Chapter 5
Algebraic Operad

The name 'operad' is a word that I coined myself, spending a week thinking about nothing else.

J.P. May in "Operads, algebras and modules"

An algebra of a certain type is usually defined by generating operations and relations, see for instance the classical definition of associative algebras, commutative algebras, Lie algebras. Given a type of algebras there is a notion of "free" algebra over a generic vector space V. Let us denote it by $\mathscr{P}(V)$. Viewed as a functor from the category Vect of vector spaces to itself, $\mathscr{P}$ is equipped with a monoid structure, that is a transformation of functors $\gamma : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$, which is associative, and another one $\eta : \mathrm{I} \to \mathscr{P}$ which is a unit. The existence of this structure follows readily from the universal properties of free algebras. Such a data $(\mathscr{P}, \gamma, \eta)$ is called an *algebraic operad*.

On the other hand, any algebraic operad $\mathscr{P}$ determines a type of algebras: the $\mathscr{P}$-algebras. The main advantage of this point of view on types of algebras is that operads look like associative algebras but in a different monoidal category (see Table 5.1).

So most of the constructions for associative algebras can be translated into this new context. This is exactly what we intend to do with Koszul duality theory in the following chapters.

Depending on further properties of the type of algebras, the associated operad might be of a special kind. For instance, if the generating operations are not supposed to satisfy any symmetry, if the relations are multilinear and if, in these relations, the variables stay in the same order, then the functor $\mathscr{P}$ is of the form

$$\mathscr{P}(V) = \bigoplus_n \mathscr{P}_n \otimes V^{\otimes n}$$

and the composition map γ is completely determined by $\mathbb{K}$-linear maps

$$\gamma : \mathscr{P}_k \otimes \mathscr{P}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \longrightarrow \mathscr{P}_{i_1+\cdots+i_k}.$$

Then $\mathscr{P}$ is called a *nonsymmetric operad*.

J.-L. Loday, B. Vallette, *Algebraic Operads*,
Grundlehren der mathematischen Wissenschaften 346,
DOI 10.1007/978-3-642-30362-3_5,

Table 5.1 Operads as monoids

	Category	Product	Unit
monoid	Set	$\times$	$\{*\}$
algebra	Vect	$\otimes$	$\mathbb{K}$
operad	$\mathsf{EndoFunct}_{\mathsf{Vect}}$	$\circ$	I

More generally, if the relations are multilinear, without any further hypothesis, then the functor $\mathscr{P}$ is completely determined by a family of $\mathbb{S}_n$-modules $\{\mathscr{P}(n)\}_{n\geq 0}$,

$$\mathscr{P}(V) := \bigoplus_n \mathscr{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$$

and the composition map γ is completely determined by $\mathbb{K}$-linear maps

$$\gamma : \mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \longrightarrow \mathscr{P}(i_1 + \cdots + i_k).$$

Then $\mathscr{P}$ is called a *symmetric operad.*

Another interesting case, leading to the study of algebras with divided powers, consists in taking

$$\Gamma\mathscr{P}(V) := \bigoplus_n \big(\mathscr{P}(n) \otimes V^{\otimes n}\big)^{\mathbb{S}_n}.$$

Of course, taking invariants instead of coinvariants leads to a different type of algebras only in positive characteristic.

In this book we are going to work mainly with symmetric operads, that we simply call operads. Since $\mathbb{S}$-modules (family of representations over all the finite symmetric groups) play a prominent role in this case, we devote Sect. 5.1 of this chapter to their study and to the Schur functors that they determine. For symmetric operads the *monoidal* definition can be made explicit in several ways.

The *classical* definition consists in describing an operad in terms of the spaces $\mathscr{P}(n)$ of n-ary operations. This family of spaces forms the $\mathbb{S}$-module, which is equipped with "compositions of operations". They satisfy some properties which reflect functoriality, associativity and unitality of the monoidal definition.

The *partial* definition is a variation of the classical definition which takes advantage of the fact that we only need to know how to compose two operations to describe the whole structure. It is a description by generators and relations.

There is also a *combinatorial* way of describing an operad. It is based on the combinatorial objects which crop up in the description of a free operad, namely the rooted trees. One can construct a monad in the monoidal category of $\mathbb{S}$-modules out of the rooted trees, and an operad is simply a representation of this monad (i.e. an algebra over the monad). It has the advantage of deserving many variations by changing these combinatorial objects. For instance, nonsymmetric operads, shuffle operads, cyclic operads, modular operads, properads, permutads and several others can be described analogously by replacing the rooted trees by some other combinatorial objects, see Sect. 13.14.

One should keep in mind that the most economical way of defining a concrete operad is, most often, by describing the type of algebras that it determines. The relationship with the monoidal definition is via the notion of "free algebra" as mentioned above. Another way is the following. A type of algebras is determined by generating operations (possibly with symmetries) and relations (supposed to be multilinear). The generating operations and their symmetries determine the $\mathbb{S}$-module. Taking all the formal compositions of operations gives the free operad on the generating operations. The relations can be translated as relators which are operations in the free operad. The relators determine an operadic ideal and the expected operad is the quotient of the free operad by this ideal. The algebras over the quotient operad are exactly the algebras of the starting type.

Historically one can say that operad theory began with composition of functions. Let us mention the seminal paper of Michel Lazard "Lois de groupes et analyseurs" [Laz55] where a system of compositions was called an "analyseur" (in French). It gave rise to the notion of formal groups. For more about the history of "operads" we refer to the first chapter of [MSS02].

Here is the content of this chapter. In Sect. 5.1 we introduce the notions of $\mathbb{S}$-module and of Schur functor, and various constructions on them. In Sect. 5.2 we give the monoidal definition of an operad, and we define the notion of algebra (and also of coalgebra) over an operad. Then we restrict ourselves for the rest of the book to symmetric operads and nonsymmetric operads. In Sect. 5.3 we give the classical and the partial definitions of a symmetric operad. In Sects. 5.4 and 5.5 we describe in detail the free operad over an $\mathbb{S}$-module. In Sect. 5.6 we give the combinatorial definition of an operad. Then we make explicit the relationship between "types of algebras" and algebraic operads in Sect. 5.7.

In Sect. 5.8 we introduce the notion of cooperad which will play a prominent role in Koszul duality theory of quadratic operads.

In Sect. 5.9 we treat the notion of nonsymmetric operad. It can be read independently of the first eight sections of this chapter. It consists in replacing the starting $\mathbb{S}$-modules by graded vector spaces. So, it is a simpler object and it can be considered as a toy-model in the operad theory.

Then we give a brief résumé of all the definitions and we end this chapter with a list of exercises.

Though we work over a ground field $\mathbb{K}$, many of the notions presented in this chapter are valid when $\mathbb{K}$ is a commutative ring.

5.1 $\mathbb{S}$-Module and Schur Functor

We introduce $\mathbb{S}$-modules upon which the notion of algebraic operad is based in this book. Composition of $\mathbb{S}$-modules is the core of operad theory. To any $\mathbb{S}$-module is associated an endofunctor of Vect, called the Schur functor, and vice versa. The interplay between both structures is a fruitful game.

5.1.1 $\mathbb{S}$-Module

By definition an $\mathbb{S}$-*module* over $\mathbb{K}$ is a family

$$M = \big(M(0), M(1), \ldots, M(n), \ldots\big),$$

of right $\mathbb{K}[\mathbb{S}_n]$-modules $M(n)$ (cf. Appendix A). It is sometimes called a "collection" in the literature. An $\mathbb{S}$-module is said to be *finite dimensional* if $M(n)$ is a finite dimensional vector space for any n. For $\mu \in M(n)$ the integer n is called the *arity* of μ. A morphism of $\mathbb{S}$-modules $f : M \to N$ is a family of $\mathbb{S}_n$-equivariant maps $f_n : M(n) \to N(n)$. When all the maps f_n are injective the $\mathbb{S}$-module M is said to be a *sub-$\mathbb{S}$-module* of N.

When $M(0) = 0$, the $\mathbb{S}$-module is called *reduced*.

5.1.2 Schur Functor

To any $\mathbb{S}$-module M we associate its *Schur functor* $\widetilde{M} : \mathsf{Vect} \to \mathsf{Vect}$ defined by

$$\widetilde{M}(V) := \bigoplus_{n\geq 0} M(n) \otimes_{\mathbb{S}_n} V^{\otimes n}.$$

Here $V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$ is viewed as a left $\mathbb{S}_n$-module under the left action

$$\sigma \cdot (v_1, \ldots, v_n) := (v_{\sigma^{-1}(1)}, \ldots, v_{\sigma^{-1}(n)}).$$

So the tensor product over $\mathbb{S}_n$ (i.e. over the ring $\mathbb{K}[\mathbb{S}_n]$) used in the definition of $\widetilde{M}$ is well-defined. Equivalently $\widetilde{M}(V)$ is the sum over n of the coinvariants of $M(n) \otimes V^{\otimes n}$ by the diagonal right action of $\mathbb{S}_n$.

Any morphism of $\mathbb{S}$-modules $\alpha : M \to N$ gives rise to a transformation of functors $\tilde{\alpha} : \widetilde{M} \to \widetilde{N}$.

Sometimes we need to work with the product instead of the sum of the components. We call *complete Schur functor* the infinite product:

$$\widehat{M}(V) := \prod_{n\geq 0} M(n) \otimes_{\mathbb{S}_n} V^{\otimes n}.$$

If the $\mathbb{S}$-module M is concentrated in arity 0 (resp. 1, resp. n), then the functor $\widetilde{M}$ is constant (resp. linear, resp. homogeneous polynomial of degree n). Observe that we get the *identity functor*, denoted $\widetilde{\mathrm{I}}$, by taking the Schur functor of $\mathrm{I} := (0, \mathbb{K}, 0, 0, \ldots)$, so $\widetilde{\mathrm{I}}(V) = \mathrm{Id}_{\mathsf{Vect}}(V) = V$.

In this subsection we use the two notations M and $\widetilde{M}$, but later on we will use only M for both notions.

There are three important constructions on endofunctors of Vect: the direct sum, the tensor product and the composition.

The *direct sum* of two functors $F, G : \mathsf{Vect} \to \mathsf{Vect}$ is given by

$$(F \oplus G)(V) := F(V) \oplus G(V).$$

The *tensor product* $(F \otimes G)$ is given by

$$(F \otimes G)(V) := F(V) \otimes G(V).$$

The *composition* of functors, denoted $F \circ G$, is given by

$$(F \circ G)(V) := F\big(G(V)\big).$$

We are going to show that in each case, if the functors F and G are Schur functors, then the resulting functor is also a Schur functor. We also unravel the $\mathbb{S}$-module from which it comes.

For the direct sum, it is immediate: for any $\mathbb{S}$-modules M and N their *direct sum* is the $\mathbb{S}$-module $M \oplus N$ defined by

$$(M \oplus N)(n) := M(n) \oplus N(n).$$

It is obvious that

$$(\widetilde{M \oplus N}) = \widetilde{M} \oplus \widetilde{N}.$$

Lemma 5.1.1. *Let M be an $\mathbb{S}$-module. For any $n \geq 0$ the multilinear part of $\widetilde{M}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$ is isomorphic, as an $\mathbb{S}_n$-module, to $M(n)$.*

Proof. First, it is clear that the multilinear part of $\widetilde{M}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$ inherits a structure of $\mathbb{S}_n$-module from the action of the symmetric group on the set of variables $\{x_1, \ldots, x_n\}$. Second, the identification of these two $\mathbb{S}_n$-modules is given by $\mu \mapsto (\mu \otimes x_1 \cdots x_n)$ for $\mu \in M(n)$. $\square$

5.1.3 Tensor Product of $\mathbb{S}$-Modules

For any $\mathbb{S}$-modules M and N their *tensor product* is the $\mathbb{S}$-module $M \otimes N$ defined by

$$(M \otimes N)(n) := \bigoplus_{i+j=n} \mathrm{Ind}_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} M(i) \otimes N(j).$$

In this formula we use the notion of induced representation, cf. Appendix A.1. Since the subset $Sh(i, j)$ of (i, j)-shuffles of $\mathbb{S}_n$ is a convenient set of representatives of the quotient $\mathbb{S}_i \times \mathbb{S}_j \backslash \mathbb{S}_n$ (cf. 1.3.2), we have:

$$(M \otimes N)(n) \cong \bigoplus_{i+j=n} M(i) \otimes N(j) \otimes \mathbb{K}\big[Sh(i, j)\big].$$

This tensor product of $\mathbb{S}$-modules is sometimes called in the literature the *Cauchy product*.

Proposition 5.1.2. *The tensor product of $\mathbb{S}$-modules is associative with unit the $\mathbb{S}$-module $(\mathbb{K}, 0, 0, \ldots)$. There is an equality of Schur functors*:

$$\widetilde{(M \otimes N)} = \widetilde{M} \otimes \widetilde{N}.$$

Proof. The first part is straightforward. The proof of the equality follows from the identities (where $i + j = n$):

$$\begin{aligned}
&\big(\mathrm{Ind}_{\mathbb{S}_i \times \mathbb{S}_j}^{\mathbb{S}_n} M(i) \otimes N(j)\big) \otimes_{\mathbb{S}_n} V^{\otimes n} \\
&\quad = \big(\big(M(i) \otimes N(j)\big) \otimes_{\mathbb{S}_i \times \mathbb{S}_j} \mathbb{K}[\mathbb{S}_n]\big) \otimes_{\mathbb{S}_n} \big(V^{\otimes i} \otimes V^{\otimes j}\big) \\
&\quad = \big(M(i) \otimes N(j)\big) \otimes_{\mathbb{S}_i \times \mathbb{S}_j} \big(V^{\otimes i} \otimes V^{\otimes j}\big) \\
&\quad = \big(M(i) \otimes_{\mathbb{S}_i} V^{\otimes i}\big) \otimes \big(N(j) \otimes_{\mathbb{S}_j} V^{\otimes j}\big).
\end{aligned}$$ □

5.1.4 Composite of $\mathbb{S}$-Modules

By definition the *composite* of the two $\mathbb{S}$-modules M and N is the $\mathbb{S}$-module

$$M \circ N := \bigoplus_{k \geq 0} M(k) \otimes_{\mathbb{S}_k} N^{\otimes k}.$$

The notation $N^{\otimes k}$ stands for the tensor product of k copies of the $\mathbb{S}$-module N. Observe that $\mathbb{S}_k$ is acting on $N^{\otimes k}$, that is $\mathbb{S}_k$ is acting on $N^{\otimes k}(n)$ for all n and this action commutes with the action of $\mathbb{S}_n$.

For instance, let $k = 2$. Then $N^{\otimes 2}(n) = (N \otimes N)(n) = \oplus_{i+j=n} N(i) \otimes N(j) \otimes \mathbb{K}[Sh(i, j)]$. The transposition $[2\ 1] \in \mathbb{S}_2$ is acting on the direct sum by sending

$$(\mu, \nu, \sigma) \in N(i) \otimes N(j) \otimes \mathbb{K}\big[Sh(i, j)\big]$$

to

$$(\nu, \mu, \sigma') \in N(j) \otimes N(i) \otimes \mathbb{K}\big[Sh(j, i)\big] \quad \text{where } \sigma' = \sigma[j+1\ \cdots\ i+j\ 1\ \cdots\ j].$$

When M and N are determined by only one representation, the operation $\circ$ is called the *plethysm* in representation theory.

Proposition 5.1.3. *The composite of the two $\mathbb{S}$-modules M and N satisfies the formula*

$$\widetilde{(M \circ N)} = \widetilde{M} \circ \widetilde{N},$$

where, on the right-hand side, the symbol $\circ$ stands for the composition of functors.

Proof. We want to prove $\widetilde{M}(\widetilde{N}(V)) = \widetilde{M \circ N}(V)$. We get:

$$\begin{aligned}
\widetilde{M} \circ \widetilde{N}(V) &= \textstyle\bigoplus_k M(k) \otimes_{\mathbb{S}_k} \widetilde{N}(V)^{\otimes k} \\
&= \textstyle\bigoplus_k M(k) \otimes_{\mathbb{S}_k} \widetilde{N^{\otimes k}}(V) && \text{by compatibility of } \otimes \\
&= \textstyle\bigoplus_{k,p} M(k) \otimes_{\mathbb{S}_k} \big(N^{\otimes k}(p) \otimes_{\mathbb{S}_p} V^{\otimes p}\big) && \text{by inspection} \\
&= \textstyle\bigoplus_{k,p} \big(M(k) \otimes_{\mathbb{S}_k} N^{\otimes k}(p)\big) \otimes_{\mathbb{S}_p} V^{\otimes p} && \text{by associativity} \\
&= \textstyle\bigoplus_p (M \circ N)(p) \otimes_{\mathbb{S}_p} V^{\otimes p} && \text{by definition of } \circ \\
&= \widetilde{M \circ N}(V).
\end{aligned}$$

□

Corollary 5.1.4. *For any two $\mathbb{S}$-modules M and N one has*

$$(M \circ N)(n) = \bigoplus_{k \geq 0} M(k) \otimes_{\mathbb{S}_k} \Big(\bigoplus \mathrm{Ind}_{\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} \big(N(i_1) \otimes \cdots \otimes N(i_k)\big)\Big)$$

where the second sum is extended, for fixed k and n, to all the nonnegative k-tuples $(i_1, \ldots, i_k)$ satisfying $i_1 + \cdots + i_k = n$.

Recall that a positive k-tuple $(i_1, \ldots, i_k)$ such that $i_1 + \cdots + i_k = n$ is called a *k-composition* of n.

The action of $\mathbb{S}_k$ on the right-hand side factor is on the set of k-tuples $\{(i_1, \ldots, i_k)\}$. This action is well-defined since the tensor product of vector spaces is associative and commutative.

Proof. It follows from the preceding propositions. □

5.1.5 Example

Suppose that $M(0) = N(0) = 0$ and $M(1) = N(1) = \mathbb{K}$. Then we get

$$\begin{gathered}
(M \circ N)(2) \cong M(2) \oplus N(2), \\
(M \circ N)(3) \cong M(3) \oplus \big(M(2) \otimes \mathrm{Ind}_{\mathbb{S}_2}^{\mathbb{S}_3}\big(N(2)\big)\big) \oplus N(3),
\end{gathered}$$

where, as a vector space, $\mathrm{Ind}_{\mathbb{S}_2}^{\mathbb{S}_3}(N(2))$ is the sum of three copies of $N(2)$. Indeed, for $k = 3$, we get the component $M(3) \otimes_{\mathbb{S}_3} N(1)^{\otimes 3}$ which is isomorphic to $M(3)$. For $k = 1$, we get the component $M(1) \otimes_{\mathbb{S}_1} N(3)$ which is isomorphic to $N(3)$. For $k = 2$, we get the component

$$M(2) \otimes_{\mathbb{S}_2} \big(\mathrm{Ind}_{\mathbb{S}_1 \times \mathbb{S}_2}^{\mathbb{S}_3}\big(N(1) \otimes N(2)\big) \oplus \mathrm{Ind}_{\mathbb{S}_2 \times \mathbb{S}_1}^{\mathbb{S}_3}\big(N(2) \otimes N(1)\big)\big).$$

Since $\mathbb{S}_2$ is exchanging the two summands, we get the expected result.

5.1.6 Notation

From now on we abandon the notation $\tilde{}$ and so we denote by the same symbol the $\mathbb{S}$-module and its associated Schur functor. Hence we freely treat an $\mathbb{S}$-module as an endofunctor of Vect. Accordingly a morphism of $\mathbb{S}$-modules $\alpha : M \to N$ is sometimes called a transformation of functors (meaning: transformation of Schur functors).

If $f : F \to F'$ and $g : G \to G'$ are two morphisms of $\mathbb{S}$-modules (equivalently natural transformations of Schur functors), then we denote sometimes the morphisms

$$f \oplus g : F \bigoplus G \to F' \bigoplus G', \tag{5.1}$$

$$f \otimes g : F \otimes G \to F' \otimes G', \tag{5.2}$$

$$f \circ g : F \circ G \to F' \circ G' \tag{5.3}$$

by (f, g) when there is no confusion.

5.1.7 On the Notation of Elements in a Composite $\mathbb{S}$-Module

As a consequence of Corollary 5.1.4 the space $(M \circ N)(n)$ is spanned by the equivalence classes of the elements

$$(\mu; \nu_1, \dots, \nu_k; \sigma)$$

(under the action of $\mathbb{S}_k$) where $\mu \in M(k)$, $\nu_1 \in N(i_1), \dots, \nu_k \in N(i_k)$, $\sigma \in Sh(i_1, \dots, i_k)$. When $\sigma = \mathrm{id}_n \in \mathbb{S}_n$ (the identity permutation), we denote the relevant class either by

$$\mu \circ (\nu_1, \dots, \nu_k)$$

or by

$$(\mu; \nu_1, \dots, \nu_k).$$

5.1.8 Associativity Isomorphism of the Composite [Sign Warning]

The composition of Schur functors is associative. It implies that the composition of $\mathbb{S}$-modules is associative too. We would like to insist on the following phenomenon which does not happen in the algebra case (versus the operad case): in the associativity isomorphism $(M \circ N) \circ P \cong M \circ (N \circ P)$ of $\mathbb{S}$-modules, the switching map τ (see Sect. 1.5.2) plays a role. Indeed, in the identification of the component

$$\big(M(a) \otimes N(b) \otimes N(c)\big) \otimes P(d) \otimes P(e) \otimes P(f) \otimes P(g)$$

in $(M \circ N) \circ P$ with the component

$$M(a) \otimes \big((N(b) \otimes P(d) \otimes P(e)\big) \otimes \big(N(c) \otimes P(f) \otimes P(g)\big)$$

in $M \circ (N \circ P)$ we need to use the switching map to carry $N(c)$ over $P(d) \otimes P(e)$. As said above this phenomenon[1] does not occur in the algebra case and on the left-hand side because the product $\otimes$ is bilinear and the product $\circ$ is linear on the left-hand side. This phenomenon is important to notice in the sign-graded case since the occurrence of τ may result in signs in the formulas.

5.1.9 Composite of Morphisms

For any pair $f : M \to M'$, $g : N \to N'$ of morphisms of $\mathbb{S}$-modules, their *composite product* $f \circ g : M \circ N \to M' \circ N'$ is given explicitly by the formula

$$(f \circ g)(\mu; \nu_1, \ldots, \nu_k) := \big(f(\mu); g(\nu_1), \ldots, g(\nu_k)\big),$$

where $(\mu; \nu_1, \ldots, \nu_k)$ represents an element of

$$M(k) \otimes_{\mathbb{S}_k} \Big(\bigoplus \mathrm{Ind}_{\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} \big(N(i_1) \otimes \cdots \otimes N(i_k)\big)\Big).$$

Beware: $f \circ g$ does not mean the composite of g and f in the sense of composition in a category, which has no meaning here anyway.

Observe that this composite is not linear in the right-hand side variable.

Proposition 5.1.5. *The category of* $\mathbb{S}$*-modules* $(\mathbb{S}\text{-}\mathsf{Mod}, \circ, \mathrm{I})$ *is a monoidal category.*

Proof. It follows from the comparison to Schur functors. □

In a first reading, the rest of this section can be bypassed and the reader can move to the beginning of Sect. 5.2.

[1] If the grandfather J wants to make a picture of his family, then he has two choices. He can put his children E, H and S on his right side, and then the grandchildren Y, B, A further right. Or, he can put the grandchildren on the right side of their parent: E, Y, B and H, A, and then put these subfamilies on his right. That gives two different pictures since JEHSYBA≠JEYBHAS. If J had only one child, the pictures would have been the same.

5.1.10 Generating Series

To any finite dimensional $\mathbb{S}$-module $M = \{M(n)\}_{n\geq 0}$ we associate its *generating series* (also called *Hilbert–Poincaré series*) defined by

$$f^M(x) := \sum_{n\geq 0} \frac{\dim M(n)}{n!} x^n.$$

Proposition 5.1.6. *Let M and N be two finite dimensional $\mathbb{S}$-modules. The following equalities hold*:

$$\begin{aligned} f^{M\oplus N}(x) &= f^M(x) + f^N(x), \\ f^{M\otimes N}(x) &= f^M(x) f^N(x), \\ f^{M\circ N}(x) &= f^M\big(f^N(x)\big), \end{aligned}$$

assuming $N(0) = 0$ in the last equality.

Proof. The first equality is immediate. The second one follows from Sect. 5.1.3. The third one (with $N(0) = 0$) follows from Corollary 5.1.4. □

5.1.11 Symmetric Function Indicator

There is a finer invariant than the generating series: the Frobenius characteristic. Starting with an $\mathbb{S}$-module M, it consists in taking the isomorphism class of the $\mathbb{S}_n$-representation $M(n)$ in the Grothendieck group of the representations of $\mathbb{S}_n$. The sum over n gives an element in $\prod_n Rep(\mathbb{S}_n)$ which is known to be isomorphic to the algebra of symmetric functions. The image of this element, denoted by F^M, is called the *Frobenius characteristic*. The operations $\oplus$, $\otimes$, $\circ$ on $\mathbb{S}$-modules commute with their counterpart in the algebra of symmetric functions, cf. [Mac95].

5.1.12 Hadamard Product of $\mathbb{S}$-Modules

By definition the *Hadamard product* of the two $\mathbb{S}$-modules $\mathscr{P}$ and $\mathscr{Q}$ is the $\mathbb{S}$-module $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ given by

$$\big(\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}\big)(n) := \mathscr{P}(n) \otimes \mathscr{Q}(n),$$

where the action of $\mathbb{S}_n$ is the diagonal action.

The generating series of $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ is not the Hadamard product of the generating series of $\mathscr{P}$ and $\mathscr{Q}$, but it will be it in the nonsymmetric framework, see Sect. 5.9.10 for a discussion on this matter.

5.1.13 Linear Species

A right $\mathbb{S}$-module M can be viewed as a functor from the groupoid $\mathbb{S}$ of symmetric groups to the category Vect (or the category of $\mathbb{K}$-modules if $\mathbb{K}$ is a commutative ring). As a consequence it can be extended as a contravariant functor from the groupoid Bij of finite sets and all bijections to the category of vector spaces. We suppose that the empty set is an object of Bij. If X is a finite set, then the extended functor, still denoted by M, is given by the coinvariant space:

$$M(X) := \Bigg(\bigoplus_{f:\underline{n}\to X} M(n)\Bigg)_{\mathbb{S}_n}$$

where the sum is over all the bijections from $\underline{n} := \{1, \dots, n\}$ to X. The right action of $\sigma \in \mathbb{S}_n$ on $(f; \mu)$ for $\mu \in M(n)$ is given by $(f; \mu)^\sigma = (f\sigma; \mu^\sigma)$.

Such a functor is sometimes called a *linear species*, cf. [Joy86], [AM10, Sect. B.1.1]. Here is the translation of the above constructions into this language. Let M and N be functors from Bij to Vect. For any set X we have:

$$
\begin{aligned}
(M \oplus N)(X) &= M(X) \oplus N(X),\\
(M \otimes N)(X) &= \bigoplus_{X=Y\sqcup Z} M(Y) \otimes N(Z),\\
&\quad \text{where the sum is over all the ordered disjoint unions} Y \sqcup Z \text{ of } X,\\
(M \circ N)(X) &= \bigoplus_{B\in \mathrm{PART}(X)} M(B) \otimes \bigotimes_{b\in \mathrm{B}} N(X_b),\\
&\quad \text{where } \mathrm{PART}(X) = \text{set of decompositions of } X \text{(see below)}\\
\big(M \underset{\mathrm{H}}{\otimes} N\big)(X) &= M(X) \otimes N(X).
\end{aligned}
$$

See for instance [AM10, Appendix B].

5.1.14 On the Notation $\bigotimes_{b\in \mathrm{B}} N(X_b)$

A *decomposition* of the finite set X is a family of subsets $\{X_b\}_{b\in B}$ of X such that their disjoint union is X. We let n be the number of elements in B. For any contravariant functor $N : \mathsf{Bij} \to \mathsf{Vect}$ we define

$$\bigotimes_{b\in B} N(X_b) := \Bigg(\bigoplus_{f:\underline{n}\to B} N(X_{f(1)}) \otimes \cdots \otimes N(X_{f(n)})\Bigg)_{\mathbb{S}_n}$$

where the sum is over all the bijections from $\underline{n}$ to B. As usual, the right action of $\mathbb{S}_n$ on the direct sum is given by

$$(f; \mu_1, \dots, \mu_n)^\sigma = (f\sigma; \mu_{\sigma(1)}, \dots, \mu_{\sigma(n)}),$$

where $\sigma \in \mathbb{S}_n$ and $\mu_i \in N(X_{f(i)})$.

5.1.15 Invariants Versus Coinvariants

The Schur functor $\mathscr{P}(V)$ can be written as a sum of coinvariant spaces

$$\mathscr{P}(V)=\bigoplus_n \big(\mathscr{P}(n)\otimes V^{\otimes n}\big)_{\mathbb{S}_n}$$

where the symmetric group is acting diagonally on the tensor product. Here we use the fact that $V^{\otimes n}$ is a right module over $\mathbb{S}_n$.

Instead of working with coinvariants we could choose to work with invariants, that is to define

$$\Gamma\mathscr{P}(V):=\bigoplus_n \big(\mathscr{P}(n)\otimes V^{\otimes n}\big)^{\mathbb{S}_n}.$$

Everything would work, because the direct sum, the tensor product and the composition of such functors are of the same type. In particular, there exists a composition $\bar{\circ}$ of $\mathbb{S}$-modules such that, for any two $\mathbb{S}$-modules $\mathscr{P}$ and $\mathscr{Q}$, one has

$$\Gamma\mathscr{P}\circ\Gamma\mathscr{Q}=\Gamma(\mathscr{P}\,\bar{\circ}\,\mathscr{Q}).$$

This composite is given by

$$(\mathscr{P}\,\bar{\circ}\,\mathscr{Q})(n):=\bigoplus_r \big(\mathscr{P}(r)\otimes \mathscr{Q}^{\otimes r}\big)^{\mathbb{S}_r}(n).$$

Recall that the norm map of an $\mathbb{S}_n$-module M is given by

$$M_{\mathbb{S}_n}\to M^{\mathbb{S}_n},\qquad x\mapsto \sum_{\sigma\in\mathbb{S}_n} x^{\sigma}.$$

The norm map induces an $\mathbb{S}$-module map $\mathscr{P}\circ\mathscr{Q}\to\mathscr{P}\,\bar{\circ}\,\mathscr{Q}$ since we took coinvariants on the left-hand side and invariants on the right-hand side. Whenever $\mathscr{Q}(0)=0$ the induced transformation of functors

$$\Phi:\Gamma(\mathscr{P}\circ\mathscr{Q})\to\Gamma(\mathscr{P}\,\bar{\circ}\,\mathscr{Q})$$

is an isomorphism since $\mathbb{S}_r$ is acting freely on $\mathscr{Q}^{\otimes r}$ (cf. [Sto93, Fre00]).

In characteristic zero the norm map is an isomorphism, so $\mathscr{P}(V)\to\Gamma\mathscr{P}(V)$ is an isomorphism (see Appendix A). However in positive characteristic we get two different functors.

5.2 Algebraic Operad and Algebra over an Operad

We define a symmetric operad as a monoid in the monoidal category of symmetric modules. Since this is a monad, that is a monoid in the category of endofunctors of

Vect, one can define the notion of an algebra over an operad. Replacing $\mathbb{S}$-modules by arity graded vector spaces we get the notion of nonsymmetric operad. Taking invariants in place of coinvariants we get the notion of divided power operad. We call them collectively "algebraic operads".

5.2.1 Monoidal Definition of an Operad

By definition a *symmetric operad* $\mathscr{P} = (\mathscr{P}, \gamma, \eta)$ is an $\mathbb{S}$-module $\mathscr{P} = \{\mathscr{P}(n)\}_{n\geq 0}$ endowed with morphisms of $\mathbb{S}$-modules

$$\gamma : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$$

called *composition map*, and

$$\eta : \mathrm{I} \to \mathscr{P}$$

called the *unit map*, which make $\mathscr{P}$ into a monoid.

Explicitly, the morphisms γ and η satisfy the classical axioms for monoids, that is associativity:

$$\begin{tikzcd}
 & \mathscr{P} \circ (\mathscr{P} \circ \mathscr{P}) \arrow[r, "\mathrm{Id}\circ\gamma"] & \mathscr{P} \circ \mathscr{P} \arrow[dd, "\gamma"] \\
(\mathscr{P} \circ \mathscr{P}) \circ \mathscr{P} \arrow[ur, "\cong"] \arrow[d, "\gamma\circ\mathrm{Id}"'] & & \\
\mathscr{P} \circ \mathscr{P} \arrow[rr, "\gamma"] & & \mathscr{P}
\end{tikzcd}$$

and unitality:

$$\begin{tikzcd}
\mathrm{I} \circ \mathscr{P} \arrow[r, "\eta\circ\mathrm{Id}"] \arrow[dr, "="'] & \mathscr{P} \circ \mathscr{P} \arrow[d, "\gamma"] & \mathscr{P} \circ \mathrm{I} \arrow[l, "\mathrm{Id}\circ\eta"'] \arrow[dl, "="] \\
 & \mathscr{P}. &
\end{tikzcd}$$

Hence for any vector space V one has linear maps

$$\gamma(V) : \mathscr{P}\big(\mathscr{P}(V)\big) \to \mathscr{P}(V) \quad \text{and} \quad \eta(V) : V \to \mathscr{P}(V).$$

In the literature a monoid structure on an endofunctor is often called a monad, cf. Appendix B.4. Interpreting the $\mathbb{S}$-module $\mathscr{P}$ as a Schur functor, the monoid structure on $\mathscr{P}$ is nothing but a monad in the category of vector spaces.

Let $\mathscr{Q}$ be another symmetric operad. A *morphism of operads* from $\mathscr{P}$ to $\mathscr{Q}$ is a morphism of $\mathbb{S}$-modules $\alpha : \mathscr{P} \to \mathscr{Q}$, which is compatible with the composition maps. In other words, the following diagrams are supposed to be commutative:

$$\begin{tikzcd} \mathscr{P} \circ \mathscr{P} \arrow[r, "(\alpha,\alpha)"] \arrow[d, "\gamma^{\mathscr{P}}"'] & \mathscr{Q} \circ \mathscr{Q} \arrow[d, "\gamma^{\mathscr{Q}}"] \\ \mathscr{P} \arrow[r, "\alpha"] & \mathscr{Q} \end{tikzcd} \qquad \begin{tikzcd} & \mathrm{I} \arrow[dl, "\eta^{\mathscr{P}}"'] \arrow[dr, "\eta^{\mathscr{Q}}"] & \\ \mathscr{P} \arrow[rr, "\alpha"] & & \mathscr{Q}. \end{tikzcd}$$

The category of operads over $\mathsf{Vect}_{\mathbb{K}}$ is denoted by $\mathsf{Op}_{\mathbb{K}}$ or Op.

In order to differentiate between the notion of composition in the operadic framework (the map γ) and the classical notion of composition of functors in category theory (denoted by $\circ$), we will sometimes say "operadic composition" for the first one.

Here we are mainly interested in the notion of operads in the category of vector spaces, or modules over a commutative ring, or in the category of chain complexes (dg spaces), but it is immediate to verify that it makes sense in any symmetric monoidal category with infinite sums such that finite sums commute with the monoidal structure.

When $\mathscr{P}(0) = 0$, the operad is called *reduced*.

5.2.2 Operadic Module

A *left module* over the symmetric operad $\mathscr{P}$ is an $\mathbb{S}$-module M together with an $\mathbb{S}$-module morphism $\mathscr{P} \circ M \to M$ satisfying associativity and unitality with respect to the operad structure on $\mathscr{P}$. The terminology varies a lot in the literature. It is sometimes called a "twisted $\mathscr{P}$-algebra", or sometimes simply a "$\mathscr{P}$-algebra" ; see [Fre09a] for a thorough study of this structure. There is a particular case which is important: when M is constant, that is concentrated in arity 0. This gives rise to the notion of algebra over an operad, see below.

Observe that the notion of *right module*, which is obvious to define, gives rise to a completely different structure.

A bimodule (or two-sided module) over $\mathscr{P}$ is an $\mathbb{S}$-module $\mathscr{M}$ together with morphisms of $\mathbb{S}$-modules $\mathscr{P} \circ \mathscr{M} \to \mathscr{M}$ and $\mathscr{M} \circ \mathscr{P} \to \mathscr{M}$ satisfying the classical axioms of associativity and unitality for two-sided modules.

5.2.3 Algebra over an Operad

By definition an *algebra over the operad* $\mathscr{P}$, or a $\mathscr{P}$*-algebra* for short, is a vector space A equipped with a linear map $\gamma_A : \mathscr{P}(A) \to A$ such that the following diagrams commute:

$$\begin{tikzcd}
 & \mathscr{P}(\mathscr{P}(A)) \arrow[r, "\mathscr{P}(\gamma_A)"] & \mathscr{P}(A) \arrow[dd, "\gamma_A"] \\
(\mathscr{P} \circ \mathscr{P})(A) \arrow[ur, "="] \arrow[d, "\gamma(A)"'] & & \\
\mathscr{P}(A) \arrow[rr, "\gamma_A"] & & A
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{I}(A) \arrow[r, "\eta(A)"] \arrow[dr, "="'] & \mathscr{P}(A) \arrow[d, "\gamma_A"] \\
 & A.
\end{tikzcd}$$

The transformation of functors γ applied to A, that is $\gamma(A)$, is not to be confused with $\mathscr{P}(\gamma_A)$ which is the functor $\mathscr{P}$ applied to the map γ_A. They have the same source and the same target, but they are different.

Let A' be another $\mathscr{P}$-algebra. A *morphism of $\mathscr{P}$-algebras* is a linear map $f : A \to A'$ which makes the following diagram commutative:

$$\begin{tikzcd}
\mathscr{P}(A) \arrow[r, "\gamma_A"] \arrow[d, "\mathscr{P}(f)"'] & A \arrow[d, "f"] \\
\mathscr{P}(A') \arrow[r, "\gamma_{A'}"] & A'.
\end{tikzcd}$$

We denote by $\mathscr{P}$-alg the category of $\mathscr{P}$-algebras.

Observe that if $\mathscr{P}$ is interpreted as a monad, then this is the classical notion of an algebra over a monad, see Appendix B.4.

5.2.4 Functors Between Categories of Algebras

Let $\alpha : \mathscr{P} \to \mathscr{Q}$ be a morphism of operads. Then there is a well-defined functor

$$\alpha^* : \mathscr{Q}\text{-}\mathsf{alg} \longrightarrow \mathscr{P}\text{-}\mathsf{alg}.$$

Indeed, the $\mathscr{P}$-algebra associated to the $\mathscr{Q}$-algebra A has the same underlying vector space structure and the composition map is the composite

$$\mathscr{P}(A) \xrightarrow{\alpha(A)} \mathscr{Q}(A) \to A.$$

Observe that the functor which assigns to an operad the category of algebras over this operad is contravariant.

We give in Proposition 5.2.2 an interpretation of a $\mathscr{P}$-algebra as a morphism of operads.

5.2.5 Free $\mathscr{P}$-Algebra

In the category of $\mathscr{P}$-algebras, a $\mathscr{P}$-algebra $\mathscr{F}(V)$, equipped with a linear map $\eta : V \to \mathscr{F}(V)$ is said to be *free* over the vector space V if it satisfies the following universal condition:

For any $\mathscr{P}$-algebra A and any linear map $f : V \to A$ there is a unique $\mathscr{P}$-algebra extension $\tilde{f} : \mathscr{F}(V) \to A$ of f:

$$\begin{tikzcd} V \arrow[r, "\eta"] \arrow[dr, "f"'] & \mathscr{F}(V) \arrow[d, "\tilde{f}"] \\ & A. \end{tikzcd}$$

Observe that a free algebra is unique up to a unique isomorphism, cf. Appendix B.2.2.

In other words, $\mathscr{F}$ is a functor $\mathsf{Vect} \to \mathscr{P}\text{-}\mathsf{alg}$ which is left adjoint to the forgetful functor:

$$\mathrm{Hom}_{\mathscr{P}\text{-}\mathsf{alg}}\big(\mathscr{F}(V), A\big) \cong \mathrm{Hom}_{\mathsf{Vect}}(V, A).$$

For any vector space V one can equip $\mathscr{P}(V)$ with a $\mathscr{P}$-algebra structure as follows. Define

$$\gamma_{\mathscr{P}(V)} := \gamma(V) : \mathscr{P}\big(\mathscr{P}(V)\big) \to \mathscr{P}(V).$$

The axioms defining the operad $\mathscr{P}$ show that $(\mathscr{P}(V), \gamma(V))$ is a $\mathscr{P}$-algebra.

Proposition 5.2.1. *The $\mathscr{P}$-algebra $(\mathscr{P}(V), \gamma(V))$ equipped with $\eta(V) : V \to \mathscr{P}(V)$ is the free $\mathscr{P}$-algebra over V.*

Proof. For any linear map $f : V \to A$, where A is a $\mathscr{P}$-algebra, we consider the composition $\tilde{f} : \mathscr{P}(V) \xrightarrow{\mathscr{P}(f)} \mathscr{P}(A) \xrightarrow{\gamma_A} A$. It extends f since the composite

$$V \xrightarrow{\eta(V)} \mathscr{P}(V) \xrightarrow{\mathscr{P}(f)} \mathscr{P}(A) \xrightarrow{\gamma_A} A$$

is f by $\mathscr{P}(f) \circ \eta(V) = \eta(A) \circ f$ and $\gamma_A \circ \eta(A) = \mathrm{I}_A$.

The following diagram is commutative by functoriality and the fact that A is a $\mathscr{P}$-algebra:

$$\begin{tikzcd} \mathscr{P}(\mathscr{P}(V)) \arrow[r] \arrow[d] & \mathscr{P}(V) \arrow[d] \arrow[dd, bend left=60, "\tilde{f}"] \\ \mathscr{P}(\mathscr{P}(A)) \arrow[r] \arrow[d] & \mathscr{P}(A) \arrow[d] \\ \mathscr{P}(A) \arrow[r] & A. \end{tikzcd}$$

It implies that the map $\tilde{f}$ is a $\mathscr{P}$-algebra morphism.

Let us show that the map $\tilde{f}$ is unique. Since we want $\tilde{f}$ to coincide with f on V and we want $\tilde{f}$ to be an algebra morphism, there is no other choice by $\tilde{f}$. $\square$

5.2.6 Endofunctors of Vect

In Sect. 5.1 we showed that any $\mathbb{S}$-module gives rise to an endofunctor of Vect, called the associated Schur functor. Similarly any graded vector space $\mathscr{P} = \{\mathscr{P}_n\}_{n\geq 0}$ gives rise to an endofunctor of Vect by the formula

$$\mathscr{P}(V) := \bigoplus_{n\geq 0} \mathscr{P}_n \otimes V^{\otimes n}.$$

We remark immediately that this endofunctor is the Schur functor associated to the $\mathbb{S}$-module, still denoted by $\mathscr{P}$, given by

$$\mathscr{P}(n) := \mathscr{P}_n \otimes \mathbb{K}[\mathbb{S}_n].$$

Writing $\mathscr{P}(n)$ or $\mathscr{P}_n$ suffices to indicate which framework we are working in. Moreover, most of the time, it is only the endofunctor which is relevant.

A third interesting case consists in starting with an $\mathbb{S}$-module and taking the invariants instead of the coinvariants when forming the endofunctor:

$$\Gamma\mathscr{P}(V) := \bigoplus_{n\geq 0} \big(\mathscr{P}(n) \otimes V^{\otimes n}\big)^{\mathbb{S}_n}.$$

5.2.7 Symmetric Operads

In a symmetric operad $(\mathscr{P}, \gamma, \eta)$ the composition map γ is made up of linear maps

$$\gamma(i_1, \ldots, i_k) : \mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \longrightarrow \mathscr{P}(i_1 + \cdots + i_k)$$

that will be studied in the next section.

If A is a $\mathscr{P}$-algebra, then the structure map γ_A determines maps

$$\mathscr{P}(n) \otimes A^{\otimes n} \twoheadrightarrow \mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n} \xrightarrow{\gamma_A(n)} A.$$

Therefore, any element $\mu \in \mathscr{P}(n)$ and any n-tuple $a_1 \ldots a_n \in A^{\otimes n}$ give rise to an element

$$\gamma_A(n)(\mu; a_1 \ldots a_n) \in A.$$

Such an element μ is called an *n-ary operation* and $\mathscr{P}(n)$ is called the space of n-ary operations. By abuse of notation we write

$$\mu(a_1 \ldots a_n) := \gamma_A(n)(\mu; a_1 \ldots a_n),$$

and we call μ an operation on A.

The unit functor $\eta : \mathrm{I} \to \mathscr{P}$ defines a particular element in $\mathscr{P}(1)$, namely the image of $1 \in \mathbb{K} = \mathrm{I}(1)$, which we denote by $\mathrm{id} \in \mathscr{P}(1)$ and call the *identity operation*.

Indeed we have $\mathrm{id}(a) = a$ for any $a \in A$. For any symmetric operad $\mathscr{P}$ the space $\mathscr{P}(1)$ inherits the structure of a unital associative algebra over $\mathbb{K}$. It is given by the map $\gamma(1) : \mathscr{P}(1) \otimes \mathscr{P}(1) \to \mathscr{P}(1)$ and id is the unit of $\mathscr{P}(1)$.

5.2.8 Nonsymmetric Operads

A *nonsymmetric operad* (ns operad for short) is an arity graded vector space $\mathscr{P} = \{\mathscr{P}_n\}_{n\geq 0}$ endowed with morphisms $\gamma : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$ and $\eta : \mathrm{I} \to \mathscr{P}$ which make it into a monoid in the monoidal category of arity graded vector spaces. The composition map γ is completely determined by maps

$$\gamma_{i_1,\dots,i_k} : \mathscr{P}_k \otimes \mathscr{P}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \longrightarrow \mathscr{P}_{i_1+\cdots+i_k}$$

for $n = i_1 + \cdots + i_k$ and η is determined by an element $\mathrm{id} \in \mathscr{P}_1$. See Sect. 5.9 for more.

A nonsymmetric operad $\mathscr{P}$ gives rise to an operad, usually still denoted by $\mathscr{P}$, such that $\mathscr{P}(n) = \mathscr{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$. The action of the symmetric group is induced by the $\mathbb{S}_n$-module structure of the regular representation $\mathbb{K}[\mathbb{S}_n]$. The composition map is the tensor product

$$\gamma(i_1, \dots, i_k) = \gamma_{i_1,\dots,i_k} \otimes \gamma^{Ass}(i_1, \dots, i_k)$$

where γ^{Ass} is the composition map of the operad *Ass* that will be described below.

Such a symmetric operad is sometimes called a *regular operad*.

5.2.9 Operads with Divided Powers

Definitions and results of this subsection come from Benoit Fresse's paper [Fre00] in which the reader will find the details. Let $\{\mathscr{P}(n)\}_{n\geq 0}$ be an $\mathbb{S}$-module with $\mathscr{P}(0) = 0$. Recall that there is defined an endofunctor $\Gamma\mathscr{P}$ by using invariants instead of coinvariants:

$$\Gamma\mathscr{P}(V) := \bigoplus_{n\geq 1} \big(\mathscr{P}(n) \otimes V^{\otimes n}\big)^{\mathbb{S}_n}.$$

An *operad with divided powers* is a monoid structure on $\Gamma\mathscr{P}$, that is a composition map $\bar{\gamma} : \Gamma\mathscr{P} \circ \Gamma\mathscr{P} \to \Gamma\mathscr{P}$ which is associative and unital.

If $\mathscr{P} = (\mathscr{P}, \gamma, \eta)$ is a symmetric operad, then it determines an operad with divided powers as follows. First, recall from Sect. 5.1.15 that the norm map permits us to construct a map $\Phi : \Gamma(\mathscr{P} \circ \mathscr{P}) \to \Gamma(\mathscr{P} \,\bar{\circ}\, \mathscr{P})$ which happens to be an isomorphism. The composition map $\bar{\gamma}$ is defined as the composite

$$\Gamma\mathscr{P} \circ \Gamma\mathscr{P} = \Gamma(\mathscr{P} \,\bar{\circ}\, \mathscr{P}) \xrightarrow{\Phi^{-1}} \Gamma(\mathscr{P} \circ \mathscr{P}) \xrightarrow{\Gamma(\gamma)} \Gamma\mathscr{P}.$$

An algebra over $\Gamma\mathscr{P}$ is called a *$\mathscr{P}$-algebra with divided powers*. It can be shown that if $\mathscr{P}(n)$ is $\mathbb{S}_n$-projective, e.g. $\mathscr{P}(n)$ is the regular representation, then a $\mathscr{P}$-algebra with divided powers is the same as a $\mathscr{P}$-algebra. It is also the case in characteristic zero since the norm map is then an isomorphism.

From this construction it follows that there is a forgetful functor from the category of $\mathscr{P}$-algebras with divided powers to the category of $\mathscr{P}$-algebras. It is often a challenge to find a presentation of the first out of a presentation of the second.

One of the interests of the notion of algebras with divided powers is the following result, proved in [Fre00]. Let $A_\bullet$ be a simplicial $\mathscr{P}$-algebra. If $A_\bullet$ is 2-reduced (that is $A_0 = A_1 = 0$), then its homotopy $\pi_*(A_\bullet)$ is a graded $\Gamma\mathscr{P}$-algebra. For $\mathscr{P} = Com$ it is a result of Henri Cartan (cf. loc.cit.).

5.2.10 First Examples of Operads

We show that a unital associative algebra can be interpreted as an operad. Then we introduce the "three graces", the operads Ass, Com and Lie. In the next section we treat the endomorphism operad which can be seen as a toy-model for the operad structure.

EXAMPLE 0. A unital associative algebra is an example of an operad. Indeed, let R be a unital associative algebra and consider the $\mathbb{S}$-module M given by $M(1) = R$ and $M(n) = 0$ otherwise. Then we have $M(V) = R \otimes V$, and an operad structure on M is equivalent to a unital associative algebra structure on R. The composition map γ is induced by the product on R:

$$\gamma(V) : M \circ M(V) \to M(V),$$
$$R \otimes R \otimes V \to R \otimes V, \qquad (r, s, v) \mapsto (rs, v).$$

The unit map η is induced by the unit of R. An algebra over this operad is simply a left R-module. So any unital associative algebra is an example of an algebraic operad.

In particular, if $R = \mathbb{K}$, then we get the *identity operad* I, that we sometimes denote by $Vect$ to emphasize the fact that its category of algebras is simply the category of vector spaces Vect.

EXAMPLE 1. Let $Ass : \mathsf{Vect} \to \mathsf{Vect}$ be the Schur functor given by $Ass(V) := \overline{T}(V) = \oplus_{n\geq 1} V^{\otimes n}$ (reduced tensor module). As an $\mathbb{S}$-module we have $Ass(n) = \mathbb{K}[\mathbb{S}_n]$ (regular representation), since $\mathbb{K}[\mathbb{S}_n] \otimes_{\mathbb{S}_n} V^{\otimes n} = V^{\otimes n}$ for $n \geq 1$, and $Ass(0) = 0$. The map $\gamma(V) : Ass(Ass(V)) \to Ass(V)$ is given by "composition of noncommutative polynomials". This is the symmetric operad encoding associative algebras since an algebra over Ass is a nonunital associative algebra. So the free Ass-algebra over the vector space V is nothing but the reduced tensor algebra $\overline{T}(V)$ (cf. Sect. 1.1.3).

The symmetric operad Ass comes from a nonsymmetric operad, denoted As, for which $As_n = \mathbb{K}\mu_n$ (one-dimensional space) for $n \geq 1$. On associative algebras, μ_n

is the n-ary operation $\mu_n(x_1, \ldots, x_n) = x_1 \cdots x_n$. This basic example is treated in more detail in Chap. 9.

The operad of unital associative algebras, denoted *uAss*, is the same except that $uAss(0) = \mathbb{K}$. The image of $1 \in uAss(0)$, in the unital associative algebra A, is the unit of A. The free algebra is the tensor algebra: $uAss(V) = T(V)$.

In the process which associates a symmetric operad to a nonsymmetric operad the composition map is given by

$$\gamma(i_1, \ldots, i_k) = \gamma^{\mathscr{P}}_{i_1,\ldots,i_k} \otimes \gamma^{Ass}(i_1, \ldots, i_k)$$

up to a reordering of the factors on the source space.

Since $Ass(n)$ is the regular representation, there is no difference between associative algebras with divided powers and associative algebras.

EXAMPLE 2. Let $Com : \mathsf{Vect} \to \mathsf{Vect}$ be the Schur functor given by

$$Com(V) := \overline{S}(V) = \bigoplus_{n \geq 1} S^n V = \bigoplus_{n \geq 1} \left(V^{\otimes n}\right)_{\mathbb{S}_n}.$$

As an $\mathbb{S}$-module we have $Com(n) = \mathbb{K}$ with trivial action, since

$$\mathbb{K} \otimes_{\mathbb{S}_n} V^{\otimes n} = \left(V^{\otimes n}\right)_{\mathbb{S}_n} = S^n V \quad \text{for } n \geq 1,$$

and $Com(0) = 0$. The map $\gamma(V) : Com(Com(V)) \to Com(V)$ is given by "composition of polynomials". This is the symmetric operad encoding commutative algebras since an algebra over *Com* is a nonunital commutative algebra (in the sense commutative and associative). So the free *Com*-algebra over the vector space V is nothing but the (nonunital) symmetric algebra $\overline{S}(V)$ (cf. Sect. 1.1.8).

Since any commutative algebra is an associative algebra, there is a functor $Com\text{-}\mathsf{alg} \longrightarrow Ass\text{-}\mathsf{alg}$. It is induced by the morphism of operads $Ass \to Com$, which, in degree n, is the augmentation map $\mathbb{K}[\mathbb{S}_n] \to \mathbb{K}$, $\sigma \mapsto 1$ (projection onto the trivial representation). This case is treated in more details in Chap. 13.

It is proved in [Fre00] that the notion of "divided power commutative algebras" is the classical one, see Sect. 13.1.12 for the precise presentation.

EXAMPLE 3. Let $Lie : \mathsf{Vect} \to \mathsf{Vect}$ be the functor such that the space $Lie(V) \subset \overline{T}(V)$ is generated by V under the *bracket* operation $[x, y] := xy - yx$. We know by Corollary 1.3.5 that this is the free Lie algebra on V. Let $Lie(n)$ be the multilinear part of degree n in the free Lie algebra $Lie(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$. One can show that there is an operad structure on the Schur functor *Lie* induced by the operad structure on *Ass* (Lie polynomials of Lie polynomials are again Lie polynomials). An algebra over the operad *Lie* is a Lie algebra.

Any associative algebra is a Lie algebra under the antisymmetrization of the product $[x, y] = xy - yx$. This functor $Ass\text{-}\mathsf{alg} \longrightarrow Lie\text{-}\mathsf{alg}$ is induced by the morphism of operads $Lie \to Ass$, which, in arity n, is the inclusion $Lie(n) \rightarrowtail Ass(n) = \mathbb{K}[\mathbb{S}_n]$ mentioned above. This case is treated in more detail in Sect. 13.2.

It is proved in [Fre00] that the notion of "divided power Lie algebras" over a field with positive characteristic coincides with the notion of restricted Lie algebras introduced by Nathan Jacobson [Jac62], see Sect. 13.2.15 for the precise presentation.

5.2.11 Endomorphism Operad

For any vector space V the endomorphism operad End_V is given by

$$\mathrm{End}_V(n) := \mathrm{Hom}\big(V^{\otimes n}, V\big),$$

where, by convention, $V^{\otimes 0} = \mathbb{K}$. The right action of $\mathbb{S}_n$ on $\mathrm{End}_V(n)$ is induced by the left action on $V^{\otimes n}$. The composition map γ is given by composition of endomorphisms:

$$\begin{tikzcd}
V^{\otimes i_1}\otimes \arrow[d, "f_1"] & \cdots & \otimes V^{\otimes i_k} \arrow[d, "f_k"] & = & V^{\otimes n} \arrow[d, "f_1\otimes\cdots\otimes f_k"] \\
V\otimes & \cdots \arrow[d, "f"] & \otimes V & = & V^{\otimes k} \arrow[d, "f"] \\
 & V & & = & V
\end{tikzcd}$$

$$\gamma(f; f_1, \ldots, f_k) := f\big(f_1 \otimes \cdots \otimes f_k\big).$$

It is immediate to verify that End_V is an algebraic operad.

Proposition 5.2.2. *A $\mathscr{P}$-algebra structure on the vector space A is equivalent to a morphism of operads $\mathscr{P} \to \mathrm{End}_A$.*

Proof. This statement follows from the natural isomorphism

$$\mathrm{Hom}_{\mathbb{S}_n}\big(\mathscr{P}(n), \mathrm{Hom}\big(A^{\otimes n}, A\big)\big) = \mathrm{Hom}\big(\mathscr{P}(n) \otimes_{\mathbb{S}_n} A^{\otimes n}, A\big). \qquad \square$$

By definition a *graded $\mathscr{P}$-algebra* over the graded operad $\mathscr{P}$ is a graded vector space A (i.e. an object in the sign-graded category gVect) and a morphism of graded operads $\mathscr{P} \to \mathrm{End}_A$. We leave it to the reader to write down the compatibility conditions in terms of the map $\gamma_A : \mathscr{P}(A) \to A$.

5.2.12 Algebras over an Operad: Functorial Properties

By abuse of notation, we often denote by $\mu : A^{\otimes n} \to A$ the image of $\mu \in \mathscr{P}(n)$ under γ_A in $\mathrm{End}_A(n)$. It follows immediately from the interpretation of an algebra over an operad given in Proposition 5.2.2 that if $\alpha : \mathscr{P} \to \mathscr{Q}$ is a morphism of operads, then any $\mathscr{Q}$-algebra A has a $\mathscr{P}$-algebra structure via the composition of operad morphisms

$$\mathscr{P} \to \mathscr{Q} \to \mathrm{End}_A.$$

Hence we get the functor

$$\alpha^* : \mathscr{Q}\text{-}\mathsf{alg} \longrightarrow \mathscr{P}\text{-}\mathsf{alg}.$$

This functor, which is analogous to the restriction functor for modules, admits a left adjoint, analogous to the induced functor for modules. It is denoted by

$$\alpha_! : \mathscr{P}\text{-}\mathsf{alg} \to \mathscr{Q}\text{-}\mathsf{alg}$$

and constructed as follows. For any $\mathscr{P}$-algebra A, the $\mathscr{Q}$-algebra $\alpha_!(A)$ is the quotient of the free $\mathscr{Q}$-algebra $\mathscr{Q}(A)$ which identifies the two different $\mathscr{P}$-algebra structures. It is a particular case the relative composite product which will appear in Sect. 11.2.1. Explicitly it is given by the coequalizer:

$$\mathscr{Q} \circ \mathscr{P} \circ A \underset{\mathrm{Id}_{\mathscr{Q}} \circ \gamma_A}{\overset{\rho \circ \mathrm{Id}_A}{\rightrightarrows}} \mathscr{Q} \circ A \twoheadrightarrow \mathscr{Q} \circ_{\mathscr{P}} A =: \alpha_!(A),$$

where the right $\mathscr{P}$-action ρ on $\mathscr{Q}$ is the composition $\mathscr{Q} \circ \mathscr{P} \xrightarrow{\mathrm{Id}_{\mathscr{Q}} \circ \alpha} \mathscr{Q} \circ \mathscr{Q} \xrightarrow{\gamma^{\mathscr{Q}}} \mathscr{Q}$. In the particular case of the morphism $\alpha : Lie \to Ass$ we obtain the universal algebra of a Lie algebra: $\alpha_!(\mathfrak{g}) = U(\mathfrak{g})$.

5.2.13 Ubiquity of the Elements of $\mathscr{P}(n)$

Let $V_n = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$ be an n-dimensional vector space with preferred basis. The element

$$x_1 \otimes \cdots \otimes x_n \in V_n^{\otimes n} \subset \mathscr{P}(V_n)^{\otimes n}$$

is called the *generic element*. Applying the n-ary operation $\mu \in \mathscr{P}(n)$ to the generic element gives an element of the free $\mathscr{P}$-algebra over V_n:

$$\gamma_{\mathscr{P}(V_n)} : \mathscr{P}(n) \otimes \mathscr{P}(V_n)^{\otimes n} \to \mathscr{P}(V_n),$$
$$\mu \otimes (x_1 \otimes \cdots \otimes x_n) \mapsto \mu(x_1, \ldots, x_n)$$

(this is a slight abuse of notation since we do not mention γ). The resulting map

$$\mathscr{P}(n) \to \mathscr{P}(V_n),$$
$$\mu \mapsto \mu(x_1, \ldots, x_n)$$

is one-to-one onto the multilinear part of degree n of $\mathscr{P}(V_n)$. The relationship with the action of the symmetric group is as follows. For $\sigma \in \mathbb{S}_n$, we have

$$\mu^\sigma(x_1, \dots, x_n) = \mu\big(\sigma \cdot (x_1, \dots, x_n)\big) = \mu(x_{\sigma^{-1}(1)}, \dots, x_{\sigma^{-1}(n)})$$

in $\mathscr{P}(V_n)$.

So, any such μ can be viewed either as an n-ary operation or as an element of some specific free $\mathscr{P}$-algebra.

In practice we will often talk about "the operation $x * y$" to mean "the operation $\mu \in \mathscr{P}(2)$ determined by $\mu(x, y) := x * y$". Similarly we will allow ourselves to say "the relation $(x * y) * z = x * (y * z)$", when we really mean "the relator $\mu \circ (\mu, \mathrm{id}) - \mu \circ (\mathrm{id}, \mu) \in \mathscr{T}(E)(3)$", see Sect. 5.5 for the notation $\mathscr{T}$.

5.2.14 Operadic Ideal and Quotient Operad

An *operadic ideal* (or simply ideal) of an operad $\mathscr{P}$ is a sub-$\mathbb{S}$-module $\mathscr{I}$ of $\mathscr{P}$ such that the operad structure of $\mathscr{P}$ passes to the quotient $\mathscr{P}/\mathscr{I}$. Explicitly it is equivalent to the following conditions. For any family of operations $\{\mu; \nu_1, \dots, \nu_k\}$, if one of them is in $\mathscr{I}$, then we require that the composite $\gamma(\mu; \nu_1, \dots, \nu_k)$ is also in $\mathscr{I}$.

5.2.15 Coalgebra over an Operad

Let V be a vector space. By definition the *co-endomorphism operad* over V, denoted coEnd_V, is given by

$$\mathrm{coEnd}_V(n) := \mathrm{Hom}\big(V, V^{\otimes n}\big).$$

The right action of $\mathbb{S}_n$ on $\mathrm{coEnd}_V(n)$ is induced by the right action on $V^{\otimes n}$. The composition map is given by composition of morphisms:

$$\begin{array}{ccccc}
 & V & & = & V \\
 & \downarrow & & & \downarrow \\
V\otimes & \cdots & \otimes V & = & V^{\otimes k} \\
\downarrow & & \downarrow & & \downarrow \\
V^{\otimes i_1}\otimes & \cdots & \otimes V^{\otimes i_k} & = & V^{\otimes n}.
\end{array}$$

By definition a *coalgebra C over the operad $\mathscr{P}$* is a vector space C and a morphism of operads

$$\mathscr{P} \longrightarrow \mathrm{coEnd}_C.$$

Explicitly, for any n, the data is an $\mathbb{S}_n$-equivariant map

$$\mathscr{P}(n) \otimes C \longrightarrow C^{\otimes n}.$$

The image of $\mu \in \mathscr{P}(n)$, that is the map $C \to C^{\otimes n}$, is called an *n-ary cooperation* and, often, still denoted by μ by abuse of notation. In order to simplify the terminology we allow ourselves to call C a $\mathscr{P}$-coalgebra.

When $\mathscr{P}$ is the associative operad *Ass*, an *Ass*-coalgebra is a coassociative coalgebra (also called associative coalgebra) as defined in Sect. 1.2.1. In the case $\mathscr{P} = \mathit{Com}$ it is a cocommutative (and coassociative) coalgebra. We simply say commutative coalgebra. When $\mathscr{P} = \mathit{Lie}$ we get the notion of *Lie coalgebra* (sometimes referred to as coLie coalgebra). Explicitly a Lie coalgebra is a vector space L equipped with a linear map $\Delta : L \to L \otimes L$ which is antisymmetric, i.e. $\tau\Delta = -\Delta$, and satisfies the co-Leibniz rule:

$$(\Delta \otimes \mathrm{Id})\Delta = (\mathrm{Id} \otimes \Delta)\Delta + (\mathrm{Id} \otimes \tau)(\Delta \otimes \mathrm{Id})\Delta.$$

5.3 Classical and Partial Definition of an Operad

From now on, by "operad" we mean symmetric operad. So we suppose that the endofunctor is in fact a Schur functor induced by an $\mathbb{S}$-module.

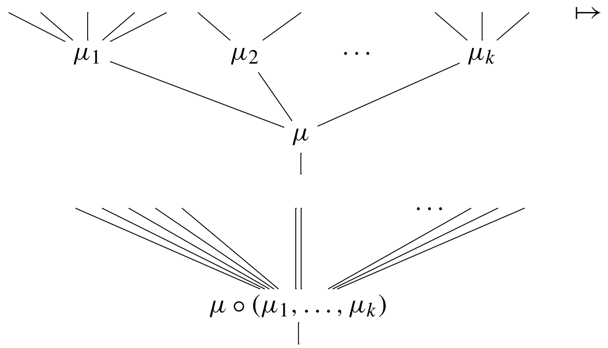
The classical definition of an operad is quite technical, but it is the most common form appearing in algebraic topology papers on operads. For the third definition one takes advantage of the fact that the operadic structure can be determined out of some elementary compositions called "partial compositions". It is very helpful in some frameworks because it has the minimal number of generators.

5.3.1 Classical Definition of an Operad (J.P. May [May72])

Let us now describe explicitly the operad structure on $\mathbb{S}$-modules. By Corollary 5.1.4 the vector space $(\mathscr{P} \circ \mathscr{P})(n)$ is a quotient of the direct sum of all the possible tensor products $\mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k)$ for $i_1 + \cdots + i_k = n$. So, the composition map γ of the operad $\mathscr{P}$ defines linear maps

$$\gamma(i_1, \ldots, i_k) : \mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \longrightarrow \mathscr{P}(i_1 + \cdots + i_k).$$

Pictorially this composition looks as follows:



The next proposition gives the conditions under which a family of linear maps $\gamma(i_1, \ldots, i_k)$ gives rise to an operad.

Proposition 5.3.1. *Let $\mathscr{P} = \{\mathscr{P}(n)\}_{n\geq 0}$ be an $\mathbb{S}$-module. The maps*

$$\gamma(i_1, \ldots, i_k) : \mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \longrightarrow \mathscr{P}(i_1 + \cdots + i_k)$$

define an operad structure on $\mathscr{P}$ if and only if they satisfy the following conditions:

(a) *for any integers k and n, the map*

$$\sum \gamma(i_1, \ldots, i_k) : \mathscr{P}(k) \otimes \Big(\bigoplus \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \Big) \longrightarrow \mathscr{P}(n),$$

where the direct sum is over all k-tuples $(i_1, \ldots, i_k)$ such that $i_1 + \cdots + i_k = n$, factors through the tensor product over $\mathbb{S}_k$. Moreover it is equivariant with respect to the action of $\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}$ (we use the natural embedding of this group product into $\mathbb{S}_n$),

(b) *for any set of indices $(j_{1,1}, \ldots, j_{1,i_1}, j_{2,1}, \ldots, j_{2,i_2}, \ldots, j_{n,1}, \ldots, j_{n,i_n})$ the following square is commutative* (*we leave out the $\otimes$ signs*):

$$\begin{tikzcd}
\mathscr{P}(n)\mathscr{P}(r_1)\cdots\mathscr{P}(r_n) \arrow[rddd, bend left] & \\
\mathscr{P}(n)\mathscr{P}(i_1)\mathscr{P}(j_{1,1})\cdots\mathscr{P}(j_{1,i_1})\mathscr{P}(i_2)\mathscr{P}(j_{2,1})\cdots\cdots\mathscr{P}(j_{n,i_n}) \arrow[u] \arrow[d, "\cong"] & \\
\mathscr{P}(n)\mathscr{P}(i_1)\cdots\mathscr{P}(i_n)\mathscr{P}(j_{1,1})\cdots\mathscr{P}(j_{1,i_1})\mathscr{P}(j_{2,1})\cdots\cdots\mathscr{P}(j_{n,i_n}) \arrow[d] & \\
\mathscr{P}(m)\mathscr{P}(j_{1,1})\cdots\mathscr{P}(j_{1,i_1})\mathscr{P}(j_{2,1})\cdots\cdots\mathscr{P}(j_{n,1})\cdots\mathscr{P}(j_{n,i_n}) \arrow[r] & \mathscr{P}(\ell)
\end{tikzcd}$$

where $r_k = j_{k,1} + \cdots + j_{k,i_k}$ for $k = 1$ to n, $m = i_1 + \cdots + i_n$ and $\ell = r_1 + \cdots + r_n$,

(c) *there is an element* id *in* $\mathscr{P}(1)$ *such that the evaluation of* $\gamma(n) : \mathscr{P}(1) \otimes \mathscr{P}(n) \to \mathscr{P}(n)$ *on* (id, μ) *is equal to* μ, *and the evaluation of* γ *on* $(\mu; \mathrm{id} \ldots, \mathrm{id})$ *is equal to* μ.

Proof. Starting with an algebraic operad $(\mathscr{P}, \gamma, \eta)$, we get the maps $\gamma(i_1, \ldots, i_k)$ by restriction to the identity shuffle. The unit map defines an inclusion $\eta : \mathbb{K} \to \mathscr{P}(1)$, whose image of $1 = 1_{\mathbb{K}}$ is the identity operation id. Then, it is clear that the axioms of functoriality, associativity and unitality of the operad data imply the properties (a), (b) and (c).

On the other hand, starting with an $\mathbb{S}$-module $\mathscr{P}$ and maps $\gamma(i_1, \ldots, i_k)$, we construct a monoid structure on the Schur functor as follows. Condition (a) provides a transformation of functors $\gamma : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$. Condition (b) ensures associativity of γ. Condition (c) ensures unitality. □

As a consequence of Proposition 5.3.1 one can define an operad as an $\mathbb{S}$-module $\{\mathscr{P}(n)\}_{n\geq 0}$ equipped with maps $\gamma(i_1, \ldots, i_k)$ for all k-tuples $(i_1, \ldots, i_k)$ satisfying the equivariance conditions (a), the associativity condition (b), and the unitality condition (c). This is what we call the *classical definition of an operad*.

5.3.2 Hadamard Product of Operads

Let $\mathscr{P}$ and $\mathscr{Q}$ be two operads. The Hadamard tensor product $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ of the underlying $\mathbb{S}$-modules (cf. Sect. 5.1.12) has a natural operad structure:

$$\begin{aligned}
&(\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})(k) \otimes (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})(n_1) \otimes \cdots \otimes (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})(n_k)\\
&\quad = \mathscr{P}(k) \otimes \mathscr{Q}(k) \otimes \mathscr{P}(n_1) \otimes \mathscr{Q}(n_1) \otimes \cdots \otimes \mathscr{P}(n_k) \otimes \mathscr{Q}(n_k)\\
&\quad \cong \mathscr{P}(k) \otimes \mathscr{P}(n_1) \otimes \cdots \otimes \mathscr{P}(n_k) \otimes \mathscr{Q}(k) \otimes \mathscr{Q}(n_1) \otimes \cdots \otimes \mathscr{Q}(n_k)\\
&\quad \longrightarrow \mathscr{P}(n) \otimes \mathscr{Q}(n) = (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})(n)
\end{aligned}$$

for $n = n_1 + \cdots + n_k$. Observe that we use the switching map in the category Vect to put the factors $\mathscr{Q}(i)$ in the correct position. Therefore, when Vect is replaced by another symmetric monoidal category (cf. Appendix B.3) signs might be involved. The operad $uCom$ is obviously a unit for this operation.

The operad $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ is the *Hadamard product* of the operads $\mathscr{P}$ and $\mathscr{Q}$.

Proposition 5.3.2. *Let* A *be a* $\mathscr{P}$*-algebra and let* B *be a* $\mathscr{Q}$*-algebra. The tensor product* $A \otimes B$ *is a* $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$*-algebra.*

Proof. Let us denote by $\gamma_A : \mathscr{P} \to \mathrm{End}_A$ and by $\gamma_B : \mathscr{Q} \to \mathrm{End}_B$ the respective actions of $\mathscr{P}$ on A and of $\mathscr{Q}$ on B. Then the action of $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ on $A \otimes B$ is given by the following composite

$$\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q} \xrightarrow{\gamma_A \underset{\mathrm{H}}{\otimes} \gamma_B} \mathrm{End}_A \underset{\mathrm{H}}{\otimes} \mathrm{End}_B \to \mathrm{End}_{A\otimes B},$$

where the last map is defined by

$$\begin{aligned}\mathrm{Hom}\big(A^{\otimes n}, A\big) \otimes \mathrm{Hom}\big(B^{\otimes n}, B\big) &\to \mathrm{Hom}\big(A^{\otimes n} \otimes B^{\otimes n}, A \otimes B\big)\\ &\cong \mathrm{Hom}\big((A \otimes B)^{\otimes n}, A \otimes B\big).\end{aligned}$$

We leave it to the reader to verify that this map is a morphism of operads. □

5.3.3 Hopf Operads

A *Hopf operad* is a reduced operad $\mathscr{P}$ with a morphism of operads $\Delta_{\mathscr{P}} : \mathscr{P} \to \mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{P}$ called the coproduct of $\mathscr{P}$ and a morphism of operads $\varepsilon_{\mathscr{P}} : \mathscr{P} \to Com$ called the counit. This structure is supposed to be coassociative and counital. Since $\Delta_{\mathscr{P}}$ and $\varepsilon_{\mathscr{P}}$ are determined by their arity n components

$$\Delta_{\mathscr{P}}(n) : \mathscr{P}(n) \to (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{P})(n) = \mathscr{P}(n) \otimes \mathscr{P}(n), \qquad \varepsilon_{\mathscr{P}} : \mathscr{P}(n) \to Com(n) = \mathbb{K},$$

a Hopf operad is equivalently defined as an operad in the category of counital coalgebras. The main purpose of this definition lies in the following result.

Proposition 5.3.3. *When $\mathscr{P}$ is a Hopf operad, the tensor product $A \otimes B$ of two $\mathscr{P}$-algebras A and B is again a $\mathscr{P}$-algebra, and there is a natural isomorphism*

$$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$$

where C is also a $\mathscr{P}$-algebra.

Proof. Proposition 5.3.2 asserts that $A \otimes B$ is a $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{P}$-algebra. Then the following composite

$$\mathscr{P} \xrightarrow{\Delta_{\mathscr{P}}} \mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{P} \to \mathrm{End}_{A\otimes B}$$

defines a $\mathscr{P}$-algebra structure on $A \otimes B$ as explained above in Sect. 5.2.11. Coassociativity of $\Delta_{\mathscr{P}}$ ensures the validity of the last assertion. □

The operads Ass and Com are Hopf operads, their diagonals are given by $\Delta_{Ass} : \sigma \in \mathbb{S}_n \mapsto \sigma \otimes \sigma \in \mathbb{K}[\mathbb{S}_n] \otimes \mathbb{K}[\mathbb{S}_n]$ and by $\Delta_{Com} : Com(n) = \mathbb{K} \xrightarrow{\sim} \mathbb{K} \otimes \mathbb{K} = Com(n) \otimes Com(n)$ respectively. (It is a good basic exercise to prove that they are morphisms of operads.) With these definitions in mind, we get an operadic interpretation of the fact that the tensor product of two associative (resp. commutative) algebras is again an associative (resp. commutative) algebra with the product given by $\mu(a \otimes b, a' \otimes b') = \mu(a, a') \otimes \mu(b, b')$.

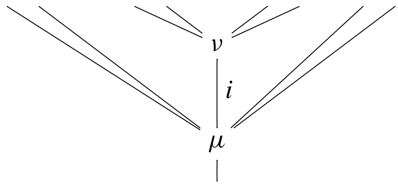
It is also a good exercise to show directly that the operad *Lie* has no nontrivial diagonal, that is *Lie* is not a Hopf operad.

5.3.4 Partial Definition of an Operad

Let $\mathscr{P}$ be an operad and let $\mu \in \mathscr{P}(m)$, $\nu \in \mathscr{P}(n)$ be two operations. By definition the *partial composition* $(\mu, \nu) \mapsto \mu \circ_i \nu \in \mathscr{P}(m-1+n)$ is defined, for $1 \leq i \leq m$, by "substitution":

$$- \circ_i - : \mathscr{P}(m) \otimes \mathscr{P}(n) \longrightarrow \mathscr{P}(m-1+n),$$
$$\mu \circ_i \nu := \gamma(\mu; \mathrm{id}, \ldots, \mathrm{id}, \nu, \mathrm{id}, \ldots, \mathrm{id}).$$

Pictorially it is represented by the following grafting of trees, where the root of ν is grafted onto the ith leaf of μ:



The relationship between this partial composition and the action of the symmetric groups is given by the following two relations. First, for any $\sigma \in \mathbb{S}_n$ we have:

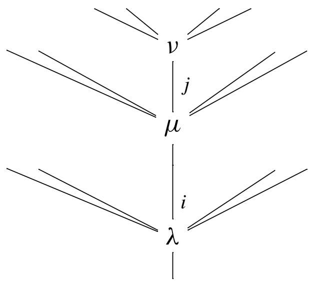
$$\mu \circ_i \nu^\sigma = (\mu \circ_i \nu)^{\sigma'}$$

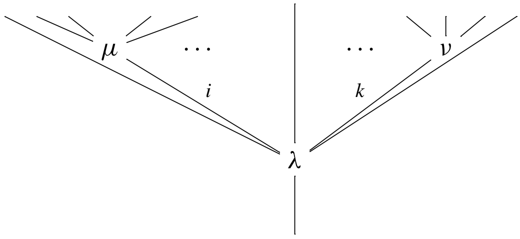
where $\sigma' \in \mathbb{S}_{m-1+n}$ is the permutation which acts by the identity, except on the block $\{i, \ldots, i-1+n\}$ on which it acts via σ. Second, for any $\sigma \in \mathbb{S}_m$ we have:

$$\mu^\sigma \circ_i \nu = (\mu \circ_{\sigma(i)} \nu)^{\sigma''}$$

where $\sigma'' \in \mathbb{S}_{m-1+n}$ is acting like σ on the block $\{1, \ldots, m-1+n\} \backslash \{i, \ldots, i-1+n\}$ with values in $\{1, \ldots, m-1+n\} \backslash \{\sigma(i), \ldots, \sigma(i)-1+n\}$ and identically on the block $\{i, \ldots, i-1+n\}$ with values in $\{\sigma(i), \ldots, \sigma(i)-1+n\}$.

There are two different cases for two-stage partial compositions, depending on the relative positions of the two graftings:





(I) (II)

In both cases associativity of the composition in an operad leads to some relation for the partial composition:

$$\begin{cases} \text{(I)} & (\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), \quad \text{for } 1 \le i \le l,\ 1 \le j \le m, \\ \text{(II)} & (\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, \quad \text{for } 1 \le i < k \le l, \end{cases}$$

for any $\lambda \in \mathscr{P}(l)$, $\mu \in \mathscr{P}(m)$, $\nu \in \mathscr{P}(n)$. Relation (I) is called the *sequential composition* axiom and relation (II) is called the *parallel composition* axiom.

Conversely an operad can be defined as being an $\mathbb{S}$-module $\mathscr{P}$ equipped with partial compositions $\circ_i$ satisfying the compatibility with the action of the symmetric groups, and the two associativity relations (I) and (II) described above. It is also assumed that there is an element id in $\mathscr{P}(1)$ satisfying $\mathrm{id} \circ_1 \nu = \nu$ and $\mu \circ_i \mathrm{id} = \mu$. This gives the *partial definition of an operad*.

In case where the $\mathbb{S}$-modules are graded, there is a sign in formula (II) (because μ and ν are exchanged):

$$(\lambda \circ_i \mu) \circ_{k-1+m} \nu = (-1)^{|\mu||\nu|} (\lambda \circ_k \nu) \circ_i \mu.$$

Proposition 5.3.4. *The partial definition of an operad is equivalent to the classical definition of an operad, and so to all the other definitions.*

Proof. We already remarked that, starting with an operad $\mathscr{P}$, we get the partial compositions which satisfy the aforementioned properties. In the other direction, starting with partial compositions $- \circ_i -$ one constructs maps

$$\gamma(i_1, \dots, i_n) : \mathscr{P}(n) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_n) \longrightarrow \mathscr{P}(i_1 + \cdots + i_n)$$

as

$$\gamma(i_1, \dots, i_n) = \big(- \circ_1 \big(\cdots \big(- \circ_{n-1} (- \circ_n -)\big) \cdot \big)\big).$$

It is a tedious, but straightforward, task to verify that the axioms of the classical definition of an operad are fulfilled. $\square$

5.3.5 Set Operads and Other Types of Operads

In this monograph we are mainly concerned with algebraic operads, that is operads in the category of vector spaces and dg vector spaces (or dg modules). The properties of the category Vect which are used here are: the tensor product is associative, commutative, unital (symmetric monoidal category) and distributive with respect to the direct sum. Notice that the classical and the partial definitions of an operad do not require that, in the underlying symmetric monoidal category, the monoidal product commutes with the coproduct.

One can define operads with values in other symmetric monoidal categories (for instance tensor categories). For instance, the category of sets (resp. simplicial sets, resp. topological sets) equipped with the cartesian product is a symmetric monoidal category. Here are some details for the category Set.

By definition a *set operad* (sometimes called *set-theoretic operad*) is a family of $\mathbb{S}_n$-sets $P(n)$ such that the functor

$$P : \mathsf{Set} \to \mathsf{Set}, \qquad X \mapsto \bigsqcup_n P(n) \times_{\mathbb{S}_n} X^n$$

is equipped with a monoid structure. Here X^n denotes the cartesian product of n copies of the set X. The composition map gives rise to maps

$$\gamma(i_1, \ldots, i_k) : P(k) \times P(i_1) \times \cdots \times P(i_k) \to P(i_1 + \cdots + i_k),$$

which satisfy properties analogous to those of the linear case (cf. Sect. 5.3.1).

To any set X we can associate the vector space $\mathbb{K}[X]$ based on X. This functor is the left adjoint to the forgetful functor from vector spaces to sets. Any set operad P gives rise to an algebraic operad $\mathscr{P}$ under this functor: $\mathscr{P}(n) := \mathbb{K}[P(n)]$. We will meet some algebraic operads coming from set operads in the sequel. For instance the operad *Ass* comes from the set operad $P(n) = \mathbb{S}_n$ and the operad *Com* comes from the set operad $P(n) = \{*\}$.

The category of $\mathbb{S}$-modules can be equipped with an associative and commutative tensor product $M \otimes N$, cf. Sect. 5.1.3. So we can define an operad in this symmetric monoidal category. Such an object is sometimes ambiguously called a *twisted operad*.

Starting with the category of topological spaces (resp. simplicial spaces) equipped with the cartesian product, one can define analogously the notion of topological operads (resp. simplicial operads). There is a large amount of literature on these objects, see for instance [May72, MSS02]. In the next section we give an example, which has the advantage of exposing the main feature of the operadic calculus.

5.3.6 The Little Discs Operad

The *little discs operad* $\mathscr{D}$ is a topological symmetric operad defined as follows. The topological space $\mathscr{D}(n)$ is made up of the unit disc (in $\mathbb{C}$) with n non-intersecting

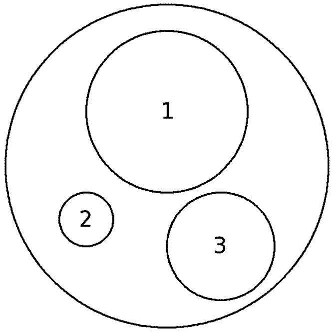


Fig. 5.1 Little discs configuration in $\mathscr{D}(3)$

subdiscs in its interior. So, an element on $\mathscr{D}(n)$ is completely determined by a family of n continuous maps $f_i : S^1 \to D^2$, $i = 1, \ldots, n$, satisfying the non-intersecting condition, see Fig. 5.1.

The enumeration of the interior discs is part of the structure. The operadic composition is given by insertion of a disc in an interior disc. The symmetric group action is given by permuting the labels. Figure 5.2 gives an example of a partial composition.

It is clear how to proceed to define the *little k-discs operad* or the *little k-cubes operad*. For $k = 1$ it is called the *little interval operad.*

The main property of the little k-discs operad is the following "recognition principle" proved by Boardman and Vogt in [BV73] and May in [May72]:

Claim. *If the connected topological space X is an algebra over the little discs operad, then it is homotopy equivalent to the k-fold loop space of some other pointed space Y:*

$$X \sim \Omega^k(Y).$$

5.4 Various Constructions Associated to an Operad

From an operad, one can construct a symmetric monoidal category, a group, a preLie algebra and a Hopf algebra.

5.4.1 Category Associated to an Operad [BV73]

Let $\mathscr{P}$ be an algebraic operad. We associate to it a symmetric monoidal category denoted $\mathsf{cat}\mathscr{P}$ as follows. The objects of $\mathsf{cat}\mathscr{P}$ are the natural numbers: $\underline{0}, \underline{1}, \ldots, \underline{n}, \ldots$. It will prove helpful to consider $\underline{n}$ as the set $\{1, 2, \ldots, n\}$, so $\underline{0} = \emptyset$.

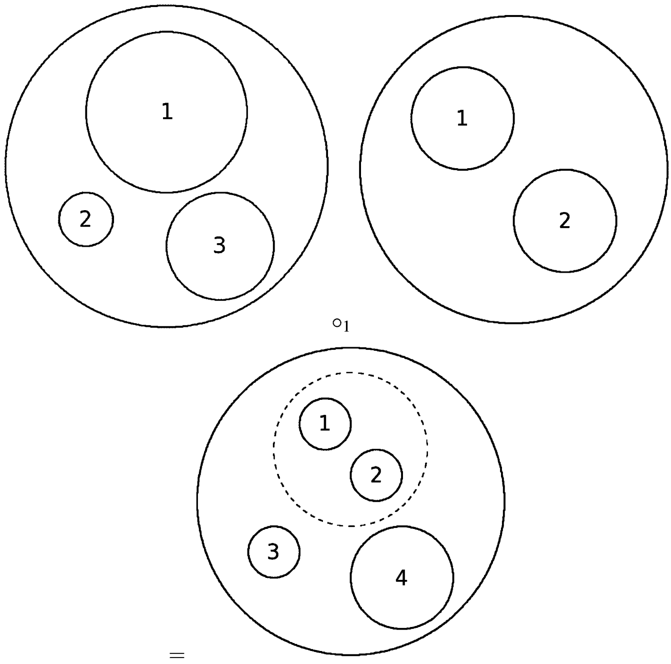


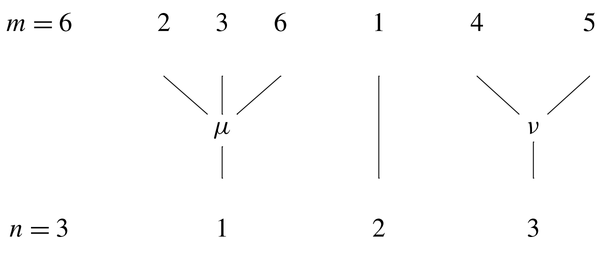
Fig. 5.2 Example of partial composition in the little discs operad

The morphisms of $\mathsf{cat}\mathscr{P}$ are defined as

$$\mathsf{cat}\mathscr{P}(\underline{m},\underline{n}) := \bigoplus_{f:\underline{m}\to\underline{n}} \bigotimes_{i=1}^{n} \mathscr{P}\big(f^{-1}(i)\big)$$

where f is a set map from $\{1, 2, \ldots, m\}$ to $\{1, 2, \ldots, n\}$. Here we use the extension of the functor $\underline{n} \mapsto \mathscr{P}(n)$ to the category of finite sets, cf. Sect. 5.1.13. Observe that for $n = 1$ we get $\mathsf{cat}\mathscr{P}(\underline{m},\underline{1}) = \mathscr{P}(m)$.

Here is an example of a morphism in $\mathsf{cat}\mathscr{P}$:



where $\mu \in \mathscr{P}(3)$, $\nu \in \mathscr{P}(2)$.

Observe that there is no harm in taking the finite sets as objects.

Proposition 5.4.1. *The operad structure of $\mathscr{P}$ induces on* $\mathsf{cat}\,\mathscr{P}$ *a structure of symmetric monoidal category which is the addition of integers on objects.*

Proof. The composition of morphisms in the category $\mathsf{cat}\mathscr{P}$ is obtained through the compositions in $\mathscr{P}$, see [MT78, Sect. 4] for an explicit formula.

Associativity of the composition in $\mathsf{cat}\mathscr{P}$ follows readily from associativity of the composition in $\mathscr{P}$.

The $\mathbb{S}_n$-module structure of $\mathscr{P}(n)$ accounts for the action of the automorphism group of $\underline{n}$.

The symmetric monoidal structure of $\mathsf{cat}\mathscr{P}$ is given by the addition of integers on objects, and therefore by concatenation of morphisms. We see by direct inspection that it is compatible with composition. □

Observe that the symmetric monoidal category $\mathsf{cat}\mathscr{P}$ is completely determined by the Hom-spaces $\mathrm{Hom}_{\mathsf{cat}\mathscr{P}}(\underline{n}, \underline{1}) = \mathscr{P}(n)$ under the composition product (and concatenation). So even among the symmetric monoidal categories based on $\mathbb{N}$ they are very special categories. There are examples of more general symmetric monoidal categories called "props", which code for the "generalized bialgebras", cf. [Lod08].

Proposition 5.4.2. *Let $\mathscr{P}$ be an algebraic operad. A $\mathscr{P}$-algebra A is equivalent to a symmetric monoidal functor* $\mathsf{cat}\mathscr{P} \to \mathsf{Vect}$ *of the form $\underline{n} \mapsto A^{\otimes n}$.*

Proof. The functor in the other direction is simply given by evaluation at $\underline{1}$. The properties are verified straightforwardly. □

EXAMPLE 1 ($\mathsf{cat}\, uCom$). Let $\mathscr{P} = uCom$ be the operad of unital commutative algebras. The category $\mathsf{cat}\, uCom$ is the linearization of the category of finite sets, denoted Fin:

$$\mathsf{cat}\, uCom = \mathbb{K}[\mathsf{Fin}].$$

Indeed, it suffices to show that $\mathsf{cat}\,\mathrm{CMon} = \mathsf{Fin}$, where *CMon* is the set operad of unital commutative monoids. Since $CMon(n) = \{*\}$ (one element), we have $\mathsf{cat}\,\mathrm{CMon}(m,n) = \{f : \underline{m} \to \underline{n}\} = \mathsf{Fin}(\underline{m}, \underline{n})$ and we are done.

EXAMPLE 2 ($\mathsf{cat}\, uAss$). Let $\mathscr{P} = uAss$ be the operad of unital associative algebras. As in the previous case we can work in the set operad framework. The operad $\mathsf{cat}\,\mathrm{Mon}$ of unital monoids admits the following description. Its objects are the integers $\underline{n}$, $n \in \mathbb{N}$, and the morphisms, elements of $\mathsf{cat}\,\mathrm{Mon}(m,n)$, are the linear maps $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ equipped with a total ordering on each fiber $f^{-1}(i)$, $1 \le i \le n$, see Appendix B.5.3. Observe that for any composite $g \circ f$ the set $(g \circ f)^{-1}(i)$ inherits a total ordering. See [Pir02a] for more details. This category is denoted by ΔS in [FL91, Lod98] because any morphism can be written uniquely as a composite of a morphism in the simplicial category Δ and an isomorphism (i.e. a permutation).

It is an example of a "matched pair of categories". Hence we have

$$\mathsf{cat}\, uAss = \mathbb{K}[\Delta S].$$

Note that there is a shift of notation: in Δ the object with $n+1$ elements is usually denoted by $[n]$.

5.4.2 Group Associated to a Symmetric Operad

We consider a partition P of $\underline{n} := \{1, \ldots, n\}$ into k subsets. We order this partition by the minimum of each subset: $\mathrm{P} = (\mathrm{P}_1, \ldots, \mathrm{P}_k)$ with

$$\min(\mathrm{P}_1) < \min(\mathrm{P}_2) < \cdots < \min(\mathrm{P}_k).$$

Let $i_j = \#\mathrm{P}_j$. We denote by $\sqcup(i_1, \ldots, i_k)$ the set of ordered partitions whose jth part has cardinal i_j (see Sect. 8.2.1 for examples). Any ordered partition $\mathrm{P} \in \sqcup(i_1, \ldots, i_k)$ defines a $(i_1, \ldots, i_k)$-unshuffle σ_{P}.

Let $\mathscr{P}$ be a reduced operad, i.e. $\mathscr{P}(0) = 0$. We consider the series

$$\underline{a} := (a_0, a_1, \ldots, a_n, \ldots)$$

where $a_n \in \mathscr{P}(n+1)$. We denote by $G(\mathscr{P})$ the set of series for which a_0 is invertible for the multiplication in $\mathscr{P}(1)$. We define a binary operation $\underline{a}\ \underline{b}$ on this set as follows:

$$(\underline{a}\ \underline{b})_n := \sum_k \sum_{\substack{(i_1,\ldots,i_k)\\ i_1+\cdots+i_k=n}} \sum_{\mathrm{P}\in\sqcup(i_1,\ldots,i_k)} \gamma > (a_{k-1}; b_{i_1-1}, \ldots, b_{i_k-1}) \circ \sigma_{\mathrm{P}}.$$

Proposition 5.4.3. [LN12] *The binary operation* $(\underline{a}, \underline{b}) \mapsto \underline{a}\ \underline{b}$ *makes* $G(\mathscr{P})$ *into a group with unit* $1 = (\mathrm{id}, 0, 0, \ldots)$.

Proof. First, we remark that the symmetric group $\mathbb{S}_k$ is acting freely on the set

$$\bigcup_{\substack{(i_1,\ldots,i_k)\\ i_1+\cdots+i_k=n}} Sh^{-1}(i_1, \ldots, i_k).$$

Second, the quotient of this set by $\mathbb{S}_k$ is precisely

$$\bigcup_{\substack{(i_1,\ldots,i_k)\\ i_1+\cdots+i_k=n}} \sqcup(i_1, \ldots, i_k).$$

Hence the associativity property of the product follows readily from the associativity property of γ.

The existence of an inverse, that is for any $\underline{a}$ there exists $\underline{b}$ such that $\underline{a}\ \underline{b} = 1$, is achieved by induction. For instance, when $a_0 = \mathrm{id}$, we get

$$b_1 = -a_1, \qquad b_2 = -a_2 + a_1 \circ_1 a_1 + (a_1 \circ_1 b_1)^{[132]} + a_1 \circ_2 a_1. \qquad \square$$

Observe that for $\mathscr{P} = Com$, the group $G(\mathscr{P})$ is isomorphic, in characteristic zero, to the group of power series in one variable with no constant term and with first coefficient invertible, under the composition. More precisely the isomorphism is given by

$$(a_0, a_1, \ldots, a_n, \ldots) \mapsto a_0 x + \frac{a_1}{2!}x^2 + \frac{a_2}{3!}x^3 + \cdots + \frac{a_{n-1}}{n!}x^n + \cdots,$$

where $a_i \in \mathbb{K}$.

5.4.3 Pre-Lie Algebra Associated to a Symmetric Operad

Let $\mathscr{P}$ be an operad with $\mathscr{P}(0) = 0$ and consider the space $\bigoplus_{n \geq 1} \mathscr{P}(n)$, resp. $\prod_{n \geq 1} \mathscr{P}(n)$. We construct a bilinear operation $\{-,-\}$ as follows:

$$\{\mu, \nu\} := \sum_{i=1}^{i=m} \sum_{\mathrm{P}} (\mu \circ_i \nu)^{\sigma_{\mathrm{P}}}$$

for $\mu \in \mathscr{P}(m)$, $\nu \in \mathscr{P}(n)$ and the sum is extended over the ordered partitions $\mathrm{P} \in \sqcup\!\sqcup(1, \ldots, 1, \underbrace{n-i+1}_{i\text{th position}}, 1, \ldots, 1)$.

Proposition 5.4.4. *The binary operation* $\{-,-\}$ *makes* $\bigoplus_n \mathscr{P}(n)$, *resp.* $\prod_n \mathscr{P}(n)$, *into a pre-Lie algebra, and hence into a Lie algebra.*

Proof. From the properties of the partial operations it follows that the binary operation $\{-,-\}$ is pre-Lie (cf. definition Sect. 1.4). Indeed, computing explicitly the associator $\{\{\lambda, \mu\}, \nu\} - \{\lambda, \{\mu, \nu\}\}$ we get a sum over trees of type II (see Sect. 5.3.4) where the vertices at the upper level are decorated by either μ or ν. So this sum is symmetric in μ and ν and, as a consequence, the associator is symmetric in the last two variables as expected. $\square$

5.4.4 Hopf Algebra Associated to a Symmetric Operad

Since the space $\bigoplus_{n \geq 1} \mathscr{P}(n)$ is a pre-Lie algebra, a fortiori it is a Lie algebra. Taking the universal enveloping algebra of this Lie algebra gives a Hopf algebra (cf.

Sect. 1.1.10). This is in fact a combinatorial Hopf algebra in the sense of [LR10], which is cofree (by PBW theorem) and left-sided. So one can recover the pre-Lie algebra structure on the primitive part from this data. A direct construction of the Hopf algebra from the operad can also be performed, see for instance [Moe01, vdLM06].

5.5 Free Operad

By definition the *free operad* over the $\mathbb{S}$-module M is an operad $\mathscr{F}(M)$ equipped with an $\mathbb{S}$-module morphism $\eta(M) : M \to \mathscr{F}(M)$ which satisfies the following universal condition:

Any $\mathbb{S}$-module morphism $f : M \to \mathscr{P}$, where $\mathscr{P}$ is an operad, extends uniquely into an operad morphism $\tilde{f} : \mathscr{F}(M) \to \mathscr{P}$:

$$\begin{tikzcd} M \arrow[r, "\eta(M)"] \arrow[dr, "f"'] & \mathscr{F}(M) \arrow[d, "\tilde{f}"] \\ & \mathscr{P}. \end{tikzcd}$$

In other words the functor $\mathscr{F} : \mathbb{S}\text{-}\mathsf{Mod} \to \mathsf{Op}_{\mathbb{K}}$ is left adjoint to the forgetful functor $\mathsf{Op}_{\mathbb{K}} \to \mathbb{S}\text{-}\mathsf{Mod}$. We will show that the free operad exists and we will construct it explicitly. Observe that the free operad over M is well-defined up to a unique isomorphism.

Another, less ad hoc, construction is given in Sect. 5.5.5.

5.5.1 The Tree Module and the Free Operad

We give an explicit construction of the free operad following [BJT97, Rez96]. We rely on the fact that the composition of $\mathbb{S}$-modules $- \circ -$ is linear on the left-hand side. The classical "tensor algebra" construction does not work here because $- \circ -$ is not linear on the right-hand side. For a more general construction which works when no linearity is assumed whatsoever, see Sect. 5.5.5 and [Val09]. In the following construction, one takes advantage of the left linearity to produce a particular colimit which gives the free operad.

Let M be an $\mathbb{S}$-module. By induction we define the functor $\mathscr{T}_n M : \mathsf{Vect} \to \mathsf{Vect}$ as follows:

$$\begin{aligned} \mathscr{T}_0 M &:= \mathrm{I}, \\ \mathscr{T}_1 M &:= \mathrm{I} \oplus M, \\ \mathscr{T}_2 M &:= \mathrm{I} \oplus \big(M \circ (\mathrm{I} \oplus M)\big), \\ &\dots \\ \mathscr{T}_n M &:= \mathrm{I} \oplus (M \circ \mathscr{T}_{n-1} M), \\ &\dots \end{aligned}$$

The transformation of functors $i_n : \mathscr{T}_{n-1}M \to \mathscr{T}_n M$ is defined inductively by $i_1 : \mathrm{I} \hookrightarrow \mathrm{I} \oplus M$ (inclusion in the first factor) for $n = 1$, and by $i_n = \mathrm{Id}_{\mathrm{I}} \oplus (\mathrm{Id}_M \circ i_{n-1})$ higher up. Observe that i_n is a split monomorphism. By definition the *tree module* $\mathscr{T}M$ over the $\mathbb{S}$-module M is:

$$\mathscr{T}M := \bigcup_n \mathscr{T}_n M = \mathrm{colim}_n \mathscr{T}_n M.$$

Observe that $\mathscr{T}_n M$ contains $M^{\circ n}$ but is strictly larger in general. In terms of trees (cf. Sect. 5.6) $\mathscr{T}_n M$ is the space of trees with at most n levels, whose vertices are labeled by I and M. We write i for any of the inclusion maps $\mathscr{T}_n M \rightarrowtail \mathscr{T}_m M$ and j_n or simply j for the inclusion of the second factor $M \circ \mathscr{T}_{n-1}M \rightarrowtail \mathscr{T}_n M$. This last map induces a transformation of functors $j : M \to \mathscr{T}M$.

Theorem 5.5.1 (The free operad construction). *There is an operad structure γ on $\mathscr{T}M$ such that $\mathscr{T}(M) := (\mathscr{T}M, \gamma, j)$ is the free operad on M, so $\mathscr{F}(M) \cong \mathscr{T}M$.*

Proof. We follow Appendix B of [BJT97] by Baues, Tonks and Jibladze word for word. In order not to confuse the composition of functors $\circ$ with the composition of transformations of functors, we denote this latter one by juxtaposition. The identity transformation of a functor F is denoted by Id_F or simply by Id.

The steps of the proof are as follows:

(1) we construct the map $\gamma : \mathscr{T}M \circ \mathscr{T}M \to \mathscr{T}M$,
(2) we prove that γ is associative and unital,
(3) we prove that $(\mathscr{T}M, \gamma, j)$ satisfies the universal property of a free operad.

(1) First, we construct the composition map $\gamma : \mathscr{T}M \circ \mathscr{T}M \to \mathscr{T}M$. For any integers n and m we construct a map

$$\gamma_{n,m} : \mathscr{T}_n M \circ \mathscr{T}_m M \to \mathscr{T}_{n+m} M$$

by induction on n as follows. For $n = 0$,

$$\gamma_{0,m} := \mathrm{Id} : \mathrm{I} \circ \mathscr{T}_m M = \mathscr{T}_m M \to \mathscr{T}_m M.$$

For higher n, $\gamma_{n,m}$ is the composite

$$\begin{aligned} \mathscr{T}_n M \circ \mathscr{T}_m M = (\mathrm{I} \oplus\ M \circ \mathscr{T}_{n-1}M) \circ \mathscr{T}_m M &\cong \mathscr{T}_m M \oplus (M \circ \mathscr{T}_{n-1}M) \circ \mathscr{T}_m M \\ &\cong \mathscr{T}_m M \oplus M \circ (\mathscr{T}_{n-1}M \circ \mathscr{T}_m M) \\ &\xrightarrow{(\mathrm{Id}, \mathrm{Id} \circ \gamma_{n-1,m})} \mathscr{T}_m M \oplus M \circ \mathscr{T}_{n+m-1} M \xrightarrow{i+j} \mathscr{T}_{n+m} M. \end{aligned}$$

Observe that, in this definition of the composite, we use the associativity isomorphism (cf. Sect. 5.1.8)

$$(M \circ \mathscr{T}_{n-1}M) \circ \mathscr{T}_m M \cong M \circ (\mathscr{T}_{n-1}M \circ \mathscr{T}_m M).$$

We prove that the map γ is compatible with the colimits on n and on m by induction. For $n=0$ it is immediate since $\gamma_{0,m}=\mathrm{Id}$. From $n-1$ to n it is a consequence of the commutativity of the following diagrams:

$$\begin{tikzcd}
\mathscr{T}_m M \oplus M \circ \mathscr{T}_{n-1} M \circ \mathscr{T}_m M \arrow[r, "i+j(\mathrm{Id}\circ\gamma_{n-1,m})"] \arrow[d, "i"] & \mathscr{T}_{n+m} M \arrow[d, "i"] \\
\mathscr{T}_m M \oplus M \circ \mathscr{T}_n M \circ \mathscr{T}_m M \arrow[r, "i+j(\mathrm{Id}\circ\gamma_{n,m})"] & \mathscr{T}_{n+m+1} M
\end{tikzcd}$$

$$\begin{tikzcd}
\mathscr{T}_m M \oplus M \circ \mathscr{T}_{n-1} M \circ \mathscr{T}_m M \arrow[r, "i+j(\mathrm{Id}\circ\gamma_{n-1,m})"] \arrow[d, "i"] & \mathscr{T}_{n+m} M \arrow[d, "i"] \\
\mathscr{T}_{m+1} M \oplus M \circ \mathscr{T}_{n-1} M \circ \mathscr{T}_{m+1} M \arrow[r, "i+j(\mathrm{Id}\circ\gamma_{n-1,m+1})"] & \mathscr{T}_{n+m+1} M.
\end{tikzcd}$$

So we have proved that $\gamma_{n+1,m}(i\circ\mathrm{Id})=i(\gamma_{n,m})=\gamma_{n,m+1}(\mathrm{Id}\circ i)$. By passing to the colimit we get a well-defined map $\gamma:\mathscr{T}M\circ\mathscr{T}M\to\mathscr{T}M$.

(2) Let us show now that γ is associative. It is sufficient to prove that for any p, q, r we have the equality

$$\gamma_{p+q,r}(\gamma_{p,q}\circ\mathrm{Id})=\gamma_{p,q+r}(\mathrm{Id}\circ\gamma_{q,r}):\mathscr{T}_p M\circ\mathscr{T}_q M\circ\mathscr{T}_r M\to\mathscr{T}_{p+q+r}M.$$

We work by induction on p. For $p=0$ it is immediate since $\gamma_{0,m}=\mathrm{Id}$. We leave it to the reader to write down the diagram which shows that associativity for $p-1$ implies associativity for p.

The map $i:\mathrm{I}=\mathscr{T}_0 M\to\mathscr{T}M$ is the unit. Indeed, it is sufficient to prove that the following diagram is commutative

$$\begin{tikzcd}
\mathrm{I}\circ\mathscr{T}_{m+1} M \arrow[r, "(i,\mathrm{Id})"] \arrow[d, "i"] & \mathscr{T}_n M \circ \mathscr{T}_{m+1} M \arrow[d, "\gamma_{n,m+1}"] \\
\mathscr{T}_{m+1} M \oplus M \circ \mathscr{T}_{n-1} M \circ \mathscr{T}_{m+1} M \arrow[r, "i+j(\mathrm{Id}\circ\gamma_{n-1,m+1})"] & \mathscr{T}_{n+m+1} M.
\end{tikzcd}$$

(3) Finally we prove that $(\mathscr{T}M,\gamma,j)$ is the free operad over M. Let $\mathscr{P}$ be an operad with composition $\gamma_{\mathscr{P}}$ and unit $\eta_{\mathscr{P}}$. It is sufficient to prove that there are morphisms of operads $\phi_{\mathscr{P}}:\mathscr{T}\mathscr{P}\to\mathscr{P}$ and $\psi_{\mathscr{P}}:\mathscr{P}\to\mathscr{T}\mathscr{P}$ natural in the variable $\mathscr{P}$ such that both composites

$$\mathscr{P}\xrightarrow{\psi_{\mathscr{P}}}\mathscr{T}\mathscr{P}\xrightarrow{\phi_{\mathscr{P}}}\mathscr{P},\qquad \mathscr{T}M\xrightarrow{\mathscr{T}(\psi_M)}\mathscr{T}\mathscr{T}M\xrightarrow{\phi_{\mathscr{T}M}}\mathscr{T}M$$

are the identity ($\mathscr{T}$ left adjoint to the forgetful functor). The first one shows the existence of the extension of $f:M\to\mathscr{P}$ to $\mathscr{T}M$ and the second one shows its uniqueness.

We construct $\phi_{\mathscr{P}}:\mathscr{T}\mathscr{P}\to\mathscr{P}$ as follows. For $n=0$, take $\phi_0=\eta_{\mathscr{P}}:\mathscr{T}_0\mathscr{P}=\mathrm{I}\to\mathscr{P}$. For $n=1$, take $\phi_1=\eta_{\mathscr{P}}+\mathrm{Id}_{\mathscr{P}}:\mathscr{T}_1\mathscr{P}=\mathrm{I}\oplus\mathscr{P}\to\mathscr{P}$. By induction, take

$\phi_n = \eta_{\mathscr{P}} + \gamma_{\mathscr{P}}(\mathrm{Id}_{\mathscr{P}} \circ \phi_{n-1}) : \mathscr{T}_n\mathscr{P} = \mathrm{I} \oplus (\mathscr{P} \circ \mathscr{T}_{n-1}\mathscr{P}) \to \mathscr{P}$. Since $\phi_n \circ i = \phi_{n-1}$, we get at the colimit a transformation of functors $\phi_{\mathscr{P}} : \mathscr{T}\mathscr{P} \to \mathscr{P}$. The expected properties of $\phi_{\mathscr{P}}$ are straightforward to prove by induction. □

5.5.2 Examples

(a) Let $M = (0, W, 0, \ldots, 0, \ldots)$ where W is a vector space. The Schur functor is $M(V) = W \otimes V$. Since M is linear, that is $M(V \oplus V') = M(V) \oplus M(V')$, it follows that

$$\mathscr{T}_n M = \mathrm{I} \oplus M \oplus M \circ M \oplus \cdots \oplus M^{\circ n},$$

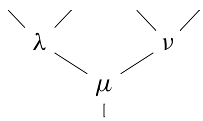
and therefore $\mathscr{T} M = (0, T(W), 0, \ldots)$. We recover that the free associative algebra is the tensor algebra, cf. Sect. 1.1.3.

(b) Let $M = (0, 0, \mathbb{K}[\mathbb{S}_2], 0, \ldots, 0, \ldots)$ where $\mathbb{K}[\mathbb{S}_2]$ is the regular representation. From the description of the free operad in terms of trees (see Sect. 5.6.1) it follows that

$$\mathscr{T}_n M = \bigoplus_{k \leq n} (\mathscr{T} M)^{(k)} \cong \bigoplus_{k \leq n} \mathbb{K}[PBT_k] \otimes \mathbb{K}[\mathbb{S}_k], \quad n \geq 1,$$

where PBT_k is the set of planar binary rooted trees with k leaves, cf. Appendix C.2. See Sect. 5.9.6 for the precise identification with trees.

(c) Let $M = (0, 0, M_2 \otimes \mathbb{K}[\mathbb{S}_2], 0, \ldots, 0, \ldots)$. The same argument as in the previous example shows that $(TM)(n) = (M_2)^{\otimes n-1} \otimes \mathbb{K}[PBT_n] \otimes \mathbb{K}[\mathbb{S}_n]$. It is helpful to think of its elements as binary operations decorating the vertices of a tree:



5.5.3 Weight-Grading of the Free Operad

We introduce a weight grading on the free operad: the weight is the number of generating operations needed in the construction of a given operation of the free operad.

Let M be an $\mathbb{S}$-module and let $\mathscr{T}(M)$ be the free operad on M. By definition the weight $w(\mu)$ of an operation μ of $\mathscr{T}(M)$ is defined as follows:

$$w(\mathrm{id}) = 0, \qquad w(\mu) = 1 \quad \text{when } \mu \in M(n),$$

and more generally

$$w(\nu; \nu_1, \ldots, \nu_n) = w(\nu) + w(\nu_1) + \cdots + w(\nu_n).$$

We denote by $\mathscr{T}(M)^{(r)}$ the $\mathbb{S}$-module of operations of weight r. So we get $\mathscr{T}(M)^{(0)} = \mathbb{K}\,\mathrm{id}$ (concentrated in arity 1) and $\mathscr{T}(M)^{(1)} = M$.

EXAMPLE 1. Suppose that $M = (0, W, 0, \ldots, 0, \ldots)$. Then $\mathscr{T}(M)$ is simply $(0, T(W), 0, \ldots, 0, \ldots)$ where $T(W)$ is the tensor algebra. The weight grading of the tensor operad corresponds to the weight grading of the tensor algebra, cf. Sect. 1.1.3.

EXAMPLE 2. Suppose that $M = (0, 0, E, 0, \ldots, 0, \ldots)$ where E is an $\mathbb{S}_2$-module. So the free operad $\mathscr{T}(M)$ is generated by binary operations. Then $\mathscr{T}(M)^{(r)}$ is concentrated in arity $r+1$ and is exactly $\mathscr{T}(M)(r+1)$. Indeed, an operation on $r+1$ variables in $\mathscr{T}(M)$ needs r binary operations to be constructed. This is the reason why, when dealing with binary operads, the weight is, in general, not mentioned.

5.5.4 Presentation of an Operad

Let M be an $\mathbb{S}$-module generated (as an $\mathbb{S}$-module) by elements μ_i. Let $\mathscr{I}$ be an ideal in the free operad $\mathscr{T}(M)$ and let r_j be generators of the ideal $\mathscr{I}$. Then a $(\mathscr{T}(M)/\mathscr{I})$-algebra is determined by the set of operations μ_i (with their symmetry) and the set of relations $r_j = 0$.

5.5.5 Another Construction of the Free Operad

In this subsection, taken out of [Val08], we give an outline of a construction of the free operad which has the advantage of working in any monoidal category satisfying some mild assumptions.

We denote by M_+ the augmented $\mathbb{S}$-module of M, that is

$$M_+ := \mathrm{I} \oplus M,$$

and we write $M^n := (M_+)^{\circ n} = (\cdots((\underbrace{M_+ \circ M_+) \circ M_+) \circ \cdots \circ M_+}_{n})$. The inclusions of I and M in M_+ are denoted respectively by $\eta : \mathrm{I} \to M_+$ and $\eta_M : M \to M_+$. The map

$$\eta_i : M^n \cong M^i \circ \mathrm{I} \circ M^{n-i} \xrightarrow{\mathrm{Id}\circ\eta\circ\mathrm{Id}} M^i \circ M_+ \circ M^{n-i} \cong M^{n+1}$$

is called the ith *degeneracy map*.

The colimit over the degeneracy maps, that is

$$\mathrm{I} = M^0 \longrightarrow M^1 \rightrightarrows M^2 \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} M^3 \cdots,$$

is too large to be the free object, essentially because, in the composition $((xy)(zt))$, the order of parenthesizing (xy first or zt first?) should be irrelevant. So we are going to make a quotient of these spaces.

Since I is a unit for $\circ$ there are isomorphisms $\lambda : \mathrm{I} \circ M \cong M$ and $\rho : M \circ \mathrm{I} \cong M$. We consider the composite

$$\tau : M \xrightarrow{\lambda^{-1} \oplus \rho^{-1}} \mathrm{I} \circ M \oplus M \circ \mathrm{I} \xrightarrow{\eta \circ \eta_M - \eta_M \circ \eta} (M_+)^{\circ 2} = M^2.$$

For any $\mathbb{S}$-modules A and B there is a well-defined $\mathbb{S}$-module

$$R_{A;B} := \mathrm{Im}\Big(A \circ \big(M \oplus M^2\big) \circ B \xrightarrow{\mathrm{Id}_A \circ (\tau + \mathrm{Id}_{M^2}) \circ \mathrm{Id}_B} A \circ M^2 \circ B\Big).$$

We put

$$\widetilde{M^n} := M^n / \sum_{i=0}^{n-2} R_{M^i ; M^{n-2-i}}.$$

It can be shown that, under this quotient, the degeneracy maps η_i, for $i = 1, \ldots, n$, are equal and define $\eta : \widetilde{M^n} \to \widetilde{M^{n+1}}$. By definition $\mathscr{F}(M)$ is the (sequential) colimit of

$$\widetilde{M} \xrightarrow{\eta} \widetilde{M^2} \xrightarrow{\eta} \widetilde{M^3} \xrightarrow{\eta} \cdots .$$

Theorem 5.5.2 [Val08] *In any monoidal category such that the monoidal product preserves sequential colimits and reflexive coequalizers, $\mathscr{F}(M)$ can be equipped with a monoid structure that is free over M.*

The advantage of this construction lies in its generalization to some types of bialgebras. In order to develop the same kind of arguments, the notion of an operad has to be replaced by the notion of properad, see Sect. 13.14.9. The aforementioned result produces the free properad, cf. [Val07b].

5.6 Combinatorial Definition of an Operad

In this section we give a fourth definition of an operad based on some combinatorial objects: the rooted trees. The main advantage of this presentation is to admit several important variations by changing the combinatorial objects and/or by decorating them. For instance, if we replace the rooted trees by ladders, then we get unital associative algebras in place of operads. If we take planar rooted trees, then we get nonsymmetric operads. If we take nonrooted trees, then we get cyclic operads, cf. Sect. 13.14 and [GK95a]. A far-reaching generalization has been given by Borisov and Manin in [BM08], see also [Lei04].

5.6.1 The Monad of Trees

A *reduced rooted tree* is a nonplanar rooted tree such that each vertex has one input or more, cf. Appendix C.3. Let X be a finite set. For any tree $t \in RT(X)$ (i.e. we are given a bijection from the set of leaves of t to X) we denote by vert(t) the set of vertices of t and by in(v) the set of inputs of the vertex $v \in \text{vert}(t)$. Let M be an $\mathbb{S}$-module with $M(0) = 0$ or, equivalently, a functor $M : \mathsf{Bij} \to \mathsf{Vect}$, $X \mapsto M(X)$, such that $M(\emptyset) = 0$, cf. Sect. 5.1.13.

We define the *treewise tensor product* $M(t)$ as follows:

$$M(t) := \bigotimes_{v \in \text{vert}(t)} M\big(\text{in}(v)\big).$$

See Sect. 5.1.14 for the precise meaning of $\otimes_{v\in \text{vert}(t)}$. Using this notation we define a functor

$$\mathbb{T} : \mathbb{S}\text{-Mod} \to \mathbb{S}\text{-Mod}$$

by

$$\mathbb{T}(M)(X) := \bigoplus_{t \in RT(X)} M(t).$$

It is helpful to think about an element of $\mathbb{T}(M)(X)$ as a sum of rooted trees where each vertex v is decorated by an element of $M(\text{in}(v))$ and each leaf is decorated by an element of X.

First, we construct a transformation of functors

$$\iota : \text{Id}_{\mathbb{S}\text{-Mod}} \to \mathbb{T}$$

as follows. For any $\mathbb{S}$-module M we have to say what is the $\mathbb{S}$-module morphism $M \to \mathbb{T}(M)$, i.e. for any finite set X a linear map $M(X) \to \mathbb{T}(M)(X)$. In the set of trees $RT(X)$, there is a particular one $t = cor$ which is the corolla. We have $M(cor) = M(X)$ by definition, since the corolla has only one vertex. Hence $M(X)$ is a direct summand of $\mathbb{T}(M)(X)$. The expected map is the corresponding inclusion.

Second, we construct a transformation of functors

$$\alpha : \mathbb{T} \circ \mathbb{T} \to \mathbb{T}$$

as follows. The *substitution* of trees consists in replacing the vertices of a tree by given trees (with matching inputs) like in Fig. 5.3.

In order to perform the substitution in the tree t we need, for any $v \in \text{vert}(t)$ a tree t_v and a bijection $\text{in}(v) \cong \text{leaves}(t_v)$.

Lemma 5.6.1. *The substitution of trees defines a transformation of functors $\alpha : \mathbb{T} \circ \mathbb{T} \to \mathbb{T}$ which is associative and unital. So $(\mathbb{T}, \alpha, \iota)$ is a monad.*

Proof. From the definition of $\mathbb{T}$ we get

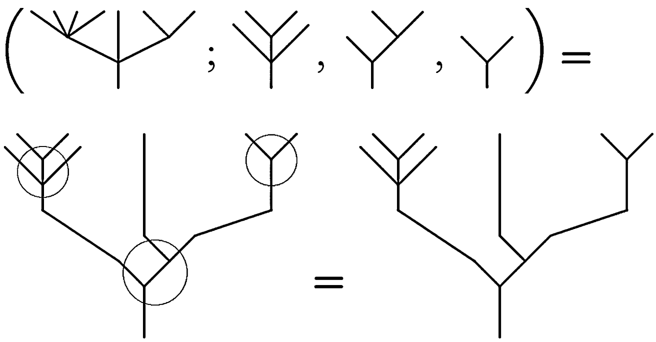


Fig. 5.3 Substitution

$$
\begin{aligned}
\mathbb{T}\big(\mathbb{T}(M)\big)(X) &= \bigoplus_{t\in RT(X)} \mathbb{T}(M)(t) \\
&= \bigoplus_{t\in RT(X)} \Big(\bigotimes_{v\in \mathrm{vert}(t)} \mathbb{T}(M)\big(\mathrm{in}(v)\big)\Big) \\
&= \bigoplus_{t\in RT(X)} \Big(\bigotimes_{v\in \mathrm{vert}(t)} \Big(\bigoplus_{s\in RT(\mathrm{in}(v))} M(s)\Big)\Big).
\end{aligned}
$$

The decoration of the vertex v in t is an element of $\mathbb{T}(M)(\mathrm{in}(v))$, that is a tree whose leaves are labeled by $\mathrm{in}(v)$. This is exactly the data which permits us to perform the substitution. So we get an element of $\mathbb{T}(M)(X)$. As a result we have defined an $\mathbb{S}$-module morphism $\alpha(M) : \mathbb{T}(\mathbb{T}(M)) \to \mathbb{T}(M)$. Obviously this morphism is functorial in M so we have constructed a transformation of functors $\alpha : \mathbb{T}\circ\mathbb{T} \to \mathbb{T}$. The substitution process is clearly associative, so α is associative.

Recall that the unit ι consists in identifying an element μ of $M(X)$ with the corolla with vertex decorated by μ. Substituting a vertex by a corolla does not change the tree. Substituting a corolla by a vertex gives the former tree. Hence α is also unital.

We have proved that $(\mathbb{T}, \alpha, \iota)$ is a monad. □

5.6.2 Combinatorial Definition

The *combinatorial definition of an operad* consists in defining it as an algebra over the monad $(\mathbb{T}, \alpha, \iota)$, cf. Appendix B.4. In other words an operad is an $\mathbb{S}$-module $\mathscr{P}$ together with an $\mathbb{S}$-module map $\mathbb{T}(\mathscr{P}) \to \mathscr{P}$ compatible with α in the obvious sense.

Proposition 5.6.2. [GJ94] *The combinatorial definition of an operad is equivalent to the partial definition of an operad, and therefore to all the other definitions.*

Proof. It suffices to verify that the partial operation is indeed a substitution and that the substitution of trees satisfies the two axioms (I) and (II) of partial operations defining an operad. □

Proposition 5.6.3. *For any $\mathbb{S}$-module M, $\mathbb{T}(M)$ is an operad which is the free operad $\mathscr{T}(M)$ over M.*

Proof. Since $\mathbb{T}(M)$ is an algebra over the monad $(\mathbb{T}, \alpha, \iota)$, by Proposition 5.6.2 it is an operad. Checking that it is free is analogous to the proof of Proposition 5.2.1. □

5.6.3 Comparison of the Two Constructions of the Free Operad

Recall that in Theorem 5.5.1 we constructed the free operad on M inductively as a colimit: $\mathscr{T}(M) = \operatorname{colim}_n \mathscr{T}_n M$, where

$$\begin{aligned} \mathscr{T}_0 M &:= \mathrm{I}, \\ \mathscr{T}_1 M &:= \mathrm{I} \oplus M, \\ \mathscr{T}_n M &:= \mathrm{I} \oplus (M \circ \mathscr{T}_{n-1} M). \end{aligned}$$

Since $\mathscr{T}(M)$ and $\mathbb{T}(M)$ are both the free operad on M we know that they are isomorphic. We make this isomorphism explicit as follows.

The map $\mathscr{T}_0 M = \mathrm{I} \to \mathbb{T}(M)$ is given by the operation $\mathrm{id} \in \mathbb{T}(M)(1)$. The map $M \to \mathbb{T}(M)$ is given by $\mu \mapsto$ corolla, where the number of leaves of the corolla is the arity of μ, and the single vertex is decorated by μ. By induction we suppose that $\mathscr{T}_{n-1} M \to \mathbb{T}(M)$ has been constructed. The map $M \circ \mathscr{T}_{n-1} M \to \mathbb{T}(M)$ is obviously given by the composition in $\mathbb{T}(M)$ of the images of M and of $\mathscr{T}_{n-1} M$. Notice that, under the above isomorphism, the $\mathbb{S}_n$-module $\mathscr{T}_n M$ corresponds to linear combination of trees with at most n levels. More details are given in Sect. 5.9.6.

5.7 Type of Algebras

We make explicit the relationship between the notion of algebraic operad and some types of algebras. We suppose that we are in characteristic zero.

5.7.1 Type of Algebras and Presentation of an Operad

Let $\mathtt{P}$-alg be a category of algebras presented as follows. An object of $\mathtt{P}$-alg is a vector space A equipped with some n-ary operations $\mu_i : A^{\otimes n} \to A$ (possibly for various n's), called the *generating operations* satisfying some relations $r_j = 0$. Let us suppose that the relations are multilinear, that is of the form

$$\sum_\phi \phi(a_1, \dots, a_n) = 0 \quad \text{for all } \ a_1, \dots, a_n \in A,$$

where ϕ is a composite of the generating operations μ_i. An element like $r = \sum \phi$ is called a *relator*. Let us denote by M the $\mathbb{S}$-module which is, in arity n, the $\mathbb{S}_n$-module spanned by the generating n-ary operations. We take into account the symmetries of these operations to determine the $\mathbb{S}_n$-module structure. A relator determines an operation in the free operad $\mathscr{T}M$. Let R be the sub-$\mathbb{S}$-module of $\mathscr{T}M$ spanned by all the relators, and let (R) be the operadic ideal of $\mathscr{T}M$ generated by R. Then we get the operad $\mathscr{T}M/(R)$.

Let us denote by $\mathscr{P}(V)$ the free $\mathtt{P}$-algebra over V. It defines a functor $\mathscr{P}$: $\mathsf{Vect} \to \mathsf{Vect}$.

Lemma 5.7.1. *The functor $\mathscr{P}$ is a Schur functor whose arity n component is the multilinear part of the free $\mathtt{P}$-algebra $\mathscr{P}(\mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n)$. Moreover $\mathscr{P}$ is an algebraic operad.*

Proof. Let us first prove that the free algebra $\mathscr{P}(V)$ of the given type $\mathtt{P}$ is equipped with a monoid structure. Since $\mathscr{P}(V)$ is free over V, it comes with a natural map $V \to \mathscr{P}(V)$ that we denote by $\eta(V)$. Consider the map $\mathrm{Id}_{\mathscr{P}(V)} : \mathscr{P}(V) \to \mathscr{P}(V)$ as a well-defined map from the vector space $W = \mathscr{P}(V)$ to the algebra $\mathscr{P}(V)$ of type $\mathtt{P}$. By the universal property, there exists a lifting of $\mathrm{Id}_{\mathscr{P}(V)}$ denoted $\gamma(V)$: $\mathscr{P}(\mathscr{P}(V)) = \mathscr{P}(W) \to \mathscr{P}(V)$. It is clear that this morphism of algebras of type $\mathtt{P}$ is functorial in V.

Again, from the universal property of free algebras, we deduce that γ is associative. From the fact that $\gamma(V)$ is a lifting of $\mathrm{Id}_{\mathscr{P}(V)}$ we deduce that γ is unital. Hence $(\mathscr{P}, \gamma, \eta)$ is a monoid in the category of endofunctors of Vect.

Let us now show that $\mathscr{P}$ is a Schur functor. Since the relations are multilinear, the free $\mathtt{P}$-algebra over V is the direct sum of its homogeneous components. By the Schur Lemma (cf. Sect. A.2.3) the homogeneous component of degree n (in characteristic zero) is of the form $\mathscr{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$ for some $\mathbb{S}_n$-module $\mathscr{P}(n)$. Taking $V = V_n = \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n$ we verify that $\mathscr{P}(n)$ is the multilinear part of the free $\mathtt{P}$-algebra over V_n as an $\mathbb{S}_n$-module, cf. Lemma 5.1.1. So it follows that the free $\mathtt{P}$-algebra over V is of the form

$$\mathscr{P}(V) = \bigoplus_{n \geq 0} \mathscr{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n},$$

as expected. $\square$

Lemma 5.7.2. *Let $\mathtt{P}$ be a type of algebras defined by the $\mathbb{S}$-module of generating operations M and the $\mathbb{S}$-module of relators $R \subset \mathscr{T}M$. Then the operad $\mathscr{P}$ constructed above coincides with the quotient operad $\mathscr{T}M/(R)$.*

Proof. By definition of the operad $\mathscr{T}M/(R)$ out of the type of algebras $\mathtt{P}$, it follows that $(\mathscr{T}M/(R))(V)$ is the free $\mathtt{P}$-algebra over V. By construction we also know that $\mathscr{P}(V)$ is a free $\mathtt{P}$-algebra over V. Since the identification $(\mathscr{T}M/(R))(V) = \mathscr{P}(V)$ is functorial in V, we are done. □

Proposition 5.7.3. *In characteristic zero, a type of algebras whose relations are multilinear determines an operad. The category of algebras over this operad is equivalent to the category of algebras of the given type.*

Proof. From the preceding results we know that the type $\mathtt{P}$ determines the operad $\mathscr{P}$. From the identification of $\mathscr{P}$ with $\mathscr{T}M/(R)$ it follows that the two categories of algebras $\mathscr{P}$-alg and $\mathtt{P}$-alg are equivalent. □

5.7.2 Examples

The associative algebras, the commutative algebras and the Lie algebras are examples of types of algebras with multilinear relations. The operad that they determine is denoted by *Ass*, *Com* and *Lie* respectively. Though they will be studied in detail later on, let us make some comments on the cases of commutative algebras and associative algebras.

Since the free commutative algebra over $\{x_1, \ldots, x_n\}$ is the polynomial algebra (modulo the constants), it follows that $Com(n) = \mathbb{K}$ for $n \geq 1$, and its generator μ_n is the n-ary operation determined by

$$\mu_n(x_1, \ldots, x_n) = x_1 \cdots x_n \in \mathbb{K}[x_1, \ldots, x_n].$$

The action of $\mathbb{S}_n$ is trivial, hence $Com(n)$ is the one-dimensional trivial representation. From the classical "composition" of polynomials, it follows that

$$\gamma(\mu_k; \mu_{i_1}, \ldots, \mu_{i_k}) = \mu_{i_1+\cdots+i_k}.$$

Since the free associative algebra over $\{x_1, \ldots, x_n\}$ is the noncommutative polynomial algebra (modulo the constants), it follows that $Ass(n) = \mathbb{K}[\mathbb{S}_n]$, and the n-ary operation μ_σ corresponding to $\sigma \in \mathbb{S}_n$ is determined by

$$\mu^\sigma(x_1, \ldots, x_n) = x_{\sigma^{-1}(1)} \ldots x_{\sigma^{-1}(n)} \in \mathbb{K}\langle x_1, \ldots, x_n\rangle.$$

The action of $\mathbb{S}_n$ is by multiplication, so $Ass(n)$ is the regular representation. From the classical "composition" of noncommutative polynomials, it follows that

$$\gamma(\sigma; \sigma_1, \ldots, \sigma_k) = \tilde{\sigma} \circ (\sigma_1, \ldots, \sigma_k),$$

where $\tilde{\sigma}$ is the block permutation associated to σ.

5.7.3 Kernels

Let $\mathscr{Q}$-alg $\to \mathscr{P}$-alg be a functor between two types of algebras, which is supposed to commute with the forgetful functor to vector spaces:

$$\begin{tikzcd} \mathscr{Q}\text{-alg} \arrow[rr] \arrow[dr] & & \mathscr{P}\text{-alg} \arrow[dl] \\ & \mathsf{Vect}_{\mathbb{K}}. & \end{tikzcd}$$

By looking at the free algebras, we check that it comes from a morphism of operads $\alpha : \mathscr{P} \to \mathscr{Q}$. We know that $\mathrm{I} \oplus \operatorname{Ker} \alpha$ is an operad, so there is a new type of algebras $(\mathrm{I} \oplus \operatorname{Ker} \alpha)$-alg. In many examples $\mathscr{P}$ and $\mathscr{Q}$ are presented by a small number of generators and relations, but no such small presentation is known for $\mathrm{I} \oplus \operatorname{Ker} \alpha$ in general. These are examples where the use of operads is a necessity.

5.7.4 A Universal Presentation

We know that a group can always be presented as follows: the generators are its elements and the relations are given by the table of multiplication. Similarly an operad can always be presented as follows. Choose a linear basis for $\mathscr{P}(n)$, $n \geq 1$, and take the composite products as relations.

5.7.5 Non-examples (?)

Here are two examples of types of algebras, which, a priori, do not fall directly into the operad theory because the relations are not multilinear. However minor changes will make them accessible.

A *Jordan algebra* A is a vector space equipped with a binary operation which satisfies the relation

$$\big(a^2 b\big)a = a^2(ba).$$

It seems to lie outside our framework since the relation fails to be multilinear. However it suffices to multilinearize it and we get the following type of algebras: one binary symmetric operation ab and one relation

$$(ab)(dc) + (ac)(db) + (bc)(da) = \big((ab)d\big)c + \big((ac)d\big)b + \big((bc)d\big)a.$$

More information is to be found in Sect. 13.10.

A *divided power algebra* is an augmented commutative algebra equipped with unary operations $\gamma_n(x)$ which bear the formal properties of the operations $x^n/n!$. A priori such a structure cannot be encoded by an operad, however taking invariants in place of coinvariants in the construction of the Schur functor permits us to solve the problem. See Sect. 13.1.12 for more details.

5.8 Cooperad

In our treatment of Koszul duality for associative algebras we put algebras and coalgebras on the same footing. In order to play the same game with operads we need to introduce the notion of cooperad.

We define the notion of cooperad as a comonoid in the monoidal category of $\mathbb{S}$-modules with the composite product. Cooperads are used to encode categories of coalgebras. We introduce the important notion of conilpotent cooperad. In the same way as for operads, we give an equivalent combinatorial definition of a conilpotent cooperad in terms of trees.

5.8.1 Algebraic Cooperad

Let $\mathscr{C}$ be an $\mathbb{S}$-module. A *cooperad* is a structure of comonoid on $\mathscr{C}$ in the monoidal category $(\mathbb{S}\textsf{-Mod}, \bar{\circ}, \mathrm{I})$, where $(\mathscr{P} \bar{\circ} \mathscr{Q})(n) := \bigoplus_r (\mathscr{P}(r) \otimes \mathscr{Q}^{\otimes r})^{\mathbb{S}_r}(n)$, see Sect. 5.1.15. Explicitly it consists into two morphisms of $\mathbb{S}$-modules (equivalently transformations of Schur functors)

$$\Delta : \mathscr{C} \to \mathscr{C} \bar{\circ} \mathscr{C} \quad \text{(decomposition)} \quad \text{and} \quad \varepsilon : \mathscr{C} \to \mathrm{I} \quad \text{(counit)},$$

which satisfy the axioms of coassociativity:

$$\begin{tikzcd}
\mathscr{C} \arrow[rr, "\Delta"] \arrow[dd, "\Delta"'] & & \mathscr{C} \bar{\circ} \mathscr{C} \arrow[d, "\mathrm{Id} \bar{\circ} \Delta"] \\
& & \mathscr{C} \bar{\circ} (\mathscr{C} \bar{\circ} \mathscr{C}) \arrow[dl, "\cong"'] \\
\mathscr{C} \bar{\circ} \mathscr{C} \arrow[r, "\Delta \bar{\circ} \mathrm{Id}"] & (\mathscr{C} \bar{\circ} \mathscr{C}) \bar{\circ} \mathscr{C} &
\end{tikzcd}$$

and counitality:

$$\begin{tikzcd}
& \mathscr{C} \arrow[dl, "\cong"'] \arrow[d, "\Delta"] \arrow[dr, "\cong"] & \\
\mathrm{I} \bar{\circ} \mathscr{C} & \mathscr{C} \bar{\circ} \mathscr{C} \arrow[l, "\varepsilon \bar{\circ} \mathrm{Id}"] \arrow[r, "\mathrm{Id} \bar{\circ} \varepsilon"'] & \mathscr{C} \bar{\circ} \mathrm{I}.
\end{tikzcd}$$

We observe that the $\mathbb{S}$-module I associated to identity functor $\textsf{Vect} \to \textsf{Vect}$ is a cooperad.

From Corollary 5.1.4, it follows that Δ is made up of $\mathbb{S}_n$-module morphisms $\Delta(n)$:

$$\mathscr{C}(n) \to \mathscr{C} \bar{\circ} \mathscr{C}(n) = \bigoplus_{k \geq 0} \Big(\mathscr{C}(k) \otimes \Big(\bigoplus \mathrm{Ind}_{\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} \big(\mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k) \big) \Big) \Big)^{\mathbb{S}_k}$$

where the second sum is extended to all the k-tuples $(i_1, \ldots, i_k)$ satisfying $i_1 + \cdots + i_k = n$. So a cooperad concentrated in arity 1 is a coassociative coalgebra.

A cooperad is said to be *coaugmented* if there is a cooperad morphism $\eta : \mathrm{I} \to \mathscr{C}$ such that $\varepsilon\eta = \mathrm{Id}_{\mathrm{I}}$. The image of $1 \in I(1) = \mathbb{K}$ is denoted id $\in \mathscr{C}(1)$ and is called the *identity cooperation*. The cokernel of η is denoted by $\overline{\mathscr{C}}$ and $\mathscr{C} \cong \mathrm{I} \oplus \overline{\mathscr{C}}$.

Observe that, by the counitality assumption, the component of $\Delta(n)(\mu)$ in the two extreme summands ($k = 1$ and $k = n$ respectively) is $(\mathrm{id}; \mu)$ and $(\mu; \mathrm{id}^{\otimes n})$ respectively. We will sometimes adopt the following abuse of notation

$$\Delta(\mu) = \sum (\nu; \nu_1, \ldots, \nu_k)$$

where $\nu \in \mathscr{C}(k)$, $\nu_j \in \mathscr{C}(i_j)$ for the image of the cooperation μ by the decomposition map, and

$$\Delta(\mu) = (\mathrm{id}; \mu) + \left(\mu; \mathrm{id}^{\otimes n}\right) + \bar{\Delta}(\mu).$$

The map $\bar{\Delta}$ is called the *reduced decomposition map*.

To define the exact dual of the notion of operad, one should instead consider the monoidal product

$$\mathscr{C} \hat{\circ} \mathscr{C}(n) = \prod_{k \geq 0} \left(\mathscr{C}(k) \otimes \left(\prod \mathrm{Ind}_{\mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k}}^{\mathbb{S}_n} \left(\mathscr{C}(i_1) \otimes \cdots \otimes \mathscr{C}(i_k)\right)\right)\right)^{\mathbb{S}_k}$$

in the category of $\mathbb{S}$-modules, where the sums are replaced by products. In that case, a cooperad is defined as a comonoid $\Delta : \mathscr{C} \to \mathscr{C} \hat{\circ} \mathscr{C}$.

When $\mathscr{C}(0) = 0$, the right-hand side product is equal to a sum. The decomposition map $\Delta : \mathscr{C} \to \mathscr{C} \bar{\circ} \mathscr{C} \subset \mathscr{C} \hat{\circ} \mathscr{C}$ of some cooperads is made up of sums of elements. In this case, we are back to the previous definition. In this book, we will mainly encounter cooperads of this first type, so we work with this definition.

Working over a field of characteristic 0, we can identify invariants with coinvariants, see Appendix A.1, and work with $\circ$ instead of $\bar{\circ}$. But this more general definition plays a key role in characteristic p, for instance.

5.8.2 From Cooperads to Operads and Vice Versa

Let $\mathscr{C}$ be a cooperad and let $\mathscr{P}(n) = \mathscr{C}(n)^* = \mathrm{Hom}(\mathscr{C}(n), \mathbb{K})$. Since $\mathscr{C}(n)$ is a right $\mathbb{S}_n$-module, its dual $\mathscr{P}(n)$ is a left $\mathbb{S}_n$-module. We make it into a right $\mathbb{S}_n$-module by the classical formula $\mu^\sigma := \sigma^{-1} \cdot \mu$, for $\mu \in \mathscr{P}(n)$ and $\sigma \in \mathbb{S}_n$. The transpose of the counit ε gives a unit η. The decomposition map Δ gives a composition map γ by dualization followed by the natural map from invariants to coinvariants, cf. Sect. 5.1.15.

In the other way round, let $\mathscr{P}$ be an operad such that each $\mathscr{P}(n)$ is finite dimensional. This condition ensures that $\mathscr{P}(n)^* \otimes \mathscr{P}(m)^* \to (\mathscr{P}(n) \otimes \mathscr{P}(m))^*$ is an isomorphism, as in Sect. 1.2.2. So the linear dual $\mathscr{C} := \mathscr{P}^*$ of $\mathscr{P}$ gives rise to a cooperad $\mathscr{C} \to \mathscr{C}\hat{\circ}\mathscr{C}$. If we further suppose that $\mathscr{P}(0) = 0$ and that the preimage under the composition map of any element in $\mathscr{P}$ is finite, then the decomposition map of $\mathscr{C}$ lives in $\mathscr{C}\bar{\circ}\mathscr{C}$.

5.8.3 Coalgebra over a Cooperad

By definition a *coalgebra* over the cooperad $\mathscr{C}$, or $\mathscr{C}$*-coalgebra* for short, is a vector space C equipped with a map $\Delta_C : C \to \widehat{\mathscr{C}}(C)$, where $\widehat{\mathscr{C}}(C) = \prod_n (\mathscr{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$, such that the following diagrams commute:

$$\begin{tikzcd} C \arrow[r, "\Delta_C"] \arrow[d, "\Delta_C"'] & \widehat{\mathscr{C}}(C) \arrow[d, "\widehat{\mathscr{C}}(\Delta_C)"] \\ \widehat{\mathscr{C}}(C) \arrow[r, "\Delta(C)"] & \widehat{\mathscr{C}}(\widehat{\mathscr{C}}(C)) \end{tikzcd} \qquad \begin{tikzcd} C \arrow[d, "\Delta(C)"'] \arrow[dr, "="] & \\ \widehat{\mathscr{C}}(C) \arrow[r, "\eta(C)"] & C. \end{tikzcd}$$

So for any n, we have a map $\Delta_n : C \to (\mathscr{C}(n) \otimes C^{\otimes n})^{\mathbb{S}_n}$.

Let $\mathscr{C}^*$ be the operad obtained by linear dualization. The map Δ_n gives rise to an $\mathbb{S}_n$-equivariant map

$$\mathscr{C}^*(n) \longrightarrow \operatorname{Hom}\big(C, C^{\otimes n}\big),$$

and it is immediate to check that C is a coalgebra over the operad $\mathscr{C}^*$ in the sense of Sect. 5.2.15.

5.8.4 Conilpotent Coalgebra, Primitive Part

Let C be a coalgebra over a coaugmented cooperad $\mathscr{C}$. We denote the image under its structure map by

$$\Delta_C(x) = (x_1, x_2, \ldots) \in \prod_{n \geq 1} \big(\mathscr{C}(n) \otimes C^{\otimes n}\big)^{\mathbb{S}_n}.$$

We define the *coradical filtration* of C as follows:

$$F_1C := \operatorname{Prim} C := \{x \in C \mid x_1 = x, \text{ and } x_k = 0 \text{ for any } k > 1\}.$$

The space $\operatorname{Prim} C$ is called the *primitive part* of C, and its elements are said to be *primitive*. Then we define the filtration by:

$$F_rC := \{x \in C \mid x_k = 0 \text{ for any } k > r\}.$$

We say that the coalgebra C is *conilpotent*, if this filtration is exhaustive $C = \bigcup_{r \geq 1} F_rC$.

Proposition 5.8.1. *The coalgebra C is conilpotent if and only if the decomposition map $\Delta_C : C \to \widehat{\mathscr{C}}(C)$ factors through $\mathscr{C}(C)$.*

Proof. By direct inspection. □

Any coaugmented coassociative coalgebra C is equivalent to an As^*-coalgebra structure on $\overline{C}$. In this case, the above definitions of filtration and conilpotent coalgebra coincide with the ones given in Sect. 1.2.4.

5.8.5 Conilpotent Cooperad

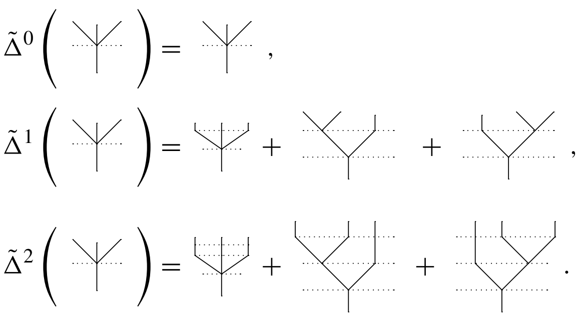
Let $(\mathscr{C}, \Delta, \varepsilon, \eta)$ be a coaugmented cooperad. Under the isomorphism $\mathscr{C} \cong \mathrm{I} \oplus \overline{\mathscr{C}}$, we consider the map $\tilde{\Delta} : \mathscr{C} \to \mathscr{C} \,\bar{\circ}\, \mathscr{C}$ defined by

$$\begin{cases} \mathrm{I} \;\to \mathrm{I} \,\bar{\circ}\, \mathrm{I}, \\ \mathrm{id} \mapsto \tilde{\Delta}(\mathrm{id}) := \mathrm{id} \,\bar{\circ}\, \mathrm{id} \end{cases} \quad \text{and} \quad \begin{cases} \overline{\mathscr{C}} \to \overline{\mathscr{C}} \,\bar{\circ}\, \mathscr{C}, \\ \mu \;\mapsto \tilde{\Delta}(\mu) := \bar{\Delta}(\mu) + \big(\mu; \mathrm{id}^{\otimes n}\big). \end{cases}$$

We iterate the map $\tilde{\Delta}$ on the right-hand side:

$$\tilde{\Delta}^0 := \mathrm{Id}_{\mathscr{C}}, \qquad \tilde{\Delta}^1 := \tilde{\Delta}, \quad \text{and} \quad \tilde{\Delta}^n := (\mathrm{Id} \circ \tilde{\Delta})\tilde{\Delta}^{n-1} : \mathscr{C} \to \mathscr{C}^{\circ(n+1)}.$$

EXAMPLE. Representing the elements of the cooperad As^* by corollas, we get:



Higher up, no new term appears. We get the same three trees but with the top levels filled with trivial trees $|$. So in this example, the iteration of $\tilde{\Delta}$ on the element of $As^*(3)$ stabilizes at rank 2.

The composite

$$(\mathrm{Id} \,\bar{\circ}\, \eta)\tilde{\Delta}^{n-1} : \mathscr{C} \to \mathscr{C}^{\bar{\circ}n} \cong \mathscr{C}^{\bar{\circ}n} \circ \mathrm{I} \to \mathscr{C}^{\bar{\circ}(n+1)}$$

amounts to adding a level made up of identities $\mathrm{id} = |$ to the trees produced by $\tilde{\Delta}^{n-1}$. So the difference

$$\hat{\Delta}^n := \tilde{\Delta}^n - (\mathrm{Id} \,\bar{\circ}\, \eta)\tilde{\Delta}^{n-1}$$

contains the new leveled trees between the nth iteration of $\tilde{\Delta}$ and the $(n-1)$th iteration. Up to identification of the target spaces, we have $\tilde{\Delta}^n = \sum_{k=0}^{n} \hat{\Delta}^k$.

We define the *coradical filtration* of a coaugmented cooperad as follows.

$$F_0\mathscr{C} := \mathrm{I} \quad \text{and} \quad F_n\mathscr{C} := \ker \hat{\Delta}^n, \quad \text{for } n \geq 1.$$

Since for $n = 1$, $\tilde{\Delta}^1 - (\mathrm{Id}\,\bar{\circ}\,\eta)\tilde{\Delta}^0 = \bar{\Delta}$, we get $F_1\mathscr{C} = \mathrm{I} \oplus \ker \bar{\Delta}$. We call the elements of $\overline{\mathscr{C}}$ which live in the kernel of $\bar{\Delta}$ the *primitive elements* of the cooperad $\mathscr{C}$. So we get a filtration of the cooperad $\mathscr{C}$:

$$F_0\mathscr{C} \subset F_1\mathscr{C} \subset F_2\mathscr{C} \subset \cdots \subset F_n\mathscr{C} \subset F_{n+1}\mathscr{C} \subset \cdots.$$

A coaugmented cooperad $\mathscr{C}$ is called *conilpotent* when the coradical filtration is exhaustive: $\mathrm{colim}_n\, F_n\mathscr{C} = \mathscr{C}$. This is equivalent to requiring that, for any element c of $\mathscr{C}$, the iteration of $\hat{\Delta}$ on c stabilizes at some point.

5.8.6 The Cofree Cooperad

By definition the cofree cooperad on M is the cooperad $\mathscr{F}^c(M)$, which is cofree in the category of conilpotent cooperads (cf. Sect. 1.2.5). Explicitly it means that for any $\mathbb{S}$-module morphism $\varphi : \mathscr{C} \to M$ sending id to 0, there exists a unique cooperad morphism $\tilde{\varphi} : \mathscr{C} \to \mathscr{F}^c(M)$ which renders the following diagram commutative

$$\begin{tikzcd} \mathscr{C} \arrow[d, "\tilde{\varphi}"'] \arrow[dr, "\varphi"] & \\ \mathscr{F}^c(M) \arrow[r, two heads] & M. \end{tikzcd}$$

By dualizing what we have done for free operads in Sect. 5.5, we can prove the existence of the cofree cooperad and give an explicit construction by induction, which we denote by $\mathscr{T}^c(M)$.

The underlying $\mathbb{S}$-module of the cofree cooperad $\mathscr{T}^c(M)$ is the $\mathbb{S}$-module $\mathscr{T}M$ constructed in Sect. 5.5.1 and called the tree module. Let us recall that $\mathscr{T}M = \mathrm{colim}_n\, \mathscr{T}_n M$, where $\mathscr{T}_n M = \mathrm{I} \oplus (M \,\bar{\circ}\, \mathscr{T}_{n-1}M)$. The decomposition map

$$\Delta : \mathscr{T}M \to \mathscr{T}M \,\bar{\circ}\, \mathscr{T}M$$

is defined inductively on $\mathscr{T}_n M$ as follows. First we put

$$\Delta(\mathrm{id}) := \mathrm{id} \,\bar{\circ}\, \mathrm{id},$$

and for any $\mu \in M$ of arity n we put

$$\Delta(\mu) := \mathrm{id} \,\bar{\circ}\, \mu + \mu \,\bar{\circ}\, \mathrm{id}^{\otimes n} \in \mathrm{I} \,\bar{\circ}\, M \oplus M \,\bar{\circ}\, \mathrm{I} \subset \mathscr{T}_1 M \,\bar{\circ}\, \mathscr{T}_1 M.$$

This formula defines Δ on $\mathscr{T}_1 M$. Then we proceed by induction. We suppose that $\Delta : \mathscr{T}_{n-1}M \to \mathscr{T}_{n-1}M \mathbin{\bar\circ} \mathscr{T}_{n-1}M$ is defined and we construct Δ on $\mathscr{T}_n M$. For $\lambda = (\mu; \nu_1, \dots, \nu_k) \in M \mathbin{\bar\circ} \mathscr{T}_{n-1}M \subset \mathscr{T}_n M$, we put

$$\Delta(\lambda) = \Delta(\mu; \nu_1, \dots, \nu_k) := \mathrm{id} \mathbin{\bar\circ} (\mu; \nu_1, \dots, \nu_k) + \Delta^+(\mu; \nu_1, \dots, \nu_k),$$

where Δ^+ is the following composite:

$$\begin{aligned} M \mathbin{\bar\circ} \mathscr{T}_{n-1}M &\xrightarrow{\mathrm{Id}_M \mathbin{\bar\circ} \Delta} M \mathbin{\bar\circ} (\mathscr{T}_{n-1}M \mathbin{\bar\circ} \mathscr{T}_{n-1}M) \\ &\cong (M \mathbin{\bar\circ} \mathscr{T}_{n-1}M) \mathbin{\bar\circ} \mathscr{T}_{n-1}M \xrightarrow{j_n \mathbin{\bar\circ} i_n} \mathscr{T}_n M \mathbin{\bar\circ} \mathscr{T}_n M. \end{aligned}$$

Adopting the operadic version $\Delta(\nu_i) = \sum \nu_i^{(1)} \mathbin{\bar\circ} \nu_i^{(2)}$ of Sweedler's notation, where $\nu_i^{(2)}$ is in fact a tensor product of elements of M: $\nu_i^{(2)} = (\nu_{i,1}^{(2)}, \dots, \nu_{i,r}^{(2)})$, we get

$$\Delta(\mu; \nu_1, \dots, \nu_k) = \mathrm{id} \mathbin{\bar\circ} (\mu; \nu_1, \dots, \nu_k) + \sum \big(\mu; \nu_1^{(1)}, \dots, \nu_k^{(1)}\big) \mathbin{\bar\circ} \big(\nu_1^{(2)}, \dots, \nu_k^{(2)}\big).$$

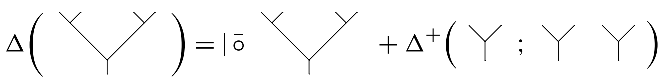
So we have written $\Delta(\lambda)$ as $\sum \lambda^{(1)} \mathbin{\bar\circ} \lambda^{(2)}$. Observe that $\Delta(\lambda)$ contains $\mathrm{id} \mathbin{\bar\circ} \lambda$ but also $\lambda \mathbin{\bar\circ} (\mathrm{id}, \dots, \mathrm{id})$ as a summand. Indeed, by induction, $\Delta(\nu_i)$ contains a summand with $\nu_i^{(1)} = \nu_i$ and $\nu_i^{(2)} = (\mathrm{id}, \dots, \mathrm{id})$, hence in the sum we get a summand of the form

$$(\mu; \nu_1, \dots, \nu_k) \mathbin{\bar\circ} (\mathrm{id}, \dots, \mathrm{id}) = \lambda \mathbin{\bar\circ} (\mathrm{id}, \dots, \mathrm{id}).$$

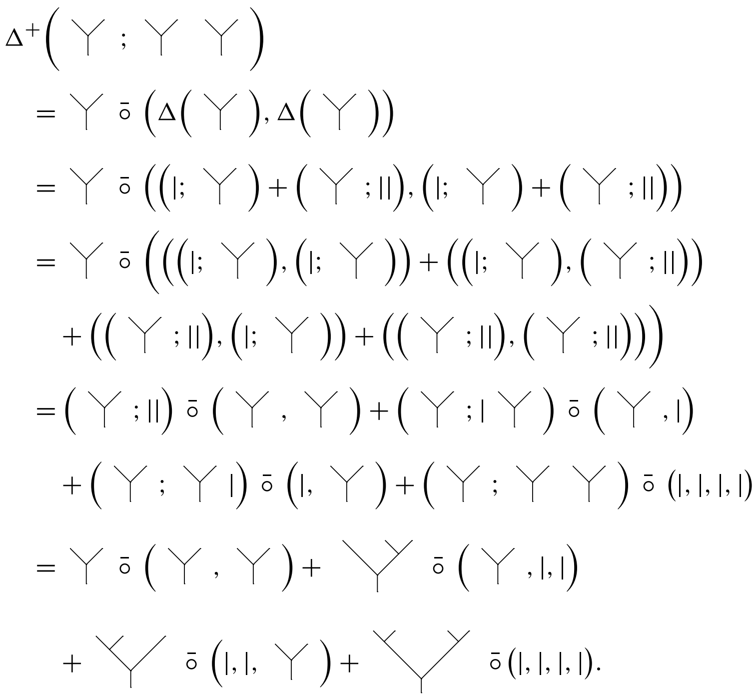
Observe that the construction of Δ involves the associativity isomorphism, see Sect. 5.1.8. So, in the graded case there is a sign appearing in front of the sum. It is (-1) to the power

$$\big|\nu_1^{(2)}\big|\big|\nu_2^{(1)}\big| + \big(\big|\nu_1^{(2)}\big| + \big|\nu_2^{(2)}\big|\big)\big|\nu_3^{(1)}\big| + \cdots + \big(\big|\nu_1^{(2)}\big| + \cdots + \big|\nu_{k-1}^{(2)}\big|\big)\big|\nu_k^{(1)}\big|.$$

Here is an example for which we allow ourselves to leave out the commas to ease the notation. Let $|$ be the tree representing the identity and let  $\in M(2)$ be a binary operation. Since $\Delta($  $) = | \mathbin{\bar\circ}$  $+$  $\mathbin{\bar\circ} (|, |)$, we get



where



The map $\varepsilon : \mathscr{T}M \to M$ is defined by $\mathscr{T}_1 M = \mathrm{I} \oplus M \twoheadrightarrow M$. The coaugmentation map $\eta : M \to \mathscr{T}M$ is equal to the map j.

Proposition 5.8.2. *The above maps induce a coaugmented cooperad structure on* $\mathscr{T}^c(M) := (\mathscr{T}M, \Delta, \varepsilon, \eta)$.

Proof. We have constructed maps $\mathscr{T}_n M \to \mathscr{T}M \,\bar{\circ}\, \mathscr{T}M$. Since they commute with the map $i_n : \mathscr{T}_n M \to \mathscr{T}_{n+1} M$, they give rise to Δ on $\mathscr{T}M = \mathrm{colim}_n\, \mathscr{T}_n M$.

Coassociativity, counitality and coaugmentation can be proved by induction, by a check similar to the one done in Theorem 5.5.1. These properties can also be proved by using the explicit form given in Proposition 5.8.4. $\square$

Theorem 5.8.3. *The conilpotent cooperad* $\mathscr{T}^c(M) := (\mathscr{T}M, \Delta, \varepsilon, \eta)$ *is cofree on* M *among conilpotent cooperads.*

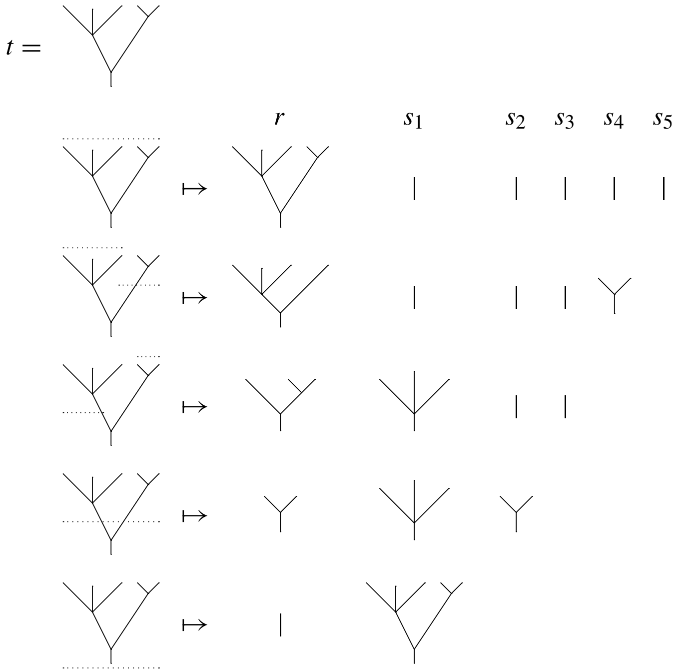
The proof is postponed to the end of the next section.

5.8.7 Description of the Cofree Cooperad in Terms of Trees

Recall from Sect. 5.6.3 that the $\mathbb{S}$-module $\mathscr{T}M$ is isomorphic to the treewise tensor module $\mathbb{T}(M)$ made up of trees with vertices labeled by elements of M.

The decomposition Δ on such a tree t is constructed by "degrafting" as follows. A *cut* of the tree is *admissible* if the grafting of the pieces gives the original tree back. The degrafting $\Delta(t)$ of t is the sum of all the admissible cuttings $(r; s_1, \dots, s_k)$, where r is the piece containing the root, and k is the number of leaves of r. Of course each vertex keeps its labeling.

EXAMPLE.



Proposition 5.8.4. *The cooperad $(\mathscr{T}^c M, \Delta, \varepsilon)$ is isomorphic to the treewise tensor module equipped with the decomposition map given by the admissible cuttings.*

Proof. Recall that $\mathscr{T}_n M$ is isomorphic to labeled trees with at most n levels. We prove the assertion by induction on n. For $n = 0$ and $n = 1$, the explicit form of Δ on $\mathscr{T}_0 M = \mathrm{I}$ and on $\mathscr{T}_0 M = \mathrm{I} \oplus M$ allows to conclude. Suppose that the result holds up to $n - 1$. Let t be a labeled tree in $\mathscr{T}_n M$. If not trivial, this tree can be written $t = (\mu; t_1, \dots, t_k)$, where $t_1, \dots, t_k$ are labeled sub-trees. By definition, $\Delta(t) := \mathrm{id}\,\bar{\circ}\,t + \Delta^+(\mu; t_1, \dots, t_k)$. The first component gives the bottom cutting. The second component is given by the cuttings $\Delta(t_i)$ of the sub-trees t_i, where the bottom part is then grafted onto μ. Finally, we get all the admissible cuttings of the tree t. $\square$

With this description of $\mathscr{T}^c(M)$, it is easy to see that M is the space of primitive elements. The coradical filtration is equal to $F_n \mathscr{T}^c(M) = \mathscr{T}_n M$, that is coincides with the defining filtration. So the coaugmented cooperad $\mathscr{T}^c(M)$ is conilpotent.

Proof of Theorem 5.8.3. Let $\mathscr{C}$ be a conilpotent cooperad and let $\varphi : \mathscr{C} \to M$ be an $\mathbb{S}$-module map, which sends id to 0. We claim that there is a unique morphism of cooperads

$$\tilde{\varphi} : \mathscr{C} \to \mathscr{T}^c(M)$$

which extends φ. We construct $\tilde{\varphi}_n : \mathscr{C} \to \mathscr{T}_n M$ by induction on n. For $n = 0$, we put $\tilde{\varphi}_0(\mathrm{id}) = \mathrm{id}$ and $\tilde{\varphi}_0$ is 0 in the other components. For $n = 1$, we put $\tilde{\varphi}_1(\mathrm{id}) = \mathrm{id}$ and $\tilde{\varphi} = \varphi : \mathscr{C} \to M \subset \mathscr{T}_1 M$. Let us suppose that $\tilde{\varphi}_{n-1}$ has been constructed. The image of id by $\tilde{\varphi}_n$ is $\mathrm{id} \in \mathrm{I} \subset \mathrm{I} \oplus M \,\bar{\circ}\, \mathscr{T}_{n-1} M = \mathscr{T}_n M$. The component in the other summand is equal to the composite

$$(\varphi \,\bar{\circ}\, \tilde{\varphi}_{n-1})\Delta : \mathscr{C} \to \mathscr{C} \,\bar{\circ}\, \mathscr{C} \to M \,\bar{\circ}\, \mathscr{T}_{n-1} M = \mathscr{T}_n M.$$

One can see that the map $\tilde{\varphi}_n$ is equal to the following composite

$$\tilde{\varphi}_n : \mathscr{C} \xrightarrow{\tilde{\Delta}^{n-1}} \mathscr{C}^{\bar{\circ} n} \xrightarrow{\tilde{\varphi}_1} (\mathrm{I} \oplus M)^{\bar{\circ} n} \twoheadrightarrow \mathscr{T}_n M,$$

where the last map is the projection of n-leveled trees into non-leveled trees.

Since the cooperad $\mathscr{C}$ is conilpotent, this process stabilizes, that is for any $c \in \mathscr{C}$, the image under the composite maps $\mathscr{C} \xrightarrow{\tilde{\varphi}_n} \mathscr{T}_n M \rightarrowtail \mathscr{T} M$ give the same image in the colimit $\mathscr{T} M$, for $n \geq N$. So the map $\tilde{\varphi}$ is well defined.

Since we want $\tilde{\varphi}$ to be a map of cooperads and to coincide with φ in the component M, we have no choice for $\tilde{\varphi}$. By the definition of $\tilde{\varphi}$ in terms of $\tilde{\Delta}$ and by the coassociativity of Δ, the map $\tilde{\varphi}$ is a morphism of cooperads. □

5.8.8 Combinatorial Definition of a Cooperad

In the same way as in Sect. 5.5, we give an equivalent definition of a conilpotent cooperad using a comonad of trees.

The adjunction

$$\mathscr{U} : \mathsf{conil\ coOp} \rightleftharpoons \mathbb{S}\text{-}\mathsf{Mod} : \mathscr{T}^c$$

of the previous sections induces a comonad denoted by $\mathbb{T}^c$. Explicitly it is a comonoid in the category of endofunctors of $\mathbb{S}$-modules, see Appendix B.4.1. The underlying endofunctor is the same as in Sect. 5.6.1: $\mathbb{T}^c : M \mapsto \mathscr{T} M$. The coproduct and the counit maps

$$\Delta : \mathbb{T}^c \to \mathbb{T}^c \circ \mathbb{T}^c \quad \text{and} \quad \varepsilon : \mathbb{T}^c \to \mathrm{Id}_{\mathbb{S}\text{-}\mathsf{Mod}}$$

are given as follows. For any $\mathbb{S}$-module M, $\mathbb{T}^c \circ \mathbb{T}^c(M) = \mathscr{T}(\mathscr{T} M)$ is made up of "trees of trees" with vertices labeled by M. Equivalently, it coincides with trees labeled by M equipped with a partition into subtrees. The map $\Delta(M)$ associates to a tree t labeled by M, the sum of all the partitioned trees coming from t. The map $\varepsilon(M) : \mathscr{T} M \to M$ is the projection onto corollas.

Similarly we consider the comonad $\overline{\mathbb{T}}^c$ made up of trees $\overline{\mathscr{T}}^c$ without the trivial tree.

Proposition 5.8.5. *Let $\overline{\mathscr{C}}$ be an $\mathbb{S}$-module. A coalgebra structure on $\overline{\mathscr{C}}$ over the comonad $\overline{\mathbb{T}}^c$ is equivalent to a conilpotent cooperad structure on $\mathscr{C} := \overline{\mathscr{C}} \oplus \mathrm{I}$.*

Proof. ($\Leftarrow$) Let $(\mathscr{C} = \overline{\mathscr{C}} \oplus \mathrm{I}, \Delta, \varepsilon, \eta)$ be a conilpotent cooperad. The map $\Delta_{\overline{\mathscr{C}}} : \overline{\mathscr{C}} \to \overline{\mathbb{T}}^c(\overline{\mathscr{C}}) = \overline{\mathscr{T}}\overline{\mathscr{C}}$ is given by the universal property of the conilpotent free cooperad applied to

$$\begin{tikzcd} \overline{\mathscr{C}} \arrow[d, "\Delta_{\overline{\mathscr{C}}}"'] \arrow[dr, "\mathrm{Id}_{\overline{\mathscr{C}}}"] & \\ \overline{\mathscr{T}}^c\overline{\mathscr{C}} \arrow[r, two heads] & \overline{\mathscr{C}}. \end{tikzcd}$$

($\Rightarrow$) In the other way round, let $\Delta_{\overline{\mathscr{C}}} : \overline{\mathscr{C}} \to \overline{\mathbb{T}}^c(\overline{\mathscr{C}}) = \overline{\mathscr{T}}\overline{\mathscr{C}}$ be a coalgebra over the comonad $\overline{\mathbb{T}}^c$. We view the trees of $\overline{\mathscr{T}}_2\overline{\mathscr{C}}$ as 2-leveled trees by adding trivial trees | if necessary. By projecting onto this summand, we get a coassociative decomposition map $\Delta : \mathscr{C} \to \mathscr{C} \bar{\circ} \mathscr{C}$, where the image of id is defined by $\mathrm{id} \circ \mathrm{id}$. The unit and the coaugmentation maps come for free. In the end, it defines a conilpotent cooperad structure on $\mathscr{C}$. □

When $\mathscr{C}$ is a conilpotent cooperad, we denote by $\Delta_{\mathscr{C}} : \mathscr{C} \to \mathscr{T}^c(\overline{\mathscr{C}})$ the morphism of cooperads $\mathrm{Id}_\mathrm{I} \oplus \Delta_{\overline{\mathscr{C}}}$. In conclusion, we get the following result, which will play a crucial role in Sect. 10.3.

Proposition 5.8.6. *Let $(\mathscr{C}, \Delta, \varepsilon, \eta)$ be a conilpotent cooperad and let $\varphi : \mathscr{C} \to M$ be a morphism of $\mathbb{S}$-modules such that $\varphi(\mathrm{id}) = 0$. Its unique extension into a morphism of cooperads $\tilde{\varphi} : \mathscr{C} \to \mathscr{T}^c(M)$ is equal to the composite*

$$\tilde{\varphi} \, : \, \mathscr{C} \xrightarrow{\Delta_{\mathscr{C}}} \mathscr{T}^c(\overline{\mathscr{C}}) \xrightarrow{\mathscr{T}^c(\varphi)} \mathscr{T}^c(M),$$

where the map $\Delta_{\mathscr{C}}$ is given by the iterations $\tilde{\Delta}^n = \sum_{k=0}^{n} \hat{\Delta}^k$.

Proof. It is direct consequence of the above results. □

5.9 Nonsymmetric Operad

Replacing the category of $\mathbb{S}$-modules by the category of arity graded vector spaces gives the notion of nonsymmetric operad (called ns operad for short). To any ns operad one can associate an operad by tensoring with the regular representation in each arity. This section can be read independently of the rest of the chapter. We work over a field $\mathbb{K}$ though most of the notions and results of this section are valid over a commutative ring.

5.9.1 More on Arity Graded Modules

Let

$$M. = \{M_n\}_{n\geq 0}$$

be a graded vector space (or graded module). We denote by $\mathbb{N}$-Mod the category of graded vector spaces (or graded $\mathbb{K}$-modules if $\mathbb{K}$ is a commutative ring). The integer n is called the arity in this framework. The *Schur functor* M : $\mathsf{Vect} \to \mathsf{Vect}$ associated to $M.$ is, by definition,

$$M(V) := \bigoplus_{n\geq 0} M_n \otimes_{\mathbb{K}} V^{\otimes n}.$$

In literature the object $M.$ is sometimes called a *collection*. We refrain to call it a nonsymmetric $\mathbb{S}$-module. Recall that the sum, tensor product, composition and Hadamard product of arity graded spaces are given by

$$\begin{aligned}
(M. \oplus N.)_n &:= M_n \oplus N_n, \\
(M. \otimes N.)_n &:= \bigoplus_{i+j=n} M_i \otimes N_j, \\
(M. \circ N.)_n &:= \bigoplus_k M_k \otimes \Big(\bigoplus N_{i_1} \otimes \cdots \otimes N_{i_k}\Big), \\
(M. \underset{\mathrm{H}}{\otimes} N.)_n &:= M_n \otimes N_n,
\end{aligned}$$

where the second sum in line 3 is over all the k-tuples $(i_1, \ldots, i_k)$ satisfying $i_1 + \cdots + i_k = n$. Observe that the associativity property of the composition of graded modules involves the switching map, cf. Sect. 5.1.8. For any vector space V we have natural isomorphisms:

$$\begin{aligned}
(M \oplus N)(V) &= M(V) \oplus N(V), \\
(M \otimes N)(V) &= M(V) \otimes N(V), \\
(M \circ N)(V) &:= M\big(N(V)\big).
\end{aligned}$$

The Hilbert–Poincaré series of the arity graded module $M.$ is:

$$f^M(x) := \sum_{n\geq 0} \dim M_n x^n.$$

The generating series of a sum (resp. product, composition, Hadamard product) of arity graded modules is the sum (resp. product, composition, Hadamard product) of their respective generating series.

In the sequel we simply write M instead of $M.$ whenever there is no ambiguity.

5.9.2 Monoidal Definition of a Nonsymmetric Operad

By definition a *nonsymmetric operad* (also called *non-Σ-operad* in the literature) $\mathscr{P} = (\mathscr{P}, \gamma, \eta)$ is an arity graded vector space $\mathscr{P} = \{\mathscr{P}_n\}_{n\geq 0}$ equipped with composition maps

$$\gamma_{i_1,\ldots,i_k} : \mathscr{P}_k \otimes \mathscr{P}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \longrightarrow \mathscr{P}_{i_1+\cdots+i_k}$$

and an element $\mathrm{id} \in \mathscr{P}_1$, such that the transformations of functors $\gamma : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$ and $\eta : \mathrm{I} \to \mathscr{P}$, deduced from this data, make $(\mathscr{P}, \gamma, \eta)$ into a monoid.

We often abbreviate "nonsymmetric operad" into "ns operad".

5.9.3 Classical Definition of a NS Operad

Obviously a nonsymmetric operad can be defined as an arity graded module $\mathscr{P}$ equipped with linear maps

$$\gamma_{i_1,\ldots,i_k} : \mathscr{P}_k \otimes \mathscr{P}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \longrightarrow \mathscr{P}_{i_1+\cdots+i_k}$$

and an element $\mathrm{id} \in \mathscr{P}_1$, such that the following diagram (in which the tensor signs are omitted) is commutative

$$\begin{tikzcd}
\mathscr{P}_n \mathscr{P}_{r_1} \cdots \mathscr{P}_{r_n} \arrow[dddr, bend left] & \\
\mathscr{P}_n \mathscr{P}_{i_1} \mathscr{P}_{j_{1,1}} \cdots \mathscr{P}_{j_{1,i_1}} \mathscr{P}_{i_2} \mathscr{P}_{j_{2,1}} \cdots\cdots \mathscr{P}_{i_n} \cdots \mathscr{P}_{j_{n,i_n}} \arrow[u] \arrow[d, "\cong"] & \\
\mathscr{P}_n \mathscr{P}_{i_1} \cdots \mathscr{P}_{i_n} \mathscr{P}_{j_{1,1}} \cdots \mathscr{P}_{j_{1,i_1}} \mathscr{P}_{j_{2,1}} \cdots\cdots \mathscr{P}_{j_{n,1}} \cdots \mathscr{P}_{j_{n,i_n}} \arrow[d] & \\
\mathscr{P}_m \mathscr{P}_{j_{1,1}} \cdots \mathscr{P}_{j_{1,i_1}} \mathscr{P}_{j_{2,1}} \cdots\cdots \mathscr{P}_{j_{n,1}} \cdots \mathscr{P}_{j_{n,i_n}} \arrow[r] & \mathscr{P}_\ell
\end{tikzcd}$$

where $r_k = j_{k,1} + \cdots + j_{k,i_k}$ for $k = 1$ to n, $m = i_1 + \cdots + i_n$ and $\ell = r_1 + \cdots + r_n$. Moreover the element id is such that the evaluation of $\gamma_n : \mathscr{P}_1 \otimes \mathscr{P}_n \to \mathscr{P}_n$ on (id, μ) is μ, and the evaluation of $\gamma_{1,\ldots,1}$ on $(\mu; \mathrm{id}, \ldots, \mathrm{id})$ is μ.

The equivalence between the monoidal definition and the classical definition is a straightforward check.

5.9.4 Partial Definition of a NS Operad

A nonsymmetric operad can be defined as an arity graded vector space $\mathscr{P}$ equipped with *partial compositions*:

$$\circ_i : \mathscr{P}_m \otimes \mathscr{P}_n \to \mathscr{P}_{m-1+n}, \quad \text{for } 1 \le i \le m,$$

satisfying the relations

$$\begin{cases} \text{(I)} \quad (\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), & \text{for } 1 \le i \le l,\ 1 \le j \le m, \\ \text{(II)} \quad (\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, & \text{for } 1 \le i < k \le l, \end{cases}$$

for any $\lambda \in \mathscr{P}_l$, $\mu \in \mathscr{P}_m$, $\nu \in \mathscr{P}_n$.

This definition (with different notations and grading) appears in Gerstenhaber's paper [Ger63] under the name "pre-Lie system".

The equivalence with the monoidal definition is given by constructing the map $\gamma_{i_1,\dots,i_n}$ as an iteration of the partial operations. In the other direction the partial operation $\circ_i$ is obtained by restriction:

$$\lambda \circ_i \mu = \gamma(\lambda; \mathrm{id}, \dots, \mathrm{id}, \mu, \mathrm{id}, \dots, \mathrm{id})$$

where μ is at the ith position.

5.9.5 Combinatorial Definition of a NS Operad

For any planar rooted tree t we denote by $\mathrm{vert}(t)$ its set of vertices and by $|v|$ the number of inputs of the vertex $v \in \mathrm{vert}(t)$, see Appendix C for details. Let M be an arity graded space with $M_0 = 0$. Recall that the integer n is called the *arity* of t and the integer $k = \#\mathrm{vert}(t)$ is called the *weight* of t. For any tree t we define

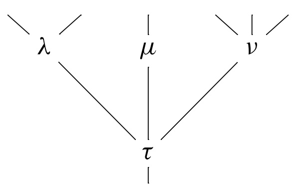
$$M_t := \bigotimes_{v \in \mathrm{vert}(t)} M_{|v|}.$$

We get a functor

$$\mathrm{P}\mathbb{T} : \mathbb{N}\text{-}\mathsf{Mod} \to \mathbb{N}\text{-}\mathsf{Mod}$$

by $\mathrm{P}\mathbb{T}(M)_n := \bigoplus_{t \in PT_n} M_t$. It is helpful to think about an element of $\mathrm{P}\mathbb{T}(M)_n$ as a planar rooted tree where each vertex v is decorated by an element of $M_{|v|}$.

In the following example we have $\tau \in M_3$, $\lambda \in M_2$, $\mu \in M_1$, $\nu \in M_3$:



In particular the corolla enables us to define the transformation of functors $\eta : \mathrm{I}_{\mathbb{N}\text{-Mod}} \to \mathrm{P}\mathbb{T}$.

The *substitution* of trees consists in replacing the vertices by given trees (with matching inputs). The substitution of trees defines a transformation of functors $\alpha : \mathrm{P}\mathbb{T} \circ \mathrm{P}\mathbb{T} \to \mathrm{P}\mathbb{T}$ as follows. From the definition of $\mathrm{P}\mathbb{T}$ we get

$$\begin{aligned}
\mathrm{P}\mathbb{T}(\mathrm{P}\mathbb{T}(M))_n &= \bigoplus_{t \in PT_n} \mathrm{P}\mathbb{T}(M)_t \\
&= \bigoplus_{t \in PT_n} \Bigl(\bigotimes_{v \in \mathrm{vert}(t)} \mathrm{P}\mathbb{T}(M)_{|v|} \Bigr) \\
&= \bigoplus_{t \in PT_n} \Bigl(\bigotimes_{v \in \mathrm{vert}(t)} \Bigl(\bigoplus_{s \in PT_{|v|}} M_s \Bigr)\Bigr).
\end{aligned}$$

Under the substitution of trees we get an element of $\mathrm{P}\mathbb{T}(M)_n$, since at any vertex v of t we have an element of $\mathrm{P}\mathbb{T}(M)_{|v|} = \bigoplus_{s \in PT_{|v|}} M_s$, that is a tree s and its decoration. We substitute this data at each vertex of t to get a new decorated tree. Therefore we have defined an $\mathbb{S}$-module morphism $\alpha(M) : \mathrm{P}\mathbb{T}(\mathrm{P}\mathbb{T}(M)) \to \mathrm{P}\mathbb{T}(M)$.

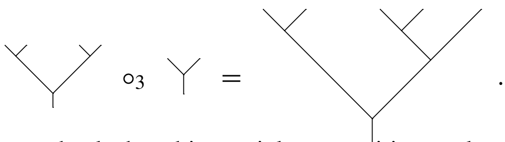
The transformation of functors α is obviously associative and unital, so $(\mathrm{P}\mathbb{T}, \alpha, \eta)$ is a monad.

The *combinatorial definition of a ns operad* consists in defining it as a unital algebra over the monad $(\mathrm{P}\mathbb{T}, \alpha, \eta)$, cf. Appendix B.4. In other words a ns operad is an arity graded module $\mathscr{P}$ together with a map $\mathrm{P}\mathbb{T}(\mathscr{P}) \to \mathscr{P}$ which is compatible with α and η in the usual sense.

The combinatorial definition of a ns operad is equivalent to the partial definition of a ns operad, and therefore to all the other definitions.

5.9.6 Free NS Operad and Planar Trees

By definition the *free nonsymmetric operad* over the arity graded module M is the ns operad $\mathscr{T}(M)$ equipped with a graded module morphism $M \to \mathscr{T}(M)$ which satisfies the classical universal property. Explicitly it can be constructed inductively as in Sect. 5.5.1, i.e. $\mathscr{T}M = \bigcup_n \mathscr{T}_n M$, where $\mathscr{T}_0 M := I$ and $\mathscr{T}_n M := \mathrm{I} \oplus (M \circ \mathscr{T}_{n-1} M)$. It can also be constructed as a quotient as in Sect. 5.5.5, or, more explicitly, by using planar trees. In fact the graded module $\mathrm{P}\mathbb{T}(M)$ constructed above is endowed with a ns operad structure as follows. Let t and s be two decorated trees. The partial composition $t \circ_i s$ is the decorated tree obtained by grafting the tree s on the ith leaf of t and keeping the decorations.

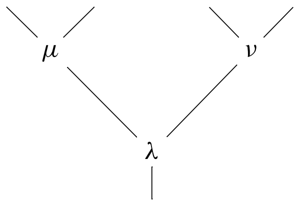


It is immediate to check that this partial composition makes $\mathrm{P}\mathbb{T}(M)$ into a ns operad, and that this ns operad is free over M. The map $\eta : M \to \mathrm{P}\mathbb{T}(M)$ consists in sending the operation $\mu \in M_n$ to the nth corolla decorated by μ. The isomorphism $\varphi : \mathscr{T}(M) \to \mathrm{P}\mathbb{T}(M)$ is made explicit as follows. First, we have $\varphi(\mathrm{id}) = |$. Second, the generating operation $\mu \in M_k$ is sent to the kth corolla decorated by μ. Third, for $\omega_i \in \mathscr{T}_{n-1}M$, $i = 1, \ldots, k$, the generic element $(\mu; \omega_1, \ldots, \omega_k) \in M \circ \mathscr{T}_{n-1}M \subset \mathscr{T}_n M$ is mapped under φ to the tree $\varphi(\mu; \omega_1, \ldots, \omega_k)$ obtained by grafting the decorated trees $\varphi(\omega_i)$ to the leaves of the kth corolla (image of μ). It is immediate to check that we get an isomorphism.

5.9.7 Free NS Operad in the Graded Framework

In the construction of the free ns operad in terms of trees in the sign-graded framework, signs show up in the computation of composition. Here is an explicit example.

Let $\lambda, \mu, \nu \in M$ be three binary graded operations. In $\mathscr{T}M$ the tree

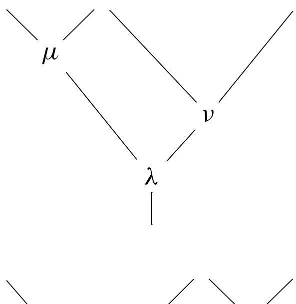


corresponds to the element $(\lambda; \mu, \nu) \in M \circ M$ which is to be interpreted as an element in $\mathscr{T}_2 M = \mathrm{I} \oplus M \circ (\mathrm{I} \oplus M)$. Viewed as an element of $\mathscr{T}_3 M$ via $i : \mathscr{T}_2 M \to \mathscr{T}_3 M$ it becomes

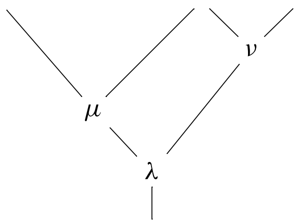
$$\big(\lambda; (\mu; \mathrm{id}, \mathrm{id}), (\nu; \mathrm{id}, \mathrm{id})\big) \in M \circ \big(\mathrm{I} \oplus M \circ (\mathrm{I} \oplus M)\big).$$

Now, let us identify the two composites from bottom to top that is

$(\lambda \circ_2 \nu) \circ_1 \mu =$



resp. $(\lambda \circ_1 \mu) \circ_3 \nu =$



corresponding to the composite $((\lambda;\mathrm{id},\nu);\mu,\mathrm{id},\mathrm{id})$, resp. $((\lambda;\mu,\mathrm{id});\mathrm{id},\mathrm{id},\nu)$. Under the associativity isomorphism, they are equal to

$$(-1)^{|\nu||\mu|}\big(\lambda;(\mathrm{id};\mu),(\nu;\mathrm{id},\mathrm{id})\big),\quad \text{resp.}\quad \big(\lambda;(\mu;\mathrm{id},\mathrm{id}),(\mathrm{id};\nu)\big),\quad \text{that is}$$
$$(-1)^{|\nu||\mu|}\big(\lambda;(\mu;\mathrm{id},\mathrm{id}),(\nu;\mathrm{id},\mathrm{id})\big),\quad \text{resp.}\quad \big(\lambda;(\mu;\mathrm{id},\mathrm{id});(\nu;\mathrm{id},\mathrm{id})\big).$$

In conclusion we get

$$(\lambda\circ_2\nu)\circ_1\mu=(-1)^{|\nu||\mu|}(\lambda;\mu,\nu)\quad\text{and}\quad(\lambda\circ_1\mu)\circ_3\nu=(\lambda;\mu,\nu).$$

5.9.8 Algebra over a Nonsymmetric Operad

For any vector space A the graded module $\mathrm{End}(A)$, defined by $\mathrm{End}(A)_n := \mathrm{Hom}(A^{\otimes n},A)$, is a ns operad for the composition of maps (cf. Sect. 5.2.11). By definition an *algebra over the ns operad* $\mathscr{P}$ is a morphism of ns operads $\mathscr{P}\to\mathrm{End}(A)$. Equivalently, a $\mathscr{P}$-algebra structure on A is a family of linear maps $\mathscr{P}_n\otimes A^{\otimes n}\to A$ compatible with the ns operad structure of $\mathscr{P}$.

5.9.9 Nonsymmetric Operad, Type of Algebras

Let us consider a type of algebras for which the generating operations have no symmetry, the relations are multilinear and, in these relations, the variables stay in the same order. Then this type of algebras can be faithfully encoded by a nonsymmetric operad.

The relationship between types of algebras and operads is slightly simpler in the nonsymmetric case, as shown by the following result.

Proposition 5.9.1. *A nonsymmetric operad $\mathscr{P}$ is completely determined by the free $\mathscr{P}$-algebra on one generator.*

Proof. For a nonsymmetric operad $\mathscr{P}$ the free algebra on one generator is

$$\mathscr{P}(\mathbb{K})=\bigoplus_{n\geq 0}\mathscr{P}_n\otimes\mathbb{K}^{\otimes n}=\bigoplus_{n\geq 0}\mathscr{P}_n.$$

Hence $\mathscr{P}_n$ is the n-multilinear part of $\mathscr{P}(\mathbb{K})$. Using the ubiquity of the operations, see Sect. 5.2.13, it follows that the composition maps are completely determined by the $\mathscr{P}$-algebra structure of $\mathscr{P}(\mathbb{K})$. $\square$

Remark that this statement is not true for symmetric operads. For instance *Ass* and *Com* have the same free algebra on one generator, namely the ideal (x) in the polynomial algebra $\mathbb{K}[x]$. It determines *As*, but not *Com*.

5.9.10 Hadamard Product of NS Operads

Let $\mathscr{P}$ and $\mathscr{Q}$ be two ns operads. The Hadamard tensor product $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ of the underlying graded modules has a natural operad structure:

$$\begin{aligned}
&(\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})_k \otimes (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})_{i_1} \otimes \cdots \otimes (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})_{i_k} \\
&\quad = \mathscr{P}_k \otimes \mathscr{Q}_k \otimes \mathscr{P}_{i_1} \otimes \mathscr{Q}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \otimes \mathscr{Q}_{i_k} \\
&\quad \cong \mathscr{P}_k \otimes \mathscr{P}_{i_1} \otimes \cdots \otimes \mathscr{P}_{i_k} \otimes \mathscr{Q}_k \otimes \mathscr{Q}_{i_1} \otimes \cdots \otimes \mathscr{Q}_{i_k} \\
&\quad \longrightarrow \mathscr{P}_n \otimes \mathscr{Q}_n = (\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q})_n
\end{aligned}$$

for $n = i_1 + \cdots + i_k$. Observe that we use the switching map in the category Vect to put the factors $\mathscr{Q}_i$ in the correct position. Therefore, when Vect is replaced by another tensor category (cf. Appendix B.3) signs might be involved. The operad uAs is obviously a unit for this operation.

The ns operad $\mathscr{P} \underset{\mathrm{H}}{\otimes} \mathscr{Q}$ is called the *Hadamard product* of the ns operads $\mathscr{P}$ and $\mathscr{Q}$.

5.9.11 From NS Operads to Symmetric Operads and Vice Versa

Let $\mathscr{P}$ be a ns operad with $\mathscr{P}_n$ as the space of n-ary operations. The category of $\mathscr{P}$-algebras can be encoded by a symmetric operad. We still denote it by $\mathscr{P}$ and the space of n-ary operations by $\mathscr{P}(n)$. It is immediate that

$$\mathscr{P}(n) = \mathscr{P}_n \otimes \mathbb{K}[\mathbb{S}_n]$$

where the action of the symmetric group on $\mathscr{P}(n)$ is given by the regular representation $\mathbb{K}[\mathbb{S}_n]$. Indeed we have $(\mathscr{P}_n \otimes \mathbb{K}[\mathbb{S}_n]) \otimes_{\mathbb{K}[\mathbb{S}_n]} V^{\otimes n} = \mathscr{P}_n \otimes V^{\otimes n}$. The composition map $\gamma(i_1, \ldots, i_k)$ in the symmetric framework is given, up to a permutation of factors, by the tensor product of the composition map $\gamma_{i_1,\ldots,i_k}$ in the ns framework with the composition map of the symmetric operad Ass. Considered as a symmetric operad $\mathscr{P}$ is sometimes called a *regular operad*. Observe that the categories of algebras over a ns operad and over its associated operad are the same, so they encode the same type of algebras. We usually take the same notation for the ns operad and its associated symmetric operad, except in the case of associative algebras where we use As and Ass respectively in this book.

In conclusion we have constructed a functor

$$\mathsf{nsOp}_{\mathbb{K}} \longrightarrow \mathsf{Op}_{\mathbb{K}}.$$

This functor admits a right adjoint:

$$\mathsf{Op}_{\mathbb{K}} \longrightarrow \mathsf{nsOp}_{\mathbb{K}}, \qquad \mathscr{P} \mapsto \widetilde{\mathscr{P}}.$$

Explicitly we have $\widetilde{\mathscr{P}}_n = \mathscr{P}(n)$, in other words we forget the $\mathbb{S}_n$-module structure. We have $\widetilde{\mathscr{P}}(n) = \mathscr{P}(n) \otimes \mathbb{K}[\mathbb{S}_n]$ where the $\mathbb{S}_n$-module structure is given by the action on $\mathbb{K}[\mathbb{S}_n]$ (not the diagonal action). The composition maps

$$\widetilde{\gamma}_{i_1,\ldots,i_k} = \gamma(i_1,\ldots,i_k) : \mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \longrightarrow \mathscr{P}(i_1 + \cdots + i_k)$$

satisfy the axioms of a ns operad.

Examples. By direct inspection we see that $\widetilde{Com} = As$. In [ST09] Salvatore and Tauraso show that the operad $\widetilde{Lie}$ is a free ns operad. In [BL10] Bergeron and Livernet show that $\widetilde{preLie}$ is also free.

5.9.12 Nonsymmetric Operads as Colored Algebras

A *colored algebra* is a graded vector space $A = \{A_n\}_{n\geq 0}$ equipped with operations which are only defined under some conditions depending on the colors (elements of an index set). For instance let us suppose that we have operations (i.e. graded linear maps) $\circ_i : A_m \otimes A \to A$ defined only when $1 \leq i \leq m+1$ and a map $\mathbb{K} \to A_0$, $1_{\mathbb{K}} \mapsto 1$. Let us suppose that they satisfy the relations:

$$\begin{cases} \text{(I)} & (x \circ_i y) \circ_{i-1+j} z = x \circ_i (y \circ_j z), \quad i \leq j \leq i+m-1, \\ \text{(II)} & (x \circ_i y) \circ_{j+m-2} z = (x \circ_j z) \circ_i y, \quad i+m \leq j \leq l+m-1, \end{cases}$$

for any $x \in A_{l-1}$, $y \in A_{m-1}$, $z \in A$ and unital relations with respect to 1. It appeared in Gerstenhaber's paper [Ger63] as a *pre-Lie system* (our notation $\circ_i$ corresponds to his notation $\circ_{i-1}$). It also appears in [Ron11] by M. Ronco, where such a colored algebra is called a *grafting algebra*. Compared to that paper we have taken the opposite products and we have shifted the numbering of the operations by 1. Then it is obvious that under the change of notation $A_n = \mathscr{P}_{n+1}$, this is nothing but the notion of nonsymmetric operad. This point of view permits us to look at variations of colored algebras as variations of operads, cf. [Ron11], Sect. 13.14, and also to introduce the notion of colored operads, cf. [vdL03].

5.9.13 Category Associated to a NS Operad

As in Sect. 5.4.1 one can associate to any ns operad a category whose objects are indexed by the natural numbers. When the operad is set-theoretic, this construction can be done in the set-theoretic framework. The category associated to uAs can be identified to the linearized simplicial category $\mathbb{K}[\Delta]$ (cf. [Pir02a]). The category associated to As can be identified with the linearized presimplicial category $\mathbb{K}[\Delta^{pre}]$ (i.e. Δ without degeneracies).

5.9.14 Group Associated to a NS Operad

Let $\mathscr{P}$ be a ns operad such that $\mathscr{P}_0 = 0$ and $\mathscr{P}_1 = \mathbb{K}\,\mathrm{id}$. We consider the series

$$\underline{a} := (a_0, a_1, \ldots, a_n, \ldots)$$

where $a_n \in \mathscr{P}_{n+1}$ for any n and $a_0 = \mathrm{id} \in \mathscr{P}_1$. We denote by $G(\mathscr{P})$ this set of series. We define a binary operation $\underline{a}\ \underline{b}$ on this set as follows:

$$(\underline{a}\ \underline{b})_n := \sum_k \sum_{i_1+\cdots+i_k=n} \gamma(a_{k-1}; b_{i_1-1}, \ldots, b_{i_k-1}).$$

Proposition 5.9.2. *The binary operation* $(\underline{a}, \underline{b}) \mapsto \underline{a}\ \underline{b}$ *makes* $G(\mathscr{P})$ *into a group with unit* $1 = (\mathrm{id}, 0, 0, \ldots)$.

Proof. The associativity property follows readily from the associativity property of γ. The existence of an inverse, that is for any $\underline{a}$ there exists $\underline{b}$ such that $\underline{a}\ \underline{b} = 1$, is achieved by induction. For instance $b_1 = -a_1, b_2 = -a_2 - a_1 \circ (\mathrm{id}, b_1) - a_1 \circ (b_1, \mathrm{id})$, etc. □

Observe that for $\mathscr{P} = As$, the group $G(\mathscr{P})$ is nothing but the group of power series in one variable with constant coefficient equal to 1. This construction has been used in several instances, cf. [Fra08, Cha01a, vdL02, LN12].

5.9.15 Pre-Lie Algebra Associated to a NS Operad

Let $\mathscr{P}$ be a ns operad with $\mathscr{P}(0) = 0$ and consider the space $\mathscr{P}(\mathbb{K}) := \bigoplus_{n\geq 1} \mathscr{P}_n$, resp. $\widehat{\mathscr{P}}(\mathbb{K}) := \prod_{n\geq 1} \mathscr{P}_n$. We construct a bilinear operation $\{-,-\}$ as follows:

$$\{\mu, \nu\} := \sum_{i=1}^{i=m} (\mu \circ_i \nu)$$

for $\mu \in \mathscr{P}_m,\ \nu \in \mathscr{P}_n$. As in the case of symmetric operads, cf. Sect. 5.4.3, the relations satisfied by the partial operations imply that the binary operation $\{-,-\}$ makes $\mathscr{P}(\mathbb{K})$, resp. $\widehat{\mathscr{P}}(\mathbb{K})$, into a pre-Lie algebra, and hence a Lie algebra by anti-symmetrization.

We check easily that in the case of the ns operad As we get the pre-Lie algebra of polynomial vector fields on the affine line, cf. Sect. 1.4.3.

When $\mathscr{P} = \mathrm{End}(A)$, this Lie bracket on $C^\bullet_{Hoch}(A, A)$ was first constructed by Gerstenhaber in [Ger63] in his study of Hochschild cohomology of an associative algebra A with coefficients into itself, cf. Sect. 13.3.11.

5.9.16 Hopf Algebra Associated to a NS Operad

Let $\mathscr{P}$ be a ns operad with $\mathscr{P}_0 = 0$ and $\mathscr{P}_1 = \mathbb{K}\,\mathrm{id}$. We put $\widehat{\mathscr{P}} := \prod_{n\geq 2} \mathscr{P}_n$. On the cofree coalgebra $T^c(\widehat{\mathscr{P}})$ we define a product, compatible in the Hopf sense with the coproduct, as follows. Since $T^c(\widehat{\mathscr{P}})$ is cofree, by Sect. 1.2.5 it suffices to construct the map

$$T^c(\widehat{\mathscr{P}}) \otimes T^c(\widehat{\mathscr{P}}) \to \widehat{\mathscr{P}}.$$

On $T^c(\widehat{\mathscr{P}})^{\geq 2} \otimes T^c(\widehat{\mathscr{P}})^{\geq 1}$ it is trivial, on $\widehat{\mathscr{P}} \otimes T^c(\widehat{\mathscr{P}})$ it is given by

$$\mu \otimes (\mu_1, \dots, \mu_k) \mapsto \sum \gamma(\mu; \mathrm{id}, \dots, \mathrm{id}, \mu_1, \mathrm{id}, \dots, \mathrm{id}, \mu_2, \mathrm{id}, \dots, \mu_k, \mathrm{id}, \dots) \in \widehat{\mathscr{P}},$$

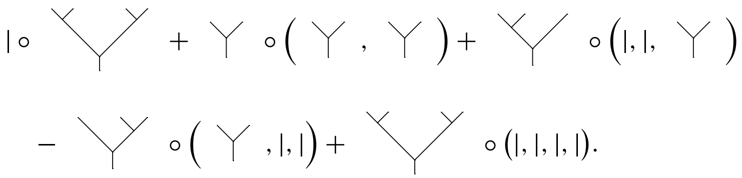
whenever $\mu \in \mathscr{P}_k$ and where the sum is over all possibilities. The associativity of this product on $T^c(\widehat{\mathscr{P}})$ follows from the associativity property of γ. One way of proving this result without too many tedious computations is to use the notion of brace algebra, see Propositions 13.11.4 and 13.11.5. As a result we get a cofree Hopf algebra. It is an example of a combinatorial Hopf algebra which is cofree and left-sided in the sense of [LR10].

Similarly, starting from a conilpotent ns cooperad one can construct a combinatorial Hopf algebra which is free and left-sided, see [vdLM02].

5.9.17 Nonsymmetric Cooperad and Cutting

It is clear that all the cooperadic definitions and constructions can be performed in the nonsymmetric framework, that is over arity graded spaces instead of $\mathbb{S}$-modules as done in Sect. 5.8.7. Let us just give some details on the free nonsymmetric cooperad over an arity graded space of the form $M = (0, 0, M_2, M_3, \dots)$. As a graded module $\mathscr{T}^c(M)$ is spanned by the planar rooted trees whose vertices are labeled by elements of M. In fact, if the vertex has k inputs (arity k), then its label is in M_k. The decomposition Δ on such a tree t is constructed by "degrafting" as follows. A *cut* of the tree is *admissible* if the grafting of the pieces gives the original tree back. The degrafting $\Delta(t)$ of t is the sum of all the admissible cuttings $(r; s_1, \dots, s_k)$, where r is the piece containing the root, and k is the number of leaves of r. Of course each vertex keeps its labeling.

The explicit formula given in Sect. 5.8.6 is valid only when M is in even degree. Indeed, since the associativity isomorphism for composition is involved (cf. Sect. 5.1.8) signs appear in the formula in the general graded case. For instance if  is in degree 1, then the formula becomes:



More generally, if t and s are elements of $\mathscr{T}^c(M)$, then the coproduct of $t \vee s =$ ( $; t, s)$ is given by

$$\Delta(t \vee s) = \Delta($$  $$; t, s)$$

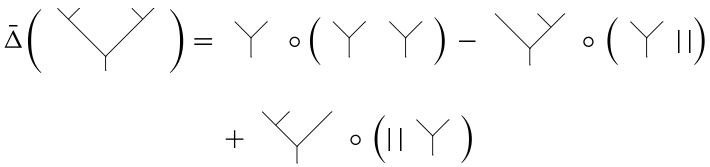
$$= (|; t \vee s) + \Big($$  $$; \big(t^{(1)}; t^{(2)}\big), \big(s^{(1)}; s^{(2)}\big)\Big)$$

$$= (|; t \vee s) + (-1)^{|t^{(2)}|\,|s^{(1)}|}\Big(\Big($$  $$; t^{(1)}, s^{(1)}\Big); t^{(2)}, s^{(2)}\Big)$$

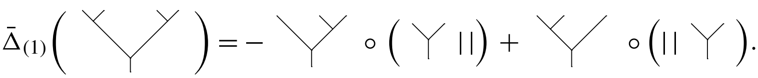
$$= (|; t \vee s) + (-1)^{|t^{(2)}|\,|s^{(1)}|}\big(t^{(1)} \vee s^{(1)}; t^{(2)}, s^{(2)}\big),$$

where $\Delta(t) = (t^{(1)}; t^{(2)})$ and $\Delta(s) = (s^{(1)}; s^{(2)})$. The sign comes from the exchange of $s^{(1)}$ and $t^{(2)}$.

For instance we get



and



5.10 Résumé

We gave several equivalent definitions of an operad, which can be summarized as follows.

Definition 0. Given a *type of algebras* the algebraic operad is given by the functor "free algebra", which is a monad in Vect. If the relations are multilinear, then the endofunctor is a Schur functor.

Definition 1. The *monoidal definition*. An algebraic operad is a monoid $(\mathscr{P}, \gamma, \eta)$ in the monoidal category of $\mathbb{S}$-modules (resp. arity graded spaces). So $\gamma : \mathscr{P} \circ \mathscr{P} \to \mathscr{P}$ is associative and $\eta : \mathrm{I} \to \mathscr{P}$ is its unit. It is called a symmetric operad (resp. nonsymmetric operad).

Definition 2. The *classical definition*. A symmetric operad is a family of $\mathbb{S}_n$-modules $\mathscr{P}(n), n \geq 1$, and linear maps

$$\gamma(i_1, \ldots, i_k) : \mathscr{P}(k) \otimes \mathscr{P}(i_1) \otimes \cdots \otimes \mathscr{P}(i_k) \longrightarrow \mathscr{P}(i_1 + \cdots + i_k)$$

satisfying some axioms expressing equivariance under the action of the symmetric group and associativity of the composition. They ensure that the associated functor $V \mapsto \mathscr{P}(V) := \bigoplus_n \mathscr{P}(n) \otimes_{\mathbb{S}_n} V^{\otimes n}$ is a monoid.

Definition 3. The *partial definition*. A symmetric operad is a family of $\mathbb{S}_n$-modules $\mathscr{P}(n)$, $n \geq 0$, and partial compositions

$$\circ_i : \mathscr{P}(m) \otimes \mathscr{P}(n) \to \mathscr{P}(m-1+n), \text{ for } 1 \leq i \leq m,$$

satisfying equivariance with respect to the symmetric groups and the axioms:

$$\begin{cases} \text{(I)} \quad (\lambda \circ_i \mu) \circ_{i-1+j} \nu = \lambda \circ_i (\mu \circ_j \nu), & \text{for } 1 \leq i \leq l,\ 1 \leq j \leq m, \\ \text{(II)} \quad (\lambda \circ_i \mu) \circ_{k-1+m} \nu = (\lambda \circ_k \nu) \circ_i \mu, & \text{for } 1 \leq i < k \leq l, \end{cases}$$

for any $\lambda \in \mathscr{P}(l)$, $\mu \in \mathscr{P}(m)$, $\nu \in \mathscr{P}(n)$. One assumes the existence of a unit element $\mathrm{id} \in \mathscr{P}(1)$.

Definition 4. The *combinatorial definition*. There exists a monad $\mathbb{T}$ over the category of $\mathbb{S}$-modules made out of rooted trees and substitution, such that a symmetric operad is an algebra (i.e. a representation) over $\mathbb{T}$. Nonsymmetric operads are obtained by replacing trees by planar trees.

Observe that Definitions 2, 3 and 4 can be thought of as various presentations of the monad $\mathbb{T}$. In Definition 2 the generators have two levels, in Definition 3 they involve only two variables, in Definition 4 every element is a generator.

Algebra over an Operad. A $\mathscr{P}$-algebra is a vector space A equipped with a linear map $\gamma_A : \mathscr{P}(A) \to A$ compatible with the operadic structure γ and η. It is equivalent to a morphism of operads

$$\mathscr{P} \to \mathrm{End}_A.$$

In order to get the analogous definitions for nonsymmetric operads, it suffices to replace the $\mathbb{S}$-modules by the arity graded modules (no action of the symmetric group anymore) where degree = arity. In definition 0 the relations should be such

Table 5.2 Various algebras over combinatorial monads

	Category	Product	Unit	Combinatorial objects
monoid	Set	$\times$	$\{*\}$	ladders
algebra	Vect	$\otimes$	$\mathbb{K}$	ladders
operad	$\mathbb{S}$-Mod	$\circ$	I	rooted trees
ns operad	$\mathbb{N}$-Mod	$\circ$	I	planar rooted trees

that the variables stay in the same order in the involved monomials. In Definition 4 the trees are supposed to be planar.

Monoids, unital associative algebras, symmetric operads, nonsymmetric operads, are all monoids in an ad hoc monoidal category. They can also be interpreted as algebras over a combinatorial monad (see Table 5.2).

5.11 Exercises

Exercise 5.11.1 (Identity operad). Let I be the $\mathbb{S}$-module corresponding to the identity functor from Vect to itself. What is $\mathrm{I}(n)$?

Exercise 5.11.2 (On $\mathrm{End}_{\mathbb{K}}$). Show that $\mathrm{End}_{\mathbb{K}} = uCom$ as an operad, and $\mathrm{End}(\mathbb{K}) = uAs$ as a ns operad.

Exercise 5.11.3 (A graded operad). Show that the category of algebras over the operad $\mathrm{End}_{s\mathbb{K}}$ can be described as follows. An $\mathrm{End}_{s\mathbb{K}}$-algebra is a graded vector space A with a bilinear map $A_n \otimes A_m \to A_{n+m+1}$, $x \otimes y \mapsto xy$ for any $n, m \geq 0$, such that

$$xy = -(-1)^{|x||y|}yx, \qquad (xy)z = (-1)^{|x|}x(yz).$$

Exercise 5.11.4 (Shifting degrees). Let M and N be two endofunctors of the category of graded vector spaces related by the formula $M(V) = N(sV)$ for any graded space V. Show that

$$s^{-1}N = (\mathrm{End}_{s^{-1}\mathbb{K}}) \underset{\mathrm{H}}{\otimes} M.$$

Exercise 5.11.5 (From *Ass* to *Com*). Show that the forgetful functor *Com*-alg $\to$ *Ass*-alg induces, on the space of n-ary operations, the augmentation map $\mathbb{K}[\mathbb{S}_n] \to \mathbb{K}, \sigma \mapsto 1$ for $\sigma \in \mathbb{S}_n$.

Exercise 5.11.6 (Explicit free operad). Show that a functor $F : \mathsf{Vect} \to \mathscr{P}$-alg gives a free $\mathscr{P}$-algebra $F(V)$ if and only if there exists $\phi_A : F(A) \to A$ for any $\mathscr{P}$-algebra A, and $\psi_V : V \to F(V)$ for any vector space V, such that ϕ_A is natural in A, ψ_V is natural in V, and both composites

$$F(V) \xrightarrow{F(\psi_V)} F\big(F(V)\big) \xrightarrow{\phi_{F(V)}} F(V) \text{ and } A \xrightarrow{\psi_A} F(A) \xrightarrow{\phi_A} A$$

give the identity.

Exercise 5.11.7 (Free operad). Show directly from the definition of a free operad that $\mathscr{T}M$ can be described in terms of planar binary trees when $M = (0, 0, M_2 \otimes \mathbb{K}[\mathbb{S}_2], 0, \ldots, 0, \ldots)$.

Exercise 5.11.8 (Plethysm). Let E, resp. F, be a representation of $\mathbb{S}_n$, resp. $\mathbb{S}_m$. Let $\widetilde{E}$, resp. $\widetilde{F}$, be the associated Schur functor. Show that $\widetilde{E} \circ \widetilde{F}$ is the Schur functor of a certain representation G of $\mathbb{S}_{m^n}$ and describe it explicitly. This representation is called the *plethysm* of E and F.

Exercise 5.11.9 (*Ass* explicit). Describe explicitly the map

$$\gamma(i_1, \ldots, i_k) : \mathbb{S}_k \times \mathbb{S}_{i_1} \times \cdots \times \mathbb{S}_{i_k} \longrightarrow \mathbb{S}_{i_1+\cdots+i_k}$$

which induces the map

$$\gamma(i_1, \ldots, i_k) : Ass(k) \otimes Ass(i_1) \otimes \cdots \otimes Ass(i_k) \longrightarrow Ass(i_1 + \cdots + i_k)$$

of the operad *Ass*.

Exercise 5.11.10 (Induction). Let E be an $\mathbb{S}_2$-module. Let $\mathbb{S}_2$ act on $E \otimes E$ via its action on the second variable only. Show that, as a vector space, $\mathrm{Ind}_{\mathbb{S}_2}^{\mathbb{S}_3}(E \otimes E) = 3E \otimes E$. Describe explicitly the action of $\mathbb{S}_3$ on $3E \otimes E$.

Exercise 5.11.11 (Arity 3). Describe explicitly the $\mathbb{S}_3$-representation $\mathscr{T}(\mathbb{K}\mu)(3)$ when μ is a binary operation, resp. a symmetric binary operation, resp. an antisymmetric binary operation.

HINT. You should obtain a space of dimension 12, resp. 3, resp. 3. The multiplicities of the isotypic components (trivial, hook, signature) are $(2, 4, 2)$, resp. $(1, 1, 0)$, resp. $(0, 1, 1)$.

Exercise 5.11.12 (Poisson algebra). A Poisson algebra is determined by a commutative product $(x, y) \mapsto x \cdot y$ and a Lie bracket $(x, y) \mapsto [x, y]$ related by the derivation property (Leibniz rule):

$$[x \cdot y, z] = x \cdot [y, z] + [x, z] \cdot y.$$

This gives a presentation of the operad *Pois* of Poisson algebras. Show that there is another presentation involving only one operation xy with no symmetry and only one relation (see Sect. 13.3.3 for the solution).

Exercise 5.11.13 (Invariants). Show that the map

$$V^{\otimes n} \to \left(\mathbb{K}[\mathbb{S}_n] \otimes V^{\otimes n}\right)^{\mathbb{S}_n}, \qquad v_1 \cdots v_n \mapsto \sum_{\sigma \in \mathbb{S}_n} \sigma \otimes (v_1 \cdots v_n)^{\sigma}$$

is an isomorphism. Deduce that the categories ΓAss-alg and Ass-alg are the same.

Exercise 5.11.14 (From operad to cooperad). Let $\mathscr{P}$ be an algebraic operad and let $\mathscr{P}^* := \{\mathscr{P}(n)^*\}_{n\geq 1}$ be the linear dual cooperad. Suppose we are given a linear basis for all the spaces $\mathscr{P}(n)$. The composition is completely determined by the constants $a^{\lambda}_{\mu\nu_1\cdots\nu_k}$ appearing in the formulas

$$\gamma(\mu; \nu_1, \dots, \nu_k) = \sum a^{\lambda}_{\mu\nu_1\cdots\nu_k}\lambda$$

where $\lambda, \mu, \nu_1, \dots, \nu_k$ are basis elements. Show that the decomposition map of the cooperad $\mathscr{P}^*$ is completely determined by the formulas:

$$\Delta(\lambda^*) = \sum a^{\lambda}_{\mu\nu_1\cdots\nu_k}\big(\mu^*; \nu_1^*, \dots, \nu_k^*\big).$$

Exercise 5.11.15 (On $\widetilde{Ass}$ ★). Find a presentation for the $\widetilde{Ass}$-algebras (notation introduced in 5.9.11).

HINT. Use [Pir03].

Exercise 5.11.16 (Möbius basis ★). Let $\{M_\sigma\}_{\sigma\in\mathbb{S}_n}$ be the basis of $\mathbb{K}[\mathbb{S}_n]$ defined as

$$M_\sigma := \sum_{\sigma\leq\tau} \mu(\sigma, \tau)\tau.$$

Here $\leq$ stands for the weak Bruhat order on the symmetric group and $\mu(\sigma, \tau)$ is the Möbius function. Show that for any integer i satisfying $1 \leq i \leq n$ there are uniquely determined permutations $(\sigma, \tau)^i$ and $(\sigma, \tau)_i$ such that

$$M_\sigma \circ_i M_\tau = \sum_{(\sigma,\tau)^i\leq\omega\leq(\sigma,\tau)_i} M_\omega.$$

Cf. [AL07].

Exercise 5.11.17 (Category associated to $uMag$ ★). Let $uMag$ be the set-theoretic ns operad with one binary operation and a unit. Give a presentation of $\mathsf{cat}\,uMag$ analogous to the classical presentation of the simplicial category $\mathbb{K}[\Delta] = \mathsf{cat}\,uAss$ (cf. [Pir02a]) in terms of faces and degeneracies.

HINT. Same but delete the relations $\sigma_j\sigma_i = \sigma_i\sigma_{j+1}, i \leq j$.

Exercise 5.11.18 (Regular $\mathbb{S}$-modules ★). Let M and N be two $\mathbb{S}$-modules such that $M(n) = M_n \otimes \mathbb{K}[\mathbb{S}_n]$ and $N(n) = N_n \otimes \mathbb{K}[\mathbb{S}_n], n \geq 1$. Show that $M \circ N$ is such that $(M \circ N)(n) = (M \circ N)_n \otimes \mathbb{K}[\mathbb{S}_n]$. Compute $(M \circ N)_n$ out of the components M_i and N_j.

Exercise 5.11.19 (Right adjoint of Schur functor ★). Let $F : \mathsf{Vect} \to \mathsf{Vect}$ be an endofunctor of the category of vector spaces. Let

$$RF_n := \mathrm{Hom}_{End}\big(T^{(n)}, F\big)$$

where the endofunctor $T^{(n)} : \mathsf{Vect} \to \mathsf{Vect}$ is given by $T^{(n)}(V) := V^{\otimes n}$. Show that R is right adjoint to the Schur functor $\mathbb{S}\text{-}mod \to \mathrm{End}(\mathsf{Vect})$.

Exercise 5.11.20 (Non-morphism ★). Show that there is a morphism of $\mathbb{S}_n$-modules $F(n) : Com(n) \to Ass(n)$, which identifies the trivial representation to its copy in the regular representation. Show that the resulting morphism of $\mathbb{S}$-modules $F : Com \to Ass$ is not a morphism of operads, i.e. $F(\mu \circ_1 \mu) \neq F(\mu) \circ_1 F(\mu)$.

HINT. It follows from the fact that, in an associative algebra, the symmetrized product $a \cdot b := ab + ba$ is not associative in general.

Exercise 5.11.21 (Composite with *Ass* ★). Let M be an $\mathbb{S}$-module and let *Reg* be the regular $\mathbb{S}$-module, that is $Reg(n) := \mathbb{K}[\mathbb{S}_n]$. This is the $\mathbb{S}$-module underlying the operad *Ass* (and several others). Show that the composite $\mathbb{S}$-module $M \circ Reg$ can be described as follows:

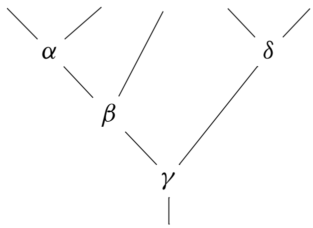
$$(M \circ Reg)(n) = \bigoplus_k M(k) \Bigg(\bigoplus_{i_1 + \cdots + i_k = n} \mathbb{K}[\mathbb{S}_n] \Bigg)$$

where the action of $\mathbb{S}_k$ on the right sum is explicitly given by

$$\sigma \cdot (i_1, \ldots, i_k; \omega) = (i_{\sigma^{-1}(1)}, \ldots, i_{\sigma^{-1}(k)}; \sigma_\omega)$$

for $\sigma \in \mathbb{S}_k$, $\omega \in \mathbb{S}_n$, $i_1 + \cdots + i_k = n$. The permutation $\sigma_\omega \in \mathbb{S}_n$ is the precomposition of ω by the action of σ on the "blocks" of size $i_1, \ldots, i_k$.

Exercise 5.11.22 (Trees and free operad ★). Let α, β, γ, δ be binary operations in M. Show that the element corresponding to the tree



is $(\gamma; (\beta; \alpha, \mathrm{id}), (\delta; \mathrm{id}, \mathrm{id})) \in \mathscr{T}_3 M$.

Exercise 5.11.23 (Free operad and automorphisms of trees ★). Extend the results of Sect. 5.6 (combinatorial definition and free operad) to the case where the $\mathbb{S}$-module M has a nontrivial component $M(0)$.

HINT. In this case, the trees are not reduced and hence they have nontrivial automorphism groups, see Appendix C.4.

Exercise 5.11.24 (Hopf operad ★). Show that any set operad gives rise to a Hopf operad.

Exercise 5.11.25 (Modules over a Hopf operad ★). Show that the tensor product of left modules over a Hopf operad is still a left module.

Exercise 5.11.26 (Explicit enveloping algebra ★). Let $\alpha : \mathscr{P} \to \mathscr{Q}$ be a morphism of operads and let (A, γ_A) be a $\mathscr{P}$-algebra. Suppose that the operad $\mathscr{P}$ comes with a presentation $\mathscr{P} = \mathscr{P}(E, R)$. Show that the relative free $\mathscr{Q}$-algebra $\alpha_!(A) = \mathscr{Q} \circ_{\mathscr{P}} A$ introduced in Sect. 5.2.12 is isomorphic to the quotient of the free $\mathscr{Q}$-algebra over the space A by the relation which identifies the two $\mathscr{P}$-algebra structures. More precisely we have

$$\mathscr{Q}(A)/\big((\alpha(\mu); a_1, \ldots, a_k) - \gamma_A(\mu; a_1, \ldots, a_k); \mu \in E(k), a_1, \ldots, a_k \in A\big),$$

where the right-hand side stands for the ideal generated by the listed elements for any k.

Exercise 5.11.27 (Convolution operad ★). Show that any symmetric, resp. ns, operad $\mathscr{P}$ is isomorphic to the convolution operad $\mathrm{Hom}_{\mathbb{S}}(uAss^c, \mathscr{P})$, resp. $\mathrm{Hom}(uAs^c, \mathscr{P})$.

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