deduce that $T^0_C\big(\overline{\mathcal{O}}\big)$ is a torsion free, hence free, $\overline{\mathcal{O}}$-module of rank 1, which equals the kernel of the map

$$\overline{\mathcal{O}}\frac{\partial}{\partial x} \oplus \overline{\mathcal{O}}\frac{\partial}{\partial y} \cong \mathrm{Hom}_{\mathcal{O}}\big(\Omega^1_{\mathbb{C}^2} \otimes \mathcal{O}, \overline{\mathcal{O}}\big) \xrightarrow{d^*} \mathrm{Hom}_{\mathcal{O}}\big(I/I^2, \overline{\mathcal{O}}\big) \cong \overline{\mathcal{O}}$$

given by the Jacobian matrix $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. By the chain rule,

$$\frac{\partial}{\partial x}(x_i, y_i)\dot{x}_i + \frac{\partial}{\partial y}(x_i, y_i)\dot{y}_i \;=\; 0\,.$$

Hence, $\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}$ is contained in $T^0_C\big(\overline{\mathcal{O}}\big)$, and it is a non-zerodivisor (in characteristic 0). Therefore, $T^0_C\big(\overline{\mathcal{O}}\big)$ is generated by $\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}$, which proves (i). (ii) follows from the braid for the normalization and from (8) (ii).

(8) (ii) follows from taking the alternating sum of dimensions in the exact sequence $\Longrightarrow$ of the cotangent braid for the normalization.

[14] The fact that the homomorphism $d^* : T^1_{\overline{C}\backslash\mathbb{C}^2} \to \mathrm{Hom}_{\mathcal{O}}(I/I^2, \overline{\mathcal{O}}/\mathcal{O})$ in the diagram (2.4.14) is the zero map is equivalent to the fact that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ annihilate $\overline{\mathcal{O}}/\mathcal{O}$ which is proved here by using deformation theory. This fact can, of course, be proved directly and gives then another proof of (5).

(i) From the cotangent braid for the normalization and from (7) (ii), we have

$$T^1_{\overline{C}/C} \cong \operatorname{Coker}\left(\overline{\mathcal{O}} \cong T^0_{\overline{C}} \xrightarrow{n'} T^0_C(\overline{\mathcal{O}}) \cong \overline{\mathcal{O}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right)\right).$$

The statement follows from the description of n', noting that, in characteristic 0, $\operatorname{ord}_{t_i}\bigl(\gcd(\dot{x}_i, \dot{y}_i)\bigr) = m_i - 1$, where m_i is the multiplicity of the i-th branch. Hence,

$$T^1_{\overline{C}/C} \cong \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}/\bigl\langle \gcd(\dot{x}_i, \dot{y}_i)\bigr\rangle,$$

which is of $\mathbb{C}$-dimension mt $-r$. □

The proof of Proposition 2.30 (5) and the footnote on page 315 yield the following lemma which is of independent interest:

Lemma 2.31. *For a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ defined by $f \in \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$, the Jacobian ideal $j(f) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \subset \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}$ satisfies*

$$j(f) \cdot \mathcal{O}_{\overline{C},\overline{0}} \subset \mathcal{O}_{C,\mathbf{0}}\,, \text{ that is, } j(f) \cdot \mathcal{O}_{C,\mathbf{0}} \subset I^{cd}(C, \mathbf{0}),$$

where $I^{cd}(C, \mathbf{0}) = \operatorname{Ann}_{\mathcal{O}_{C,\mathbf{0}}}\bigl(\mathcal{O}_{\overline{C},\overline{0}}/\mathcal{O}_{C,\mathbf{0}}\bigr)$ is the conductor ideal.

Next, we give an independent proof of Deligne's formula, used in the proof of Proposition 2.30, for plane curve singularities:

Lemma 2.32. *For a reduced plane curve singularity $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$, we have*

$$\dim_{\mathbb{C}}\bigl(T^0_{\overline{C}}/T^0_C\bigr) = 3\delta(C, \mathbf{0}) - \tau(C, \mathbf{0}).$$

Proof. We use the notations of Proposition 2.30. As each derivation of $\mathcal{O}$ lifts uniquely to $\overline{\mathcal{O}}$, the modules T^0_C and $T^0_{\overline{C}\to C}$ have the same image in $T^0_{\overline{C}}$. The latter image consists of derivations $\xi = \sum_{i=1}^{r} h_i \frac{\partial}{\partial t_i} \in \operatorname{Der}_{\mathbb{C}}(\overline{\mathcal{O}}, \overline{\mathcal{O}})$ such that there exists an $\eta \in \operatorname{Der}_{\mathbb{C}}(\mathcal{O}, \mathcal{O})$ satisfying $\xi \circ n^* = n^* \circ \eta$.

η is of the form $g_1 \frac{\partial}{\partial x} + g_2 \frac{\partial}{\partial y}$, $g_1, g_2 \in \mathcal{O}$, such that $g_1 \frac{\partial f}{\partial x} + g_2 \frac{\partial f}{\partial y} = 0$. Evaluating $\xi \circ n^*$ and $n^* \circ \eta$ at x and at y, we obtain

$$h_i \cdot \dot{x}_i = g_1(x_i, y_i)\,, \quad h_i \cdot \dot{y}_i = g_2(x_i, y_i)\,, \quad i = 1, \dots, r\,.$$

The condition $h_i\left(\dot{x}_i \frac{\partial f}{\partial x} + \dot{y}_i \frac{\partial f}{\partial y}\right) = 0$ is fulfilled as $\dot{x}_i \frac{\partial f}{\partial x} + \dot{y}_i \frac{\partial f}{\partial y} = 0$ by the chain rule. Hence, identifying $T^0_{\overline{C}}$ with $\overline{\mathcal{O}}$, we get

$$\operatorname{Im}\bigl(T^0_{\overline{C}\to C} \to T^0_{\overline{C}}\bigr) \cong \bigl\{h \in \overline{\mathcal{O}} \bigm| h \cdot \dot{\boldsymbol{x}} \in \mathcal{O},\ h \cdot \dot{\boldsymbol{y}} \in \mathcal{O}\bigr\}.$$

Now, we have to use local duality. Let ω denote the *dualizing module* (or *canonical module* of $\mathcal{O}$), see [HeK1]. The dualizing module may be realized as a *fractional ideal*, that is, an $\mathcal{O}$-ideal in $\operatorname{Quot}\bigl(\overline{\mathcal{O}}\bigr)$, such that

$$I \mapsto \mathrm{Hom}_{\mathcal{O}}(I,\omega) = \omega : I = \left\{ h \in \mathrm{Quot}\left(\overline{\mathcal{O}}\right) \,\middle|\, hI \subset \omega \right\}$$

defines an inclusion preserving functor on the set of fractional ideals satisfying, in particular,

$$\omega : \omega = \mathcal{O}\,, \quad \omega : (\omega : I) = I\,, \quad \dim_{\mathbb{C}} I/J = \dim_{\mathbb{C}} (\omega : J)/(\omega : I)$$

for each fractional ideals I, J. Since $(C,\mathbf{0})$ is a plane curve singularity, hence Gorenstein, we have $\omega \cong \mathcal{O}$.

An explicit description of the dualizing module ω can be given by means of meromorphic differential forms. Let $\Omega^1_{\overline{C},\overline{0}}(\overline{0})$ denote the germs of meromorphic 1-forms on $\left(\overline{C},\overline{0}\right)$ with poles only at $\overline{0}$. Set

$$\omega^R_{C,\mathbf{0}} := n_* \left\{ \alpha \in \Omega^1_{\overline{C},\overline{0}}(\overline{0}) \,\middle|\, \sum_{i=1}^{r} \mathrm{res}_{\overline{0}_i}(f\alpha) = 0 \text{ for all } f \in \mathcal{O} \right\},$$

which are Rosenlicht's regular differential forms (see [Ser3, IV.9]). We have canonical mappings

$$\mathcal{O} \xrightarrow{\ d\ } \Omega^1_{C,\mathbf{0}} \longrightarrow n_*\Omega^1_{\overline{C},\overline{0}} \hookrightarrow \omega^R_{C,\mathbf{0}}$$

with d the exterior derivation. Exterior multiplikation with df provides (for plane curve singularities) an isomorphism

$$\wedge\, df : \omega^R_{C,\mathbf{0}} \xrightarrow{\ \cong\ } \mathcal{O} dx \wedge dy$$

(see [Ser3, Ch. II]). Let $\Omega_{C,\mathbf{0}}$ ($\cong \Omega^1_{C,\mathbf{0}}/\text{torsion}$) denote the image of $\Omega^1_{C,\mathbf{0}}$ in $\omega^R_{C,\mathbf{0}}$. Then

$$\wedge\, df : \Omega_{C,\mathbf{0}} \xrightarrow{\ \cong\ } \left\langle \frac{\partial f}{\partial x} dx \wedge dy, \frac{\partial f}{\partial y} dx \wedge dy \right\rangle \subset \mathcal{O} dx \wedge dy\,,$$

and, hence,

$$\dim_{\mathbb{C}} \omega^R_{C,\mathbf{0}}/\Omega_{C,\mathbf{0}} = \dim_{\mathbb{C}} \mathcal{O} \Big/ \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \tau\,.$$

All this can be understood in terms of fractional ideals. We can identify the meromorphic differential forms on $\left(\overline{C},\overline{0}\right)$ with $\mathrm{Quot}\big(\overline{\mathcal{O}}\big)$ by mapping $g(t_i)dt_i \mapsto g(t_i)t_i$. Under this identification, we get ideals in $\mathrm{Quot}\big(\overline{\mathcal{O}}\big)$ corresponding to $\omega^R_{C,\mathbf{0}}$, to $\Omega^1_{\overline{C},\overline{0}}$, respectively to $\Omega_{C,\mathbf{0}}$. We denote these fractional ideals by ω, $\overline{\Omega}$, respectively Ω. Note that, as $\Omega_{C,\mathbf{0}}$ is generated by dx and dy, we obtain $\Omega = \langle \dot{\boldsymbol{x}}, \dot{\boldsymbol{y}} \rangle \mathcal{O} \subset \overline{\mathcal{O}}$, and, hence,

$$\mathrm{Im}\left(T^0_{\overline{C}\to C} \to T^0_{\overline{C}}\right) = \left\{ h \in \mathrm{Quot}\big(\overline{\mathcal{O}}\big) \,\middle|\, h\Omega \subset \mathcal{O} \right\} = \mathcal{O} : \Omega\,.$$

To compute the dimension of $T^0_{\overline{C}}/T^0_C = \overline{\mathcal{O}}/(\mathcal{O} : \Omega)$, we use that

$$\dim_{\mathbb{C}}\big(\overline{\mathcal{O}}/(\mathcal{O}:\Omega)\big) = \dim_{\mathbb{C}}\big(\mathcal{O}:(\mathcal{O}:\Omega)/(\mathcal{O}:\overline{\mathcal{O}})\big) = \dim_{\mathbb{C}}(\Omega/I^{cd}),$$

where $I^{cd} = \mathcal{O}:\overline{\mathcal{O}}$ is the conductor ideal. Furthermore, we use that the residue map $\operatorname{res}: \overline{\mathcal{O}} \times \omega \to \mathbb{C}$, $(h,\alpha) \mapsto \sum_{i=1}^{r} \operatorname{res}_{\overline{\mathcal{O}}_i}(h\alpha)$ induces a non-degenerate pairing between $\overline{\mathcal{O}}/\mathcal{O}$ and $\omega/\overline{\mathcal{O}}$. In particular,

$$\dim_{\mathbb{C}}\big(\omega/\overline{\mathcal{O}}\big) = \dim_{\mathbb{C}}\big(\overline{\mathcal{O}}/\mathcal{O}\big) = \delta\,.$$

We have the inclusions $I^{cd} \subset \Omega \subset \overline{\mathcal{O}} \subset \omega$. As $(C,\mathbf{0})$ is a plane curve singularity, $\dim_{\mathbb{C}}\big(\overline{\mathcal{O}}/I^{cd}\big) = 2\delta$ (see I.(3.4.12)), and we get

$$\begin{aligned}\dim_{\mathbb{C}}\big(\Omega/I^{cd}\big) &= \dim_{\mathbb{C}}\big(\overline{\mathcal{O}}/I^{cd}\big) + \dim_{\mathbb{C}}\big(\omega/\overline{\mathcal{O}}\big) - \dim_{\mathbb{C}}\big(\omega/\Omega\big)\\ &= 2\delta + \delta - \tau \;=\; 3\delta - \tau\,,\end{aligned}$$

proving the statement of the lemma. □

We continue by describing the vector space homomorphisms

$$T^1_{\overline{C}\to\mathbb{C}^2} \xrightarrow{\alpha'} T^1_C\,, \qquad T^{1,sec}_{\overline{C}\to\mathbb{C}^2} \xrightarrow{\beta'} T^{1,sec}_C$$

(see page 302) in explicit terms, see (2.4.16) on page 319.

Let $(C,\mathbf{0})$ be given by the local equation $f \in \mathbb{C}\{x,y\}$ with irreducible decomposition $f = f_1 \cdot \ldots \cdot f_r$. Let $(C_i,\mathbf{0})$ be the branch of $(C,\mathbf{0})$ defined by f_i, $i = 1,\ldots,r$. Further, let $x_i(t_i), y_i(t_i) \in \mathbb{C}\{t_i\}$ define a parametrization $\varphi_i : (\mathbb{C},\mathbf{0}) \to (C_i,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ of $(C_i,\mathbf{0})$. In addition to the notations introduced before in this section, we set

$$\mathfrak{m} := \mathfrak{m}_{C,\mathbf{0}}\,, \quad \overline{\mathfrak{m}} = \mathfrak{m}_{\overline{C},\bar{0}} = \bigoplus_{i=1}^{r} t_i\mathbb{C}\{t_i\}\,.$$

Every deformation of the parametrization $\varphi = (\varphi_1,\ldots,\varphi_r)$ of $(C,\mathbf{0})$ is given by a deformation of the φ_i. Over T_ε, it is defined by

$$\begin{aligned}X_i(t_i,\varepsilon) &= x_i(t_i) + \varepsilon a_i(t_i)\,,\\ Y_i(t_i,\varepsilon) &= y_i(t_i) + \varepsilon b_i(t_i)\,,\end{aligned}$$

with

$$\boldsymbol{a} = \begin{pmatrix} a_1\\ \vdots\\ a_r\end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} b_1\\ \vdots\\ b_r\end{pmatrix} \in \overline{\mathcal{O}} = \mathcal{O}_{\overline{C},\bar{0}} = \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}\,.$$

If we consider deformations with (trivial) sections, we assume that $\boldsymbol{a},\boldsymbol{b} \in \overline{\mathfrak{m}}$. As each section can be trivialized (Proposition 2.2), this no loss of generality.

Lemma 2.33. *Let $x_i(t_i) + \varepsilon a_i(t_i)$, $y_i(t_i) + \varepsilon b_i(t_i)$, $i = 1,\ldots,r$, define a deformation of the parametrization of $(C,\mathbf{0})$ over T_ε. Then the induced deformation of the equation is given by*

$$f - \varepsilon(g + hf)$$

for some $h \in \mathbb{C}\{x, y\}$ *and for* $g \in \mathbb{C}\{x, y\}$ *a representative of*

$$\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y} \in \mathcal{O} = \mathbb{C}\{x, y\}/\langle f\rangle\,.$$

Moreover, $g \in \langle x, y\rangle\mathbb{C}\{x, y\}$ *if* $\boldsymbol{a}, \boldsymbol{b} \in \overline{\mathfrak{m}}$*.*

Here, $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$ has to be interpreted as an element of $\overline{\mathcal{O}}$ via

$$\boldsymbol{a}\frac{\partial f}{\partial x} = \begin{pmatrix} a_1(t_1)\frac{\partial f}{\partial x}\big(x_1(t_1), y_1(t_1)\big) \\ \vdots \\ a_r(t_r)\frac{\partial f}{\partial x}\big(x_r(t_r), y_r(t_r)\big) \end{pmatrix}$$

and similarly for $\boldsymbol{b}\frac{\partial f}{\partial y}$. By Lemma 2.31, we know that $\frac{\partial f}{\partial x} \cdot \overline{\mathcal{O}} \subset \mathcal{O}$ and $\frac{\partial f}{\partial x} \cdot \overline{\mathfrak{m}} \subset \mathfrak{m}$ and that the analogous statements hold for $\frac{\partial f}{\partial y}$. Hence, a representative g can be chosen as in Lemma 2.33.

If we write $f = f_i \cdot \widehat{f}_i$ then

$$a_i(t_i)\frac{\partial f}{\partial x}\big(x_i(t_i), y_i(t_i)\big) = a_i(t_i)\frac{\partial f_i}{\partial x}\big(x_i(t_i), y_i(t_i)\big) \cdot \widehat{f}_i\big(x_i(t_i), y_i(t_i)\big)$$

(since $f_i\big(x_i(t_i), y_i(t_i)\big) = 0$) and similarly for $\frac{\partial f_i}{\partial y}$.

In Proposition 2.30, we computed $T^1_{\overline{C}\to\mathbb{C}^2}$ and $T^{1,sec}_{\overline{C}\to\mathbb{C}^2}$, and in Proposition 1.25 we showed that $T^1_{C/\mathbb{C}^2} \cong \mathrm{Hom}_{\mathbb{C}\{x,y\}}(\langle f\rangle, \mathcal{O}) \cong \mathcal{O}$. The same argument yields $T^{1,sec}_{C/\mathbb{C}^2} \cong \mathfrak{m}$.

It follows that the homomorphism α', resp. β', is given by the class mod $\mathcal{O}\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, resp. mod $\mathfrak{m}\langle\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, of

$$\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \longmapsto \boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}\,. \tag{2.4.16}$$

Proof of Lemma 2.33. Let $F_i = f_i + \varepsilon g_i$ define the deformation of $(C_i, \mathbf{0})$ induced by $x_i(t_i) + \varepsilon a_i(t_i)$, $y_i(t_i) + \varepsilon b_i(t_i)$. Then

$$\begin{aligned} 0 &= F_i(x_i + \varepsilon a_i,\, y_i + \varepsilon b_i) \\ &= F_i(x_i, y_i) + \varepsilon\left(a_i\frac{\partial F_i}{\partial x}(x_i, y_i) + b_i\frac{\partial F_i}{\partial y}(x_i, y_i)\right) \\ &= \varepsilon g_i(x_i, y_i) + \varepsilon \cdot \left(a_i\frac{\partial f_i}{\partial x}(x_i, y_i) + b_i\frac{\partial f_i}{\partial y}(x_i, y_i)\right)\,. \end{aligned}$$

It follows that the right-hand side vanishes on the branch $(C_i, \mathbf{0})$. Hence, we get, for some $h_i \in \mathbb{C}\{x, y\}$,

$$-g_i = k_i + h_i f_i\,,$$

where $k_i \in \mathbb{C}\{x,y\}$ is a representative of

$$a_i \frac{\partial f_i}{\partial x} + b_i \frac{\partial f_i}{\partial y} \in \mathcal{O}_{C_i,\mathbf{0}} = \mathbb{C}\{x,y\}/\langle f_i \rangle \,.$$

This shows already the claim in the unibranch case. For the case of several branches, the deformation of $(C,\mathbf{0})$ is given by

$$F = F_1 \cdot \ldots \cdot F_r = f_1 \cdot \ldots \cdot f_r + \varepsilon \cdot \sum_{i=1}^{r} g_i \widehat{f}_i \,.$$

Consider the image of $g_i \widehat{f}_i$ in $\bigoplus_{j=1}^r \mathbb{C}\{t_j\}$. Since $\widehat{f}_i\big(x_j(t_j), y_j(t_j)\big) = 0$ for $j \neq i$, only the i-th component is non-zero and we get

$$\begin{aligned} g_i \widehat{f}_i\big(x_i(t_i), y_i(t_i)\big) = -a_i(t_i) \frac{\partial f_i}{\partial x}\big(x_i(t_i), y_i(t_i)\big) \cdot \widehat{f}_i\big(x_i(t_i), y_i(t_i)\big) \\ + b_i(t_i) \frac{\partial f_i}{\partial y}\big(x_i(t_i), y_i(t_i)\big) \cdot \widehat{f}_i\big(x_i(t_i), y_i(t_i)\big) \end{aligned}$$

which is the i-th component of $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$. □

We close this section by computing T^1 for deformations with section. In addition to the above short hand notations, we introduce

$$J := \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle \cdot \mathcal{O} \,,$$

where $\mathcal{O} = \mathbb{C}\{x,y\}/\langle f \rangle$.

Proposition 2.34. *(1) We have the following isomorphisms of $\mathcal{O}$-modules:*

(i) $T^{1,sec}_{\overline{C}\to C} \cong \left((\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial x} \oplus (\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial y} \right) \Big/ (\overline{\mathfrak{m}}/\mathfrak{m}) \left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y} \right)$,

(ii) $T^{1,sec}_{C} \cong \mathfrak{m}/\mathfrak{m}J$,

(iii) $T^{1,sec}_{\overline{C}/C} \cong \overline{\mathfrak{m}} \left(\dot{\overline{\boldsymbol{x}}}\frac{\partial}{\partial x} + \dot{\overline{\boldsymbol{y}}}\frac{\partial}{\partial y} \right) \Big/ \overline{\mathfrak{m}} \left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y} \right)$, *where*

$$(\dot{\overline{\boldsymbol{x}}}, \dot{\overline{\boldsymbol{y}}}) = t^{-\boldsymbol{m}+\boldsymbol{1}}(\dot{\boldsymbol{x}}, \dot{\boldsymbol{y}})) \,, \quad t^{-\boldsymbol{m}+\boldsymbol{1}} = (t_1^{-m_1+1}, \ldots, t_r^{-m_r+1}) \,,$$

with $m_i = \min\{\mathrm{ord}_{t_i} x_i(t_i), \mathrm{ord}_{t_i} y_i(t_i)\}$.

(2) There are exact sequences of $\mathcal{O}$-modules

$$0 \to T^{1,sec}_{\overline{C}/C} \to T^{1,sec}_{\overline{C}\to C} \to T^{1,sec}_{C} \to \mathfrak{m}/\overline{\mathfrak{m}}J \to 0 \,,$$
$$0 \to T^{1}_{\overline{C}/C} \to T^{1}_{\overline{C}\to C} \to T^{1}_{C} \to \mathcal{O}/\overline{\mathcal{O}}J \to 0 \,.$$

With respect to the isomorphisms in (1), the map $T^{1,sec}_{\overline{C}\to C} \to T^{1,sec}_{C}$ *maps the class of* $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in (\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial x} \oplus (\overline{\mathfrak{m}}/\mathfrak{m})\frac{\partial}{\partial y}$ *to the class of* $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$ *mod* $\mathfrak{m}J$ *and similar for the second sequence.*

(3) (i) $\dim_{\mathbb{C}} T^{1,sec}_{\overline{C}\to C} = \dim_{\mathbb{C}} T^1_{\overline{C}\to C} + \dim_{\mathbb{C}} J/\mathfrak{m}J - r\,,$
(ii) $\dim_{\mathbb{C}} T^{1,sec}_{C} = \dim_{\mathbb{C}} T^1_{C} + \dim_{\mathbb{C}} J/\mathfrak{m}J - 1\,,$
(iii) $\dim_{\mathbb{C}} T^{1,sec}_{\overline{C}/C} = \dim_{\mathbb{C}} T^1_{\overline{C}/C} = \mathrm{mt} - r\,.$
Moreover, if $(C,\mathbf{0})$ *is not smooth, then* $\dim_{\mathbb{C}} T^{1,sec}_{\overline{C}\to C} = \tau - \delta - r + 2$ *and* $\dim_{\mathbb{C}} T^{1,sec}_{C} = \tau + 1$.

Proof. (1) Since $T^{1,sec}_{\overline{C}\to C} \cong T^{1,sec}_{\overline{C}\to\mathbb{C}^2}$ (Proposition 2.23), the first isomorphism follows from Proposition 2.27. The proof of Proposition 1.25 shows that $T^{1,sec}_{C} \cong \mathfrak{m}/\mathfrak{m}J$. The third isomorphism follows in the same way as the isomorphism in Proposition 2.30 (8)(i) (or from the exact sequence in statement (2)).

(2) The exactness at the first three places is given by the exact sequence $\dashrightarrow$ in the braid of Figure 2.14 on page 311 (for deformations with, resp. without, section). The statement about the map $T^{1,sec}_{\overline{C}\to C} \to T^{1,sec}_{C}$ was proved in Lemma 2.33, the cokernel being obviously $\mathfrak{m}/\overline{\mathfrak{m}}J$. The same argument works for $T^1_{\overline{C}\to C} \to T^1_{C}$.

(3) The formula for the dimension of $T^{1,sec}_{\overline{C}/C}$ follows from Proposition 2.30 and using that multiplication with $(t_1,\dots,t_r) \in \overline{\mathfrak{m}}$ induces an isomorphism $T^1_{\overline{C}/C} \cong T^{1,sec}_{\overline{C}/C}$. Since $T^1_C \cong \mathcal{O}/J$, the dimension formula for $T^{1,sec}_{C}$ follows from the inclusions $\mathfrak{m}J \subset J \subset \mathfrak{m} \subset \mathcal{O}$ (for a singular germ $(C,\mathbf{0})$).

To prove the formula in (i), we use the exact sequence in (2). Using that $\dim_{\mathbb{C}} T^1_{\overline{C}\to C} = \tau - \delta$ by Proposition 2.30 and using the exact sequence for deformations without sections, we get $\dim_{\mathbb{C}} \mathcal{O}/\overline{\mathcal{O}}J = \delta + \mathrm{mt} - r$ and, hence, $\dim_{\mathbb{C}} \mathfrak{m}/\overline{\mathfrak{m}}J = \delta + \mathrm{mt} - 1$. Taking into account the dimension formulas for $T^{1,sec}_{\overline{C}/C}$ and for $T^{1,sec}_{C}$, we obtain the formula for $T^{1,sec}_{\overline{C}\to C}$.

To show that $\dim_{\mathbb{C}} J/\mathfrak{m}J = 2$ if $(C,\mathbf{0})$ is singular, we assume to the contrary that $\dim_{\mathbb{C}} J/\mathfrak{m}J = 1$. Then the Tjurina ideal $\langle f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle \subset \mathbb{C}\{x,y\}$ can be generated by f and some $\mathbb{C}\{x,y\}$-linear combination $a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}$ of the partials. But then the definition of the intersection multiplicity together with Propositions I.3.12 and I.3.38 imply that

$$\tau(f) = \dim_{\mathbb{C}} \mathbb{C}\{x,y\} \Big/ \Big\langle f,\, a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y}\Big\rangle \geq \kappa(f) = \mu(f) + \mathrm{mt}(f) - 1\,.$$

But this is impossible if $\mathrm{mt}(f) > 1$.

Corollary 2.35. *The composed map* $T^{1,sec}_{\overline{C}\to C} \to T^1_{\overline{C}\to C} \to T^1_C$ *sending an element* $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in \overline{\mathfrak{m}}\frac{\partial}{\partial x} + \overline{\mathfrak{m}}\frac{\partial}{\partial y}$ *to* $\boldsymbol{a}\frac{\partial f}{\partial x} + \boldsymbol{b}\frac{\partial f}{\partial y}$ *is injective on the vector subspace*

$$T^{1,em}_{\overline{C}\to C} = T^{1,em}_{\overline{C}\to\mathbb{C}^2} = \left\{ \boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \;\middle|\; \begin{array}{c} \min\{\mathrm{ord}_{t_i} a_i, \mathrm{ord}_{t_i} b_i\} \geq \mathrm{mt}\, f_i \\ \text{for each } i = 1,\dots,r \end{array} \right\}.$$

Proof. If $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in T^{1,em}_{\overline{C}\to C}$ is mapped to zero, then the exact sequence (2) together with (1)(iii) of Proposition 2.34 implies that, for some $h_i \in \mathbb{C}\{t_i\}$,

$$a_i\frac{\partial}{\partial x} + b_i\frac{\partial}{\partial y} = h_i t_i^{-m_i+1}\left(\dot{x}_i\frac{\partial}{\partial x} + \dot{y}_i\frac{\partial}{\partial y}\right) \bmod \overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right).$$

By the equimultiplicity assumption, $\operatorname{ord}_{t_i}(h_i t_i^{-m_i+1}) \geq 1$. This shows that $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y}$ is an element of $\overline{\mathfrak{m}}(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y})$ which is zero in $T^{1,sec}_{\overline{C}/C} \subset T^{1,sec}_{\overline{C}\to C}$. $\square$

2.5 Equisingular Deformations of the Parametrization

We define now equisingular deformations of the parametrization. In this context, (embedded) equisingular deformations of the plane curve germ $(C, \mathbf{0})$ as defined in Section 2.1 are referred to as *equisingular deformations of the equation*. In contrast to the semiuniversal equisingular deformation of the equation, the semiuniversal equisingular deformation of the parametrization has an easy explicit description. This description shows that its base space is smooth. We use this to give a new proof of the result of Wahl [Wah] that the base space of the semiuniversal equisingular deformation of the equation is smooth. This implies that the μ-constant stratum in the semiuniversal deformation of $(C, \mathbf{0})$ is smooth.

In order to define equisingular deformations of the parametrization

$$\varphi : (\overline{C}, \overline{0}) = \coprod_{i=1}^{r} (\overline{C}_i, \overline{0}_i) \to (\mathbb{C}^2, \mathbf{0})$$

of the reduced plane curve singularity $(C, \mathbf{0}) = \bigcup_{i=1}^{r}(C_i, \mathbf{0})$, we fix some notations that will be in force for the rest of this section.

If x, y are local coordinates of $(\mathbb{C}^2, \mathbf{0})$, and if t_i are local coordinates of $(\overline{C}_i, \overline{0}_i)$, then $\varphi = (\varphi_i)_{i=1}^{r}$ is given by

$$t_i \overset{\varphi_i}{\longmapsto} \big(x_i(t_i), y_i(t_i)\big), \quad i = 1, \ldots, r,$$

where $x_i, y_i \in \mathbb{C}\{t_i\}$. Let $C \subset M$ be a representative of $(C, \mathbf{0})$, and let $M \subset \mathbb{C}^2$ be an open neighbourhood of $\mathbf{0}$. Let $\pi : \widetilde{M} \to M$ be a finite sequence of point blowing ups, let $\widetilde{C}$, $\widetilde{C}_i$ be the strict transforms of C and C_i, respectively, and let $\widetilde{p} := \widetilde{C} \cap \pi^{-1}(\mathbf{0})$.

Any point $p \in \widetilde{p}$ arising this way, including $\mathbf{0} \in C$, is called an *infinitely near point belonging to* $(C, \mathbf{0})$. For $p \in \widetilde{p}$, we set

$$\begin{aligned}
\Lambda_p &:= \big\{i \,\big|\, 1 \leq i \leq r,\ \widetilde{C}_i \text{ passes through } p\big\},\\
(C_p, \mathbf{0}) &:= \bigcup_{i\in\Lambda_p} (C_i, \mathbf{0}),\ \text{the corresponding subgerm of } C \text{ at } \mathbf{0},\\
(\widetilde{C}, p) &:= \bigcup_{i\in\Lambda_p} \big(\widetilde{C}_i, p\big),\ \text{the germ of } \widetilde{C} \text{ at } p,\\
(\overline{C}, \overline{p}) &:= \coprod_{i\in\Lambda_p} (\overline{C}_i, \overline{0}_i),\ \text{the multigerm of } \overline{C} \text{ at } \overline{p}.
\end{aligned}$$

Of course, $\{\Lambda_p \mid p \in \widetilde{p}\}$, is a partition of $\{1, \ldots, r\}$. $\big(\widetilde{M}, \widetilde{p}\big)$ denotes the multigerm $\coprod_{p\in\widetilde{p}}\big(\widetilde{M}, p\big)$, and $\big(\widetilde{C}, \widetilde{p}\big)$ denotes the multigerm $\coprod_{p\in\widetilde{p}}\big(\widetilde{C}, p\big)$. The restriction of φ,

$$\varphi_p : \big(\overline{C}, \overline{p}\big) \longrightarrow \big(\mathbb{C}^2, \mathbf{0}\big)$$

is a parametrization of $(C_p, \mathbf{0})$. Since $(C_p, \mathbf{0})$ and $\big(\widetilde{C}, p\big)$ have the same normalization, φ_p factors through $\big(\widetilde{M}, p\big)$. The induced map

$$\widetilde{\varphi}_p : \big(\overline{C}, \overline{p}\big) \longrightarrow \big(\widetilde{M}, p\big)$$

is a parametrization of $(\widetilde{C}, p)$. Furthermore, $\pi_p : \big(\widetilde{M}, p\big) \to (M, \mathbf{0})$ denotes the germ of π at p.

Let, for the moment, $\pi : \widetilde{M} \to M$ be the single blowing up of the point $\mathbf{0} \in M$. Then we identify $\pi^{-1}(\mathbf{0})$, the *first infinitely near neighbourhood* of $\mathbf{0}$, with $\mathbb{P}^1$, and we have for a point $p = (\beta : \alpha) \in \mathbb{P}^1$

$$\Lambda_p = \big\{ i \,\big|\, 1 \leq i \leq r\,,\ (C_i, \mathbf{0}) \text{ has tangent direction } p = (\beta : \alpha) \big\}\,.$$

We want to describe $\widetilde{\varphi}_p$ for p belonging to the first infinitely near neighbourhood of $(C, \mathbf{0})$, in terms of local coordinates u, v for $\big(\widetilde{M}, p\big)$. We can assume that π_p is given by

$$\pi_p(u, v) = \begin{cases} \big(u, u(v + \alpha)\big) & \text{if } p = (1 : \alpha)\,, \\ \big(uv, v\big) & \text{if } p = (0 : 1)\,, \end{cases} \tag{2.5.17}$$

(see Remark I.3.16.1) and that $\widetilde{\varphi}_p$ is given by

$$\widetilde{\varphi}_i(t_i) = \big(u_i(t_i), v_i(t_i)\big)\,, \quad i \in \Lambda_p\,,$$

for some $u_i, v_i \in t_i\mathbb{C}\{t_i\}$. As $\varphi_p = \pi_p \circ \widetilde{\varphi}_p$, we get, for all $i \in \Lambda_p$,

$$(x_i, y_i) = \begin{cases} \big(u_i, u_i(v_i + \alpha)\big) & \text{if } p = (1 : \alpha)\,, \\ \big(u_i v_i, v_i\big) & \text{if } p = (0 : 1)\,. \end{cases} \tag{2.5.18}$$

Now, consider a deformation $\phi : \big(\overline{\mathscr{C}}, \overline{0}\big) \to (\mathscr{M}, \mathbf{0})$ of φ over $(T, \mathbf{0})$, with compatible sections $\sigma : (T, \mathbf{0}) \to (\mathscr{M}, \mathbf{0})$ and $\overline{\sigma} = (\overline{\sigma}_i)_{i=1}^r : (T, \mathbf{0}) \to \big(\overline{\mathscr{C}}, \overline{0}\big)$. For an arbitrary infinitely near point $p \in \widetilde{p}$ consider the restriction of ϕ,

$$\phi_p : \big(\overline{\mathscr{C}}, \overline{p}\big) := \coprod_{i\in\Lambda_p} \big(\overline{\mathscr{C}}_i, \overline{0}_i\big) \longrightarrow (\mathscr{M}, \mathbf{0})\,,$$

given by

$$t_i \longmapsto \big(X_i(t_i), Y_i(t_i)\big)\,, \quad i \in \Lambda_p\,,$$

$X_i, Y_i \in \mathscr{O}_{\overline{\mathscr{C}}_i, \overline{0}_i} = \mathscr{O}_{T,\mathbf{0}}\{t_i\}$. Together with σ and $\overline{\sigma}_p = \big(\overline{\sigma}_i\big)_{i\in\Lambda_p}$, ϕ_p is a deformation with compatible sections of φ_p over $(T, \mathbf{0})$.

Let T be a representative of $(T,\mathbf{0})$, and let $\mathscr{M}=M\times T$. Assume that $\pi:\widetilde{\mathscr{M}}\to\mathscr{M}$ is a finite sequence of blowing ups of sections over T such that the restriction over $M\times\{\mathbf{0}\}$ induces the blowing up $\widetilde{M}\to M$ considered before (and which was denoted by the same letter π).

For equisingularity, we require that ϕ_p factors through $\big(\widetilde{\mathscr{M}},p\big)$, that is, there exists

$$\widetilde{\phi}_p:\big(\overline{\mathscr{C}},\overline{p}\big)\longrightarrow\big(\widetilde{\mathscr{M}},p\big)\,,\quad p\in\widetilde{p}\,,$$

such that $\phi_p=\pi_p\circ\widetilde{\phi}_p$, $\pi_p:\big(\widetilde{\mathscr{M}},p\big)\to(\mathscr{M},\mathbf{0})$ being the germ of π at p.

The existence of $\widetilde{\phi}_p$ is in general not sufficient, since it is not necessarily a deformation of the parametrization of $\big(\widetilde{C},p\big)$. In fact, the special fibre of $\big(\widetilde{\mathscr{C}},p\big)\to(T,\mathbf{0})$ is in general the union of $\big(\widetilde{C},p\big)$ with some exceptional divisors. This will be clear from the following considerations.

Let $\pi:\widetilde{M}\to M$ be the blowing up of $\mathbf{0}\in M$, and let $\pi:\widetilde{\mathscr{M}}\to\mathscr{M}$ be the blowing up of the trivial section $\{\mathbf{0}\}\times T$ in $\mathscr{M}$. The above coordinates u,v of $\big(\widetilde{M},p\big)$ induce an isomorphism $\big(\widetilde{\mathscr{M}},p\big)\cong(\mathbb{C}^2,\mathbf{0})\times(T,\mathbf{0})$, and with respect to these coordinates, $\widetilde{\phi}_p:\big(\overline{\mathscr{C}},\overline{p}\big)\to\big(\widetilde{\mathscr{M}},p\big)$ is given by

$$\widetilde{\phi}_p:t_i\longmapsto\big(U_i(t_i),V_i(t_i)\big)\,,\quad i\in\Lambda_p\,,$$

with $U_i,V_i\in\mathcal{O}_{\overline{\mathscr{C}}_i,\overline{0}_i}=\mathcal{O}_{T,\mathbf{0}}\{t_i\}$. Moreover, for all $i\in\Lambda_p$, we have the relation

$$(X_i,Y_i)=\begin{cases}\big(U_i,U_i(V_i+\alpha)\big) & \text{if } p=(1:\alpha)\,,\\ \big(U_iV_i,V_i\big) & \text{if } p=(0:1)\,,\end{cases}$$

where X_i,Y_i define $\phi_p:\big(\overline{\mathscr{C}},\overline{p}\big)\to(\mathscr{M},\mathbf{0})$. Now, let $f\in\mathbb{C}\{x,y\}$ define $(C,\mathbf{0})$, let $F\in\mathcal{O}_{T,\mathbf{0}}\{x,y\}$ define $(\mathscr{C},\mathbf{0})\subset(\mathscr{M},\mathbf{0})$, and let $\widetilde{F}\in\mathcal{O}_{T,\mathbf{0}}\{u,v\}$ define $\big(\widetilde{\mathscr{C}},p\big)\subset\big(\widetilde{\mathscr{M}},p\big)$.

If the (x,y)-order of F is not constant, that is, if $\operatorname{ord}_{x,y}F=\operatorname{ord}_{x,y}f-n$ for some n, then $\big(\widetilde{F}\bmod\mathfrak{m}_{T,\mathbf{0}}\big)\in\mathbb{C}\{u,v\}$ and $\widetilde{f}$, defining the strict transform $\big(\widetilde{C},p\big)$, satisfy the relation $(\widetilde{F}\bmod\mathfrak{m}_{T,\mathbf{0}})=e^n\widetilde{f}$ with $e\in\mathbb{C}\{u,v\}$ defining the exceptional divisor of π_p ($e=u$ if $p=(1:\alpha)$, and $e=v$ if $p=(0:1)$). That is, the special fibre of $\big(\widetilde{\mathscr{C}},p\big)\to(T,\mathbf{0})$ is given by the germ of $\{e^n=0\}\cup\widetilde{C}$ at p.

The definition below forbids this for each infinitely near point belonging to $(C,\mathbf{0})$ if the deformation is equisingular.

Definition 2.36. A deformation $(\phi,\overline{\sigma},\sigma)\in\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ of the parametrization $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$,

$$
\begin{tikzcd}
 & (\overline{C},\overline{0}) \arrow[r, hook] \arrow[d, "\varphi"'] \arrow[dr, phantom, "\square"] & (\overline{\mathscr{C}},\overline{0}) \arrow[d, "\widetilde{\phi}_p"] \\
(\mathbb{C}^2,\mathbf{0}) = & (M,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{M},\mathbf{0}) \arrow[d, shift right] \\
 & \{\mathbf{0}\} \arrow[r, hook] & (T,\mathbf{0}) \arrow[u, shift right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"']
\end{tikzcd}
$$

is called *equisingular* if it is equimultiple and if the following holds

(i) For each infinitely near point $p \in \widetilde{M}$ belonging to $(C,\mathbf{0})$, there exists a germ $\big(\widetilde{\mathscr{M}},p\big)$ and morphisms $\widetilde{\phi}_p$, σ_p, fitting in the commutative diagram with Cartesian squares:

$$
\begin{tikzcd}
(\overline{C},\overline{p}) \arrow[r, hook] \arrow[d, "\widetilde{\varphi}_p"'] \arrow[dr, phantom, "\square"] & (\overline{\mathscr{C}},\overline{p}) \arrow[d, "\widetilde{\phi}_p"] \\
\big(\widetilde{M},p\big) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & \big(\widetilde{\mathscr{M}},p\big) \arrow[d] \\
(M,\mathbf{0}) \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\square"] & (\mathscr{M},\mathbf{0}) \arrow[d, shift right] \\
\{\mathbf{0}\} \arrow[r, hook] & (T,\mathbf{0}) \arrow[u, shift right, "\sigma"'] \arrow[uu, bend right=60, "\sigma_p"'] \arrow[uuu, bend right=70, "\overline{\sigma}_p"']
\end{tikzcd}
$$

such that $\big(\widetilde{\phi}_p,\overline{\sigma}_p,\sigma_p\big)$ is an equimultiple deformation of the parametrization φ_p of $\big(\widetilde{C},p\big)$ over $(T,\mathbf{0})$, with compatible sections $\overline{\sigma}_p$, σ_p.

(ii) The system of such diagrams is compatible: if the germ $\big(\widetilde{M}',q\big)$ dominates $\big(\widetilde{M},p\big)$ (that is, if there is a morphism $\big(\widetilde{M}',q\big) \to \big(\widetilde{M},p\big)$ with dense image), then there exists a morphism $\big(\widetilde{\mathscr{M}}',q\big) \to \big(\widetilde{\mathscr{M}},p\big)$ such that the obvious diagram commutes.

(iii) If $\big(\widetilde{\mathscr{M}}',q\big)$ is consecutive to $\big(\widetilde{\mathscr{M}},p\big)$ (that is, if there is no infinitely near point between the dominating relation) then $\big(\widetilde{\mathscr{M}}',q\big)$ is the blow up of $\big(\widetilde{\mathscr{M}},p\big)$ along the section σ_p.

Remark 2.36.1. (1) In order to check equisingularity of a deformation of the parametrization $\varphi : (\overline{C},\overline{0}) \to (\mathbb{C}^2,\mathbf{0})$, we need only consider infinitely near points appearing in a minimal embedded resolution of $(C,\mathbf{0})$. Since, if $\pi' : (M',p') \to (\mathbb{C}^2,\mathbf{0})$ is any infinitely near neighbourhood of $(C,\mathbf{0})$, then there is an isomorphism $(M',p') \xrightarrow{\cong} (\widetilde{M},p)$ commuting with π' and π, where $\pi : (\widetilde{M},p) \to (\mathbb{C}^2,\mathbf{0})$ is an infinitely near neighbourhood of $(C,\mathbf{0})$ belonging to the minimal embedded resolution of $(C,\mathbf{0})$.

(2) If $(C,\mathbf{0})$ is an ordinary singularity, then $(\widetilde{C},p)$ is smooth for each infinitely near point $p\neq\mathbf{0}$ belonging to $(C,\mathbf{0})$. Then $\widetilde{\varphi}_p:(\overline{C},\overline{p})\to(\widetilde{C},p)$ is an isomorphism and it follows that $(\phi,\overline{\sigma},\sigma)$ is equisingular iff it is equimultiple.

We denote by $\mathcal{D}ef^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ the *category of equisingular deformations of the parametrization* $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$, and by $\underline{\mathcal{D}ef}^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ the corresponding functor of isomorphism classes. Moreover, we introduce

$$T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}:=\underline{\mathcal{D}ef}^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T_\varepsilon)\,,$$

the tangent space to this functor.

Note that $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ is a subspace of $T^{1,\boldsymbol{m}}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ for each vector $\boldsymbol{m}$ satisfying $1\leq m_i\leq\operatorname{ord}\varphi_i$ for all i.

Recall the notation $\varphi=(\varphi_i)_{i=1}^r$, with $\varphi_i(t_i)=\big(x_i(t_i),y_i(t_i)\big)$, and

$$\dot\varphi=\left(\frac{\partial x_i}{\partial t_i}\right)_{i=1}^r\frac{\partial}{\partial x}+\left(\frac{\partial y_i}{\partial t_i}\right)_{i=1}^r\frac{\partial}{\partial y}$$

In view of Proposition 2.27, p. 305, we obviously have the following statement:

Lemma 2.37. *There is an isomorphism of* $\mathbb{C}$*-vector spaces,*

$$T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}\cong I^{es}_\varphi\Big/\left(\dot\varphi\cdot\mathfrak{m}_{\overline{C},\overline{0}}+\varphi^\sharp(\mathfrak{m}_{\mathbb{C}^2,\mathbf{0}})\frac{\partial}{\partial x}\oplus\varphi^\sharp(\mathfrak{m}_{\mathbb{C}^2,\mathbf{0}})\frac{\partial}{\partial y}\right)\,,$$

where $I^{es}_\varphi:=I^{es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ *denotes the set of all elements*

$$\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix}\cdot\frac{\partial}{\partial x}+\begin{pmatrix}b_1\\ \vdots\\ b_r\end{pmatrix}\cdot\frac{\partial}{\partial y}\in\mathfrak{m}_{\overline{C},\overline{0}}\cdot\frac{\partial}{\partial x}\oplus\mathfrak{m}_{\overline{C},\overline{0}}\cdot\frac{\partial}{\partial y}$$

such that $\big\{\big(x_i(t_i)+\varepsilon a_i(t_i),y_i(t_i)+\varepsilon b_i(t_i)\big)\,\big|\,i=1,\dots,r\big\}$ *is an equisingular deformation of the parametrization* $\varphi:\big(\overline{C},\overline{0}\big)\to(\mathbb{C}^2,\mathbf{0})$ *along the trivial sections over* T_ε*.*

We call I^{es}_φ the *equisingularity module of the parametrization* of $(C,\mathbf{0})$. It is an $\mathcal{O}_{C,\mathbf{0}}$-submodule of $\varphi^*\Theta_{\mathbb{C}^2,\mathbf{0}}=\mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial x}\oplus\mathcal{O}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$, as will be shown in Proposition 2.40. Here, $\Theta_{\mathbb{C}^2,\mathbf{0}}=\operatorname{Der}_{\mathbb{C}}\big(\mathcal{O}_{\mathbb{C}^2,\mathbf{0}},\mathcal{O}_{\mathbb{C}^2,\mathbf{0}}\big)$.

The natural map $\Theta_{\overline{C},\overline{0}}\to\varphi^*\Theta_{\mathbb{C}^2,\mathbf{0}}$ maps $\frac{\partial}{\partial t_i}$ to $\dot x_i\frac{\partial}{\partial x}+\dot y_i\frac{\partial}{\partial y}$. Hence, in invariant terms, we see that I^{es}_φ is a submodule of

$$\varphi^*\Theta_{\mathbb{C}^2,\mathbf{0}}\big/\big(\mathfrak{m}_{\overline{C},\overline{0}}\Theta_{\overline{C},\overline{0}}+\varphi^{-1}(\mathfrak{m}_{\mathbb{C}^2,\mathbf{0}}\Theta_{\mathbb{C}^2,\mathbf{0}})\big)\,.$$

Remark 2.37.1. (1) If $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ is smooth, then each deformation $(\phi,\overline{\sigma},\sigma)\in\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T,\mathbf{0})$ is equisingular. This follows as each deformation is equimultiple and the lifting to the blow up of σ is a deformation of the strict transform by the considerations before Definition 2.36. As the strict transform is again smooth, we can continue, and the conditions of Definition 2.36 are fulfilled. It follows that, for a smooth germ $(C,\mathbf{0})$, $I^{es}_{\varphi} = \mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial x} \oplus \mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$ and $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})} = \{0\}$.

(2) If $(T,\mathbf{0}) \subset (\mathbb{C}^n,\mathbf{0})$ and if ϕ is given by $X_i(t_i), Y_i(t_i) \in \mathcal{O}_{T,\mathbf{0}}\{t_i\}$, then we can lift the non-zero coefficients of X_i and Y_i to $\mathcal{O}_{\mathbb{C}^n,\mathbf{0}}$, getting in this way $\widetilde{X}_i(t_i)$, $\widetilde{Y}_i(t_i) \in \mathcal{O}_{\mathbb{C}^n,\mathbf{0}}\{t_i\}$ having the same t_i-order as X_i, Y_i. The same holds after blowing up the trivial section. Hence, as there is no flatness requirement (Remark 2.21.1), we can extend $\boldsymbol{m}$-multiple, respectively equisingular, deformations over $(\mathbb{C}^n,\mathbf{0})$. In particular, when considering $\boldsymbol{m}$-multiple, respectively equisingular, deformations of the parametrization, we may always assume that the base $(T,\mathbf{0})$ is smooth.

Example 2.37.2. (Continuation of Example 2.24.1.) The deformation of the parametrization $(t,s) \mapsto (t^3 - s^2t, t^2 - s^2, s)$, $s \in (\mathbb{C},0)$, of the cusp to a node is not equisingular along any section, since it is not equimultiple for any choice of compatible sections $(\overline{\sigma},\sigma)$ (note that $\overline{\sigma}$ must be a single section, not a multisection, since the cusp is unibranch).

The first order deformation of the parametrization

$$(t,\varepsilon) \mapsto (t^3 - \varepsilon t, t^2 - \varepsilon, \varepsilon)\,, \quad \varepsilon^2 = 0\,,$$

is also not equisingular. However, the corresponding deformation of the equation, given by $x^2 - y^3 - \varepsilon y^2$, is equisingular (along the section σ with $I_\sigma = \langle x, y + \frac{\varepsilon}{3}\rangle$). The same deformation of the equation is induced by the equisingular deformation of the parametrization $(t,\varepsilon) \mapsto (t^3, t^2 - \frac{\varepsilon}{3}, \varepsilon)$.

This shows that an equisingular deformation of the equation (over T_ε) can be induced by several deformations of the parametrization. Exactly one of the inducing deformations of the parametrization is equisingular. This example illustrates the existence, resp. uniqueness, statements of Proposition 2.23 and Theorem 2.64.

The following theorem shows that $\underline{\mathcal{D}ef}^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$ is a "linear" subfunctor of $\underline{\mathcal{D}ef}^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}$. As such, it is already completely determined by its tangent space. We use the notation

$$\boldsymbol{a}^j = \begin{pmatrix} a_1^j \\ \vdots \\ a_r^j \end{pmatrix}\,, \quad \boldsymbol{b}^j = \begin{pmatrix} b_1^j \\ \vdots \\ b_r^j \end{pmatrix} \in \bigoplus_{i=1}^{r} \mathbb{C}\{t_i\}\,, \quad j = 1, \dots, k\,.$$

Theorem 2.38. *Let $\varphi : \big(\overline{C},\overline{0}\big) \to (\mathbb{C}^2,\mathbf{0})$ be a parametrization of the reduced plane curve singularity $(C,\mathbf{0})$, and let $\boldsymbol{s} = (s_1,\dots,s_k)$ be local coordinates of $(\mathbb{C}^k,\mathbf{0})$. Then the following holds:*

(1) Let $\phi : (\overline{C},\overline{0}) \times (\mathbb{C}^k,\mathbf{0}) \to (\mathbb{C}^2,\mathbf{0}) \times (\mathbb{C}^k,\mathbf{0})$ be a deformation of φ with trivial sections over $(\mathbb{C}^k,\mathbf{0})$, given by $\phi_i = (X_i, Y_i, \boldsymbol{s})$ with

$$X_i(t_i,\boldsymbol{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i)s_j\,, \quad a_i^j \in t_i\mathbb{C}\{t_i\}\,,$$

$$Y_i(t_i,\boldsymbol{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i)s_j\,, \quad b_i^j \in t_i\mathbb{C}\{t_i\}\,,$$

$i = 1,\dots,r$. Then ϕ is equisingular iff $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in I_\varphi^{es}$ for all $j = 1,\dots,k$.

(2) Let $\phi = \{(X_i, Y_i, \boldsymbol{s}) \mid i = 1,\dots,r\}$, $X_i, Y_i \in \mathcal{O}_{\mathbb{C}^k,\boldsymbol{0}}\{t_i\}$, be an equisingular deformation of φ with trivial sections over $(\mathbb{C}^k,\boldsymbol{0})$. Then ϕ is a versal (respectively semiuniversal) object of $\mathcal{D}ef^{\,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})}$ iff

$$\begin{pmatrix} \frac{\partial X_1}{\partial s_j}(t_1,\boldsymbol{0}) \\ \vdots \\ \frac{\partial X_r}{\partial s_j}(t_r,\boldsymbol{0}) \end{pmatrix} \cdot \frac{\partial}{\partial x} + \begin{pmatrix} \frac{\partial Y_1}{\partial s_j}(t_1,\boldsymbol{0}) \\ \vdots \\ \frac{\partial Y_r}{\partial s_j}(t_r,\boldsymbol{0}) \end{pmatrix} \cdot \frac{\partial}{\partial y}\,, \quad j = 1,\dots,k\,,$$

represent a system of generators (respectively a basis) of the $\mathbb{C}$-vector space $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})}$.

(3) Let $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in I_\varphi^{es}$, $j = 1,\dots,k$, represent a basis (respectively a system of generators) of $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})}$. Then $\phi = \{(X_i, Y_i, \boldsymbol{s}) \mid i = 1,\dots,r\}$ with

$$X_i(t_i,\boldsymbol{s}) = x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i)s_j\,,$$

$$Y_i(t_i,\boldsymbol{s}) = y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i)s_j\,,$$

$\boldsymbol{s} = (s_1,\dots,s_k) \in (\mathbb{C}^k,\boldsymbol{0})$, is a semiuniversal (respectively versal) equisingular deformation of φ with trivial sections over $(\mathbb{C}^k,\boldsymbol{0})$. In particular, equisingular deformations of the parametrization are unobstructed, and the semiuniversal deformation has a smooth base space of dimension $\dim_{\mathbb{C}} T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})}$.

For the proof, we need some preparations. We fix local coordinates x, y of $(\mathbb{C}^2,\boldsymbol{0})$ and t_i of $\big(\overline{C}_i,\overline{0}_i\big)$. Assume that

$$(\phi,\overline{\sigma},\sigma) = \{(\phi_i,\overline{\sigma}_i,\sigma) \mid i = 1,\dots,r\} \in \mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\boldsymbol{0})}(T,\boldsymbol{0})$$

is given as

$$\big(\overline{\mathscr{C}},\overline{0}\big) = \big(\overline{C}\times T,\overline{0}\big) \xrightarrow{\phi} \big(\mathbb{C}^2\times T,\boldsymbol{0}\big) \xrightarrow{\mathrm{pr}} (T,\boldsymbol{0})\,, \quad \phi = \big(\phi_i,\mathrm{id}_T\big)_{i=1}^r,$$

(see Remark 2.21.1), with $\phi_i = (X_i, Y_i)$, $X_i, Y_i \in \mathcal{O}_{\overline{C}_i\times T,(\overline{0}_i,\boldsymbol{0})}$, and with σ, $\overline{\sigma} = \big(\overline{\sigma}_i\big)_{i=1}^r$ the trivial sections.

We have to consider *small extensions* $(T',\mathbf{0}) \subset (T,\mathbf{0})$ of base spaces, that is, we assume that the surjective map $\mathcal{O}_{T,\mathbf{0}} \to \mathcal{O}_{T',\mathbf{0}}$ has one-dimensional kernel (whose generator is denoted by ε). To shorten notation, we set

$$A := \mathcal{O}_{T,\mathbf{0}}\,, \quad A' := \mathcal{O}_{T',\mathbf{0}}\,.$$

Then we have the analytic A-algebras (respectively analytic A'-algebras)

$$\begin{aligned} A\{t_i\} &= \mathcal{O}_{\overline{C}_i \times T,(\overline{0}_i,\mathbf{0})}\,, \quad & A'\{t_i\} &= \mathcal{O}_{\overline{C}_i \times T',(\overline{0}_i,\mathbf{0})}\,, \\ A\{x,y\} &= \mathcal{O}_{\mathbb{C}^2 \times T,(\mathbf{0},\mathbf{0})}\,, \quad & A'\{x,y\} &= \mathcal{O}_{\mathbb{C}^2 \times T',(\mathbf{0},\mathbf{0})}\,. \end{aligned}$$

Note that, as complex vector spaces, $A = A' \oplus \varepsilon\mathbb{C}$, and that $\varepsilon\mathfrak{m}_A = 0$. The deformation $(\phi,\overline{\sigma},\sigma)$ over $(T,\mathbf{0})$ is given by

$$X_i(t_i) = X_i'(t_i) + \varepsilon a_i\,, \quad Y_i(t_i) = Y_i'(t_i) + \varepsilon b_i\,,$$

with $X_i, Y_i \in A\{t_i\}$ and $a_i, b_i \in \mathbb{C}\{t_i\}$, where $X_i', Y_i' \in A'\{t_i\}$ define a deformation of the parametrization with compatible sections $\overline{\sigma}', \sigma'$ over $(T',\mathbf{0})$. On the ring level, ϕ_i is given by

$$\phi_i^\sharp : A\{x,y\} \to A\{t_i\}\,, \quad x \mapsto X_i\,, \;\; y \mapsto Y_i\,, \quad i = 1,\dots,r\,.$$

Furthermore, the residue classes $x_i(t_i)$, respectively $y_i(t_i)$, of $X_i(t_i)$, respectively $Y_i(t_i)$ modulo $\mathfrak{m}_A$ define the parametrization of the i-th branch $(C_i,\mathbf{0})$, $i = 1,\dots,r$.

Proposition 2.39. *Consider the diagram with given solid arrows*

$$\begin{tikzcd}
(\overline{\mathscr{C}}',\overline{0}) \arrow[r, hook, dotted] \arrow[d, "\widetilde{\phi}'"] \arrow[dr, phantom, "\Box"] & (\overline{\mathscr{C}},\overline{0}) \arrow[d, dotted, "\widetilde{\phi}"'] \\
(\widetilde{\mathscr{M}}',\widetilde{p}) \arrow[r, hook, dotted] \arrow[d, "\pi'"] \arrow[dr, phantom, "\Box"] & (\widetilde{\mathscr{M}},\widetilde{p}) \arrow[d, dotted, "\pi"'] \\
(\mathscr{M}',\mathbf{0}) \arrow[r, hook, dotted] \arrow[d, shift left] \arrow[dr, phantom, "\Box"] & (\mathscr{M},\mathbf{0}) \arrow[d, dotted, shift right] \\
(T',\mathbf{0}) \arrow[r, hook] \arrow[u, shift left, "\sigma'"] \arrow[uuu, bend left=70, "\overline{\sigma}'"] \arrow[uu, bend left=45, "\widetilde{\sigma}'"] & (T,\mathbf{0})\,. \arrow[u, dotted, shift right, "\sigma"'] \arrow[uu, dotted, bend right=45, "\widetilde{\sigma}"'] \arrow[uuu, dotted, bend right=70, "\overline{\sigma}"']
\end{tikzcd}$$

where $(T',\mathbf{0}) \hookrightarrow (T,\mathbf{0})$ is a small extension of complex germs. Assume that

(i)' $\big(\phi' = \pi' \circ \widetilde{\phi}', \overline{\sigma}', \sigma'\big) \in \mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T',\mathbf{0})$.

(ii)' $\pi' : \widetilde{\mathscr{M}}' \to \mathscr{M}'$ is the blowing up of the section σ'. Let $\big(\widetilde{\mathscr{M}}',\widetilde{p}\big)$ be the multigerm at the set $\widetilde{p}$ of infinitely near points belonging to $(C,\mathbf{0})$ in the blow up $\widetilde{M}$ of $\mathbf{0} \in M$.

(iii)' $\widetilde{\sigma}' = \{\widetilde{\sigma}'_p \mid p \in \widetilde{p}\}$ *is a (multi-)section such that* $\big(\widetilde{\phi}', \overline{\sigma}', \widetilde{\sigma}'\big)$ *is an object of* $\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\widetilde{M},\widetilde{p})}(T', \mathbf{0})$.

Then the following holds: The data given in (i)' – (iii)' can be extended over $(T, \mathbf{0})$ *as indicated in the diagram. More precisely, there exist dotted arrows making the above diagram commutative, respectively Cartesian, such that*

(i) $\big(\phi = \pi \circ \widetilde{\phi}, \overline{\sigma}, \sigma\big) \in \mathcal{D}ef^{\,em}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$,

(ii) $\pi : \widetilde{\mathscr{M}} \to \mathscr{M}$ *is the blowing up of* σ,

(iii) $\widetilde{\sigma} = \big\{\widetilde{\sigma}_p \,\big|\, p \in \widetilde{p}\big\}$ *is a (multi-)section such that* $\big(\widetilde{\phi}, \overline{\sigma}, \widetilde{\sigma}\big)$ *is an element of* $\mathcal{D}ef^{\,sec}_{(\overline{C},\overline{0})\to(\widetilde{M},\widetilde{p})}(T, \mathbf{0})$.

Furthermore, $\big(\widetilde{\phi}, \widetilde{\sigma}\big)$ *satisfying (iii) is uniquely determined by* $(\phi, \overline{\sigma}, \sigma)$ *and* $\big(\widetilde{\phi}', \widetilde{\sigma}'\big)$.

Proof. We use the notations introduced above. Since we consider (multi-) germs at $\mathbf{0}, \overline{0}$ and $\widetilde{p}$, we may assume that all sections σ', $\overline{\sigma}'_i$, $i = 1, \ldots, r$, and $\widetilde{\sigma}'_p$, $p \in \widetilde{p}$ are trivial. Let $\varphi : \big(\overline{C}, \overline{0}\big) \to (\mathbb{C}^2, \mathbf{0})$ be given by $x_i, y_i \in t_i\mathbb{C}\{t_i\}$, and let ϕ' be given by X'_i, Y'_i which are elements of $t_i A'\{t_i\}$ as the sections are trivial.

Step 1: Uniqueness. Assume we have extensions $\widetilde{\phi}, \sigma, \overline{\sigma}, \widetilde{\sigma}$ over $(T, \mathbf{0})$ as claimed, with $\sigma, \overline{\sigma}$ the trivial sections. Then $\phi_p : \big(\overline{\mathscr{C}}, \overline{0}\big) \to (\mathscr{M}, \mathbf{0})$, $p \in \widetilde{p}$, is given, on the ring level, by a map

$$\phi_p^\sharp : A\{x, y\} \to \bigoplus_{i \in \Lambda_p} A\{t_i\}\,, \quad x \mapsto (X_i)_{i \in \Lambda_p}\,, \quad y \mapsto (Y_i)_{i \in \Lambda_p}\,,$$

where

$$X_i = X'_i + \varepsilon a_i\,, \quad Y_i = Y'_i + \varepsilon b_i\,, \quad a_i, b_i \in t_i\mathbb{C}\{t_i\}\,.$$

Further, $\widetilde{\phi}_p : \big(\overline{\mathscr{C}}, \overline{p}\big) \to \big(\widetilde{\mathscr{M}}, p\big)$, $p \in \widetilde{p}$, is given by a map

$$\widetilde{\phi}_p^\sharp : A\{u, v\} \to \bigoplus_{i \in \Lambda_p} A\{t_i\}\,, \quad u \mapsto (U_i)_{i \in \Lambda_p}\,, \quad v \mapsto (V_i)_{i \in \Lambda_p}\,,$$

where

$$U_i = U'_i + \varepsilon \widetilde{a}_i\,, \quad V_i = V'_i + \varepsilon \widetilde{b}_i\,,$$

$\widetilde{a}_i, \widetilde{b}_i \in \mathbb{C}\{t_i\}$, and $\widetilde{\phi}'_p$ is given by U'_i, V'_i, $i \in \Lambda_p$. Since σ is the trivial section, the blowing up of σ, $\pi : (\widetilde{\mathscr{M}}, \widetilde{p}) \to (\mathscr{M}, \mathbf{0})$, is given by

$$\pi_p^\sharp : A\{x, y\} \to A\{u, v\}\,, \quad p \in \widetilde{p}\,,$$

with $\pi_p^\sharp(a) = a$ for $a \in A$ and $(u, v) \mapsto \big(u, u(v + \alpha)\big)$ if $p = (1 : \alpha)$, respectively $(u, v) \mapsto (uv, v)$ if $p = (0 : 1)$, where, as usually, we identify the exceptional divisor in $\widetilde{M}$ with $\mathbb{P}^1$. The condition $\phi_p = \pi_p \circ \widetilde{\phi}_p$ implies $X_i = U_i$, $Y_i = U_i(V_i + \alpha)$, hence,

$$X_i' + \varepsilon a_i = U_i' + \varepsilon \widetilde{a}_i\,, \quad Y_i' + \varepsilon b_i = \big(U_i' + \varepsilon \widetilde{a}_i\big)\big(V_i' + \varepsilon \widetilde{b}_i + \alpha\big)$$

for $p = (1:\alpha)$. Moreover, $X_i = U_iV_i$, $Y_i = V_i$, hence

$$X_i' + \varepsilon a_i = \big(U_i' + \varepsilon \widetilde{a}_i\big)\big(V_i' + \varepsilon \widetilde{b}_i\big)\,, \quad Y_i' + \varepsilon b_i = V_i' + \varepsilon \widetilde{b}_i$$

for $p = (0:1)$.

Comparing the coefficients of ε, we obtain for $p = (1:\alpha)$

$$a_i = \widetilde{a}_i \quad b_i = \widetilde{b}_i u_i + \widetilde{a}_i(v_i + \alpha)\,,$$

where $u_i = (U_i' \bmod \mathfrak{m}_{A'})$ and $v_i = (V_i' \bmod \mathfrak{m}_{A'})$ (recall that $\varepsilon \cdot \mathfrak{m}_{A'} = 0$). Equivalently, since $x_i = u_i$,

$$\widetilde{a}_i = a_i\,, \quad \widetilde{b}_i = \frac{b_i - a_i(v_i + \alpha)}{x_i}\,, \quad i \in \Lambda_p\,. \tag{2.5.19}$$

For $p = (0:1)$, we get

$$a_i = \widetilde{a}_i v_i + \widetilde{b}_i u_i\,, \quad b_i = \widetilde{b}_i\,,$$

or, equivalently $(y_i = v_i)$,

$$\widetilde{b}_i = b_i\,, \quad \widetilde{a}_i = \frac{a_i - b_i u_i}{y_i}\,, \quad i \in \Lambda_p\,. \tag{2.5.20}$$

In particular, $\widetilde{\phi}$ is uniquely determined by ϕ, $\overline{\sigma}$ and $\widetilde{\phi}'$.

The condition $\widetilde{\sigma} = \widetilde{\phi} \circ \overline{\sigma}$ implies that $\widetilde{\sigma}$ is uniquely determined with

$$\widetilde{\sigma}_p^\sharp(u) = \overline{\sigma}_i^\sharp(U_i' + \varepsilon \widetilde{a}_i) = (\widetilde{\sigma}_p')^\sharp(u) + \varepsilon \overline{\sigma}_i^\sharp(\widetilde{a}_i) \text{ for } i \in \Lambda_p\,.$$

As $\widetilde{\sigma}_p'$ and $\overline{\sigma}_i$ are trivial sections, the right-hand side equals $\varepsilon \widetilde{a}_i(0)$, where $\widetilde{a}_i(0)$ is the constant term of $\widetilde{a}_i$. In the same way, we have $\widetilde{\sigma}_p^\sharp(v) = \varepsilon \widetilde{b}_i(0)$ for all $i \in \Lambda_p$. In particular, we get the equalities

$$\big(\widetilde{a}_i(0), \widetilde{b}_i(0)\big) = \big(\widetilde{a}_j(0), \widetilde{b}_j(0)\big)\,, \quad \text{for all } i, j \in \Lambda_p\,, p \in \widetilde{p}\,, \tag{2.5.21}$$

which is a necessary and sufficient condition for the (multi-)sections $\widetilde{\sigma}$ and $\overline{\sigma}$ to be compatible. Moreover, $\widetilde{\sigma}$ is trivial iff $\widetilde{a}_i(0) = \widetilde{b}_i(0) = 0$ for all $i = 1, \dots, r$.

Step 2: Existence. We can define the extensions $\phi, \widetilde{\phi}, \sigma, \overline{\sigma}, \widetilde{\sigma}$ over $(T, \mathbf{0})$ using the above conditions. We choose σ and $\overline{\sigma}$ as trivial sections, and we define ϕ by

$$X_i := X_i' + \varepsilon a_i\,, \quad Y_i := Y_i' + \varepsilon b_i\,,$$

with $a_i, b_i \in \mathbb{C}\{t_i\}$ satisfying the following conditions:

$$\operatorname{ord}_{t_i}(a_i), \operatorname{ord}_{t_i}(b_i) \geq \operatorname{mt}(C_i, \mathbf{0})\,, \tag{2.5.22}$$

and, if $p=(1:\alpha)$,

$$\frac{b_i}{x_i}(0)-\alpha\frac{a_i}{x_i}(0)=\frac{b_j}{x_j}(0)-\alpha\frac{a_j}{x_j}(0)\,,\quad \text{for all } i,j\in\Lambda_p\,, \tag{2.5.23}$$

while for $p=(0:1)$

$$\frac{a_i}{y_i}(0)=\frac{a_j}{y_j}(0)\,,\quad \text{for all } i,j\in\Lambda_p\,. \tag{2.5.24}$$

Note that for $p=(1:\alpha)$ and $i\in\Lambda_p$, we have $\operatorname{ord}_{t_i}(x_i)=\operatorname{mt}(C_i,\mathbf{0})$, while for $p=(0:1)$ and $i\in\Lambda_p$, we have $\operatorname{ord}_{t_i}(y_i)=\operatorname{mt}(C_i,\mathbf{0})$, showing that $\frac{b_i}{x_i}$ and $\frac{a_i}{y_i}$ are power series.

By (2.5.22), $(\phi,\overline{\sigma},\sigma)$ is equimultiple and, defining $\widetilde{a}_i,\widetilde{b}_i$ as in (2.5.19), respectively as in (2.5.20), they are well-defined power series in $\mathbb{C}\{t_i\}$. We define $\widetilde{\phi}_p$ by $U_i=U_i'+\varepsilon\widetilde{a}_i$, $V_i=V_i'+\varepsilon\widetilde{b}_i$. Then, using (2.5.19) and (2.5.23), the condition (2.5.21) is satisfied, since for $p=(1:\alpha)$ and $i\in\Lambda_p$ we have $\widetilde{a}_i(0)=0$ and $v_i(0)=0$. For $p=(0:1)$, we can argue similarly using condition (2.5.24). Hence, we can define a section $\widetilde{\sigma}_p$ satisfying $\widetilde{\sigma}_p=\widetilde{\phi}\circ\overline{\sigma}_p$ by setting

$$\widetilde{\sigma}_p^\sharp(u):=\varepsilon\widetilde{a}_i(0)\,,\quad \widetilde{\sigma}_p^\sharp(v):=\varepsilon\widetilde{b}_i(0)\,,$$

for some $i\in\Lambda_p$. The condition $\sigma=\pi\circ\widetilde{\sigma}$ is automatically fulfilled. □

Remark 2.39.1. (1) Note that for $p\neq q\in\widetilde{p}$ and for $i\in\Lambda_p$, $j\in\Lambda_q$ there is no relation between (a_i,b_i) and (a_j,b_j).

(2) If $\sigma,\overline{\sigma}$ and $\widetilde{\sigma}'$ are the trivial sections, then the extension $\widetilde{\sigma}$ is trivial iff, for all $p\in\widetilde{p}$ and all $i\in\Lambda_p$, we have $\frac{b_i}{x_i}(0)=\alpha\frac{a_i}{x_i}(0)$ if $p=(1:\alpha)$ and $\frac{a_i}{y_i}(0)=0$ if $p=(0:1)$.

(3) The extension σ of σ' in Proposition 2.39 has only to satisfy (2.5.22)–(2.5.24). Hence, it is not unique.

We describe now the behaviour of the equisingularity module I^{es}_φ under blowing up.

Let $\varphi:(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})$ be a parametrization of $(C,\mathbf{0})=\bigcup_{i=1}^r(C_i,\mathbf{0})$, let $\pi:(\widetilde{M},\widetilde{p})\to(\mathbb{C}^2,\mathbf{0})$ be the blowing up of $\mathbf{0}$, let $(\widetilde{C},\widetilde{p})=\coprod_{p\in\widetilde{p}}(\widetilde{C},p)$ be the strict transform of $(C,\mathbf{0})$, and let $\widetilde{\varphi}:(\overline{C},\overline{0})\to(\widetilde{C},\widetilde{p})$ be the induced parametrization of $(\widetilde{C},\widetilde{p})$. Further, let x,y be local coordinates for $(\mathbb{C}^2,\mathbf{0})$, and let u,v be local coordinates for $(\widetilde{M},p)$, satisfying $\pi(u,v)=\big(u,u(v+\alpha)\big)$ if $p=(1:\alpha)\in\pi^{-1}(\mathbf{0})=\mathbb{P}^1$, and $\pi(u,v)=\big(uv,v\big)$ if $p=(0:1)$.

Recall that, for $p\in\widetilde{p}$, we have $i\in\Lambda_p$ iff the strict transform $\widetilde{C}_i$ of C_i passes through p, and that Λ_p, $p\in\widetilde{p}$, is a partition of $\{1,\dots,r\}$.

Then $\widetilde{\varphi}$ is a multigerm $(\widetilde{\varphi}_p)_{p\in\widetilde{p}}$, with

$$\widetilde{\varphi}_p:(\overline{C},\overline{p})=\coprod_{i\in\Lambda_p}(\overline{C}_i,\overline{0}_i)\longrightarrow(\widetilde{M},p)\,,\quad t_i\longmapsto\big(u_i(t_i),v_i(t_i)\big)\,,$$

a parametrization of the germ $(\widetilde{C},p)$. Furthermore, for $\widetilde{a}_i,\widetilde{b}_i,a_i,b_i\in\mathbb{C}\{t_i\}$, we set

$$(\widetilde{\boldsymbol{a}},\widetilde{\boldsymbol{b}})=\left(\begin{pmatrix}\widetilde{a}_1\\ \vdots\\ \widetilde{a}_r\end{pmatrix},\begin{pmatrix}\widetilde{b}_1\\ \vdots\\ \widetilde{b}_r\end{pmatrix}\right),\quad(\boldsymbol{a},\boldsymbol{b})=\left(\begin{pmatrix}a_1\\ \vdots\\ a_r\end{pmatrix},\begin{pmatrix}b_1\\ \vdots\\ b_r\end{pmatrix}\right).$$

Proposition 2.40. *With the above notations, the following holds:*

(1) Let $(\widetilde{a}_i,\widetilde{b}_i)\in t_i\mathbb{C}\{t_i\}\oplus t_i\mathbb{C}\{t_i\}$, $i=1,\dots,r$, be given. For $i\in\Lambda_p$ set

$$(a_i,b_i)=\begin{cases}\big(\widetilde{a}_i,\widetilde{b}_iu_i+\widetilde{a}_i(v_i+\alpha)\big) & \textit{if } p=(1:\alpha)\,,\\ \big(\widetilde{a}_iv_i+\widetilde{b}_iu_i,\widetilde{b}_i\big) & \textit{if } p=(0:1)\,.\end{cases}$$

Then $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ iff $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}$ and $\min\{\operatorname{ord}_{t_i}a_i,\operatorname{ord}_{t_i}b_i\}\geq\operatorname{mt}(C_i,\mathbf{0})$ for each $i=1,\dots,r$.

(2) Given $a_i,b_i\in t_i\mathbb{C}\{t_i\}$ such that $\min\{\operatorname{ord}_{t_i}a_i,\operatorname{ord}_{t_i}b_i\}\geq\operatorname{mt}(C_i,\mathbf{0})$ for each $i=1,\dots,r$. For $i\in\Lambda_p$ set

$$(\widetilde{a}_i,\widetilde{b}_i)=\begin{cases}\left(a_i,\dfrac{b_i-a_i(v_i+\alpha)}{x_i}-\dfrac{b_i-a_i}{x_i}(0)\right) & \textit{if } p=(1:\alpha)\,,\\ \left(\dfrac{a_i}{y_i}-\dfrac{a_i}{y_i}(0),b_i\right) & \textit{if } p=(0:1)\,.\end{cases}$$

Then $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}$ iff $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$.

(3) I^{es}_{φ} is an $\mathcal{O}_{C,\mathbf{0}}$-submodule of $\mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial x}\oplus\mathfrak{m}_{\overline{C},\overline{0}}\frac{\partial}{\partial y}$.

Proof. (1) By definition, $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ iff $x_i(t_i)+\varepsilon a_i(t_i)$, $y_i(t_i)+\varepsilon b_i(t_i)$ defines an equisingular deformation ϕ of $\varphi:t_i\mapsto\big(x_i(t_i),y_i(t_i)\big)$ over T_ε along the trivial sections σ, $\overline{\sigma}_i$. Similarly for $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial v}\in I^{es}_{\widetilde{\varphi}}=\bigoplus_{p\in\widetilde{p}}I^{es}_{\widetilde{\varphi}_p}$, where $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial u}=\big(\widetilde{a}_p\frac{\partial}{\partial u}\big)_{p\in\widetilde{p}}$ and $\big(\widetilde{a}_p\frac{\partial}{\partial u}\big)=(a_i)_{i\in\Lambda_p}\frac{\partial}{\partial u}$.

We apply (the proof of) Proposition 2.39 with $T'=\{0\}$, $T=T_\varepsilon$ and with ϕ given by $x_i+\varepsilon a_i$, $y_i+\varepsilon b_i$. If ϕ is equisingular, it is equimultiple. Then, after blowing up σ, the induced deformation $\widetilde{\phi}_p$ of $\widetilde{\varphi}_p$ over T_ε along the trivial section is given by $\widetilde{u}_i+\varepsilon\widetilde{a}_i$, $\widetilde{v}_i+\varepsilon\widetilde{b}_i$ (see (2.5.19), (2.5.20)).

Since any infinitely near point belonging to $(\widetilde{C},p)$ belongs also to $(C,\mathbf{0})$, we get: If ϕ is equisingular, then blowing up σ induces, by definition, an equisingular deformation $\widetilde{\phi}_p$ of $\widetilde{\varphi}_p$, for each $p\in\widetilde{p}$. Conversely, if, for each $p\in\widetilde{p}$, $\widetilde{\phi}_p$ is equisingular, and if ϕ is equimultiple, then ϕ is equisingular, too. Thus, $\boldsymbol{a}\frac{\partial}{\partial x}+\boldsymbol{b}\frac{\partial}{\partial y}\in I^{es}_{\varphi}$ iff ϕ is equimultiple and $\widetilde{\boldsymbol{a}}\frac{\partial}{\partial \boldsymbol{u}}+\widetilde{\boldsymbol{b}}\frac{\partial}{\partial \boldsymbol{v}}\in I^{es}_{\widetilde{\varphi}}$.

(2) Given a_i,b_i, we can argue as in (1) if the section $\widetilde{\sigma}$ in the proof of Proposition 2.39 is trivial. The result follows by applying (2.5.19), respectively (2.5.20).

(3) To see that I_φ^{es} is an $\mathcal{O}_{C,\mathbf{0}}$-module, let $\boldsymbol{g} = (g_i)_{i=1}^r \in \mathcal{O}_{C,\mathbf{0}} \subset \mathcal{O}_{(\overline{C},\overline{0})}$, and let $(\boldsymbol{a}, \boldsymbol{b})$ define an element of I_φ^{es}. We argue by induction on the number of blowing ups needed to resolve the singularity $(C, \mathbf{0})$. We start with a smooth germ. Remark 2.37.1 gives $I_\varphi^{es} = \mathfrak{m}_{\overline{C},\overline{0}} \frac{\partial}{\partial x} + \mathfrak{m}_{\overline{C},\overline{0}} \frac{\partial}{\partial y}$, which is an $\mathcal{O}_{C,\mathbf{0}}$-module. By induction hypothesis, we may assume that $I_{\widetilde{\varphi}}^{es}$ containing $\widetilde{\boldsymbol{a}} \frac{\partial}{\partial \boldsymbol{u}} + \widetilde{\boldsymbol{b}} \frac{\partial}{\partial \boldsymbol{v}}$ is an $\mathcal{O}_{C,\mathbf{0}}$-module. That is, $\boldsymbol{g} \cdot (\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}) \in I_{\widetilde{\varphi}}^{es}$, where $\widetilde{\boldsymbol{a}}, \widetilde{\boldsymbol{b}}$ are defined as in (2). We notice that $\big(\widetilde{g_i a_i}, \widetilde{g_i b_i}\big) - \big(g_i \widetilde{a}_i, g_i \widetilde{b}_i\big)$ equals $\big(0, g_i\big(\frac{b_i - a_i}{x_i}(0)\big) - \frac{g_i(a_i - b_i)}{x_i}(0)\big)$ if $p = (1 : \alpha)$, respectively $\big(g_i\big(\frac{a_i}{y_i}(0)\big) - \frac{g_i a_i}{y_i}(0), 0\big)$ if $p = (0 : 1)$. In any case, it has no constant term. It follows that

$$\left(\widetilde{\boldsymbol{g}\boldsymbol{a}} \frac{\partial}{\partial u} + \widetilde{\boldsymbol{g}\boldsymbol{b}} \frac{\partial}{\partial v}\right) - \boldsymbol{g} \left(\widetilde{\boldsymbol{a}} \frac{\partial}{\partial u} + \widetilde{\boldsymbol{b}} \frac{\partial}{\partial v}\right) \in \mathfrak{m}_{\widetilde{C},\widetilde{p}} \frac{\partial}{\partial u} \oplus \mathfrak{m}_{\widetilde{C},\widetilde{p}} \frac{\partial}{\partial v} \subset I_{\widetilde{\varphi}}^{es}$$

and, hence, $\widetilde{\boldsymbol{g}\boldsymbol{a}} \frac{\partial}{\partial u} + \widetilde{\boldsymbol{g}\boldsymbol{b}} \frac{\partial}{\partial v} \in I_{\widetilde{\varphi}}^{es}$. By (1), we conclude $\boldsymbol{g}\boldsymbol{a} \frac{\partial}{\partial x} + \boldsymbol{g}\boldsymbol{b} \frac{\partial}{\partial y} \in I_\varphi^{es}$ which proves the claim. □

Lemma 2.41. *Let $(T', \mathbf{0}) \subset (T, \mathbf{0})$ be a small extension of germs with ε a vector space generator of* $\ker(\mathcal{O}_{T,\mathbf{0}} \twoheadrightarrow \mathcal{O}_{T',\mathbf{0}})$. *Let*

$$(\phi', \overline{\sigma}', \sigma') \in \mathcal{D}ef^{es}_{(\overline{C},\overline{0}) \to (\mathbb{C}^2, \mathbf{0})}(T', \mathbf{0}),$$

with $\overline{\sigma}', \sigma'$ the trivial sections, and with ϕ' given by $X_i', Y_i' \in t_i \mathcal{O}_{T',\mathbf{0}}\{t_i\}$, $i = 1, \ldots, r$. Furthermore, let $(\boldsymbol{a}, \boldsymbol{b}) \in \mathfrak{m}_{\overline{C},\overline{0}} \oplus \mathfrak{m}_{\overline{C},\overline{0}}$, and let ϕ be the deformation over $(T, \mathbf{0})$ given by $X_i = X_i' + \varepsilon a_i$, $Y_i = Y_i' + \varepsilon b_i$, with trivial sections $\overline{\sigma}, \sigma$. Then

$$(\phi, \overline{\sigma}, \sigma) \in \mathcal{D}ef^{es}_{(\overline{C},\overline{0}) \to (\mathbb{C}^2, \mathbf{0})}(T, \mathbf{0}) \iff \boldsymbol{a} \frac{\partial}{\partial x} + \boldsymbol{b} \frac{\partial}{\partial y} \in I_\varphi^{es}.$$

Proof. Let $(\phi, \overline{\sigma}, \sigma)$ be equisingular, let $p \in \widetilde{M}$ be an infinitely near point belonging to $(C, \mathbf{0})$, and let $\widetilde{\phi}_p : (\overline{\mathscr{C}}, \overline{p}) \to (\widetilde{\mathscr{M}}, p)$, $\overline{\sigma}_p, \sigma_p$ be as in Definition 2.36. With respect to local coordinates of $(\widetilde{\mathscr{M}}, p)$, $\widetilde{\phi}_p$ is given by $U_i' + \varepsilon \widetilde{a}_i$, $V_i' + \varepsilon \widetilde{b}_i$, and its restriction to $(T', \mathbf{0})$, $\widetilde{\phi}_p'$, is given by U_i', V_i'.

Then U_i', V_i' is equimultiple and, hence, $\operatorname{ord} \widetilde{a}_i, \operatorname{ord} \widetilde{b}_i \geq \min\{\operatorname{ord} u_i, \operatorname{ord} v_i\}$ with u_i, v_i a parametrization of $(\widetilde{C}, p)$, that is, $u_i + \varepsilon \widetilde{a}_i$, $v_i + \varepsilon \widetilde{b}_i$ is equimultiple over T_ε. It follows that $x_i + \varepsilon a_i$, $y_i + \varepsilon b_i$ is equisingular over T_ε. Hence, $\boldsymbol{a} \frac{\partial}{\partial x} + \boldsymbol{b} \frac{\partial}{\partial y} \in I_\varphi^{es}$.

Conversely, let $\boldsymbol{a} \frac{\partial}{\partial x} + \boldsymbol{b} \frac{\partial}{\partial y} \in I_\varphi^{es}$. We argue again by induction on the number of blowing ups needed to resolve $(C, \mathbf{0})$, the case of a smooth germ $(C, \mathbf{0})$ being trivial. As X_i', Y_i' is equisingular over $(T', \mathbf{0})$, it is equimultiple, hence, X_i, Y_i is equimultiple, too. Blowing up the trivial section, we get that

$$U_i = U_i' + \varepsilon \widetilde{a}_i\,, \quad V_i = V_i' + \varepsilon \widetilde{b}_i\,,$$

with (a_i, b_i) and $\big(\widetilde{a}_i, \widetilde{b}_i\big)$ related as in Proposition 2.40, defines a deformation of $\big(\widetilde{C}, p\big)$ over $(T, \mathbf{0})$ by (2.5.19), respectively (2.5.20), in the proof of Proposition 2.39. By induction, this deformation is equisingular and, hence, as $\operatorname{ord} a_i, \operatorname{ord} b_i \geq \min\{\operatorname{ord} x_i, \operatorname{ord} y_i\}$, X_i, Y_i define an equisingular deformation of $(C, \mathbf{0})$. $\square$

Lemma 2.42. *Let* $(\phi, \overline{\sigma}, \sigma) \in \mathcal{D}ef^{sec}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(T, \mathbf{0})$*, and let* T_N *denote the fat point given by* $\mathcal{O}_{T,\mathbf{0}}/\mathfrak{m}^{N+1}_{T,\mathbf{0}}$*,* $N \geq 0$*. Then* $(\phi, \overline{\sigma}, \sigma)$ *is equisingular iff it is* formally equisingular*, that is, iff, for each* $N \geq 1$*, the restriction to* T_N*,* $(\phi_N, \overline{\sigma}_N, \sigma_N)$ *is equisingular over* T_N*.*

Proof. Since the necessity is obvious, let $(\phi_N, \overline{\sigma}_N, \sigma_N)$ be equisingular over T_N, for all $N \geq 0$. It follows that, for all $N \geq 0$, ϕ_N is equimultiple along σ_N, hence ϕ itself is equimultiple along σ. Therefore, we can blow up σ and obtain, for each point p in the first infinitely near neighbourhood of $\mathbf{0}$ belonging to $(C, \mathbf{0})$, a deformation $\widetilde{\phi}_p : \big(\overline{\mathscr{C}}, \overline{p}\big) \to \big(\widetilde{\mathscr{M}}, p\big)$ of the strict transform $\big(\widetilde{C}, p\big)$ of $(C, \mathbf{0})$ at p along the sections $\overline{\sigma}_p, \sigma_p$ (see the considerations before Definition 2.36).

The restriction to T_N, $\big(\widetilde{\phi}_{p,N}, \overline{\sigma}_{p,N}, \sigma_{p,N}\big)$, is equisingular, hence equimultiple for all $N \geq 0$. Hence, $\widetilde{\phi}_p$ is equimultiple along σ_p, and we can continue in the same manner. Since an arbitrary infinitely near point belonging to $(C, \mathbf{0})$ is obtained by a finite number of blowing ups, the result follows by induction on this number. $\square$

Proof of Theorem 2.38. (1) Since $X_i, Y_i \bmod \langle s_1, \dots, s_j^2, \dots, s_k\rangle$, $j = 1, \dots, k$, define an equisingular deformation over T_ε, and since we can apply Lemma 2.41, the necessity is obvious.

For the sufficiency, let $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in I^{es}_\varphi$. Since each extension of Artinian local rings factors through small extensions, it follows from Lemma 2.41 that $\phi \bmod \langle \boldsymbol{s}\rangle^{N+1}$ is equisingular over the fat point $T_N = \big(\{\mathbf{0}\}, \mathcal{O}_{T,\mathbf{0}}/\langle \boldsymbol{s}\rangle^{N+1}\big)$. Now, apply Lemma 2.42.

As (3) is an immediate consequence of (2), it remains to prove (2): Let ϕ be versal (respectively semiuniversal), and let $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y} \in I^{es}_\varphi$. Then the equisingular deformation $(x_i + \varepsilon a_i, y_i + \varepsilon b_i)_{i=1}^r$ can be induced (respectively uniquely induced) from ϕ. Hence, the class of $\boldsymbol{a}\frac{\partial}{\partial x} + \boldsymbol{b}\frac{\partial}{\partial y}$ in $T^{1,es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,0)}$ is a linear combination (respectively a unique linear combination) of $\big(\frac{\partial X_i}{\partial s_j}(t_i, \mathbf{0})\big)_{i=1}^r \frac{\partial}{\partial x} + \big(\frac{\partial Y_i}{\partial s_j}(t_i, \mathbf{0})\big)_{i=1}^r \frac{\partial}{\partial y}$, $j = 1, \dots, r$. This shows that the condition is indeed necessary.

For the other direction, we have only to show that $(\phi, \overline{\sigma}, \sigma)$ with $\sigma, \overline{\sigma}$ denoting the trivial sections, is formally versal by [Fle1, Satz 5.2] (see also Theorem 1.13).

Thus, it is sufficient to consider a small extension $(Z', \mathbf{0}) \subset (Z, \mathbf{0})$, with $\varepsilon\mathbb{C}$ being the kernel of $A = \mathcal{O}_{Z,\mathbf{0}} \twoheadrightarrow \mathcal{O}_{Z',\mathbf{0}} =: A'$.

Let $(\psi, \overline{\tau}, \tau) \in \mathcal{D}ef^{es}_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(Z,\mathbf{0})$ with trivial sections $\tau, \overline{\tau}$, such that the restriction $(\psi', \overline{\tau}', \tau') \in \mathcal{D}ef_{(\overline{C},\overline{0})\to(\mathbb{C}^2,\mathbf{0})}(Z',\mathbf{0})$ is induced from $(\psi, \overline{\tau}, \tau)$ by some morphism $\eta' : (Z',\mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$. We have to show that $(\psi, \overline{\tau}, \tau)$ is isomorphic to the pull-back of $(\phi, \overline{\sigma}, \sigma)$ by some morphism $\eta : (Z,\mathbf{0}) \to (\mathbb{C}^k,\mathbf{0})$ extending η'. By Remark 2.37.1 (2), we may assume that $(Z',\mathbf{0})$ is smooth, that is, we may assume that $A' = \mathbb{C}\{\boldsymbol{z}\}$, $\boldsymbol{z} = (z_1, \dots, z_n)$, and that

$$A = \mathbb{C}\{\boldsymbol{z}, \varepsilon\}/\langle z_1\varepsilon, \dots, z_n\varepsilon, \varepsilon^2\rangle .$$

The pull-back map $\eta'^*\phi : \big(\overline{C} \times Z', \mathbf{0}\big) \to (\mathbb{C}^2 \times Z', \mathbf{0})$ is then given by the power series $X_i\big(t_i, \eta'(\boldsymbol{z})\big)$, $Y_i\big(t_i, \eta'(\boldsymbol{z})\big)$.

Let ψ' be given by $U_i'\big(t_i, \boldsymbol{z}\big), V_i'\big(t_i, \boldsymbol{z}\big) \in A'\{t_i\}$, and let ψ be given by

$$U_i = U_i' + \varepsilon u_i\,, \quad V_i = V_i' + \varepsilon v_i \in A\{t_i\}\,, \quad u_i, v_i \in \mathbb{C}\{t_i\}\,,$$

with

$$\big(U_i(t_i), V_i(t_i)\big) \equiv \big(x_i(t_i), y_i(t_i)\big) \bmod \mathfrak{m}_A\,.$$

The morphism $\eta' : (Z',\mathbf{0}) \to (\mathbb{C}^k, \mathbf{0})$ is given by $\eta' = (\eta_1, \dots, \eta_k)$, $\eta_i \in \mathbb{C}\{\boldsymbol{z}\}$, and the extension $\eta : (Z,\mathbf{0}) \to (\mathbb{C}^k,\mathbf{0})$ is then given by

$$\eta = \eta' + \varepsilon\eta^0\,, \quad \eta^0 = (\eta_1^0, \dots, \eta_k^0) \in \mathbb{C}^k\,.$$

The assumption says that there is

- an A'-automorphism H' of $A'\{x,y\} = \mathbb{C}\{x,y,\boldsymbol{z}\}$, $x \mapsto H_1'$, $y \mapsto H_2'$, with $H_1', H_2' \in \langle x,y\rangle A'\{x,y\}$, and
- an A'-automorphism h' of $\bigoplus_{i=1}^r A'\{t_i\}$, $t_i \mapsto h_i' \in t_iA'\{t_i\} = t_i\mathbb{C}\{t_i, \boldsymbol{z}\}$,

with H' and h' being the identity modulo $\mathfrak{m}_{A'}$, such that the following holds for $i = 1, \dots, r$:

$$X_i(t_i, \eta') = H_1'\big(U_i'(h_i'), V_i'(h_i')\big)\,, \quad Y_i(t_i, \eta') = H_2'\big(U_i'(h_i'), V_i'(h_i')\big)\,. \qquad (2.5.25)$$

We have to extend η', H' and h' over $(Z,\mathbf{0})$ such that these equations extend, too. That is, we have to show the existence of $\eta^0 = (\eta_1^0, \dots, \eta_k^0) \in \mathbb{C}^k$, $H_1^0, H_2^0 \in \langle x,y\rangle\mathbb{C}\{x,y\}$, $h^0 = (h_1^0, \dots h_r^0) \in \bigoplus_{i=1}^r t_i\mathbb{C}\{t_i\}$, such that

$$X_i(t_i, \eta' + \varepsilon\eta^0) = (H_1' + \varepsilon H_1^0)\big(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0)\big)\,, \qquad (2.5.26)$$

$$Y_i(t_i, \eta' + \varepsilon\eta^0) = (H_2' + \varepsilon H_2^0)\big(U_i(h_i' + \varepsilon h_i^0), V_i(h_i' + \varepsilon h_i^0)\big)\,. \qquad (2.5.27)$$

Applying Taylor's formula, and using that $\varepsilon\mathfrak{m}_A = 0$, we obtain

$$\begin{aligned} X_i(t_i, \eta' + \varepsilon\eta^0) &= X_i(t_i, \eta') + \varepsilon\sum_{j=1}^k \frac{\partial X_i}{\partial s_j}(t_i, \eta') \cdot \eta_j^0 \\ &= X_i'(t_i, \eta') + \varepsilon\sum_{j=1}^k \frac{\partial X_i}{\partial s_j}(t_i, \mathbf{0}) \cdot \eta_j^0\,, \end{aligned} \qquad (2.5.28)$$

$$Y_i(t_i, \eta' + \varepsilon\eta^0) = Y_i'(t_i, \eta') + \varepsilon\sum_{j=1}^k \frac{\partial Y_i}{\partial s_j}(t_i, \mathbf{0}) \cdot \eta_j^0\,.$$

Moreover, with $\dot{}$ denoting the derivative with respect to t_i,

$$\begin{aligned} U_i(h'_i + \varepsilon h^0_i) &= U_i(h'_i) + \varepsilon \dot{U}_i(h'_i) \cdot h^0_i \;=\; U'_i(h'_i) + \varepsilon\big(\dot{x}_i h^0_i + u_i\big)\,, \\ V_i(h'_i + \varepsilon h^0_i) &= V'_i(h'_i) + \varepsilon\big(\dot{y}_i h^0_i + v_i\big)\,. \end{aligned}$$

Since H' is the identity mod $\mathfrak{m}_{A'}$, we have

$$\frac{\partial H'_1}{\partial x} = 1 \bmod \mathfrak{m}_{A'}\,, \qquad \frac{\partial H'_1}{\partial y} \in \mathfrak{m}_{A'} A'\{x,y\}\,.$$

In particular, $\varepsilon \cdot \frac{\partial H'_1}{\partial y} = 0$. Applying again Taylor's formula, and using that $h' = \mathrm{id} \bmod \mathfrak{m}_{A'}$, the right-hand side of (2.5.26) equals

$$\begin{aligned} &(H'_1 + \varepsilon H^0_1)\Big(U'_i(h'_i) + \varepsilon\big(\dot{x}_i h^0_i + u_i\big),\, V'_i(h'_i) + \varepsilon\big(\dot{y}_i h^0_i + v_i\big)\Big) \\ &\quad = H'_1\big(U'_i(h'_i), V'_i(h'_i)\big) + \varepsilon\,\big(H^0_1\big(U'_i(h'_i), V'_i(h'_i)\big) + 1 \cdot \big(\dot{x}_i h^0_i + u_i\big)\big) \\ &\quad = H'_1\big(U'_i(h'_i), V'_i(h'_i)\big) + \varepsilon\big(H^0_1(x_i, y_i) + \dot{x}_i h^0_i + u_i\big)\,, \end{aligned} \tag{2.5.29}$$

and similar for the right-hand side of (2.5.27).

Using (2.5.25), (2.5.28) and (2.5.29), we have to find $(\eta^0_1, \dots, \eta^0_k) \in \mathbb{C}^k$, $H^0_1, H^0_2 \in \langle x, y\rangle \mathbb{C}\{x,y\}$, and $h^0_i \in t_i \mathbb{C}\{t_i\}$, such that

$$\begin{aligned} \big(u_i(t_i), v_i(t_i)\big) = \sum_{j=1}^{k} \eta^0_j \cdot \left(\frac{\partial X_i}{\partial s_j}(t_i, \mathbf{0}), \frac{\partial Y_i}{\partial s_j}(t_i, \mathbf{0})\right) - h^0_i(t_i) \cdot \Big(\dot{x}_i(t_i), \dot{y}_i(t_i)\Big) \\ -\Big(H^0_1\big(x_i(t_i), y_i(t_i)\big),\, H^0_2\big(x_i(t_i), y_i(t_i)\big)\Big)\,. \end{aligned} \tag{2.5.30}$$

Since $(\psi, \overline{\tau}, \tau)$, with ψ given by $U'_i + \varepsilon u_i$, $V'_i + \varepsilon v_i$, is equisingular, Lemma 2.41 gives that $(u_i)_{i=1}^r \frac{\partial}{\partial x} + (v_i)_{i=1}^r \frac{\partial}{\partial y} \in I^{es}_\varphi$. But then the assumption implies that (2.5.30) can be solved (respectively solved with unique $\eta^0_1, \dots, \eta^0_k$). This proves that $(\phi, \overline{\sigma}, \sigma)$ is versal (respectively semiuniversal). □

The fact that I^{es}_φ is a module provides an easy proof of the *openness of versality* for equisingular deformations. Consider an *equisingular family of parametrizations* of reduced plane curve singularities over some complex space S. That is, we have morphisms of complex spaces

$$\overline{\mathscr{C}} \xrightarrow{\phi} \mathscr{M} \xrightarrow{\mathrm{pr}} S$$

with pr and $\mathrm{pr} \circ \phi$ being flat, ϕ being finite, together with a section $\sigma : S \to \mathscr{M}$ and a multisection $\overline{\sigma} = \big(\overline{\sigma}_i\big)_{i=1}^r : S \to \overline{\mathscr{C}}$, such that, for each $s \in S$, and $\mathscr{M}_s := pr^{-1}(s)$, $\overline{\mathscr{C}}_s := (pr \circ \phi)^{-1}(s)$, the following holds:

- $\big(\mathscr{M}_s, \sigma(s)\big) \cong (\mathbb{C}^2, \mathbf{0})$,
- $\phi\big(\overline{\sigma}_i(s)\big) = \sigma(s)$, $\big(\overline{\mathscr{C}}_s, \overline{\sigma}_i(s)\big) \cong (\mathbb{C}, 0)$, $i = 1, \dots, r$,

- the restriction $\phi_s : \big(\overline{\mathscr{C}}_s, \overline{\sigma}(s)\big) = \coprod_{i=1}^r \big(\overline{\mathscr{C}}_s, \overline{\sigma}_i(s)\big) \to \big(\mathscr{M}_s, \sigma(s)\big)$ is the parametrization of a reduced plane curve singularity $\big(\mathscr{C}_s, \sigma(s)\big)$ with r branches, and
- $\phi : \big(\overline{\mathscr{C}}, \overline{\sigma}(s)\big) \to \big(\mathscr{M}, \sigma(s)\big)$ is an equisingular deformation of the parametrization ϕ_s.

We say that a family $\overline{\mathscr{C}} \xrightarrow{\phi} \mathscr{M} \xrightarrow{\mathrm{pr}} S$ as above is *equisingular* (resp. *equisingular-versal*) along $(\sigma, \overline{\sigma})$ at s, if $\phi : \big(\overline{\mathscr{C}}, \overline{\sigma}(s)\big) \to \big(\mathscr{M}, \sigma(s)\big)$, together with the germs of $\overline{\sigma}$ and σ, is an equisingular (resp. a versal equisingular) deformation of ϕ_s.

More generally, let $\sigma = (\sigma^{(1)}, \dots, \sigma^{(\ell)})$ be a finite set of disjoint sections, $\sigma^{(i)} : S \to \mathscr{M}$, and $\overline{\sigma} = (\overline{\sigma}^{(1)}, \dots, \overline{\sigma}^{(\ell)})$ be disjoint multisections, $\overline{\sigma}^{(i)} : S \to \overline{\mathscr{C}}$, $\overline{\sigma}^{(i)} = (\overline{\sigma}_j^{(i)})_{j=1}^{r_i}$. If $\overline{\mathscr{C}} \to \mathscr{M} \to S$ is equisingular (resp. equisingular-versal) along $(\sigma^{(i)}, \overline{\sigma}^{(i)})$ for $i = 1, \dots, \ell$ at each $s \in S$, then we say that $\overline{\mathscr{C}} \to \mathscr{M} \to S$ is an *equisingular* (resp. *equisingular-versal*) *family of parametrizations* of reduced plane curve singularities (along $(\sigma, \overline{\sigma})$).

Theorem 2.43. *Let $\overline{\mathscr{C}} \xrightarrow{\phi} \mathscr{M} \xrightarrow{\mathrm{pr}} S$ be an equisingular family of parametrizations of reduced plane curve singularities over some complex space S. Then the set of points $s \in S$ such that the family is equisingular-versal at s is analytically open in S.*

Proof. Since the set in question for several sections is the intersection of the corresponding sets for each section, we may assume that σ is just one section. Let $I_{\overline{\sigma}} \subset \mathcal{O}_{\overline{\mathscr{C}}}$ denote the ideal sheaf of the section $\overline{\sigma}$. Then we define a subsheaf $\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ of $I_{\overline{\sigma}} \cdot \phi^* \operatorname{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathscr{M}}, \mathcal{O}_{\mathscr{M}}) = I_{\overline{\sigma}} \cdot \phi^* \Theta_{\mathscr{M}/S}$, as follows: For $s \in S$ and local coordinates x, y of $\mathscr{M}_s$ at $\sigma(s)$ and t_i of $\overline{\mathscr{C}}_s$ at $\overline{\sigma}_i(s)$, ϕ is given near $\overline{\sigma}(s)$ by $X_i, Y_i \in \mathcal{O}_{S,s}\{t_i\}$. Moreover, a local section of $I_{\overline{\sigma}} \cdot \phi^* \operatorname{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathscr{M}}, \mathcal{O}_{\mathscr{M}})$ is given by $\big(a_i\big)_{i=1}^r \frac{\partial}{\partial x} + \big(b_i\big)_{i=1}^r \frac{\partial}{\partial y}$, $a_i, b_i \in \mathcal{O}_{S,s}\{t_i\}$. The local sections of the sheaf $\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ are, by definition, those local sections of $I_{\overline{\sigma}} \cdot \phi^* \operatorname{Der}_{\mathcal{O}_S}(\mathcal{O}_{\mathscr{M}}, \mathcal{O}_{\mathscr{M}})$, for which $X_i + a_i, Y_i + b_i$ defines an equisingular deformation of ϕ_s over the germ (S, s). Since equimultiplicity in the infinitely near points of $\big(\mathscr{M}, \sigma(s)\big)$ belonging to $\big(\mathscr{C}_s, \sigma(s)\big)$ is preserved near s, $X_i + a_i, Y_i + b_i$ also define an equisingular deformation of $\phi_{s'}$ over (S, s') for s' in some open neighbourhood of s. It follows that $\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ is, indeed, a sheaf, and that the stalk at s generates the stalks at s' close to s. Hence, $\phi_* \mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ is a coherent $\mathcal{O}_{\mathscr{M}}$-module by Proposition 2.40 and A.7.

Consider the quotient sheaf

$$\mathcal{T}^{1,es}_{\overline{\mathscr{C}} \to \mathscr{M}} = (\mathrm{pr} \circ \phi)_* \Big(\mathcal{I}^{es}_{\overline{\mathscr{C}} \to \mathscr{M}} \Big/ \big(I_{\overline{\sigma}} \Theta_{\overline{\mathscr{C}}/S} + \phi^{-1}(I_{\overline{\sigma}} \Theta_{\mathscr{M}/S}) \big) \Big) ,$$

which is a coherent $\mathcal{O}_S$-sheaf, since the support of the sheaf to which $(\mathrm{pr} \circ \phi)_*$ is applied is finite over S. In local coordinates x, y and t_i, the image of $(\mathrm{pr} \circ \phi)_* \Theta_{\overline{\mathscr{C}}/S} = (\mathrm{pr} \circ \phi)_* \bigoplus_{i=1}^r \mathcal{O}_{\overline{\mathscr{C}}} \frac{\partial}{\partial t_i}$ in $(\mathrm{pr} \circ \phi)_* \phi^* \Theta_{\mathscr{M}/S}$ is generated by $\big(\dot{X}_i \frac{\partial}{\partial x} + \dot{Y}_i \frac{\partial}{\partial y}\big)_{i=1}^r$. Hence, the stalk at s of $\mathcal{T}^{1,es}_{\overline{\mathscr{C}} \to \mathscr{M}}$ equals $T^{1,es}_{(\overline{\mathscr{C}}_s, \overline{\sigma}(s)) \to (\mathscr{M}, \sigma(s))}$.

Moreover, we have the *"Kodaira-Spencer map"*

$$\Theta_S \longrightarrow \mathcal{T}^{1,es}_{\mathscr{C}\to\mathscr{M}},$$

which maps $\delta \in \Theta_{S,s}$ to $\left(\delta(X_i)\frac{\partial}{\partial x} + \delta(X_i)\frac{\partial}{\partial y}\right)_{i=1}^r$ in local coordinates. Theorem 2.38 (2) implies that the cokernel of this map has support at points $s \in S$, where ϕ is not equisingular-versal. But since the cokernel is coherent, this support is analytically closed, which proves the theorem. $\square$

To compute a semiuniversal equisingular deformation of $\varphi : (\overline{C}, \overline{0}) \to (\mathbb{C}^2, \mathbf{0})$, we only need to compute a basis of $T^{1,es}_{\varphi}$ by Theorem 2.38. Moreover, if all branches of $(C, \mathbf{0})$ have different tangents, Remark 2.11.1 gives

$$T^{1,es}_{\varphi} = \bigoplus_{i=1}^r T^{1,es}_{\varphi_i},$$

where φ_i is the parametrization of the i-th branch of $(C, \mathbf{0})$. In general, $T^{1,es}_{\varphi}$ can be computed, as a subspace of M^{em}_{φ}, by following the lines of the proof of Proposition 2.39.

Example 2.43.1. (1) Consider the parametrization $\varphi : t \mapsto (t^2, t^7)$ of an A_6-singularity. By Example 2.27.1, M^{em}_{φ} has the basis $\left\{t^3\frac{\partial}{\partial y}, t^5\frac{\partial}{\partial y}\right\}$. Blowing up the trivial section of $X(t, \boldsymbol{s}) = t^2$, $Y(t, \boldsymbol{s}) = t^7 + s_1 t^3 + s_2 t^5$, we get

$$U(t, \boldsymbol{s}) = t^2, \quad V(t, \boldsymbol{s}) = \frac{Y(t, \boldsymbol{s})}{X(t, \boldsymbol{s})} = t^5 + s_1 t + s_2 t^3,$$

which is equimultiple along the trivial section iff $s_1 = 0$. Blowing up once more, we get the necessary condition $s_2 = 0$ for equisingularity. Hence, $T^{1,es}_{\varphi} = 0$, as it should be, since A_6 is a simple singularity.

(2) For $\varphi : t \mapsto (t^3, t^7)$, a basis for the $\mathbb{C}$-vector space M^{em}_{φ} is given by $\left\{t^4\frac{\partial}{\partial y}, t^5\frac{\partial}{\partial y}, t^8\frac{\partial}{\partial y}\right\}$, respectively by $\left\{t^4\frac{\partial}{\partial x}, t^4\frac{\partial}{\partial y}, t^5\frac{\partial}{\partial y}\right\}$. Blowing up the trivial section, only $t^8\frac{\partial}{\partial y}$, respectively $t^4\frac{\partial}{\partial x}$, survives for an equimultiple deformation. It also survives in further blowing ups. Hence, $X(t, s) = t^3$, $Y(t, s) = t^5 + st^8$ (respectively $X(t, s) = t^3 + st^4$, $Y(t, s) = t^5$) is a semiuniversal equisingular deformation of φ.

(3) Reconsider the parametrization given in Example 2.27.1 (2):

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 \end{pmatrix}, \quad \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}.$$

A semiuniversal equimultiple deformation is given by

$$\begin{pmatrix} X_1(t, \boldsymbol{s}) \\ X_2(t, \boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 + s_1 t^3 + s_2 t^4 + s_3 t^6 + s_4 t^9 \end{pmatrix}, \quad \begin{pmatrix} Y_1(t, \boldsymbol{s}) \\ Y_2(t, \boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}.$$

Blowing up the trivial section shows that only the parameters s_3 and s_4 survive. These survive also in subsequent blowing up steps. Hence,

$$\begin{pmatrix} X_1(t,\boldsymbol{s}) \\ X_2(t,\boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} 0 \\ t^5 + s_3t^6 + s_4t^9 \end{pmatrix}, \quad \begin{pmatrix} Y_1(t,\boldsymbol{s}) \\ Y_2(t,\boldsymbol{s}) \end{pmatrix} = \begin{pmatrix} t \\ t^3 \end{pmatrix}.$$

is a semiuniversal equisingular deformation.

2.6 Equinormalizable Deformations

We show in this section that each deformation of the normalization of a reduced plane curve singularity $(\overline{C},\overline{0}) \to (C,\mathbf{0})$ induces a δ-constant deformation of $(C,\mathbf{0})$. Conversely, if the base space of a δ-constant deformation of $(C,\mathbf{0})$ is normal, then the deformation is equinormalizable, that is, it lifts to a deformation of the normalization $(\overline{C},\overline{0}) \to (C,\mathbf{0})$ and, as such, it induces a simultaneous normalization of each fibre. Hence, over a normal base space, a deformation of $(C,\mathbf{0})$ admits a simultaneous normalization of each of its fibres iff the total δ-invariant of the fibres is constant.

The study of equinormalizable deformations has been initiated by Teissier in the 1970's. The main results of this section are Theorem 2.54 and Theorem 2.56 due to Teissier, resp. to Teissier and Raynaud [Tei]. A generalization to families with (projective) fibres of arbitrary positive dimension was recently given by Chiang-Hsieh and Lipman [ChL]. They also give a complete treatment for families of curve singularities, clarifying some points in the proof given in [Tei]. We follow closely the presentation in [ChL] which is basically algebraic. By working directly in the complex analytic setting, we can avoid technical complications that appear when working with general schemes.

We consider first arbitrary morphisms $f : X \to S$ of complex spaces. Such a morphism f is called *reduced* (resp. *normal*) if it is flat and if all non-empty fibres are reduced (resp. normal).

Definition 2.44. Let $f : X \to S$ be a reduced morphism of complex spaces.

A *simultaneous normalization* of f is a finite morphism $\nu : Z \to X$ of complex spaces such that $\overline{f} = f \circ \nu : Z \to S$ is normal and that, for each $s \in f(X)$, the induced map on the fibres $\nu_s : Z_s = \overline{f}^{-1}(s) \to X_s = f^{-1}(s)$ is the normalization of X_s.

We say that an arbitrary morphism $f : X \to S$ of complex spaces *admits a simultaneous normalization* if it is reduced and if there exists a simultaneous normalization of f.

The morphism f is called *equinormalizable* if X is reduced and if the normalization $\nu : \overline{X} \to X$ of X is a simultaneous normalization of f. We call f *equinormalizable at* $x \in X$, if $\overline{f} = f \circ \nu : \overline{X} \to S$ is flat at each point of the fibre $\nu^{-1}(x)$ and if, for $s = f(x)$, the induced map $\nu_s : \overline{f}^{-1}(s) =: \overline{X}_s \to X_s$ is the normalization. A morphism $(X,x) \to (S,s)$ of complex space germs is *equinormalizable* if it has a representative which is equinormalizable at x. We shall show below that, under some mild assumptions, equinormalizability is an open property.

Remark 2.44.1. If $\nu : Z \to S$ is a simultaneous normalization of f, then, for each $s \in S$ and $x \in X_s = f^{-1}(s)$, the diagram

$$\begin{array}{ccc} (Z_s, \overline{z}) & \hookrightarrow & (Z, \overline{z}) \\ {\scriptstyle \nu_s}\downarrow & & \downarrow{\scriptstyle \nu} \\ (X_s, x) & \hookrightarrow & (X, x) \\ \downarrow & & \downarrow{\scriptstyle f} \\ \{s\} & \hookrightarrow & (S, s) \end{array}$$

is a deformation of the normalization map $\nu_s : (Z_s, \overline{z}) \to (X_s, x)$, that is, an object of $\mathcal{D}ef_{(Z_s,\overline{z})\to(X_s,x)}(S, s)$ in the sense of Definition 2.20. Here, $(Z, \overline{z})$ denotes the multigerm of Z at $\overline{z} = \nu^{-1}(x)$.

First, we show that a simultaneous normalization does not modify X at normal points of the fibres.

Lemma 2.45. *Let $\nu : Z \to X$ be a simultaneous normalization of the reduced morphism $f : X \to S$, and let*

$$\mathrm{NNor}(f) := \{x \in X \mid x \textit{ is a non-normal point of the fibre } f^{-1}(f(x))\}\,.$$

Then $N := \mathrm{NNor}(f)$ is analytic and nowhere dense in X, and the restriction $\nu : Z \setminus \nu^{-1}(N) \to X \setminus N$ is biholomorphic.

Proof. Since f is flat, $\mathrm{NNor}(f)$ is the set of non-normal points of f, which is analytic by Theorem I.1.100. Since the fibres of f are reduced, hence generically smooth, every component of X contains points outside of N. It follows that N is nowhere dense in X.

For $x \in X$, $s = f(x)$, the restriction $\nu_s : Z_s \to X_s$ is the normalization of the fibre X_s by assumption. For $x \notin N$, the germ (X_s, x) is normal and, therefore, $\nu^{-1}(x)$ consists of exactly one point $z \in Z$ and $\nu_s : (Z_s, z) \to (X_s, x)$ is an isomorphism of germs. Since $f \circ \nu$ is flat, $\nu : (Z, z) \to (X, x)$ is an isomorphism, too, by Lemma I.1.86. This shows that $\nu : Z \setminus \nu^{-1}(N) \to X \setminus N$ is bijective and locally an isomorphism, hence biholomorphic. □

Proposition 2.46. *Let $f : X \to S$ be a morphism of complex spaces.*

(1) If f is reduced, then X is reduced iff S is reduced at each point of the image $f(X)$.
(2) If f is normal, then X is normal iff S is normal at each point of the image $f(X)$.

Proof. Since each reduced (resp. normal) morphism is flat, the statement follows immediately from Theorem B.8.20. □

In particular, if f admits a simultaneous normalization $\nu : Z \to X$, then Z is normal iff S is normal at each point of $f(X)$ (apply Proposition 2.46 to $f \circ \nu$).

Corollary 2.47. *Let $f : X \to S$ be a reduced morphism of complex spaces. If f is equinormalizable at $x \in X$, then S is normal at $f(x)$.*

The corollary implies that a normal morphism need not be equinormalizable: If $f : X \to S$ is a normal morphism, then id_X is a simultaneous normalization of f, independent of S. However, if S is not normal at some point $f(x)$, then X is not normal at x. If $\nu : \overline{X} \to X$ is the normalization then $f \circ \nu$ is not normal by Proposition 2.46 and, hence, f is not equinormalizable at x. For example, by Theorem I.1.100, each small representative $f : X \to S$ of a deformation of a normal singularity X_0 (e.g., an isolated hypersurface singularity of dimension at least 2) is a normal morphism and, hence, equinormalizable if and only if S is normal.

The following example shows that for a non-normal base space S strange things can happen:

Example 2.47.1. Let $F(x,y,u,v) = x^3 + y^2 + ux + v$ be the semiuniversal unfolding of the cusp $C = \{x^3 + y^2 = 0\}$, let $D = 4u^3 + 27v^2$ be the discriminant equation of the projection $\pi : V(F) \to \mathbb{C}^2$, $(x,y,u,v) \mapsto (u,v)$, and consider

$$f : X = V(F,D) \to \Delta = V(D) \subset \mathbb{C}^2.$$

That is, f is the restriction of the semiuniversal deformation π of $(C,\mathbf{0})$ over the discriminant Δ which is, in this case, the δ-constant stratum (see page 355).

Then f is reduced, but f is not equinormalizable because otherwise Δ has to be normal by Proposition 2.46. To see what happens, note that the normalization map is $\nu : \overline{X} = \mathbb{C}^2 \to X \subset \mathbb{C}^4$, given by

$$\big(T_1, T_2\big) \stackrel{\nu}{\longmapsto} (x,y,u,v) = \big(-\tfrac{1}{81}T_2^2 - \tfrac{4}{3}T_1, \tfrac{1}{729}T_2^2 + \tfrac{2}{27}T_1T_2, -\tfrac{4}{27}T_1^2, \tfrac{16}{729}T_1^3\big)\,.$$

The map $\overline{f} = f \circ \nu$, given by the last two components of ν, has $V(T_1^2)$ as special fibre, which is not reduced. The morphism $\overline{f}$ is also not flat, since $\mathcal{O}_{\overline{X},\mathbf{0}} \otimes_{\mathcal{O}_{\Delta,\mathbf{0}}} \mathfrak{m}_{\Delta,\mathbf{0}} \to \mathcal{O}_{\overline{X},\mathbf{0}} = \mathbb{C}\{T_1, T_2\}$ is not injective (the non-zero element $\frac{27}{4}T_1 \otimes u + \frac{729}{16} \otimes v$ is mapped to zero).

This family does not even admit a simultaneous normalization $\nu : Z \to X$ with Z non-normal. Otherwise, the corresponding deformation of the normalization of the cusp (see Remark 2.44.1) could be induced by a morphism $(\Delta, \mathbf{0}) \to (\mathbb{C}, 0)$, where $(\mathbb{C}, 0)$ is the base space of the semiuniversal deformation of the normalization of Δ (see Proposition 2.27). By the semiuniversality of the deformations, the tangent map of the composition $(\Delta, \mathbf{0}) \to (\mathbb{C}, 0) \to (\Delta, \mathbf{0})$ must be the identity, contradicting the fact that the tangent map of the normalization $(\mathbb{C}, 0) \to (\Delta, \mathbf{0})$ is the zero map.

Exercise 2.6.1. Recompute Example 2.47.1 by using SINGULAR. First compute the singular locus, then the discriminant by eliminating x and y, and finally the normalization of X using the library `normal.lib`.

We start by studying the equinormalizability condition locally.

Lemma 2.48. *Let $f:(X,x)\to(S,s)$ be a flat morphism of complex space germs with reduced fibre (X_s,x) and with reduced base (S,s). Further, let $(\overline{X},\overline{x})\xrightarrow{\nu}(X,x)$ be the normalization of (X,x), let $(\overline{X}_s,\overline{x})$ be the fibre of $\overline{f}=f\circ\nu$, let $\nu_s:(\overline{X}_s,\overline{x})\to(X_s,x)$ be the restriction of ν, and let $n:\big(\widetilde{X}_s,\widetilde{x}\big)\to(X_s,x)$ be the normalization of (X_s,x). Set*

$$\begin{aligned}\mathcal{O}&:=\mathcal{O}_{X,x}\,, \quad \overline{\mathcal{O}}:=\nu_*\mathcal{O}_{\overline{X},\overline{x}}\,,\\ \mathcal{O}_s&:=\mathcal{O}_{X_s,x}\,,\ \overline{\mathcal{O}}_s:=\nu_{s*}\mathcal{O}_{\overline{X}_s,\overline{x}}\,,\ \widetilde{\mathcal{O}}_s:=n_*\mathcal{O}_{\widetilde{X}_s,\widetilde{x}}\,.\end{aligned}$$

Then there is an $h\in\mathcal{O}$ such that the following holds:

(1) h is a non-zerodivisor of $\mathcal{O}$, $\overline{\mathcal{O}}$, $\mathcal{O}_s$, $\widetilde{\mathcal{O}}_s$ and $h\widetilde{\mathcal{O}}_s\subset\mathcal{O}_s$. Moreover, $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat.

(2) If $(\overline{X}_s,\overline{x})$ is reduced, then n factors as $n:(\widetilde{X}_s,\widetilde{x})\xrightarrow{\overline{n}}(\overline{X}_s,\overline{x})\xrightarrow{\nu_s}(X_s,x)$ where $\overline{n}$ is the normalization of the multigerm $(\overline{X}_s,\overline{x})$. Hence, there are inclusions $\mathcal{O}_s\hookrightarrow\overline{\mathcal{O}}_s\hookrightarrow\widetilde{\mathcal{O}}_s$ and h is a non-zerodivisor of $\overline{\mathcal{O}}_s$.

(3) If (S,s) is normal, then $h\overline{\mathcal{O}}\subset\mathcal{O}$ and the $\mathcal{O}_{S,s}$-module $\mathcal{O}/h\overline{\mathcal{O}}$ is torsion free.

Proof. Let $(N,x)\subset(X,x)$ denote the non-normal locus of f, which is an analytic subgerm by Theorem I.1.100. Since f is flat, the intersection $(N\cap X_s,x)$ is the non-normal locus of (X_s,x), which is nowhere dense as the fibre (X_s,x) is reduced. Again by Theorem I.1.100, the nearby fibres of f are also reduced, hence (N,x) is nowhere dense in (X,x).

Therefore, there exists some $h\in\mathcal{O}$ which vanishes along (N,x) but not along any irreducible component of (X,x) or of (X_s,x). Thus, h is a non-zerodivisor of $\mathcal{O}$ and of $\mathcal{O}_s$. Since h is invertible in the total ring of fractions of $\mathcal{O}$ and of $\mathcal{O}_s$, it is a non-zerodivisor of $\overline{\mathcal{O}}$ and of $\widetilde{\mathcal{O}}_s$. Further, since h vanishes on the support $(N\cap X_s,x)$ of the conductor $I_s^{cd}=\mathrm{Ann}_{\mathcal{O}_s}(\widetilde{\mathcal{O}}_s/\mathcal{O}_s)$, some power of h is contained in I_s^{cd} by the Hilbert-Rückert Nullstellensatz. Replacing h by some power of h, we get the first part of (1). Applying Proposition B.5.3 to $\mathcal{O}_{S,s}\hookrightarrow\mathcal{O}$ and $h:\mathcal{O}\to\mathcal{O}$, it follows that $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat.

A similar argument shows that (X_s,x) and $(\overline{X}_s,\overline{x})$ have the same normalization if $(\overline{X},\overline{x})$ is reduced, which shows (2).

Finally, we prove (3). Since (S,s) is normal, the non-normal locus of (X,x) is contained in the non-normal locus of f. Thus, h vanishes along the non-normal locus of (X,x). As above, it follows that some power of h is contained in $\mathrm{Ann}_{\mathcal{O}}(\overline{\mathcal{O}}/\mathcal{O})$. Replacing h by an appropriate power, h satisfies $h\overline{\mathcal{O}}\subset\mathcal{O}$.

To show that $\mathcal{O}/h\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-torsion free, we have to show that each non-zero element of $\mathcal{O}_{S,s}$ is a non-zerodivisor of $\mathcal{O}/h\overline{\mathcal{O}}$, that is, $\mathcal{O}_{S,s}\cap P=\{0\}$

for each associated prime P of the $\mathcal{O}$-ideal $h\overline{\mathcal{O}}$ (see Appendix B.1). Since $\overline{\mathcal{O}}$ is the integral closure of $\mathcal{O}$ in $\mathrm{Quot}(\mathcal{O})$, the ideal $h\overline{\mathcal{O}}$ is the integral closure of the ideal $h\mathcal{O}$ in $\mathrm{Quot}(\mathcal{O})$. Hence, $h\mathcal{O} \subset h\overline{\mathcal{O}} \subset \sqrt{h\mathcal{O}}$ and $h\mathcal{O}$ and $h\overline{\mathcal{O}}$ have the same associated prime ideals. Since $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat by (1), $\mathcal{O}_{S,s} \cap P = \{0\}$ as required. □

Proposition 2.49. *Let $f:(X,x) \to (S,s)$ and $\nu:(\overline{X},\overline{x}) \to (X,x)$ be as in Lemma 2.48. Denote by $(\overline{X}_s,\overline{x})$ the fibre of $\overline{f} = f\circ\nu$, and by $\nu_s:(\overline{X}_s,\overline{x}) \to (X_s,x)$ the restriction of ν to this fibre.*

(1) If $(\overline{X}_s,\overline{x})$ is reduced, then $\overline{f}:(\overline{X},\overline{x}) \to (S,s)$ is flat.
(2) Let (S,s) be normal. If $(\overline{X}_s,\overline{x})$ is normal and if (X,x) is equidimensional, then ν_s is the normalization of (X_s,x) and f is equinormalizable.

Note that, under the assumptions of Proposition 2.49, (X,x) is equidimensional iff there exists a representative $f:X\to S$ such that every fibre of f is equidimensional. This follows from Proposition B.8.13 since f is flat and (S,s) is normal (hence, equidimensional).

Proof. (1) We use the notations of Lemma 2.48. The element h is invertible in the total ring of fractions of $\mathcal{O}$, and we have inclusions $\overline{\mathcal{O}} \hookrightarrow h^{-1}\mathcal{O}$ and $\widetilde{\mathcal{O}}_s \hookrightarrow h^{-1}\mathcal{O}_s$. Tensoring $\overline{\mathcal{O}} \hookrightarrow h^{-1}\mathcal{O}$ with $\mathbb{C}$, we get a long exact Tor-sequence

$$\begin{aligned}\ldots \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\overline{\mathcal{O}},\mathbb{C}) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O},\mathbb{C}) \longrightarrow \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}},\mathbb{C})\\ \longrightarrow \overline{\mathcal{O}}\otimes_{\mathcal{O}_{S,s}}\mathbb{C} \longrightarrow h^{-1}\mathcal{O}\otimes_{\mathcal{O}_{S,s}}\mathbb{C}\,.\end{aligned}$$

Since $h^{-1}\mathcal{O}\cong\mathcal{O}$ is flat over $\mathcal{O}_{S,s}$, we have $\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O},\mathbb{C})=0$ for each $i\geq 1$ (Proposition B.3.2). Further, by assumption, $\overline{\mathcal{O}}\otimes_{\mathcal{O}_{S,s}}\mathbb{C} = \overline{\mathcal{O}}_s$ is reduced and, hence, injects into $\widetilde{\mathcal{O}}_s$ by Lemma 2.48. Thus, the last arrow displayed in the above sequence is injective. Altogether, this shows that $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}},\mathbb{C})=0$, and the local criterion of flatness (Theorem B.5.1) implies that $h^{-1}\mathcal{O}/\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-flat and that $\mathrm{Tor}_i^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}},\mathbb{C})=0$ for $i\geq 1$. From the Tor-sequence, we read that $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\overline{\mathcal{O}},\mathbb{C})=0$, whence $\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-flat by the local criterion of flatness.

For (2), we choose sufficiently small representatives of the involved morphisms and spaces. Let N_s be the (analytic) set of non-normal points of $X_s = f^{-1}(s)$. If $x'\in X_s\setminus N_s$, then X is normal at x' by Proposition 2.46 (since (S,s) is normal). Hence, the fibre $\nu^{-1}(x')$ consists of a unique point $z'\in\overline{X}$, and $\nu:(\overline{X},z')\to(X,x')$ is an isomorphism. It follows that $\nu_s:(\overline{X}_s,z')\to(X_s,x')$ is an isomorphism, too. Thus, $\nu_s:\overline{X}_s\setminus\nu_s^{-1}(N_s)\to X_s\setminus N_s$ is bijective and locally an isomorphism, hence biholomorphic. To show that $\nu_s:\overline{X}_s\to X_s$ is the normalization, it suffices to show that $\nu_s^{-1}(N_s)$ is nowhere dense in $\overline{X}_s$.

Choose $z\in\overline{x}=\nu^{-1}(x)$. Then $\nu:(\overline{X},z)\to(X,x)$ normalizes some component of (X,x) and, since (X,x) is equidimensional, $\dim(\overline{X},z)=\dim(X,x)$. Since the germ $(\overline{X},z)$ is normal, it is irreducible. Applying Theorem B.8.13 to $\overline{f}$ and to f, we get

$$\begin{aligned}\dim(\overline{X}_s, z) &\geq \dim(\overline{X}, z) - \dim(S, s)\\ &= \dim(X, x) - \dim(S, s) = \dim(X_s, x)\,.\end{aligned}$$

Since ν_s is a finite morphism, $\nu_s(\overline{X}_s, z)$ is an analytic subgerm of (X_s, x) of dimension $\dim(\overline{X}_s, z)$. It follows that $\dim \nu(\overline{X}_s, z)$ must be equal to $\dim(X_s, z)$ and, therefore, $\nu(\overline{X}_s, z)$ is an irreducible component of (X_s, x). As N_s is nowhere dense in X_s, it follows that $\nu^{-1}(N_s)$ is nowhere dense in $\overline{X}_s$. □

The following corollary shows that equinormalizability of $f : X \to S$ at a point $x \in X$ is an open property, provided that X is equidimensional at x and S is normal at $f(x)$.

Corollary 2.50. *Let $f : (X, x) \to (S, s)$ be a flat morphism of complex space germs, where (X, x) is equidimensional, (S, s) is normal, and the fibre $(X_s, x) = (f^{-1}(s), x)$ is reduced. Let $(\overline{X}, \overline{x}) \xrightarrow{\nu} (X, x)$ be the normalization of (X, x), and assume that the fibre $(\overline{X}_s, \overline{x})$ of $\overline{f} = f \circ \nu$ is normal. Then there exists a representative $f : X \to S$ which is equinormalizable.*

Proof. We may choose sufficiently small representatives $f : X \to S$ such that f is reduced (Theorem I.1.100), S is normal and X is reduced (Proposition I.1.93) and equidimensional. Let $\nu : \overline{X} \to X$ be the normalization. Then we may assume that the special fibre $\overline{X}_s$ of $\overline{f} = f \circ \nu$ is normal at each point $z \in \overline{x} = \nu^{-1}(x)$. By Proposition 2.49 (1), $\overline{f}$ is flat, hence normal at each point $z \in \nu^{-1}(x)$. By Theorem I.1.100, the set of normal points of $\overline{f}$ is open, hence we may assume (after shrinking X and $\overline{X}$ if necessary) that $\overline{f}$ is normal. Since X is equidimensional at each point, we can apply Proposition 2.49 (2) to every non-empty fibre of f which shows that ν normalizes every fibre of f. Hence, $\nu : \overline{X} \to X$ is a simultaneous normalization of $f : X \to S$. □

We turn back to global morphisms and show, in particular, that a reduced morphism $f : X \to S$ with X equidimensional and S normal is equinormalizable iff all non-empty fibres of $\overline{f} = f \circ \nu$ are normal.

Theorem 2.51. *Let $f : X \to S$ be a reduced morphism of complex spaces, where S is normal.*

(1) If f admits a simultaneous normalization $\nu : Z \to X$, then ν is necessarily the normalization of X.
(2) Let $\nu : \overline{X} \to X$ be the normalization of X and $\overline{f} = f \circ \nu$. Then the following holds:
   (i) ν is a simultaneous normalization of f iff for each $s \in f(X)$ the map $\nu_s : \overline{f}^{-1}(s) \to f^{-1}(s)$ is the normalization of the fibre $f^{-1}(s)$.
   (ii) If X is locally equidimensional, then ν is a simultaneous normalization of f iff for each $s \in f(X)$, the fibre $\overline{f}^{-1}(s)$ is normal.

Proof. (1) Since S is normal, hence reduced, X is also reduced (Proposition 2.46 (1)) and the normalization of X exists. Moreover, since $\overline{f} = f \circ \nu$ is normal by assumption and, since S is normal, Z is normal, too (Proposition

2.46 (2)). To show that $\nu : Z \to X$ is the normalization, it suffices (since ν is finite and surjective by assumption) that $\nu : Z \setminus \nu^{-1}(N) \to X \setminus N$ is biholomorphic, where N denotes the set of non-normal points of f. But this was shown in Lemma 2.45.

(2) If ν is a simultaneous normalization, all non-empty fibres of $\overline{f}$ are normal and ν induces a normalization of all non-empty fibres of f by definition. The converse is a direct consequence of Proposition 2.49 (1), resp. Corollary 2.50. $\square$

Next, we consider families of curves and prove the δ-constant criterion for equinormalizability. In order to shorten notation, we introduce the following notion:

Definition 2.52. A morphism $f : \mathscr{C} \to S$ of complex spaces is a *family of reduced curves* if f is reduced, if the restriction $f : \mathrm{Sing}(f) \to S$ is finite and if all non-empty fibres $\mathscr{C}_s = f^{-1}(s)$ are purely one-dimensional.

Recall that for a reduced curve singularity (C, x) the δ-invariant is defined as $\delta(C, x) = \dim_{\mathbb{C}}\big(n_*\mathcal{O}_{\widetilde{C},\widetilde{x}}/\mathcal{O}_{C,x}\big)$, where $n : (\widetilde{C}, \widetilde{x}) \to (C, x)$ is the normalization of (C, x). For a family of reduced curves $f : \mathscr{C} \to S$ and $s \in S$, we define

$$\delta(\mathscr{C}_s) := \sum_{x \in \mathscr{C}_s} \delta(\mathscr{C}_s, x)\,.$$

This is a finite number, since the fibre $\mathscr{C}_s$ has only finitely many singularities and since $\delta(\mathscr{C}_s, x)$ is zero if (and only if) $(\mathscr{C}_s, x)$ is smooth. The family $f : \mathscr{C} \to S$ is called *(locally) δ-constant* if the function $s \mapsto \delta(\mathscr{C}_s)$ is (locally) constant on S.

If $f : (\mathscr{C}, x) \to (S, s)$ is a flat map germ with reduced and one-dimensional fibre $(\mathscr{C}_s, x)$, then there exists a representative $f : \mathscr{C} \to S$ which is a family of reduced curves such that $\mathscr{C}_s \setminus \{x\}$ is smooth. If there exists such a representative which is δ-constant, we call the germ $f : (\mathscr{C}, x) \to (S, s)$ *δ-constant* or a *δ-constant deformation* of $(\mathscr{C}_s, x)$.

Lemma 2.53. *Let $f : \mathscr{C} \to S$ be a family of reduced curves with reduced base S. If f is equinormalizable, then f is locally δ-constant.*

Proof. Let $\nu : \overline{\mathscr{C}} \to \mathscr{C}$ be the normalization. By assumption, the composition $\overline{f} = f \circ \nu$ is normal, hence flat, and the direct image sheaf $\nu_*\mathcal{O}_{\overline{\mathscr{C}}}$ is also flat over $\mathcal{O}_S$. Moreover, since $\nu_s : \overline{\mathscr{C}}_s = \overline{f}^{-1}(s) \to \mathscr{C}_s$ is the normalization, the induced map $\mathcal{O}_{\mathscr{C}_s} \to \nu_*\mathcal{O}_{\overline{\mathscr{C}}_s}$ is injective. Thus, Proposition B.5.3 gives that the quotient $\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}$ is a flat $\mathcal{O}_S$-module. Since this quotient is concentrated on $\mathrm{Sing}(f)$, which is finite over S, the direct image $f_*(\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}})$ is locally free on S. Since ν normalizes the fibres, we get that

$$\dim_{\mathbb{C}}\big(f_*\big(\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}\big) \otimes_{\mathcal{O}_{S,s}} \mathbb{C}\big) = \dim_{\mathbb{C}}\big(\nu_{s*}\mathcal{O}_{\overline{\mathscr{C}}_s}/\mathcal{O}_{\mathscr{C}_s}\big) = \delta(\mathscr{C}_s)$$

is locally constant on S. $\square$

We want to show the converse implication under the assumption that S is normal. We start with the case that S is a smooth curve:

Theorem 2.54 (Teissier). *Let $f : (\mathscr{C}, \mathbf{0}) \to (\mathbb{C}, 0)$ be a flat morphism such that the fibre $(\mathscr{C}_0, \mathbf{0})$ is a reduced curve singularity. If $\nu : (\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{C}, \mathbf{0})$ is the normalization and $\overline{f} = f \circ \nu$, then the fibre $(\overline{\mathscr{C}}_0, \overline{0}) = (\overline{f}^{-1}(0), \overline{0})$ is reduced. Moreover:*

(1) For each sufficiently small representative $f : \mathscr{C} \to S \subset \mathbb{C}$, we have

$$\delta(\mathscr{C}_0, \mathbf{0}) = \delta(\mathscr{C}_s) + \delta(\overline{\mathscr{C}}_0, \overline{0}) \quad \textit{for each} \quad s \in S \setminus \{0\}\,.$$

In particular, δ is upper semicontinuous on S.

(2) $f : (\mathscr{C}, \mathbf{0}) \to (\mathbb{C}, 0)$ is equinormalizable iff it is δ-constant.

Proof. Note that $(\overline{\mathscr{C}}, \overline{0})$ has only isolated singularities since the germ $(\mathscr{C}, \mathbf{0})$ is purely two-dimensional. Moreover, by Remark B.8.10.1 (2), $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$ is Cohen-Macaulay, thus $\operatorname{depth}(\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}) = 2$ and $\overline{f}$ is a non-zerodivisor of $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$. The latter shows that $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$ is flat over $\mathcal{O}_{\mathbb{C},0}$ (Theorem B.8.11). Since $\mathcal{O}_{\mathscr{C},\mathbf{0}}$ is also $\mathcal{O}_{\mathbb{C},0}$-flat, and since the fibre $(\mathscr{C}_0, \mathbf{0})$ is reduced, the quotient $\nu_* \mathcal{O}_{\overline{\mathscr{C}},\overline{0}} / \mathcal{O}_{\mathscr{C},\mathbf{0}}$ is $\mathcal{O}_{\mathbb{C},0}$-flat (Proposition B.5.3), hence free. This shows that there exists a sufficiently small representative $\overline{f} : \overline{\mathscr{C}} \xrightarrow{\nu} \mathscr{C} \xrightarrow{f} S \subset \mathbb{C}$ such that

$$\overline{\delta}(\mathscr{C}_s) := \dim_{\mathbb{C}} \left(f_* \left(\nu_* \mathcal{O}_{\overline{\mathscr{C}}} / \mathcal{O}_{\mathscr{C}} \right) \otimes_{\mathcal{O}_{S,s}} \mathbb{C} \right) = \dim_{\mathbb{C}} \left(\nu_{s*} \mathcal{O}_{\overline{\mathscr{C}}_s} / \mathcal{O}_{\mathscr{C}_s} \right)$$

is constant on S.

Since $\overline{f}$ is a non-zerodivisor of $\mathcal{O}_{\overline{\mathscr{C}},\overline{0}}$, $\operatorname{depth} \mathcal{O}_{\overline{\mathscr{C}}_0,\overline{0}} = 1$ and the fibre $\overline{\mathscr{C}}_0$ is reduced at $\overline{0}$. After shrinking the chosen representatives, we may assume that each fibre $\overline{\mathscr{C}}_s$, $s \in S$, is reduced at each of its points. Hence, $\mathscr{C}_s$ and $\overline{\mathscr{C}}_s$ have the same normalization $\widetilde{\mathscr{C}}_s$.

By Proposition 2.55 below, we may assume that $0 \in S$ is the only critical value of $\overline{f}$. Therefore, $\overline{\mathscr{C}}_s$ is smooth, that is, $\widetilde{\mathscr{C}}_s = \overline{\mathscr{C}}_s$ for $s \neq 0$, which implies that

$$\delta(\mathscr{C}_s) = \overline{\delta}(\mathscr{C}_s) \quad \text{for} \quad s \neq 0\,.$$

For $s = 0$, we have inclusions $\mathcal{O}_{\mathscr{C}_0} \hookrightarrow \mathcal{O}_{\overline{\mathscr{C}}_0} \hookrightarrow \mathcal{O}_{\widetilde{\mathscr{C}}_0}$, hence

$$\delta(\mathscr{C}_0) = \overline{\delta}(\mathscr{C}_0) + \delta(\overline{\mathscr{C}}_0)\,,$$

which proves (1), because $\overline{\delta}(\mathscr{C}_0) = \overline{\delta}(\mathscr{C}_s)$ for $s \neq 0$.

For (2), note that if f is δ-constant, then $\delta(\mathscr{C}_0, \mathbf{0}) = \delta(\mathscr{C}_s)$ for each s and, by (1), $\delta(\overline{\mathscr{C}}_0, \overline{0}) = 0$, which shows that $(\overline{\mathscr{C}}_0, \overline{0})$ is smooth. Hence, f is equinormalizable by Proposition 2.49 (2). The converse implication was shown in Lemma 2.53. □

Proposition 2.55. *Let $f : \mathscr{C} \to S$ be a family of reduced curves with S reduced. Then there is an analytically open dense subset $U \subset S$ such that the restriction $f : f^{-1}(U) \to U$ is equinormalizable.*

Proof. Since S is reduced, $S \setminus \mathrm{Sing}(S)$ is open and dense in S and, replacing S by $S \setminus \mathrm{Sing}(S)$, we may assume that S is smooth.

Let $\nu : \overline{\mathscr{C}} \to \mathscr{C}$ be the normalization of $\mathscr{C}$ and $\overline{f} := f \circ \nu$. For a point $x \in \mathscr{C} \setminus \mathrm{Sing}(f)$, we have $(\mathscr{C}, x) \cong (\mathscr{C}_s, x) \times (S, s)$ with f being the projection to the second factor under this isomorphism (Theorem I.1.115). Since (S, s) is smooth and $\overline{f}$ (in particular, $\overline{\mathscr{C}}$) is smooth at $\overline{x} = \nu^{-1}(x)$, it follows that $\overline{x}$ consists of one point and that $\nu : (\overline{\mathscr{C}}, \overline{x}) \xrightarrow{\cong} (\mathscr{C}, x)$ is an isomorphism. Thus, $\nu(\mathrm{Sing}(\overline{f})) \subset \mathrm{Sing}(f)$, the restriction $\overline{f} : \mathrm{Sing}(\overline{f}) \to S$ is finite as composition of the finite maps ν and $f|_{\mathrm{Sing}(f)}$, and the set of critical values $\Sigma := \overline{f}(\mathrm{Sing}(\overline{f})) \subset S$ is a closed analytic subset of S.

Since the fibres $\overline{f}^{-1}(s)$, $s \in U := S \setminus \Sigma$, are smooth, the restriction $\nu : \overline{f}^{-1}(U) \to f^{-1}(U)$ is a simultaneous normalization of $f^{-1}(U)$ (Theorem 2.51 (2)(i)). We have to show that U is dense in S.

Since $\overline{\mathscr{C}}$ is normal, the singular locus $\mathrm{Sing}(\overline{\mathscr{C}})$ has codimension at least 2 in $\overline{\mathscr{C}}$. Since f is flat and its fibres have dimension 1, the image $\overline{f}(\mathrm{Sing}(\overline{\mathscr{C}})) = f(\nu(\mathrm{Sing}(\overline{\mathscr{C}}))) \subset \Sigma$ has codimension at least 1 in S, too (Theorem I.B.8.13). Hence, $\overline{f} : \overline{f}^{-1}(U) \to U$ is a morphism of complex manifolds and $\mathrm{Sing}(\overline{f}|_{\overline{f}^{-1}(U)})$ is nowhere dense in U (Sard's Theorem I.1.103). Therefore, Σ is nowhere dense in S and its complement U is open and dense in S. $\square$

We turn now to the general theorem due to Teissier and Raynaud [Tei] (see the proof given by Chiang-Hsieh and Lipman [ChL]):

Theorem 2.56 (Teissier, Raynaud). *Let $f : \mathscr{C} \to S$ be a family of reduced curves with normal base S. Then f is equinormalizable iff f is locally δ-constant.*

Proof. By Lemma 2.53, it suffices to show that f equinormalizable implies that f is locally δ-constant.

Step 1. Let $\nu : \overline{\mathscr{C}} \to \mathscr{C}$ be the normalization and $\overline{f} = f \circ \nu$. For each $s \in S$, let $\mathscr{C}_s = f^{-1}(s)$ and $\overline{\mathscr{C}}_s = \overline{f}^{-1}(s)$. By Theorem 2.51, we have to show that, for every fixed $s \in S$, the fibre $\overline{\mathscr{C}}_s$ is normal. Hence, the problem is local on S and we may (and will) replace $f : \mathscr{C} \to S$ by the restriction over a sufficiently small (connected) neighbourhood of s in S. Let $n : \widetilde{\mathscr{C}}_s \to \mathscr{C}_s$ be the normalization of $\mathscr{C}_s$ and denote by $\nu_s : \overline{\mathscr{C}}_s \to \mathscr{C}_s$ the restriction of ν. By the universal property of the normalization, the map $\widetilde{\mathscr{C}}_s \xrightarrow{n} \mathscr{C}_s \hookrightarrow \mathscr{C}$ factors through ν and, hence, n factors through ν_s,

$$n : \widetilde{\mathscr{C}}_s \to \overline{\mathscr{C}}_s \xrightarrow{\nu_s} \mathscr{C}_s \,.$$

For $x \in \mathscr{C}_s$, let $\overline{x} := \nu_s^{-1}(x)$ and $\widetilde{x} := n^{-1}(x)$. We have, on the ring level, morphisms of (semilocal) algebras

$$\mathcal{O}_s := \mathcal{O}_{\mathscr{C}_s, x} \to \overline{\mathcal{O}}_s := \mathcal{O}_{\overline{\mathscr{C}}_s, \overline{x}} \to \widetilde{\mathcal{O}}_s := \mathcal{O}_{\widetilde{\mathscr{C}}_s, \widetilde{x}}\,,$$

where the composition $\mathcal{O}_s \to \widetilde{\mathcal{O}}_s$ is injective.

We have to show that, for $s \in S$ and $x \in \mathscr{C}_s$, the map $\overline{\mathcal{O}}_s \to \widetilde{\mathcal{O}}_s$ is an isomorphism.

Step 2. We show that $\overline{\mathcal{O}}_s \cong \widetilde{\mathcal{O}}_s$ holds for s outside a one-codimensional analytic subset of S.

Since normalization is a local operation, we have $\overline{f}^{-1}(U) = \overline{f^{-1}(U)}$ for each open subset $U \subset S$. Hence, the claim follows from Proposition 2.55. That is, there is a closed analytic subset $\Sigma \subset S$ of codimension at least 1 containing the set of critical values $\overline{f}(\mathrm{Sing}(\overline{f}))$. Then

$$\overline{f} : \overline{f}^{-1}(U) \xrightarrow{\nu} f^{-1}(U) \xrightarrow{f} U := S \setminus \Sigma\,.$$

is smooth and $\nu : \overline{f}^{-1}(U) \to f^{-1}(U)$ is a simultaneous normalization, hence $\overline{\mathcal{O}}_s \cong \widetilde{\mathcal{O}}_s$ for $s \in U$.

Step 3. Let $s \in S$ be arbitrary, $x \in \mathscr{C}_s$, $\overline{x} = \nu^{-1}(x)$, and set

$$\mathcal{O} := \mathcal{O}_{\mathscr{C},x}, \quad \overline{\mathcal{O}} = \nu_* \mathcal{O}_{\overline{\mathscr{C}},\overline{x}}\,.$$

We choose $h \in \mathcal{O}$ as in Lemma 2.48. Since h is a non-zerodivisor of $\mathcal{O}_s$, and since $\mathcal{O}_s$ has dimension 1, the quotient $\mathcal{O}_s/h\mathcal{O}_s$ is Artinian. Therefore, $\mathcal{O}/h\mathcal{O}$ is a quasifinite, hence finite, $\mathcal{O}_{S,s}$-module. By Lemma 2.48 (1), $\mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-flat, hence free of some rank d. Since h is invertible in the total ring of fractions of $\mathcal{O}$, $h^{-1}\mathcal{O}/\mathcal{O} \cong \mathcal{O}/h\mathcal{O}$ is $\mathcal{O}_{S,s}$-free of rank d.

The question whether $\overline{\mathcal{O}}_s \to \widetilde{\mathcal{O}}_s$ is an isomorphism is local in x and s. Thus, we fix x and $s = f(x)$ and we can assume that $\mathscr{C}$ and S are sufficiently small neighbourhoods of x and s such that h is a global section of $\mathcal{O}_{\mathscr{C}}$ and such that

$$\mathcal{E} := f_*(h^{-1}\mathcal{O}_{\mathscr{C}}/\mathcal{O}_{\mathscr{C}})$$

is a locally free $\mathcal{O}_S$-sheaf of rank d. Moreover, $f : \mathscr{C} \to S$ is δ-constant with $\delta := \delta(\mathscr{C}_s, x)$. Since the quotient $\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}}$ is concentrated on $\mathrm{Sing}(f)$ (by Step 2), which is finite over S, we get that

$$\mathcal{L} := f_*(\nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}})$$

is a coherent $\mathcal{O}_S$-module. Since $h\overline{\mathcal{O}} \subset \mathcal{O}$, we have $\nu_*\mathcal{O}_{\overline{\mathscr{C}}} \subset h^{-1}\mathcal{O}_{\mathscr{C}}$, which induces an exact sequence

$$0 \to \nu_*\mathcal{O}_{\overline{\mathscr{C}}}/\mathcal{O}_{\mathscr{C}} \to h^{-1}\mathcal{O}_{\mathscr{C}}/\mathcal{O}_{\mathscr{C}} \to h^{-1}\mathcal{O}_{\mathscr{C}}/\nu_*\mathcal{O}_{\overline{\mathscr{C}}} \to 0$$

of coherent $\mathcal{O}_{\mathscr{C}}$-modules whose support is finite over S. Hence, applying f_*, we obtain an exact sequence of coherent $\mathcal{O}_S$-modules

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0 \qquad (2.6.31)$$

with $\mathcal{E}$ being locally free of rank d.

Let U be as in Step 2. Then, for $s' \in U \subset S$, we have $\nu_{s*}\mathcal{O}_{\overline{\mathscr{C}}_{s'}} \cong n_*\mathcal{O}_{\widetilde{\mathscr{C}}_{s'}}$, and

$$\mathcal{L} \otimes_{\mathcal{O}_{S,s'}} \mathbb{C} \cong n_*\mathcal{O}_{\widetilde{\mathscr{C}}_{s'}}/\mathcal{O}_{\mathscr{C}_{s'}} = \bigoplus_{y \in \mathrm{Sing}(\mathscr{C}_{s'})} \left(n_*\mathcal{O}_{\widetilde{\mathscr{C}}_{s'}}\right)_y/\mathcal{O}_{\mathscr{C}_{s'},y}$$

has complex dimension $\delta(\mathscr{C}_{s'})$, which coincides with δ since f is δ-constant. Therefore, $\mathcal{L}|_U$ is locally free of rank δ.

By Lemma 2.48, $\widetilde{\mathcal{O}}_{s'} \subset h^{-1}\mathcal{O}_{s'}$ and, hence, $\overline{\mathcal{O}}_{s'}/\mathcal{O}_{s'} \to h^{-1}\mathcal{O}_{s'}/\mathcal{O}_{s'}$ is injective. It follows that the sequence (2.6.31) stays exact if we tensor it with $\mathbb{C}$ over $\mathcal{O}_{S,s'}$, $s' \in U$. As a consequence, the restriction $\mathcal{E}/\mathcal{L}|_U$ is locally free of rank $d-\delta$.

Step 4. Assume for the moment that the quotient $\mathcal{E}/\mathcal{L}$ is everywhere locally free on S. Then $\mathrm{Tor}_1^{\mathcal{O}_{S,s}}(\mathcal{E}/\mathcal{L}, \mathbb{C}) = \mathrm{Tor}_1^{\mathcal{O}_{S,s}}(h^{-1}\mathcal{O}/\overline{\mathcal{O}}) = 0$ and, applying $\otimes_{\mathcal{O}_{S,s}}\mathbb{C}$ to the exact sequence $0 \to \overline{\mathcal{O}} \to h^{-1}\mathcal{O} \to h^{-1}\mathcal{O}/\overline{\mathcal{O}} \to 0$, we get that $\overline{\mathcal{O}}_s \to h^{-1}\mathcal{O}_s$ is injective. Hence, $\overline{\mathcal{O}}_s$ is reduced and we have inclusions $\mathcal{O}_s \hookrightarrow \overline{\mathcal{O}}_s \hookrightarrow \widetilde{\mathcal{O}}_s$.

Since $\mathcal{E}/\mathcal{L}$ is locally free of rank $d-\delta$, $\mathcal{L}$ is locally free of rank δ and, hence, $\overline{\mathcal{O}}_s/\mathcal{O}_s \cong \mathcal{L} \otimes_{\mathcal{O}_{S,s}} \mathbb{C}$ has complex dimension δ. This proves that $\overline{\mathcal{O}}_s = \widetilde{\mathcal{O}}_s$ which, by Step 1, implies that f is equinormalizable.

Hence, it remains to show that $\mathcal{E}/\mathcal{L}$ is locally free on S.

Step 5. Assume that there exists a coherent subsheaf $\widetilde{\mathcal{L}}$ of $\mathcal{E}$ with $\widetilde{\mathcal{L}}|_U = \mathcal{L}|_U$ for some open dense subset $U \subset S$, such that the quotient $\mathcal{E}/\widetilde{\mathcal{L}}$ is locally free on S. We show that $\widetilde{\mathcal{L}} \cong \mathcal{L}$.

By Lemma 2.48 (3), we know that $h^{-1}\mathcal{O}/\overline{\mathcal{O}} \cong \mathcal{O}/h\overline{\mathcal{O}}$ is $\mathcal{O}_{S,s}$-torsion free. Hence, the quotient $\mathcal{E}/\mathcal{L}$ is torsion free for S sufficiently small. Consider the subsheaf $\mathcal{L} + \widetilde{\mathcal{L}}$ of $\mathcal{E}$, which coincides with $\mathcal{L}$ on U. Thus, the quotient $(\mathcal{L} + \widetilde{\mathcal{L}})/\mathcal{L}$ is a torsion subsheaf of the torsion free sheaf $\mathcal{E}/\mathcal{L}$. It follows that $(\mathcal{L} + \widetilde{\mathcal{L}})/\mathcal{L}$ is the zero sheaf, that is, $\widetilde{\mathcal{L}} \subset \mathcal{L}$. Similarly, $\mathcal{L} \subset \widetilde{\mathcal{L}}$.

Thus, it remains to show that the sheaf $\mathcal{L}|_U$ has an extension to a coherent subsheaf $\widetilde{\mathcal{L}}$ of $\mathcal{E}$ on S such that the quotient $\mathcal{E}/\widetilde{\mathcal{L}}$ is a locally free $\mathcal{O}_S$-sheaf.

Step 6. To show the existence of $\widetilde{\mathcal{L}}$, we use the quot scheme of $\mathcal{E}$. Let $\mathscr{G} = \mathrm{Grass}_{d-\delta}(\mathcal{E})$ be the Grassmannian of locally free $\mathcal{O}_S$-module quotients of $\mathcal{E}$ of rank $d-\delta$ (see [GrD, 9.7]). That is, $\mathscr{G}$ is a complex space, projective over[15] S, such that, for each complex spaces T over S, there is a bijection

$$\mathrm{Mor}_S(T, \mathscr{G}) \longleftrightarrow \left\{ \begin{array}{c} \mathcal{O}_T\text{-subsheaves } \mathcal{F} \text{ of } \mathcal{E}_T \text{ such that } \mathcal{E}_T/\mathcal{F} \text{ is} \\ \text{a locally free } \mathcal{O}_T\text{-module of rank } d-\delta \end{array} \right\}.$$

which is functorial in T. Here, for an $\mathcal{O}_S$-module $\mathscr{M}$, we denote by $\mathscr{M}_T$ the pull-back to T.

[15] We say that X is *projective over* S if the morphism $X \to S$ is *projective*, that is, if it factors through a closed immersion $X \hookrightarrow \mathbb{P}^N \times S$ for some N, followed by the projection $\mathbb{P}^N \times S \to S$.

By Step 3, the quotient $\mathcal{E}_U/\mathcal{L}_U$ is a locally free $\mathcal{O}_U$-module of rank $d-\delta$. Hence, under the above bijection, $\mathcal{L}_U$ corresponds to an S-morphism $\psi : U \to \mathscr{G}$. Any extension $\widetilde{\psi} : S \to \mathscr{G}$ of ψ corresponds to an $\mathcal{O}_S$-submodule $\widetilde{\mathcal{L}} \subset \mathcal{E}$ such that $\widetilde{\mathcal{L}}_U = \mathcal{L}_U$ and $\mathcal{E}/\widetilde{\mathcal{L}}$ is a locally free $\mathcal{O}_S$-module of rank $d-\delta$.

To see that ψ has such an extension $\widetilde{\psi} : S \to \mathscr{G}$, consider the graph of ψ, $\varGamma_\psi \subset U \times \mathscr{G}$, and let $\overline{\varGamma_\psi}$ be the analytic closure[16] of $\varGamma_\psi$ in $S \times \mathscr{G}$. We have to show that in the commutative diagram

$$\begin{tikzcd}
S & & \overline{\varGamma_\psi} \subset S \times \mathscr{G} \arrow[ll, "p"'] \arrow[dd, "q"] \\
U \arrow[u, hook] \arrow[drr, "\psi"'] & \varGamma_\psi \arrow[l, "\cong", "p_0"'] \arrow[ur, hook] \arrow[dr, "q_0"] & \\
& & \mathscr{G}
\end{tikzcd}$$

the map p is an isomorphism. Here, p, p_0, resp. q, q_0 are induced by the projections to the first, resp. second, factor.

Step 7. Since S is normal and the restriction of p to a dense open subset of $\overline{\varGamma_\psi}$ is an isomorphism onto a dense open subset of S, we only have to show that p is a homeomorphism (Theorem 1.102). Since $\mathscr{G} \to S$ is projective and since $\overline{\varGamma_\psi}$ is closed in $S \times \mathscr{G}$, the projection p is projective, hence closed. It follows that p is surjective and that p^{-1} is continuous if it exists (that is, if p is injective). Thus, it remains to show that, for each $s \in S \setminus U$, the fibre $p^{-1}(s)$ consists of only one point.

Let $z = (s, L) \in p^{-1}(s) \subset \overline{\varGamma_\psi}$ be any point. Then $z \in \overline{\varGamma_\psi} \setminus \varGamma_\psi$, where, by our assumptions, $\overline{\varGamma_\psi} \setminus \varGamma_\psi$ is of codimension at least 1 in $\overline{\varGamma_\psi}$ and $\varGamma_\psi$ is smooth. Thus, we can choose an irreducible germ of a curve C in $\overline{\varGamma_\psi}$ such that $C \cap (\overline{\varGamma_\psi} \setminus \varGamma_\psi) = \{z\}$ and $C \setminus \{z\}$ is smooth (we may, for instance, intersect $(\overline{\varGamma_\psi}, z)$ after a local embedding in some $(\mathbb{C}^N, \mathbf{0})$ with a general linear subspace of dimension $\dim(\overline{\varGamma_\psi}, z) - 1$ and taking an irreducible component if necessary). Then the image $p(C)$ is an irreducible curve in S such that $p(C) \cap (S \setminus U) = \{s\}$ and $p(C) \setminus \{s\}$ is smooth. Consider the normalization of $p(C)$, $\varphi : D \to p(C) \subset S$, $\varphi(0) = s$, where $D \subset \mathbb{C}$ is a small disc with centre 0. The map $\psi \circ (\varphi|_{D\setminus\{0\}}) : D \setminus \{0\} \to \mathscr{G}$ corresponds to a subsheaf $\mathcal{L}_{D\setminus\{0\}} = \varphi^*(\mathcal{L}_U)$ of $\mathcal{E}_{D\setminus\{0\}} = \varphi^*(\mathcal{E}_U)$ such that $\mathcal{E}_{D\setminus\{0\}}/\mathcal{L}_{D\setminus\{0\}}$ is locally free of rank $d-\delta$.

By Theorem 2.54, the submodule $\mathcal{L}_{D\setminus\{0\}} \subset \mathcal{E}_{D\setminus\{0\}}$ extends over D to a submodule $\mathcal{L}' \subset \mathcal{E}_D := \varphi^*\mathcal{E}$ such that $\mathcal{E}_D/\mathcal{L}'$ is locally free of rank $d-\delta$ and $\mathcal{L}' \otimes_{\mathcal{O}_{D,0}} \mathbb{C} = \widetilde{\mathcal{O}}_s/\mathcal{O}_s \subset \mathcal{E}_D \otimes_{\mathcal{O}_{D,0}} \mathbb{C} = h^{-1}\mathcal{O}_s/\mathcal{O}_s$. The $\mathcal{O}_D$-module $\mathcal{L}'$ corresponds to an extension $\chi : D \to \mathscr{G}$ of $\psi \circ (\varphi|_{D\setminus\{0\}})$ and $\chi(0)$ corresponds to the vector subspace $\mathcal{O}_s$ of $\widetilde{\mathcal{O}}_s$.

[16] The *analytic closure* of a set M in a complex space X is the intersection of all closed analytic subsets of X containing M.

The graph $\Gamma_D \subset D \times \mathscr{G}$ is mapped under $\varphi \times \mathrm{id}$ onto $C \times \mathscr{G} \subset \overline{\Gamma_\psi} \times \mathscr{G}$ such that $(0, \mathcal{O}_s)$ is mapped to (s, L). Hence $(s, \mathcal{O}_s)$ is the unique point of the fibre $p^{-1}(s)$. □

2.7 δ-Constant and μ-Constant Stratum

In the previous sections, we considered equisingular, respectively equinormalizable deformations. Here, we study arbitrary deformations of a reduced plane curve singularity $(C,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$ and we analyse the maximal strata in the base space such that the restriction to these strata is equisingular, resp. equinormalizable (possibly after base change). Recall that $\delta(C,\mathbf{0}) = \dim_{\mathbb{C}} n_*\mathcal{O}_{(\overline{C},\overline{0})}/\mathcal{O}_{C,\mathbf{0}}$, where $n : (\overline{C},\overline{0}) \to (C,\mathbf{0})$ is the normalization, and that $\mu(C,\mathbf{0}) = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}/\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\rangle$, where $f = 0$ is a local equation of $(C,\mathbf{0})$.

Let $F : \mathscr{C} \to S$ be a family of reduced curves (see Definition 2.52). If F is equinormalizable, then F is locally δ-constant by Lemma 2.53. We show now that, for each given k, the set of points $s \in S$ such that $\delta(\mathscr{C}_s) = k$ is a locally closed analytic subset of S. Here, $\mathscr{C}_s = F^{-1}(s)$ and $\delta(\mathscr{C}_s) = \sum_{x\in\mathscr{C}_s} \delta(\mathscr{C}_s, x)$.

Let us introduce the notation

$$\Delta := F\big(\mathrm{Sing}(F)\big) \subset S\,,$$

the set of *critical values* of F, also called the *discriminant* of F. Since $F : \mathrm{Sing}(F) \to S$ is finite, the discriminant is a closed analytic subset of S (by the finite mapping Theorem I.1.68). We endow Δ with the Fitting structure of Definition I.1.45.

For $k \geq 0$, we define

$$\begin{aligned} \Delta^\delta_F(k) &:= \Delta^\delta(k) := \{s \in S \mid \delta(\mathscr{C}_s) \geq k\}\,,\\ \Delta^\mu_F(k) &:= \Delta^\mu(k) := \{s \in S \mid \mu(\mathscr{C}_s) \geq k\}\,, \end{aligned}$$

where $\mu(\mathscr{C}_s) = \sum_{x\in\mathscr{C}_s} \mu(\mathscr{C}_s, x) < \infty$ (since $\mu(\mathscr{C}_s, x) = 0$ for x a smooth point of $\mathscr{C}_s$). We show below that $\Delta^\delta(k)$ and $\Delta^\mu(k)$ are closed analytic subsets of S (Proposition 2.57) which we endow with its reduced structure. In particular, $\Delta^\delta(0) = \Delta^\mu(0) = S_{red}$ and $\Delta^\delta(1) = \Delta^\mu(1) = \Delta_{red}$.

If $T \to S$ is any morphism, we use the notation

$$F_T : \mathscr{C}_T \to T$$

to denote the pull-back of $F : \mathscr{C} \to S$ to T.

Proposition 2.57. *Let $F : \mathscr{C} \to S$ be a family of reduced curves and let k be a non-negative integer. Then $\Delta^\delta(k)$ and $\Delta^\mu(k)$ are closed analytic subsets of S.*

Proof. Since $\Delta^\delta(k)$ and $\Delta^\mu(k)$ are defined set-theoretically and since the induced map $F_{S_{red}} : \mathscr{C}_{S_{red}} \to S_{red}$ is a family of reduced curves, too, we may assume that S is reduced.

We start with $\Delta^\delta(k)$ for some fixed k. Since the question is local in S, we may shrink S if necessary. If $\Delta^\delta(k) = S$, we are done. Otherwise, there exists an irreducible component S' of S such that $\delta(\mathscr{C}_{s_0}) < k$ for at least one $s_0 \in S'$. By Proposition 2.55 and Lemma 2.53, there exists an analytically open dense subset $U' \subset S'$ such that $\delta(\mathscr{C}_s)$ is constant, say k', for $s \in U'$ and some integer $k' \leq k$ (U' is connected since S' is irreducible).

We claim that $k' \leq \delta(\mathscr{C}_{s_0}) < k$. Indeed, if $k' \neq \delta(\mathscr{C}_{s_0})$, choose a curve germ $(D, s_0) \subset (S, s_0)$ which meets $S' \setminus U'$ only in s_0 and apply Teissier's Theorem 2.54 to the pull-back of $F_{(D,s_0)}$ to the normalization of (D, s_0) (see Step 7 in the proof of Theorem 2.56) to obtain that $k' \leq \delta(\mathscr{C}_{s_0})$. Hence, $\Delta^\delta(k) \cap S' \subset S' \setminus U'$ which is closed in S.

We see that if $\Delta^\delta(k) \neq S$ then there exists a closed analytic subset $S_1 \subsetneq S$ such that $\Delta^\delta(k) \subset S_1$. Applying the same argument to F_{S_1}, we get that either $\Delta^\delta(k) = S_1$ or there exists a closed analytic subset $S_2 \subsetneq S_1$ such that $\Delta^\delta(k) \subsetneq S_2$, etc.. In this way, we obtain a sequence $S \supsetneq S_1 \supsetneq S_2 \supsetneq \ldots$ of closed analytic subsets containing $\Delta^\delta(k)$. This sequence cannot be infinite, since the intersection of all the S_i is locally finite. Hence, $\Delta^\delta(k) = S_\ell$ for some ℓ which proves the proposition.

For $\Delta^\mu(k)$ we may argue similarly, using Theorem I.2.6 and Remark I.2.7.1 (and its proof), to show the existence of U' as above such that $\mu(\mathscr{C}_s) < k$ if $\Delta^\mu(k) \subsetneq S$. □

Exercise 2.7.1. Call a morphism $F : \mathscr{X} \to S$ of complex spaces a *family of hypersurfaces with isolated singularities* if F is reduced, if the restriction $F : \mathrm{Sing}(F) \to S$ is finite and all non-empty fibres $\mathscr{X}_s = F^{-1}(s)$ are pure dimensional and satisfy $\mathrm{edim}(\mathscr{X}_s, x) = \dim(\mathscr{X}_s, x) + 1$ for each $x \in \mathscr{X}_s$. Show that, locally, $\mathscr{X}_s$ is isomorphic to a hypersurface in some $\mathbb{C}^n$ having only isolated singularities. Moreover, show that the sets

$$\Delta^\mu_F(k) := \{s \in S \mid \mu(\mathscr{X}_s) \geq k\},$$
$$\Delta^\tau_F(k) := \{s \in S \mid \tau(\mathscr{X}_s) \geq k\},$$

are closed analytic subsets of S. Here, $\tau(\mathscr{X}_s) = \sum_{x \in \mathscr{X}_s} \tau(\mathscr{X}_s, x)$ is the total Tjurina number of $\mathscr{X}_s$.

HINT: For $\Delta^\mu_F(k)$ you may proceed as in the proof of Proposition 2.57 and for $\Delta^\tau_F(k)$ as in Theorem I.2.6.

We continue by studying in more detail the relation between deformations of the normalization $(\overline{C}, \overline{0}) \to (C, \mathbf{0})$ and deformations of the equation of $(C, \mathbf{0})$. To simplify notations, we omit the base points of the germs, resp. multigerms, in the notation and work with sufficiently small representatives.

This understood, let $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C} \to C}$ denote the semiuniversal deformation of the normalization $\overline{C} \to C$, and let $\mathscr{D} \to B_C$ be the semiuniversal deformation of C. By versality of $\mathscr{D} \to B_C$, there exists a morphism

$$\alpha : (B_{\overline{C}\to C}, \mathbf{0}) \to (B_C, \mathbf{0})$$

such that the pull-back of $\mathscr{D} \to B_C$ via α is isomorphic to $\mathscr{C} \to B_{\overline{C}\to C}$. A priori, α is not unique (only its tangent map is unique due the semiuniversality of $\mathscr{D} \to B_C$). However, in our situation, α itself is unique (see Theorem 2.59). The statements about the δ-constant stratum, resp. the μ-constant stratum, of $(C, \mathbf{0})$ in $(B_C, \mathbf{0})$ established below then follow from properties of α.

We first study deformations of $\overline{C}/C$, that is, deformations of the normalization which fix C. In terms of the notation introduced in Definition 1.21, we study objects of $\mathcal{D}ef_{(\overline{C},\overline{0})/(C,\mathbf{0})}$, resp. their isomorphism classes. Recall that $\mathrm{mt} := \mathrm{mt}(C, \mathbf{0})$ denotes the multiplicity, $r := r(C, \mathbf{0})$ the number of branches and $\delta := \delta(C, \mathbf{0})$ the δ-invariant of $(C, \mathbf{0})$.

Proposition 2.58. *With the above notations, the following holds:*

(1) The restriction of $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ *to* $\alpha^{-1}(\mathbf{0})$ *represents a semiuniversal deformation of* $\overline{C}/C$.
(2) The map α *is finite; it is a closed embedding iff* $\mathrm{mt} = r$.
(3) In particular, the functor $\underline{\mathcal{D}ef}_{(\overline{C},\overline{0})/(C,\mathbf{0})}$ *has a semiuniversal deformation whose base space* $B_{\overline{C}/C}$ *consists of a single point of embedding dimension* $\mathrm{mt} - r$. *This point is reduced iff* $(C, \mathbf{0})$ *consists of* r *smooth branches.*

Proof. (1) Since each object in $\mathcal{D}ef_{\overline{C}\to C}(S)$, S any complex space germ, maps to the trivial deformation in $\mathcal{D}ef_C(S)$ iff it is an object of $\mathcal{D}ef_{\overline{C}/C}(S)$, we get that the restriction of the semiuniversal deformation of the normalization to $B_{\overline{C}/C} := \alpha^{-1}(\mathbf{0})$ is a versal element of $\mathcal{D}ef_{\overline{C}/C}$. By Lemma 2.28 (1), the map $T^0_{\overline{C}\to C} \to T^0_C$ induced by α is an isomorphism. From the braid for the normalization (see Figure 2.14 on page 311), we get an exact sequence

$$0 \to T^1_{\overline{C}/C} \to T^1_{\overline{C}\to C} \to T^1_C\,. \tag{2.7.32}$$

Thus, the pull-back of the semiuniversal object of $\mathcal{D}ef_{\overline{C}\to C}$ to $B_{\overline{C}/C}$ satisfies the uniqueness condition on the tangent level to ensure that it is a semiuniversal object for $\mathcal{D}ef_{\overline{C}/C}$.

(2) Assume to the contrary that α is not finite, that is, $\dim B_{\overline{C}/C} > 0$. Then there exists a reduced curve germ $(D, \mathbf{0}) \subset (B_{\overline{C}/C}, \mathbf{0})$ such that, for each $s \in D \setminus \{\mathbf{0}\}$, the germ of D at s is smooth (D sufficiently small). The restriction of $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ to (D, s) is a family $(\overline{\mathscr{C}}_D, \overline{x}) \to (\mathscr{C}_D, x) \to (D, s)$ such that $(\mathscr{C}_D, x) \cong (C, \mathbf{0}) \times (D, s) \to (D, s)$ is the projection (since $\mathscr{C}_D \to D$ is trivial). By Theorem 2.51 (1), $(\overline{\mathscr{C}}_D, \overline{x}) \to (\mathscr{C}_D, x)$ is the normalization of $(\mathscr{C}_D, x)$. Hence, $(\overline{\mathscr{C}}_D, \overline{x}) \cong (\overline{C}, \overline{0}) \times (D, s)$ and $(\overline{\mathscr{C}}_D, \overline{x}) \to (\mathscr{C}_D, x) \to (D, s)$ is a trivial deformation of the normalization $\overline{C} \to C$.

By openness of versality [Fle1, Satz 4.3], $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ is versal over $(B_{\overline{C}\to C}, s)$. However, it is not semiuniversal as it contains the trivial subfamily over (D, s).

By Propositions 2.30 and Theorem 2.38, $B_{\overline{C}\to C}$ is a smooth complex space germ of dimension $\tau(C,\mathbf{0})-\delta(C,\mathbf{0})$. Hence, $\dim(B_{\overline{C}\to C},s)=\tau(C,\mathbf{0})-\delta(C,\mathbf{0})$. Because $(B_{\overline{C}\to C},s)$ is a versal, but not a semiuniversal base space for the deformation of the normalization of the fibre $(\mathscr{C}_s,x)$, its dimension is bigger than $\tau(\mathscr{C}_s,x)-\delta(\mathscr{C}_s,x)$. However, $(\mathscr{C}_s,x)\cong(C,\mathbf{0})$ and, therefore, $\tau(\mathscr{C}_s,x)=\tau(C,\mathbf{0})$ and $\delta(\mathscr{C}_s,x)=\delta(C,\mathbf{0})$, which is a contradiction.

This shows that α is finite and, hence, $\alpha^{-1}(\mathbf{0})$ is a single point, which is of embedding dimension $\dim_{\mathbb{C}} T^1_{\overline{C}/C}=\mathrm{mt}-r$ by Proposition 2.30.

Thus, we proved statement (2) and, at the same time, (3) since $\mathrm{mt}=r$ iff $(C,\mathbf{0})$ has r smooth branches. □

The next theorem relates deformations of the parametrization to the δ-constant stratum in the base space of the semiuniversal deformation of the reduced plane curve singularity $(C,\mathbf{0})$.

Let $\Psi:\mathscr{D}\to B_C$ denote a sufficiently small representative of the semiuniversal deformation of $(C,\mathbf{0})$, $\mathscr{D}_s=\Psi^{-1}(s)$ the fibre over s, and call

$$\Delta^\delta := \{s\in B_C \mid \delta(\mathscr{D}_s)=\delta(C,\mathbf{0})\}\,,$$

respectively the germ $(\Delta^\delta,\mathbf{0})\subset(B_C,\mathbf{0})$, the *$\delta$-constant stratum* of Ψ. Since $\delta(\mathscr{D}_s)\le\delta(C,\mathbf{0})$ by Theorem 2.54, $\Delta^\delta=\Delta^\delta_\Psi(\delta(C,\mathbf{0}))$ and $\Delta^\delta\subset B_C$ is a closed analytic subset (Proposition 2.57). We set $\delta:=\delta(C,\mathbf{0})$ and $\tau:=\tau(C,\mathbf{0})$. Using these notations, we have the following theorem:

Theorem 2.59. *Let $\Psi:\mathscr{D}\to B_C$, resp. $\overline{\mathscr{C}}\to\mathscr{C}\to B_{\overline{C}\to C}$, be sufficiently small representatives of the semiuniversal deformation of $(C,\mathbf{0})$, resp. of the semiuniversal deformation of the normalization $(\overline{C},\overline{0})\to(C,\mathbf{0})$. Then the following holds:*

(0) B_C, resp. $B_{\overline{C}\to C}$ are smooth of dimension τ, resp. $\tau-\delta$.
(1) The δ-constant stratum $\Delta^\delta\subset B_C$ has the following properties:
  (a) Δ^δ is irreducible of dimension $\tau-\delta$.
  (b) $s\in\Delta^\delta$ is a smooth point of Δ^δ iff each singularity of the fibre $\mathscr{D}_s=\Psi^{-1}(s)$ has only smooth branches.
  (c) There exists an open dense set $U\subset\Delta^\delta$ such that each fibre $\mathscr{D}_s$, $s\in U$, of Ψ has only ordinary nodes as singularities.
(2) Each map $\alpha:B_{\overline{C}\to C}\to B_C$ induced by versality of Ψ satisfies:
  (a) $\alpha(B_{\overline{C}\to C})=\Delta^\delta$.
  (b) $\alpha:B_{\overline{C}\to C}\to\Delta^\delta$ is the normalization of Δ^δ, hence unique.
  (c) The pull-back of $\Psi:\mathscr{D}\to B_C$ to $B_{\overline{C}\to C}$ via α is isomorphic to $\mathscr{C}\to B_{\overline{C}\to C}$ and, hence, lifts to the semiuniversal deformation $\overline{\mathscr{C}}\to\mathscr{C}\to B_{\overline{C}\to C}$ of the normalization.

Corollary 2.60 (Diaz, Harris). *The δ-constant stratum $(\Delta^\delta,\mathbf{0})$ has a smooth normalization. It is smooth iff $(C,\mathbf{0})$ is the union of smooth branches.*

Proof of Theorem 2.59. Recall that we work with a sufficiently small representative Ψ of the semiuniversal deformation of $(C,\mathbf{0})$.

(0) follows from Corollary 1.17, p. 239, resp. from Theorem 2.38, p. 327, and Proposition 2.30, p. 312.

(1) Let s be any point of Δ^δ. Then, by openness of versality, the restriction of Ψ over a sufficiently small neighbourhood of s in B_C is a joint versal deformation of all singular points of $\mathscr{D}_s$. It is known ([Gus, Lemma 1], [ACa, Théorème 1], [Tei1, Proposition II.5.2.1, Lemma II.5.2.8]) that each reduced plane curve singularity can be deformed, with total δ-invariant being constant, to a plane curve with only nodes as singularities. Hence, arbitrarily close to s, there exists $s' \in \Delta^\delta$ such that the fibre $\mathscr{D}_{s'}$ has only nodes as singularities. For a node, the δ-constant stratum consists of a (reduced) point in the one-dimensional base space of the semiuniversal deformation. Since Ψ induces over some neighbourhood of s' a versal deformation of the nodal curve $\mathscr{D}_{s'}$ with δ nodes, it follows that (Δ^δ, s') is smooth of codimension δ in (B_C, s').

It follows that the set $U \subset \Delta^\delta$ of all $s \in \Delta^\delta$ such that $\mathscr{D}_s$ is a nodal curve is open and dense in Δ^δ of dimension $\tau(C,\mathbf{0}) - \delta(C,\mathbf{0})$.

To show the irreducibility of the δ-constant stratum, we have to prove that U is connected (see Remark (B) on page 62). Let $s_0, s_1 \in U$ be two points. Although the fibres $\mathscr{D}_{s_i}$ are not germs, they appear as fibres in a δ-constant deformation of $(C,\mathbf{0})$ and, hence, can be parametrized: if Δ_i^δ is the irreducible component of Δ^δ to which s_i belongs, let $\widetilde{\Delta}_i^\delta \to \Delta_i^\delta$ be the normalization and apply Theorem 2.56 to the pull-back of Ψ to $\widetilde{\Delta}_i^\delta$.

In particular, there exist parametrizations

$$\varphi^{(i)} = \bigl(\varphi_j^{(i)}\bigr)_{j=1}^r : \coprod_{j=1}^r D_j \to \mathscr{D}_{s_i} \subset B\,, \quad i = 0,1\,,$$

of $\mathscr{D}_{s_i}$, where the $D_j \subset \mathbb{C}$ are small discs, $B \subset \mathbb{C}^2$ is a small ball, and where r is the number of branches of $(C,\mathbf{0})$.

Now, join the two parametrizations by the family

$$\phi = \bigl(\phi_j\bigr)_{j=1}^r : \coprod_{j=1}^r D_j \times D \to B \times D\,,$$

where $D \subset \mathbb{C}$ is a disc containing 0 and 1, and where

$$\phi_j : (t_j, s) \mapsto (1-s)\varphi_j^{(0)}(t_j) + s\varphi_j^{(1)}(t_j)\,, \quad j = 1, \dots, r\,.$$

Being a nodal curve is an open property. Hence, for almost all s, ϕ parametrizes a nodal curve. That is, there is an open set $V \subset D$, being the complement of finitely many points, such that $0, 1 \in V$ and the restriction $\phi' : \coprod_{j=1}^r D_j \times V \to B \times V$ is finite and $\phi(D_j \times \{s\})$ is a nodal curve for $s \in V$. Applying Proposition 2.9 to ϕ', we see that the image of ϕ' defines a family of nodal curves which is δ-constant (being induced by a deformation of the normalization) and which connects $\mathscr{D}_{s_0}$ and $\mathscr{D}_{s_1}$. This show that U is connected.

To complete the proof of (1), we have to show that $(\varDelta^\delta, s)$ is smooth iff the singularities of $\mathscr{D}_s$ have only smooth branches. By openness of versality, (B_C, s) is the base space of a versal deformation for each of the singularities of $\mathscr{D}_s$. By Proposition 1.14, $(\varDelta^\delta, s)$ is smooth if the germs of the δ-constant strata in the semiuniversal deformations of all singularities of $\mathscr{D}_s$ are smooth. Hence, it suffices to show that $(\varDelta^\delta, \mathbf{0})$ is smooth iff $(C, \mathbf{0})$ has only smooth branches. This is shown now when we prove (2).

(2) By (0), $B_{\overline{C}\to C}$ is smooth of dimension $\tau(C,\mathbf{0}) - \delta(C,\mathbf{0})$. Its image under α is contained in $\varDelta^\delta$ by Lemma 2.53.

Since $\varDelta^\delta$ is irreducible and of the same dimension (as shown in part (1) above), α surjects onto $\varDelta^\delta$. Let $n : \widetilde{\varDelta}^\delta \to \varDelta^\delta$ denote the normalization of $\varDelta^\delta$. By the universal property of the normalization, α factors as $\alpha = n \circ \widetilde{\alpha}$ for a unique morphism $\widetilde{\alpha} : B_{\overline{C}\to C} \to \widetilde{\varDelta}^\delta$. By the first part of the proof, we know already that $(\varDelta^\delta, s)$ is a smooth germ (hence, $\widetilde{\varDelta}^\delta \cong \varDelta^\delta$ locally at s) for $\mathscr{D}_s$ a nodal curve. Further, α is finite and bijective over the locus of nodal fibres by Proposition 2.58. Hence, $\widetilde{\alpha} : B_{\overline{C}\to C} \to \widetilde{\varDelta}^\delta$ is surjective and finite and an isomorphism outside a nowhere dense analytic subset. It follows that $\widetilde{\alpha}$ is the normalization of $\widetilde{\varDelta}^\delta$ (Remark I.1.94.1) and, hence, an isomorphism (since $\widetilde{\varDelta}^\delta$ is normal).

Finally, we show the smoothness statement of (1)(b). The epimorphism Theorem I.1.20 implies that $\alpha : (B_{\overline{C}\to C}, \mathbf{0}) \to (B_C, \mathbf{0})$ is a closed embedding (hence, an isomorphism onto $(\varDelta^\delta, \mathbf{0})$) iff the induced map of the cotangent spaces is surjective, that is, iff the dual map $T^1_{\overline{C}\to C} \to T^1_C$ is injective. However, from the exact sequence (2.7.32), we know that the kernel of this map is $T^1_{\overline{C}/C}$ which has dimension $m - r$ by Proposition 2.30. This shows that $(\varDelta^\delta, \mathbf{0})$ is smooth iff $m = r$, which means that $(C, \mathbf{0})$ has only smooth branches. □

We turn now to the μ-constant stratum. As before, let $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$, resp. $\mathscr{D} \to B_C$, denote the semiuniversal deformation of the normalization, resp. of the equation, of $(C, \mathbf{0})$. Moreover, let the right vertical sequence of the diagram

$$
\begin{tikzcd}
\overline{\mathscr{C}}^{es} \arrow[r, hook] \arrow[d] & \overline{\mathscr{C}}^{sec} \arrow[d] \\
\mathscr{C}^{es} \arrow[r, hook] \arrow[d] & \mathscr{C}^{sec} \arrow[d, shift right] \\
B^{es}_{\overline{C}\to C} \arrow[r, hook] & B^{sec}_{\overline{C}\to C} \arrow[u, shift right, "\sigma"'] \arrow[uu, bend right=60, "\overline{\sigma}"']
\end{tikzcd}
$$

be the semiuniversal deformation with section of the normalization which contains as a subfamily the *semiuniversal equisingular deformation (with section) of the normalization*, given by the left vertical sequence.

Here and in what follows, we identify the semiuniversal deformations (with section) of the normalization and of the parametrization according to Proposition 2.23.

Forgetting the sections, we get a (non-unique) morphism $B^{sec}_{\overline{C}\to C} \to B_{\overline{C}\to C}$ which we compose with the map α defined above to obtain a morphism

$$\alpha^{sec} : B^{sec}_{\overline{C}\to C} \to B_{\overline{C}\to C} \xrightarrow{\alpha} B_C\,.$$

We can formulate now the main result about the μ-constant stratum:

Theorem 2.61. *Let $B^{sec}_{\overline{C}\to C}$, resp. B_C, be sufficiently small representatives of the base spaces of the semiuniversal deformation with section of the normalization, resp. of the semiuniversal deformation of the equation, of the reduced plane curve singularity $(C,\mathbf{0})$. Let $\alpha^{sec} : B^{sec}_{\overline{C}\to C} \to B_C$ be any morphism induced by versality as above. Then the following holds:*

(1) The tangent map of α^{sec} restricted to the tangent space of $B^{es}_{\overline{C}\to C}$ is injective.
(2) α^{sec} maps the base space $B^{es}_{\overline{C}\to C}$ of the semiuniversal equisingular deformation of the normalization isomorphically onto the μ-constant stratum $\Delta^\mu \subset B_C$.
(3) In particular, Δ^μ is smooth of dimension $\dim_{\mathbb{C}} T^{1,es}_{\overline{C}\to\mathbb{C}^2}$.

Before giving the proof of this theorem, we recall the explicit description of the maps $\overline{\mathscr{C}}^{sec} \to B^{sec}_{\overline{C}\to\mathbb{C}^2}$ and $\overline{\mathscr{C}}^{es} \to B^{es}_{\overline{C}\to\mathbb{C}^2}$ from Proposition 2.27 and Theorem 2.38: Let $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y} \in \overline{\mathfrak{m}}\frac{\partial}{\partial x} \oplus \overline{\mathfrak{m}}\frac{\partial}{\partial y}$, $j = 1,\dots,k$, represent a basis of

$$T^{1,sec}_{\overline{C}\to\mathbb{C}^2} = \overline{\mathfrak{m}}\frac{\partial}{\partial x} \oplus \overline{\mathfrak{m}}\frac{\partial}{\partial y} \Big/ \left(\overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right) + \left(\mathfrak{m}\frac{\partial}{\partial x} \oplus \mathfrak{m}\frac{\partial}{\partial y}\right)\right)\,.$$

Then the deformation

$$\begin{aligned} X_i(t_i,\boldsymbol{s}) &= x_i(t_i) + \sum_{j=1}^{k} a_i^j(t_i)s_j\,,\\ Y_i(t_i,\boldsymbol{s}) &= y_i(t_i) + \sum_{j=1}^{k} b_i^j(t_i)s_j\,, \end{aligned} \tag{2.7.33}$$

represents a semiuniversal deformation of the normalization over

$$(B^{sec}_{\overline{C}\to C},\mathbf{0}) \cong (T^{1,sec}_{\overline{C}\to\mathbb{C}^2},\mathbf{0}) \cong (\mathbb{C}^k,\mathbf{0})\,.$$

If the $\boldsymbol{a}^j \frac{\partial}{\partial x} + \boldsymbol{b}^j \frac{\partial}{\partial y}$, $j = 1,\dots,\ell$, $\ell \le k$, are chosen from $I^{es}_{\overline{C}\to\mathbb{C}^2}$ such that they represent a basis of the vector subspace

$$T^{1,es}_{\overline{C}\to\mathbb{C}^2} = I^{es}_{\overline{C}\to\mathbb{C}^2} \Big/ \left(\overline{\mathfrak{m}}\left(\dot{\boldsymbol{x}}\frac{\partial}{\partial x} + \dot{\boldsymbol{y}}\frac{\partial}{\partial y}\right) + \left(\mathfrak{m}\frac{\partial}{\partial x} \oplus \mathfrak{m}\frac{\partial}{\partial y}\right)\right)$$

of $T^{1,sec}_{\overline{C}\to\mathbb{C}^2}$, then (2.7.33) with k replaced by ℓ represents a semiuniversal equisingular deformation of the normalization over

$$(B^{es}_{\overline{C}\to C},\mathbf{0}) \cong (T^{1,es}_{\overline{C}\to\mathbb{C}^2},\mathbf{0}) \subset (T^{1,sec}_{\overline{C}\to\mathbb{C}^2},\mathbf{0}) \cong (B^{sec}_{\overline{C}\to C},\mathbf{0})\,.$$

For the proof of Theorem 2.61, we make now use of the following results of Lazzeri, Lê and Teissier:

Proposition 2.62. *Let $\phi:\mathscr{C}\to S$ be a sufficiently small representative of an arbitrary deformation of the reduced plane curve singularity $(C,\mathbf{0})$ with S reduced. Then the following holds:*

(1) If $\mu(\mathscr{C}_s)=\mu(C,\mathbf{0})$ for each $s\in S$, then there exists a unique section $\sigma:S\to\mathscr{C}$ of ϕ such that $\mathscr{C}_s\setminus\{\sigma(s)\}$ is smooth and, hence, $\mu(\mathscr{C}_s)=\mu(\mathscr{C}_s,\sigma(s))$ for each $s\in S$.
(2) Let $\sigma:S\to\mathscr{C}$ be a section of ϕ. Then $\mu(\mathscr{C}_s,\sigma(s))$ is independent of $s\in S$ iff $\delta(\mathscr{C}_s,\sigma(s))$ and $r(\mathscr{C}_s,\sigma(s))$ are independent of $s\in S$.
(3) If $\sigma:S\to\mathscr{C}$ is a section of ϕ such that $\mu(\mathscr{C}_s,\sigma(s))$ is independent of $s\in S$ then the multiplicity $\mathrm{mt}(\mathscr{C}_s,\sigma(s))$ is independent of $s\in S$.

Proof. (1) is due to C. Has Bey [Has] and Lazzeri [Laz] (for arbitrary isolated hypersurface singularities); the existence of the section to Teissier [Tei]. For a proof of (2), see e.g. [Tei]. (3) is due to Lê [Le, LeR]. □

Proof of Theorem 2.61. (1) The tangent map of $\alpha^{sec}:B^{sec}_{\overline{C}\to C}\to B_C$ is the map $\alpha':T^{1,sec}_{\overline{C}\to\mathbb{C}^2}\to T^1_C$ described in Lemma 2.33. By Corollary 2.35, we know that $\alpha'|_{T^{1,es}_{\overline{C}\to\mathbb{C}^2}}$ is injective since $T^{1,es}_{\overline{C}\to\mathbb{C}^2}\subset T^{1,em}_{\overline{C}\to\mathbb{C}^2}$ by construction.

This proves already (applying the epimorphism Theorem I.1.20) that $\alpha^{sec}|_{B^{es}_{\overline{C}\to C}}$ is a closed embedding mapping $B^{es}_{\overline{C}\to C}$ isomorphically onto a smooth closed analytic subset $\Delta^{es}\subset B_C$ (for sufficiently small representatives).

(2) We prove that $\Delta^{es}=\Delta^{\mu}$. For the inclusion $\Delta^{es}\subset\Delta^{\mu}$ note that the deformation $\mathscr{C}^{es}\to B^{es}_{\overline{C}\to C}$ is δ-constant along the given section $\sigma:B^{es}_{\overline{C}\to C}\to\mathscr{C}^{es}$ since it has a simultaneous normalization $\overline{\mathscr{C}}^{es}\to\mathscr{C}^{es}$ (see Lemma 2.53).

Moreover, we claim that $r(\mathscr{C}_s,\sigma(s))=r(C,\mathbf{0})$ for all $s\in S$. If we assume the contrary, then $r(\mathscr{C}^{es}_s,\sigma(s))>r(C,\mathbf{0})=:r$ for $s\in U\setminus\{\mathbf{0}\}$, U some open neighbourhood of $\mathbf{0}\in S$ (since r branches are given by the parametrization). The extra branches of $(\mathscr{C}^{es}_s,\sigma(s))$ split off in some strict transform $\mathscr{C}'_s$ obtained by successively blowing up equimultiple sections. Hence, they are not in the image of $(\overline{\mathscr{C}}^{es}_s,\overline{0})$. This implies that the deformation of the parametrization of $(\mathscr{C}^{es}_s,\sigma(s))$ is not equimultiple (see Example 2.26.1), contradicting the definition of equisingularity.

From the relation $\mu=2\delta-r+1$ (Proposition I.3.35), we get that the Milnor number $\mu(\mathscr{C}^{es}_s,\sigma(s))$ is constant in s and, hence, $\Delta^{es}\subset\Delta^{\mu}$.

To show the opposite inclusion, $\Delta^{es} \supset \Delta^{\mu}$, we apply Proposition 2.62. It yields the existence of a section $\rho : \Delta^{\mu} \to \mathscr{D}^{\mu}$ of the restriction of $\mathscr{D} \to B_C$ to Δ^{μ} such that $\delta(\mathscr{D}_s, \rho(s))$, $r(\mathscr{D}_s, \rho(s))$ and $\mu(\mathscr{D}_s, \rho(s))$ are constant for $s \in \Delta^{\mu}$.

Hence, $\Delta^{\mu} \subset \Delta^{\delta}$ and, therefore, Δ^{μ} is in the image of $\alpha : B_{\overline{C}\to C} \to B_C$. Moreover, being r-constant and mt-constant implies as in the proof of (1) that the restriction of $\overline{\mathscr{C}} \to \mathscr{C} \to B_{\overline{C}\to C}$ to $\alpha^{-1}(\Delta^{\mu})$ admits uniquely determined compatible sections $\sigma : \alpha^{-1}(\Delta^{\mu}) \to \mathscr{C}$ and $\overline{\sigma}_i : \alpha^{-1}(\Delta^{\mu}) \to \overline{\mathscr{C}}$, $i = 1, \ldots, r = r(C, \mathbf{0})$, such that the deformation of the parametrization $\coprod_{i=1}^{r} \big(\overline{\mathscr{C}}, \overline{\sigma}_i(s)\big) \to \big(\mathscr{C}, \sigma(s)\big)$ is equimultiple for $s \in \alpha^{-1}(\Delta^{\mu})$. This shows that $\alpha^{-1}(\Delta^{\mu})$ is in the image of the morphism $B^{em}_{\overline{C}\to C} \hookrightarrow B^{sec}_{\overline{C}\to C} \to B_{\overline{C}\to C}$, that is, $(\alpha^{sec})^{-1}(\Delta^{\mu}) \subset B^{em}_{\overline{C}\to C}$.

Now, we blow up $\mathscr{C}$ along the equimultiple section $\sigma : \alpha^{-1}(\Delta^{\mu}) \to \mathscr{C}$ to get a family $\mathscr{C}' = \coprod_{i=1}^{r'} \big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$ of (multi)germs. Since $r\big(\mathscr{C}_s, \sigma(s)\big)$ is constant, the number of branches of $\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$ is constant for $i = 1, \ldots, r'$. By Proposition I.3.34, we have

$$\delta\big(\mathscr{C}, \sigma(s)\big) = \delta(\mathscr{C}') + \frac{\mathrm{mt}(\mathrm{mt} - 1)}{2},$$

where $\mathrm{mt} = \mathrm{mt}\big(\mathscr{C}, \sigma(s)\big) = \mathrm{mt}(C, \mathbf{0})$. Hence, for each $i = 1, \ldots, r'$, the map germ $\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big) \to (B_{\overline{C}\to C}, s)$, is a δ-constant family. Applying, again, the relation $\mu = 2\delta - r + 1$, we get that $\mu\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$ and, hence, $\mathrm{mt}\big(\mathscr{C}', \widetilde{\sigma}_i(s)\big)$, is constant for $s \in \alpha^{-1}(\Delta^{\mu})$. Therefore, we can argue by induction on the number of blowing ups needed to resolve $(C, \mathbf{0})$, to show that after blowing up there exist always equimultiple sections. We conclude that $(\alpha^{sec})^{-1}(\Delta^{\mu}) \subset B^{es}_{\overline{C}\to C}$.
$\square$

2.8 Comparison of Equisingular Deformations

The main purpose of this section is to prove the equivalence of the functors of equisingular deformations of the parametrization and of equisingular deformations of the equation. Moreover, we discuss related deformations.

We start by reconsidering the constructions and results of this chapter, describe their relation and discuss computational aspects.

In Section 2.1, we introduced equisingular deformations of $(C, \mathbf{0})$, also denoted equisingular deformations of the equation, and proved that equisingular deformations of $(C, \mathbf{0})$ induce equisingular deformations of the branches (Proposition 2.11). We defined (Definition 2.7) the equisingular deformation functor $\underline{\mathcal{D}ef}^{\,es}_{(C,\mathbf{0})}$ as a subfunctor of $\underline{\mathcal{D}ef}_{(C,\mathbf{0})}$, where we required the existence of an equimultiple section $\sigma = \sigma^{(0)}$ and of equimultiple sections $\sigma^{(\ell)}$ through the infinitely near points of successive blow ups of $(C, \mathbf{0})$. By Proposition 2.8, these sections are unique if $(C, \mathbf{0})$ is singular (which we assume in this discussion).

We can also consider equisingular deformations as deformations with section, where the section σ is part of the data (Definition 2.6). The set of isomorphism classes of equisingular deformations with section over (T, t_0) is denoted by $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}(T, t_0)$ and the functor

$$\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})} : (\text{complex germs}) \to (\text{sets}), \quad (T, t_0) \mapsto \underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}(T, t_0)$$

is called the *equisingular deformation functor with section*. By definition, $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}$ is a subfunctor of $\underline{\mathcal{D}ef}^{sec}_{(C,\mathbf{0})}$.

Since σ is uniquely determined by Proposition 2.8, $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$ and $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}$ are isomorphic functors, but they are not equal. In particular, in concrete calculations, we have to distinguish them carefully.

In Section 2.2, we defined the equisingularity ideals $I^{es}(f)$, $I^{es}_{\mathit{fix}}(f)$ and gave explicit descriptions for semiquasihomogeneous and Newton non-degenerate singularities. For Newton-degenerate singularities, these ideals are quite complicated and no other description, besides their definition, is available.

We show now how $I^{es}(f)$ and $I^{es}_{\mathit{fix}}(f)$ are related to the functors $\underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}$ and $\underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}$.

Proposition 2.63. *Let* $T^{1,es}_{(C,\mathbf{0})} = \underline{\mathcal{D}ef}^{es}_{(C,\mathbf{0})}(T_\varepsilon)$*, resp.* $T^{1,es,sec}_{(C,\mathbf{0})} = \underline{\mathcal{D}ef}^{es,sec}_{(C,\mathbf{0})}(T_\varepsilon)$ *be the vector spaces of infinitesimal equisingular deformations (resp. with section) of* $(C, \mathbf{0})$*. Then we have*

$$T^{1,es}_{(C,\mathbf{0})} \cong I^{es}(f)/\langle f, j(f)\rangle \subset \mathcal{O}_{\mathbb{C}^2,\mathbf{0}}/\langle f, j(f)\rangle = T^1_{(C,\mathbf{0})},$$
$$T^{1,es,sec}_{(C,\mathbf{0})} \cong I^{es}_{\mathit{fix}}(f)/\langle f, \mathfrak{m}j(f)\rangle \subset \mathfrak{m}/\langle f, \mathfrak{m}j(f)\rangle = T^{1,sec}_{(C,\mathbf{0})},$$

where $\mathfrak{m} = \mathfrak{m}_{\mathbb{C}^2,\mathbf{0}}$.

The statement follows from Proposition 2.14, noting that the ideals $\langle f, j(f)\rangle$ (resp. $\langle f, \mathfrak{m}j(f)\rangle$) describe the infinitesimally trivial (embedded) deformations (resp. with trivial section) of $(C, \mathbf{0})$ (see Remark 1.25.1 and Corollary 2.3).

For Newton degenerate singularities, the vector spaces $T^{1,es}_{(C,\mathbf{0})}$ and $T^{1,es,sec}_{(C,\mathbf{0})}$ cannot be easily described. In particular, they are, in general, not generated by monomials (see the example below). However, in [CGL1], an algorithm to compute both vector spaces is given. This algorithm is implemented in SINGULAR and can be used to compute explicit examples:

Example 2.63.1 (Continuation of Example 2.17.2). The following SINGULAR session computes a list `Ies` whose first entry is the ideal $I^{es}(f)$ (given by a list of generators), whose second entry is the ideal $I^{es}_{\mathit{fix}}(f)$, and whose third entry is the ideal $\langle j(f), I^s\rangle$:

```
LIB "equising.lib";
ring R = 0,(x,y),ds;
poly f = (x-2y)^2*(x-y)^2*x2y2+x9+y9;
list Ies = esIdeal(f,1);
```

We make SINGULAR display ideal generators for the quotient $I^{es}(f)/\langle f, j(f)\rangle$ and for $I^{es}_{fix}(f)/\langle f, \mathfrak{m}j(f)\rangle$:

```
ideal J = f,jacob(f);
ideal IesQ = reduce(Ies[1],std(J));
simplify(IesQ,11);
//-> _[1]=x3y5-3x2y6+2xy7
//-> _[2]=y9
//-> _[3]=x2y7
//-> _[4]=xy8
ideal mJ = f,maxideal(1)*jacob(f);
ideal IesfixQ = reduce(Ies[2],std(mJ));
simplify(IesfixQ,11);
//-> _[1]=x3y5-xy7+9/4y9
//-> _[2]=x2y7-y9
//-> _[3]=xy8+y9
//-> _[4]=y10
```

From the output, we read that $I^{es}(f)$ is generated as an ideal by the Tjurina ideal $\langle f, j(f)\rangle$ and the polynomials $x^3y^5 - 3x^2y^6 + 2xy^7$, y^9, x^2y^7 and xy^8, and similarly for $I^{es}_{fix}(f)$. Finally, we check that

$$x^5y^3 - 6x^4y^4 + 13x^3y^5 - 12x^2y^6 + 4xy^7 \in I^{es}(f) \setminus \langle f, j(f), I^s(f)\rangle$$

as claimed in Example 2.17.2:

```
poly g=x5y3-6x4y4+13x3y5-12x2y6+4xy7;
reduce(g,std(Ies[1]));
//-> 0
reduce(g,std(Ies[3]));
//-> 1/3x3y5-x2y6+2/3xy7
```

In order to prove properties of equisingular deformations of $(C, \mathbf{0})$, we introduced in Section 2.3 (equimultiple) deformations of the parametrization $\varphi : (\overline{C}, \overline{0}) \to (\mathbb{C}^2, \mathbf{0})$, and we computed the vector spaces T^1 and T^2 for several related deformation functors in Section 2.4. In Section 2.5, we defined equisingular deformations of φ and showed that they have a rather simple description. In particular, the functor of equisingular deformations of φ is a linear subfunctor of the functor of (arbitrary) deformations with section of φ and, thus, each versal equisingular deformation of φ has a smooth base (Theorem 2.38).

The link between deformations of the parametrization and deformations of the equation is given in Proposition 2.23 which is based on Proposition 2.9. It says that each deformation of the parametrization induces a unique (up to isomorphism) deformation of the equation. By Lemma 2.53, such deformations of $(C, \mathbf{0})$ are δ-constant. Conversely, if the base space $(T, \mathbf{0})$ is normal, then a δ-constant deformation of $(C, \mathbf{0})$ over $(T, \mathbf{0})$ is induced by a deformation of the parametrization (Theorem 2.56).

If $(B_C,\mathbf{0})$, resp. $(B_{\overline{C}\to C},\mathbf{0})$, is the base space of the semiuniversal deformation of $(C,\mathbf{0})$, resp. of the parametrization φ of $(C,\mathbf{0})$, then $(B_C,\mathbf{0})$ and $(B_{\overline{C}\to C},\mathbf{0})$ are smooth, the natural map $\alpha:(B_{\overline{C}\to C},\mathbf{0})\to(B_C,\mathbf{0})$ maps $(B_{\overline{C}\to C},\mathbf{0})$ onto the δ-constant stratum and $(B_{\overline{C}\to C},\mathbf{0})\to(\Delta^\delta,\mathbf{0})$ is the normalization (Theorem 2.59, using that deformations of the parametrization and of the normalization coincide by Proposition 2.23).

The base space $(B^{es}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})$ of the semiuniversal equisingular deformation of the parametrization is a subspace of the base space $(B^{sec}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})$ of the semiuniversal deformation of the parametrization with section. Theorem 2.38 yields that $(B^{es}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})$ is smooth. Moreover, the natural map $\alpha^{sec}:(B^{sec}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})\to(B_C,\mathbf{0})$ takes $(B^{es}_{\overline{C}\to\mathbb{C}^2},\mathbf{0})$ isomorphically onto the μ-constant stratum $(\Delta^\mu,\mathbf{0})\subset(B_C,\mathbf{0})$ (Theorem 2.61).

It still remains to complete the relation between equisingular deformations of the parametrization (Definition 2.36) and equisingular deformations of the equation (Definition 2.7):

Theorem 2.64. *Let $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity.*

(1) Every equisingular deformation of the parametrization of $(C,\mathbf{0})$ induces a unique equisingular deformation of the equation, providing a functor $\mathcal{D}ef^{es}_{\overline{C}\to\mathbb{C}^2}\to\mathcal{D}ef^{es}_{C}$.

(2) Every equisingular embedded deformation of the equation of $(C,\mathbf{0})$ comes from an equisingular deformation of the parametrization (which is induced by the equisingular deformation of the resolution); that is, $\mathcal{D}ef^{es}_{\overline{C}\to\mathbb{C}^2}\to\mathcal{D}ef^{es}_{C}$ is surjective.

(3) The functor $\mathcal{D}ef^{es}_{\overline{C}\to\mathbb{C}^2}\to\mathcal{D}ef^{es}_{C}$ induces a natural equivalence between the functors $\underline{\mathcal{D}ef}^{es}_{\overline{C}\to\mathbb{C}^2}$ and $\underline{\mathcal{D}ef}^{es}_{C}$.

The proof of this theorem is less evident than one might think, in particular for non-reduced base spaces.

Before giving the proof, we need some preparations. If

$$\psi:(\mathscr{C},\mathbf{0})\hookrightarrow(\mathscr{M},\mathbf{0})\to(T,\mathbf{0})$$

is an embedded equisingular deformation of the reduced plane curve singularity $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ along a section $\sigma:(T,\mathbf{0})\to(\mathscr{C},\mathbf{0})$, then we consider the associated equisingular deformation of the resolution (see Definition 2.6, p. 271, and Remark 2.6.1 (6)),

$$\begin{tikzcd}
(\mathscr{C}^{(N)},p^{(N)}) \arrow[r] \arrow[d, hook] & \cdots \arrow[r] & (\mathscr{C}^{(1)},p^{(1)}) \arrow[r] \arrow[d, hook] & (\mathscr{C},\mathbf{0}) \arrow[d, hook] \arrow[dr, "\psi"] & \\
(\mathscr{M}^{(N)},p^{(N)}) \arrow[r, "\pi_N"] & \cdots \arrow[r, "\pi_2"] & (\mathscr{M}^{(1)},p^{(1)}) \arrow[r, "\pi_1"] & (\mathscr{M},\mathbf{0}) \arrow[r] & (T,\mathbf{0})
\end{tikzcd} \tag{2.8.34}$$

with (multi-)sections $\sigma^{(\ell)} : (T, \mathbf{0}) \to (\mathscr{C}^{(\ell)}, p^{(\ell)})$, $\ell = 1, \ldots, N$, which are unique by Proposition 2.8, p. 275.

If we restrict the diagram to $\{\mathbf{0}\} \subset T$, we obtain an embedded (minimal) resolution of $(C, \mathbf{0})$ with $(C^{(\ell)}, p^{(\ell)}) \subset (M^{(\ell)}, p^{(\ell)})$ the strict transform of $(C, \mathbf{0})$. We denote by $E^{(\ell)} \subset M^{(\ell)}$, resp. $\mathscr{E}^{(\ell)} \subset \mathscr{M}^{(\ell)}$, the exceptional divisor of the successive blowing ups of the points $\mathbf{0}$, $p^{(i)}$, resp. of the (multi-)sections σ and $\sigma^{(j)}$, $j < \ell$, such that $C^{(\ell)} \cup E^{(\ell)} \subset M^{(\ell)}$, resp. $\mathscr{C}^{(\ell)} \cup \mathscr{E}^{(\ell)} \subset \mathscr{M}^{(\ell)}$, are the reduced total transforms of $(C, \mathbf{0})$, resp. the deformations of the reduced total transforms.

The composition $(\overline{\mathscr{C}}, \overline{0}) := (\mathscr{C}^{(N)}, p^{(N)}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ together with the section $(T, \mathbf{0}) \xrightarrow{\sigma} (\mathscr{M}, \mathbf{0}) \hookrightarrow (T, \mathbf{0})$, which we denote also by σ, and the (multi-)section $\sigma^{(N)}$, which we denote by $\overline{\sigma}$, is a deformation of the parametrization. The deformation $(\phi, \overline{\sigma}, \sigma) \in \mathcal{D}ef^{sec}_{\overline{C} \to C}(T, \mathbf{0})$ is uniquely determined (up to isomorphism) by $(\psi, \sigma) \in \mathcal{D}ef^{es}_C(T, \mathbf{0})$. We call it the *deformation of the parametrization induced by the equisingular deformation of the resolution of* (ψ, σ).

Theorem 2.64 implies that $(\phi, \overline{\sigma}, \sigma)$ is equisingular, that is, an object of $\mathcal{D}ef_{\overline{C} \to C}(T, \mathbf{0})$.

We have to generalize the concept of constant intersection multiplicity (see page 281) to families with non-reduced base spaces.

Let (M, p) be a germ of a two-dimensional complex manifold and let $(C, p) \subset (M, p)$ be a reduced curve singularity given by $f \in \mathcal{O}_{M,p}$. Consider an embedded deformation

$$\psi : (\mathscr{C}, p) \hookrightarrow (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$$

of (C, p) with section $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, p)$. Then $(\mathscr{C}, p) \subset (\mathscr{M}, p)$ is defined by a holomorphic germ $F \in \mathcal{O}_{\mathscr{M},p}$.

Consider a second reduced curve singularity $(D, p) \subset (M, p)$ given by a parametrization $\varphi : (\overline{D}, \overline{0}) \to (M, p)$ such that (D, p) and (C, p) have no common component. Let

$$\phi : (\overline{\mathscr{D}}, \overline{0}) \hookrightarrow (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$$

be a deformation of φ with compatible sections $\overline{\sigma} : (T, \mathbf{0}) \to (\overline{\mathscr{D}}, \overline{0})$ and $\sigma : (T, \mathbf{0}) \to (\mathscr{M}, p)$. We assume that the section σ coincides with the composition $(T, \mathbf{0}) \xrightarrow{\sigma} (\mathscr{C}, p) \hookrightarrow (\mathscr{M}, p)$, where $\sigma : (T, \mathbf{0}) \to (\mathscr{C}, p)$ is the section for the embedded deformation $(\mathscr{C}, p) \to (T, \mathbf{0})$ from above.

If (D, p) has r branches (D_i, p), $i = 1, \ldots, r$, then $(\overline{\mathscr{D}}, \overline{0}) = \coprod_{i=1}^r (\overline{\mathscr{D}}_i, \overline{0}_i)$ and $\overline{\sigma} = (\overline{\sigma}_i)_{i=1..r}$. We may (and do) assume that $(\overline{\mathscr{D}}_i, \overline{0}_i) = (\mathbb{C} \times T, \mathbf{0})$, $i = 1, \ldots, r$, that $(\mathscr{M}, p) = (\mathbb{C}^2 \times T, \mathbf{0})$, and that $\overline{\sigma}_i$ and σ are the trivial sections. Then the deformation ϕ is given by maps $\phi_i : \mathbb{C} \times T \to \mathbb{C}^2$, $i = 1, \ldots, r$,

$$\begin{aligned} (\overline{\mathscr{D}}_i, \overline{0}_i) = (\mathbb{C} \times T, \mathbf{0}) &\longrightarrow (\mathbb{C}^2 \times T, \mathbf{0}) = (\mathscr{M}, p)\,, \\ (t_i, s) &\longmapsto \big(\phi_i(t_i, s), s\big)\,. \end{aligned}$$

Definition 2.65. With the above notations, we say that the deformation of the equation $\psi : (\mathscr{C}, p) \hookrightarrow (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$ of $(C, \mathbf{0})$ with section σ and the deformation of the parametrization $(\overline{\mathscr{D}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, p) \xrightarrow{\pi} (T, \mathbf{0})$ of (D, p) with compatible sections $\overline{\sigma}$ and σ are *equiintersectional (along σ)* if

$$\mathrm{ord}_{t_i}(F \circ \phi_i) = \mathrm{ord}_{t_i}(f \circ \varphi_i), \quad i = 1, \dots, r\,.$$

We call $\mathrm{ord}_{t_i}(F \circ \phi_i)$ the *intersection multiplicity of the deformations* (ψ, σ) and $(\phi, \overline{\sigma}, \sigma)$.

Remark 2.65.1. Let the base space $(T, \mathbf{0})$ be reduced. Then, for sufficiently small representatives,

$$\mathrm{ord}_{t_i}\big(F \circ \phi_i(t_i, s)\big) = i_{\sigma(s)}(\mathscr{C}_s, \mathscr{D}_{i,s})\,, \quad s \in T\,.$$

Here, $\mathscr{C}_s = \psi^{-1}(s)$ and $\mathscr{D}_{i,s} = \phi_i(\overline{\mathscr{D}}_i \cap (\mathbb{C} \times \{s\})$ are the fibres of $\psi : \mathscr{C} \to T$ and $\mathscr{D} \to T$ over s, where $\mathscr{D} = \phi(\overline{\mathscr{D}}) \to T$ is the induced deformation of the equation (Corollary 2.24).

Hence, for reduced base spaces, equiintersectional along σ means that the intersection number of $\mathscr{C}_s$ and $\mathscr{D}_{i,s}$ at $\sigma(s)$ is independent of $s \in T$ for $i = 1, \dots, r$.

Proposition 2.66. *Let $(D, \mathbf{0})$, $(L, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ be reduced curve singularities with $(L, \mathbf{0})$ smooth and not a component of $(D, \mathbf{0})$. Let*

$$\chi : (\mathscr{D}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) = (\mathbb{C}^2 \times T, \mathbf{0}) \to (T, \mathbf{0})$$

be an equisingular deformation of the equation of $(D, \mathbf{0})$ along the trivial section σ and let $(\overline{\mathscr{D}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be the deformation of the parametrization of $(D, \mathbf{0})$ with trivial (multi-)section $\overline{\sigma} : (T, \mathbf{0}) \to (\overline{\mathscr{D}}, \overline{0})$ induced by the equisingular deformation of the resolution of $(D, \mathbf{0})$ associated to (χ, σ). Assume that $(\phi, \overline{\sigma}, \sigma)$ is equisingular as deformation of the parametrization. Further, denote by $\psi : (\mathscr{L}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ the (trivial) deformation of $(L, \mathbf{0})$ along σ and by

$$\chi_L : (\mathscr{C}, \mathbf{0}) := (\mathscr{D}, \mathbf{0}) \cup (\mathscr{L}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$$

the induced deformation of the equation of $(C, \mathbf{0}) := (D, \mathbf{0}) \cup (L, \mathbf{0})$ along σ.

Then (χ_L, σ) is equisingular iff (ψ, σ) and $(\phi, \overline{\sigma}, \sigma)$ are equiintersectional along σ.

Proof. (1) Let (χ_L, σ) be equisingular. Since the statement is about the branches of $(D, \mathbf{0})$, we may assume that $(D, \mathbf{0})$ is irreducible. Choosing local analytic coordinates x, y of $(\mathbb{C}^2, \mathbf{0})$ and t of $(\mathbb{C}, 0)$, the map $\phi : (\overline{\mathscr{C}}, \overline{0}) = (\mathbb{C} \times T, \mathbf{0}) \to (\mathbb{C}^2 \times T, \mathbf{0})$ is given by

$$t \mapsto \big(X(t), Y(t)\big) \;\text{ with }\; X, Y \in \mathcal{O}_{T,\mathbf{0}}\{t\}$$

such that $\big(x(t),y(t)\big) := \big(X(t),Y(t)\big) \bmod \mathfrak{m}_{T,\mathbf{0}}$ parametrize $(D,\mathbf{0})$.

Since $(\phi,\overline{\sigma},\sigma)$ is equisingular, it is equimultiple along σ, that is,

$$\min\big\{\operatorname{ord}_t X(t), \operatorname{ord}_t Y(t)\big\} = m\,,$$

where $m = \min\big\{\operatorname{ord}_t x(t), \operatorname{ord}_t y(t)\big\}$ is the multiplicity of $(D,\mathbf{0})$.

We may choose the coordinates such that $x=0$ is an equation for $(L,\mathbf{0}) \subset (\mathbb{C}^2,\mathbf{0})$. Then $\operatorname{ord}_t X(t)$ is the intersection multiplicity of (ψ,σ) and $(\phi,\overline{\sigma},\sigma)$ and we have to show that $\operatorname{ord}_t X(t) = \operatorname{ord}_t x(t)$.

We prove this by induction on the number n of blowing ups needed to separate $(D,\mathbf{0})$ and $(L,\mathbf{0})$.

If $n=1$, then the germs $(D,\mathbf{0})$ and $(L,\mathbf{0})$ intersect transversally so that $m = \operatorname{mt}(D,\mathbf{0}) = i_{\mathbf{0}}(D,L) = \operatorname{ord}_t x(t)$. Since $(\phi,\overline{\sigma},\sigma)$ is equimultiple along σ, $\operatorname{ord}_t X(t) \geq m$ and, hence, $\operatorname{ord}_t X(t) = m = \operatorname{ord}_t x(t)$.

Now, let $n>1$ and consider the blowing up $\mathscr{M}^{(1)} \to \mathscr{M}$ of the trivial section σ (for a small representative $\mathscr{M}$ of $(\mathscr{M},\mathbf{0})$). Since $n>1$, there is a unique point $p = p^{(1)} \in \mathscr{M}^{(1)}$ belonging to $D \cap L$, and $(\mathscr{M}^{(1)},p) \cong (M^{(1)},p) \times (T,\mathbf{0})$ in the notation introduced right after Theorem 2.64.

We choose local coordinates u,v identifying $(M^{(1)},p)$ with $(\mathbb{C}^2,\mathbf{0})$. Then the (germ of the) blowing up

$$(\mathbb{C}^2 \times T,\mathbf{0}) \cong (\mathscr{M}^{(1)},p) \xrightarrow{\pi} (\mathscr{M},\mathbf{0}) = (\mathbb{C}^2 \times T,\mathbf{0})$$

is given by $(x,y) = (uv,v)$ and the identity on $(T,\mathbf{0})$. We assume again that $\sigma^{(1)} : (T,\mathbf{0}) \to (\mathscr{M}^{(1)},p)$ is the trivial section.

Let $(C^{(1)},p) = (D^{(1)},p) \cup (L^{(1)},p)$ be the strict transform of $(C,\mathbf{0})$. Then, by Remark 2.6.1 (5), p. 273,

$$(\mathscr{C}^{(1)},p) = (\mathscr{D}^{(1)} \cup \mathscr{L}^{(1)},p) \hookrightarrow (\mathscr{M}^{(1)},p) \to (T,\mathbf{0})$$

is an equisingular embedded deformation of $(C^{(1)},p)$ along $\sigma^{(1)}$. The induced deformation $\psi^{(1)} : (\mathscr{L}^{(1)},p) \hookrightarrow (\mathscr{M}^{(1)},p) \to (T,\mathbf{0})$ of $(L^{(1)},p)$ along $\sigma^{(1)}$ is denoted by $(\psi^{(1)},\sigma^{(1)})$.

Since $(\overline{\mathscr{D}},\overline{0}) \xrightarrow{\phi} (\mathscr{M},\mathbf{0}) \to (T,\mathbf{0})$ is induced by the equisingular deformation of the resolution, we have an induced map

$$(\overline{\mathscr{D}},\overline{0}) \xrightarrow{\phi^{(1)}} (\mathscr{M}^{(1)},p) \to (T,\mathbf{0})$$

such that $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ is an equisingular deformation of the parametrization of $(D^{(1)},p)$. Using the coordinates u,v, the map $\phi^{(1)} : (\mathbb{C} \times T,\mathbf{0}) \to (\mathbb{C}^2 \times T,\mathbf{0})$ is given by

$$t \mapsto \big(U(t),V(t)\big) \;\text{ with }\; U,V \in \mathcal{O}_{T,\mathbf{0}}\{t\}$$

such that $\big(u(t),v(t)\big) := \big(U(t),V(t)\big) \bmod \mathfrak{m}_{T,\mathbf{0}}$ parametrize $(D^{(1)},p)$.

Now, $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ satisfy the assumptions of the lemma and $(\mathscr{D}^{(1)},p)$ and $(\mathscr{L}^{(1)},p)$ are separated after $n-1$ blowing ups. Hence, by induction assumption, $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ are equiintersectional along $\sigma^{(1)}$.

Since we assumed that L is given as $x=0$, we get that $L^{(1)}$ and $D^{(1)}$ meet in the chart given by $(x,y)=(uv,v)$ and $L^{(1)}$ is given by $u=0$. Moreover, the exceptional divisor $\mathscr{E}^{(1)}$ in $(\mathbb{C}^2\times T,\mathbf{0})$ is given by $v=0$ and we have

$$X(t)=U(t)V(t)\,,\quad Y(t)=V(t)\,.$$

The assumption $n>1$ implies that $i_{\mathbf{0}}(L,D)=\mathrm{ord}_t\,x(t)>\mathrm{ord}_t\,y(t)=m$. Thus, $\mathrm{ord}_t\,V(t)=\mathrm{ord}_t\,Y(t)=m$. Since $\mathrm{ord}_t\,X(t)=\mathrm{ord}_t\,U(t)+\mathrm{ord}_t\,V(t)$ and $\mathrm{ord}_t\,x(t)=\mathrm{ord}_t\,u(t)+\mathrm{ord}_t\,v(t)$, we have to show that $\mathrm{ord}_t\,U(t)=\mathrm{ord}_t\,u(t)$.

Since $L^{(1)}$ is given by $u=0$, the intersection multiplicity of $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ is $\mathrm{ord}_t\,U(t)$. Since $(\psi^{(1)},\sigma^{(1)})$ and $(\phi^{(1)},\overline{\sigma},\sigma^{(1)})$ are equiintersectional along $\sigma^{(1)}$, we have $\mathrm{ord}_t\,U(t)=\mathrm{ord}_t\,u(t)$ as claimed.

(2) Let (ψ,σ) and $(\phi,\overline{\sigma},\sigma)$ be equiintersectional. Since $(L,\mathbf{0})$ is smooth, (ψ,σ) is equimultiple. Since (χ,σ) is equisingular, (χ_L,σ) is equimultiple, too.

Consider the equisingular deformation of the minimal embedded resolution of $(D,\mathbf{0})$ associated to (χ,σ). Then the deformation

$$(\mathscr{D}^{(\ell)}\cup\mathscr{E}^{(\ell)},p^{(\ell)})\hookrightarrow(\mathscr{M}^{(\ell)},p^{(\ell)})\to(T,\mathbf{0})$$

of the reduced total transform $(D^{(\ell)}\cup E^{(\ell)},p^{(\ell)})$ of $(D,\mathbf{0})$ is equimultiple along the (multi-)section $\sigma^{(\ell)}$. It remains to show that the deformation

$$(\mathscr{D}^{(\ell)}\cup\mathscr{L}^{(\ell)}\cup\mathscr{E}^{(\ell)},p^{(\ell)})\hookrightarrow(\mathscr{M}^{(\ell)},p^{(\ell)})\to(T,\mathbf{0})\tag{2.8.35}$$

of the reduced total transform $(D^{(\ell)}\cup L^{(\ell)}\cup E^{(\ell)},p^{(\ell)})$ of $(D\cup L,\mathbf{0})$ is equimultiple along $\sigma^{(\ell)}$ for $\ell\geq 1$.

We prove this claim again by induction on n, the number of blowing ups needed to separate $(D,\mathbf{0})$ and $(L,\mathbf{0})$.

If $n=1$, then $D^{(\ell)}$ and $L^{(\ell)}$ do not meet in $M^{(\ell)}$ for $\ell\geq 1$ and the claim is trivially true.

Let $n>1$ and $p\in M^{(1)}$ the unique intersection point of $L^{(1)}$ and $E^{(1)}$. Denote by $\Lambda_p\subset\{1,\dots,r\}$ the set of indices such that the strict transform $D_i^{(1)}$ of the i-th branch $(D_i,\mathbf{0})$ of $(D,\mathbf{0})$ passes through p for $i\in\Lambda_p$. Set

$$(D_p,\mathbf{0}):=\bigcup_{i\in\Lambda_p}(D_i,\mathbf{0})\,,$$

and let $(D^{(1)},p)$ be the strict transform of $(D_p,\mathbf{0})$.

Choose coordinates x,y of $(\mathbb{C}^2,\mathbf{0})$ and u,v of $(M^{(1)},p)$, and let $x=0$ be the equation of $(L,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$. Since $n>1$, we have the relation $x=uv$ and $y=v$ and $(L^{(1)},p)\subset(M^{(1)},p)$ is given by $u=0$.

Let $\phi_i:(\overline{\mathscr{D}}_i,\overline{0}_i)\to(\mathscr{M},\mathbf{0})$, resp. $\phi_i^{(1)}:(\overline{\mathscr{D}}_i,\overline{0}_i)\to(\mathscr{M}^{(1)},p)$, be the deformation of the parametrization of $(D_i,\mathbf{0})$, resp. $(D_i^{(1)},p)$ (along the trivial section $\sigma_p^{(1)}$), $i\in\Lambda_p$, given by $t_i\mapsto\big(X_i(t_i),Y_i(t_i)\big)$, resp. $t_i\mapsto\big(U_i(t_i),V_i(t_i)\big)$. We have the relations

$$X_i(t_i) = U_i(t_i)V_i(t_i)\,, \quad Y_i(t_i) = V_i(t_i)\,, \quad i \in \Lambda_p\,,$$

and the same for the reductions mod $\mathfrak{m}_{(T,\mathbf{0})}$, $\big(x_i(t_i), y_i(t_i)\big)$, resp. $\big(u_i(t_i), v_i(t_i)\big)$, which are the paramerizations of $(D_i, \mathbf{0})$, resp. $(D_i^{(1)}, p)$.

For $i \in \Lambda_p$, the smooth germ $(L, \mathbf{0})$ is tangent to $(D_i, \mathbf{0})$ and, hence,

$$i_{\mathbf{0}}(L, D_i) = \operatorname{ord}_{t_i} x_i(t_i) > \operatorname{ord}_{t_i} y_i(t_i) =: m_i = \operatorname{mt}(D_i, \mathbf{0})\,.$$

Since ϕ_i is equimultiple along σ, $\operatorname{ord}_{t_i} Y_i(t_i) = m_i$ and, since (ψ, σ) and $(\phi, \overline{\sigma}, \sigma)$ are equiintersectional, $\operatorname{ord}_{t_i} X_i(t_i) = \operatorname{ord}_{t_i} x_i(t_i)$. Since, by the above relations, $\operatorname{ord}_{t_i} X_i(t_i) = \operatorname{ord}_{t_i} U_i(t_i) + m_i$ and $\operatorname{ord}_{t_i} x_i(t_i) = \operatorname{ord}_{t_i} u_i(t_i) + m_i$, we get $\operatorname{ord}_{t_i} U_i(t_i) = \operatorname{ord}_{t_i} u_i(t_i)$ for all $i \in \Lambda_p$.

Since $u = 0$ is the equation of the trivial deformation

$$\psi^{(1)} \colon (\mathscr{L}^{(1)}, p) \hookrightarrow (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})\,,$$

it follows that $(\psi^{(1)}, \sigma_p^{(1)})$ and $\big(\phi_p^{(1)} = \coprod_{i \in \Lambda_p} \phi_i^{(1)}, \overline{\sigma}_p, \sigma_p^{(1)}\big)$ are equiintersectional. Hence, we can apply the induction hypothesis to $(D^{(1)} \cup L^{(1)}, p)$ and it follows that the deformation (2.8.35) is equimultiple along $\sigma^{(\ell)}$ for all $\ell \geq 1$ as claimed. $\square$

Proof of Theorem 2.64. (2) Let $\psi : (\mathscr{C}, \mathbf{0}) \hookrightarrow (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be an embedded equisingular deformation of the equation along σ, and let

$$(\overline{\mathscr{C}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$$

be the deformation of the parametrization induced by the equisingular deformation of the resolution along sections $\overline{\sigma}$, σ.

We prove that $(\phi, \overline{\sigma}, \sigma)$ is equisingular by induction on the number $N = N(C, \mathbf{0})$ of blowing ups needed to obtain a minimal embedded resolution of $(C, \mathbf{0})$.

If $N = 0$, then $(C, \mathbf{0})$ is smooth and every deformation is equisingular. Thus, let $N > 0$.

We consider the blowing up $(\mathscr{M}^{(1)}, p^{(1)}) \to (\mathscr{M}, \mathbf{0})$ of $(\mathscr{M}, \mathbf{0})$ along σ, with the uniquely determined equimultiple sections $\sigma_p^{(1)} : (T, \mathbf{0}) \to (\mathscr{M}^{(1)}, p)$ for each $p \in p^{(1)}$ (see Proposition 2.8, p. 275). Let $(\mathscr{C}^{(1)}, p)$, resp. $(\mathscr{E}^{(1)}, p)$, denote the strict transform of $(\mathscr{C}, \mathbf{0})$, resp. the exceptional divisor. By Remark 2.6.1 (5), $(\mathscr{C}^{(1)} \cup \mathscr{E}^{(1)}, p) \hookrightarrow (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})$ is an equisingular embedded deformation of the reduced total transform $(C^{(1)} \cup E^{(1)}, p)$ of $(C, \mathbf{0})$.

Moreover, by induction hypothesis ($N(C^{(1)}, p) < N$), the induced map

$$(\overline{\mathscr{C}}, \overline{p}) \xrightarrow{\phi^{(1)}} (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})$$

together with the sections $\overline{\sigma}_p$ and $\sigma_p^{(1)}$ defines an equisingular deformation of the parametrization of $(C^{(1)}, p)$. To show that $(\phi, \overline{\sigma}, \sigma)$ is equisingular, it remains to show that $\phi = (\phi_i)_{i=1..r}$ is equimultiple along $\overline{\sigma}, \sigma$ (see Remark 2.36.1 (1)).

Choosing coordinates and using the notations as in the proof of Proposition 2.66 with all sections trivial,

$$\phi_i : (\mathbb{C} \times T, \mathbf{0}) \cong (\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{M}, \mathbf{0}) \cong (\mathbb{C}^2 \times T, \mathbf{0})$$

is given by $X_i(t_i)$, $Y_i(t_i)$ and we have to show that

$$\min\{\mathrm{ord}_{t_i} X_i(t_i), \mathrm{ord}_{t_i} Y_i(t_i)\} = \min\{\mathrm{ord}_{t_i} x_i(t_i), \mathrm{ord}_{t_i} y_i(t_i)\} =: m_i .$$

Let $(C_i^{(1)}, p_i) \subset (M^{(1)}, p_i)$ be the strict transform of $(C_i, \mathbf{0})$.

Choosing coordinates u, v of $(M^{(1)}, p_i) \cong (\mathbb{C}^2, \mathbf{0})$,

$$\phi_i^{(1)} : (\mathbb{C} \times T, \mathbf{0}) = (\overline{\mathscr{C}}_i, \overline{0}_i) \to (\mathscr{M}^{(1)}, p_i) \cong (\mathbb{C}^2 \times T, \mathbf{0})$$

is given by $U_i(t_i), V_i(t_i) \in \mathcal{O}_{T,\mathbf{0}}\{t_i\}$ defining an equimultiple deformation of the parametrization of $(C_i^{(1)}, p_i)$.

In the two charts covering $\mathscr{M}^{(1)}$, we have $(x, y) = (u, uv)$, resp. $(x, y) = (uv, v)$, depending on $p_i \in E^{(1)} = \mathbb{P}^1$. We may assume that $\{x = 0\}$ is tangent to the branch $(C_i, \mathbf{0})$. Then $\{y = 0\}$ is transversal to $(C_i, \mathbf{0})$, hence $m_i = \mathrm{ord}_{t_i} y_i(t_i)$. Since $\{x = 0\}$ and $(C_i, \mathbf{0})$ are not separated by blowing up $\mathbf{0}$ in $(\mathbb{C}^2, \mathbf{0})$, $p_i = \mathbf{0}$ in the chart given by $(x, y) = (uv, v)$ and $\mathscr{E}^{(1)}$ is given by $v = 0$ in $(\mathbb{C}^2 \times T, \mathbf{0})$.

Now, we apply Proposition 2.66 with $L = E^{(1)}$ to the deformation of the parametrization $(\phi_i^{(1)}, \overline{\sigma}_i, \sigma_i^{(1)})$ of $(C_i^{(1)}, p_i)$ and to the embedded deformation

$$(\mathscr{C}_i^{(1)} \cup \mathscr{E}^{(1)}, p) \hookrightarrow (\mathscr{M}^{(1)}, p) \to (T, \mathbf{0})$$

of $(C_i^{(1)} \cup E^{(1)}, p)$ and get that they are equiintersectional.

Since $\mathscr{E}^{(1)}$ is defined by v, equiintersectional means that

$$\mathrm{ord}_{t_i} V_i(t_i) = \mathrm{ord}_{t_i} v_i(t_i) .$$

Since we have the relations $x_i(t_i) = u_i(t_i) v_i(t_i)$, $y_i(t_i) = v_i(t_i)$, and

$$X_i(t_i) = U_i(t_i) V_i(t_i) , \quad Y_i(t_i) = V_i(t_i) ,$$

we get $m_i = \mathrm{ord}_{t_i} y_i(t_i) = \mathrm{ord}_{t_i} v_i(t_i) = \mathrm{ord}_{t_i} Y_i(t_i) \leq \mathrm{ord}_{t_i} X_i(t_i)$. This proves that ϕ_i is equimultiple along $\overline{\sigma}_i$, $\sigma_i^{(1)}$, $i = 1, \ldots, r$, which had to be shown.

(1) Let $(\overline{\mathscr{C}}, \overline{0}) \xrightarrow{\phi} (\mathscr{M}, \mathbf{0}) \to (T, \mathbf{0})$ be an equisingular deformation of the parametrization with section $\overline{\sigma}, \sigma$ and $(\overline{\mathscr{C}}, \overline{0}) \to (\mathscr{C}, \mathbf{0}) \xrightarrow{\psi} (T, \mathbf{0})$ the induced deformation of the normalization, which yields a functor (by Proposition 2.23, p. 301). We have to show that $\psi : (\mathscr{C}, \mathbf{0}) \to (T, \mathbf{0})$ together with the section σ is equisingular in the sense of Definition 2.6.

We may assume that $\overline{\sigma}$ and σ are trivial sections. We argue by induction on the number of blowing ups needed to resolve the singularity $(C,\mathbf{0})$. The case $(C,\mathbf{0})$ being smooth is trivial. In the general case, Lemma 2.26, p. 303 yields that $(\mathscr{C},\mathbf{0})\to(T,\mathbf{0})$ is equimultiple and we may consider the blowing up of $(\mathscr{M},\mathbf{0})$ along σ,

$$\begin{tikzcd}
(\widetilde{\mathscr{C}},\widetilde{p}) \arrow[r, "\pi"] \arrow[d, hook] & (\mathscr{C},\mathbf{0}) \arrow[d, hook] \arrow[dr] & \\
(\widetilde{\mathscr{M}},\widetilde{p}) \arrow[r] & (\mathscr{M},\mathbf{0}) \arrow[r] & (T,\mathbf{0})\,,
\end{tikzcd}$$

where $(\widetilde{\mathscr{C}},\widetilde{p})$ is the (multi)germ of the strict transform of $(\mathscr{C},\mathbf{0})$. By Definition 2.36 and Proposition 2.23, there is a morphism $\widetilde{\phi}:(\overline{\mathscr{C}},\overline{0})\to(\widetilde{\mathscr{C}},\widetilde{p})\to(T,\mathbf{0})$ and a (multi)section $\widetilde{\sigma}:(T,\mathbf{0})\to(\widetilde{\mathscr{C}},\widetilde{p})$ such that $(\widetilde{\phi},\overline{\sigma},\widetilde{\sigma})$ is an equisingular deformation of the parametrization of $(\widetilde{\mathscr{C}},\widetilde{p})$. By induction hypothesis, for every $p\in\widetilde{p}=\pi^{-1}(\mathbf{0})$, $(\widetilde{\mathscr{C}},p)\to(T,\mathbf{0})$ is an equisingular deformation of the equation of the strict transform $(\widetilde{C},p)$ of $(C,\mathbf{0})$ along $\widetilde{\sigma}$ by Lemma 2.26.

Let $\mathscr{E}\subset\widetilde{\mathscr{M}}$ be the exceptional divisor. We have to show that, for each $p\in\widetilde{p}$,

$$(\widetilde{\mathscr{C}}\cup\mathscr{E},p)\hookrightarrow(\widetilde{\mathscr{M}},p)\to(T,\mathbf{0})$$

is an equisingular embedded deformation of the reduced total transform $(\widetilde{C}\cup E,p)$ along $\widetilde{\sigma}_p:(T,\mathbf{0})\to(\widetilde{\mathscr{M}},p)$. By Proposition 2.66, we have to show that the deformations $(\widetilde{\phi},\overline{\sigma},\sigma)$ and (ψ,σ) with $\psi:(\mathscr{E},p)\hookrightarrow(\widetilde{\mathscr{M}},p)\to(T,\mathbf{0})$ are equiintersectional along σ.

We choose coordinates x,y of $(M,\mathbf{0})=(\mathbb{C}^2,\mathbf{0})$ and u,v of $(\widetilde{M},p)=(\mathbb{C}^2,\mathbf{0})$ as in the proof of (2) and consider a branch $(\widetilde{C}_i,p_i)$ of $(\widetilde{C},p)$. Assuming that $\{x=0\}$ is tangent to $(C_i,\mathbf{0})$, we have $m_i:=\mathrm{mt}(C_i,\mathbf{0})=\mathrm{ord}_{t_i}\,y_i(t_i)$. As in the proof of (2), we have that the deformations of the parametrization $\widetilde{\phi}_i$, resp. ϕ_i, of $(\widetilde{C}_i,p_i)$, resp. $(C_i,\mathbf{0})$, are given by $U_i(t_i)$, $V_i(t_i)$, resp. by $X_i(t_i)$, $Y_i(t_i)$, satisfying the relations $X_i(t_i)=U_i(t_i)V_i(t_i)$ and $Y_i(t_i)=V_i(t_i)$. Since ϕ_i is equimultiple along the trivial sections $\overline{\sigma}_i,\sigma$, we have

$$\mathrm{ord}_{t_i}\,Y_i(t_i)=\mathrm{ord}_{t_i}\,y_i(t_i)=\mathrm{ord}_{t_i}\,v_i(t_i)=\mathrm{ord}_{t_i}\,V_i(t_i)\,.$$

This proves that (ψ,σ) and $(\widetilde{\phi}_i,\overline{\sigma}_i,\sigma_i)$ are equiintersectional along σ and hence (1).

(3) By (1), we have a natural transformation $\underline{\mathcal{D}ef}^{\,es}_{\overline{C}\to C}\to\underline{\mathcal{D}ef}^{\,es}_{C}$. It is easy to see that the equisingular deformation of the parametrization in (2) is unique up to isomorphism. This proves the claim. □

Corollary 2.67. *Let $(C,\mathbf{0})\subset(\mathbb{C}^2,\mathbf{0})$ be a reduced plane curve singularity and let $i:(\Delta^\mu,\mathbf{0})\hookrightarrow(B_C,\mathbf{0})$ be the inclusion of the μ-constant stratum in the base space of the semiuniversal deformation of $(C,\mathbf{0})$. Then the restriction of* *the semiuniversal deformation* $(\mathscr{C}, \mathbf{0}) \to (B_C, \mathbf{0})$ *to* $(\Delta^\mu, \mathbf{0})$ *is an equisingular semiuniversal deformation of* $(C, \mathbf{0})$*, that is,* $i^*(\mathscr{C}, \mathbf{0}) \to (\Delta^\mu, \mathbf{0})$ *is isomorphic to* $(\mathscr{C}^{es}_C, \mathbf{0}) \to (B^{es}_C, \mathbf{0})$.

Proof. By Theorem 2.61, $i^*(\mathscr{C}, \mathbf{0}) \to (\Delta^\mu, \mathbf{0})$ lifts to a semiuniversal equisingular deformation of the parametrization $(\overline{\mathscr{C}}^{es}, \overline{0}) \to i^*(\mathscr{C}, \mathbf{0}) \to (\Delta^\mu, \mathbf{0})$ and, therefore, the result follows from Theorem 2.64. □

As an immediate consequence, we obtain:

Corollary 2.68. *A deformation of the equation of* $(C, \mathbf{0})$ *over a reduced base* $(T, \mathbf{0})$ *is equisingular iff, for sufficiently small representatives, the Milnor number is constant (along the unique singular section).*

For a reduced plane curve singularity $(C, \mathbf{0})$ with local equation $f \in \mathbb{C}\{x, y\}$, we introduce

$$\tau^{es}(C, \mathbf{0}) := \tau(C, \mathbf{0}) - \dim_{\mathbb{C}} T^{1,es}(C, \mathbf{0}) = \dim_{\mathbb{C}}(\mathbb{C}\{x, y\}/I^{es}(f)),$$

which is equal to the codimension of the μ-constant stratum $(\Delta^\mu, \mathbf{0})$ in the base of the semiuniversal deformation of $(C, \mathbf{0})$ (Theorem 2.64 and Proposition 2.63).

One of the reasons why equisingular deformations of the parametrization are so easy is that they form a linear subspace in the base space of the semiuniversal deformation of the parametrization (Theorem 2.38). This is in general not the case for equisingular deformation of the equation (see Example 2.71.1 below). Hence, the question arises whether there are singularities for which the μ-constant stratum is linear. The answer was given in [Wah]:

Proposition 2.69 (Wahl). *Let* $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ *be a reduced plane curve singularity with local equation* f*. Then the following are equivalent:*

(a) There are $\tau' = \tau(C, \mathbf{0}) - \tau^{es}(C, \mathbf{0})$ *elements* $g_1, \ldots, g_{\tau'} \in I^{es}(f)$ *such that*

$$\varphi^{es} : V\Big(f + \sum\nolimits_i t_i g_i\Big) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'}, \mathbf{0}) \xrightarrow{\mathrm{pr}} (\mathbb{C}^{\tau'}, \mathbf{0})$$

is a semiuniversal equisingular deformation for $(C, \mathbf{0})$.

(b) Let $g_1, \ldots, g_{\tau'} \in I^{es}(f)$ *induce a basis for* $I^{es}(f)/\langle f, j(f)\rangle$*. Then*

$$\varphi^{es} : V\Big(f + \sum\nolimits_i t_i g_i\Big) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'}, \mathbf{0}) \xrightarrow{\mathrm{pr}} (\mathbb{C}^{\tau'}, \mathbf{0}),$$

is a semiuniversal equisingular deformation for $(C, \mathbf{0})$.

(c) Each equisingular deformation of $(C, \mathbf{0})$ *is isomorphic to an equisingular deformation where all the equimultiple sections* $\sigma^{(\ell)}_j$ *through non-nodes of*

the reduced total transform $C^{(\ell)} \cup E^{(\ell)} \subset M^{(\ell)}$, $\ell = 1, \dots, N$, *of* $(C, \mathbf{0})$ *are globally trivial sections.*[17]

(d) *Each locally trivial deformation of the reduced exceptional divisor* E *of a minimal embedded resolution of* $(C, \mathbf{0}) \subset (\mathbb{C}^2 \times \{\mathbf{0}\}, \mathbf{0})$ *is trivial.*

(e) $I^{es}(f) = \langle f, j(f), I^s(f) \rangle$.[18]

Our construction implies the following "openness of versality" result for equisingular deformations: Call a flat morphism $\phi : \mathscr{C} \to S$ of complex spaces a *family of reduced plane curve singularities* if the restriction of ϕ to $\mathrm{Sing}(\phi)$ is finite and if, for each $s \in S$ and each $x \in \mathscr{C}_s := \phi^{-1}(s)$, there is an isomorphism of germs $(\mathscr{C}, x) \cong (\mathbb{C}^2, \mathbf{0})$ mapping $(\mathscr{C}_s, x)$ to the germ of a reduced plane curve singularity in $(\mathbb{C}^2, \mathbf{0})$.

If $\sigma = (\sigma^{(1)}, \dots, \sigma^{(\ell)})$ is a system of disjoint sections $\sigma^{(i)} : S \to \mathscr{C}$ of ϕ, then we call the family ϕ *equisingular* (resp. *equisingular-versal*) *at* $s \in S$ *along* σ if the induced morphism of germs $\phi : \big(\mathscr{C}, \sigma^{(i)}(s)\big) \to (S, s)$ is an equisingular (resp. equisingular-versal) deformation of $\big(\mathscr{C}_s, \sigma^{(i)}(s)\big)$ for $i = 1, \dots, \ell$.

Combining Theorems 2.64 and 2.43, we get openness of equisingular-versality:

Theorem 2.70. *Let* $\phi : \mathscr{C} \to S$ *be a family of reduced plane curve singularities which is an equisingular family at* s *along* σ *for all* $s \in S$. *Then the set of points* $s \in S$ *such that* ϕ *is equisingular-versal at* s *is analytically open in* S.

Let us conclude with formulating explicitely how equisingular deformations look like for semiquasihomogeneous and Newton non-degenerate singularities. In fact, Propositions 2.17 and 2.69 imply that for semiquasihomogeneous and for Newton non-degenerate plane curve singularities, the semiuniversal equisingular deformation of the equation is completely determined by its tangent space:

Corollary 2.71. *Let* $(C, \mathbf{0}) \subset (\mathbb{C}^2, \mathbf{0})$ *be a reduced plane curve singularity with local equation* $f \in \mathbb{C}\{x, y\}$, *and let* $\tau' = \tau(C, \mathbf{0}) - \tau^{es}(C, \mathbf{0})$.

(a) *If* $f = f_0 + f'$ *is semiquasihomogeneous with principal part* f_0 *being quasihomogeneous of type* $(w_1, w_2; d)$, *then a semiuniversal equisingular deformation for* $(C, \mathbf{0})$ *is given by*

$$\varphi^{es} : V\left(f + \sum\nolimits_{i=1}^{\tau'} t_i g_i\right) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'}, \mathbf{0}) \xrightarrow{\mathrm{pr}} (\mathbb{C}^{\tau'}, \mathbf{0}),$$

[17] See Definition 2.6, p. 271, for notations. Since the reduced total transform contains the (compact) exceptional divisors, there are obstructions against the global trivialization (that is, by an isomorphism of a neighbourhood of the exceptional divisors) of the sections, for instance by the cross-ratio of more than three sections through one exceptional component.

[18] For the definition of $I^s(f)$, see Remark 2.17.1, p. 288.

where $g_1, \ldots, g_{\tau'}$ *represent a* $\mathbb{C}$*-basis for the quotient*

$$\langle j(f),\, x^\alpha y^\beta \mid w_1\alpha + w_2\beta \geq d\rangle / j(f)\,.$$

(b) If f *is Newton non-degenerate with Newton diagram* $\Gamma(f,0)$ *at the origin, then a semiuniversal equisingular deformation for* $(C,\mathbf{0})$ *is given by*

$$\varphi^{es} : V\left(f + \sum\nolimits_{i=1}^{\tau'} t_i g_i\right) \subset (\mathbb{C}^2 \times \mathbb{C}^{\tau'}, \mathbf{0}) \xrightarrow{\text{pr}} (\mathbb{C}^{\tau'}, \mathbf{0})\,,$$

where $g_1, \ldots, g_{\tau'}$ *represent a* $\mathbb{C}$*-basis for the quotient*

$$\langle j(f),\, x^\alpha y^\beta \mid x^\alpha y^\beta \text{ has Newton order } \geq 1\rangle / j(f)\,.$$

Moreover, in both cases each equisingular deformation of $(C,0)$ *is isomorphic to an equisingular deformation where all the equimultiple sections through non-nodes of the reduced total transform of* $(C,0)$ *are trivial sections.*

A.N. Varchenko proved that the last statement holds for equisingular deformations of isolated semiquasihomogeneous hypersurface singularities of arbitrary dimension [Var, Thm. 2]. In particular, if $f \in \mathbb{C}\{\boldsymbol{x}\} = \mathbb{C}\{x_1, \ldots, x_n\}$ is a convenient semiquasihomogeneous power series, then Varchenko's result says that all fibres of a μ-constant deformation of the singularity defined by f are semiquasihomogeneous of the same type (see [Var]). The analogous statement for Newton non-degenerate hypersurface singularities does not hold for $n \geq 3$. In fact, the Newton diagram of the fibres may vary in a μ-constant deformation of a Newton non-degenerate singularity (see, for instance, [Dim, Example 2.14]).

Remarks and Exercises

Using Gabrielov's result ([Gab1]) which states that the *modality* of the function f with respect to right equivalence is equal to the dimension of the μ-constant stratum of f in the (μ-dimensional) semiuniversal unfolding of f, we get

$$\tau^{es}(C,\mathbf{0}) = \mu(C,\mathbf{0}) - \text{modality}(f)\,.$$

In fact, the semiuniversal unfolding of f being a versal deformation of $(C,\mathbf{0})$, this formula follows from Theorem 2.64, since the codimension of the μ-constant stratum in any versal deformation of $(C,\mathbf{0})$ is the same.

Alternatively, in terms of the minimal free resolution of $(C,\mathbf{0})$, the codimension $\tau^{es}(C,\mathbf{0})$ can be computed as

$$\tau^{es}(C,\mathbf{0}) = \sum_q \frac{m_q(m_q+1)}{2} - \#\{q \mid q \text{ is a free point}\} - 1\,, \qquad (2.8.36)$$

where the sum extends over all infinitely near points to $\mathbf{0}$ belonging to $(C,\mathbf{0})$ which appear when resolving the plane curve singularity $(C,\mathbf{0})$, and m_q denotes the multiplicity of the strict transform of $(C,\mathbf{0})$ at q. Here, an infinitely near point is called *free* if it lies on at most one component of the exceptional divisor. The computation of $\tau^{es}(C,\mathbf{0})$ by Formula (2.8.36) is implemented in SINGULAR and accessible via the `tau_es` command. For instance, continuing the SINGULAR session of Example 2.63.1, we get:

```
tau_es(f);           // compute tau^es by the formula
//-> 38
vdim(std(Ies[1])); // compute tau^es as codimension of I^es(f)
//-> 38
```

Comparing the equisingularity ideal $I^{es}(f)$ with the equiclassical ideal $I^{ec}(f)$ and the equigeneric ideal $I^{eg}(f)$ (see [DiH]), we can give an estimate for $\tau^{es}(C,\mathbf{0})$ in terms of the "classical" invariants δ and κ:

$$\kappa(C,\mathbf{0}) - \delta(C,\mathbf{0}) \leq \tau^{es}(C,\mathbf{0}) \leq \kappa(C,\mathbf{0}) \leq 2\tau^{es}(C,\mathbf{0})\,.$$

In fact, the vector spaces $I^{ec}(f)/\langle f, j(f)\rangle$ and $I^{eg}(f)/\langle f, j(f)\rangle$ are isomorphic to the tangent cones of the germs of the (κ,δ)-constant stratum (that is, the stratum where κ and δ are both constant), and the δ-constant stratum, respectively. Since equisingular deformations preserve the multiplicities of the successive strict transforms, δ and $\kappa = \mu - \mathrm{mt} + 1$ (Propositions I.3.34 and I.3.38) are constant under such deformations. Therefore, the equisingularity stratum is contained in the equiclassical stratum and the same holds for the tangent cones. For a smooth germ, the tangent cone is the same as the tangent space and therefore we have

$$j(f) \subset \langle f, j(f)\rangle \subset I^{es}(f) \subset I^{ec}(f) \subset I^{eg}(f)\,. \tag{2.8.37}$$

The above estimate follows then from the dimension formulas

$$\begin{aligned}
\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/I^{es}(f) &= \tau^{es}(C,0)\,,\\
\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/I^{ec}(f) &= \kappa(C,0) - \delta(C,0)\,,\\
\dim_{\mathbb{C}} \mathbb{C}\{x,y\}/I^{eg}(f) &= \delta(C,0)\,.
\end{aligned}$$

Exercise 2.8.1. Compute the Milnor number, Tjurina number, τ^{es}, and modality for the singularities at the origin of $\{x^m + y^n = 0\}$, $\{x^m y + y^n = 0\}$, resp. $\{x^m y + xy^n = 0\}$, $m, n \geq 2$.

In [CGL1], we give an algorithm which, given a deformation with section of a reduced plane curve singularity, computes equations for the equisingularity stratum (that is, the μ-constant stratum in characteristic 0) in the parameter space of the deformation. The algorithm works for any, not necessarily reduced, parameter space and for algebroid curve singularities C defined over an algebraically closed field of characteristic 0 (or of characteristic $p > \mathrm{ord}(C)$). It has been implemented in the SINGULAR library `equising.lib`. The following example shows the implemented algorithm at work.

Example 2.71.1. Consider the reduced (Newton degenerate) plane curve singularity with local equation $f = (y^4 - x^4)^2 - x^{10}$. We compute equations for the μ-constant stratum in the base space of the semiuniversal deformation with section of $(C, \mathbf{0})$ where the section is trivialized (for more details see [CGL1]):

```
LIB "equising.lib";      //loads deform.lib, sing.lib, too
ring R = 0, (x,y), ls;
poly f = (y4-x4)^2 - x10;
ideal J = f, maxideal(1)*jacob(f);
ideal KbJ = kbase(std(J));
int N = size(KbJ);
N;                       //number of deformation parameters
//-> 50
ring Px = 0, (a(1..N),x,y), ls;
matrix A[N][1] = a(1..N);
poly F = imap(R,f)+(matrix(imap(R,KbJ))*A)[1,1];
list M = esStratum(F);   //compute the stratum of equisingularity
                         //along the trivial section
def ESSring = M[1]; setring ESSring;
option(redSB);
ES = std(ES);
size(ES);                //number of equations for ES stratum
//-> 44
```

Inspecting the elements of `ES`, we see that 42 of the 50 deformation parameters must vanish. Additionally, there are two non-linear equations, showing that the equisingularity (μ-constant) stratum is smooth (of dimension 6) but not linear:

```
ES[9];
//-> 8*a(42)+a(2)*a(24)-a(2)^2
ES[26];
//-> 8*a(24)+8*a(2)+a(2)^3
```

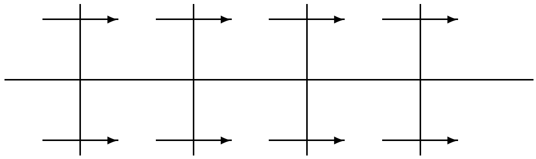
The correctness of the computed equations can be checked by choosing a random point $\boldsymbol{p}$ satisfying the equations and computing the system of Hamburger-Noether expansions for the evaluation of F at $\boldsymbol{s} = \boldsymbol{p}$. From the system of Hamburger-Noether expansions, we can read a complete set of numerical invariants of the equisingularity type (such as the Puiseux pairs and the intersection numbers) which have to coincide with the respective invariants of f. In characteristic 0, it suffices to compare the two Milnor numbers. To do this, we reduce `F` by `ES` and evaluate the result at a random point satisfying the above two non-linear conditions:

```
poly F = reduce(imap(Px,F),ES);  //a(2),a(24) both appear in F
poly g = subst(F, a(24), -a(2)-(1/8)*a(2)^3);
for (int ii=1; ii<=44; ii++){ g = subst(g,a(ii),random(1,100)); }
setring R;
```

```
milnor(f);                        //Milnor number of f
//-> 57
milnor(imap(ESSring,g));          //Milnor number of g
//-> 57
```

Finally, we show that for the reduced plane curve singularity with local equation $f = (y^4 - x^4)^2 - x^{10}$ none of the properties (a) – (e) of Proposition 2.69 is satisfied.

Its reduced total transform has the form



(lines and arrows indicating components of the exceptional divisor and the strict transform, respectively). In particular, since the cross-ratio of the 4 intersection points of components of the exceptional divisor E is preserved by a trivial deformation, (d) is not satisfied.

To see the failure of (c), consider the equisingular deformation

$$F = \left(y^4 - x^4 + t \cdot x^2 y^2\right)^2 - x^{10}.$$

Since F induces a locally trivial deformation of E which varies the cross-ratio of the four intersection points, it cannot be isomorphic to an equisingular deformation with trivial equimultiple sections $\sigma_j^{(i)}$.

Property (e) fails, too:

```
LIB "equising.lib";
ring R = 0,(x,y),ds;
poly f = (y4-x4)^2-x10;
list Ies = esIdeal(f,1);
Ies[3];                         // the ideal <f,j(f),I^s(f)>
//-> _[1]=x3y7
//-> _[2]=x2y8
//-> _[3]=xy9
//-> _[4]=y10
//-> _[5]=x8-2x4y4+y8-x10
//-> _[6]=8x7-8x3y4-10x9
//-> _[7]=-8x4y3+8y7
ideal J = std(Ies[3]);          // compute standard basis
size(reduce(maxideal(10),J));   // m^10 in <f,j(f),I^s(f)>?
//-> 0
vdim(J);                        // dim_C C{x,y}/<f,j(f),I^s(f)>
//-> 43
vdim(std(Ies[1]));              // dim_C C{x,y}/<f,j(f),I^es(f)>
//-> 42
simplify(reduce(Ies[1],J),10);
//-> _[1]=x6y2-x2y6
```

From the output, we read that $\langle f, j(f), I^s(f)\rangle = \langle f, j(f), \mathfrak{m}^{10}\rangle$, while, as complex vector space, the equisingularity ideal is generated by $\langle f, j(f), \mathfrak{m}^{10}\rangle$ and the polynomial $x^2y^2(y^4 - x^4) \notin \langle f, j(f), \mathfrak{m}^{10}\rangle$.

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