

Contents

| | | |
|----------|--|-----|
| 1 | Introduction | 1 |
| 1.1 | An Overview of Variable Lebesgue Spaces | 2 |
| 1.2 | A Brief History of Variable Lebesgue Spaces | 4 |
| 1.3 | The Organization of this Book | 8 |
| 1.4 | Prerequisites and Notation | 11 |
| 2 | Structure of Variable Lebesgue Spaces | 13 |
| 2.1 | Exponent Functions | 13 |
| 2.2 | The Modular | 17 |
| 2.3 | The Space $L^{p(\cdot)}(\Omega)$ | 18 |
| 2.4 | Hölder's Inequality and the Associate Norm | 26 |
| 2.5 | Embedding Theorems | 35 |
| 2.6 | Convergence in $L^{p(\cdot)}(\Omega)$ | 43 |
| 2.7 | Completeness and Dense Subsets of $L^{p(\cdot)}(\Omega)$ | 54 |
| 2.8 | The Dual Space of a Variable Lebesgue Space | 62 |
| 2.9 | The Lebesgue Differentiation Theorem | 66 |
| 2.10 | Notes and Further Results | 68 |
| 2.10.1 | References | 68 |
| 2.10.2 | Musiélak-Orlicz Spaces and Modular Spaces | 70 |
| 2.10.3 | Banach Function Spaces | 72 |
| 2.10.4 | Alternative Definitions of the Modular | 74 |
| 2.10.5 | Variable Lebesgue Spaces and Orlicz Spaces | 75 |
| 2.10.6 | More on Convergence | 75 |
| 2.10.7 | Variable Sequence Spaces | 77 |
| 3 | The Hardy-Littlewood Maximal Operator | 79 |
| 3.1 | Basic Properties | 79 |
| 3.2 | The Calderón-Zygmund Decomposition | 82 |
| 3.3 | The Maximal Operator on Variable Lebesgue Spaces | 88 |
| 3.4 | The Proof of Theorem 3.16 | 93 |
| 3.5 | Modular Inequalities | 107 |
| 3.6 | Interpolation and Convexity | 113 |

| | | |
|----------|--|------------|
| 3.7 | Notes and Further Remarks | 117 |
| 3.7.1 | References | 117 |
| 3.7.2 | More on Modular Inequalities | 118 |
| 3.7.3 | L log L Inequalities in Variable Lebesgue Spaces | 119 |
| 3.7.4 | The Fractional Maximal Operator | 120 |
| 3.7.5 | Hardy Operators on Variable Lebesgue Spaces | 122 |
| 3.7.6 | Other Maximal Operators | 124 |
| 3.7.7 | Decreasing Rearrangements | 126 |
| 3.7.8 | Real and Complex Interpolation | 127 |
| 4 | Beyond Log-Hölder Continuity | 129 |
| 4.1 | Control at Infinity: The N_∞ Condition | 130 |
| 4.2 | A Useful Tool: Muckenhoupt A_p Weights | 142 |
| 4.3 | Applications of Weights to the Maximal Operator | 152 |
| 4.4 | Local Control: The K_0 Condition | 160 |
| 4.5 | A Necessary and Sufficient Condition | 177 |
| 4.6 | Notes and Further Results | 180 |
| 4.6.1 | References | 180 |
| 4.6.2 | More on the K_0 Condition | 181 |
| 4.6.3 | Discontinuous Exponents | 183 |
| 4.6.4 | Perturbation of Exponents | 184 |
| 4.6.5 | Weighted Variable Lebesgue Spaces | 185 |
| 5 | Extrapolation in the Variable Lebesgue Spaces | 191 |
| 5.1 | Basic Properties of Convolutions | 191 |
| 5.2 | Approximate Identities on Variable Lebesgue Spaces | 197 |
| 5.3 | The Failure of Young's Inequality | 202 |
| 5.4 | Rubio de Francia Extrapolation | 205 |
| 5.5 | Applications of Extrapolation | 213 |
| 5.6 | Notes and Further Results | 227 |
| 5.6.1 | References | 227 |
| 5.6.2 | Pointwise Estimates | 228 |
| 5.6.3 | More on Approximate Identities | 229 |
| 5.6.4 | Applications of Extrapolation | 230 |
| 5.6.5 | Sharp Maximal Operator Estimates | 230 |
| 5.6.6 | Local to Global Estimates | 231 |
| 5.6.7 | The Variable Riesz Potential | 233 |
| 5.6.8 | Vector-Valued Maximal Operators | 234 |
| 5.6.9 | Two Classical PDEs | 235 |
| 5.6.10 | The Fourier Transform | 236 |
| 6 | Basic Properties of Variable Sobolev Spaces | 239 |
| 6.1 | The Space $W^{k,p(\cdot)}(\Omega)$ | 239 |
| 6.2 | Density of Smooth Functions | 243 |
| 6.3 | The Poincaré Inequalities | 249 |
| 6.4 | Sobolev Embedding Theorems | 252 |

- 6.5 Notes and Further Results 260
 - 6.5.1 References 260
 - 6.5.2 An Alternative Definition of the Norm 262
 - 6.5.3 Boundary Regularity 262
 - 6.5.4 Extension Theorems 263
 - 6.5.5 More on the Density of Smooth Functions 264
 - 6.5.6 More on the Poincaré Inequalities 265
 - 6.5.7 More on the Sobolev Embedding Theorem 266
 - 6.5.8 Compact Embeddings 268
 - 6.5.9 Mean Continuity 269
 - 6.5.10 Gagliardo-Nirenberg Inequalities 270

- A Appendix: Open Problems 271**

- Bibliography 279**

- Symbol Index 295**

- Author Index 301**

- Subject Index 305**



<http://www.springer.com/978-3-0348-0547-6>

Variable Lebesgue Spaces
Foundations and Harmonic Analysis
Cruz-Uribe, D.; Fiorenza, A.
2013, IX, 312 p., Hardcover
ISBN: 978-3-0348-0547-6
A product of Birkhäuser Basel