2

Counting

2.1 Basic Counting Principles

Inclusion and Exclusion

Suppose S and T are any two sets, and you want to list all members of $S \cup T$. If you list all members of S, then list all the members of T, you will cover all members of $S \cup T$, but those in $S \cap T$ will be listed twice. To count all members of $S \cup T$, you could count all members of both lists, then subtract the number of duplicates. In other words,

$$|S \cup T| = |S| + |T| - |S \cap T|. \tag{2.1}$$

This is the simplest case of a rule called the *principle of inclusion and exclusion*, and you can remember it in the following way. To list all members of the union of two sets, list all members of the first set and all members of the second set. This ensures that all members are included. However, some elements—those in $S \cap T$—will be listed twice, so it is necessary to exclude the duplicates.

Methods for finding the sizes of sets are often used to count the number of members of a universal set that have a particular property. If S is the set of all the objects that have some property A and T is the set of all the objects with property B, then (2.1) expresses the way to count the objects that have *either* property A *or* property B:

(i) count the objects with property A;

(ii) count the objects with property B;

(iii) count the objects with both properties;

(iv) subtract the third answer from the sum of the other two.

As an example, we repeat Sample Problem 1.24, and solve it using (2.1).

W.D. Wallis, *A Beginner's Guide to Finite Mathematics*,
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Sample Problem 2.1. *Recall that there are* 40 *students in Dr. Brown's finite mathematics course and* 50 *in his calculus section. If these are his only classes, and if* 20 *of the students are taking both subjects, use* (2.1) *to calculate how many students he has altogether.*

Solution. Using the notation F for the set of students in the finite mathematics class and C for calculus, (2.1) is

$$|F \cup C| = |F| + |C| - |F \cap C| = 40 + 50 - 20 = 70,$$

so he has 70 students.

The Rule of Sum

Another rule, sometimes called the *rule of sum*, can be simply expressed by saying "the number of objects with property A equals the number that have both property A and property B, plus the number that have property A but not property B"; if properties A and B are related to sets S and T as before, this is

$$|S| = |S \cap T| + |S \backslash T|. \tag{2.2}$$

If we rewrite (2.2) as

$$|S \backslash T| = |S| - |S \cap T|$$

and substitute into (2.1), we obtain

$$|S \cup T| = |T| + |S \backslash T|. \tag{2.3}$$

Sample Problem 2.2. *Suppose the set S has* 25 *elements, T has* 17 *elements, and $S \cap T$ has seven elements. Find $|S \cup T|$ and $|S \backslash T|$.*

Solution. From (2.1) we see

$$|S \cup T| = |S| + |T| - |S \cap T| = 25 + 17 - 7 = 35.$$

From (2.2) we get

$$|S \backslash T| = |S| - |S \cap T| = 25 - 7 = 18.$$

Your Turn. Suppose $|S| = 42$, $|T| = 32$, and $|S \cap T| = 22$. Find $|S \cup T|$ and $|S \backslash T|$.

The Multiplication Principle

It is sometimes useful to break an event down into several parts, forming what we shall call a *compound event*. For example, suppose you are planning a trip from Los Angeles to Paris, with a stopover in New York. You have two options for the flight to New York: a direct flight with United or an American flight that stops in Chicago. For the second leg, you consider the direct flight with Air France, a British Airways flight through London, and Lufthansa stopping in Frankfurt. There are two ways to make the first flight and three to make the second, for a total of six combinations.

Suppose S is the set of available flights for the first leg and T is the set of available flights for the second leg. Then the flight combinations correspond to the members of the Cartesian product $S \times T$, and in this example

$$|S| = 2, \qquad |T| = 3, \qquad |S \times T| = |S| \cdot |T| = |2 \cdot 3| = 6.$$

The correspondence between compound events and Cartesian products applies in general. If S is the set of cases where A occurs, and T is the set of cases where B occurs, then the possible combinations correspond to the set $S \times T$, which has $|S| \cdot |T|$ elements. This idea is usually applied without mentioning the sets S and T. Suppose the event A can occur in a ways, and the event B can occur in b ways, then the combination of events A and B can occur in ab ways. This very obvious principle is sometimes called the *multiplication principle* or *rule of product*. It can be extended to three or more sets.

Sample Problem 2.3. *To open a bicycle lock you must know a three-number combination. You must first turn to the left until the first number is reached, then back to the right until the second number, then left to the third number. Any number from* 1 *to* 36 *can be used. How many combinations are possible?*

Solution. There are 36 ways to choose the first number, 36 ways to choose the second, and 36 ways to choose the third. So there are $36 \cdot 36 \cdot 36$ combinations.

Your Turn. Your debit card has a 4-digit PIN. If you can use any digits, how many PINs are possible?

The multiplication principle only works when the events are performed independently—if the result of A is somehow used to affect the performance of B, some combined results may be impossible. In the airline example, if the United flight leaves too late to connect with the Air France flight, then your choices are not independent, and only five combinations would be available.

An Extension of the Multiplication Principle

Suppose a class of 20 students has to elect a main representative and an alternate representative to attend faculty meetings. The alternate will attend only if the main appointee is unavailable, so the two students selected must be different.

There are 20 candidates for the main position. No matter which one is chosen, there will be 19 candidates for alternate. So the total number of possible choices is $20 \cdot 19 = 380$.

The second choice is not independent of the first choice. The set of candidates for the second election will depend on the choice made in the first election. So this is not an application of the multiplication principle. But it uses a similar idea—treating the occurrence, whose possible outcomes are to be counted, as though it were a compound event.

Sample Problem 2.4. *A high school class contains* 10 *boys and* 13 *girls. They have boys' and girls' charity drives, and wish to elect a chair and treasurer for each. The same person cannot be both chair and treasurer. How many combinations are possible?*

Solution. There are 10 ways to select the boys' chair, and when this is done there are nine ways to select their treasurer. So there are 90 ways to select the boys' committee. In the same way, there are $13 \cdot 12 = 156$ ways to select the girls' committee. So the total is $90 \cdot 156 = 14040$.

Your Turn. What is the answer if each committee also has a secretary?

Sometimes the order of the elements in a set is unimportant, but sometimes the order is significant. Suppose S is a set with n elements. How many different ways are there to order the elements of S?

We solve this by treating the ordering as a compound event with n parts. There are n ways to choose the first element of the ordered set. Whichever element is chosen, there remain $n-1$ possible choices for the second element. When two elements have been selected, there are $n-2$ choices for the third element.

In this way, we see that there are $n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$ ways to order S. This number is called n factorial, and denoted $n!$. So

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1.$$

For convenience, we define $0!$ to equal 1.

The different ways of ordering the set S are called *permutations of* S.

Sample Problem 2.5. *Evaluate* $10!$.

Solution. $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362880$.

Your Turn. Evaluate $6!$.

Sample Problem 2.6. *A committee of three people—chair, secretary, and treasurer—is to be elected by a club with* 11 *members. If every member is eligible to stand for each position, how many different committees are possible?*

Solution. We can treat the selection of the committee as a compound event with three parts: choose the chair, choose the secretary, and choose the treasurer. These parts can be performed in 11, 10, and 9 ways, respectively. So there are $11 \cdot 10 \cdot 9$ committees possible.

Your Turn. What is the number of committees if there are 9 members?

This more general method is often combined with the regular multiplication principle.

Sample Problem 2.7. *Three boys and four girls are to sit along a bench. Three boys must sit together, as must the girls. How many ways can this be done.*

Solution. We treat this as a compound event with three parts. First, it is decided whether the boys are to be on the left or on the right. This can be done in two ways. Then the ordering of the boys is chosen. This can be done in $3! = 6$ ways. Finally, the girls are ordered. This can be accomplished in $4! = 24$ ways. So there are $2 \cdot 6 \cdot 24 = 288$ arrangements.

Your Turn. How many ways could five boys and four girls be seated on two benches, if the boys must sit on the back bench and the girls on the front?

Exercises 2.1 A

1. Suppose $|S| = 19$, $|T| = 11$, and $|S \cap T| = 8$. Find $|S \cup T|$ and $|S \backslash T|$.
2. Suppose $|S| = 22$, $|T| = 12$, and $|S \cup T| = 28$. Find $|S \cap T|$ and $|S \backslash T|$.
3. Suppose $|S| = 30$, $|T| = 24$, and $|S \backslash T| = 12$. Find $|S \cap T|$ and $|S \cup T|$.
4. Suppose $|S| = 52$, $|T| = 50$, and $|S \cup T| = 62$. Find $|S \backslash T|$ and $|T \backslash S|$.
5. Suppose $|S| = 20$, $|T| = 18$, and $|S \cap T| = 13$. Find $|S \backslash T|$ and $|T \backslash S|$.
6. A survey of 1100 voters it was found that 275 of them will vote in favor of a $\frac{1}{2}\%$ increase in sales tax for Public Safety funding, 550 will vote in favor of a $\frac{1}{4}\%$ increase in income tax for school funding, and 200 of these will vote for both tax increases.
   (i) How many favor the school tax but not the Public Safety tax?
   (ii) How many will vote for neither option?
7. In a survey of 500 people, 300 said they intended to holiday in another state within the next twelve months, and 100 said they would holiday overseas. Of these, 50 planned to take both types of holiday. Use (2.1) and the rule of sum to calculate: How many people planned to take an overseas holiday but not an interstate holiday? How many did not plan to holiday outside their state?

8. A true–false test consists of eight questions. Assuming you answer all questions, how many ways are there to answer the test?

9. There are three roads from town X to town Y, four roads from town Y to town Z, and two roads from town X to town Z.

(i) How many routes are there from town X to town Z with a stopover in town Y?

(ii) How many routes are there in total from town X to town Z?

(Assume that no road is traveled twice.)

10. One form of Illinois license plate consists of three letters followed by three digits.

(i) How many different licence plates are possible?

(ii) If the letters O and I are never used, how many licence plates are possible?

11. List all different permutations of the set $\{A, B, C\}$.

12. The three boys and four girls in the choir are to sit on two benches. The boys must sit on the back bench and the girls on the front. How many different ways can they be seated?

13. A multiple-choice quiz contains 8 questions. Each has three possible answers.

(i) If you must answer every question, how many different answer sheets are possible?

(ii) If you may either answer a question or leave a blank, how many different answer sheets are possible?

14. In how many different ways can you order the letters of the word “BREAK”?

15. How many different ways can five people stand in line?

16. Ten midsize cars are available for rental. Three customers arrive, and each chooses a midsize. In how many different ways can the choice be made?

17. You have six textbooks for your courses and four notebooks.

(i) In how many ways can you stack your books?

(ii) In how many ways can you stack your books, if all the notebooks are to go on the bottom?

Exercises 2.1 B

1. Suppose $|S| = 20$, $|T| = 14$, and $|S \cap T| = 8$. Find $|S \cup T|$ and $|S \backslash T|$.

2. Suppose $|S| = 22$, $|T| = 23$, and $|S \cup T| = 34$. Find $|S \cap T|$ and $|S \backslash T|$.

3. Suppose $|T| = 37$, $|S \cap T| = 7$, and $|S \backslash T| = 14$. Find $|S|$ and $|S \cup T|$.

4. Suppose $|S| = 44$, $|T| = 18$, and $|S \cap T| = 12$. Find $|S \cup T|$ and $|S \backslash T|$.

5. Suppose $|S| = 17$, $|T| = 19$, and $|S \cup T| = 24$. Find $|S \cap T|$ and $|S \backslash T|$.

6. Suppose $|S| = 34$, $|T| = 34$, and $|S\backslash T| = 18$. Find $|S \cap T|$ and $|S \cup T|$.

7. Suppose $|S| = 24$, $|S \cap T| = 6$, and $|S \cup T| = 28$. Find $|T|$ and $|S\backslash T|$.

8. Suppose $|S \cup T| = 28$, $|S \cap T| = 6$, and $|S\backslash T| = 12$. Find $|S|$ and $|T|$.

9. Suppose $|S \cup T| = 26$, $|S\backslash T| = 2$ and $|T\backslash S| = 3$. Find $|S|$ and $|T|$.

10. Among 440 telephone subscribers, 240 have blocked anonymous callers and 300 have blocked calls from telemarketers. In total, 360 have blocked at least one of these types of call.

(i) How many have blocked both anonymous callers and telemarketers?
(ii) How many have blocked neither?

11. Among 1000 telephone subscribers it was found that 475 have answering machines and 250 call waiting. Moreover 150 had both options.

(i) How many had either an answering machine or call waiting?
(ii) How many had neither option?

12. Out of 400 people surveyed, 100 said they plan to buy a new house within the next three years, and 200 expected to buy a new car in that period. Of these, 50 planned to make both kinds of expenditure. Use (2.1) and the rule of sum to find out how many planned to buy a new house but not buy a new car, and how many planned to buy a new car but not buy a new house.

13. The menu in a restaurant lists three appetizers, five entrees and three desserts. In how many ways can you order a three-course meal of appetizer, entree and dessert?

14. International airport codes consist of three letters. How many codes are possible?

15. There are 13 contestants. No ties are allowed. In how many different ways can the judges award first, second, and third prize?

16. In a multiple-choice test, each question has four different possible answers. If there are five questions, and all questions must be answered, how many different answer sheets are possible?

17. The ACME company uses serial numbers consisting of three letters followed by five digits. How many possible serial numbers are there?

18. Dave's Pizza sells eight different pizzas by the slice and five different brands of soda. If you want to order a pizza slice and a soda for lunch, how many different combinations could you choose?

19. Your PIN number consists of four digits; the first cannot be zero.

(i) How many such PIN numbers are possible?
(ii) How many allowable PIN numbers have no zeros anywhere?
(iii) How many allowable PIN numbers have all digits different?

20. For your soup-and-salad lunch, you must choose between five soups, three salads, and six dressings. Assuming you choose a soup, a salad, and a dressing, how many different possible lunches could you order?

21. List all different permutations of the set $\{1, 2, 3, 4\}$.

22. On your bookshelf you have five novels and five textbooks.
 (i) How many different ways can you arrange your books?
 (ii) How many different ways can you arrange your books, if all the novels go on the left and all the textbooks go on the right?

23. On your bookshelf you have five Mathematics textbooks and four Chemistry textbooks.
 (i) How many different ways can you arrange your books?
 (ii) How many different ways can you arrange your books, if all the Chemistry books go on the left?

24. Derive the formula

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ + |A \cap B \cap C|.$$

25. Prove that $(2n)!/n! = 2^n(1 \cdot 3 \cdot 5 \cdots (2n-1))$, and consequently that, for any positive integer n,

$$2 \cdot 6 \cdot 10 \cdots (4n-6) \cdot (4n-2) = (n+1) \cdot (n+2) \cdots (2n-1) \cdot 2n.$$

2.2 Arrangements

Sequences on a Set

Suppose S is a set with s elements. We often need to know how many k-element *ordered* sets or *k-sequences* or *arrangements of size k* can be chosen from S. This number is denoted $P(s, k)$. In particular, $P(s, s)$ denotes the number of s-sequences that can be chosen from an s-set, which is the same as the number of permutations of S. It follows that $P(s, s) = s!$. For this reason arrangements of size k are often called *permutations of size k*.

Given an s-set $S = \{x_1, x_2, \ldots, x_s\}$, there are s different sequences of length 1 on S, namely (x_1), (x_2), ..., and (x_s). So $P(s, 1) = s$. There are $s \cdot (s-1)$ sequences of length 2, because each sequence of length 1 can be extended to length 2 in $s - 1$ different ways, and no two of these $s \cdot (s-1)$ extensions will ever be equal. So $P(s, 2) = s(s-1)$. Similarly, we find

$$\begin{aligned}
P(s,3) &= s\cdot(s-1)\cdot(s-2),\\
P(s,4) &= s\cdot(s-1)\cdot(s-2)\cdot(s-3),\\
&\vdots\\
P(s,k) &= s\cdot(s-1)\cdot(s-2)\cdots(s-k+1).
\end{aligned}$$

So $P(s, k)$ is calculated by multiplying, s, $s-1$, $s-2$, ... until there are k factors.

It follows that

$$P(s,k) = s!/(s-k)!. \tag{2.4}$$

Sample Problem 2.8. *Calculate* $P(10, 3)$.

Solution. There are two ways to calculate $P(10, 3)$. We could say $P(10, 3) = 10\cdot 9\cdot 8 = 720$. Or we could use the formula:

$$P(10,3) = 10!/7! = 362880/5040 = 720.$$

The first way is easier.

Your Turn. Calculate $P(6, 4)$.

Several of the problems in the preceding section, such as those on selecting a committee, asked for the number of sequences of a certain length, and their solutions can sometimes be stated compactly by using arrangements.

Sample Problem 2.9. *A committee of three people—chair, secretary, and treasurer—is to be elected by a club with* 14 *members. If every member is eligible to stand for each position, how many different committees are possible?*

Solution. We can treat the committee as an ordered set of three elements chosen from the 14-element set of members. So the answer is $P(14, 3)$, or 2184.

Your Turn. What is the number of committees if there are 12 members?

Sometimes an added condition makes the solution of a problem easier, not harder. For example, arranging people around a circular table is no more difficult than arranging them in a line, and sometimes easier.

Suppose n people are to sit around a circular table. We start by arbitrarily labeling one seat at the table as "1", the one to its left as "2", and so on. Then there are $n!$ different ways of putting the n people into the n seats. However, we have counted two arrangements as different if one is obtained from the other by shifting every person one place to the left because these two arrangements put different people in "1"; but they are clearly the same arrangement for the purposes of the question. Each arrangement is one of a set of n, all obtained from the others by shifting in a circular fashion. So the number of truly different arrangements is $n!/n$, which equals $(n-1)!$.

Sample Problem 2.10. *How many ways can you make a necklace by threading together seven different beads?*

Solution. Suppose you put the beads on a table before threading them. There would be $(n-1)! = 6! = 720$ ways to arrange them in a circle. However, after the beads are threaded, the necklace could be flipped over, so every necklace has been counted twice (for example, $abcdefg$ and $agfedcb$ are the same necklace). Therefore, the total number is $6!/2 = 360$.

Your Turn. How many ways could the three boys and four girls be arranged around a circular table if the boys must sit together and the girls as well?

Sample Problem 2.11. *Kirsten's Kopying Kompany has eight photocopying machines and seven employees who can operate them. There are four copying jobs to be done simultaneously. How many ways are there to allocate these jobs to operators and machines?*

Solution. Call the jobs A, B, C, D. Choose two arrangements: first, which four operators should do the jobs; second, which machines should be used. The operator choice can be made in $P(7, 4)$ ways, and the machines in $P(8, 4)$ ways. In each case, the first member of the sequence is the one allocated to job A, the second to job B, and so on. There are $P(7, 4) \cdot P(8, 4) = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 1411200$ ways.

Sample Problem 2.12. *The club in Sample Problem 2.9 wishes to elect a by-laws committee with three members—Chair, Secretary, and Legal Officer—and requires that no members of the main club committee be members of the by-laws committee. In how many different ways can the two committees be chosen?*

Solution. Suppose the main committee is chosen first. There are $P(14, 3) = 2184$ ways to do this. After the election, there are 11 members eligible for election to the by-laws committee, so it can be chosen in $P(11, 3) = 990$ ways. So there are a grand total of $2184 \cdot 990 = 2162160$.

In the preceding Sample Problem we see that, even in small problems, the numbers get quite large. It might be better to report the answer in its factored form, as $14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9$. This form of answer also makes it clear that we could have solved the problem by treating the two committees as one six-member sequence, with $P(14, 6)$ possible solutions.

Arrangements with Repetitions

If repeated elements are allowed, the number of sequences that can be formed from a set is far larger. If there is no restriction, then s^k k-sequences can be formed from an

k	$P(6,k)$	6^k	$P(6,k)/6^k$
1	6	6	1.00000
2	30	36	0.83333
3	120	216	0.55556
4	360	1296	0.27778
5	720	7776	0.09259
6	720	46656	0.01543

Table 2.1. Numbers of sequences without and with repetitions

s-set. To show just how great this difference is, even in small cases, Table 2.1 shows the relative values of $P(6, k)$ and 6^k.

In some cases, repetitions are allowed but limited. Problems of this kind can be modeled by talking about sets with a certain number of copies of each element. For example, if you want to count the number of sequences of length five based on the set $\{A, B\}$ that contain no more than three A's and no more than four B's, you could instead talk about 5-sequences selected from the set $\{A, A, A, B, B, B, B\}$.

To avoid the problems that can arise by having multiple copies of the same element, we often talk of *distinguishable* and *indistinguishable* elements. For example, many problems can be modeled in terms of selecting marbles from an urn; the usual convention is that two sequences are distinguishable only if they differ in color sequence: any two blue marbles are indistinguishable.

One way to tackle these problems is to assume the "indistinguishable" objects can be distinguished, and then take this into account. For example, consider the letters in the word $ASSESS$. In how many distinguishable ways can you order these letters?

Suppose the letters were written on tiles with numbers as subscripts, like scrabble tiles. Label them so that no two copies of the same letter get the same subscript, for example, $A_1S_1S_2E_1S_3S_4$. Then all the 6 letters are different, and there are 6! orderings. Say you have each of these orderings written on slips of paper.

Now collect together into one pile all the slips that differ only in their subscripts. For example, $A_1E_1S_1S_2S_3S_4$ and $A_1E_1S_2S_1S_3S_4$ will be in the same pile, as will $A_1E_1S_2S_3S_1S_4$, $A_1E_1S_4S_2S_3S_1$, and several others. In fact, we can work out how many slips there are in a pile. There are four letters S, and one each of the others. Two slips will be in the same pile when they have the letters in the same order, but the subscripts on the S's are in different order. There are $4! = 24$ ways to order the four subscripts, so there are 4! slips in each pile. Therefore, there are $6!/4! = 30$ piles.

Two orderings can be distinguished if and only if their slips are in different piles, so there are $6!/4! = 30$ distinguishable orderings of $ASSESS$.

The same principle can be applied with several repeated letters. For example, if $SUCCESS$ is written as $S_1U_1C_1C_2E_1S_2S_3$, we see that there are 2! ways of ordering the C's and 3! ways if ordering the S's, so each pile will contain $3! \cdot 2!$ slips, and the number of distinguishable orderings is $7!/(3! \cdot 2!) = 420$.

Sample Problem 2.13. *In how many distinguishable ways can you order the letters of the word $MISSISSIPPI$?*

Solution. There are one M, four I's, four S's and two P's, for a total of 11 letters. So the number of orderings is $11!/(4! \cdot 4! \cdot 2!)$ or 34650.

Your Turn. In how many distinguishable ways can you order the letters of the word $BANANA$?

The problems are significantly harder if not all the available repetitions are used.

Sample Problem 2.14. *An experimenter has an urn containing* 12 *marbles: five red, two blue, and five green. Assuming that marbles of the same color are indistinguishable, how many different sequences of length* 4 *can be chosen from the set?*

Solution. If unlimited numbers of each color were available, this would be the same as the problem of selecting sequences of length 4 from a 3-set with unlimited repetitions allowed, and there would be $3^4 = 81$ solutions. It is necessary to exclude those with three or four blue marbles. In the obvious notation, these are $BBBR$, $BBRB$, $BRBB$, $RBBB$, $BBBG$, $BBGB$, $BGBB$, $GBBB$, and $BBBB$, nine in total. So the answer is $81 - 9 = 72$.

Your Turn. What is the answer if there are four red, three blue, and five green marbles?

Exercises 2.2 A

1. Calculate:

(i) $P(8,3)$; (ii) $P(4,4)$;

(iii) $P(5,4)$; (iv) $P(9,2)$.

2. A *palindrome* is a "word" (any string of letters) that reads the same forwards or backwards, such as $CIVIC$ or $AABAA$. How many 5-letter palindromes are there (using the ordinary, 26-letter alphabet)?

3. Five people are to sit at a round table.

(i) How many ways can they be seated?

(ii) How many ways can they be seated if two specific people must sit together?

4. What are the answers to the preceding Exercise if the five people sit along a straight bench, rather than a round table?

5. Three men and four women sit in a row. How many different ways can they do it if:

(i) the men must sit together;
(ii) the women must sit together?

6. In how many ways can you arrange the letters of the following words?

(i) $PROFESSOR$; (ii) $ACCESSORY$;
(iii) $RUBBLE$; (iv) $BOOKKEEPER$.

7. John likes to arrange his books. He has four western, five mystery and six science fiction books, all different.

(i) In how many ways can he arrange them on a shelf?
(ii) In how many ways can he arrange them if all the books on the same subject must be grouped together?

8. Suppose your student ID consists of six digits, of which the first cannot be 0 or 9.

(i) How many different ID numbers are possible?
(ii) How many different ID numbers are possible if no digit may appear more than once?

9. Your PIN number consists of four digits. No repetitions are allowed, and 0 is not to be used.

(i) How many PIN numbers are possible?
(ii) How many PIN numbers are smaller than 4000?
(iii) How many PIN numbers are even?
(iv) How many PIN numbers contain no number greater than 7?

Exercises 2.2 B

1. Calculate:

(i) $P(8, 5)$; (ii) $P(7, 2)$; (iii) $P(4, 1)$;
(iv) $P(7, 5)$; (v) $P(5, 2)$; (vi) $P(9, 4)$.

2. Calculate:

(i) $P(8, 4)$; (ii) $P(8, 2)$; (iii) $P(6, 1)$;
(iv) $P(6, 3)$; (v) $P(5, 3)$; (vi) $P(9, 3)$.

3. (i) How many ways are there of seating six people at a round table?
(ii) How many ways are there of seating six people at a round table so that two specific people sit together?

4. The deck for a card game consists of 24 different cards. You deal a sequence of three cards from such a deck. How many different sequences are possible?

5. A football league consists of nine teams. Each team must play each other team twice: once as home team, once as visitors.

(i) How many games must be played?

(ii) If each pair plays only once (that is, you don't care which is the home team), how many games must be played?

6. Five stereo systems are to be arranged in a line against the wall of the appliance department.

(i) How many ways can this be done?

(ii) How many ways can they be arranged if the most expensive model must be in the middle?

7. A basketball league consists of eight teams. Each team must play each other team twice: once as home team, once as visitors. How many games must be played?

8. You have a deck of nine cards: three (identical) aces of spades, three (identical) Kings of hearts, and three (identical) Queens of clubs. In how many different ways can you deal a sequence of four cards?

9. A company assigns serial numbers to its computers. Each number consists of four digits followed by three letters.

(i) How many possible serial numbers are there?

(ii) How many possible serial numbers are there, if no digit can be repeated?

(iii) How many, if no repetitions of digits or letters are allowed?

10. Four men and four women are to be seated at a round table, with men and women seated alternately. In how many different ways can they be seated?

11. Five men and four women are to be seated in a row, with men and women seated alternately. In how many different ways can they be seated?

12. In how many ways can you arrange the letters of the following words?

(i) $TODDLER$; (ii) $OFFERED$; (iii) $BORROW$;
(iv) $ARROWROOT$; (v) $MOOSEWOOD$; (vi) $APPLESEED$.

13. In how many ways can you arrange the letters of the following words?

(i) $TOPPLE$; (ii) $PEPPERED$; (iii) $BOBBIN$;
(iv) $LIFELINE$; (v) $HOOSEGOW$; (vi) $BROWBEATEN$.

14. There are 10 speakers in a debate, five on each side. It is agreed that the first speaker must speak in favor of the proposition, followed by a speaker against it, then one in favor, then one against. The remaining six speakers may speak in any order. In how many different ways can the debate be scheduled?

2.3 Selections

Selections

Given a set S, we are often interested in knowing how many different subsets of a given size are contained in S. This depends only on the size of S. We shall write $C(s, k)$ or $\binom{s}{k}$ for the number of k-subsets of an s-set; it is usual to read the symbol as "s choose k". The form $C(s, k)$ is used more frequently in specific practical cases, while we write $\binom{s}{k}$ in general or theoretical discussions. You should be familiar with both notations.

Obviously $\binom{s}{k}$ depends on the values s and k. We often call $\binom{s}{k}$ or $C(s, k)$ the *choice function* (of s and k).

We can use the formula (2.4) to derive expressions for the numbers $C(s, k)$. Suppose S is a set with s elements. It is clear that every k-set that we choose from S gives rise to exactly $k!$ distinct k-sequences on S and that the same k-sequence never arises from different k-sets. So the number of k-sequences on S is $k!$ times the number of k-sets on S, or

$$\binom{s}{k} = \frac{P(s,k)}{k!} = \frac{s!}{(s-k)!k!}. \tag{2.5}$$

When calculating $\binom{s}{k}$ in practice, you would usually calculate $P(s, k)$, then divide by $k!$. So

$$\binom{s}{k} = \frac{s \cdot (s-1) \cdot (s-2) \cdots (s-k+1)}{1 \cdot 2 \cdot 3 \cdots k}.$$

There are k factors in the denominator and in the numerator.

Recall that we agreed to say $0! = 1$. In combination with (2.5) this yields $\binom{s}{0} = 1$. This makes sense: it is possible to choose *no* elements from a set, but one cannot imagine different ways of doing so. We also define $\binom{s}{k} = 0$ if $k > s$. Again this makes sense—there is no way to choose more than s elements from an s-set.

Sample Problem 2.15. *Calculate $C(8, 5)$ and $\binom{6}{6}$.*

Solution.

$$C(8,5) = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 56,$$
$$\binom{6}{6} = \frac{6!}{0! \times 6!} = 1.$$

There is no need for calculation: the terms 6! in the numerator and denominator cancel.

Your Turn. Calculate $C(9, 5)$ and $\binom{6}{0}$.

Sample Problem 2.16. *A student must answer five of the eight questions on a test. How many different ways can she answer, assuming there is no restriction on her choice and the order in which she answers them is unimportant?*

Solution. $\binom{8}{5} = 56$ ways.

Your Turn. How many ways can she answer if she must choose five, one of which is Question 1?

Sample Problem 2.17. *Computers read strings consisting of the digits* 0 *and* 1. *Such a string with k entries is called a k-bit string. How many* 8*-bit strings are there that contain exactly five* 1*'s?*

Solution. To specify a string, it is sufficient to say which positions have 1's. There are $C(8, 5)$ choices, so the answer is $C(8, 5) = 56$.

Your Turn. How many 8-bit strings contain exactly four 1's?

Sample Problem 2.18. *How many ways can a committee of three men and two women be chosen from six men and four women?*

Solution. The three men can be chosen in $\binom{6}{3}$ ways; the two women can be chosen in $\binom{4}{2}$ ways. Using the multiplication principle, the total number of committees possible with no restrictions is

$$\begin{aligned}\binom{6}{3} \cdot \binom{4}{2} &= \frac{6!}{3!3!} \cdot \frac{4!}{2!2!} \\ &= 120.\end{aligned}$$

Your Turn. You wish to borrow two mystery books and three westerns from your friend. He owns five mysteries and seven westerns. How many different selections can you make?

Sample Problem 2.19. *How many different "words" of five letters can you make from the letters of the word*

$$REPUBLICAN,$$

if every word must contain two different vowels and three different consonants?

Solution. The three consonants can be chosen in $\binom{6}{3} = 20$ ways, and the vowels in $\binom{4}{2} = 6$ ways. After the choice is made, the letters can be arranged in $5! = 120$ ways. So there are $20 \cdot 6 \cdot 120 = 14400$ "words."

Your Turn. What is the answer if you use the word

$$DEMOCRAT?$$

An Illustrative Example

The following example illustrates two important facts. First, the numbers that arise in selection problems can be very large—sometimes the intermediate steps are much greater than the answer. And second, there are often two (or more) ways to attack the same problem.

Suppose the sixteen members of the girls' track team are to be divided into four teams of four members each, to train for the relay. Assuming that you don't care who runs in which position, how many ways can they be divided?

Start by arbitrarily ordering the four teams I, II, III, IV. The members of Team I can be chosen in $C(16, 4)$ ways. After they are chosen, there are 12 girls left, so there are $C(12, 4)$ ways to choose Team II. Team III can be chosen in $C(8, 4)$ ways. Then Team IV is decided (there are only four girls left.) We have

$$C(16, 4) \cdot C(12, 4) \cdot C(8, 4)$$

arrangements. But the order of the teams did not really matter, so every arrangement is one of 4! that all give the same solution. (4! is the number of ways of arbitrarily assigning the labels I, II, III, IV to the four teams.) So the answer is

$$\frac{C(16, 4) \cdot C(12, 4) \cdot C(8, 4)}{4!} = \frac{16!}{12! \cdot 4!} \cdot \frac{12!}{8! \cdot 4!} \cdot \frac{8!}{4! \cdot 4!} = \frac{16!}{(4!)^4}$$

(the 12!'s and 8!'s cancel).

Equation (2.5) was useful because of the way the factorials 12! and 8! canceled.

If you try to work out the exact answer to the problem, you will find that 16! is greater than 2×10^{13}. On many calculators this large number will only be represented approximately, and the divisions will give you at best an approximate answer. But if you write out

$$\frac{16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{(4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1) \cdot (4 \cdot 3 \cdot 2 \cdot 1)}$$

and cancel you'll get $15 \cdot 14 \cdot 13 \cdot 11 \cdot 10 \cdot 7 \cdot 6 \cdot 5$, which comes to 63063000. This selection problem involved intermediate numbers nearly a million times as great as the solution. If, instead of canceling the 12! and 8! we had calculated the three C-numbers, the multiplications would have been very large.

As we promised, there is another way to handle this problem. This time, select one girl at random. There are 15 other girls from whom her three teammates can be chosen, so this can be done in $\binom{15}{3}$ ways. For each possible choice here, suppose you select one of the 12 remaining girls. Her team can be completed in $\binom{11}{3}$ ways. Again choose a girl from those remaining; her team can be completed in $\binom{7}{3}$ ways. And the remaining girls must make up the other team. The number of choices is

$$\binom{15}{3} \cdot \binom{11}{3} \cdot \binom{7}{3} = \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} \cdot \frac{11 \cdot 10 \cdot 9}{3 \cdot 2 \cdot 1} \cdot \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1}$$

which comes to the same answer. This time the intermediate numbers were considerably smaller.

Problems Combining Different Methods

Some problems can be solved by combining the methods we have been using with the rule of sum.

Sample Problem 2.20. *In Sample Problem 2.18, how many ways are there to select the committee if one particular couple, say Mr. and Mrs. Smith, do not wish to be on the committee together?*

Solution. The total number of committees possible with no restrictions was 120. If Mr. and Mrs. Smith are both on the committee, then the rest of the committee is found by choosing two of the five remaining men and one of the three remaining women. This can be done in

$$\binom{5}{2} \cdot \binom{3}{1} = \frac{5!}{3!2!} \cdot \frac{3!}{2!1!} = 30$$

ways. So 30 of the 120 possible committees contain both Smiths; when these are excluded there remain 90 ways of choosing the committee.

(We have in fact applied the rule of sum: interpret S as the set of all possible committees, and T as the set of committees not containing both Smiths. Then we found $|S| = 120$ and $|S \cap T| = 30$, we required $|S \backslash T|$, which the rule tells us equals 90.)

Your Turn. A group contains three men and seven women. In how many ways can a committee of three people be chosen if it can contain no more than one man?

Sometimes both arrangement and selection techniques are applied to the same problem.

Sample Problem 2.21. *A copying company has eight photocopying machines and seven employees who can operate them. There are four identical copying jobs to be done.* (*There are* 4000 *booklets to be made; each person makes* 1000 *copies.*) *How many ways are there to allocate these jobs to operators and machines?*

Solution. We shall give two solutions to this problem.

(i) This is like Sample Problem 2.11, but the jobs are now identical. In that problem there were $P(7, 4) \cdot P(8, 4)$ arrangements. As the jobs are indistinguishable, the ordering of A, B, C, D is irrelevant, so we divide the number of arrangements by 4!. The answer is

$$\frac{P(7,4)\cdot P(8,4)}{4!} = \frac{(7\cdot 6\cdot 5\cdot 4)\cdot(8\cdot 7\cdot 6\cdot 5)}{4\cdot 3\cdot 2\cdot 1},$$

which works out to 58800.

(ii) There are $C(7,4)$ ways to choose four operators and $C(8,4)$ ways to choose four machines. When the choice is made, there are 4! ways to assign the four workers to four machines. So the answer is

$$P(7,4)\cdot P(8,4)\cdot 4! = \frac{7\cdot 6\cdot 5\cdot 4}{4\cdot 3\cdot 2\cdot 1}\cdot\frac{8\cdot 7\cdot 6\cdot 5}{4\cdot 3\cdot 2\cdot 1}\cdot(4\cdot 3\cdot 2\cdot 1),$$

the same as before. (There is an extra factor 4! in the numerator and in the denominator.)

Exercises 2.3 A

1. List all the selections of size 3 that can be made from the set $\{A, B, C, D, E\}$.

2. Calculate the following quantities:

(i) $C(8,3)$; (ii) $C(9,4)$; (iii) $\binom{6}{3}$;

(iv) $C(7,3)$; (v) $C(7,7)$; (vi) $\binom{8}{6}$.

3. The math department wishes to select four of its 16 members to teach the finite math course. In how many ways can this selection be made?

4. The Student Council consists of six juniors and 12 seniors. A committee of two juniors and three seniors is to be formed. How many ways can this be done?

5. A test has 12 questions, and you must answer nine of them.

(i) How many ways can you choose which questions to answer?

(ii) In how many ways can you make your choice if you must include Question 1 or Question 2 (or maybe both)?

6. How many 12-bit binary strings with five 1's are possible? How many of them start 101?

7. A state lottery requires you to choose five different numbers from $\{1, 2, \dots, 49\}$. The order in which the numbers are chosen does not matter.

(i) How many possible choices are there?

(ii) The state then draws six different numbers. You win if all five of your numbers are chosen. How many of the possible choices are winners?

8. The directors of a mutual fund select a portfolio of three speculative stocks and three blue-chip stocks. If they must choose from five speculative stocks and seven blue-chip stocks, how many different portfolios could be formed?

9. A businessman wishes to pack three different ties for a business trip. If he has six ties available, how many different selections could be made?

10. There are 18 undergraduates and 14 graduates in the Math club. A committee of five members is to be selected. Calculate how many ways this can be done, if:

(i) There is no restriction.
(ii) There must be exactly two graduates.
(iii) There must be at least two graduates and two undergraduates.

11. A Euchre deck of cards contains 25 cards: 6 spades, 6 hearts, 6 diamonds, 6 clubs and a joker. A five-card hand is dealt.

(i) How many different hands are possible?
(ii) How many of those hands contain only spades?
(iii) How many hands contain three spades and two clubs?
(iv) How many hands contain the joker?
(v) How many hands contain only spades and clubs, at least one of each?

Exercises 2.3 B

1. Calculate the following quantities:

(i) $C(7, 7)$; (ii) $\binom{6}{5}$; (iii) $\binom{6}{1}$;

(iv) $C(10, 2)$; (v) $\binom{8}{7}$; (vi) $\binom{8}{0}$.

2. Calculate the following quantities:

(i) $C(8, 4)$; (ii) $C(10, 0)$; (iii) $C(8, 3)$;

(iv) $\binom{6}{4}$; (v) $\binom{2}{2}$; (vi) $\binom{7}{2}$.

3. A test has ten questions, five in part A and five in part B.

(i) A student has to choose five questions, two from part A and three from part B. How many ways can she make her choice?

(ii) How many ways can she make her choice if she must choose five questions, at least two from each part?

4. There are eight seniors, six juniors, five sophomores, and five freshmen on the student senate. A bylaws committee with four members is to be selected. Calculate how many ways this can be done, if:

(i) There must be one member from each class.
(ii) There must be exactly two seniors.
(iii) There must be at least two seniors.

5. The homecoming committee wishes to decorate twelve tables in the school colors of red and blue. The tables are in two rows of six and there are to be six red and six blue tables.

(i) How many different arrangements are possible?
(ii) In how many arrangements are there three red tables in each row?

6. A club with 20 members wants to elect a committee consisting of President, Secretary, and two Ordinary Members. How many committees are possible?

7. Ruth wishes to choose five guests for a dinner party from among her nine closest friends.

(i) In how many different ways can she choose her guest list?
(ii) Suppose two of the group are husband and wife and Ruth must include either both of them or neither. How many possible choices are there?
(iii) Suppose instead that two of the group are bitter enemies and Ruth cannot include both. How many possible guest lists are there?

8. Suppose Ruth, in the preceding problem, wishes to choose five guests for a dinner party from among eleven friends, six men, and five women. She wishes the final party to consist of three men and three women (including herself). How many different guest lists are possible?

9. A regular deck of cards contains 52 cards: 13 spades, 13 hearts, 13 diamonds, and 13 clubs. A five-card hand is dealt.

(i) How many hands contain only spades?
(ii) How many hands contain three spades and two clubs?
(iii) How many hands contain only spades and clubs, at least one of each?

10. There are 12 appetizers on the menu, five cold and seven hot. How many ways can you choose four different appetizers for your table? In how many cases does this include two hot and two cold appetizers?

11. A club with 24 members wants to elect a committee consisting of President, Secretary, and three Ordinary Members. How many committees are possible?

12. The motor pool has eight drivers and nine vehicles. Four drivers must be assigned to four vehicles for today's duty. How many ways can it be done?

13. It is required to select, from the set of numbers $\{1, 2, 3, 4, 5, 6, 7, 8\}$, a subset of four that will contain either 1 or 2, but not both. How many selections are possible?

14. A baseball squad contains three pitchers, two catchers and 12 players who can play at any other position. How many different teams of nine players, with exactly one pitcher and exactly one catcher, can be formed?

15. Suppose the Senate contains 52 Democrats and 48 Republicans. A committee of six must be chosen.

(i) How many different committees can be chosen?

(ii) How many of these possible committees contain exactly three Democrats and three Republicans?

2.4 More About Selections

Identities Concerning the Choice Function

We observe two important properties of the choice function $\binom{s}{k}$.

Theorem 4. *When* $0 \le k \le s$,

$$\binom{s}{k} = \binom{s}{s-k}.$$

Proof. From (2.5),

$$\binom{s}{k} = \frac{s!}{(s-k)!k!} = \frac{s!}{k!(s-k)!} = \binom{s}{s-k}.$$ □

This equality can also be derived as follows. If we wish to choose k things from a set of s, we could just as easily say which $k - s$ will *not* be included. In other words, to specify a subset T of S, we could just as easily specify the complement of T in the set S.

Sample Problem 2.22. *Calculate* $\binom{100}{98}$.

Solution. From Theorem 4,

$$\binom{100}{98} = \binom{100}{2} = \frac{100 \cdot 99}{1 \cdot 2} = 4950.$$

Your Turn. Calculate $\binom{81}{79}$.

Theorem 5. *For all integers k and s such that* $1 \le k \le s$,

$$\binom{s}{k} = \binom{s-1}{k} + \binom{s-1}{k-1}.$$

Proof. Suppose S is an s-set, and suppose x is one particular member of S. Each subset of S falls into one of two categories: those that *do* contain x, and those that *do not* contain x. So we calculate the number of k-sets on S that contain x, and then the number that do not. Adding these numbers together, we get the total number of sets.

The k-sets on S that contain x each have $k-1$ other elements, and we get a different k-set whenever we choose a different $(k-1)$-set from those other objects. So their number equals the number of $(k-1)$-sets that can be chosen from $S\backslash\{x\}$, namely $\binom{s-1}{k-1}$.

The k-sets on S that do not contain x are precisely the k-sets of $S\backslash\{x\}$, and there are $\binom{s-1}{k}$ of them. Adding, we get the result. $\square$

Theorem 5 can be used to generate a table of values of the numbers $\binom{s}{k}$. For convenience, we call the top row of the table row 0. In row s we write the values of

$$\binom{s}{0}\binom{s}{1}\binom{s}{2}\cdots\binom{s}{s}$$

in that order. We write the table in a triangular form: $\binom{s}{0}$ occurs a half-space to the left of $\binom{s-1}{0}$. So the table is as shown. Observe that $\binom{s}{k}$ lies in the table with $\binom{s-1}{k}$ and $\binom{s-1}{k}$ above it, a half-step to the left and a half-step to the right, respectively. So Theorem 5 tells us that every term can be constructed by adding together the two numbers above it.

$$\begin{array}{ccccccc}
 & & & \binom{0}{0} & & & \\
 & & \binom{1}{0} & & \binom{1}{1} & & \\
 & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\
\binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

This table is called *Pascal's triangle*. When the values are substituted in we get the following array.

$$\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\multicolumn{13}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots}
\end{array}$$

The Binomial Theorem

The choice function also arises when an expression like $x+y$, a *binomial* function, is raised to a positive integer power.

It is easy to verify formulae like

$$\begin{aligned}(x+y)^2 &= x^2+2xy+y^2,\\ (x+y)^3 &= x^3+3x^2y+3xy^2+y^3,\\ (x+y)^4 &= x^4+4x^3y+6x^2y^2+4xy^3+y^4.\end{aligned}$$

Now suppose n is any positive integer. The following theorem provides an expression for $(x+y)^n$, as a function of x and y. The theorem is called the *binomial theorem* for positive integer index n,

Theorem 6. *If x and y are any numbers and n is any positive integer, then*

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

For example, if $n=4$, the formula is

$$\begin{aligned}(x+y)^4 &= \binom{4}{0}x^{4-0}y^0 + \binom{4}{1}x^{4-1}y^1 + \binom{4}{2}x^{4-2}y^2 + \binom{4}{3}x^{4-3}y^3 + \binom{4}{4}x^{4-4}y^4\\ &= \binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4.\end{aligned}$$

$\binom{4}{0}=1$, so the first term is x^4. As $\binom{4}{1}=4$, the second term is $4x^3y$. Since $\binom{4}{2}=6$, the next term is $6x^2y^2$. In the same way, it is easy to verify the remaining terms in the previously stated expression for $(x+y)^4$.

The proof of the binomial theorem occurs later in this section.

The theorem is useful when one of the two variables is replaced by 1, or when a negative sign or a numerical coefficient is included. Some examples are:

$$\begin{aligned}(1+x)^3 &= 1+3x+3x^2+x^3,\\ (x-y)^3 &= x^3-3x^2y+3xy^2-y^3,\\ (2x+y)^3 &= 8x^3+12x^2y+6xy^2+y^3.\end{aligned}$$

Sample Problem 2.23. *Write down an expression for* $(1+x)^6$.

Solution. The relevant coefficients are $\binom{6}{0}=\binom{6}{6}=1$, $\binom{6}{1}=\binom{6}{5}=6$, $\binom{6}{2}=\binom{6}{4}=15$, $\binom{6}{3}=20$. So

$$\begin{aligned}(1+x)^6 &= \binom{6}{0} + \binom{6}{1}x + \binom{6}{2}x^2 + \binom{6}{3}x^3 + \binom{6}{4}x^4 + \binom{6}{5}x^5 + \binom{6}{6}x^6\\ &= 1+6x+15x^2+20x^3+15x^4+6x^5+x^6.\end{aligned}$$

Your Turn. Write down an expression for $(x-y)^6$.

Sample Problem 2.24. *What is the coefficient of y^3 in $(2x - 3y)^5$?*

Solution. The term involving y^3 is $\binom{5}{3}(2x)^2(-3y)^3$, so the numerical coefficient is

$$\binom{5}{3}2^2(-3)^3 = 10 \cdot 4 \cdot (-27) = -1080$$

and the coefficient is $-1080x^2$.

Your Turn. What is the coefficient of y^2 in $(3x + 2y)^4$?

The binomial theorem can be used to find the approximate values of powers of numbers close to 1.

Sample Problem 2.25. *Find the approximate value* (to within 10^{-3}) *of* 1.02^{10}.

Solution. We write $1.02^{10} = (1 + 2 \cdot 10^{-2})^{10}$. Then it equals

$$1 + 10 \cdot 2 \cdot 10^{-2} + 45 \cdot 4 \cdot 10^{-4} + 120 \cdot 8 \cdot 10^{-6} + \cdots$$

Every subsequent term is at most one-tenth of the one before it. So the approximate value is

$$1 + 0.2 + 0.018 + 0.00096 + \cdots = 1.219 \text{ approx.}$$

Your Turn. Find the approximate value of 0.99^8.

You should observe the relationship between Pascal's triangle and the binomial theorem. The coefficients in row s of the triangle are precisely the coefficients in the expression for $(x + y)^s$. For example, row 6 is

$$1, \quad 6, \quad 15, \quad 20, \quad 15, \quad 6, \quad 1.$$

(Remember, the first row is row 0.)

Proof of the Binomial Theorem

The following proof can be omitted at a first reading.

We consider the following product of n factors:

$$(x_1 + y_1) \cdot (x_2 + y_2) \cdots (x_n + y_n),$$

where $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ are some $2n$ variables. On expanding the product, we obtain 2^n terms, and each will have n factors, with either x_i or y_i as the ith factor. For example, if $n = 2$ we get

$$x_1x_2 + y_1x_2 + x_1y_2 + y_1y_2,$$

and if $n = 3$ the product is

$$x_1x_2x_3 + y_1x_2x_3 + x_1y_2x_3 + y_1y_2x_3 + x_1x_2y_3 + y_1x_2y_3 + x_1y_2y_3 + y_1y_2y_3.$$

In constructing the terms, we made n decisions:

- from the first factor, take either x_1 or y_1;
- from the second factor, take either x_2 or y_2;

and so on.

In the general case, how many terms are there with exactly k y's? The term with $y_1y_2\cdots y_kx_{k+1}x_{k+2}\cdots x_n$ can arise in one and only one way in the product: you must choose the y term at steps $1, 2, \ldots, k$ and the x term at every later step. Similarly, if $i_1, i_2, \ldots, i_k$ are any k of the numbers $1, 2, \ldots, n$, then the term with $y_{i_1}y_{i_2}\cdots y_{i_k}$ and all other terms x's occurs if and only if you choose y at steps $i_1, i_2, \ldots, i_k$, and x at the other $n - k$ steps. So the number of terms that contain precisely k of the y's will equal the number of ways of selecting k indices $i_1, i_2, \ldots,$ i_k from $\{1, 2, \ldots, n\}$, namely $\binom{n}{k}$.

Now put $x_1 = x_2 = \cdots = x_n = x$ and $y_1 = y_2 = \cdots = y_n = y$. There will be exactly $\binom{n}{k}$ terms equal to $x^{n-k}y^k$. So

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}y^k.$$

Consequences of the Theorem

An interesting summation formula can be obtained by putting $x = y = 1$ in Theorem 6:

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n. \tag{2.6}$$

Equation (2.6) gives us an interesting way to work out the number of subsets of a set.

Theorem 7. *Suppose S is any set with n elements. The number of subsets of S is 2^s.*

Proof. The number of k-element subsets of S is $\binom{n}{k}$. The total number of subsets equals the number of 0-element subsets (1, for the empty set), plus the number of 1-element subsets, plus the number of 2-element subsets, and so on up to S itself. So the number is $\sum_{k=0}^{n} \binom{n}{k}$, and the result follows from (2.6). $\square$

Sample Problem 2.26. *How many ways are there to choose three or more people from a set of* 11 *people?*

Solution. There are 2^{11} possible ways to choose a subset of the 11 people. However, the subsets with 0, 1, or 2 elements are not allowed. So the number is

$$2^{11} - \binom{11}{0} - \binom{11}{1} - \binom{11}{2} = 2048 - 1 - 11 - 55 = 1981.$$

Your Turn. How many ways are there to choose at most seven books from a selection of ten books?

Exercises 2.4 A

1. Construct the first ten rows of Pascal's triangle.

2. What is the coefficient of:

(i) y^2 in $(1-2y)^6$? (ii) x^3 in $(x+2y)^5$?
(iii) y^5 in $(2x-y)^7$? (iv) y^5 in $(x-y)^6$?

3. Use the binomial theorem to calculate:

(i) $(x-1)^4$; (ii) $(1-2z)^5$;
(iii) $(x+x^{-1})^4$; (iv) $(x+y+z)^3$;
(v) $(x+y-z)^2$.

4. Evaluate 1.01^3.

5. Find an approximation to 1.01^5, using four terms of the binomial theorem.

6. Use the binomial theorem with $x=-1$ and $y=1$ to prove that

$$\sum_{k=0}^{s}(-1)^k\binom{s}{k} = 0.$$

7. How many subsets are there in a set of seven elements?

Exercises 2.4 B

1. Calculate $\binom{97}{94}$.

2. What is the coefficient of:

(i) x^2 in $(1+3x)^6$? (ii) y^3 in $(5x+2y)^4$?
(iii) y^4 in $(3x-2y)^6$? (iv) x^3 in $(2x-2y)^4$?

(v) x^2 in $(1+2x)^5$? (vi) x^2y^2 in $(x+y)^4$?

3. Use the binomial theorem to calculate:

(i) $(x-3y)^3$; (ii) $(1+x^2)^3$;
(iii) $(2x+y)^4$; (iv) $(2x-5y)^3$.

4. Use the binomial theorem to calculate:

(i) $(x+1)^3$; (ii) $(2-4x)^3$;
(iii) $(2x-x^{-1})^3$; (iv) $(x^2-x^{-1})^3$;
(v) $(1+x^2)^2$; (vi) $(1+2x-y)^3$.

5. Use the binomial theorem to evaluate 1.02^2.

6. Find an approximation to 1.01^8, using four terms of the binomial theorem.

7. Find an approximation to 0.99^5, using four terms of the binomial theorem.

8. What is the number of non-empty subsets of a nine-element set?

9. How many ways are there to select a subset of at least two elements from a set with eight elements?

10. Use Theorem 5 to prove that

$$\binom{n}{r}+2\binom{n}{r-1}+\binom{n}{r-2}=\binom{n+2}{r}$$

for any integers satisfying $2 \le r \le n$.

11. Use the binomial theorem with $x=-1$ and $y=1$ to prove that

$$\binom{s}{0}+\binom{s}{2}+\binom{s}{4}+\cdots=\binom{s}{1}+\binom{s}{3}+\binom{s}{5}+\cdots$$

and consequently that

$$\binom{s}{0}+\binom{s}{2}+\binom{s}{4}+\cdots=2^{s-1}.$$

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