

## Subdifferentials of Lower Semicontinuous Functionals

### 9.1 Fréchet Subdifferentials: First Properties

In this section we study another kind of derivative-like concepts.

**Definition 9.1.1** Assume that  $E$  is a Banach space,  $f : E \rightarrow \overline{\mathbb{R}}$  is proper and l.s.c., and  $\bar{x} \in \text{dom } f$ .

- (a) The functional  $f$  is said to be *Fréchet subdifferentiable* (*F-subdifferentiable*) at  $\bar{x}$  if there exists  $x^* \in E^*$ , the *F-subderivative* of  $f$  at  $\bar{x}$ , such that

$$\liminf_{y \rightarrow 0} \frac{f(\bar{x} + y) - f(\bar{x}) - \langle x^*, y \rangle}{\|y\|} \geq 0. \quad (9.1)$$

- (b) The functional  $f$  is said to be *viscosity subdifferentiable* at  $\bar{x}$  if there exist  $x^* \in E^*$ , the *viscosity subderivative* of  $f$  at  $\bar{x}$ , and a  $C^1$ -function  $g : E \rightarrow \mathbb{R}$  such that  $g'(\bar{x}) = x^*$  and  $f - g$  attains a local minimum at  $\bar{x}$ . If, in particular,

$$g(x) = \langle x^*, x - \bar{x} \rangle - \sigma \|x - \bar{x}\|^2$$

with some positive constant  $\sigma$ , then  $x^*$  is called *proximal subgradient* of  $f$  at  $\bar{x}$ . The sets

$$\partial_F f(\bar{x}) := \text{set of all F-subderivatives of } f \text{ at } \bar{x},$$

$$\partial_V f(\bar{x}) := \text{set of all viscosity subderivatives of } f \text{ at } \bar{x},$$

$$\partial_P f(\bar{x}) := \text{set of all proximal subgradients of } f \text{ at } \bar{x}$$

are called *Fréchet subdifferential* (*F-subdifferential*), *viscosity subdifferential*, and *proximal subdifferential* of  $f$  at  $\bar{x}$ , respectively.

**Remark 9.1.2** Observe that the function  $g$  in Definition 9.1.1(b) can always be chosen such that  $(f - g)(\bar{x}) = 0$  (cf. Fig. 9.1).

We study the relationship between the different notions.

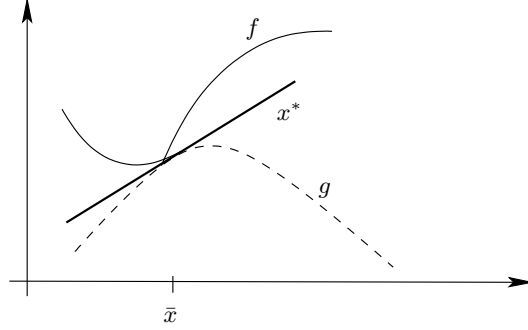


Fig. 9.1

**Proposition 9.1.3** Assume that  $E$  is a Banach space,  $f : E \rightarrow \overline{\mathbb{R}}$  is proper and l.s.c., and  $\bar{x} \in \text{dom } f$ . Then  $\partial_V f(\bar{x}) \subseteq \partial_F f(\bar{x})$ .

*Proof.* See Exercise 9.8.1. □

**Remark 9.1.4** Notice that  $\partial_F f(\bar{x})$  and  $\partial_V f(\bar{x})$  can be defined as above for any proper, not necessarily l.s.c. functional  $f$ . However, if  $\partial_F f(\bar{x})$  (in particular,  $\partial_V f(\bar{x})$ ) is nonempty, then in fact  $f$  is l.s.c. at  $\bar{x}$  (see Exercise 9.8.2).

The next result is an immediate consequence of the definition of the viscosity F-subdifferential and Proposition 9.1.3.

**Proposition 9.1.5 (Generalized Fermat Rule)** If the proper l.s.c. functional  $f : E \rightarrow \overline{\mathbb{R}}$  attains a local minimum at  $\bar{x}$ , then  $0 \in \partial_V f(\bar{x})$  and in particular  $0 \in \partial_F f(\bar{x})$ .

We shall now show that we even have  $\partial_V f(\bar{x}) = \partial_F f(\bar{x})$  provided  $E$  is a Fréchet smooth Banach space. We start with an auxiliary result.

**Lemma 9.1.6** Let  $E$  be a Fréchet smooth Banach space and  $\|\cdot\|$  be an equivalent norm on  $E$  that is F-differentiable on  $E \setminus \{o\}$ . Then there exist a functional  $d : E \rightarrow \mathbb{R}_+$  and a number  $\alpha > 1$  such that:

- (a)  $d$  is bounded,  $L$ -continuous on  $E$  and continuously differentiable on  $E \setminus \{o\}$ .
- (b)  $\|x\| \leq d(x) \leq \alpha\|x\|$  if  $\|x\| \leq 1$  and  $d(x) = 2$  if  $\|x\| \geq 1$ .

*Proof.* Let  $b : E \rightarrow \mathbb{R}$  be the bump functional of Lemma 8.4.1. Define  $d : E \rightarrow \mathbb{R}_+$  by  $d(o) := 0$  and

$$d(x) := \frac{2}{s(x)}, \quad \text{where} \quad s(x) := \sum_{n=0}^{\infty} b(nx) \quad \text{for } x \neq o.$$

We show that  $d$  has the stated properties:

Ad (b). First notice that the series defining  $s$  is locally a finite sum. In fact, if  $\bar{x} \neq o$ , then we have

$$b(nx) = 0 \quad \forall x \in B(\bar{x}, \|\bar{x}\|/2) \quad \forall n \geq 2\|\bar{x}\|. \quad (9.2)$$

Moreover,  $s(x) \geq b(o) = 1$  for any  $x \neq o$ . Hence  $d$  is well defined. We have

$$d(E) \subseteq [0, 2] \quad \text{and} \quad d(x) = 2 \text{ whenever } \|x\| \geq 1.$$

Further it is clear that

$$[x \neq o \text{ and } b(nx) \neq 0] \implies n < 1/\|x\| \quad (9.3)$$

and so, since  $0 \leq b \leq 1$ , we conclude that  $s(x) \leq 1 + 1/\|x\|$ . Hence  $d(x) \geq 2\|x\|/(1+\|x\|)$ , which shows that  $d(x) \geq \|x\|$  whenever  $\|x\| \leq 1$ . Since  $b(o) = 1$  and  $b$  is continuous at  $o$ , there exists  $\eta > 0$  such that  $b(x) \geq 1/2$  whenever  $\|x\| \leq \eta$ . Let  $x \in E$  and  $m \geq 1$  be such that  $\eta/(m+1) < \|x\| \leq \eta/m$ . It follows that

$$s(x) \geq \sum_{n=1}^m b(nx) \geq \frac{m+1}{2} > \frac{\eta}{2\|x\|}$$

and so  $d(x) < (4/\eta)\|x\|$  whenever  $\|x\| \leq \eta$ . This and the boundedness of  $d$  imply that  $d(x)/\|x\|$  is bounded on  $E \setminus \{o\}$ . This verifies (b).

Ad (a). Since by (9.2) the sum defining  $s$  is locally finite, the functional  $d$  is continuously differentiable on  $E \setminus \{o\}$ . For any  $x \neq o$  we have

$$d'(x) = -2 \left( \sum_{n=0}^{\infty} nb'(nx) \right) \left( \sum_{n=0}^{\infty} b(nx) \right)^{-2} = -\frac{(d(x))^2}{2} \sum_{n=0}^{\infty} nb'(nx).$$

Since  $b$  is L-continuous,  $\lambda := \sup\{\|b'(x)\| \mid x \in E\}$  is finite and we obtain for any  $x \neq o$ ,

$$\left\| \sum_{n=0}^{\infty} nb'(nx) \right\| \leq \lambda \sum_{n=0}^{\lceil \|x\|^{-1} \rceil} n \leq \lambda \left( 1 + \frac{1}{\|x\|} \right)^2;$$

here the first inequality holds by (9.3). This estimate together with (b) yields

$$\|d'(x)\| \leq \lambda \max\{\alpha, 2\}^2 (\|x\| + 1)^2,$$

showing that  $d'$  is bounded on  $B(o, 1) \setminus \{o\}$ . Since  $d'$  is zero outside  $B(o, 1)$ , it follows that  $d'$  is bounded on  $E \setminus \{o\}$ . Hence  $d$  is L-continuous on  $E$ . This verifies (a).  $\square$

Now we can supplement Proposition 9.1.3.

**Theorem 9.1.7** *Let  $E$  be a Fréchet smooth Banach space,  $f : E \rightarrow \overline{\mathbb{R}}$  be a proper l.s.c. functional, and  $\bar{x} \in \text{dom } f$ . Then  $\partial_V f(\bar{x}) = \partial_F f(\bar{x})$ .*

*Proof.* In view of Proposition 9.1.3 it remains to show that  $\partial_F f(\bar{x}) \subseteq \partial_V f(\bar{x})$ . Thus let  $x^* \in \partial_F f(\bar{x})$ . Replacing  $f$  with the functional  $\tilde{f} : E \rightarrow \mathbb{R}$  defined by

$$\tilde{f}(y) := \sup\{f(\bar{x} + y) - f(\bar{x}) - \langle x^*, y \rangle, -1\}, \quad y \in E,$$

we have  $o \in \partial_F \tilde{f}(o)$ . We show that  $o \in \partial_V \tilde{f}(o)$ . Notice that  $\tilde{f}(\bar{x}) = 0$  and  $\tilde{f}$  is bounded below. By (9.1) we obtain

$$\liminf_{y \rightarrow o} \frac{\tilde{f}(y)}{\|y\|} \geq 0. \quad (9.4)$$

Define  $\rho : \mathbb{R}_+ \rightarrow \mathbb{R}$  by  $\rho(t) := \inf\{\tilde{f}(y) \mid \|y\| \leq t\}$ . Then  $\rho$  is nonincreasing,  $\rho(0) = 0$  and  $\rho \leq 0$ . This and (9.4) give

$$\lim_{t \rightarrow 0} \frac{\rho(t)}{t} = 0. \quad (9.5)$$

Define  $\rho_1$  and  $\rho_2$  on  $(0, +\infty)$  by

$$\rho_1(t) := \int_t^{et} \frac{\rho(s)}{s} ds, \quad \rho_2(t) := \int_t^{et} \frac{\rho_1(s)}{s} ds.$$

Since  $\rho$  is nonincreasing, we have

$$\rho_1(et) = \int_{et}^{e^2 t} \frac{\rho(s)}{s} ds \geq \rho(e^2 t) \int_{et}^{e^2 t} \frac{1}{s} ds = \rho(e^2 t). \quad (9.6)$$

Since  $\rho_1$  is also nonincreasing, we obtain analogously  $\rho_1(et) \leq \rho_2(t) \leq 0$ . This and (9.5) yield

$$\lim_{t \downarrow 0} \frac{\rho_2(t)}{t} = \lim_{t \downarrow 0} \frac{\rho_1(t)}{t} = \lim_{t \downarrow 0} \frac{\rho(t)}{t} = 0. \quad (9.7)$$

Now define  $\tilde{g} : E \rightarrow \mathbb{R}$  by  $\tilde{g}(x) := \rho_2(d(x))$  for  $x \neq o$  and  $\tilde{g}(o) := 0$ , where  $d$  denotes the functional in Lemma 9.1.6. Recall that  $d(x) \neq 0$  whenever  $x \neq o$ . Since  $\rho_1$  is continuous on  $(0, +\infty)$  and so  $\rho_2$  is continuously differentiable on  $(0, +\infty)$ , the chain rule implies that  $\tilde{g}$  is continuously differentiable on  $E \setminus \{o\}$  with derivative

$$\tilde{g}'(x) = \frac{\rho_1(ed(x)) - \rho_1(d(x))}{d(x)} \cdot d'(x), \quad x \neq o.$$

The properties of  $d$  and (9.7) further imply that  $\lim_{x \rightarrow o} \|\tilde{g}'(x)\| = 0$ . Therefore it follows as a consequence of the mean value theorem that  $\tilde{g}$  is also F-differentiable at  $o$  with  $\tilde{g}'(o) = o$ , and  $\tilde{g}'$  is continuous at  $o$ . Since  $\rho$  is nonincreasing, we have  $\rho_2(t) \leq \rho_1(t) \leq \rho(t)$ ; here, the second inequality follows analogously as (9.6) and the first is a consequence of the second. Let  $\|x\| \leq 1$ . Then  $\|x\| \leq d(x)$ , and since  $\rho_2$  is nonincreasing (as  $\rho_1$  is nonincreasing), we obtain

$$(\tilde{f} - \tilde{g})(x) = \tilde{f}(x) - \rho_2(d(x)) \geq \tilde{f}(x) - \rho_2(\|x\|) \geq \tilde{f}(x) - \rho(\|x\|) \geq 0.$$

Since  $0 = (\tilde{f} - \tilde{g})(o)$ , we see that  $\tilde{f} - \tilde{g}$  attains a local minimum at  $o$ . Hence  $o \in \partial_V \tilde{f}(o)$  and so  $x^* \in \partial_V f(\bar{x})$ .  $\square$

**Remark 9.1.8** Let  $E, f$ , and  $\bar{x}$  be as in Theorem 9.1.7. Further let  $x^* \in \partial_V f(\bar{x})$ , which by Theorem 9.1.7 is equivalent to  $x^* \in \partial_F f(\bar{x})$ . Then there exists a *concave*  $C^1$  function  $g : E \rightarrow \mathbb{R}$  such that  $g'(\bar{x}) = x^*$  and  $f - g$  attains a local minimum at  $\bar{x}$  (cf. Fig. 9.1); see Exercise 9.8.4.

In order to have both the limit definition and the viscosity definition of F-subderivatives at our disposal, we shall in view of Theorem 9.1.7 assume that  $E$  is a Fréchet smooth Banach space and we denote the common F-subdifferential of  $f$  at  $\bar{x}$  by  $\partial_F f(\bar{x})$ .

The relationship to classical concepts is established in Proposition 9.1.9. In this connection recall that

$$\partial_P f(\bar{x}) \subseteq \partial_F f(\bar{x}). \quad (9.8)$$

**Proposition 9.1.9** Assume that  $E$  is a Fréchet smooth Banach space and  $f : E \rightarrow \overline{\mathbb{R}}$  is proper and l.s.c.

- (a) If the directional G-derivative  $f_G(\bar{x}, \cdot)$  of  $f$  at  $\bar{x} \in \text{dom } f$  exists on  $E$ , then for any  $x^* \in \partial_F f(\bar{x})$  (provided there exists one),

$$\langle x^*, y \rangle \leq f_G(\bar{x}, y) \quad \forall y \in E.$$

If, in particular,  $f$  is G-differentiable at  $\bar{x} \in \text{dom } f$ , then  $\partial_F f(\bar{x}) \subseteq \{f'(\bar{x})\}$ .

- (b) If  $f \in C^1(U)$ , where  $U \subseteq E$  is nonempty and open, then  $\partial_F f(x) = \{f'(x)\}$  for any  $x \in U$ .  
(c) If  $f \in C^2(U)$ , where  $U \subseteq E$  is nonempty and open, then  $\partial_P f(x) = \partial_F f(x) = \{f'(x)\}$  for any  $x \in U$ .  
(d) If  $f$  is convex, then  $\partial_P f(x) = \partial_F f(x) = \partial f(x)$  for any  $x \in \text{dom } f$ .  
(e) If  $f$  is locally L-continuous on  $E$ , then  $\partial_F f(x) \subseteq \partial_o f(x)$  for any  $x \in E$ .

*Proof.*

- (a) Let  $x^* \in \partial_F f(\bar{x})$  be given. Then there exist a  $C^1$  function  $g$  and a number  $\epsilon > 0$  such that  $g'(\bar{x}) = x^*$  and for each  $x \in B(\bar{x}, \epsilon)$  we have

$$(f - g)(x) \geq (f - g)(\bar{x}) \quad \forall x \in B(\bar{x}, \epsilon). \quad (9.9)$$

Now let  $y \in E$ . Then for each  $\tau > 0$  sufficiently small we have  $\bar{x} + \tau y \in B(\bar{x}, \epsilon)$  and so

$$\frac{1}{\tau} (f(\bar{x} + \tau y) - f(\bar{x})) \geq \frac{1}{\tau} (g(\bar{x} + \tau y) - g(\bar{x})).$$

Letting  $\tau \downarrow 0$  it follows that  $f_G(\bar{x}, y) \geq \langle g'(\bar{x}), y \rangle = \langle x^*, y \rangle$ . If  $f$  is G-differentiable at  $\bar{x}$ , then by linearity the latter inequality passes into  $f'(\bar{x}) = x^*$ .

- (b) It is obvious that  $f'(x) \in \partial_F f(x)$  for each  $x \in U$ . This and (a) imply  $\partial_F f(x) = \{f'(x)\}$  for each  $x \in U$ .
- (c) By Proposition 3.5.1 we have  $f'(x) \in \partial_P f(x)$ , which together with (a) and (9.8) verifies the assertion.
- (d) It is evident that  $\partial f(\bar{x}) \subseteq \partial_P f(\bar{x}) \subseteq \partial_F f(\bar{x})$  for each  $\bar{x} \in \text{dom } f$ . Now let  $x^* \in \partial_F f(\bar{x})$  be given. As in the proof of (a) let  $g$  and  $\epsilon$  be such that (9.9) holds. Further let  $x \in E$ . If  $\tau \in (0, 1)$  is sufficiently small, then  $(1 - \tau)\bar{x} + \tau x \in B(\bar{x}, \epsilon)$  and we obtain using the convexity of  $f$ ,

$$(1 - \tau)f(\bar{x}) + \tau f(x) \geq f((1 - \tau)\bar{x} + \tau x) \underset{(9.9)}{\geq} f(\bar{x}) + g((1 - \tau)\bar{x} + \tau x) - g(\bar{x}).$$

It follows that

$$f(x) - f(\bar{x}) \geq \frac{g(\bar{x} + \tau(x - \bar{x})) - g(\bar{x})}{\tau}.$$

Letting  $\tau \downarrow 0$ , we see that  $f(x) - f(\bar{x}) \geq \langle g'(\bar{x}), x - \bar{x} \rangle = \langle x^*, x - \bar{x} \rangle$ . Since  $x \in E$  was arbitrary, we conclude that  $x^* \in \partial f(\bar{x})$ .

- (e) See Exercise 9.8.5.

□

In Sect. 9.5 we shall establish the relationship between the Fréchet subdifferential and the Clarke subdifferential.

## 9.2 Approximate Sum and Chain Rules

**Convention.** Throughout this section, we assume that  $E$  is a Fréchet smooth Banach space, and  $\|\cdot\|$  is a norm on  $E$  that is F-differentiable on  $E \setminus \{o\}$ .

Recall that we write  $\omega_{\bar{x}}(x) := \|x - \bar{x}\|$ , and in particular  $\omega(x) := \|x\|$ ,  $x \in E$ .

One way to develop subdifferential analysis for l.s.c. functionals is to start with sum rules. It is an easy consequence of the definition of the F-subdifferential that we have

$$\partial_F f_1(\bar{x}) + \partial_F f_2(\bar{x}) \subseteq \partial_F (f_1 + f_2)(\bar{x}).$$

But the reverse inclusion

$$\partial_F (f_1 + f_2)(\bar{x}) \subseteq \partial_F f_1(\bar{x}) + \partial_F f_2(\bar{x}) \tag{9.10}$$

does not hold in general.



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