Chapter 2
Counting Distributions with Recursion of Order One

Summary

In Chap. 1, we defined compound distributions, presented some of their properties, and mentioned their importance in modelling aggregate claims distributions in an insurance setting. The main topic of the present chapter is recursions for compound distributions, mainly with severity distribution in $\mathcal{P}_{11}$, but in Sect. 2.7, we extend the theory to severity distributions in $\mathcal{P}_{10}$; as a special case, we consider thinning in Sect. 2.7.2.

Section 2.3 is devoted to the Panjer class of distributions $p \in \mathcal{P}_{1\underline{0}}$ that satisfy a recursion in the form

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1) \qquad (n = 1, 2, \dots)$$

for some constants a and b. This class is characterised in Sect. 2.3.2. The key result of the present chapter is the Panjer recursion for compound distributions with counting distribution in the Panjer class. This recursion is motivated and deduced in Sect. 2.3.1, where we also give a continuous version. Section 2.3.3 discusses an alternative recursion that for some severity distributions is more efficient than the Panjer recursion.

To motivate the Panjer recursion and the sort of deductions that we shall mainly apply in the present book, we first discuss two special cases, geometric counting distribution in Sect. 2.1 and Poisson counting distribution in Sect. 2.2. Within the Poisson case, in Sect. 2.2.2 we also discuss an alternative way of deduction based on generating functions as well as present an alternative recursion that for some severity distributions can be more efficient than the Panjer recursion.

Section 2.5 is devoted to an extension of the Panjer class, and that class is further extended in Sect. 2.6.

Although the main emphasis is on compound distributions in the present chapter, Sect. 2.4 is devoted to recursions for convolutions of a distribution on the integers with range bounded on at least one side as these recursions are closely related to the Panjer recursion.

2.1 Geometric Distribution

Let N be a random variable with distribution $p \in \mathcal{P}_{10}$ and $Y_1, Y_2, \dots$ independent and identically distributed random variables with distribution $h \in \mathcal{P}_{11}$. It is assumed

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that the Y_js are independent of N. We want to evaluate the distribution f of $X = Y_{\bullet N}$, that is, $f = p \vee h$. From (1.9), we obtain the initial value

$$f(0) = p(0). \tag{2.1}$$

The simplest case is when p is the *geometric distribution* $\text{geo}(\pi)$ given by

$$p(n) = (1-\pi)\pi^n. \qquad (n = 0, 1, 2, \dots; 0 < \pi < 1) \tag{2.2}$$

Theorem 2.1 *When p is the geometric distribution* $\text{geo}(\pi)$ *and $h \in \mathcal{P}_{11}$, then $f = p \vee h$ satisfies the recursion*

$$f(x) = \pi \sum_{y=1}^{x} h(y) f(x-y) \qquad (x = 1, 2, \dots) \tag{2.3}$$

$$f(0) = 1 - \pi. \tag{2.4}$$

Proof The initial condition (2.4) follows immediately from (2.1) and (2.2).

From (2.2), we see that

$$p(n) = \pi p(n-1). \qquad (n = 1, 2, \dots) \tag{2.5}$$

Insertion in (1.6) gives that for $x = 1, 2, \dots$, we have

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} p(n) h^{n*}(x) = \pi \sum_{n=1}^{\infty} p(n-1)(h * h^{(n-1)*})(x) \\ &= \pi \left(h * \left(\sum_{n=1}^{\infty} p(n-1) h^{(n-1)*} \right) \right)(x) = \pi (h * f)(x) = \pi \sum_{y=1}^{x} h(y) f(x-y), \end{aligned}$$

which proves (2.3).

This completes the proof of Theorem 2.1. □

2.2 Poisson Distribution

2.2.1 General Recursion

We now assume that the claim number distribution p is the *Poisson distribution* $\text{Po}(\lambda)$ given by

$$p(n) = \frac{\lambda^n}{n!} e^{-\lambda}, \qquad (n = 0, 1, 2, \dots; \lambda > 0) \tag{2.6}$$

but keep the other assumptions and notation of Sect. 2.1.

Theorem 2.2 *When p is the Poisson distribution* $\mathrm{Po}(\lambda)$ *and $h \in \mathcal{P}_{11}$, then $f = p \vee h$ satisfies the recursion*

$$f(x) = \frac{\lambda}{x} \sum_{y=1}^{x} y h(y) f(x-y) \qquad (x = 1, 2, \dots) \tag{2.7}$$

$$f(0) = \mathrm{e}^{-\lambda}. \tag{2.8}$$

Proof The initial condition (2.8) follows immediately from (2.1) and (2.6).

For the recursion for the compound geometric distribution, we utilised a recursion for the counting distribution, so let us try to do something similar in the Poisson case. From (2.6), we obtain

$$p(n) = \frac{\lambda}{n} p(n-1). \qquad (n = 1, 2, \dots) \tag{2.9}$$

Insertion in (1.6) gives that for $x = 1, 2, \dots$, we have

$$f(x) = \sum_{n=1}^{\infty} \frac{\lambda}{n} p(n-1) h^{n*}(x). \tag{2.10}$$

This one looks more awkward, but let us see what we can do. This $h^{n*}(x)$, the probability that $Y_{\bullet n} = x$, might lead to something. If we condition on that event, then the conditional expectation of each Y_j must be x/n, that is,

$$\frac{1}{n} = \mathrm{E}\left[\frac{Y_1}{x} \middle| Y_{\bullet n} = x\right] = \sum_{y=1}^{x} \frac{y}{x} \frac{h(y) h^{(n-1)*}(x-y)}{h^{n*}(x)}. \tag{2.11}$$

Insertion in (2.10) gives

$$\begin{aligned} f(x) &= \lambda \sum_{n=1}^{\infty} p(n-1) \sum_{y=1}^{x} \frac{y}{x} \frac{h(y) h^{(n-1)*}(x-y)}{h^{n*}(x)} h^{n*}(x) \\ &= \frac{\lambda}{x} \sum_{y=1}^{x} y h(y) \sum_{n=1}^{\infty} p(n-1) h^{(n-1)*}(x-y) = \frac{\lambda}{x} \sum_{y=1}^{x} y h(y) f(x-y), \end{aligned}$$

which proves (2.7).

This completes the proof of Theorem 2.2. □

2.2.2 *Application of Generating Functions*

The proofs we have given for Theorems 2.1 and 2.2, introduce a technique we shall apply to deduce many recursions in this book. However, the results can often also be proved by using generating functions. Some authors do that with great elegance. However, in the opinion of the present authors, when working on the distributions themselves instead of through generating functions, you get a more direct feeling of what is going on. Using generating functions seems more like going from one place to another by an underground train; you get where you want, but you do not have any feeling of how the landscape gradually changes on the way.

To illustrate how generating functions can be used as an alternative to the technique that we shall normally apply, we shall now first give an alternative proof of Theorem 2.2 based on such functions. After that, we shall deduce an alternative recursion for f based on the form of τ_h.

Alternative Proof of Theorem 2.2 We have

$$\tau_p(s) = \sum_{n=0}^{\infty} s^n p(n) = \sum_{n=0}^{\infty} s^n \frac{\lambda^n}{n!} \mathrm{e}^{-\lambda} = \mathrm{e}^{-\lambda} \sum_{n=0}^{\infty} \frac{(s\lambda)^n}{n!},$$

that is,

$$\tau_p(s) = \mathrm{e}^{\lambda(s-1)}. \tag{2.12}$$

By application of (1.30), we obtain

$$\tau_f(s) = \tau_p(\tau_h(s)) = \mathrm{e}^{\lambda(\tau_h(s)-1)}. \tag{2.13}$$

Differentiation with respect to s gives

$$\tau_f'(s) = \lambda \tau_h'(s) \tau_f(s), \tag{2.14}$$

that is,

$$\sum_{x=1}^{\infty} x s^{x-1} f(x) = \lambda \sum_{y=1}^{\infty} y s^{y-1} h(y) \sum_{x=0}^{\infty} s^x f(x), \tag{2.15}$$

from which we obtain

$$\begin{aligned}\sum_{x=1}^{\infty} s^x x f(x) &= \lambda \sum_{y=1}^{\infty} \sum_{x=0}^{\infty} y s^{x+y} h(y) f(x) = \lambda \sum_{y=1}^{\infty} \sum_{x=y}^{\infty} y s^x h(y) f(x-y) \\ &= \sum_{x=1}^{\infty} s^x \lambda \sum_{y=1}^{x} y h(y) f(x-y).\end{aligned}$$

Comparison of coefficients gives (2.7). □

This proof still holds when $h \in \mathcal{P}_{10}$. From (2.12) and (1.33), we then get the initial value $f(0) = \mathrm{e}^{-\lambda(1-h(0))}$.

With some experience, one would see (2.7) immediately from (2.15).

Let us now assume that $h \in \mathcal{P}_{10}$ satisfies the relation

$$\tau_h'(s) = \frac{\sum_{y=1}^r \eta(y)s^{y-1}}{1 - \sum_{y=1}^r \chi(y)s^y} \tag{2.16}$$

with r being a positive integer or infinity. Then

$$\begin{aligned}\sum_{y=1}^{\infty} yh(y)s^{y-1} = \tau_h'(s) &= \sum_{y=1}^{r} \eta(y)s^{y-1} + \sum_{y=1}^{r} \chi(y)s^y \tau_h'(s) \\ &= \sum_{y=1}^{r} \eta(y)s^{y-1} + \sum_{z=1}^{r} \chi(z)s^z \sum_{u=1}^{\infty} us^{u-1}h(u) \\ &= \sum_{y=1}^{\infty} \left(\eta(y) + \sum_{z=1}^{r} (y-z)\chi(z)h(y-z) \right) s^{y-1}.\end{aligned}$$

Comparison of coefficients gives

$$h(y) = \frac{\eta(y)}{y} + \sum_{z=1}^{r} \left(1 - \frac{z}{y}\right) \chi(z)h(y-z). \qquad (y = 1, 2, \dots) \tag{2.17}$$

Theorem 2.3 *If p is the Poisson distribution* $\mathrm{Po}(\lambda)$ *and $h \in \mathcal{P}_{10}$ satisfies the recursion* (2.17) *for functions η and χ on* $\{1, 2, \dots, r\}$ *with r being a positive integer or infinity, then $f = p \vee h$ satisfies the recursion*

$$f(x) = \sum_{y=1}^{r} \left(\frac{\lambda}{x}\eta(y) + \left(1 - \frac{y}{x}\right)\chi(y) \right) f(x-y). \quad (x = 1, 2, \dots) \tag{2.18}$$

Proof Insertion of (2.17) in (2.7) gives that for $x = 1, 2, \dots,$

$$\begin{aligned}f(x) &= \frac{\lambda}{x} \sum_{y=1}^{x} \left(\eta(y) + \sum_{z=1}^{r} (y-z)\chi(z)h(y-z) \right) f(x-y) \\ &= \frac{\lambda}{x} \left(\sum_{y=1}^{r} \eta(y)f(x-y) + \sum_{z=1}^{r} \chi(z) \sum_{y=z+1}^{x} (y-z)h(y-z)f(x-y) \right),\end{aligned}$$

and by application of (2.7), we get (2.18). □

We see that the conditions of the theorem are always satisfied with $r = \infty$, $\eta = \Phi h$, and $\chi \equiv 0$. In that case, (2.18) reduces to (2.7).

Let us now look at three examples where Theorem 2.3 gives some simplification.

Example 2.1 Let h be the *logarithmic distribution* $\mathrm{Log}(\pi)$ given by

$$h(y) = \frac{1}{-\ln(1-\pi)} \frac{\pi^y}{y}. \qquad (y = 1, 2, \dots; 0 < \pi < 1) \tag{2.19}$$

Then

$$h(y) = \frac{\pi}{-\ln(1-\pi)} I(y=1) + \left(1 - \frac{1}{y}\right) \pi h(y-1), \qquad (y = 1, 2, \dots)$$

that is, h satisfies the conditions of Theorem 2.3 with

$$r = 1; \qquad \eta(1) = \frac{\pi}{-\ln(1-\pi)}; \qquad \chi(1) = \pi.$$

Insertion in (2.18) gives that for $x = 1, 2, \dots$, we have

$$f(x) = \frac{\pi}{x}\left(\frac{\lambda}{-\ln(1-\pi)} + x - 1\right) f(x-1) = \frac{\alpha + x - 1}{x} \pi f(x-1)$$

with

$$\alpha = \frac{\lambda}{-\ln(1-\pi)}, \tag{2.20}$$

that is,

$$f(x) = \frac{(\alpha + x - 1)^{(x)}}{x!} \pi^x f(0) = \binom{\alpha + x - 1}{x} \pi^x f(0).$$

From (2.8) and (2.20), we obtain $f(0) = \mathrm{e}^{-\lambda} = (1-\pi)^\alpha$. Hence,

$$f(x) = \binom{\alpha + x - 1}{x} \pi^x (1-\pi)^\alpha.$$

This is the *negative binomial distribution* $\mathrm{NB}(\alpha, \pi)$. Hence, we have shown that a compound Poisson distribution with logarithmic severity distribution can be expressed as a negative binomial distribution. In Example 4.1, we shall show this in another way. □

Example 2.2 Let h be the shifted geometric distribution given by

$$h(y) = (1-\pi)\pi^{y-1}. \qquad (y = 1, 2, \dots; 0 < \pi < 1) \tag{2.21}$$

In this case, the compound distribution f is called a *Pólya–Aeppli distribution*. We have

$$\tau_h(s) = \sum_{y=1}^{\infty} (1-\pi)\pi^{y-1} s^y = \frac{(1-\pi)s}{1-\pi s}, \tag{2.22}$$

from which we obtain

$$\tau_h'(s) = \frac{1-\pi}{1-2\pi s+\pi^2 s^2},$$

that is, h satisfies the conditions of Theorem 2.3 with

$$r=2; \quad \eta(1)=1-\pi; \quad \eta(2)=0; \quad \chi(1)=2\pi; \quad \chi(2)=-\pi^2. \tag{2.23}$$

Insertion in (2.18) gives

$$f(x) = \frac{1}{x}((\lambda(1-\pi)+2(x-1)\pi)f(x-1)-(x-2)\pi^2 f(x-2)).$$
$$(x=1,2\dots) \qquad \square$$

Example 2.3 Let h be the *uniform distribution* on the integers $0, 1, 2, \dots, k$, that is,

$$h(y) = \frac{1}{k+1}. \qquad (y=0,1,2,\dots,k) \tag{2.24}$$

Then

$$\tau_h(s) = \sum_{y=0}^{k} \frac{1}{k+1} s^y = \frac{1}{k+1}\frac{1-s^{k+1}}{1-s}, \tag{2.25}$$

from which we obtain

$$\tau_h'(s) = \frac{\frac{1}{k+1}-s^k+\frac{k}{k+1}s^{k+1}}{1-2s+s^2}, \tag{2.26}$$

that is, h satisfies the conditions of Theorem 2.3 with

$$r=k+2$$
$$\eta(1)=\frac{1}{k+1}; \qquad \eta(k+1)=-1; \qquad \eta(k+2)=\frac{k}{k+1}$$
$$\chi(1)=2; \qquad \chi(2)=-1$$

and $\eta(y)$ and $\chi(y)$ equal to zero for all other values of y. Insertion in (2.18) gives

$$f(x) = \frac{1}{x}\left(\left(\frac{\lambda}{k+1}+2(x-1)\right)f(x-1)-(x-2)f(x-2)\right.$$
$$\left.-\lambda\left(f(x-k-1)-\frac{k}{k+1}f(x-k-2)\right)\right). \qquad (x=1,2,\dots) \qquad \square$$

2.3 The Panjer Class

2.3.1 Panjer Recursions

Let us now compare the proof of Theorem 2.1 and the first proof of Theorem 2.2. In both cases, we utilised that the claim number distribution $p \in \mathcal{P}_{1\underline{0}}$ satisfied a recursion in the form

$$p(n) = v(n)p(n-1). \qquad (n = 1, 2, \dots) \tag{2.27}$$

In the Poisson case, we found a function t such that

$$\mathrm{E}[t(Y_1, x)|Y_{\bullet n} = x] = v(n) \qquad (x = 1, 2, \dots; n = 1, 2, \dots) \tag{2.28}$$

was independent of x; we had

$$t(y, x) = \lambda\frac{y}{x}; \qquad v(n) = \frac{\lambda}{n}.$$

In the geometric case, we actually did the same with $t(y, x) = v(n) = \pi$. In both cases, (2.28) was satisfied for any choice of $h \in \mathcal{P}_{11}$.

For any $p \in \mathcal{P}_{10}$, if (2.28) is satisfied, then by proceeding like in the first proof of Theorem 2.2, using that

$$\mathrm{E}[t(Y_1, x)|Y_{\bullet n} = x] = \sum_{y=1}^{x} t(y, x)\frac{h(y)h^{(n-1)*}(x-y)}{h^{n*}(x)} \tag{2.29}$$
$$(x = 1, 2, \dots; n = 1, 2, \dots)$$

like in (2.11), we obtain

$$\sum_{n=1}^{x} v(n)p(n-1)h^{n*}(x) = \sum_{y=1}^{x} t(y, x)h(y)f(x-y). \tag{2.30}$$
$$(x = 1, 2, \dots)$$

For $n = 1, 2, \dots$, we have $p(n) = q(n) + v(n)p(n-1)$ with

$$q(n) = p(n) - v(n)p(n-1), \tag{2.31}$$

so that

$$f(x) = (q \vee h)(x) + \sum_{n=1}^{x} v(n)p(n-1)h^{n*}(x). \qquad (x = 1, 2, \dots)$$

Insertion of (2.30) gives

$$
\begin{aligned}
f(x) &= (q \vee h)(x) + \sum_{y=1}^{x} t(y,x)h(y)f(x-y) \\
&= \sum_{n=1}^{x} (p(n) - v(n)p(n-1))h^{n*}(x) \\
&\quad + \sum_{y=1}^{x} t(y,x)h(y)f(x-y). \qquad (x = 1, 2, \dots)
\end{aligned} \tag{2.32}
$$

We have $t(x,x) = v(1)$ because in the conditional distribution of Y_1 given $\sum_{j=1}^{n} Y_j = x$ we have $Y_1 = x$ iff $n = 1$ as the Y_js are strictly positive. Insertion in (2.32) gives

$$
\begin{aligned}
f(x) &= p(1)h(x) + \sum_{n=2}^{x} (p(n) - v(n)p(n-1))h^{n*}(x) \\
&\quad + \sum_{y=1}^{x-1} t(y,x)h(y)f(x-y). \qquad (x = 1, 2, \dots)
\end{aligned} \tag{2.33}
$$

If $p \in \mathcal{P}_{1\underline{l}}$ and $h \in \mathcal{P}_{1\underline{r}}$, then we have $f(x) = 0$ for all $x < lr$. Thus, $f \in \mathcal{P}_{1\underline{lr}}$, and the initial value of the recursion is

$$
f(lr) = \begin{cases} p(l)h^{l*}(r) & (l = 1, 2, \dots) \\ p(0). & (l = 0) \end{cases}
$$

If p satisfies (2.27), then $q \equiv 0$, so that (2.32) reduces to

$$
f(x) = \sum_{y=1}^{x} t(y,x)h(y)f(x-y). \qquad (x = 1, 2, \dots) \tag{2.34}
$$

If both (t_1, v_1) and (t_2, v_2) satisfy (2.28), then $(t, v) = (at_1 + bt_2, av_1 + bv_2)$ also satisfies (2.28) for any constants a and b. In particular, this gives that for all $h \in \mathcal{P}_{11}$, (2.28) is satisfied for all linear combinations of the ts of Theorems 2.1 and 2.2, that is, for

$$
t(y,x) = a + b\frac{y}{x}; \qquad v(n) = a + \frac{b}{n}. \tag{2.35}
$$

Insertion in (2.31)–(2.33) gives

$$
q(n) = p(n) - \left(a + \frac{b}{n}\right)p(n-1) \qquad (n = 1, 2, \dots) \tag{2.36}
$$

and

$$
\begin{aligned}
f(x) &= (q \vee h)(x) + \sum_{y=1}^{x} \left(a + b\frac{y}{x}\right) h(y) f(x-y) \\
&= \sum_{n=1}^{x} \left(p(n) - \left(a + \frac{b}{n}\right) p(n-1)\right) h^{n*}(x) \\
&\quad + \sum_{y=1}^{x} \left(a + b\frac{y}{x}\right) h(y) f(x-y) \\
&= p(1)h(x) + \sum_{n=2}^{x} \left(p(n) - \left(a + \frac{b}{n}\right) p(n-1)\right) h^{n*}(x) \\
&\quad + \sum_{y=1}^{x-1} \left(a + b\frac{y}{x}\right) h(y) f(x-y), \qquad (x = 1, 2, \dots) \qquad (2.37)
\end{aligned}
$$

from which we immediately obtain the following theorem.

Theorem 2.4 *If $p \in \mathcal{P}_{1\underline{0}}$ satisfies the recursion*

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1) \qquad (n = 1, 2, \dots) \tag{2.38}$$

for some constants a and b, then

$$f(x) = \sum_{y=1}^{x} \left(a + b\frac{y}{x}\right) h(y) f(x-y) \qquad (x = 1, 2, \dots) \tag{2.39}$$

for any $h \in \mathcal{P}_{11}$.

The following theorem is a continuous version of Theorem 2.4.

Theorem 2.5 *The compound distribution with continuous severity distribution on $(0, \infty)$ with density h and counting distribution $p \in \mathcal{P}_{1\underline{0}}$ that satisfies the recursion (2.38), has mass $p(0)$ at zero and for $x > 0$ density f that satisfies the integral equation*

$$f(x) = p(1)h(x) + \int_0^x \left(a + b\frac{y}{x}\right) h(y) f(x-y)\, dy. \tag{2.40}$$

Proof We immediately see that the compound distribution has mass $p(0)$ at zero.

For $x > 0$, we have

$$f(x) = \sum_{n=1}^{\infty} p(n) h^{n*}(x) = p(1)h(x) + \sum_{n=2}^{\infty} \left(a + \frac{b}{n}\right) p(n-1) h^{n*}(x)$$

$$= p(1)h(x) + \sum_{n=2}^{\infty} \int_0^x \left(a + b\frac{y}{x}\right) \frac{h(y)h^{(n-1)*}(x-y)}{h^{n*}(x)} \mathrm{d}y \; p(n-1)h^{n*}(x)$$

$$= p(1)h(x) + \int_0^x \left(a + b\frac{y}{x}\right) h(y) \sum_{n=2}^{\infty} p(n-1)h^{(n-1)*}(x-y)\,\mathrm{d}y$$

$$= p(1)h(x) + \int_0^x \left(a + b\frac{y}{x}\right) h(y) f(x-y)\,\mathrm{d}y.$$

This completes the proof of Theorem 2.5. □

The Volterra integral equation (2.40) can be solved by numerical methods. However, in practice it seems more natural to approximate the continuous severity distribution with a discrete one, perhaps using an approximation that gives an upper or lower bound for the exact distribution.

Analogously, other recursions presented in this book can be modified to integral equations when the severity distribution is continuous.

2.3.2 Subclasses

The class of counting distributions satisfying the recursion (2.38) is often called the *Panjer class*. We already know that this class contains the geometric distribution and the Poisson distribution. The following theorem gives a complete characterisation of the Panjer class.

Theorem 2.6 *If* $p \in \mathcal{P}_{1\underline{0}}$ *satisfies the recursion* (2.38), *then we must have one of the following four cases*:

1. *Degenerate distribution in zero*:

$$p(n) = I(n = 0). \tag{2.41}$$

2. *Poisson distribution* $\mathrm{Po}(\lambda)$.
3. *Negative binomial distribution* $\mathrm{NB}(\alpha, \pi)$:

$$p(n) = \binom{\alpha + n - 1}{n} \pi^n (1-\pi)^\alpha. \qquad (n = 0, 1, 2, \ldots; 0 < \pi < 1; \alpha > 0) \tag{2.42}$$

4. *Binomial distribution* $\mathrm{bin}(M, \pi)$:

$$p(n) = \binom{M}{n} \pi^n (1-\pi)^{M-n}. \quad (n = 0, 1, 2, \ldots, M; 0 < \pi < 1; M = 1, 2, \ldots) \tag{2.43}$$

Proof To avoid negative probabilities, we must have $a + b \geq 0$.

If $a + b = 0$, we obtain $p(n) = 0$ for all $n > 0$, so that we get the degenerate distribution given by (2.41).

For the rest of the proof, we assume that $a + b > 0$.

From (2.9), we see that if $a = 0$, then p satisfies (2.6) with $\lambda = b$.

We now assume that $a > 0$. Then Theorem 1.11 gives that $a < 1$. With $\alpha = (a + b)/a$ and $\pi = a$, we obtain

$$p(n) = p(0)\prod_{i=1}^{n}\left(a + \frac{b}{i}\right) = p(0)\pi^n \prod_{i=1}^{n}\left(1 + \frac{\alpha - 1}{i}\right) = p(0)\pi^n \prod_{i=1}^{n}\frac{\alpha + i - 1}{i}$$

$$= p(0)\pi^n \frac{(\alpha + n - 1)^{(n)}}{n!} = p(0)\pi^n \binom{\alpha + n - 1}{n}.$$

Comparison with (2.42) gives that p must now be the negative binomial distribution $\mathrm{NB}(\alpha, \pi)$.

Let us finally consider the case $a < 0$. To avoid negative probabilities, there must then exist an integer M such that

$$a + \frac{b}{M + 1} = 0,$$

that is,

$$M = \frac{a + b}{-a}; \qquad b = -a(M + 1).$$

In that case, we have $p(n) = 0$ for all $n > M$. For $n = 0, 1, 2, \ldots, M$, we obtain

$$p(n) = p(0)\prod_{i=1}^{n}\left(a + \frac{b}{i}\right) = p(0)a^n \prod_{i=1}^{n}\left(1 - \frac{M + 1}{i}\right)$$

$$= p(0)(-a)^n \prod_{i=1}^{n}\frac{M - i + 1}{i} = p(0)(-a)^n \frac{M^{(n)}}{n!} = p(0)(-a)^n \binom{M}{n},$$

which gives (2.43) when $-a = \pi/(1 - \pi)$, that is, $\pi = -a/(1 - a)$.

This completes the proof of Theorem 2.6. $\square$

In Fig. 2.1, the four cases of Theorem 2.6 are illustrated in an (a, b) diagram. Table 2.1 presents the recursion of Theorem 2.4 for the three non-degenerate cases of Theorem 2.6.

As a special case of the negative binomial distribution, we obtain the geometric distribution of Theorem 2.1 with $\alpha = 1$.

It is well known that both the binomial class and the negative binomial class satisfy the property that the convolution of two distribution within the class with the same value of the parameter π is the distribution in the same class with the same value of π and the other parameter being the sum of that parameter from the two

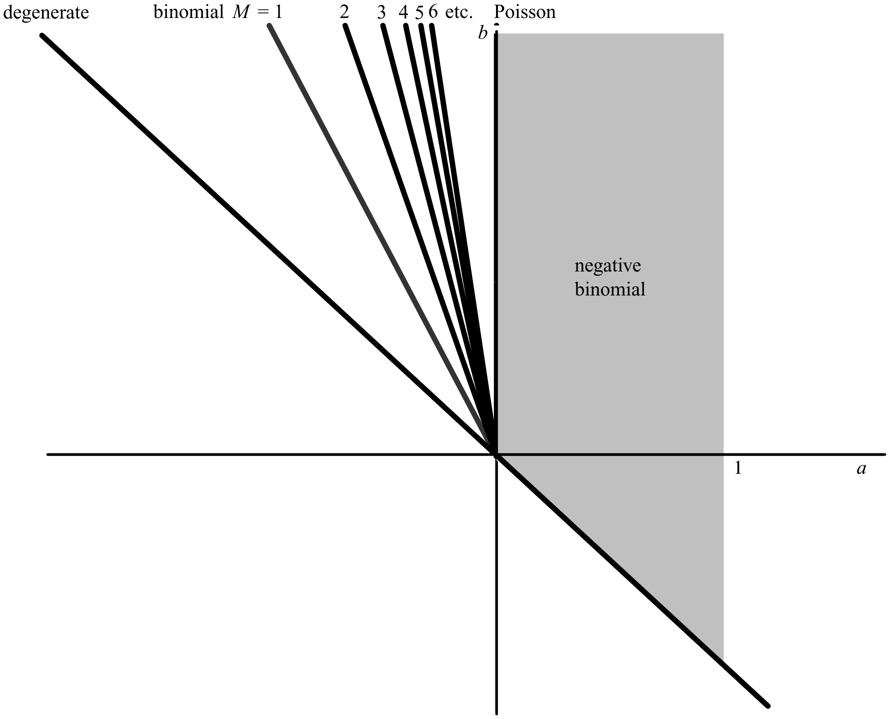

Fig. 2.1 (a, b) for the Panjer class

Table 2.1 Recursions for compound Panjer distributions

Distribution	a	b	$f(x)$	$f(0)$
$\mathrm{NB}(\alpha, \pi)$	π	$(\alpha - 1)\pi$	$\pi \sum_{y=1}^{x} (1 + (\alpha - 1)\frac{y}{x}) h(y) f(x - y)$	$(1 - \pi)^{\alpha}$
$\mathrm{Po}(\lambda)$	0	λ	$\frac{\lambda}{x} \sum_{y=1}^{x} y h(y) f(x - y)$	$e^{-\lambda}$
$\mathrm{bin}(M, \pi)$	$-\frac{\pi}{1-\pi}$	$\frac{(M+1)\pi}{1-\pi}$	$\frac{\pi}{1-\pi} \sum_{y=1}^{x} ((M + 1)\frac{y}{x} - 1) h(y) f(x - y)$	$(1 - \pi)^{M}$

original distributions. When looking at the expressions for a and b for these two classes, we see that the two original distributions and their convolutions are in the Panjer class and have the same value of a, and the b of their convolution is a plus the sum of the bs of the two original distributions. As the convolution of two Poisson distributions is a Poisson distribution with parameter equal to the sum of the parameters of the two original distributions, this property also holds for the Poisson distributions, and, hence, for the whole Panjer class. We formulate this result as a theorem.

Theorem 2.7 *The convolution of two distributions that satisfy the recursion* (2.38) *with the same value of a, satisfies* (2.38) *with the same value of a and b equal to a plus the sum of the bs of the two original distributions.*

In Sect. 5.3.4, we shall prove a more general version of this theorem.

Let us now consider the moments of a distribution $p \in \mathcal{P}_{1\underline{0}}$ that satisfies the recursion (2.38). For $j = 1, 2, \dots$, we have

$$\begin{aligned}\mu_p(j) &= \sum_{n=1}^{\infty} n^j p(n) = \sum_{n=1}^{\infty} n^j \left(a + \frac{b}{n}\right) p(n-1) = \sum_{n=1}^{\infty} n^{j-1}(an+b)p(n-1) \\ &= \sum_{n=0}^{\infty} (n+1)^{j-1}(a(n+1)+b)p(n),\end{aligned}$$

that is,

$$\mu_p(j) = a\mu_p(j;-1) + b\mu_p(j-1;-1). \tag{2.44}$$

By using that

$$\mu_p(k;-1) = \sum_{i=0}^{k} \binom{k}{i} \mu_p(i) \qquad (k = 0, 1, \dots)$$

and solving (2.44) for $\mu_p(j)$, we obtain a recursion for $\mu_p(j)$; we shall return to that in Sect. 9.2.2. In particular, we get

$$\mu_p(1) = a\mu_p(1;-1) + b\mu_p(0;-1) = a\mu_p(1) + a + b,$$

which gives

$$\mu_p(1) = \frac{a+b}{1-a}.$$

Furthermore,

$$\begin{aligned}\mu_p(2) &= a\mu_p(2;-1) + b\mu_p(1;-1) \\ &= a(\mu_p(2) + 2\mu_p(1) + 1) + b(\mu_p(1) + 1) \\ &= a\mu_p(2) + (a+b)\mu_p(1) + a\mu_p(1) + a + b,\end{aligned}$$

from which we obtain

$$\begin{aligned}\mu_p(2) &= \frac{1}{1-a}((a+b)\mu_p(1) + a\mu_p(1) + a + b) \\ &= \mu_p(1)^2 + \frac{1}{1-a}\left(a\frac{a+b}{1-a} + a + b\right) = \mu_p(1)^2 + \frac{a+b}{(1-a)^2},\end{aligned}$$

which gives

$$\kappa_p(2) = \frac{a+b}{(1-a)^2}.$$

Table 2.2 Moments of distributions in the Panjer class

p	$\mu_p(1)$	$\kappa_p(2)$	$\kappa_p(2)/\mu_p(1)$
$\mathrm{NB}(\alpha, \pi)$	$\alpha\pi/(1-\pi)$	$\alpha\pi/(1-\pi)^2$	$1/(1-\pi)$
$\mathrm{Po}(\lambda)$	λ	λ	1
$\mathrm{bin}(M, \pi)$	$M\pi$	$M\pi(1-\pi)$	$1-\pi$

Hence,

$$\frac{\kappa_p(2)}{\mu_p(1)} = \frac{1}{1-a}.$$

From this we see that the variance is greater than the mean when $a > 0$, that is, negative binomial distribution, equal to the mean when $a = 0$, that is, Poisson distribution, and less than the mean when $a < 0$, that is, binomial distribution. This makes the Panjer class flexible for fitting counting distributions by matching of moments. In Table 2.2, we display the mean and the variance and their ratio for the three non-degenerate cases of Theorem 2.6.

2.3.3 An Alternative Recursion

For evaluation of $f = p \vee h$ with $h \in \mathcal{P}_{11}$ and $p \in \mathcal{P}_{1\underline{0}}$ satisfying the recursion (2.38), we shall deduce an alternative recursive procedure that can sometimes be more efficient than Theorem 2.4. We assume that h satisfies the relation

$$\tau_h(s) = \frac{\sum_{y=1}^m \alpha(y)s^y}{1 - \sum_{y=1}^m \beta(y)s^y} \tag{2.45}$$

with m being a positive integer or infinity. Rewriting this as $\tau_h = \tau_\alpha + \tau_\beta \tau_h$ and using (1.20), we obtain that

$$h = \alpha + \beta * h, \tag{2.46}$$

which gives the recursion

$$h(y) = \alpha(y) + \sum_{z=1}^m \beta(z)h(y-z). \qquad (y = 1, 2, \ldots)$$

We shall need the following lemma.

Lemma 2.1 *If $w \in \mathcal{F}_{10}$ and $h \in \mathcal{P}_{11}$ satisfies the relation (2.45) with m being a positive integer or infinity, then $h * w$ satisfies the recursion*

$$(h * w)(x) = \sum_{y=1}^m (\alpha(y)w(x-y) + \beta(y)(h * w)(x-y)). \quad (x = 1, 2, \ldots) \tag{2.47}$$

Proof Application of (2.46) gives $h * w = \alpha * w + \beta * h * w$, from which (2.47) follows. $\square$

We see that (2.45) is always satisfied with $m = \infty$, $\alpha = h$, and $\beta \equiv 0$. In that case, (2.47) gives

$$(h * w)(x) = \sum_{y=1}^{x} h(y)w(x-y), \qquad (x = 1, 2, \dots)$$

which we already know.

We can express (2.39) in the form

$$f(x) = (a+b)(h * f)(x) - \frac{b}{x}(h * \Phi f)(x). \qquad (x = 1, 2, \dots) \tag{2.48}$$

For $x = 1, 2, \dots,$ we can first evaluate $(h * f)(x)$ and $(h * \Phi f)(x)$ by (2.47) and then $f(x)$ by insertion in (2.48).

Example 2.4 Let p be the geometric distribution $\mathrm{geo}(\pi)$. Then $a = \pi$ and $b = 0$ so that (2.48) reduces to $f = \pi(h * f)$. Application of (2.47) gives that for $x = 1, 2, \dots,$

$$\begin{aligned} f(x) &= \pi \sum_{y=1}^{m} (\alpha(y) f(x-y) + \beta(y)(h * f)(x-y)) \\ &= \sum_{y=1}^{m} (\pi\alpha(y) + \beta(y)) f(x-y). \end{aligned}$$

This recursion can be considered as a parallel to the recursion (2.18). $\square$

Example 2.5 Let h be the shifted geometric distribution given by (2.21). From (2.22), we see that (2.45) is satisfied with

$$m = 1; \qquad \alpha(1) = 1 - \pi; \qquad \beta(1) = \pi.$$

Insertion in (2.47) gives

$$(h * w)(x) = (1-\pi)w(x-1) + \pi(h * w)(x-1). \qquad (x = 1, 2, \dots) \qquad \square$$

By differentiating (2.45), we obtain

$$\tau_h'(s) = \frac{\sum_{y=1}^{m} y\alpha(y)s^{y-1}(1 - \sum_{z=1}^{m} \beta(z)s^z) + \sum_{y=1}^{m} \alpha(y)s^y \sum_{z=1}^{m} z\beta(z)s^{z-1}}{(1 - \sum_{z=1}^{m} \beta(z)s^z)^2},$$

which can be written in the form (2.16) with $r = 2m$. Hence, when p is the Poisson distribution $\mathrm{Po}(\lambda)$, we can also evaluate f by the recursion (2.18). In this case, $a = 0$

and $b = \lambda$ so that we can write (2.48) as

$$f(x) = \lambda\left((h * f)(x) - \frac{(h * \Phi f)(x)}{x}\right). \qquad (x = 1, 2, \dots) \tag{2.49}$$

As $r = 2m$, the number of terms in the summation in (2.18) is twice the number of terms in the summation in (2.47). On the other hand, for each value of x in (2.49), we have to apply (2.47) twice, whereas in the recursion of Theorem 2.3, it suffices with one application of (2.18). As it seems to be an advantage to have the recursion expressed in one formula, we tend to go for the recursion of Theorem 2.3 in this case.

2.4 Convolutions of a Distribution

Let us now for a moment leave compound distributions and instead let $f = g^{M*}$ with $g \in \mathcal{P}_{1\underline{0}}$, that is, f is the distribution of $X = Y_{\bullet M}$, where $Y_1, Y_2, \dots, Y_M$ are independent and identically distributed with distribution g. Then we have the following result.

Theorem 2.8 *The M-fold convolution $f = g^{M*}$ of $g \in \mathcal{P}_{1\underline{0}}$ satisfies the recursion*

$$f(x) = \frac{1}{g(0)} \sum_{y=1}^{x} \left((M+1)\frac{y}{x} - 1\right) g(y) f(x-y) \quad (x = 1, 2, \dots) \tag{2.50}$$

$$f(0) = g(0)^M. \tag{2.51}$$

Proof Formula (2.51) follows immediately from (1.4).

Let us now prove (2.50). We introduce an auxiliary random variable Y_0, which is independent of X and has distribution g. Then, because of symmetry, we easily see that for $x = 1, 2, \dots$

$$\mathrm{E}\left((M+1)\frac{Y_0}{x} - 1\right) I(Y_0 + X = x) = 0, \tag{2.52}$$

that is,

$$\sum_{y=0}^{x} \left((M+1)\frac{y}{x} - 1\right) g(y) f(x-y) = 0.$$

Solving for $f(x)$ gives (2.50).

This completes the proof of Theorem 2.8. □

Example 2.6 If g is the discrete uniform distribution given by (2.24), then the recursion (2.50) reduces to

$$f(x) = \sum_{y=1}^{k} \left((M+1)\frac{y}{x} - 1 \right) f(x-y), \qquad (x = 1, 2, \ldots)$$

and (2.51) gives the initial condition $f(0) = (k+1)^{-M}$. In Example 5.2, we shall deduce a simpler recursion for f in the present situation. □

The simplest special case of a non-degenerate distribution g in Theorem 2.8 is the Bernoulli distribution Bern(π) given by

$$g(1) = \pi = 1 - g(0), \tag{2.53}$$

that is, the binomial distribution bin$(1, \pi)$. By Theorem 2.7 and Table 2.1, we obtain that then f is bin(M, π). Insertion of (2.53) in (2.50) gives

$$f(x) = \frac{\pi}{1-\pi} \left(\frac{M+1}{x} - 1 \right) f(x-1), \qquad (x = 1, 2, \ldots) \tag{2.54}$$

which is (2.38) with a and b given by Table 2.1 for the binomial distribution.

More generally, for any $g \in \mathcal{P}_{1\underline{0}}$, it follows from (1.8) that $g = q \vee h$ with q being Bern(π) with

$$\pi = 1 - g(0) \tag{2.55}$$

and $h \in \mathcal{P}_{11}$ given by

$$h(y) = \frac{g(y)}{\pi}. \qquad (y = 1, 2, \ldots) \tag{2.56}$$

Then

$$f = g^{M*} = (q \vee h)^{M*} = q^{M*} \vee h = p \vee h$$

with $p = q^{M*}$, that is bin(M, π). Insertion of (2.55) and (2.56) in the recursion for the compound binomial distribution in Table 2.1 gives Theorem 2.8.

If

$$k = \max(x : g(x) > 0) < \infty, \tag{2.57}$$

then $f(x) = 0$ for all integers $x > Mk$, and we can turn the recursion (2.50) around and start it from $f(Mk)$. This can be convenient if we are primarily interested in $f(x)$ for high values of x. Furthermore, in this converted recursion, we can also allow for negative integers in the range of g.

Theorem 2.9 *If the distribution $g \in \mathcal{P}_1$ satisfies the condition* (2.57), *then $f = g^{M*}$ satisfies the recursion*

$$f(x) = \frac{1}{g(k)} \sum_{y=1}^{Mk-x} \left(\frac{(M+1)y}{Mk-x} - 1 \right) g(k-y) f(x+y)$$

$$(x = Mk-1, Mk-2, \dots, 0)$$

$$f(Mk) = g(k)^M.$$

Proof Let $\tilde{Y}_j = k - Y_j$ $(j = 1, 2, \dots, M)$ and

$$\tilde{X} = \sum_{j=1}^{M} \tilde{Y}_j = \sum_{j=1}^{M} (k - Y_j) = Mk - X,$$

and denote the distributions of $\tilde{Y}_j$ and $\tilde{X}$ by $\tilde{g}$ and $\tilde{f}$ respectively. Then $\tilde{g}, \tilde{f} \in \mathcal{P}_{1\underline{0}}$. Thus, they satisfy the recursion of Theorem 2.8, and we obtain

$$f(Mk) = \tilde{f}(0) = \tilde{g}(0)^M = g(k)^M.$$

For $x = Mk-1, Mk-2, \dots, 0$, (2.50) gives

$$f(x) = \tilde{f}(Mk-x) = \frac{1}{\tilde{g}(0)} \sum_{y=1}^{Mk-x} \left(\frac{(M+1)y}{Mk-x} - 1 \right) \tilde{g}(y) \tilde{f}(Mk-x-y)$$

$$= \frac{1}{g(k)} \sum_{y=1}^{Mk-x} \left(\frac{(M+1)y}{Mk-x} - 1 \right) g(k-y) f(x+y).$$

This completes the proof of Theorem 2.9. □

If $g \in \mathcal{P}_{1\underline{l}}$ for some non-zero integer l, then we can also obtain a recursion for $g = f^{M*}$ from Theorem 2.8 by shifting g and f to $\mathcal{P}_{1\underline{0}}$.

Theorem 2.10 *If $g \in \mathcal{P}_{1\underline{l}}$ for some integer l, then $f = g^{M*}$ satisfies the recursion*

$$f(x) = \frac{1}{g(l)} \sum_{y=1}^{x-Ml} \left(\frac{(M+1)y}{x-Ml} - 1 \right) g(l+y) f(x-y)$$

$$(x = Ml+1, Ml+2, \dots)$$

$$f(Ml) = g(l)^M.$$

Proof Let $\tilde{Y}_j = Y_j - l$ $(j = 1, 2, \dots, M)$ and

$$\tilde{X} = \sum_{j=1}^{M} \tilde{Y}_j = \sum_{j=1}^{M} (Y_j - l) = X - Ml,$$

and denote the distributions of $\tilde{Y}_j$ and $\tilde{X}$ by $\tilde{g}$ and $\tilde{f}$ respectively. Then $\tilde{g}$ and $\tilde{f}$ satisfy the recursion of Theorem 2.8, and we obtain

$$f(Ml) = \tilde{f}(0) = \tilde{g}(0)^M = g(l)^M.$$

For $x = Ml + 1, Ml + 2, \dots,$

$$f(x) = \tilde{f}(x - Ml) = \frac{1}{\tilde{g}(0)} \sum_{y=1}^{x-Ml} \left(\frac{(M+1)y}{x - Ml} - 1 \right) \tilde{g}(y) \tilde{f}(x - Ml - y)$$

$$= \frac{1}{g(l)} \sum_{y=1}^{x-Ml} \left(\frac{(M+1)y}{x - Ml} - 1 \right) g(l + y) f(x - y).$$

This completes the proof of Theorem 2.10. $\square$

2.5 The Sundt–Jewell Class

2.5.1 Characterisation

Let us now return to the situation of Sect. 2.3.1. There we showed that if there exist functions t and v such that (2.28) holds, then (2.30) holds. If, in addition, the counting distribution p satisfies the recursion (2.27), then the compound distribution $f = p \vee h$ satisfies the recursion (2.34). Furthermore, we showed that for all severity distributions $h \in \mathcal{P}_{11}$, (2.28) is satisfied for t and v given by (2.35). A natural question is then for what other couples (t, v) (2.28) is satisfied for all h. The following theorem gives the answer.

Theorem 2.11 *There exists a function t that satisfies the relation* (2.28) *for all $h \in \mathcal{P}_{11}$ iff there exist constants a and b such that*

$$v(n) = a + \frac{b}{n}. \qquad (n = 2, 3, \dots) \tag{2.58}$$

Proof If the function v satisfies (2.58), then (2.28) is satisfied for all $h \in \mathcal{P}_{11}$ with

$$t(y, x) = \begin{cases} a + b\frac{y}{x} & (x \neq y) \\ v(1). & (x = y) \end{cases} \tag{2.59}$$

The reason that it works with a different value when $x = y$, is that in the conditional distribution of Y_1 given $Y_{\bullet n} = x$, we have $Y_1 = x$ iff $n = 1$ as the Y_js are strictly positive.

Let us now assume that there exists a function t that satisfies (2.28) for all $h \in \mathcal{P}_{11}$. We want to prove that then v must satisfy (2.58). It is then sufficient to show that for a particular choice of h (2.58) must be satisfied. We let

$$h(1) = h(2) = \frac{1}{2}.$$

By using that h^{n*} is a shifted binomial distribution, we obtain

$$h^{n*}(y) = \binom{n}{y-n} 2^{-n} \qquad (y = n, n+1, n+2, \dots, 2n; n = 1, 2, \dots)$$

from (2.43). Letting

$$h_n(y|x) = \Pr(Y_1 = y | Y_{\bullet n} = x)$$

for $n = 1, 2, \dots; x = n, n+1, n+2, \dots, 2n$, and $y = 1, 2$, we obtain

$$h_n(1|x) = \frac{h(1)h^{(n-1)*}(x-1)}{h^{n*}(x)} = \frac{\frac{1}{2}\binom{n-1}{x-n}2^{-(n-1)}}{\binom{n}{x-n}2^{-n}} = 2 - \frac{x}{n}$$

$$h_n(2|x) = 1 - h_n(1|x) = \frac{x}{n} - 1.$$

Insertion in (2.28) gives

$$\begin{aligned} v(n) &= \mathrm{E}[t(Y_1, x) | Y_{\bullet n} = x] = t(1, x)h_n(1|x) + t(2, x)h_n(2|x) \\ &= t(1, x)\left(2 - \frac{x}{n}\right) + t(2, x)\left(\frac{x}{n} - 1\right). \end{aligned}$$

With $x = n$ and $x = 2n$ respectively, we obtain

$$v(n) = t(1, n) = t(2, 2n).$$

Letting $x = 2z$ be an even number, we obtain

$$\begin{aligned} v(n) &= t(1, 2z)\left(2 - \frac{2z}{n}\right) + t(2, 2z)\left(\frac{2z}{n} - 1\right) \\ &= v(2z)\left(2 - \frac{2z}{n}\right) + v(z)\left(\frac{2z}{n} - 1\right), \end{aligned}$$

that is,

$$v(n) = A(z) + \frac{B(z)}{n} \qquad (1 \le n \le 2z \le 2n)$$

with

$$A(z) = 2v(2z) - v(z); \qquad B(z) = 2z(v(z) - v(2z)).$$

In particular, for $z \geq 2$, we obtain

$$v(z+1) = A(z+1) + \frac{B(z+1)}{z+1} = A(z) + \frac{B(z)}{z+1}$$

$$v(z+2) = A(z+1) + \frac{B(z+1)}{z+2} = A(z) + \frac{B(z)}{z+2},$$

which gives

$$A(z+1) = A(z); \qquad B(z+1) = B(z),$$

that is, (2.58) must be satisfied for some a and b.

This completes the proof of Theorem 2.11. □

2.5.2 Recursions

From (2.37) we immediately obtain that if $p \in \mathcal{P}_{1\underline{0}}$ satisfies the recursion

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1) \qquad (n = l+1, l+2, \dots) \tag{2.60}$$

for some positive integer l and $h \in \mathcal{P}_{11}$, then the compound distribution $f = p \vee h$ satisfies the recursion

$$\begin{aligned} f(x) &= \sum_{n=1}^{l} \left(p(n) - \left(a + \frac{b}{n}\right) p(n-1) \right) h^{n*}(x) \\ &\quad + \sum_{y=1}^{x} \left(a + b\frac{y}{x}\right) h(y) f(x-y) \\ &= p(1)h(x) + \sum_{n=2}^{l} \left(p(n) - \left(a + \frac{b}{n}\right) p(n-1) \right) h^{n*}(x) \\ &\quad + \sum_{y=1}^{x-1} \left(a + b\frac{y}{x}\right) h(y) f(x-y). \qquad (x = 1, 2, \dots) \end{aligned} \tag{2.61}$$

In particular, if

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1), \qquad (n = 2, 3, \dots) \tag{2.62}$$

then

$$f(x) = p(1)h(x) + \sum_{y=1}^{x-1}\left(a + b\frac{y}{x}\right)h(y)f(x-y). \quad (x = 1, 2, \ldots) \tag{2.63}$$

The class of counting distributions given by (2.62) is sometimes called the *Sundt–Jewell class*.

2.5.3 Subclasses

The Sundt–Jewell class obviously contains the Panjer class. In the proof of Theorem 2.6, we pointed out that to avoid negative probabilities, we had to have $a + b \geq 0$; this is also illustrated in Fig. 2.1. In the Sundt–Jewell class, the recursion (2.62) starts at $n = 2$, so that we need only $2a + b \geq 0$ when $a > 0$.

Let us now look at what sort of distributions we have in the Sundt–Jewell class:

1. *The Panjer class.*
2. *Degenerate distribution concentrated in one*. Here we have $2a + b = 0$. In this case, $f = h$.
3. *Logarithmic distribution* $\mathrm{Log}(\pi)$. Here we have

$$a = \pi; \qquad b = -\pi, \tag{2.64}$$

so that

$$f(x) = \pi\left(\frac{h(x)}{-\ln(1-\pi)} + \sum_{y=1}^{x-1}\left(1 - \frac{y}{x}\right)h(y)f(x-y)\right). \qquad (x = 1, 2, \ldots) \tag{2.65}$$

4. *Extended truncated negative binomial distribution* $\mathrm{ETNB}(\alpha, \pi)$. Let

$$p(n) = \frac{1}{\sum_{j=1}^{\infty}\binom{\alpha+j-1}{j}\pi^j}\binom{\alpha+n-1}{n}\pi^n. \tag{2.66}$$

$$(n = 1, 2, \ldots; 0 < \pi \leq 1; -1 < \alpha < 0)$$

Then

$$a = \pi; \qquad b = (\alpha - 1)\pi \tag{2.67}$$

so that

$$f(x) = \pi\left(\frac{\alpha h(x)}{\sum_{j=1}^{\infty}\binom{\alpha+j-1}{j}\pi^j} + \sum_{y=1}^{x-1}\left(1 - (1-\alpha)\frac{y}{x}\right)h(y)f(x-y)\right).$$

$$(x = 1, 2, \ldots) \tag{2.68}$$

When $0 < \pi < 1$, then

$$\sum_{j=1}^{\infty} \binom{\alpha + j - 1}{j} \pi^j = (1 - \pi)^{-\alpha} - 1.$$

5. *Truncated Panjer distributions*. Let $\tilde{p}$ be a distribution in the Panjer class and define the distribution $p \in \mathcal{P}_{11}$ by

$$p(n) = \frac{\tilde{p}(n)}{1 - \tilde{p}(0)}. \qquad (n = 1, 2, \dots)$$

6. *Zero-modification of distributions in the* Sundt–Jewell *class*. If $\tilde{p}$ is in the Sundt–Jewell class and $0 \le \rho \le 1$, then the mixed distribution p given by

$$p(n) = \rho I(n = 0) + (1 - \rho)\tilde{p}(n) \qquad (n = 0, 1, 2, \dots)$$

is also in the Sundt–Jewell class with the same a and b. We can also have $\rho \notin [0, 1]$ as long as $p(n) \in [0, 1]$ for all n. If $f = p \vee h$ and $\tilde{f} = \tilde{p} \vee h$, then we also have

$$f(x) = \rho I(x = 0) + (1 - \rho)\tilde{f}(x). \qquad (x = 0, 1, 2, \dots)$$

The following theorem shows that there are no other members in the Sundt–Jewell class than those in these classes.

Theorem 2.12 *The six classes described above contain all distributions in the Sundt–Jewell class.*

Proof Because of the mixtures in the class 6, there is an infinite number of counting distributions p with the same a and b. On the other hand, for each distribution in the Sundt–Jewell class, the corresponding distribution with lower truncation at one also belongs to the class, and any distribution in the Sundt–Jewell class can be obtained as a mixture between one of these truncated distributions and a degenerate distribution concentrated in zero. Hence, it suffices to study the distributions in the intersection between the Sundt–Jewell class and $\mathcal{P}_{11}$, and for each admissible pair (a, b) there exist only one such $p \in \mathcal{P}_{11}$.

We know that the class of Panjer distributions is contained in the Sundt–Jewell class, and, hence, that also goes for the truncated Panjer distributions. In Theorem 2.6 and Fig. 2.1, we have characterised the classes of (a, b) for these distributions.

Now, what other admissible values of (a, b) have we got with the extension from the Panjer class to the Sundt–Jewell class? We must now have $2a + b \ge 0$, and we have already considered the distributions with $a + b > 0$ in Theorem 2.6 and Fig. 2.1. From Theorem 1.11, we see that we still cannot have $a > 1$, and that we can have $a = 1$ only when $b < -1$. Hence, it suffices to check all possibilities within the closed triangle bounded by the lines $2a + b = 0$, $a + b = 0$, and $a = 1$, apart from the point with $a = 1$ and $b = -1$. This set is illustrated in Fig. 2.2.

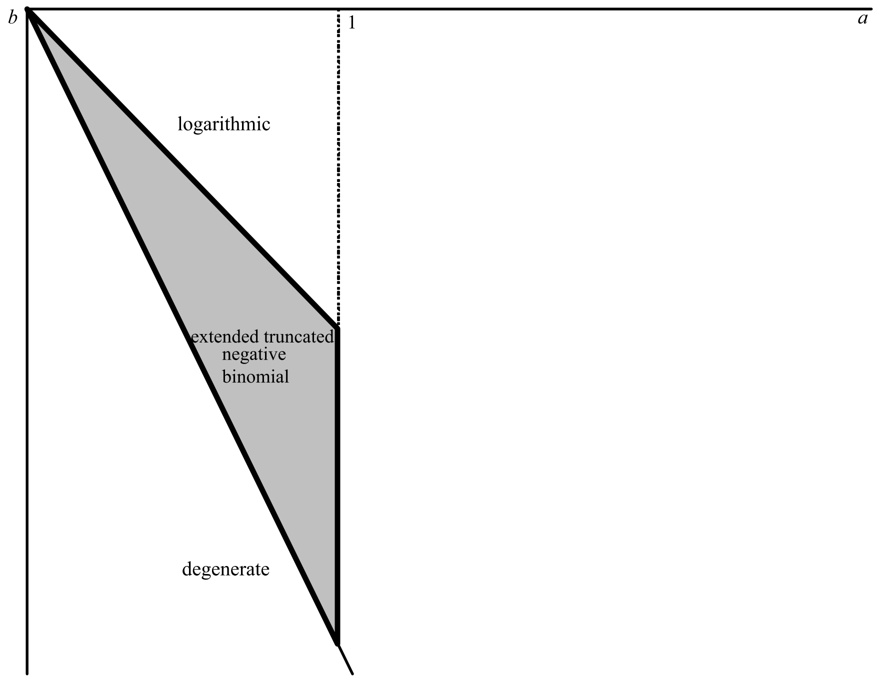

Fig. 2.2 (a, b) diagram for the complement of the Panjer class in the Sundt–Jewell class

When $2a + b = 0$, we obtain the degenerate distribution concentrated in one, and, when $a + b = 0$ with $a < 1$, the logarithmic distribution $\mathrm{Log}(a)$.

For any (a, b) in the remaining area, we define π and α by (2.67), that is,

$$\pi = a; \qquad \alpha = \frac{a+b}{a}.$$

We then obviously have $0 < \pi \leq 1$. Furthermore, as $a + b < 0$,

$$\alpha = \frac{a+b}{a} < 0,$$

and, as $2a + b > 0$,

$$\alpha = \frac{2a+b-a}{a} > -1.$$

Hence, for each (a, b) in our remaining area, there exists an extended truncated negative binomial distribution $\mathrm{ETNB}(\alpha, \pi)$.

We have now allocated distributions in $\mathcal{P}_{11}$ from the classes 2–5 to all admissible pairs (a, b).

This completes the proof of Theorem 2.12. $\square$

2.6 Higher Order Panjer Classes

2.6.1 Characterisation

For $l = 0, 1, 2, \dots$, let $\mathcal{S}_l$ denote the class of counting distributions $p \in \mathcal{P}_{1\underline{l}}$ that satisfy the recursion (2.60). We call this class the *Panjer class of order l*. Like we have done earlier, we often call the Panjer class of order zero simply the Panjer class.

The following theorem is proved analogous to Theorem 2.12.

Theorem 2.13 *If $p \in \mathcal{S}_l$ with $l = 2, 3, \dots$, then p belongs to one of the four classes*:

$$p(n) = \frac{\tilde{p}(n)}{1 - \tilde{p}(l-1)} \qquad (\tilde{p} \in \mathcal{S}_{l-1})$$

$$p(n) = I(n = l) \tag{2.69}$$

$$p(n) = \frac{1}{\sum_{j=l}^{\infty} \pi^j \binom{j}{l}^{-1}} \pi^n \binom{n}{l}^{-1} \qquad (0 < \pi \le 1) \tag{2.70}$$

$$p(n) = \frac{1}{\sum_{j=l}^{\infty} \binom{\alpha+j-1}{j} \pi^j} \binom{\alpha+n-1}{n} \pi^n \qquad (0 < \pi \le 1; -l < \alpha < -l+1) \tag{2.71}$$

for $n = l, l+1, l+2, \dots$.

With the distribution (2.70), we have $a = \pi$ and $b = -l\pi$, so that in the (a, b) plane, we cover the line $la + b = 0$ with $a \in (0, 1]$, and we obtain the distribution (2.69) when $(l+1)a + b = 0$. With the distribution (2.71), we have a and b given by (2.67), that is, in the (a, b) plane, we cover the triangle given by the restrictions $0 < a \le 1$, $la + b < 0$, and $a(l+1) + b \ge 0$.

If $p \in \mathcal{S}_l$ and $h \in \mathcal{P}_{11}$, then $f = p \vee h \in \mathcal{P}_{1l}$, and from (2.61), we obtain that f satisfies the recursion

$$f(x) = p(l)h^{l*}(x) + \sum_{y=1}^{x-l} \left(a + b\frac{y}{x}\right) h(y) f(x-y). \qquad (x = l, l+1, l+2, \dots) \tag{2.72}$$

We can evaluate h^{l*} recursively by Theorem 2.10.

2.6.2 Shifted Counting Distribution

Let us now assume that $p \in \mathcal{P}_{1\underline{0}}$ satisfies the recursion

$$p(n) = \left(a + \frac{b}{n+l}\right) p(n-1), \qquad (n = 1, 2, \dots) \tag{2.73}$$

and that we want to evaluate the compound distribution $f = p \vee h$ with $h \in \mathcal{P}_{11}$. Then the shifted distribution p_l given by $p_l(n) = p(n-l)$ for $n = l, l+1, l+2, \dots$ is the distribution in $\mathcal{S}_l$ given by (2.60). Thus, we can evaluate the compound distribution $f_l = p_l \vee h$ recursively by (2.72). Furthermore, we have $f_l = h^{l*} * f$, so that for $x = lr, lr+1, lr+2, \dots$

$$f_l(x) = \sum_{y=lr}^{x} h^{l*}(y) f(x-y)$$

if $h \in \mathcal{P}_{1\underline{r}}$. By solving this equation for $f(x-lr)$, using that $h^{l*}(lr) = h(r)^l$, we obtain

$$f(x-lr) = \frac{1}{h(r)^l}\left(f_l(x) - \sum_{y=lr+1}^{x} h^{l*}(y) f(x-y)\right).$$

Change of variable gives

$$f(x) = \frac{1}{h(r)^l}\left(f_l(x+lr) - \sum_{y=1}^{x} h^{l*}(y+lr) f(x-y)\right). \qquad (x = 0, 1, 2, \dots)$$

Let us look at shifting the opposite way. We want to evaluate $f = p \vee h$ with $h \in \mathcal{P}_{11}$ and $p \in \mathcal{P}_{1\underline{l}}$ satisfying the recursion

$$p(n) = \left(a + \frac{b}{n-l}\right) p(n-1). \qquad (n = l+1, l+2, \dots)$$

Then the shifted distribution p_{-l} given by $p_{-l}(n) = p(n+l)$ satisfies the recursion (2.38), and, thus, the compound distribution $f_{-l} = p_{-l} \vee h$ satisfies the recursion (2.39). As $f = h^{l*} * f_{-l}$, we can evaluate f by

$$f(x) = \sum_{y=l}^{x} h^{l*}(y) f_{-l}(x-y). \qquad (x = l, l+1, l+2, \dots)$$

2.6.3 Counting Distribution with Range Bounded from Above

Let us now consider a distribution p on a range of non-negative integers $\{l, l+1, l+2, \dots, r\}$ obtained from a distribution $\tilde{p} \in \mathcal{S}_l$ by

$$p(n) = \frac{\tilde{p}(n)}{\sum_{j=l}^{r} \tilde{p}(j)}. \qquad (n = l, l+1, \dots, r)$$

Then p satisfies a recursion in the form

$$p(n) = \left(a + \frac{b}{n}\right) p(n-1). \qquad (n = l+1, l+2, \dots, r) \tag{2.74}$$

Application of (2.37) gives

$$f(x) = p(l)h^{l*}(x) - \left(a + \frac{b}{r+1}\right)p(r)h^{(r+1)*}(x) + \sum_{y=1}^{x}\left(a + b\frac{y}{x}\right)h(y)f(x-y).$$
$$(x = l, l+1, l+2, \dots) \qquad (2.75)$$

The recursions (2.74) and (2.75) are satisfied for more general pairs (a, b) than what follows from the construction from distributions in $\mathcal{S}_l$, as for $n > r$, $a + b/n$ does not need to be non-negative.

Example 2.7 Let $p \in \mathcal{P}_{\underline{1}0}$ be given by

$$p(n) = \frac{\binom{M}{n}(\frac{\pi}{1-\pi})^n}{\sum_{j=0}^{r}\binom{M}{j}(\frac{\pi}{1-\pi})^j}.$$
$$(n = 0, 1, 2 \dots, r; 0 < \pi < 1; r = 1, 2, \dots; M \geq r)$$

Then

$$a = -\frac{\pi}{1-\pi}; \qquad b = (M+1)\frac{\pi}{1-\pi},$$

and (2.75) gives

$$f(x) = \frac{\pi}{1-\pi}\left(\sum_{y=1}^{x}\left((M+1)\frac{y}{x} - 1\right)h(y)f(x-y) - \frac{M-r}{r+1}p(r)h^{(r+1)*}(x)\right).$$
$$(x = 1, 2, \dots) \qquad (2.76)$$

If $r = M$, then p is the binomial distribution $\text{bin}(M, \pi)$, and (2.76) reduces to the recursion for compound binomial distributions given in Table 2.1. $\square$

2.7 Extension to Severity Distributions in $\mathcal{P}_{10}$

2.7.1 Recursions

Apart from Sect. 2.2.2, till now, we have always assumed that the severity distribution belongs to $\mathcal{P}_{11}$ when discussing recursions for compound distributions. We shall now relax this assumption by allowing the severities to be equal to zero, so let $h \in \mathcal{P}_{10}$. In (2.29) and (2.30), we must then sum from $y = 0$ instead of $y = 1$, so that (2.32) becomes

$$f(x) = (q \vee h)(x) + \sum_{y=0}^{x} t(y, x)h(y)f(x-y). \qquad (x = 1, 2, \dots)$$

As $f(x)$ appears with $y=0$ in summation, this does not yet give an explicit expression for $f(x)$, so we solve for $f(x)$ and obtain

$$\begin{aligned}
f(x) &= \frac{1}{1-t(0,x)h(0)}\left((q \vee h)(x) + \sum_{y=1}^{x} t(y,x)h(y)f(x-y)\right) \\
&= \frac{1}{1-t(0,x)h(0)}\left(\sum_{n=1}^{\infty}(p(n)-v(n)p(n-1))h^{n*}(x)\right. \\
&\quad \left. + \sum_{y=1}^{x} t(y,x)h(y)f(x-y)\right) \\
&= \frac{1}{1-t(0,x)h(0)}\left(p(1)h(x) + \sum_{n=2}^{\infty}(p(n)-v(n)p(n-1))h^{n*}(x)\right. \\
&\quad \left. + \sum_{y=1}^{x-1} t(y,x)h(y)f(x-y)\right). \qquad (x=1,2,\dots) \qquad (2.77)
\end{aligned}$$

From (1.33) we obtain that $f(0)=\tau_p(h(0))$. If h and/or p belong to $\mathcal{P}_{1\underline{0}}$, then we can use this as initial value for the recursion (2.77).

If $p \in \mathcal{P}_{10}$ satisfies (2.60), then (2.77) gives the recursion

$$\begin{aligned}
f(x) &= \frac{1}{1-ah(0)}\left(\sum_{n=1}^{l}\left(p(n)-\left(a+\frac{b}{n}\right)p(n-1)\right)h^{n*}(x)\right. \\
&\quad \left. + \sum_{y=1}^{x}\left(a+b\frac{y}{x}\right)h(y)f(x-y)\right) \\
&= \frac{1}{1-ah(0)}\left(p(1)h(x) + \sum_{n=2}^{l}\left(p(n)-\left(a+\frac{b}{n}\right)p(n-1)\right)h^{n*}(x)\right. \\
&\quad \left. + \sum_{y=1}^{x-1}\left(a+b\frac{y}{x}\right)h(y)f(x-y)\right). \qquad (x=1,2,\dots) \qquad (2.78)
\end{aligned}$$

In particular, if p is in the Panjer class, we obtain

$$f(x) = \frac{1}{1-ah(0)}\sum_{y=1}^{x}\left(a+b\frac{y}{x}\right)h(y)f(x-y). \qquad (x=1,2,\dots) \qquad (2.79)$$

Table 2.3 presents this recursion and its initial value $f(0)$ for the three subclasses of non-degenerate distributions in the Panjer class as given by Theorem 2.6. We have already encountered the Poisson case in Sect. 2.2.2.

Table 2.3 Recursions for compound Panjer distributions

Distribution	$f(x)$	$f(0)$
$\mathrm{NB}(\alpha,\pi)$	$\frac{\pi}{1-\pi h(0)}\sum_{y=1}^{x}(1+(\alpha-1)\frac{y}{x})h(y)f(x-y)$	$(\frac{1-\pi}{1-\pi h(0)})^{\alpha}$
$\mathrm{Po}(\lambda)$	$\frac{\lambda}{x}\sum_{y=1}^{x}yh(y)f(x-y)$	$\mathrm{e}^{-\lambda(1-h(0))}$
$\mathrm{bin}(M,\pi)$	$\frac{\pi}{1-\pi+\pi h(0)}\sum_{y=1}^{x}((M+1)\frac{y}{x}-1)h(y)f(x-y)$	$(1-\pi(1-h(0)))^{M}$

2.7.2 Thinning

The recursions introduced in Sect. 2.7.1 can be used to study the effect of thinning. Let N be the number of observations and Y_j the size of the jth of these. We assume that the Y_js are mutually independent and identically distributed with distribution h and independent of N which has distribution p. We also introduce $X = Y_{\bullet N}$ and its distribution $f = p \vee h$. Let us assume that we are interested in the number of observations that satisfy a certain criterion. In insurance, this could e.g. be the number of claims that exceed some retention. In this context, we can let Y_j be an indicator variable equal to one if the observation satisfies the criterion, and zero otherwise. Thus, we let h be the Bernoulli distribution $\mathrm{Bern}(\pi)$ with $0 < \pi < 1$. Then, for $n = 1, 2, \ldots, h^{n*}$ is the binomial distribution $\mathrm{bin}(n, \pi)$, and insertion of (2.43) in (2.78) gives that for $x = 1, 2, \ldots,$

$$\begin{aligned}
f(x) &= \frac{1}{1-a(1-\pi)}\left(\sum_{n=x}^{l}\left(p(n)-\left(a+\frac{b}{n}\right)p(n-1)\right)\binom{n}{x}\pi^x(1-\pi)^{n-x}\right. \\
&\quad \left. + \left(a+\frac{b}{x}\right)\pi f(x-1)\right) \\
&= \frac{1}{1-a(1-\pi)}\sum_{n=x}^{l}\left(p(n)-\left(a+\frac{b}{n}\right)p(n-1)\right)\binom{n}{x}\pi^x(1-\pi)^{n-x} \\
&\quad + \left(a_\pi+\frac{b_\pi}{x}\right)f(x-1)
\end{aligned}$$

with

$$a_\pi = \frac{a\pi}{1-a+a\pi}; \qquad b_\pi = \frac{b\pi}{1-a+a\pi}. \tag{2.80}$$

For $x > l$, the first term vanishes, so that

$$f(x) = \left(a_\pi + \frac{b_\pi}{x}\right)f(x-1). \qquad (x = l+1, l+2, \ldots)$$

It is interesting to note that (a_π, b_π) is on the line between $(0,0)$ and (a,b) in Fig. 2.1. As each of the classes of distributions given by (2.6), (2.42), (2.43), (2.70), and (2.71) satisfy the property that for any point in its area in the (a, b) diagram, all points on the line between that point and $(0, 0)$ also belong to the same class, we see that all these classes, in particular the Panjer class, are closed under thinning.

The thinned distribution f is called the π*-thinning* of p, that is, p is thinned with thinning probability π. Analogously, X is called a π-thinning of N.

2.7.3 Conversion to Severity Distributions in $\mathcal{P}_{11}$

Let $h \in \mathcal{P}_{1\underline{0}}$. When discussing the connection between the recursions for the M-fold convolutions and compound binomial distributions after the proof of Theorem 2.8, we showed how any distribution in $\mathcal{P}_{1\underline{0}}$ can be expressed as a compound Bernoulli distribution with severity distribution in $\mathcal{P}_{11}$. Let us now do this with a distribution $h \in \mathcal{P}_{1\underline{0}}$, denoting the counting distribution by q, its Bernoulli parameter by π, and the severity distribution by $\tilde{h}$, so that $h = q \vee \tilde{h}$. We want to evaluate the compound distribution $f = p \vee h$ with $p \in \mathcal{P}_{10}$ satisfying the recursion (2.60). Then

$$f = p \vee h = p \vee (q \vee \tilde{h}) = (p \vee q) \vee \tilde{h} = \tilde{p} \vee \tilde{h}$$

with $\tilde{p} = p \vee q$. Hence, we have now transformed a compound distribution with severity distribution in $\mathcal{P}_{1\underline{0}}$ to a compound distribution with severity distribution in $\mathcal{P}_{11}$. Furthermore, from the discussion above, we know that the counting distribution satisfies a recursion of the same type as the original counting distribution.

Further Remarks and References

With a different parameterisation, the Panjer class was studied by Katz (1945, 1965) and is sometimes referred to as the *Katz class*. In particular, Katz (1965) gave a characterisation of this class similar to Theorem 2.6 and visualised it in a diagram similar to Fig. 2.1. However, he seems to believe that when $a < 0$, we obtain a proper distribution even when b/a is not an integer; as we have indicated in the proof of Theorem 2.6, we then get negative probabilities.

Even earlier traces of the Panjer class are given by Carver (1919), Guldberg (1931), and Ottestad (1939); see Johnson et al. (2005, Sect. 2.3.1).

Luong and Garrido (1993) discussed parameter estimation within the Panjer class, and Katz (1965) and Fang (2003a, 2003b) discussed testing the hypothesis that a distribution within the Panjer class is Poisson.

In the actuarial literature, Theorem 2.4 is usually attributed to Panjer (1981). However, there are earlier references both within and outside the actuarial area. In the actuarial literature, the Poisson case was presented by Panjer (1980) and Williams (1980) and the Poisson and negative binomial cases by Stroh (1978). The Poisson, binomial, and negative binomial cases were deduced separately by Tilley in a discussion to Panjer (1980). Outside the actuarial literature, the Poisson case was treated by Neyman (1939), Beall and Rescia (1953), Katti and Gurland (1958), Shumway and Gurland (1960), Adelson (1966), Kemp (1967), and Plackett (1969); Khatri and Patel (1961) treat the Poisson, binomial, and negative binomial cases separately. Other proofs for Theorem 2.2 are given by Gerber (1982) and Hürlimann (1988).

Panjer (1981) also proved the continuous case given in Theorem 2.5. The negative binomial case was presented by Seal (1971). Ströter (1985) and den Iseger et al. (1997) discussed numerical solution of the integral equation (2.40). Another approach is to approximate the severity distribution by an arithmetic distribution; references for such approximations are given in Chap. 1.

Panjer (1981) was followed up by Sundt and Jewell (1981). They discussed various aspects of Panjer's framework. In particular, they proved Theorem 2.6 and visualised it in an (a, b) diagram like Fig. 2.1. They also introduced the framework with (2.27) and (2.28) and proved a slightly different version of Theorem 2.11. Furthermore, they presented the recursion (2.61) and its special case (2.72) as well as the recursion (2.75). They also extended the recursions to severity distributions in $\mathcal{P}_{1\underline{0}}$ like in Sect. 2.7.1.

Compound geometric distributions often appear in ruin theory and queuing theory. For applications of Theorem 2.1 in ruin theory, see e.g. Goovaerts and De Vylder (1984), Dickson (1995, 2005), Willmot (2000), and Cossette et al. (2004), and for applications in queuing theory, Hansen (2005) and Hansen and Pitts (2006). Reinhard and Snoussi (2004) applied Theorem 2.2 in ruin theory.

From Theorem 2.2, we obtain that if $f = p \vee h$ with $h \in \mathcal{P}_{11}$ and p being the Poisson distribution $\mathrm{Po}(\lambda)$, then

$$\lambda = -\ln f(0) \tag{2.81}$$

$$h(x) = \frac{1}{f(0)}\left(-\frac{xf(x)}{\ln f(0)} - \sum_{y=1}^{x-1} yh(y)f(x-y)\right). \quad (x = 1, 2, \dots) \tag{2.82}$$

Hence, λ and h are uniquely determined by f. Buchmann and Grübel (2003) proposed estimating λ and h by replacing f in (2.81) and (2.82) with the empirical distribution of a sample of independent observations from the distribution f. It should be emphasised that as a compound Poisson distribution with severity distribution in $\mathcal{P}_{11}$ always has infinite support, such an estimate of h based on the empirical distribution of a finite sample from f, can never be a distribution itself.

Such an estimation procedure can also be applied for other counting distributions p as long as p has only one unknown parameter. When the counting distribution has more parameters, we can still estimate h by replacing f in (2.82) with its empirical counterpart if we consider the parameters of p as given. Such estimation procedures were studied by Hansen and Pitts (2009).

Panjer and Willmot (1992) used generating functions extensively for deduction of recursions for aggregate claims distributions.

Theorem 2.3 was proved by De Pril (1986a), who also gave some examples. Chadjiconstantinidis and Pitselis (2008) present results based on that theorem.

In July 2006, Georgios Pitselis kindly gave us an early version of that paper and was positive to us to use material from it in our book. Since then, there has been some exchange of ideas between him and us, and this has influenced results both later in this book and in the paper.

The representation of a negative binomial distribution as a compound Poisson distribution with a logarithmic severity distribution was presented by Ammeter (1948, 1949) and Quenouille (1949).

The discussion on moments in Sect. 2.3.2 is based on Jewell (1984).

Lemma 2.1 and the related algorithm for recursive evaluation of compound distributions was presented by Hipp (2006) within the framework of phase distributions. He also discussed continuous and mixed severity distributions.

Willmot and Woo (2007) applied Panjer recursions in connection with evaluating discrete mixtures of Erlang distributions.

Reinsurance applications of Panjer recursions are discussed by Panjer and Willmot (1984), Sundt (1991a, 1991b), Mata (2000), Walhin (2001, 2002a), and Walhin et al. (2001).

McNeil et al. (2005) presented Theorem 2.4 within the framework of quantitative risk management.

Douligeris et al. (1997) applied Panjer recursions in connection with oil transportation systems.

For M-fold convolutions, De Pril (1985) deduced the recursions in Theorems 2.8 and 2.10. However, in pure mathematics, the recursion in Theorem 2.8 is well known for evaluation of the coefficients of powers of power series; Gould (1974) traces it back to Euler (1748). Sundt and Dickson (2000) compared the recursion of Theorem 2.8 with other methods for evaluation of M-fold convolutions of distributions in $\mathcal{P}_{1\underline{0}}$.

Willmot (1988) characterised the Sundt–Jewell class; see also Panjer and Willmot (1992, Sect. 7.2) and Johnson et al. (2005, Sect. 2.3.2). The higher order Panjer classes were characterised by Hess et al. (2002). Recursive evaluation of compound distributions with counting distribution satisfying (2.70) or (2.71) and severity distribution in $\mathcal{P}_{10}$ have been discussed by Gerhold et al. (2008). Sundt (2002) presented the procedure for recursive evaluation of a compound distribution with counting distribution given by (2.73).

Thinning in connection with the recursions has been discussed by Milidiu (1985), Willmot (1988), and Sundt (1991b). For more information, see also Willmot (2004) and Grandell (1991).

Panjer and Willmot (1982) and Hesselager (1994) discussed recursive evaluation of compound distributions with severity distribution in $\mathcal{P}_{10}$ and counting distribution $p \in \mathcal{P}_{1\underline{0}}$ that satisfies a recursion is the form

$$p(n) = \frac{\sum_{i=0}^{t} c(i)n^i}{\sum_{i=0}^{t} d(i)n^i} p(n-1). \qquad (n = 1, 2, \dots)$$

The Panjer class appears as a special case with $t = 1$ and $d(0) = 0$.

Ambagaspitiya (1995) discussed a class of distributions $p_{a,b} \in \mathcal{P}_{10}$ that satisfy a relation in the form

$$p_{a,b}(n) = \left(u(a,b) + \frac{v(a,b)}{n}\right) p_{a+b,b}(n-1). \qquad (n = l+1, l+2, \dots)$$

In particular, he discussed recursive evaluation of compound distributions with such a counting distribution and severity distribution in $\mathcal{P}_{11}$. The special case where $p_{a,b}$ satisfies the relation

$$p_{a,b}(n) = \frac{a}{a+b}\left(a + \frac{b}{n}\right) p_{a+b,b}(n-1), \qquad (n = 1, 2, \dots)$$

was treated by Ambagaspitiya and Balakrishnan (1994).

Hesselager (1997) deduced recursions for a compound Lagrange distribution and a compound shifted Lagrange distribution with kernel in the Panjer class. Recursions in connection with Lagrange distributions have also been studied by Sharif (1996) and Sharif and Panjer (1998). For more information on Lagrange distributions, see Johnson et al. (2005).

A special case of compound Lagrange distributions is the generalised Poisson distribution. Recursions in connection with this distribution have been studied by Goovaerts and Kaas (1991), Ambagaspitiya and Balakrishnan (1994), and Sharif and Panjer (1995).

By counting the number of dot operations (that is, multiplications and divisions), Bühlmann (1984) compared the recursive method of Theorem 2.2 with a method presented by Bertram (1981) (see also Feilmeier and Bertram 1987) based on the Fast Fourier Transform. Such comparison of methods presented in this book with each other or other methods have been performed by Kuon et al. (1987), Waldmann (1994), Dhaene and Vandebroek (1995), Sundt and Dickson (2000), Dickson and Sundt (2001), Dhaene et al. (2006), Sundt and Vernic (2006), and, in a bivariate setting, Walhin and Paris (2001c), some of them also counting bar operations (that is, additions and subtractions). Where both dot and bar operations are treated, these two classes are usually considered separately. The reason for distinguishing between these classes and sometimes dropping the bar operations, is that on computers, dot operations are usually more time-consuming than bar operations. Counting arithmetic operations is not a perfect criterion of comparing methods. There are also other aspects that should be taken into account. This is discussed by Sundt and Dickson (2000). It should be emphasised that when doing such counting, one should not just count the operations mechanically from the recursion formulae, but also consider how one could reduce the number of operations by introduction of auxiliary functions. For instance, in the recursion (2.7), one can save a lot of multiplications by first evaluating Φh instead of multiplying y by $h(y)$ at each occurrence, and if h has a finite range, we can further reduce the number of multiplications by instead evaluating $\lambda \Phi h$. How one sets up the calculations, can also affect the numerical accuracy of the evaluation. This aspect has been discussed by Waldmann (1995). We shall not pursue these issues further in this book.

Operators like $\vee$, Φ, and Ψ can be used to make our formulae more compact. However, it has sometimes been difficult to decide on how far to stretch this. As an example, let us look at the first part of (2.37), that is,

$$f(x) = (q \vee h)(x) + \sum_{y=1}^{x} \left(a + b\frac{y}{x} \right) h(y) f(x-y). \tag{2.83}$$

This formula can be made more compact as

$$f(x) = (q \vee h)(x) + a(h \vee f)(x) + b\Psi(\Phi h \vee f)(x). \tag{2.84}$$

On the other hand, it can be made less compact as

$$f(x) = \sum_{n=1}^{x} q(n) h^{n*}(x) + \sum_{y=1}^{x} \left(a + b\frac{y}{x} \right) h(y) f(x-y). \tag{2.85}$$

So why have we then used something between these two extremes? The reason that in (2.83) we have not written the last summation in the compact form we use in (2.84), is that with the compact form, the recursive nature of the formula becomes less clear; we do not immediately see how $f(x)$ depends on $f(0)$, $f(1), \dots,$ $f(x-1)$ like in (2.83). On the other hand, in such a respect, we do not gain anything by writing the first term in (2.83) in the less compact form of (2.85), so it seems appropriate to use the compact form of that term. This reasoning may lead to apparent notational inconsistencies even within the same formula.

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