Preface

Harmonic analysis is one of the most active and fastest growing parts of both pure and applied mathematics. It has gone far beyond its primary goal, which was to study the representation of functions or signals as superpositions of trigonometric functions (Fourier series). The interest in harmonic analysis has always been great because of the wealth of its applications, and it plays nowadays a central role in the study of signal theory and time-frequency analysis. Its interest in pure mathematics (especially in functional analysis) has been revived by the introduction of new functional spaces which are tools of choice for studying regularity properties of pseudo-differential operators and their applications to mathematical physics. Methods from symplectic geometry add power and scope to modern harmonic analysis; historically these methods were perhaps for the first time systematically used in Folland's seminal book [59].

The aim of the present book is to give a rigorous and modern treatment of various objects from harmonic analysis with a strong emphasis on the underlying symplectic structure (for instance symplectic and metaplectic covariance properties). More specifically we have in mind two audiences: the time-frequency community, and mathematical physicists interested in applications to quantum mechanics. The concepts and methods are presented in such a way that they should be easily accessible to students at the upper-undergraduate level (a certain familiarity with basic Fourier analysis and the elementary theory of distributions is assumed). Needless to say, this book can also be read with profit by more advanced readers, and can be used as a reference work by researchers in partial differential equations, harmonic analysis, and mathematical physics. (Several chapters are part of ongoing research and contain material that is usually not addressed in introductory texts. For instance Gromov's non-squeezing theorem from symplectic topology and its applications, or the theory of phase space pseudodifferential operators.)

Description of the book

This book is divided into parts and chapters, each devoted to a particular topic. They have been designed in such a way that the material of each chapter can be covered in a 90 minutes lecture (but this, of course, very much depends on the student's background). The parts can be read independently.

Part I: Symplectic Mechanics

- **Chapter 1** is intended to be a review of the main concepts from Hamiltonian mechanics; while it can be skipped by the reader wanting to advance rapidly in the mathematics of harmonic analysis on symplectic spaces, it is recommended as a reference for a better understanding of the reasons for which many concepts are introduced. For instance, the Hamiltonian approach leads to a very natural and "obvious" motivation for consideration of the Heisenberg–Weyl operators, and of the Weyl pseudo-differential calculus. Also, deformation quantization does not really make sense unless one understands the mechanical reasons which lie behind it. The main result of this first chapter is that Hamiltonian flows consist of symplectomorphisms (the physicist's canonical transformations). This is proven in detail using an elementary method, that of the "variational equation" (which is a misnomer, because there is per se nothing variational in that equation!). We also discuss other topics, such as Poisson brackets (which is helpful to understand the first steps of deformation quantization; Hamilton–Jacobi theory is also briefly discussed).

- In **Chapter 2** the basics of the theory of the symplectic group are developed in a self-contained way. Only an elementary knowledge of linear algebra is required for understanding of the topics of this chapter; the few parts where we invoke more sophisticated material such as differential forms can be skipped by the beginner. A particular emphasis is put on the machinery of free symplectic matrices and their generating functions, which are usually ignored in first courses. The consideration of this topic simplifies many calculations, and has the advantage of yielding the easiest approach to the theory of the metaplectic group. We also discuss classical topics, such as the identification of the unitary group with a subgroup of the symplectic group.

- In **Chapter 3** we refine our study of the symplectic group by introducing the notion of free symplectic matrix and its generating functions. Free symplectic matrices can be defined in several different ways. Their importance comes from the fact that they are in a sense the building blocks of the symplectic group: every symplectic matrix is the product (in infinitely many ways) of exactly two free symplectic matrices. This property in turn allows an easy construction of simple sets of generators for the symplectic group. Last – but certainly not least! – the notion of free symplectic matrix will be instrumental for our definition in Chapter 7 of the metaplectic representation.

- In **Chapter 4** we discuss the notion of symplectomorphism, which is a generalization to the non-linear case of the symplectic transformations introduced in the previous chapters. This leads us to define two very interesting groups $\mathrm{Symp}(2n, \mathbb{R})$ and $\mathrm{Ham}(2n, \mathbb{R})$, respectively the group of all symplectomorphisms, and that of all Hamiltonian symplectomorphisms. These groups, which are of great interest in current research in symplectic topology, are non-linear generalizations of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The group

$\mathrm{Ham}(2n, \mathbb{R})$ will play an important role in our derivation of Schrödinger's equation for arbitrary Hamiltonian functions.

- In **Chapter 5** we introduce new and very powerful tools from symplectic geometry and topology: Gromov's symplectic non-squeezing theorem, and the associated notion of symplectic capacity. The importance of these concepts (which go back to the mid 1980s, and for which Gromov got the Abel Prize in 2009) in applications has probably not yet been fully realized in mathematical analysis, and even less in mathematical physics.

- In **Chapter 6** we address a topic which belongs to both classical and quantum mechanics, namely uncertainties principles, and we do this from a topological point of view. We begin by discussing uncertainty principles associated with a quasi-probability distribution from a quite general point of view (hence applicable both to the classical and quantum cases), and introduce the associated notion of covariance matrix. This enables us to reformulate the strong version of the uncertainty principle in terms of symplectic capacities. This approach to both classical and quantum uncertainties is new and due to the author. It seems to be promising because it allows us to analyze uncertainties which are more general than those usually considered in the literature, and has led to the definition of "quantum blobs", which are symplectically invariant subsets of phase space with minimum symplectic capacity one-half of Planck's constant h. We also prove a multi-dimensional Hardy uncertainty principle, which says that a function and its Fourier transform cannot be simultaneously dominated by too sharply peaked Gaussians.

Part II: Harmonic Analysis in Symplectic Spaces

- **Chapter 7** is devoted to a detailed study of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$ as a unitary representation in $L^2(\mathbb{R}^n)$ of the two-fold covering of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The properties of $\mathrm{Sp}(2n, \mathbb{R})$, as exposed in Chapters 2 and 3, allow us to identify the generators of the metaplectic group as "quadratic Fourier transforms", generalizing the usual Fourier transforms. We construct with great care the projection (covering) mapping from the metaplectic group to the symplectic group, having in mind our future applications to the Wigner transform and the Schrödinger equation. In the forthcoming chapters we will use systematically the properties of the metaplectic group, in particular when establishing symplectic/metaplectic covariance formulas in Weyl calculus and the theory of the Wigner transform.

- In **Chapter 8** we study two companions, the Heisenberg–Weyl and Grossmann–Royer operators. These operators can in a sense be viewed as "quantized" versions of, respectively, translation and reflection operators and are symplectic Fourier transforms of each other. We also discuss the related notion of Heisenberg group and algebra which play such an important role in harmonic analysis in phase space; our approach starts with the canonical commutation relations of quantum mechanics. We also define and briefly dis-

cuss the affine variant of the metaplectic group, namely the inhomogeneous metaplectic group $\mathrm{AMp}(2n, \mathbb{R})$ which is an extension by the Heisenberg–Weyl operators of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$.

- In **Chapter 9** we study in great detail various algebraic and functional properties of the cross-ambiguity and cross-Wigner functions, which are concisely defined using the Heisenberg–Weyl and Grossmann–Royer operators introduced in the previous chapter. We discuss the relations with the short-time Fourier transform used in signal and time-frequency analysis. We also prove a useful inversion formula for the cross-Wigner transform; this formula plays an important role in the theory of Feichtinger's modulation spaces which will be studied later in this book.

Part III: Pseudo-differential Operators and Function Spaces

- In **Chapter 10** we present the basics of Weyl calculus, in particular the definition of the Weyl correspondence which plays such an important role both in the theory of pseudodifferential operators and in modern quantum mechanics of which it is one of the pillars. The chapter begins with an introductory section where the need for "quantization" is briefly discussed. We prove various formulas (in particular formulas for the adjoint of a Weyl operator, and that for the twisted symbol of the composition of two operators).

- In **Chapter 11** we take a close look at the notion of coherent states (they are elementary Gaussian functions); the properties of the metaplectic group allow us to give very explicit formulas for their natural extension, the squeezed coherent states, which play a pivotal role both in harmonic analysis and in quantum mechanics (especially in the subdiscipline known as quantum optics). This leads us naturally to the consideration of anti-Wick operators (also called Toeplitz or Berezin operators), of which we give the main properties.

- In **Chapter 12** we review two venerable topics from functional analysis: the theory of Hilbert–Schmidt and the associated theory of trace class operators. This will allow us to give a precise meaning to the notion of mixed quantum state in Chapter 13. We discuss in some detail the delicate procedure of calculating the trace. In particular we state and prove a result making legitimate the integration of the kernel when the operator is a Weyl pseudodifferential operator.

- In **Chapter 13** we give a rigorous definition of the notion of mixed quantum state, and of the associated density operators (called density matrices in quantum mechanics). The relation between density operators and the Wigner transform is made clear and fully exploited. We discuss the very delicate notion of positivity for the density operator. This is done by introducing the Kastler–Loupias–Miracle-Sole conditions, which we relate to the uncertainty principle. We also apply Hardy's uncertainty principle in its multi-dimensional form to the characterization of sub-Gaussian mixed states; the results are stated concisely using the notion of symplectic capacity.

- **Chapter 14** is of a rather technical nature. We introduce Shubin's global symbol classes, and the associated pseudo-differential operators. Shubin classes are of a greater use in quantum-mechanics than the ordinary Hörmander classes because their definition takes into account global properties of polynomial decrease in phase space. We discuss the notion of asymptotic expansion of the symbols, and show that operators which are at first sight much more general can be reduced to the case of ordinary pseudo-differential operators. We also study the notion of τ-symbol of a pseudo-differential, and give formulas allowing one to switch between different values of the parameter τ.

Part IV: Applications

- In **Chapter 15** we study a great classic of quantum mechanics, in fact one of its pillars: Schrödinger's time-dependent equation. We begin by showing that this equation can be derived from the theory of the metaplectic group when associated to a quadratic Hamiltonian function. In the second part of the chapter we generalize our construction to arbitrary Hamiltonian functions by using Stone's theorem on strongly continuous one-parameter groups of unitary operators together with the characteristic property of symplectic covariance of Weyl pseudo-differential calculus.

- **Chapters 16 and 17** are an introduction to Feichtinger's theory of modulation spaces. The elements ψ of these spaces are functions (or distributions) on $\mathbb{R}^n$ characterized by the property that the cross-Wigner transform $W(\psi, \phi)$ belongs to some weighted Banach space of integrable functions on $\mathbb{R}^{2n}$ for every "window" ϕ. The simplest example is provided by the Feichtinger algebra $M^1(\mathbb{R}^n)$ which is the smallest Banach algebra containing the Schwartz functions and being invariant under the action of the inhomogeneous group (Chapter 16). Since Feichtinger's algebra is a Banach algebra it can be used with profit as a substitute for the Schwartz space; it allows in particular, together with its dual, to define a Gelfand triple. Modulation spaces play a crucial role in time-frequency analysis and in the theory of pseudodifferential operators. Their importance in quantum mechanics has only been recently realized, and is being very actively investigated.

- **Chapter 18** is an introduction to a new topic, which we have called *Bopp calculus*. Bopp operators are pseudodifferential operators of a certain type acting on phase space functions or distributions. They are associated in a natural way to the usual Weyl operators by "Bopp quantization rules", $x \longrightarrow x + \frac{1}{2}i\hbar\partial_p$, $p \longrightarrow p - \frac{1}{2}i\hbar\partial_x$. These rules are often used heuristically by physicists working in the area of deformation quantization; this chapter gives a rigorous justification of these manipulations. We note that the theory of Bopp operators certainly has many applications in pure mathematics and physics (Schrödinger equation in phase space).

- In **Chapter 19** we give a few applications of Bopp calculus. We begin by studying spectral properties of Bopp operators, which we relate to those of

the corresponding standard Weyl operators. As an example we derive the energy levels and eigenfunctions of the magnetic operator (also called Landau operator). We thereafter show that Bopp pseudodifferential operators allow one to express deformation quantization in terms of a pseudodifferential theory; this has of course many technical and conceptual advantages since it allows us to easily prove deep results on "stargenvalues" and "stargenvectors". The book ends on a beginning: the application of Bopp operators to an emerging subfield of mathematics called "noncommutative quantum mechanics" (NCQM), which has its origins in the quest for quantum gravity.

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