Chapter 2
Structure of Variable Lebesgue Spaces

In this chapter we give a precise definition of the variable Lebesgue spaces and establish their structural properties as Banach function spaces. Throughout this chapter we will generally assume that Ω is a Lebesgue measurable subset of $\mathbb{R}^n$ with positive measure. Occasionally we will have to assume more, but we make it explicit if we do.

2.1 Exponent Functions

We begin with a fundamental definition.

Definition 2.1. Given a set Ω, let $\mathcal{P}(\Omega)$ be the set of all Lebesgue measurable functions $p(\cdot) : \Omega \to [1,\infty]$. The elements of $\mathcal{P}(\Omega)$ are called exponent functions or simply exponents. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(\cdot)$.

Some examples of exponent functions on $\Omega = \mathbb{R}$ include $p(x) = p$ for some constant p, $1 \leq p \leq \infty$, or $p(x) = 2 + \sin(x)$. Exponent functions can be unbounded: for instance, if $\Omega = (1,\infty)$, let $p(x) = x$, and if $\Omega = (0,1)$, let $p(x) = 1/x$. We will consider these last two frequently, as they will provide good examples of the differences between bounded and unbounded exponent functions.

We define some notation to describe the range of exponent functions. Given $p(\cdot) \in \mathcal{P}(\Omega)$ and a set $E \subset \Omega$, let

$$p_-(E) = \operatorname*{ess\,inf}_{x \in E} p(x), \qquad p_+(E) = \operatorname*{ess\,sup}_{x \in E} p(x).$$

If the domain is clear we will simply write $p_- = p_-(\Omega)$, $p_+ = p_+(\Omega)$. As is the case for the classical Lebesgue spaces, we will encounter different behavior depending on whether $p(x) = 1$, $1 < p(x) < \infty$, or $p(x) = \infty$. Therefore, we define three canonical subsets of Ω:

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$$\Omega_{\infty}^{p(\cdot)} = \{x \in \Omega : p(x) = \infty\},$$

$$\Omega_{1}^{p(\cdot)} = \{x \in \Omega : p(x) = 1\},$$

$$\Omega_{*}^{p(\cdot)} = \{x \in \Omega : 1 < p(x) < \infty\}.$$

Again, for simplicity we will omit the superscript $p(\cdot)$ if there is no possibility of confusion. Since $p(\cdot)$ is a measurable function, these sets are only defined up to sets of measure zero; however, in practice this will have no effect. Below, the value of certain constants will depend on whether these sets have positive measure; if they do we will use the fact that, for instance, $\|\chi_{\Omega_1^{p(\cdot)}}\|_\infty = 1$.

Given $p(\cdot)$, we define the conjugate exponent function $p'(\cdot)$ by the formula

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \qquad x \in \Omega,$$

with the convention that $1/\infty = 0$. Since $p(\cdot)$ is a function, the notation $p'(\cdot)$ can be mistaken for the derivative of $p(\cdot)$, but we will never use the symbol "$'$" in this sense.

The notation p' will also be used to denote the conjugate of a constant exponent. The operation of taking the supremum/infimum of an exponent does not commute with forming the conjugate exponent. In fact, a straightforward computation shows that

$$\big(p'(\cdot)\big)_+ = \big(p_-\big)', \quad \big(p'(\cdot)\big)_- = \big(p_+\big)'.$$

For simplicity we will omit one set of parentheses and write the left-hand side of each equality as $p'(\cdot)_+$ and $p'(\cdot)_-$. We will always avoid ambiguous expressions such as p'_+.

Though the basic theory of variable Lebesgue spaces only requires that $p(\cdot)$ be a measurable function, in many applications in subsequent chapters we will often assume that $p(\cdot)$ has some additional regularity. In particular, there are two continuity conditions that are of such importance that we want to establish notation for them.

Definition 2.2. Given Ω and a function $r(\cdot) : \Omega \to \mathbb{R}$, we say that $r(\cdot)$ is locally log-Hölder continuous, and denote this by $r(\cdot) \in LH_0(\Omega)$, if there exists a constant C_0 such that for all $x, y \in \Omega$, $|x - y| < 1/2$,

$$|r(x) - r(y)| \leq \frac{C_0}{-\log(|x - y|)}.$$

We say that $r(\cdot)$ is log-Hölder continuous at infinity, and denote this by $r(\cdot) \in LH_\infty(\Omega)$, if there exist constants C_∞ and r_∞ such that for all $x \in \Omega$,

$$|r(x) - r_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.$$

If $r(\cdot)$ is log-Hölder continuous locally and at infinity, we will denote this by writing $r(\cdot) \in LH(\Omega)$. If there is no confusion about the domain we will sometimes write LH_0, LH_∞ or LH.

In practice we will often assume that $p(\cdot)$ or $1/p(\cdot)$ is contained in one of the log-Hölder continuity classes. In the latter case, if $p(\cdot)$ is unbounded at infinity we let $p_\infty = \infty$ and use the convention $1/p_\infty = 0$.

The next result is an immediate consequence of Definition 2.2.

Proposition 2.3. *Given a domain* Ω*:*

1. *If* $r(\cdot) \in LH_0(\Omega)$*, then* $r(\cdot)$ *is uniformly continuous and* $r(\cdot) \in L^\infty(E)$ *for every bounded subset* $E \subset \Omega$.
2. *If* $r(\cdot) \in LH_\infty(\Omega)$*, then* $r(\cdot) \in L^\infty(\Omega)$.
3. *If* Ω *is bounded and* $r(\cdot) \in L^\infty(\Omega)$*, then* $r(\cdot) \in LH_\infty(\Omega)$*, with a constant* C_∞ *depending on* $\|r(\cdot)\|_\infty$*, the diameter of* Ω*, and its distance from the origin.*
4. *The inclusion* $r(\cdot) \in LH_\infty(\Omega)$ *is equivalent to the existence of a constant* C *such that for all* $x,\ y \in \Omega$, $|y| \geq |x|$,
$$|r(x) - r(y)| \leq \frac{C}{\log(e + |x|)}.$$
5. *If* $p_+ < \infty$*, then* $p(\cdot) \in LH_0(\Omega)$ *is equivalent to assuming* $r(\cdot) = 1/p(\cdot) \in LH_0(\Omega)$*: in fact, given* $x,\ y \in \Omega$,
$$\left|\frac{p(x) - p(y)}{(p_+)^2}\right| \leq \left|\frac{1}{p(x)} - \frac{1}{p(y)}\right| \leq \left|\frac{p(x) - p(y)}{(p_-)^2}\right|.$$
Similarly, $p(\cdot) \in LH_\infty(\Omega)$ *if and only if* $r(\cdot) = 1/p(\cdot) \in LH_\infty(\Omega)$.

Given two domains $\widetilde{\Omega} \subset \Omega$, we clearly have that if $p(\cdot) \in LH_0(\Omega)$, then $\tilde{p}(\cdot) = p(\cdot)\big|_{\tilde{\Omega}} \in LH_0(\widetilde{\Omega})$, and similarly for the class LH_∞. In applications, we will be concerned with the converse: given an exponent function in $LH(\widetilde{\Omega})$, can it be extended to a function in $LH(\Omega)$? The answer is yes as the next result shows.

Lemma 2.4. *Given a set* $\Omega \subset \mathbb{R}^n$ *and* $p(\cdot) \in \mathcal{P}(\Omega)$ *such that* $p(\cdot) \in LH(\Omega)$*, there exists a function* $\tilde{p}(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ *such that:*

1. $\tilde{p} \in LH$;
2. $\tilde{p}(x) = p(x)$, $x \in \Omega$;
3. $\tilde{p}_- = p_-$ *and* $\tilde{p}_+ = p_+$.

Remark 2.5. It follows from the proof below that if we only have that $p(\cdot) \in LH_0(\Omega)$ or $LH_\infty(\Omega)$ we can extend it to a function in the same class on $\mathbb{R}^n$.

Proof. Since $p(\cdot)$ is bounded and uniformly continuous, by a well-known result it extends to a continuous function on $\overline{\Omega}$; denote this extension by $p(\cdot)$ as well. Then it is immediate that $p(\cdot) \in LH(\overline{\Omega})$, $p_-(\Omega) = p_-(\overline{\Omega})$, and $p_-(\Omega) = p_-(\overline{\Omega})$.

To extend $p(\cdot)$ from $\overline{\Omega}$ to all of $\mathbb{R}^n$ we first consider the case when $\overline{\Omega}$ is unbounded; the case when $\overline{\Omega}$ is bounded is simpler and will be sketched below. Define a new function $r(\cdot)$ by $r(x) = p(x) - p_\infty$. Then $r(\cdot)$ is still bounded (though no longer necessarily positive) and $r(\cdot) \in LH(\overline{\Omega})$.

We will extend $r(\cdot)$ to all of $\mathbb{R}^n$. If we define $\omega(t) = 1/\log(e/2t)$, $0 < t \leq 1/2$, and $\omega(t) = 1$ for $t \geq 1/2$, then a straightforward calculation shows that $\omega(t)/t$ is a decreasing function and $\omega(2t) \leq C\,\omega(t)$. Further, since $\log(e/2t) \approx \log(1/t)$, $0 < t < 1/2$, and since $r(\cdot)$ is bounded, $|r(x) - r(y)| \leq C\omega(|x - y|)$ for all $x, y \in \overline{\Omega}$. Therefore, there exists a function $\tilde{r}(\cdot)$ on $\mathbb{R}^n$ such that $\tilde{r}(x) = r(x)$, $x \in \overline{\Omega}$, and such that $\tilde{r}(\cdot) \in LH_0(\mathbb{R}^n)$, with a constant that depends only on $p(\cdot)$ and the LH_0 constant, and not on Ω. For a proof, see Stein [339, Corollary 2.2.3, p. 175]. Briefly, and using the terminology of this reference, the function $\tilde{r}(\cdot)$ is defined as follows. Form the Whitney decomposition $\{Q_k\}$ of $\mathbb{R}^n \setminus \overline{\Omega}$ and let $\{\phi_k^*\}$ be a partition of unity subordinate to this decomposition. In each cube Q_k, fix a point $p_k \in \overline{\Omega}$ such that $\operatorname{dist}(p_k, Q_k) = \operatorname{dist}(\overline{\Omega}, Q_k)$. Then for $x \in \mathbb{R}^n \setminus \overline{\Omega}$,

$$\tilde{r}(x) = \sum_k r(p_k)\phi_k^*(x).$$

It follows immediately from this definition that for all $x \in \mathbb{R}^n$, $r_- \leq \tilde{r}(x) \leq r_+$. However, $\tilde{r}(\cdot)$ need not be in LH_∞, so we must modify it slightly. To do so we need the following observation: if f_1, f_2 are functions such that $|f_i(x) - f_i(y)| \leq C\omega(|x - y|)$, $x, , y \in \mathbb{R}^n$, $i = 1, 2$, then $\min(f_1, f_2)$ and $\max(f_1, f_2)$ satisfy the same inequality. The proof of this observation consists of a number of very similar cases. For instance, suppose $\min(f_1(x), f_2(x)) = f_1(x)$ and $\min(f_1(y), f_2(y)) = f_2(y)$. Then

$$f_1(x) - f_2(y) \leq f_2(x) - f_2(y) \leq C\omega(|x - y|),$$
$$f_2(y) - f_1(x) \leq f_1(y) - f_1(x) \leq C\omega(|x - y|).$$

Hence,

$$|\min(f_1(x), f_2(x)) - \min(f_1(y), f_2(y))| = |f_1(x) - f_2(y)| \leq C\omega(|x - y|).$$

It follows immediately from this observation that

$$s(x) = \max(\min(\tilde{r}(x), C_\infty/\log(e + |x|)), -C_\infty/\log(e + |x|))$$

is in $LH(\mathbb{R}^n)$. Therefore, if we define

$$\tilde{p}(x) = s(x) + p_\infty,$$

then (1)–(3) hold.

Finally, if Ω is bounded, we define $r(x) = p(x) - p_+$ and repeat the above argument essentially without change. □

2.2 The Modular

Intuitively, given an exponent function $p(\cdot) \in \mathcal{P}(\Omega)$, we want to define the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions f such that

$$\int_{\Omega} |f(x)|^{p(x)}\, dx < \infty.$$

There are problems with this approach, the most obvious being that it does not work when Ω_∞ has positive measure. To remedy them, we begin with the following definition.

Definition 2.6. Given Ω, $p(\cdot) \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function f, define the modular functional (or simply the modular) associated with $p(\cdot)$ by

$$\rho_{p(\cdot),\Omega}(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

If f is unbounded on Ω_∞ or if $f(\cdot)^{p(\cdot)} \not\in L^1(\Omega\setminus\Omega_\infty)$, we define $\rho_{p(\cdot),\Omega}(f) = +\infty$. When $|\Omega_\infty| = 0$, in particular when $p_+ < \infty$, we let $\|f\|_{L^\infty(\Omega_\infty)} = 0$; when $|\Omega \setminus \Omega_\infty| = 0$, then $\rho_{p(\cdot),\Omega}(f) = \|f\|_{L^\infty(\Omega_\infty)}$. In situations where there is no ambiguity we will simply write $\rho_{p(\cdot)}(f)$ or $\rho(f)$.

We will use the modular to define the space $L^{p(\cdot)}(\Omega)$ in the next section. In preparation, we give here its fundamental properties.

Proposition 2.7. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$:*

1. *For all f, $\rho(f) \geq 0$ and $\rho(|f|) = \rho(f)$.*
2. *$\rho(f) = 0$ if and only if $f(x) = 0$ for almost every $x \in \Omega$.*
3. *If $\rho(f) < \infty$, then $f(x) < \infty$ for almost every $x \in \Omega$.*
4. *ρ is convex: given α, $\beta \geq 0$, $\alpha + \beta = 1$,*

$$\rho(\alpha f + \beta g) \leq \alpha\rho(f) + \beta\rho(g).$$

5. *ρ is order preserving: if $|f(x)| \geq |g(x)|$ a.e., then $\rho(f) \geq \rho(g)$.*
6. *ρ has the continuity property: if for some $\Lambda > 0$, $\rho(f/\Lambda) < \infty$, then the function $\lambda \mapsto \rho(f/\lambda)$ is continuous and decreasing on $[\Lambda, \infty)$. Further, $\rho(f/\lambda) \to 0$ as $\lambda \to \infty$.*

An immediate consequence of the convexity of ρ is that if $\alpha > 1$, then $\alpha\rho(f) \leq \rho(\alpha f)$, and if $0 < \alpha < 1$, then $\rho(\alpha f) \leq \alpha\rho(f)$. We will often invoke this property by referring to the convexity of the modular.

Proof. Property (1) is immediate from the definition of the modular, and Properties (2), (3) and (5) follow from the properties of the L^1 and L^∞ norms.

Property (4) follows since the L^∞ norm is convex and since for almost every $x \in \Omega \setminus \Omega_\infty$, the function $t \mapsto t^{p(x)}$ is convex.

To prove (6), note that by Property (5), if $\lambda \geq \Lambda$, then $\rho(f/\lambda)$ is a decreasing function, and by the dominated convergence theorem (applied to the integral) it is continuous and tends to 0 as $\lambda \to \infty$. □

Remark 2.8. The modular does not satisfy the triangle inequality, i.e., $\rho(f+g) \leq \rho(f)+\rho(g)$. However, there is a substitute that is sometimes useful. For $1 \leq p < \infty$ and $a, b \geq 0$, $(a+b)^p \leq 2^{p-1}(a^p+b^p)$. Therefore, for almost every $x \in \Omega \setminus \Omega_\infty$,

$$|f(x)+g(x)|^{p(x)} \leq 2^{p(x)-1}(|f(x)|^{p(x)}+|g(x)|^{p(x)});$$

in particular, if $p_+ < \infty$,

$$\rho(f+g) \leq 2^{p_+-1}\big(\rho(f)+\rho(g)\big).$$

We will refer to this as the modular triangle inequality.

2.3 The Space $L^{p(\cdot)}(\Omega)$

The most basic property of the classical Lebesgue space L^p is that it is a Banach space: a normed vector space that is complete with respect to the norm. Here we define $L^{p(\cdot)}(\Omega)$ and use the properties of the modular to show that it is a normed vector space; we defer the proof that it is complete until Sect. 2.7, after we establish the requisite convergence properties of the norm.

Definition 2.9. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, define $L^{p(\cdot)}(\Omega)$ to be the set of Lebesgue measurable functions f such that $\rho(f/\lambda) < \infty$ for some $\lambda > 0$. Define $L^{p(\cdot)}_{\mathrm{loc}}(\Omega)$ to be the set of measurable functions f such that $f \in L^{p(\cdot)}(K)$ for every compact set $K \subset \Omega$.

Remark 2.10. By Proposition 2.7, Property (3), if $f \in L^{p(\cdot)}(\Omega)$, then f is finite almost everywhere.

Since we are dealing with measurable functions, we will adopt the usual convention that two functions are the same if they are equal almost everywhere; in particular, we will say $f \equiv 0$ if $f(x) = 0$ except on a set of measure 0.

In defining $L^{p(\cdot)}(\Omega)$ we do not restrict ourselves to a single value of λ: for instance, we do not take $L^{p(\cdot)}(\Omega)$ to be the set of all f such that $\rho(f) < \infty$. We do so in order to make the space homogeneous when $p_+(\Omega \setminus \Omega_\infty) = \infty$.

Example 2.11. Let $\Omega = (1,\infty)$, $p(x) = x$, and $f(x) = 1$. Then $\rho(f) = \infty$, but for all $\lambda > 1$,

$$\rho(f/\lambda) = \int_1^\infty \lambda^{-x}\,dx = \frac{1}{\lambda\log(\lambda)} < \infty.$$

Similarly, if we let $\Omega = (0,1)$ and $p(x) = 1/x$, and again let $f(x) = 1$, then $\rho(f) < \infty$, but $\rho(f/\lambda) = \infty$ for all $\lambda < 1$.

However, this technicality is only necessary if $p(\cdot)$ is unbounded: more precisely, if $p_+(\Omega \setminus \Omega_\infty) < \infty$, then $L^{p(\cdot)}(\Omega)$ coincides with the set of functions such that $\rho(f)$ is finite.

Proposition 2.12. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, then the property that $f \in L^{p(\cdot)}(\Omega)$ if and only if*

$$\rho(f) = \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega)} < \infty$$

is equivalent to assuming that $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$.

Proof. We first assume that $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$. Clearly, if $\rho(f) < \infty$, then $f \in L^{p(\cdot)}(\Omega)$. Conversely, if $f \in L^{p(\cdot)}(\Omega)$, then by Property (5) in Proposition 2.7 we have that $\rho(f/\lambda) < \infty$ for some $\lambda > 1$. But then

$$\rho(f) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|\lambda}{\lambda}\right)^{p(x)} dx + \lambda\|f/\lambda\|_{L^\infty(\Omega_\infty)} \le \lambda^{p_+(\Omega\setminus\Omega_\infty)}\rho(f/\lambda) < \infty.$$

Now suppose that $p_- < \infty$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$. We will construct a function f such that $\rho(f) = \infty$ but $f \in L^{p(\cdot)}(\Omega)$. By the definition of the essential supremum, there exists a sequence of sets $\{E_k\}$ with finite measure such that:

1. $E_k \subset \Omega \setminus \Omega_\infty$,
2. $E_{k+1} \subset E_k$ and $|E_k \setminus E_{k+1}| > 0$,
3. $|E_k| \to 0$,
4. If $x \in E_k$, $p(x) \ge p_k > k$.

Define the function f by

$$f(x) = \left(\sum_{k=1}^{\infty} \frac{1}{|E_k \setminus E_{k+1}|}\chi_{E_k\setminus E_{k+1}}(x)\right)^{1/p(x)}.$$

Then for any $\lambda > 1$,

$$\rho(f/\lambda) = \sum_{k=1}^{\infty} \fint_{E_k\setminus E_{k+1}} \lambda^{-p(x)}\,dx \le \sum_{k=1}^{\infty} \lambda^{-k} < \infty,$$

and the same computation shows that $\rho(f) = \infty$. □

Remark 2.13. The construction in the second half of the proof of Proposition 2.12 will be used frequently to prove that there are essential differences among the variable Lebesgue spaces that depend on whether $p_+(\Omega \setminus \Omega_\infty)$ is finite or infinite.

This ability to "pull" a constant out of the modular when $p_+ < \infty$ is very useful, and makes the study of variable Lebesgue spaces in this case much simpler. The proof of Proposition 2.12 is easily modified to prove the following inequalities.

Proposition 2.14. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+(\Omega \setminus \Omega_\infty) < \infty$, then for all $\lambda \geq 1$,*

$$\rho(\lambda f) \leq \lambda^{p_+(\Omega\setminus\Omega_\infty)}\rho(f).$$

Moreover, if $p_+ < \infty$ and $\lambda \geq 1$, then

$$\lambda^{p_-}\rho(f) \leq \rho(\lambda f) \leq \lambda^{p_+}\rho(f),$$

and if $0 < \lambda < 1$, the reverse inequalities are true.

Theorem 2.15. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is a vector space.*

Proof. Since the set of all Lebesgue measurable functions is itself a vector space, and since $0 \in L^{p(\cdot)}(\Omega)$, it will suffice to show that for all $\alpha, \beta \in \mathbb{R}$, not both 0, if $f, g \in L^{p(\cdot)}(\Omega)$, then $\alpha f + \beta g \in L^{p(\cdot)}(\Omega)$. By Property (5) in Proposition 2.7, there exists $\lambda > 0$ such that $\rho(f/\lambda), \rho(g/\lambda) < \infty$. Therefore, by Properties (1), (3) and (4) of the same proposition, if we let $\mu = \big(|\alpha| + |\beta|\big)\lambda$, then

$$\begin{aligned}\rho\left(\frac{\alpha f + \beta g}{\mu}\right) = \rho\left(\frac{|\alpha f + \beta g|}{\mu}\right) &\leq \rho\left(\frac{|\alpha|}{|\alpha|+|\beta|}\frac{|f|}{\lambda} + \frac{|\beta|}{|\alpha|+|\beta|}\frac{|g|}{\lambda}\right) \\ &\leq \frac{|\alpha|}{|\alpha|+|\beta|}\rho(f/\lambda) + \frac{|\beta|}{|\alpha|+|\beta|}\rho(g/\lambda) < \infty.\end{aligned}$$

□

On the classical Lebesgue spaces, if $1 \leq p < \infty$, then the norm is gotten directly from the modular:

$$\|f\|_{L^p(\Omega)} = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}.$$

Such a definition obviously fails since we cannot replace the constant exponent $1/p$ outside the integral with the exponent function $1/p(\cdot)$. The solution is a more subtle approach which is similar to that used to define the Luxemburg norm on Orlicz spaces.

Definition 2.16. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if f is a measurable function, define

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot),\Omega}(f/\lambda) \leq 1\right\}.$$

If the set on the right-hand side is empty we define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. If there is no ambiguity over the domain Ω, we will often write $\|f\|_{p(\cdot)}$ instead of $\|f\|_{L^{p(\cdot)}(\Omega)}$.

By Property (6) of Proposition 2.7, $\|f\|_{L^{p(\cdot)}(\Omega)} < \infty$ for all $f \in L^{p(\cdot)}(\Omega)$; equivalently, $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$ when $f \notin L^{p(\cdot)}(\Omega)$. When $p(\cdot) = p$, $1 \leq p \leq \infty$, Definition 2.16 is equivalent to the classical norm on $L^p(\Omega)$: if $p < \infty$ and

$$\int_\Omega \left(\frac{|f(x)|}{\lambda}\right)^p dx = 1,$$

then $\lambda = \|f\|_{L^p(\Omega)}$; the same is true if $p = \infty$.

Given two domains Ω and $\widetilde{\Omega}$, if $\widetilde{\Omega} \subset \Omega$ and $p(\cdot) \in \mathcal{P}(\Omega)$, then $\tilde{p}(\cdot) = p(\cdot)\big|_{\tilde{\Omega}} \in \mathcal{P}(\widetilde{\Omega})$ and it is immediate from the definition of the norm that for $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{L^{\tilde{p}(\cdot)}(\tilde{\Omega})} = \|f\chi_{\tilde{\Omega}}\|_{L^{p(\cdot)}(\Omega)}.$$

Hereafter we will implicitly make these restrictions without comment and simply write $\|f\|_{L^{p(\cdot)}(\tilde{\Omega})}$, etc. Conversely, given $p(\cdot) \in \mathcal{P}(\widetilde{\Omega})$ and $f \in L^{p(\cdot)}(\widetilde{\Omega})$, we can extend both to Ω by defining $f(x) = 0$ for $x \in \Omega \setminus \widetilde{\Omega}$ and defining $p(\cdot)$ arbitrarily on $\Omega \setminus \widetilde{\Omega}$. If we do so, then $\|f\|_{L^{p(\cdot)}(\tilde{\Omega})} = \|f\|_{L^{p(\cdot)}(\Omega)}$. Moreover, if $p(\cdot) \in LH(\widetilde{\Omega})$, by Lemma 2.4 we may assume that $p(\cdot) \in LH(\Omega)$ as well.

Theorem 2.17. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, the function $\|\cdot\|_{L^{p(\cdot)}(\Omega)}$ defines a norm on $L^{p(\cdot)}(\Omega)$.*

Proof. We will prove that $\|\cdot\|_{p(\cdot)}$ has the following properties:

1. $\|f\|_{p(\cdot)} = 0$ if and only if $f \equiv 0$;
2. (Homogeneity) for all $\alpha \in \mathbb{R}$, $\|\alpha f\|_{p(\cdot)} = |\alpha|\|f\|_{p(\cdot)}$;
3. (Triangle inequality) $\|f + g\|_{p(\cdot)} \leq \|f\|_{p(\cdot)} + \|g\|_{p(\cdot)}$.

If $f \equiv 0$, then $\rho(f/\lambda) = 0 \leq 1$ for all $\lambda > 0$, and so $\|f\|_{p(\cdot)} = 0$. Conversely, if $\|f\|_{p(\cdot)} = 0$, then for all $\lambda > 0$,

$$1 \geq \rho(f/\lambda) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \|f/\lambda\|_{L^\infty(\Omega_\infty)}.$$

We consider each term of the modular separately. It is immediate that we have $\|f\|_{L^\infty(\Omega_\infty)} \leq \lambda$; hence, $f(x) = 0$ for almost every $x \in \Omega_\infty$. Similarly, if $\lambda < 1$, by Proposition 2.14 we have

$$1 \geq \lambda^{-p_-} \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\, dx.$$

Therefore, $\|f(\cdot)^{p(\cdot)}\|_{L^1(\Omega\setminus\Omega_\infty)} = 0$, and so $f(x) = |f(x)|^{p(x)} = 0$ for almost every $x \in \Omega \setminus \Omega_\infty$. Thus $f \equiv 0$ and we have proved (1).

To prove (2), note that if $\alpha = 0$, this follows from (1). Fix $\alpha \neq 0$; then by a change of variables,

$$\begin{aligned}\|\alpha f\|_{p(\cdot)} &= \inf\{\lambda > 0 : \rho(|\alpha| f/\lambda) \le 1\} \\ &= |\alpha| \inf\{\lambda/|\alpha| > 0 : \rho(f/(\lambda/|\alpha|)) \le 1\} \\ &= |\alpha| \inf\{\mu > 0 : \rho(f/\mu)) \le 1\} = |\alpha| \|f\|_{p(\cdot)}.\end{aligned}$$

Finally, to prove (3), fix $\lambda_f > \|f\|_{p(\cdot)}$ and $\lambda_g > \|g\|_{p(\cdot)}$; then $\rho(f/\lambda_f) \le 1$ and $\rho(g/\lambda_g) \le 1$. Now let $\lambda = \lambda_f + \lambda_g$. Then by Property (3) of Proposition 2.7,

$$\rho\left(\frac{f+g}{\lambda}\right) = \rho\left(\frac{\lambda_f}{\lambda}\frac{f}{\lambda_f} + \frac{\lambda_g}{\lambda}\frac{g}{\lambda_g}\right) \le \frac{\lambda_f}{\lambda}\rho(f/\lambda_f) + \frac{\lambda_g}{\lambda}\rho(g/\lambda_g) \le 1.$$

Hence, $\|f + g\|_{p(\cdot)} \le \lambda_f + \lambda_g$. If we now take the infimum over all such λ_f and λ_g, we get the desired inequality. □

An immediate consequence of the order preserving property of the modular (Property (6) of Proposition 2.7) is that the norm itself is order preserving: if $|f(x)| \ge |g(x)|$ almost everywhere, then $\|f\|_{p(\cdot)} \ge \|g\|_{p(\cdot)}$.

Another elementary but useful property of the classical Lebesgue norm is that it is homogeneous in the exponent: more precisely, for $1 < s < \infty$, $\|f\|_{sp}^s = \||f|^s\|_p$. This property extends to variable Lebesgue spaces.

Proposition 2.18. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $|\Omega_\infty| = 0$, then for all s, $1/p_- \le s < \infty$,*

$$\||f|^s\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s.$$

Proof. This follows at once from the definition of the norm: since $|\Omega_\infty| = 0$, if we let $\mu = \lambda^{1/s}$,

$$\begin{aligned}\||f|^s\|_{p(\cdot)} &= \inf\left\{\lambda > 0 : \int_\Omega \left(\frac{|f(x)|^s}{\lambda}\right)^{p(x)} dx \le 1\right\} \\ &= \inf\left\{\mu^s > 0 : \int_\Omega \left(\frac{|f(x)|}{\mu}\right)^{sp(x)} dx \le 1\right\} = \|f\|_{sp(\cdot)}^s.\end{aligned}$$

□

Example 2.19. If $|\Omega \setminus \Omega_\infty| = 0$, then $\|f\|_{p(\cdot)} = \|f\|_\infty$ and Proposition 2.18 is still true. However, if $|\Omega_\infty| > 0$ but $p(\cdot)$ is not identically infinite, then it need not hold. To see this, let $\Omega = [-1, 1]$, and define

$$p(x) = \begin{cases} 1 & -1 \le x \le 0 \\ \infty & 0 < x \le 1, \end{cases}$$

and

$$f(x) = \begin{cases} 1 & -1 \le x \le 0 \\ 2 & 0 < x \le 1. \end{cases}$$

Then

$$\rho_{p(\cdot)}(f^2/\lambda) = \int_{-1}^{0} \lambda^{-1}\,dx + 2^2\lambda^{-1} = 5\lambda^{-1},$$

and so $\|f^2\|_{p(\cdot)} = 5$. On the other hand, a similar computation shows that $\rho_{2p(\cdot)}(f/\lambda) = \lambda^{-2}+2\lambda^{-1}$; thus, if we solve the quadratic equation $\lambda^{-2}+2\lambda^{-1}-1 = 0$, we get that $\|f\|^2_{2p(\cdot)} = (\sqrt{2}-1)^{-2} \neq 5$.

We conclude this section by considering more closely the relationship between the norm and the modular. Though the norm is defined as the infimum of the set $\{\lambda : \rho(f/\lambda) \le 1\}$, there may be an explicit value λ for which the infimum is attained. For instance, in Example 2.11 we see that if $\Omega = (1,\infty)$, $p(x) = x$ and $f \equiv 1$, then the infimum of $\rho(f/\lambda)$ is attained when λ is such that $\lambda \log(\lambda) = 1$. In fact, if f is non-trivial, then the infimum is always attained. (If $f \equiv 0$, then clearly the infimum is zero and is not attained.) In Proposition 2.21 below we will prove that $\rho(f/\|f\|_{p(\cdot)}) \le 1$, so $\lambda = \|f\|_{p(\cdot)}$ is always an element of the set $\{\lambda : \rho(f/\lambda) \le 1\}$. However, even though the infimum is attained it is possible that $\rho(f/\|f\|_{p(\cdot)}) < 1$.

Example 2.20. Let $\Omega = (1,\infty)$ and $p(x) = x$. Then there exists a function $f \in L^{p(\cdot)}(\Omega)$ such that $\rho(f/\|f\|_{p(\cdot)}) < 1$.

Proof. We will construct a function f such that $\rho(f) < 1$ but for any $\lambda < 1$, $\rho(f/\lambda) = \infty$. Then $\|f\|_{p(\cdot)} = 1$ and $\rho(f/\|f\|_{p(\cdot)}) = \rho(f) < 1$.

For $k \ge 2$ let $I_k = [k, k+k^{-2}]$ and define the function f by

$$f(x) = \sum_{k=2}^{\infty} \chi_{I_k}(x).$$

Then

$$\rho(f) = \sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1 < 1.$$

On the other hand, for any $\lambda < 1$,

$$\rho(f/\lambda) = \sum_{k=2}^{\infty} \int_{k}^{k+k^{-2}} \lambda^{-x}\,dx \ge \sum_{k=2}^{\infty} \frac{1}{\lambda^k k^2} = \infty.$$

□

This example can be adapted to any space such that $p_+(\Omega \setminus \Omega_\infty) = \infty$; otherwise, equality must hold.

Proposition 2.21. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{p(\cdot)} > 0$, then $\rho(f/\|f\|_{p(\cdot)}) \le 1$. Further, $\rho(f/\|f\|_{p(\cdot)}) = 1$ for all non-trivial $f \in L^{p(\cdot)}(\Omega)$ if and only if $p_+(\Omega \setminus \Omega_\infty) < \infty$.*

Proof. Fix a decreasing sequence $\{\lambda_k\}$ such that $\lambda_k \to \|f\|_{p(\cdot)}$. Then by Fatou's lemma and the definition of the modular,

$$\rho(f/\|f\|_{p(\cdot)}) \le \liminf_{k\to\infty} \rho(f/\lambda_k) \le 1.$$

Now suppose that $p_+(\Omega \setminus \Omega_\infty) < \infty$ but assume to the contrary that $\rho(f/\|f\|_{p(\cdot)}) < 1$. Then for all λ, $0 < \lambda < \|f\|_{p(\cdot)}$, by Proposition 2.14,

$$\rho(f/\lambda) = \rho\left(\frac{\|f\|_{p(\cdot)}}{\lambda}\frac{f}{\|f\|_{p(\cdot)}}\right) \le \left(\frac{\|f\|_{p(\cdot)}}{\lambda}\right)^{p_+(\Omega\setminus\Omega_\infty)} \rho\left(\frac{f}{\|f\|_{p(\cdot)}}\right).$$

Therefore, we can find λ sufficiently close to $\|f\|_{p(\cdot)}$ such that $\rho(f/\lambda) < 1$. But by the definition of the norm, we must have $\rho(f/\lambda) \ge 1$. From this contradiction we see that equality holds.

Now suppose that $p_+(\Omega \setminus \Omega_\infty) = \infty$. Form the sets $\{E_k\}$ as in the proof of Proposition 2.12 and define the function f by

$$f(x) = \left(\sum_{k=2}^{\infty} \frac{k^{-2}}{|E_k \setminus E_{k+1}|}\chi_{E_k\setminus E_{k+1}}(x)\right)^{1/p(x)}.$$

Then for all $\lambda < 1$,

$$\rho(f/\lambda) = \sum_{k=2}^{\infty} k^{-2} \fint_{E_k\setminus E_{k+1}} \lambda^{-p(x)}\,dx \ge \sum_{k=2}^{\infty} k^{-2}\lambda^{-k} = \infty.$$

On the other hand, essentially the same computation shows that

$$\rho(f) = \sum_{k=2}^{\infty} k^{-2} < 1.$$

Therefore, $f \in L^{p(\cdot)}(\Omega)$ and $\|f\|_{p(\cdot)} = 1$, but $\rho(f/\|f\|_{p(\cdot)}) < 1$. □

Corollary 2.22. *Fix Ω and $p(\cdot) \in \mathcal{P}(\Omega)$. If $\|f\|_{p(\cdot)} \le 1$, then $\rho(f) \le \|f\|_{p(\cdot)}$; if $\|f\|_{p(\cdot)} > 1$, then $\rho(f) \ge \|f\|_{p(\cdot)}$.*

Proof. If $\|f\|_{p(\cdot)} = 0$, then $f \equiv 0$ and so $\rho(f) = 0$. If $0 < \|f\|_{p(\cdot)} \le 1$, then by the convexity of the modular (Property (4) of Proposition 2.7) and Proposition 2.21,

$$\rho(f) = \rho(\|f\|_{p(\cdot)}\, f/\|f\|_{p(\cdot)}) \le \|f\|_{p(\cdot)}\rho(f/\|f\|_{p(\cdot)}) \le \|f\|_{p(\cdot)}.$$

If $\|f\|_{p(\cdot)} > 1$, then $\rho(f) > 1$: for if $\rho(f) \le 1$, then by the definition of the norm we would have $\|f\|_{p(\cdot)} \le 1$. But then we have that

$$\rho\big(f/\rho(f)\big) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\rho(f)}\right)^{p(x)} dx + \rho(f)^{-1}\|f\|_{L^\infty(\Omega_\infty)}$$
$$\le \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\rho(f)^{-1}\,dx + \rho(f)^{-1}\|f\|_{L^\infty(\Omega_\infty)} = 1.$$

It follows that $\|f\|_{p(\cdot)} \le \rho(f)$. □

The previous result can be strengthened as follows.

Corollary 2.23. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega_\infty| = 0$. If $\|f\|_{p(\cdot)} > 1$, then*

$$\rho(f)^{1/p_+} \le \|f\|_{p(\cdot)} \le \rho(f)^{1/p_-}.$$

If $0 < \|f\|_{p(\cdot)} \le 1$, then

$$\rho(f)^{1/p_-} \le \|f\|_{p(\cdot)} \le \rho(f)^{1/p_+}.$$

If $p(\cdot)$ is constant, Corollary 2.23 reduces to the identity

$$\|f\|_p = \left(\int_\Omega |f(x)|^p\,dx\right)^{1/p}.$$

The first inequality makes sense if $p_+ = \infty$ and $\rho(f) = \infty$ provided we define $\infty^0 = 1$. The second inequality makes sense if $\|f\|_{p(\cdot)} = 0$, since in this case $\rho(f) = 0$; if $p_+ = \infty$, then we need to interpret 0^0 as 1.

Proof. We prove the first pair of inequalities; the proof of the second is essentially the same. If $p_+ < \infty$, by Proposition 2.14,

$$\frac{\rho(f)}{\|f\|_{p(\cdot)}^{p_+}} \le \rho\left(\frac{f}{\|f\|_{p(\cdot)}}\right) \le \frac{\rho(f)}{\|f\|_{p(\cdot)}^{p_-}}.$$

By Proposition 2.21, $\rho(f/\|f\|_{p(\cdot)}) = 1$, so the desired result follows.

If $p_+ = \infty$, then $\rho(f)^{1/p_+} = 1$, so we only need to prove the right-hand inequality. By Corollary 2.22, $\rho(f) > 1$; hence, since $|\Omega_\infty| = 0$,

$$\rho\big(f/\rho(f)^{1/p_-}\big) = \int_\Omega \left(\frac{|f(x)|}{\rho(f)^{1/p_-}}\right)^{p(x)} dx \le \int_\Omega |f(x)|^{p(x)}\rho(f)^{-1}\,dx = 1.$$

It follows that $\|f\|_{p(\cdot)} \le \rho(f)^{1/p_-}$. □

Remark 2.24. If $|\Omega_\infty| > 0$, then Corollary 2.23 does not hold. Fix $p(\cdot)$ such that $p_- > 1$ and $|\Omega_\infty| > 0$, and take $f \in L^{p(\cdot)}(\Omega)$ such that $\operatorname{supp}(f) \subset \Omega_\infty$ and $\|f\|_{p(\cdot)} = \|f\|_{L^\infty(\Omega_\infty)} = \rho(f) \neq 1$. Then neither inequality comparing $\|f\|_{p(\cdot)}$ to $\rho(f)^{1/p_-}$ can hold in general.

As an application of the above results we will give an equivalent norm on $L^{p(\cdot)}(\Omega)$ that is usually referred to as the Amemiya norm.

Proposition 2.25. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, define*

$$\|f\|^A_{p(\cdot)} = \inf\{\lambda > 0 : \lambda + \lambda\rho_{p(\cdot)}(f/\lambda)\}.$$

Then for all $f \in L^{p(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot)} \leq \|f\|^A_{p(\cdot)} \leq 2\|f\|_{p(\cdot)}.$$

Proof. Since both $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|^A_{p(\cdot)}$ are homogeneous, it will suffice to prove that if $\|f\|_{p(\cdot)} = 1$, then

$$1 \leq \|f\|^A_{p(\cdot)} \leq 2.$$

The second inequality is immediate: by the definition and Corollary 2.22,

$$\|f\|^A_{p(\cdot)} \leq 1 + \rho(f) \leq 1 + \|f\|_{p(\cdot)} = 2.$$

To prove the first inequality, note that if $\lambda \geq 1$, then

$$\lambda + \lambda\rho(f/\lambda) \geq \lambda \geq 1.$$

On the other hand, if $0 < \lambda < 1$, then arguing as in the proof of Proposition 2.14,

$$\lambda + \lambda\rho(f/\lambda) \geq \lambda^{1-p_-}\int_{\Omega\setminus\Omega_\infty} |f(x)|\,dx + \|f\|_{L^\infty(\Omega_\infty)} \geq \rho(f) = 1.$$

Therefore, if we take the infimum over all $\lambda > 0$ we get the desired inequality. □

2.4 Hölder's Inequality and the Associate Norm

In this section we show that the variable Lebesgue space norm satisfies a generalization of Hölder's inequality, and then use this to define an equivalent norm, the associate norm, on $L^{p(\cdot)}(\Omega)$. The classical Hölder's inequality is that for all p, $1 \leq p \leq \infty$, given $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$, then $fg \in L^1(\Omega)$ and

$$\int_\Omega |f(x)g(x)|\,dx \leq \|f\|_p\|g\|_{p'}.$$

This inequality is true for variable exponents with a constant on the right-hand side.

Theorem 2.26. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, for all $f \in L^{p(\cdot)}(\Omega)$ and $g \in L^{p'(\cdot)}(\Omega)$, $fg \in L^1(\Omega)$ and*

$$\int_\Omega |f(x)g(x)|\,dx \leq K_{p(\cdot)}\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

where

$$K_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty.$$

Remark 2.27. Each of the last three terms in the definition of $K_{p(\cdot)}$ is equal to 0 or 1, and at least one of them must equal 1. Therefore, if $p(\cdot)$ is not constant, $1 < K_{p(\cdot)} \leq 4$.

Proof. If $\|f\|_{p(\cdot)} = 0$ or $\|g\|_{p'(\cdot)} = 0$, then $fg \equiv 0$ so there is nothing to prove. Therefore, we may assume that $\|f\|_{p(\cdot)}, \|g\|_{p'(\cdot)} > 0$; moreover, by homogeneity we may assume $\|f\|_{p(\cdot)} = \|g\|_{p'(\cdot)} = 1$.

We consider the integral of $|fg|$ on the disjoint sets Ω_∞, Ω_1 and Ω_*. If $x \in \Omega_\infty$, then $p(x) = \infty$ and $p'(x) = 1$, so

$$\begin{aligned}\int_{\Omega_\infty} |f(x)g(x)|\,dx &\leq \|f\chi_{\Omega_\infty}\|_\infty \|g\chi_{\Omega_\infty}\|_1 \\ &= \|f\chi_{\Omega_\infty}\|_{p(\cdot)}\|g\chi_{\Omega_\infty}\|_{p'(\cdot)} \leq \|f\|_{p(\cdot)}\|g\|_{p'(\cdot)} = 1.\end{aligned}$$

Similarly, if we reverse the roles of $p(\cdot)$ and $p'(\cdot)$, we have that

$$\int_{\Omega_1} |f(x)g(x)|\,dx \leq 1.$$

To estimate the integral on Ω_* we use Young's inequality:

$$\begin{aligned}&\int_{\Omega_*} |f(x)g(x)|\,dx \\ &\quad\leq \int_{\Omega_*} \frac{1}{p(x)}|f(x)|^{p(x)} + \frac{1}{p'(x)}|g(x)|^{p'(x)}\,dx \\ &\quad\leq \frac{1}{p_-}\rho_{p(\cdot)}(f) + \frac{1}{p'(\cdot)_-}\rho_{p'(\cdot)}(g).\end{aligned}$$

Since

$$\frac{1}{p'(\cdot)_-} = \frac{1}{(p_+)'} = 1 - \frac{1}{p_+},$$

and by Proposition 2.21, $\rho_{p(\cdot)}(f),\ \rho_{p'(\cdot)}(g) \leq 1$, we have that

$$\int_{\Omega_*} |f(x)g(x)|\,dx \leq \frac{1}{p_-} + 1 - \frac{1}{p_+}.$$

Combining the above terms, and using the fact that each is needed precisely when the L^∞ norm of the corresponding characteristic function equals 1, we have that

$$\int_{\Omega} |f(x)g(x)|\,dx$$
$$\leq \left(\left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty\right)\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

which is the desired inequality. □

In the classical Lebesgue case, an immediate consequence of Hölder's inequality is that for $p,\ q,\ r$ such that $1 \leq p,\ q,\ r \leq \infty$, and $r^{-1} = p^{-1} + q^{-1}$, if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then $fg \in L^r(\Omega)$ and

$$\|fg\|_r \leq \|f\|_p\|g\|_q.$$

The same result holds in variable Lebesgue spaces; the proof again depends on Hölder's inequality, but is somewhat more complicated.

Corollary 2.28. *Given Ω and exponent functions $r(\cdot),\ q(\cdot) \in \mathcal{P}(\Omega)$ define $p(\cdot) \in \mathcal{P}(\Omega)$ by*

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Then there exists a constant K such that for all $f \in L^{q(\cdot)}(\Omega)$ and $g \in L^{r(\cdot)}(\Omega)$, $fg \in L^{p(\cdot)}(\Omega)$ and

$$\|fg\|_{p(\cdot)} \leq K\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

Proof. Fix $p(\cdot),\ q(\cdot),\ r(\cdot)$ as in the statement of the theorem, and take $f \in L^{q(\cdot)}(\Omega)$ and $g \in L^{r(\cdot)}(\Omega)$. If $\|f\|_{q(\cdot)} = 0$ or if $\|g\|_{r(\cdot)} = 0$, then $fg \equiv 0$ so there is nothing to prove. Therefore, we may assume that these quantities are positive; further, by homogeneity we may assume that $\|f\|_{q(\cdot)} = \|g\|_{r(\cdot)} = 1$.

By the definition of $p(\cdot)$, $\Omega_\infty^{p(\cdot)} = \Omega_\infty^{q(\cdot)} \cap \Omega_\infty^{r(\cdot)}$. Therefore, we can define the exponent function $s(\cdot) \in \mathcal{P}(\Omega \setminus \Omega_\infty^{p(\cdot)})$ by

$$s(x) = \begin{cases} \frac{q(x)}{p(x)} & x \notin \Omega_\infty^{q(\cdot)} \cup \Omega_\infty^{r(\cdot)} \\ 1 & x \in \Omega_\infty^{r(\cdot)} \setminus \Omega_\infty^{q(\cdot)} \\ \infty & x \in \Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{r(\cdot)}. \end{cases}$$

Suppose for the moment that

$$|f(\cdot)|^{p(\cdot)} \in L^{s(\cdot)}(\Omega \setminus \Omega_\infty^{p(\cdot)}) \quad \text{and} \quad |g(\cdot)|^{p(\cdot)} \in L^{s'(\cdot)}(\Omega \setminus \Omega_\infty^{p(\cdot)}), \tag{2.1}$$

and $\||f(\cdot)|^{p(\cdot)}\|_{L^{s(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})}$, $\||g(\cdot)|^{p(\cdot)}\|_{L^{s'(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})} \leq 1$. Then by the generalized Hölder's inequality (Theorem 2.26),

$$\begin{aligned}
\rho_{p(\cdot)}(fg) &= \int_{\Omega\setminus\Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}|g(x)|^{p(x)}\,dx + \|fg\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \\
&\leq K_{s(\cdot)}\||f(\cdot)|^{p(\cdot)}\|_{L^{s(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})}\||g(\cdot)|^{p(\cdot)}\|_{L^{s'(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})} \\
&\qquad + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})}\|g\|_{L^\infty(\Omega_\infty^{r(\cdot)})} \\
&\leq K_{s(\cdot)} + \|f\|_{q(\cdot)}\|g\|_{r(\cdot)} \\
&= K_{s(\cdot)} + 1.
\end{aligned}$$

Then by the convexity of the modular (Property (4) of Proposition 2.7) $fg \in L^{p(\cdot)}(\Omega)$ and

$$\|fg\|_{p(\cdot)} \leq K_{s(\cdot)} + 1 = (K_{s(\cdot)} + 1)\|f\|_{q(\cdot)}\|g\|_{r(\cdot)}.$$

Therefore, to complete the proof we need to show (2.1) and estimate the norms. We first consider $|f(\cdot)|^{p(\cdot)}$. Since $\|f\|_{q(\cdot)} = 1$, by Corollary 2.22, $\|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} \leq \rho_{q(\cdot)}(f) \leq 1$. Further, $\Omega_\infty^{s(\cdot)} \subset \Omega_\infty^{q(\cdot)}$ and $\Omega \setminus \Omega_\infty^{s(\cdot)} \subset \Omega \setminus \Omega_\infty^{q(\cdot)}$, and on $\Omega_1^{s(\cdot)}$, $p(x) = q(x) < \infty$. Hence,

$$\begin{aligned}
\rho_{s(\cdot)}(f(\cdot)^{p(\cdot)}\chi_{\Omega\setminus\Omega_\infty^{p(\cdot)}}) &\leq \int_{\Omega\setminus\Omega_\infty^{s(\cdot)}} |f(x)|^{p(x)s(x)}\,dx + \||f(\cdot)|^{p(\cdot)}\|_{L^\infty(\Omega_{s(\cdot)}^\infty)} \\
&\leq \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx + \||f(\cdot)|^{p(\cdot)}\|_{L^\infty(\Omega_{q(\cdot)}^\infty)} \\
&\leq \int_{\Omega\setminus\Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\,dx + \|f\|_{L^\infty(\Omega_{q(\cdot)}^\infty)} \\
&\leq 1.
\end{aligned}$$

Therefore, by the definition of the norm, $\||f(\cdot)|^{p(\cdot)}\|_{L^{s(\cdot)}(\Omega\setminus\Omega_\infty^{p(\cdot)})} \leq 1$. The same argument, with $s(\cdot)$ replaced by $s'(\cdot)$ and $q(\cdot)$ replaced by $r(\cdot)$ gives the corresponding bound for $|g(\cdot)|^{p(\cdot)}$. This completes the proof. □

Remark 2.29. It follows from the proof that we can take $K = K_{s(\cdot)} + 1$; by an abuse of notation we can write this as $K_{q(\cdot)/p(\cdot)} + 1$.

As a consequence of Corollary 2.28 we can generalize Theorem 2.26 to three or more exponents.

Corollary 2.30. *Given Ω, suppose $p_1(\cdot), p_2(\cdot), \dots, p_k(\cdot) \in \mathcal{P}(\Omega)$ is a collection of exponents that satisfy*

$$\sum_{i=1}^{k} \frac{1}{p_i(x)} = 1, \qquad x \in \Omega.$$

Then there exists a constant C, depending on the p_i, such that for all $f_i \in L^{p_i(\cdot)}(\Omega)$, $1 \le i \le k$,

$$\int_\Omega |f_1(x) f_2(x) \cdots f_k(x)|\, dx \le C \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \cdots \|f_k\|_{p_k(\cdot)}.$$

Proof. We prove this by induction. When $k = 2$, this is just Theorem 2.26. Now suppose that for some $k \ge 2$ the inequality holds; we will prove it true for $k+1$ exponents. Given exponents $p_1(\cdot), \dots, p_{k+1}(\cdot)$, define $r(\cdot)$ by

$$\frac{1}{r(x)} = \frac{1}{p_k(x)} + \frac{1}{p_{k+1}(x)}.$$

Fix functions $f_i \in L^{p_i(\cdot)}(\Omega)$; then by Corollary 2.28, $f_k f_{k+1} \in L^{r(\cdot)}(\Omega)$ and

$$\|f_k\|_{p_k(\cdot)} \|f_{k+1}\|_{p_{k+1}(\cdot)} \ge c \|f_k f_{k+1}\|_{r(\cdot)}.$$

Therefore, by our induction hypothesis applied to $p_1(\cdot), \dots, p_{k-1}(\cdot), r(\cdot)$,

$$\begin{aligned} &\|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \cdots \|f_{k+1}\|_{p_{k+1}(\cdot)} \\ &\qquad \ge c \|f_1\|_{p_1(\cdot)} \|f_2\|_{p_2(\cdot)} \cdots \|f_{k-1}\|_{p_{k-1}(\cdot)} \|f_k f_{k+1}\|_{r(\cdot)} \\ &\qquad\qquad \ge c \int_\Omega |f_1(x) \cdots f_{k+1}(x)|\, dx. \end{aligned}$$

□

In the classical Lebesgue space $L^p(\Omega)$, $1 \le p \le \infty$, the norm can be computed using the identity

$$\|f\|_p = \sup \int_\Omega f(x) g(x)\, dx,$$

where the supremum is taken over all $g \in L^{p'}(\Omega)$ with $\|g\|_{p'} \le 1$. Indeed, g can be taken from any dense subset of $L^{p'}(\Omega)$—for example, $C_c(\Omega)$ if $p > 1$. A slightly weaker analog of this equality is true for variable Lebesgue spaces.

Definition 2.31. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a measurable function f, define

$$\|f\|'_{p(\cdot)} = \sup \int_\Omega f(x) g(x)\, dx, \tag{2.2}$$

where the supremum is taken over all $g \in L^{p'(\cdot)}(\Omega)$ with $\|g\|_{p'(\cdot)} \le 1$.

Temporarily denote by $M^{p(\cdot)}(\Omega)$ the set of all measurable functions f such that $\|f\|'_{p(\cdot)} < \infty$.

Proposition 2.32. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, the set $M^{p(\cdot)}(\Omega)$ is a normed vector space with respect to the norm $\|\cdot\|'_{p(\cdot)}$. Furthermore, the norm is order preserving: given f, $g \in M^{p(\cdot)}(\Omega)$ such that $|f| \le |g|$, then $\|f\|'_{p(\cdot)} \le \|g\|'_{p(\cdot)}$.*

Proof. It is immediate that $M^{p(\cdot)}(\Omega)$ is a vector space. The fact that $\|\cdot\|'_{p(\cdot)}$ is an order preserving norm is a consequence of the properties of integrals and supremums and the following equivalent characterization of $\|\cdot\|'_{p(\cdot)}$. First note that it is immediate from this definition that for all measurable functions f,

$$\|f\|'_{p(\cdot)} \le \sup_{\|g\|_{p'(\cdot)} \le 1} \left| \int_\Omega f(x)g(x)\,dx \right| \le \sup_{\|g\|_{p'(\cdot)} \le 1} \int_\Omega |f(x)g(x)|\,dx,$$

but in fact all of these are equal. To see this, it suffices to note that for any $g \in L^{p'(\cdot)}(\Omega)$, $\|g\|_{p'(\cdot)} \le 1$, $|f(x)g(x)| = f(x)h(x)$, where $h(x) = \operatorname{sgn} f(x)|g(x)|$ and $\|h\|_{p'(\cdot)} \le \|g\|_{p'(\cdot)} \le 1$; hence,

$$\int_\Omega |f(x)g(x)|\,dx = \int_\Omega f(x)h(x)\,dx \le \|f\|'_{p(\cdot)}.$$

□

Remark 2.33. As a consequence of the proof of Proposition 2.32 we get another version of Hölder's inequality:

$$\int_\Omega |f(x)g(x)|\,dx \le \|f\|_{p(\cdot)}\|g\|'_{p'(\cdot)}.$$

In the next result we show that $M^{p(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)$ and that the norms $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|'_{p(\cdot)}$ are equivalent. We will refer to the norm $\|\cdot\|'_{p(\cdot)}$ as the associate norm on $L^{p(\cdot)}(\Omega)$.

Theorem 2.34. *Given Ω, $p(\cdot) \in \mathcal{P}(\Omega)$, and a measurable f, then $f \in L^{p(\cdot)}(\Omega)$ if and only if $f \in M^{p(\cdot)}(\Omega)$; furthermore,*

$$k_{p(\cdot)}\|f\|_{p(\cdot)} \le \|f\|'_{p(\cdot)} \le K_{p(\cdot)}\|f\|_{p(\cdot)},$$

where

$$K_{p(\cdot)} = \left(\frac{1}{p_-} - \frac{1}{p_+} + 1\right)\|\chi_{\Omega_*}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty,$$
$$\frac{1}{k_{p(\cdot)}} = \|\chi_{\Omega_\infty}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_*}\|_\infty.$$

Remark 2.35. For every variable Lebesgue space we have that $K_{p(\cdot)} \leq 4$ and $k_{p(\cdot)} \geq 1/3$.

To motivate the proof of Theorem 2.34, recall the proof of (2.2) if $1 < p < \infty$. By Hölder's inequality, $\|f\|'_p \leq \|f\|_p$. To prove the reverse inequality, let

$$g(x) = \left(\frac{|f(x)|}{\|f\|_p}\right)^{p/p'} \operatorname{sgn} f(x).$$

Then $\|g\|_{p'} = 1$, and

$$\int_\Omega f(x)g(x)\,dx = \|f\|_p,$$

and so in fact the supremum is attained.

Our proof will be based on a similar but more complicated function g; first we need to prove a lemma.

Lemma 2.36. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $\|f\chi_{\Omega_*}\|'_{p(\cdot)} \leq 1$ and $\rho(f\chi_{\Omega_*}) < \infty$, then $\rho(f\chi_{\Omega_*}) \leq 1$.*

Proof. Suppose to the contrary that $\rho(f\chi_{\Omega_*}) > 1$. Then by the continuity of the modular (Proposition 2.7, (6)) there exists $\lambda > 1$ such that $\rho(f\chi_{\Omega_*}/\lambda) = 1$. Let

$$g(x) = \left(\frac{|f(x)|}{\lambda}\right)^{p(x)-1} \operatorname{sgn} f(x)\chi_{\Omega_*}(x).$$

Then $\rho_{p'(\cdot)}(g) = \rho_{p(\cdot)}(f\chi_{\Omega_*}/\lambda) = 1$, so $\|g\|_{p'(\cdot)} \leq 1$. Therefore, by the definition of the associate norm,

$$\|f\chi_{\Omega_*}\|'_{p(\cdot)} \geq \int_\Omega f(x)\chi_{\Omega_*}(x)\,g(x)\,dx = \lambda \int_{\Omega_*} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx = \lambda\rho(f\chi_{\Omega_*}/\lambda) > 1.$$

This contradicts our hypothesis on f, so the desired inequality holds. □

Proof of Theorem 2.34. One implication is immediate: given $f \in L^{p(\cdot)}(\Omega)$, by Hölder's inequality for variable Lebesgue spaces (Theorem 2.26),

$$\|f\|'_{p(\cdot)} \leq K_{p(\cdot)}\|f\|_{p(\cdot)}.$$

To prove the converse, we will assume that

$$|\Omega_\infty^{p(\cdot)}|,\ |\Omega_1^{p(\cdot)}|,\ |\Omega_*^{p(\cdot)}| > 0.$$

If any of these sets has measure 0, then the proof can be readily adapted by omitting the terms associated with them. Further, by the definition of the norms we only have to prove this for non-negative functions f.

We will prove that if $\|f\|'_{p(\cdot)} \le 1$ and $\rho_{p(\cdot)}(f\chi_{\Omega_*}) < \infty$, then

$$\rho_{p(\cdot)}(k_{p(\cdot)} f) \le 1. \tag{2.3}$$

Given this, the desired inequality follows by an approximation argument. Fix any non-negative $f \in M^{p(\cdot)}(\Omega)$. By homogeneity we may assume that $\|f\|'_{p(\cdot)} = 1$. For each $k \ge 1$, define the sets

$$E_k = B_k(0) \cap \big(\Omega \setminus \Omega_* \cup \{x \in \Omega_* : p(x) < k\}\big),$$

and define the functions $f_k = \min(f, k)\chi_{E_k}$. Then $f_k \le f$, so by Proposition 2.32, $\|f_k\|'_{p(\cdot)} \le \|f\|'_{p(\cdot)} = 1$. Furthermore, the sequence $\{f_k\}$ increases to f pointwise. Finally, $\rho(f_k\chi_{\Omega_*}) < \infty$, and so we can apply (2.3) with f replaced by f_k. Therefore, by Fatou's lemma on the classical Lebesgue spaces and (2.3),

$$\rho_{p(\cdot)}(k_{p(\cdot)} f/\|f\|'_{p(\cdot)}) = \rho_{p(\cdot)}(k_{p(\cdot)} f) \le \liminf_{k\to\infty} \rho_{p(\cdot)}(k_{p(\cdot)} f_k) \le 1.$$

Thus, we have that

$$\|f\|_{p(\cdot)} \le k_{p(\cdot)}^{-1} \|f\|'_{p(\cdot)}.$$

To complete the proof, fix f with $\|f\|'_{p(\cdot)} \le 1$ and $\rho(f\chi_{\Omega_*}) < \infty$; we will show that (2.3) holds. First note that by Proposition 2.32, $\|f\chi_{\Omega_*^{p(\cdot)}}\|'_{p(\cdot)} \le 1$. Now fix ϵ, $0 < \epsilon < 1$; then there exists a set $E_\epsilon \subset \Omega_\infty^{p(\cdot)}$ such that $0 < |E_\epsilon| < \infty$, and for each $x \in E_\epsilon$,

$$|f(x)| \ge (1-\epsilon)\|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})}.$$

Now define the function g_ϵ by

$$g_\epsilon(x) = \begin{cases} k_{p(\cdot)}|f(x)|^{p(x)-1}\,\mathrm{sgn}\, f(x) & x \in \Omega_*^{p(\cdot)} = \Omega_*^{p'(\cdot)}, \\ k_{p(\cdot)}\,\mathrm{sgn}\, f(x) & x \in \Omega_1^{p(\cdot)} = \Omega_\infty^{p'(\cdot)}, \\ k_{p(\cdot)}|E_\epsilon|^{-1}\chi_{E_\epsilon}(x)\,\mathrm{sgn}\, f(x) & x \in \Omega_\infty^{p(\cdot)} = \Omega_1^{p'(\cdot)}. \end{cases}$$

We claim that $\rho_{p'(\cdot)}(g_\epsilon) \le 1$, so $\|g_\epsilon\|_{p'(\cdot)} \le 1$. To see this, note that

$$\begin{aligned} &\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \\ &\quad \le \int_{\Omega_*^{p'(\cdot)}} |f(x)|^{p(x)}\,dx + \|\mathrm{sgn}\, f\|_{L^\infty(\Omega_\infty^{p'(\cdot)})} + |E_\epsilon|^{-1}\int_{\Omega_1^{p'(\cdot)}} \chi_{E_\epsilon}(x)\,dx \\ &\quad = \int_{\Omega_*^{p(\cdot)}} |f(x)|^{p(x)}\,dx + \|\mathrm{sgn}\, f\|_{L^\infty(\Omega_1^{p(\cdot)})} + |E_\epsilon|^{-1}\int_{\Omega_\infty^{p(\cdot)}} \chi_{E_\epsilon}(x)\,dx. \end{aligned}$$

By Lemma 2.36, the first term on the right-hand side is dominated by 1; the second term equals 0 or 1, and the third term always equals 1. Therefore,

$$\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \le \|\chi_{\Omega_*^{p(\cdot)}}\|_\infty + \|\chi_{\Omega_1^{p(\cdot)}}\|_\infty + \|\chi_{\Omega_\infty^{p(\cdot)}}\|_\infty = \frac{1}{k_{p(\cdot)}}.$$

Since $k_{p(\cdot)} \leq 1$, by the convexity of the modular (Proposition 2.7),

$$\rho_{p'(\cdot)}(g_\epsilon) \leq k_{p(\cdot)}\rho_{p'(\cdot)}(g_\epsilon/k_{p(\cdot)}) \leq 1,$$

which is what we claimed to be true.

Furthermore, we have that

$$\begin{aligned}
&\int_\Omega f(x)g_\epsilon(x)\,dx \\
&\qquad = k_{p(\cdot)}\int_{\Omega_*^{p(\cdot)}} |f(x)|^{p(x)}\,dx + k_{p(\cdot)}\int_{\Omega_1^{p(\cdot)}} |f(x)|\,dx + k_{p(\cdot)}\fint_{E_\epsilon} |f(x)|\,dx \\
&\qquad \geq k_{p(\cdot)}\int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx + (1-\epsilon)k_{p(\cdot)}\|f\|_{L^\infty(\Omega_\infty)} \\
&\qquad \geq (1-\epsilon)k_{p(\cdot)}\rho_{p(\cdot)}(f).
\end{aligned}$$

Therefore, by the definition of the associate norm, since $\|g_\epsilon\|_{p'(\cdot)} \leq 1$,

$$1 \geq \|f\|'_{p(\cdot)} \geq \int_\Omega f(x)g_\epsilon(x)\,dx \geq (1-\epsilon)k_{p(\cdot)}\rho_{p(\cdot)}(f).$$

Since $\epsilon > 0$ was arbitrary, again by the convexity of the modular we have that

$$1 \geq k_{p(\cdot)}\rho_{p(\cdot)}(f) \geq \rho_{p(\cdot)}(k_{p(\cdot)}f).$$

□

In the notation introduced above, given an exponent $p(\cdot)$, the Banach space $M^{p'(\cdot)}$ of measurable functions f such that

$$\|f\|'_{p'(\cdot)} = \sup\left\{\int_\Omega f(x)g(x)\,dx, g \in L^{p(\cdot)}(\Omega), \|g\|_{p(\cdot)} \leq 1\right\} < \infty,$$

is called the associate space of $L^{p(\cdot)}(\Omega)$. As an immediate consequence of Theorem 2.34 we have the following result.

Proposition 2.37. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, the associate space of $L^{p(\cdot)}(\Omega)$ is equal to $L^{p'(\cdot)}(\Omega)$, and $\|\cdot\|_{p'(\cdot)}$ and $\|\cdot\|'_{p'(\cdot)}$ are equivalent norms.*

Finally, as a corollary to Theorem 2.34 we prove a version of Minkowski's integral inequality for variable Lebesgue spaces.

Corollary 2.38. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, let $f : \Omega \times \Omega \to \mathbb{R}$ be a measurable function (with respect to product measure) such that for almost every $y \in \Omega$, $f(\cdot, y) \in L^{p(\cdot)}(\Omega)$. Then*

$$\left\|\int_\Omega f(\cdot,y)\,dy\right\|_{p(\cdot)} \leq k_{p(\cdot)}^{-1}K_{p(\cdot)}\int_\Omega \|f(\cdot,y)\|_{p(\cdot)}\,dy. \tag{2.4}$$

Proof. If the right-hand side of (2.4) is infinite, then there is nothing to prove, so we may assume that this integral is finite. Define the function

$$g(x) = \int_{\Omega} f(x, y)\, dy,$$

and take any $h \in L^{p'(\cdot)}(\Omega)$, $\|h\|_{p'(\cdot)} \leq 1$. Then by Fubini's theorem (see Royden [301]) and Hölder's inequality on the variable Lebesgue spaces (Theorem 2.26),

$$\begin{aligned}
\int_{\Omega} |g(x)h(x)|\, dx &\leq \int_{\Omega}\int_{\Omega} |f(x, y)|\, dy\, |h(x)|\, dx \\
&= \int_{\Omega}\int_{\Omega} |f(x, y)h(x)|\, dx\, dy \\
&\leq K_{p(\cdot)} \int_{\Omega} \|f(\cdot, y)\|_{p(\cdot)} \|h\|_{p'(\cdot)}\, dy \\
&\leq K_{p(\cdot)} \int_{\Omega} \|f(\cdot, y)\|_{p(\cdot)}\, dy.
\end{aligned}$$

Therefore, we have that

$$\|g\|'_{p(\cdot)} \leq K_{p(\cdot)} \int_{\Omega} \|f(\cdot, y)\|_{p(\cdot)}\, dy,$$

and inequality (2.4) follows by Theorem 2.34. □

2.5 Embedding Theorems

In this section we consider the embeddings of classical and variable Lebesgue spaces into one another. We begin by showing that every function in a variable Lebesgue space is locally integrable. To do so we prove a simple but useful lemma.

Lemma 2.39. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $E \subset \Omega$ is such that $|E| < \infty$, then $\chi_E \in L^{p(\cdot)}(\Omega)$ and $\|\chi_E\|_{p(\cdot)} \leq |E| + 1$.*

Proof. Fix $\lambda = |E| + 1$. Then

$$\begin{aligned}
\rho(\chi_E/\lambda) = \int_{E\setminus\Omega_\infty} \lambda^{-p(x)}\, dx + \lambda^{-1}\|\chi_{E\cap\Omega_\infty}\|_\infty \\
\leq \lambda^{-p_-}|E| + \lambda^{-1}. \leq \lambda^{-1}(|E| + 1) = 1.
\end{aligned}$$

By the definition of the norm we get the desired result. □

Remark 2.40. If $|\Omega_\infty| = 0$, then by Corollary 2.23 we get a sharper bound that depends on E and $p(\cdot)$:

$$\|\chi_E\|_{p(\cdot)} \leq \max\left(|E|^{1/p_-}, |E|^{1/p_+}\right).$$

Proposition 2.41. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $f \in L^{p(\cdot)}(\Omega)$, then f is locally integrable.*

Proof. Let $E \subset \Omega$ be a set of finite measure. Then by the generalized Hölder's inequality (Theorem 2.26) and Lemma 2.39,

$$\int_E |f(x)|\,dx \leq C\|f\|_{p(\cdot)}\|\chi_E\|_{p'(\cdot)} < \infty.$$

□

We now consider the embedding of $L^\infty(\Omega)$ into $L^{p(\cdot)}(\Omega)$. It follows from the proof of Lemma 2.39 that if $|\Omega \backslash \Omega_\infty| < \infty$, then $\chi_\Omega \in L^{p(\cdot)}(\Omega)$, which immediately implies that $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$. However, unlike in the case of classical Lebesgue spaces, this embedding can hold even if $|\Omega \setminus \Omega_\infty| = \infty$.

Example 2.42. Let $\Omega = (1, \infty)$ and $p(x) = x$. By Example 2.11, $1 \in L^{p(\cdot)}(\Omega)$, and so if $f \in L^\infty(\Omega)$,

$$\|f\|_{p(\cdot)} \leq \|f\|_\infty \|1\|_{p(\cdot)} < \infty.$$

More generally, we have the following characterization of when this embedding holds.

Proposition 2.43. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $1 \in L^{p(\cdot)}(\Omega)$, which in turn is true if and only if for some $\lambda > 1$,*

$$\int_{\Omega \backslash \Omega_\infty} \lambda^{-p(x)}\,dx < \infty. \tag{2.5}$$

In particular, the embedding holds if $|\Omega| < \infty$ or if $1/p(\cdot) \in LH_\infty(\Omega)$ and $p(x) \to \infty$ as $|x| \to \infty$.

Proof. We repeat the above argument: $L^\infty(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $1 \in L^{p(\cdot)}(\Omega)$, and by the definition of $L^{p(\cdot)}(\Omega)$ and Proposition 2.7 this is true if and only if there exists $\lambda > 1$ such that

$$\rho(1/\lambda) = \int_{\Omega \backslash \Omega_\infty} \lambda^{-p(x)}\,dx + \lambda^{-1}\|1\|_{L^\infty(\Omega_\infty)} < \infty.$$

This in turn is equivalent to (2.5).

If $|\Omega| < \infty$, then the integral in (2.5) is clearly dominated by $|\Omega|$. If $1/p(\cdot) \in LH_\infty$ and $p(x) \to \infty$ as $|x| \to \infty$, then we have that

$$\frac{1}{p(x)} \le \frac{C_\infty}{\log(e + |x|)}.$$

Therefore, for $\lambda > 1$ sufficiently large,

$$\int_{\Omega\setminus\Omega_\infty} \lambda^{-p(x)}\,dx \le \int_{\Omega\setminus\Omega_\infty} \lambda^{-C_\infty^{-1}\log(e+|x|)}\,dx$$
$$\le \int_{\Omega\setminus\Omega_\infty} (e + |x|)^{-C_\infty^{-1}\log(\lambda)}\,dx < \infty.$$

□

The smoothness condition LH_∞ in Proposition 2.43 is in some sense sharp, as the next example shows.

Example 2.44. Let $\Omega = (e, \infty)$, and let $p(x) = \phi(x)\log(x)$, where ϕ is a decreasing function such that $\phi(x) \to 0$ as $x \to \infty$, and $p(\cdot)$ is increasing and $p(x) \to \infty$ as $x \to \infty$. Then $L^\infty(\Omega)$ is not contained in $L^{p(\cdot)}(\Omega)$.

A simple example of such a function ϕ is $\phi(x) \approx \log\log(x)^{-1}$.

Proof. We will show that for any $\lambda > 1$,

$$\int_e^\infty \lambda^{-p(x)}\,dx = \infty.$$

Fix $\lambda > 1$; since $\phi(x)$ decreases to 0, there exists $N > 0$ such that if $k \ge N$, then $\log(\lambda)\phi(e^{k+1}) < 1/2$. Then, since $p(\cdot)$ is increasing,

$$\int_e^\infty \lambda^{-p(x)}\,dx \ge \sum_{k\ge N} \int_{e^k}^{e^{k+1}} \lambda^{-p(x)}\,dx \ge \sum_{k\ge N} e^k \cdot \lambda^{-\phi(e^{k+1})\log(e^{k+1})}$$
$$\ge \sum_{k\ge N} e^k e^{-\phi(e^{k+1})\log(\lambda)(k+1)} \ge \sum_{k\ge N} e^k e^{-\frac{1}{2}(k+1)} = \infty.$$

□

As a consequence of Proposition 2.43 we can completely characterize the exponents $p(\cdot)$ and $q(\cdot)$ such that $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$. Unlike in the case of classical Lebesgue spaces, this embedding is possible even when $|\Omega| = \infty$.

Theorem 2.45. *Given Ω and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ and there exists $K > 1$ such that for all $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{p(\cdot)} \le K\|f\|_{q(\cdot)}$, if and only if:*

1. $p(x) \le q(x)$ for almost every $x \in \Omega$;
2. There exists $\lambda > 1$ such that

$$\int_D \lambda^{-r(x)}\,dx < \infty, \tag{2.6}$$

where $D = \{x \in \Omega : p(x) < q(x)\}$ and $r(\cdot)$ is the defect exponent defined by

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

Remark 2.46. If $1/p(\cdot),\ 1/q(\cdot) \in LH_\infty(\Omega)$, then $1/r(\cdot) \in LH_\infty(\Omega)$ and arguing as we did in the proof of Proposition 2.43 we have that (2.6) holds if $r(x) \to \infty$ as $|x| \to \infty$.

Proof. Suppose first that Conditions (1) and (2) hold. By Proposition 2.43 we have that $1 \in L^{r(\cdot)}(\Omega)$. Therefore, by Corollary 2.28, given any $f \in L^{q(\cdot)}(\Omega)$,

$$\|f\|_{p(\cdot)} = \|1 \cdot f\|_{p(\cdot)} \le K\|1\|_{r(\cdot)}\|f\|_{q(\cdot)}.$$

To prove the converse, we will show that if either Condition (1) or (2) do not hold, then the embedding also does not hold.

Suppose first that Condition (1) does not hold. Then there exists a set $E \subset \Omega$, $|E| > 0$, such that if $x \in E$, $p(x) > q(x)$. We will construct $f \in L^{q(\cdot)}(\Omega) \setminus L^{p(\cdot)}(\Omega)$. There are two cases.

Case 1: $|\boldsymbol{\Omega}_\infty^{\mathbf{p}(\cdot)} \cap \mathbf{E}| > \mathbf{0}$. Since $q(\cdot)$ is finite on E, there exists a set $F \subset E \cap \Omega_\infty^{p(\cdot)}$, $0 < |F| < \infty$, and r, $1 < r < \infty$, such that if $x \in F$, $q(x) \le r$. Partition F as the union of disjoint sets F_j, $j \ge 1$, such that $|F_j| = 2^{-j}|F|$ and define the function f by

$$f(x) = \sum_{j=1}^{\infty} \left(\frac{3}{2}\right)^{j/r} \chi_{F_j}(x).$$

Then f is unbounded, and so

$$\|f\|_{p(\cdot)} \ge \|f\chi_F\|_{p(\cdot)} = \|f\chi_F\|_\infty = \infty.$$

On the other hand, $f \in L^{q(\cdot)}(\Omega)$ since

$$\rho_{q(\cdot)}(f) = \int_F |f(x)|^{q(\cdot)}\,dx = \sum_{j=1}^{\infty} \int_{F_j} \left(\frac{3}{2}\right)^{jq(x)/r} dx$$
$$\le \sum_{j=1}^{\infty} \left(\frac{3}{2}\right)^{j} 2^{-j}|F| = 3|F| < \infty.$$

Case 2: $|\boldsymbol{\Omega}_{\infty}^{\mathbf{p}(\cdot)} \cap \mathbf{E}| = \mathbf{0}$. In this case, $1 \le q(x) < p(x) < \infty$ almost everywhere on E. Therefore, there exists a set $F \subset E$, $0 < |F| < \infty$, and constants $\epsilon > 0$ and $r > 1$ such that if $x \in F$,

$$q(x) + \epsilon \le p(x) \le r < \infty.$$

In particular,

$$\frac{p(x)}{q(x)} \ge 1 + \frac{\epsilon}{r}.$$

Again partition F into disjoint sets F_j, $|F_j| = 2^{-j}|F|$, and define f by

$$f(x) = \sum_{j=1}^{\infty} \left(\frac{2^j}{j^2}\right)^{1/q(x)} \chi_{F_j}(x).$$

Then

$$\rho_{q(\cdot)}(f) = \sum_{j=1}^{\infty} 2^j j^{-2}|F_j| = |F| \sum_{j=1}^{\infty} j^{-2} < \infty.$$

On the other hand, since for $j \ge 4$, $2^j/j^2 \ge 1$,

$$\begin{aligned}
\rho_{p(\cdot),F}(f) &= \sum_{j=1}^{\infty} \int_{F_j} \left(\frac{2^j}{j^2}\right)^{p(x)/q(x)} dx \\
&\ge \sum_{j=4}^{\infty} \left(\frac{2^j}{j^2}\right)^{1+\epsilon/r} |F_j| = |F| \sum_{j=4}^{\infty} 2^{\epsilon j/r} j^{-2(1+\epsilon/r)} = \infty.
\end{aligned}$$

Since $p_+(F) \le r < \infty$, by Proposition 2.12,

$$\|f\|_{L^{p(\cdot)}(\Omega)} \ge \|f\|_{L^{p(\cdot)}(F)} = \infty.$$

This completes the proof.

Now suppose that Condition (2) does not hold. Again there are two cases. Define the sets

$$D_\infty = \{x \in D : q(x) = \infty\}, \quad D_0 = \{x \in D : p(x) < q(x) < \infty\}.$$

Then (2.6) must fail to hold for all $\lambda > 1$ with D replaced by D_∞ or it fails to hold for all $\lambda > 1$ with D replaced by D_0.

Case 1: Suppose first that for any $\lambda > 1$,

$$\int_{D_\infty} \lambda^{-r(x)}\, dx = \infty.$$

We will construct $f \in L^{q(\cdot)}(\Omega) \setminus L^{p(\cdot)}(\Omega)$. Let $f = \chi_{D_\infty}$; since $D_\infty \subset \Omega_\infty^{q(\cdot)}$, $\|f\|_{q(\cdot)} = \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} = 1$, so $f \subset L^{q(\cdot)}(\Omega)$. On the other hand, by the definition of the defect exponent $r(\cdot)$, for $x \in D_\infty$, $p(x) = r(x)$. Hence, for all $\lambda > 1$

$$\rho_{p(\cdot)}(f/\lambda) = \int_{D_\infty} \lambda^{-r(x)}\,dx = \infty.$$

Since the same is obviously true for $\lambda \le 1$, it follows that $f \notin L^{p(\cdot)}(\Omega)$.

Case 2: Now suppose that for any $\lambda > 1$,

$$\int_{D_0} \lambda^{-r(x)}\,dx = \infty. \tag{2.7}$$

We will construct a sequence of functions $\{f_k\} \subset L^{q(\cdot)}(\Omega)$ such that $\|f_k\|_{q(\cdot)} \to 0$ as $k \to \infty$, but $\|f_k\|_{p(\cdot)} \ge 1$. It follows immediately that the embedding cannot hold.

Given (2.7), for any compact set $K \subset D_0$ and any $\lambda > 1$ we have that

$$\int_{D_0 \setminus K} \lambda^{-r(x)}\,dx = \infty.$$

Therefore, by the continuity of the integral we can construct a sequence of disjoint sets $D_j \subset D_0$, $j \ge 1$, such that

$$\int_{D_j} 2^{-jr(x)}\,dx = 1.$$

For each $k \ge 1$ define the function f_k by

$$f_k(x) = \sum_{j>k} 2^{-j\frac{r(x)}{p(x)}} \chi_{D_j}(x).$$

Then

$$\rho_{p(\cdot)}(f_k) = \sum_{j>k} \int_{D_j} 2^{-jr(x)}\,dx = \sum_{j>k} 1 = \infty.$$

Thus $\|f_k\|_{p(\cdot)} \ge 1$. On the other hand, by the definition of the defect exponent $r(\cdot)$, we have that for $x \in D_0$,

$$q(x) - \frac{q(x)r(x)}{p(x)} = -r(x).$$

Hence,

$$\begin{aligned}\rho_{q(\cdot)}(2^k f_k) = \sum_{j>k} \int_{D_j} 2^{kq(x)} 2^{-j\frac{q(x)r(x)}{p(x)}}\,dx &\le \sum_{j>k} 2^{k-j} \int_{D_j} 2^{j\left(q(x) - \frac{q(x)r(x)}{p(x)}\right)}\,dx \\ &= \sum_{j>k} 2^{k-j} \int_{D_j} 2^{-jr(x)}\,dx = \sum_{j>k} 2^{k-j} = 1.\end{aligned}$$

Therefore, $\|f_k\|_{q(\cdot)} \le 2^{-k}$ and so $\|f_k\|_{q(\cdot)} \to 0$ as $k \to \infty$. □

As a corollary to the construction in the second half of the proof of Theorem 2.45 we have that the spaces $L^{p(\cdot)}(\Omega)$ are different for different exponent functions $p(\cdot)$.

Corollary 2.47. *Given Ω and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, if there exists a set $E \subset \Omega$, such that $|E| > 0$ and $p(x) \neq q(x)$, $x \in E$, then the set $\big(L^{p(\cdot)}(\Omega) \setminus L^{q(\cdot)}(\Omega)\big) \cup \big(L^{q(\cdot)}(\Omega) \setminus L^{p(\cdot)}(\Omega)\big)$ is not empty.*

If $|\Omega^{p(\cdot)} \setminus \Omega_\infty^{p(\cdot)}| < \infty$, then condition (2.6) is true for any $\lambda > 1$, so a necessary and sufficient condition for the embedding $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ is that $p(x) \leq q(x)$. Thus the next result is a corollary of Theorem 2.45. However, we give a direct proof of one implication since by doing so we get a sharper constant.

Corollary 2.48. *Given Ω and $p(\cdot), q(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega \setminus \Omega_\infty^{p(\cdot)}| < \infty$. Then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if $p(x) \leq q(x)$ almost everywhere. Furthermore, in this case we have that*

$$\|f\|_{p(\cdot)} \leq (1 + |\Omega \setminus \Omega_\infty^{p(\cdot)}|)\|f\|_{q(\cdot)}. \tag{2.8}$$

Proof. We will assume that $p(x) \leq q(x)$ almost everywhere and prove (2.8). By the homogeneity of the norm, it will suffice to show that if $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{q(\cdot)} \leq 1$, then $\|f\|_{p(\cdot)} \leq 1 + |\Omega \setminus \Omega_\infty^{p(\cdot)}|$. By the definition of the norm,

$$1 \geq \rho_{q(\cdot)}(f) = \int_{\Omega \setminus \Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})}.$$

In particular, $|f(x)| \leq 1$ almost everywhere on $\Omega_\infty^{q(\cdot)}$. Further, since $p(x) \leq q(x)$, $\Omega_\infty^{p(\cdot)} \subset \Omega_\infty^{q(\cdot)}$ up to a set of measure zero. Therefore,

$$\begin{aligned}
\rho_{p(\cdot)}(f) &= \int_{\Omega \setminus \Omega_\infty^{q(\cdot)}} |f(x)|^{p(x)}\, dx + \int_{\Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{p(\cdot)}} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty^{p(\cdot)})} \\
&\leq |\{x \in \Omega \setminus \Omega_\infty^{q(\cdot)} : |f(x)| \leq 1\}| + \int_{\Omega \setminus \Omega_\infty^{q(\cdot)}} |f(x)|^{q(x)}\, dx \\
&\qquad + |\Omega_\infty^{q(\cdot)} \setminus \Omega_\infty^{p(\cdot)}| + \|f\|_{L^\infty(\Omega_\infty^{q(\cdot)})} \\
&\leq |\Omega \setminus \Omega_\infty^{p(\cdot)}| + \rho_{q(\cdot)}(f) \\
&\leq |\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1.
\end{aligned}$$

Hence, by the convexity of the modular,

$$\rho_{p(\cdot)}\left(\frac{f}{|\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1}\right) \leq \frac{\rho_{p(\cdot)}(f)}{|\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1} \leq 1,$$

and so $\|f\|_{p(\cdot)} \leq |\Omega \setminus \Omega_\infty^{p(\cdot)}| + 1$. □

Remark 2.49. A variant of this result is used in Chap. 3 to prove norm inequalities for the maximal operator: see Lemma 3.28 below.

Corollary 2.48 is commonly applied with the stronger hypothesis $|\Omega| < \infty$. In particular, as an immediate consequence we get the following relationship between the classical and variable Lebesgue spaces on bounded domains.

Corollary 2.50. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $|\Omega| < \infty$. Then there exist constants $c_1, c_2 > 0$ such that*

$$c_1\|f\|_{p_-} \leq \|f\|_{p(\cdot)} \leq c_2\|f\|_{p_+}.$$

Finally, we give an embedding that will be very useful in applications. For $1 \leq p < q < \infty$, define

$$L^p(\Omega) + L^q(\Omega) = \big\{f = g + h : g \in L^p(\Omega), h \in L^q(\Omega)\big\};$$

this is a Banach space with norm

$$\|f\|_{L^p(\Omega)+L^q(\Omega)} = \inf_{f=g+h}\{\|g\|_{L^p(\Omega)} + \|h\|_{L^q(\Omega)}\}.$$

Theorem 2.51. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, then*

$$L^{p(\cdot)}(\Omega) \subset L^{p_+}(\Omega) + L^{p_-}(\Omega)$$

and

$$\|f\|_{L^{p_+}(\Omega)+L^{p_-}(\Omega)} \leq 2\|f\|_{L^{p(\cdot)}(\Omega)}.$$

Further, this embedding is proper if and only if $p(\cdot)$ is non-constant.

Proof. By the homogeneity of the norms we may assume without loss of generality that $\|f\|_{p(\cdot)} = 1$. This implies that $\|f\|_{L^\infty(\Omega_\infty)} \leq 1$. Decompose f as $f_1 + f_2$, where

$$f_1 = f\chi_{\{x\in\Omega:|f(x)|\leq 1\}}, \qquad f_2 = f\chi_{\{x\in\Omega\setminus\Omega_\infty:|f(x)|>1\}}. \tag{2.9}$$

If $p_+ < \infty$, $|\Omega_\infty| = 0$, so by Corollary 2.22,

$$\int_\Omega |f_1(x)|^{p_+}\,dx \leq \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx \leq \|f\|_{p(\cdot)} = 1,$$

$$\int_\Omega |f_2(x)|^{p_-}\,dx \leq \int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx \leq \|f\|_{p(\cdot)} = 1.$$

Hence,

$$\|f\|_{L^{p_+}(\Omega)+L^{p_-}(\Omega)} \leq \|f_1\|_{p_+} + \|f_2\|_{p_-} \leq 2 = 2\|f\|_{p(\cdot)}.$$

If $p_+ = \infty$, then we argue as before for f_2 and for f_1 we note that $\|f_1\|_\infty \leq 1 = \|f\|_{p(\cdot)}$.

Now assume that $p(\cdot)$ is non-constant. Then there exists q, $p_- < q < p_+$, such that $E = \{x \in \Omega : p(x) > q\}$ has positive measure. Then by (the proof of) Corollary 2.47, there exists a function $f \in L^{p_-}(\Omega) \subset L^{p_-}(\Omega) + L^{p_+}(\Omega)$ but $f \notin L^{p(\cdot)}(\Omega)$.

Conversely, if $p(\cdot)$ is constant then $p_- = p_+$ and equality clearly holds. □

Remark 2.52. In applying Theorem 2.51 we will often use the explicit decomposition $f = f_1 + f_2$ given by (2.9).

If we assume that the exponent $p(\cdot)$ is log-Hölder continuous at infinity, then we can give a different decomposition of f that reflects this fact.

Proposition 2.53. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$ and $p(\cdot) \in LH_\infty(\Omega)$. Then*

$$L^{p(\cdot)}(\Omega) \subset L^{p_\infty}(\Omega) + L^{p_-}(\Omega).$$

Proof. Fix $f \in L^{p(\cdot)}(\Omega)$. By homogeneity we may assume without loss of generality that $\|f\|_{p(\cdot)} = 1$. Decompose f as $f_1 + f_2$ as in (2.9). Then $f_2 \in L^{p_-}(\Omega)$, so it will suffice to prove that $f_1 \in L^{p_\infty}(\Omega)$. Let $q(x) = \max(p(x), p_\infty)$; then $|f_1(x)|^{q(x)} \leq |f_1(x)|^{p(x)}$. Hence, by Proposition 2.12, $f_1 \in L^{q(\cdot)}(\Omega)$. By the definition of $q(\cdot)$,

$$\frac{1}{r(x)} = \frac{1}{p_\infty} - \frac{1}{q(x)} \leq \left| \frac{1}{p_\infty} - \frac{1}{p(x)} \right|.$$

Since $p(\cdot) \in LH_\infty(\Omega)$, by Theorem 2.45 and Remark 2.46, $L^{q(\cdot)}(\Omega) \subset L^{p_\infty}(\Omega)$. This completes the proof. □

2.6 Convergence in $L^{p(\cdot)}(\Omega)$

In this section we consider three types of convergence in the variable Lebesgue spaces: convergence in modular, in norm, and in measure.

Definition 2.54. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a sequence of functions $\{f_k\} \subset L^{p(\cdot)}(\Omega)$, we say that $f_k \to f$ in modular if for some $\beta > 0$, $\rho(\beta(f - f_k)) \to 0$ as $k \to \infty$. We say that $f_k \to f$ in norm if $\|f - f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$.

In defining modular convergence it might seem more natural to assume that $\rho(f - f_k) \to \infty$. As in the definition of the norm, we introduce the constant β to preserve the homogeneity of convergence: if $f_k \to f$ in modular, then we want $2f_k \to 2f$ in modular. With this alternative definition this is not always the case.

Example 2.55. Let $\Omega = (0, 1)$ and $p(x) = 1/x$. Let $f_k = \chi_{(0,1/k)}$. Then $\rho(f_k) = 1/k \to 0$, but for all k, $\rho(2f_k) = \infty$.

We can reformulate norm convergence in a way that highlights the connection with modular convergence.

Proposition 2.56. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, the sequence $\{f_k\}$ converges to f in norm if and only if for every $\beta > 0$, $\rho(\beta(f - f_k)) \to 0$ as $k \to \infty$. In particular, convergence in norm implies convergence in modular.*

Proof. Suppose first that $f_k \to f$ in norm. Fix $\beta > 0$. Then by the homogeneity of the norm,

$$\|\beta(f - f_k)\|_{p(\cdot)} = \beta\|f - f_k\|_{p(\cdot)} \to 0.$$

Hence, by Corollary 2.22, for all k sufficiently large,

$$\rho(\beta(f - f_k)) \leq \|\beta(f - f_k)\|_{p(\cdot)} \leq 1,$$

and so $\rho(\beta(f - f_k)) \to 0$.

To prove the converse, fix $\lambda > 0$ and let $\beta = \lambda^{-1}$. Then for all k sufficiently large, $\rho((f - f_k)/\lambda) \leq 1$, and so $\|f - f_k\|_{p(\cdot)} \leq \lambda$. Since this is true for any λ, $\|f - f_k\|_{p(\cdot)} \to 0$. □

While convergence in norm implies convergence in modular, the converse does not always hold.

Example 2.57. Let $\Omega = (1, \infty)$ and $p(x) = x$. Define $f \equiv 1$ and $f_k = \chi_{(1,k)}$. Then $f_k \to f$ in modular since

$$\rho((f - f_k)/2) = \int_k^\infty 2^{-x}\, dx \to 0$$

as $k \to \infty$. On the other hand, f_k does not converge to f in norm because for all $k \geq 1$,

$$\rho(f - f_k) = \int_k^\infty 1^x\, dx = \infty,$$

which in turn implies that $\|f - f_k\|_{p(\cdot)} \geq 1$.

This example can be generalized to any space $L^{p(\cdot)}(\Omega)$ such that $\Omega \setminus \Omega_\infty$ has positive measure and $p(\cdot)$ is unbounded on $\Omega \setminus \Omega_\infty$.

Theorem 2.58. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, convergence in norm is equivalent to convergence in modular if and only if either $p_- = \infty$ or $p_+(\Omega \setminus \Omega_\infty) < \infty$.*

Proof. By Proposition 2.56, convergence in norm always implies convergence in modular. Therefore, we need only consider whether modular convergence implies norm convergence.

Suppose first that $p_- = \infty$. Then the modular and the norm are the same and the result is trivially true.

Now suppose that $p_- < \infty$ and $p_+(\Omega \setminus \Omega_\infty) < \infty$ and fix a sequence $\{f_k\}$ such that $f_k \to f$ in modular. Then there exist $\beta > 0$ such that $\rho(\beta(f - f_k)) \to 0$. Fix λ, $0 < \lambda < \beta^{-1}$. Then by Proposition 2.14,

$$\rho((f - f_k)/\lambda) \leq \left(\frac{1}{\beta\lambda}\right)^{p_+(\Omega\setminus\Omega_\infty)} \rho(\beta(f - f_k)).$$

Hence, for all k sufficiently large we have that

$$\rho\left(\frac{f - f_k}{\lambda}\right) \leq 1.$$

Equivalently, for all such k, $\|f - f_k\|_{p(\cdot)} \leq \lambda$. Since λ was arbitrary, $f_k \to f$ in norm.

Now suppose $p_- < \infty$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$. We will construct a sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ such that $\rho(f_k) \to 0$ but $\|f_k\|_{p(\cdot)} \geq 1/2$ for all k. Let $\{E_k\}$ be the sequence of sets constructed in the proof of Proposition 2.12. Define the function f by

$$f(x) = \left(\sum_{k=1}^{\infty} \frac{1}{2^k|E_k \setminus E_{k+1}|}\chi_{E_k\setminus E_{k+1}}(x)\right)^{1/p(x)},$$

and for each k let $f_k = f\chi_{E_k}$. Then for all $k \geq 1$,

$$\rho(f_k) = \sum_{j=k}^{\infty} \int_{E_j\setminus E_{j+1}} \frac{1}{2^j|E_j \setminus E_{j+1}|}\,dx = \sum_{j=k}^{\infty} 2^{-j} = 2^{-k+1};$$

hence, $f_k \in L^{p(\cdot)}(\Omega)$ and $\rho(f_k) \to 0$ as $k \to \infty$. On the other hand, for all $k \geq 1$,

$$\rho\left(\frac{f_k}{1/2}\right) = \sum_{j=k}^{\infty} \int_{E_j\setminus E_{j+1}} \frac{2^{p(x)}}{2^j|E_j \setminus E_{j+1}|}\,dx \geq \sum_{j=k}^{\infty} 2^{p_j - j} = \infty.$$

Thus, $\|f_k\|_{p(\cdot)} \geq 1/2$. This completes the proof. □

In the classical Lebesgue spaces the three ubiquitous convergence theorems are the monotone convergence theorem, Fatou's lemma, and the dominated convergence theorem. Versions of the first two are always true in variable Lebesgue spaces, but the third is only true when the exponent function is bounded. We prove each of these results in turn.

Theorem 2.59. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a sequence of non-negative functions such that f_k increases to a function f pointwise almost everywhere. Then either $f \in L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$, or $f \notin L^{p(\cdot)}(\Omega)$ and $\|f_k\|_{p(\cdot)} \to \infty$.*

Remark 2.60. If $f \notin L^{p(\cdot)}(\Omega)$, we have defined $\|f\|_{p(\cdot)} = \infty$, so in every case we may write the conclusion as $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$.

Theorem 2.59 is sometimes referred to as the Fatou property of the norm. To avoid confusion with the variable Lebesgue space version of Fatou's lemma and to stress the parallels with the classical Lebesgue spaces, we will always refer to it as the monotone convergence theorem.

Proof. Since $\{f_k\}$ is an increasing sequence, so is $\{\|f_k\|_{p(\cdot)}\}$; thus, it either converges or diverges to ∞. If $f \in L^{p(\cdot)}(\Omega)$, since $f_k \leq f$, $\|f_k\|_{p(\cdot)} \leq \|f\|_{p(\cdot)}$; otherwise, since $f_k \in L^{p(\cdot)}(\Omega)$, $\|f_k\|_{p(\cdot)} < \infty = \|f\|_{p(\cdot)}$. In either case it will suffice to show that for any $\lambda < \|f\|_{p(\cdot)}$, for all k sufficiently large $\|f_k\|_{p(\cdot)} > \lambda$.

Fix such a λ; by the definition of the norm, $\rho(f/\lambda) > 1$. Therefore, by the monotone convergence theorem on the classical Lebesgue spaces and the definition of the L^∞ norm,

$$\begin{aligned}
\rho(f/\lambda) &= \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f\|_{L^\infty(\Omega_\infty)} \\
&= \lim_{k\to\infty} \left(\int_{\Omega\setminus\Omega_\infty} \left(\frac{|f_k(x)|}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|f_k\|_{L^\infty(\Omega_\infty)}\right) \\
&= \lim_{k\to\infty} \rho(f_k/\lambda).
\end{aligned}$$

(In this calculation we allow the possibility that $\rho(f/\lambda), \rho(f_k/\lambda) = \infty$.) Hence, for all k sufficiently large, $\rho(f_k/\lambda) > 1$, and so $\|f_k\|_{p(\cdot)} > \lambda$. □

Theorem 2.61. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ is such that $f_k \to f$ pointwise almost everywhere. If*

$$\liminf_{k\to\infty} \|f_k\|_{p(\cdot)} < \infty,$$

then $f \in L^{p(\cdot)}(\Omega)$ and

$$\|f\|_{p(\cdot)} \leq \liminf_{k\to\infty} \|f_k\|_{p(\cdot)}.$$

In the classical version of Fatou's lemma it is necessary to assume that each f_k is non-negative. However, since we are taking the norm this hypothesis is not necessary in Theorem 2.61.

Proof. Define a new sequence

$$g_k(x) = \inf_{m\geq k} |f_m(x)|.$$

Then for all $m \geq k$, $g_k(x) \leq |f_m(x)|$, and so $g_k \in L^{p(\cdot)}(\Omega)$. Further, by definition $\{g_k\}$ is an increasing sequence and

$$\lim_{k\to\infty} g_k(x) = \liminf_{m\to\infty} |f_m(x)| = |f(x)|, \quad \text{a.e. } x \in \Omega.$$

Therefore, by Theorem 2.59,

$$\|f\|_{p(\cdot)} = \lim_{k\to\infty} \|g_k\|_{p(\cdot)} \le \lim_{k\to\infty} \Big(\inf_{m\ge k} \|f_m\|_{p(\cdot)} \Big) = \liminf_{k\to\infty} \|f_k\|_{p(\cdot)} < \infty,$$

and $f \in L^{p(\cdot)}(\Omega)$. □

Theorem 2.62. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. If the sequence $\{f_k\}$ is such that $f_k \to f$ pointwise almost everywhere, and there exists $g \in L^{p(\cdot)}(\Omega)$ such that $|f_k(x)| \le g(x)$ almost everywhere, then $f \in L^{p(\cdot)}(\Omega)$ and $\|f - f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$.*

Further, if $p_+ = \infty$, then this result is always false.

Remark 2.63. It follows at once from the triangle inequality that the dominated convergence theorem implies that $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$.

As an immediate corollary to the dominated convergence theorem we can give a stronger version of the monotone convergence theorem.

Corollary 2.64. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $p_+ < \infty$, suppose the sequence of non-negative functions f_k increases pointwise almost everywhere to a function $f \in L^{p(\cdot)}(\Omega)$. Then $\|f - f_k\|_{p(\cdot)} \to 0$.*

Proof of Theorem 2.62. Suppose first that $p_+ < \infty$. Then by Proposition 2.12,

$$|f(x) - f_k(x)|^{p(x)} \le 2^{p(x)-1}(|f(x)|^{p(x)} + |f_k(x)|^{p(x)}) \le 2^{p_+}|g(x)|^{p(x)} \in L^1(\Omega).$$

Therefore, by the classical dominated convergence theorem, $\rho(f - f_k) \to 0$ as $k \to 0$, and so by Theorem 2.58, $\|f - f_k\|_{p(\cdot)} \to 0$.

Now suppose that $p_+ = \infty$; then either $|\Omega_\infty| = 0$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$, or $|\Omega_\infty| > 0$. In the first case, let f and $\{f_k\}$ be the functions constructed in the second half of the proof of Theorem 2.58. Then $f(\cdot)^{p(\cdot)} \in L^1(\Omega)$, so $f \in L^{p(\cdot)}(\Omega)$. Further, $f_k \le f$ and $f_k \to 0$ pointwise. However, $\|f_k\|_{p(\cdot)} \ge 1/2$, so the dominated convergence theorem does not hold.

If $|\Omega_\infty| > 0$, let $\{E_k\}$ be a sequence of sets such that for each k, $|E_k| > 0$ and $E_{k+1} \subset E_k \subset \Omega_\infty$, and $|E_k| \to 0$ as $k \to \infty$. Let $f_k = \chi_{E_k}$; then $f_k \le f_1$ and $f_k \to 0$ pointwise, but $\|f_k\|_{p(\cdot)} = \|f_k\|_\infty = 1$. □

As in the classical Lebesgue spaces, norm convergence need not imply that the sequence converges pointwise almost everywhere unless $p_- = \infty$.

Example 2.65. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $|\Omega \setminus \Omega_\infty| > 0$, then there exists a sequence $\{f_k\}$ in $L^{p(\cdot)}(\Omega)$ such that $f_k \to 0$ in norm but not pointwise almost everywhere.

Proof. Since $|\Omega \setminus \Omega_\infty| > 0$, there exists a set $E \subset \Omega \setminus \Omega_\infty$ such that $0 < |E| < \infty$ and $p_+(E) < \infty$. Form a "dyadic" decomposition of E as follows. Let $E = E_1^1 \cup E_2^1$, where the sets E_1^1 and E_2^1 are disjoint and have measure $|E|/2$. Repeat this decomposition. Then by induction, we get a collection of sets $\{E_j^i : i \geq 1, 1 \leq j \leq 2^i\}$ such that for each i, the sets E_j^i are pairwise disjoint, $E = \bigcup_{j=1}^{2^i} E_j^i$, and $|E_j^i| = |E|/2^i$. Define the collection of functions $\{g_j^i\}$ by $g_j^i = \chi_{E_j^i}$. Then by Corollary 2.50,

$$\|g_j^i\|_{L^{p(\cdot)}(\Omega)} = \|g_j^i\|_{L^{p(\cdot)}(E)} \leq C\|g_j^i\|_{p_+(E)} = C(|E|/2^i)^{1/p_+(E)}. \tag{2.10}$$

Define the sequence $\{f_k\}$ by $\{g_1^1, g_2^1, g_1^2, g_2^2, g_3^2, g_4^2, \ldots\}$. Then (2.10) shows that $\|f_k\|_{p(\cdot)} \to 0$ as $k \to \infty$. On the other hand, given any point $x \in E$, for every i there exists j such that $x \in E_j^i$, so there exists an infinite number of functions g_j^i such that $g_j^i(x) = 1$. Thus the sequence $\{f_k\}$ does not converge to 0 pointwise almost everywhere. □

Despite this example, we can always find a subsequence of a norm convergent sequence that converges pointwise almost everywhere. To show this we will consider the slightly stronger property of convergence in measure. Given a domain Ω and a sequence of functions $\{f_k\}$, recall that $f_k \to f$ in measure if for every $\epsilon > 0$, there exists $K > 0$ such that if $k \geq K$,

$$|\{x \in \Omega : |f(x) - f_k(x)| \geq \epsilon\}| < \epsilon.$$

If $\{f_k\}$ converges to f in measure, then there exists a subsequence $\{f_{k_j}\}$ that converges to f pointwise almost everywhere. (See Royden [301].) Norm convergence implies convergence in measure in the classical Lebesgue spaces, and the same is true for variable Lebesgue spaces.

Theorem 2.66. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ converges to f in norm, then it converges to f in measure.*

Proof. Suppose to the contrary that there exists a sequence $\{f_k\}$ that converges to f in norm but not in measure. Then by passing to a subsequence we may assume that there exists ϵ, $0 < \epsilon < 1$, such that for all k,

$$|\{x \in \Omega : |f(x) - f_k(x)| \geq \epsilon\}| \geq \epsilon.$$

Denote the set on the left-hand side by A_k; since for each k either $|A_k \cap \Omega_\infty| \geq \epsilon/2$ or $|A_k \setminus \Omega_\infty| \geq \epsilon/2$, by passing to another subsequence we may assume that one of these inequalities holds for all k.

If $|A_k \cap \Omega_\infty| \geq \epsilon/2$ for all k, then

$$\|f - f_k\|_{p(\cdot)} \geq \|(f - f_k)\chi_{\Omega_\infty}\|_{p(\cdot)} = \|f - f_k\|_{L^\infty(\Omega_\infty)} \geq \epsilon,$$

contradicting our assumption that f_k converges to f in norm. If $|A_k \setminus \Omega_\infty| \geq \epsilon/2$ for all k, then

$$\rho\left(\frac{f-f_k}{\epsilon^2/2}\right) \geq \int_{\Omega\setminus\Omega_\infty} \left(\frac{|f(x)-f_k(x)|}{\epsilon^2/2}\right)^{p(x)} dx$$
$$\geq \int_{A_k\setminus\Omega_\infty} \left(\frac{2}{\epsilon}\right)^{p(x)} dx \geq \left(\frac{2}{\epsilon}\right)^{p-} |A_k \setminus \Omega_\infty| \geq 1.$$

Hence, $\|f - f_k\|_{p(\cdot)} \geq \epsilon^2/2 > 0$, again contradicting our assumption that f_k converges to f in norm. □

As an immediate corollary we get that every norm convergent sequence has a subsequence that converges pointwise almost everywhere. We record this fact as part of a somewhat stronger result which is a partial converse to the dominated convergence theorem.

Proposition 2.67. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ converges in norm to $f \in L^{p(\cdot)}(\Omega)$. Then there exists a subsequence $\{f_{k_j}\}$ and $g \in L^{p(\cdot)}(\Omega)$ such that the subsequence converges pointwise almost everywhere to f, and for almost every $x \in \Omega$, $|f_{k_j}(x)| \leq g(x)$.*

Proof. By Theorem 2.66 we immediately have the existence of a subsequence $\{f_{k_j}\}$ that converges pointwise almost everywhere to f. Further, since convergent sequences are Cauchy sequences, we may choose the k_j large enough that for each j, $\|f_{k_{j+1}} - f_{k_j}\|_{p(\cdot)} \leq 2^{-j}$. For simplicity, we will write f_j instead of f_{k_j}.

For each j, define the function h_j by

$$h_j(x) = \sum_{i=1}^{j-1} |f_{i+1}(x) - f_i(x)|.$$

Then $\{h_j\}$ is an increasing sequence and so converges pointwise to a function h. By our choice of the functions f_j,

$$\|h_j\|_{p(\cdot)} \leq \sum_{i=1}^{j-1} 2^{-i} \leq 1.$$

Hence, by the monotone convergence theorem (Theorem 2.59), $h \in L^{p(\cdot)}(\Omega)$. But then, for every j and almost every $x \in \Omega$,

$$|f_j(x) - f_1(x)| \leq \sum_{i=1}^{j-1} |f_{i+1} - f_i(x)| = h_j(x) \leq h(x).$$

Thus, if we let $g = h + |f_1|$, we have that $g \in L^{p(\cdot)}(\Omega)$ and $|f_j(x)| \le g(x)$ almost everywhere. □

We conclude this section by considering more carefully the relationship between convergence in norm, convergence in modular and convergence in measure.

Theorem 2.68. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, if $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ is such that $\|f_k\|_{p(\cdot)} \to 0$ (or ∞), then the sequence $\rho(f_k) \to 0$ (or ∞). The converse holds if and only if $p_+(\Omega \setminus \Omega_\infty) < \infty$.*

Proof. Suppose first that $\|f_k\|_{p(\cdot)} \to 0$ (or ∞). Then the fact that $\rho(f_k) \to 0$ (or ∞) follows immediately from Corollary 2.22.

Now suppose that $p_+(\Omega \setminus \Omega_\infty) < \infty$. Given a sequence $\{f_k\}$ such that $\rho(f_k) \to 0$, there exists a sequence $\{a_k\}$ such that $a_k \le 1$, $a_k \to 0$, but $a_k^{-p_+(\Omega\setminus\Omega_\infty)}\rho(f_k) \le 1$. Then by Proposition 2.14,

$$\rho(f_k/a_k) \le a_k^{-p_+(\Omega\setminus\Omega_\infty)}\rho(f_k) \le 1.$$

Therefore, $\|f_k\|_{p(\cdot)} \le a_k$ and so $\|f_k\|_{p(\cdot)} \to 0$.

If $\rho(f_k) \to \infty$, then the proof is nearly the same: there exists a sequence $\{a_k\}$ such that $a_k \ge 1$, $a_k \to \infty$ but such that, again by Proposition 2.14,

$$\rho(f_k/a_k) \ge a_k^{-p_+(\Omega\setminus\Omega_\infty)}\rho(f_k) > 1,$$

and so $\|f_k\|_{p(\cdot)} \ge a_k$.

Now suppose that $p_+(\Omega \setminus \Omega_\infty) = \infty$; we will show that neither convergence result holds. First, the example constructed in Theorem 2.58 shows that there is always a sequence $\{f_k\}$ such that $\rho(f_k) \to 0$ but $\|f_k\|_{p(\cdot)} \ge 1/2$. For the other case, form the sets $\{E_k\}$ as in the proof of Proposition 2.12 and define

$$f_k(x) = \left(\sum_{j=1}^{k} \frac{1}{|E_j \setminus E_{j+1}|}\chi_{E_j\setminus E_{j+1}}(x)\right)^{1/p(x)}.$$

Then arguing as in that proof, we have $\rho(f_k) = k$ but

$$\rho(f_k/2) = \sum_{j=1}^{k} \fint_{E_j\setminus E_{j+1}} 2^{-p(x)}\,dx \le \sum_{j=1}^{\infty} 2^{-j} = 1.$$

Hence, $\rho(f_k) \to \infty$ but $\|f_k\|_{p(\cdot)} \le 2$. □

Theorem 2.69. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. Then for $f \in L^{p(\cdot)}(\Omega)$ and a sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$, the following are equivalent:*

1. *$f_k \to f$ in norm,*

2. *$f_k \to f$ in modular,*
3. *$f_k \to f$ in measure and for some $\gamma > 0$, $\rho(\gamma f_k) \to \rho(\gamma f)$.*

Proof. The equivalence of (1) and (2) was proved in Theorem 2.58; here we will prove the equivalence of (2) and (3).

To show that (2) implies (3), first note that by Theorem 2.66 norm convergence implies convergence in measure, so modular convergence also implies convergence in measure. To complete the proof of this implication we will show that convergence in modular implies that $\rho(\gamma f_k) \to \rho(\gamma f)$ for $\gamma = 1$.

We begin with an elementary inequality. By the mean value theorem, if $1 \le p < \infty$ and $a, b \ge 0$, then

$$|a^p - b^p| \le p \max(a^{p-1}, b^{p-1})|a-b| \le p(a^{p-1} + b^{p-1})|a-b|.$$

Therefore,

$$\begin{aligned}|\rho(f) - \rho(f_k)| &\le \int_\Omega \big| |f(x)|^{p(x)} - |f_k(x)|^{p(x)} \big|\, dx \\ &\le p_+ \int_\Omega \big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\, dx.\end{aligned}$$

To estimate the right-hand side we write the domain of integration as $\Omega_1 \cup \Omega_*$. The integral on Ω_1 is straightforward to estimate:

$$\begin{aligned}p_+ \int_{\Omega_1} &\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\, dx \\ &= 2p_+ \int_{\Omega_1} |f(x) - f_k(x)|^{p(x)}\, dx \le 2p_+\rho(f - f_k).\end{aligned}$$

Since modular convergence and norm convergence are equivalent, by Proposition 2.56 the right-hand side tends to 0 as $k \to \infty$.

To estimate the integral on Ω_*, fix ϵ, $0 < \epsilon < 1/4$, and apply Young's inequality to get

$$\begin{aligned}p_+ \int_{\Omega_*} &\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\, dx \\ &\le p_+ \int_{\Omega_*} \frac{\epsilon^{p'(x)}}{p'(x)} \big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)^{p'(x)}\, dx \\ &\quad + p_+ \int_{\Omega_*} \frac{\epsilon^{-p(x)}}{p(x)} |f(x) - f_k(x)|^{p(x)}\, dx \\ &= I_1 + I_2.\end{aligned}$$

We estimate I_1 and I_2 separately. Since $p(x) > 1$ for all $x \in \Omega_*$,

$$I_2 \le p_+ \rho(\epsilon^{-1}(f - f_k)).$$

To estimate I_1 we need two additional inequalities: for $p > 0$ and $a, b > 0$, we have by elementary calculus that

$$a^p + b^p \le \max(1, 2^{1-p})(a+b)^p,$$
$$(a+b)^p \le \max(1, 2^{p-1})(a^p + b^p).$$

Hence, since $1 < p'(x) < \infty$ on Ω_*,

$$\begin{aligned}
I_1 &\le p_+ \int_{\Omega_*} \epsilon^{p'(x)} \max(1, 2^{2-p(x)})^{p'(x)} \big(|f(x)| + |f_k(x)|\big)^{p(x)}\, dx \\
&\le p_+ \int_{\Omega_*} (4\epsilon)^{p'(x)} \big(2|f(x)| + |f(x) - f_k(x)|\big)^{p(x)}\, dx \\
&\le 4\epsilon p_+ \int_{\Omega_*} 2^{p(x)-1} \big(2^{p(x)}|f(x)|^{p(x)} + |f(x) - f_k(x)|^{p(x)}\big)\, dx \\
&\le \epsilon p_+ 2^{2p_+ + 1} \rho(f) + p_+ \epsilon 2^{p_+ + 1} \rho(f - f_k).
\end{aligned}$$

Combining this with the previous estimate, we see that

$$\begin{aligned}
&p_+ \int_{\Omega_*} \big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big) |f(x) - f_k(x)|\, dx \\
&\qquad \le \epsilon p_+ 2^{2p_+ + 1} \rho(f) + \epsilon p_+ 2^{p_+ + 1} \rho(f - f_k) + p_+ \rho(\epsilon^{-1}(f - f_k)).
\end{aligned}$$

Therefore, by Proposition 2.56,

$$\begin{aligned}
&\limsup_{k\to\infty} p_+ \int_{\Omega_*} \big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big) |f(x) - f_k(x)|\, dx \\
&\qquad \le \epsilon p_+ 2^{2p_+ + 1} \rho(f).
\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, we conclude that $|\rho(f) - \rho(f_k)| \to 0$.

Now suppose that $f_k \to f$ in measure and that for some $\gamma > 0$, $\rho(\gamma f_k) \to \rho(\gamma f)$. Since we also have that $\gamma f_k \to \gamma f$ in measure, we may assume without loss of generality that $\gamma = 1$. Then for each ϵ, $0 < \epsilon < 1$,

$$\begin{aligned}
|\{x \in \Omega : |f(x) - f_k(x)|^{p(x)} > \epsilon\}| &\le |\{x \in \Omega : |f(x) - f_k(x)| > \epsilon^{1/p_-}\}| \\
&\le |\{x \in \Omega : |f(x) - f_k(x)| > \epsilon\}| \le \epsilon.
\end{aligned}$$

Hence, $|f(\cdot) - f_k(\cdot)|^{p(\cdot)} \to 0$ in measure.

Further, arguing as we did above, we have that

$$\begin{aligned}
&\big||f(x)|^{p(x)} - |f_k(x)|^{p(x)}\big| \qquad (2.11)\\
&\quad \le p_+\big(|f(x)|^{p(x)-1} + |f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\\
&\quad \le p_+|f(x)|^{p(x)-1}|f(x) - f_k(x)|\\
&\qquad + p_+ \max(1, 2^{p(x)-2})\\
&\qquad\quad \times \big(|f(x)|^{p(x)-1} + |f(x) - f_k(x)|^{p(x)-1}\big)|f(x) - f_k(x)|\\
&\quad \le p_+(2^{p_+} + 1)|f(x)|^{p(x)-1}|f(x) - f_k(x)| + p_+2^{p_+}|f(x) - f_k(x)|^{p(x)}.
\end{aligned}$$

Now fix ϵ, $0 < \epsilon < 1$. Since $|f(\cdot)|^{p(\cdot)} \in L^1(\Omega)$, there exists $M \ge 1$ such that

$$|\{x : |f(x)|^{p(x)-1} > M\}| \le |\{x : |f(x)|^{p(x)} > M\}| \le \epsilon/2.$$

By inequality (2.11), since $f_k \to f$ and $|f(\cdot) - f_k(\cdot)|^{p(\cdot)} \to 0$ in measure, for all k sufficiently large,

$$\begin{aligned}
&|\{x : \big||f(x)|^{p(x)} - |f_k(x)|^{p(x)}\big| > \epsilon\}|\\
&\quad \le |\{x : |f(x)|^{p(x)-1} > M\}|\\
&\qquad + |\{x : p_+(2^{p_+} + 1)M|f(x) - f_k(x)| > \epsilon/2\}|\\
&\qquad + |\{x : p_+2^{p_+}|f(x) - f_k(x)|^{p(x)} > \epsilon/2\}|\\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2p_+(2^{p_+} + 1)M} + \frac{\epsilon}{p_+2^{p_+ + 1}}\\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4}\\
&\quad = \epsilon.
\end{aligned}$$

Therefore, $|f_k(\cdot)|^{p(\cdot)} \to |f(\cdot)|^{p(\cdot)}$ in measure.

Now define

$$h_k(x) = 2^{p_+-1}|f_k(x)|^{p(x)} + 2^{p_+-1}|f(x)|^{p(x)} - |f(x) - f_k(x)|^{p(x)} \ge 0;$$

then $h_k \to 2^{p_+}|f(\cdot)|^{p(\cdot)}$ in measure. Therefore, by Fatou's lemma on the classical Lebesgue spaces with respect to convergence in measure (see Royden [301]),

$$\begin{aligned}
&2^{p_+}\int_\Omega |f(x)|^{p(x)}\,dx\\
&\quad \le \liminf_{k\to\infty}\int_\Omega 2^{p_+-1}|f_k(x)|^{p(x)} + 2^{p_+-1}|f(x)|^{p(x)} - |f(x) - f_k(x)|^{p(x)}\,dx.
\end{aligned}$$

Rearranging terms and using the fact that $\rho(f_k) \to \rho(f)$ we get that

$$\limsup_{k\to\infty} \int_\Omega |f(x) - f_k(x)|^{p(x)}\, dx \le 0.$$

Therefore, $f_k \to f$ in modular and our proof is complete. □

2.7 Completeness and Dense Subsets of $L^{p(\cdot)}(\Omega)$

In this section we prove that $L^{p(\cdot)}(\Omega)$ is a Banach space—that is, a complete normed vector space—and determine some canonical dense subsets of $L^{p(\cdot)}(\Omega)$. Since we proved that $L^{p(\cdot)}(\Omega)$ is a normed vector space in Sect. 2.3, to see that it is a Banach space we only have to show that it is complete.

Our proof of completeness follows one of the standard proofs for classical Lebesgue spaces and so makes heavy use of the convergence theorems proved in the previous section. We begin with a result that is of independent interest and is referred to as the Riesz-Fischer property.

Theorem 2.70. *Given Ω and $p(\cdot) \in L^{p(\cdot)}(\Omega)$, let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be such that*

$$\sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

Then there exists $f \in L^{p(\cdot)}(\Omega)$ such that

$$\sum_{k=1}^{i} f_k \to f$$

in norm as $i \to \infty$, and

$$\|f\|_{p(\cdot)} \le \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)}.$$

Proof. Define the function F on Ω by

$$F(x) = \sum_{k=1}^{\infty} |f_k(x)|,$$

and define the sequence $\{F_i\}$ by

$$F_i(x) = \sum_{k=1}^{i} |f_k(x)|.$$

Then the sequence $\{F_i\}$ is non-negative and increases pointwise almost everywhere to F. Further, for each i, $F_i \in L^{p(\cdot)}(\Omega)$, and its norm is uniformly bounded, since

$$\|F_i\|_{p(\cdot)} \leq \sum_{k=1}^{i} \|f_k\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

Therefore, by the monotone convergence theorem (Theorem 2.59), $F \in L^{p(\cdot)}(\Omega)$.

In particular, by Remark 2.10, F is finite almost everywhere, so the sequence $\{F_i\}$ converges pointwise almost everywhere. Hence, if we define the sequence of functions $\{G_i\}$ by

$$G_i(x) = \sum_{k=1}^{i} f_k(x),$$

then this sequence also converges pointwise almost everywhere since absolute convergence implies convergence. Denote its sum by f.

Now let $G_0 = 0$; then for any $j \geq 0$, $G_i - G_j \to f - G_j$ pointwise almost everywhere. Furthermore,

$$\liminf_{i\to\infty} \|G_i - G_j\|_{p(\cdot)} \leq \liminf_{i\to\infty} \sum_{k=j+1}^{i} \|f_k\|_{p(\cdot)} = \sum_{k=j+1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

By Fatou's lemma (Theorem 2.61), if we take $j = 0$, then

$$\|f\|_{p(\cdot)} \leq \liminf_{i\to\infty} \|G_i\|_{p(\cdot)} \leq \sum_{k=1}^{\infty} \|f_k\|_{p(\cdot)} < \infty.$$

More generally, for each j the same argument shows that

$$\|f - G_j\|_{p(\cdot)} \leq \liminf_{i\to\infty} \|G_i - G_j\|_{p(\cdot)} \leq \sum_{k=j+1}^{\infty} \|f_k\|_{p(\cdot)};$$

since the sum on the right-hand side tends to 0, we see that $G_j \to f$ in norm, which completes the proof. □

The completeness of $L^{p(\cdot)}(\Omega)$ now follows from the Riesz-Fischer property.

Theorem 2.71. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is complete: every Cauchy sequence in $L^{p(\cdot)}(\Omega)$ converges in norm.*

Proof. Let $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ be a Cauchy sequence. Choose k_1 such that $\|f_i - f_j\|_{p(\cdot)} < 2^{-1}$ for $i, j \geq k_1$, choose $k_2 > k_1$ such that $\|f_i - f_j\|_{p(\cdot)} < 2^{-2}$ for $i, j \geq k_2$, and so on. This construction yields a subsequence $\{f_{k_j}\}$, $k_{j+1} > k_j$, such that

$$\|f_{k_{j+1}} - f_{k_j}\|_{p(\cdot)} < 2^{-j}.$$

Define a new sequence $\{g_j\}$ by $g_1 = f_{k_1}$ and for $j > 1$, $g_j = f_{k_j} - f_{k_{j-1}}$. Then for all j we get the telescoping sum

$$\sum_{i=1}^{j} g_i = f_{k_j};$$

further, we have that

$$\sum_{j=1}^{\infty} \|g_j\|_{p(\cdot)} \leq \|f_{k_1}\|_{p(\cdot)} + \sum_{j=1}^{\infty} 2^{-j} < \infty.$$

Therefore, by the Riesz-Fischer property (Theorem 2.70), there exists $f \in L^{p(\cdot)}(\Omega)$ such that $f_{k_j} \to f$ in norm.

Finally, by the triangle inequality we have that

$$\|f - f_k\|_{p(\cdot)} \leq \|f - f_{k_j}\|_{p(\cdot)} + \|f_{k_j} - f_k\|_{p(\cdot)};$$

since $\{f_k\}$ is a Cauchy sequence, for k sufficiently large we can choose k_j to make the right-hand side as small as desired. Hence, $f_k \to f$ in norm. □

We now consider the question of dense subsets of $L^{p(\cdot)}(\Omega)$. To simplify our exposition, we will assume that all domains Ω are open.

Theorem 2.72. *Given an open set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose that $p_+ < \infty$. Then the set of bounded functions of compact support with* $\operatorname{supp}(f) \subset \Omega$ *is dense in $L^{p(\cdot)}(\Omega)$.*

Proof. Let K_k be a nested sequence of compact subsets of Ω such that $\Omega = \bigcup_k K_k$. (For instance, let $K_k = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 1/k\} \cap \overline{B_k(0)}$.) Fix $f \in L^{p(\cdot)}(\Omega)$ and define the sequence $\{f_k\}$ by

$$f_k(x) = \begin{cases} k & f_k(x) > k \\ f(x) & -k \leq f(x) \leq k \\ -k & f_k(x) < -k, \end{cases}$$

and let $g_k(x) = f_k(x)\chi_{K_k}(x)$. Since f is finite almost everywhere, $g_k \to f$ pointwise almost everywhere; since $f \in L^{p(\cdot)}(\Omega)$ and $|g_k(x)| \leq |f(x)|$, $g_k \in L^{p(\cdot)}(\Omega)$. Therefore, since $p_+ < \infty$, by the dominated convergence theorem (Theorem 2.62), $g_k \to f$ in norm. □

As a corollary to Theorem 2.72 we get two additional dense subsets. Recall that $C_c(\Omega)$ denotes the set of all continuous functions whose support is compact and contained in Ω. We define $S(\Omega)$ to be the collection of all simple functions, that is, functions whose range is finite: $s \in S(\Omega)$ if

$$s(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x),$$

where the numbers a_j are distinct and the sets $E_j \subset \Omega$ are pairwise disjoint. The family $S_0(\Omega)$ is the collection of $s \in S$ with the additional property that

$$\left| \bigcup_{j=1}^{n} E_j \right| < \infty.$$

Corollary 2.73. *Given an open set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose $p_+ < \infty$. Then the sets $C_c(\Omega)$ and $S_0(\Omega)$ are dense in $L^{p(\cdot)}(\Omega)$.*

Proof. We will prove this for $C_c(\Omega)$; the proof for $S_0(\Omega)$ is identical. Fix $f \in L^{p(\cdot)}(\Omega)$ and fix $\epsilon > 0$; we will find a function $h \in C_c(\Omega)$ such that $\|f - h\|_{p(\cdot)} < \epsilon$.

By Theorem 2.72 there exists a bounded function of compact support, g, such that $\|f - g\|_{p(\cdot)} < \epsilon/2$. Let $\operatorname{supp}(g) \subset B \cap \Omega$ for some open ball B. Then since $p_+ < \infty$, $C_c(B \cap \Omega)$ is dense in $L^{p_+}(B \cap \Omega)$; thus there exists $h \in C_c(B \cap \Omega) \subset C_c(\Omega)$ such that

$$\|g - h\|_{L^{p_+}(\Omega)} = \|g - h\|_{L^{p_+}(B\cap\Omega)} < \frac{\epsilon}{2(1 + |B \cap \Omega|)}.$$

Therefore, by Corollary 2.48,

$$\|g - h\|_{L^{p(\cdot)}(\Omega)} = \|g - h\|_{L^{p(\cdot)}(B\cap\Omega)} \le (1 + |B \cap \Omega|)\|g - h\|_{L^{p_+}(B\cap\Omega)} < \epsilon/2,$$

and so

$$\|f - h\|_{p(\cdot)} \le \|f - g\|_{p(\cdot)} + \|g - h\|_{p(\cdot)} < \epsilon.$$

□

Remark 2.74. If $p_+ < \infty$, then the set $\bigcap_{p>1} L^p(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$ since this intersection contains $C_c(\Omega)$. This observation will be useful in Chap. 5 below.

Theorem 2.72 need not be true if $p_+ = \infty$. This is clearly the case if Ω_∞ is open and $|\Omega_\infty| > 0$, since bounded functions of compact support with $\operatorname{supp}(f) \subset \Omega_\infty$ are not dense in $L^\infty(\Omega_\infty)$. But it still fails even if $p(\cdot)$ is simply unbounded. First, we will show that bounded functions are not dense, and then show that under certain conditions functions of compact support are not dense.

Theorem 2.75. *Given Ω open and $p(\cdot) \in \mathcal{P}(\Omega)$, if $p_+(\Omega \setminus \Omega_\infty) = \infty$, then bounded functions are not dense in $L^{p(\cdot)}(\Omega)$.*

Remark 2.76. It follows from Theorem 2.75 that if $p_+(\Omega \setminus \Omega_\infty) = \infty$, then $C_c(\Omega)$ and $S_0(\Omega)$ are not dense in $L^{p(\cdot)}(\Omega)$.

Proof. We will construct a function $f \in L^{p(\cdot)}(\Omega)$ that cannot be approximated by bounded functions. To do so we will modify the construction given in the proof of Proposition 2.12.

Since $p_+(\Omega \setminus \Omega_\infty) = \infty$, there exists an increasing sequence $\{p_i\}$, $p_i > i$, such that the sets

$$F_i = \{x \in \Omega \setminus \Omega_\infty : p_i < p(x) < p_{i+1}\}$$

have positive measure. For each i, choose $G_i \subset F_i$ such that

$$0 < |G_i| < \left(\frac{1}{2^i}\right)^{p_{i+1}} < 1.$$

Then for all $\lambda > 0$,

$$\begin{aligned}\rho(\chi_{G_i}/\lambda) = \int_{\Omega\setminus\Omega_\infty} \left(\frac{\chi_{G_i}(x)}{\lambda}\right)^{p(x)} dx + \lambda^{-1}\|\chi_{G_i}\|_{L^\infty(\Omega_\infty)} \\ = \int_{G_i} \lambda^{-p(x)}\, dx \le |G_i| \max(\lambda^{-p_i}, \lambda^{-p_{i+1}}).\end{aligned}$$

Hence,

$$\begin{aligned}\|\chi_{G_i}\|_{p(\cdot)} &\le \inf\{\lambda > 0 : |G_i| \max(\lambda^{-p_i}, \lambda^{-p_{i+1}}) \le 1\} \\ &\le \inf\{\lambda > 0 : |G_i| \le \min(\lambda^{p_i}, \lambda^{p_{i+1}})\} \\ &\le \max(|G_i|^{1/p_i}, |G_i|^{1/p_{i+1}}) = |G_i|^{1/p_{i+1}} < 2^{-i}.\end{aligned}$$

Now define the sets $\{E_k\}$ by

$$E_k = \bigcup_{i=k}^{\infty} G_i.$$

Then we have that

1. $E_k \subset \Omega \setminus \Omega_\infty$;
2. $E_{k+1} \subset E_k$ and $|E_k \setminus E_{k+1}| = |G_k| > 0$;
3. $|E_k| \to 0$ since
$$|E_k| = \sum_{i=k}^{\infty} |G_i| < \sum_{i=k}^{\infty} (2^{-i})^{p_{i+1}};$$
4. If $x \in E_k$, then $p(x) \ge p_k > k$;
5. $\|\chi_{E_k}\|_{p(\cdot)} \to 0$ since
$$\|\chi_{E_k}\|_{p(\cdot)} \le \sum_{i=k}^{\infty} \|\chi_{G_i}\|_{p(\cdot)} < \sum_{i=k}^{\infty} 2^{-i}.$$

Properties (1)–(4) are exactly the properties from the proof of Proposition 2.12 used in the proof of Theorem 2.58 to construct the function f and show that $f \in L^{p(\cdot)}(\Omega)$ and $\|f\chi_{E_k}\|_{p(\cdot)} \geq 1/2$; repeat this construction using these sets.

For any $h \in L^\infty(\Omega)$, by Property (5) fix k sufficiently large such that

$$\|h\chi_{E_k}\|_{p(\cdot)} \leq \|h\|_{L^\infty}\|\chi_{E_k}\|_{p(\cdot)} < \frac{1}{4}.$$

Then by the triangle inequality we have that

$$\|f-h\|_{p(\cdot)} \geq \|(f-h)\chi_{E_k}\|_{p(\cdot)} \geq \|f\chi_{E_k}\|_{p(\cdot)} - \|h\chi_{E_k}\|_{p(\cdot)} \geq \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

Since h is an arbitrary bounded function, we see that bounded functions are not dense in $L^{p(\cdot)}(\Omega)$. □

Intuitively, the next result shows that if $p(\cdot)$ is unbounded at the boundary of Ω, then functions of compact support are not dense.

Theorem 2.77. *Given Ω open and $p(\cdot) \in \mathcal{P}(\Omega)$, suppose that for every compact set $K \subset \Omega$, $p_+(\Omega \setminus K) = \infty$. Then functions with compact support and* $\operatorname{supp}(f) \subset \Omega$ *are not dense in $L^{p(\cdot)}(\Omega)$.*

Proof. Define the sequence $K_k = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) \geq 1/k\} \cap \overline{B_k(0)}$. Then the sets K_k are compact, nested, and their union is Ω. By our hypothesis there exists a sequence of disjoint sets $E_k \subset \Omega \setminus K_k$ such that $|E_k| > 0$ and $p_-(E_k) > k$. Let $E_k^* = E_k \setminus \Omega_\infty$ and $E_k^\infty = E_k \cap \Omega_\infty$. By passing to a subsequence and renumbering, we may assume without loss of generality that either $|E_k^\infty| > 0$ for every k or $|E_k^*| > 0$ for every k. In the first case, define

$$f(x) = \sum_{k=1}^\infty \chi_{E_k^\infty}(x).$$

Since the sets E_k^∞ are disjoint, $f \in L^\infty(\Omega_\infty) \subset L^{p(\cdot)}(\Omega)$. Further, given any function g such that $\operatorname{supp}(g)$ is compact and contained in Ω, there exists k_0 such that $\operatorname{supp}(g) \subset K_{k_0}$. But then,

$$\|f-g\|_{p(\cdot)} \geq \|\chi_{E_{k_0+1}^\infty}\|_{p(\cdot)} = \|\chi_{E_{k_0+1}^\infty}\|_\infty = 1.$$

Hence, functions of compact support are not dense.

If, on the other hand, $|E_k^*| > 0$ for every k, define

$$f(x) = \sum_{k=1}^\infty |E_k^*|^{-1/p(x)}\chi_{E_k^*}(x).$$

Then for any $\lambda > 1$,

$$\rho(f/\lambda) = \sum_{k=1}^{\infty} \int_{E_k^*} \lambda^{-p(x)}\,dx \le \sum_{k=1}^{\infty} \lambda^{-k} < \infty.$$

Thus $f \in L^{p(\cdot)}(\Omega)$. But given g as before,

$$\rho(f-g) \ge \sum_{k=k_0+1}^{\infty} \int_{E_k^*} f(x)^{p(x)}\,dx = \sum_{k=k_0+1}^{\infty} 1 = \infty.$$

Therefore, $\|f-g\|_{p(\cdot)} \ge 1$, so again functions of compact support are not dense in $L^{p(\cdot)}(\Omega)$. □

We conclude this section with an important characterization of the dense subsets of $L^{p(\cdot)}$. Recall that a Banach space is separable if it has a countable dense subset.

Theorem 2.78. *Given an open set Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, then $L^{p(\cdot)}(\Omega)$ is separable if and only if $p_+ < \infty$.*

Proof. Suppose first that $p_+ < \infty$. Then the proof of separability is almost identical to the proof of Corollary 2.73 so we sketch only the key details. We can write

$$\Omega = \bigcup_{k=1}^{\infty} B_k(0) \cap \Omega.$$

Since $B_k(0) \cap \Omega$ is open, $L^{p_+}(B_k(0) \cap \Omega)$ is separable and so contains a countable dense subset. The union of all of these sets is a countable set contained in $L^{p(\cdot)}(\Omega)$. Arguing exactly as we did before we see that this set is also dense in $L^{p(\cdot)}(\Omega)$.

Now suppose that $p_+ = \infty$. We will show that no countable set is dense. If $|\Omega_\infty| > 0$, then this follows from the same construction that shows that $L^\infty(\Omega_\infty)$ is non-separable, since the restriction of any dense subset of $L^{p(\cdot)}(\Omega)$ will be dense in $L^\infty(\Omega_\infty)$. (See, for example, Brezis [37].)

Now let $|\Omega_\infty| = 0$ and $p_+(\Omega \setminus \Omega_\infty) = \infty$, and suppose to the contrary that there exists a countable dense set $\{h_j\}$. Let the sets E_k and the function f be as in the proof of Theorem 2.75. Then for all k, $\|f\chi_{E_k}\|_{p(\cdot)} \ge 1/2$, so by Theorem 2.34, there exist functions $g_k \in L^{p'(\cdot)}(\Omega)$, $\|g_k\|_{p'(\cdot)} \le 1$, and $\epsilon > 0$ such that

$$\int_\Omega f(x)\chi_{E_k(x)} g_k(x)\,dx \ge \epsilon.$$

By Hölder's inequality (Theorem 2.26), for each j,

$$\left| \int_\Omega h_j(x) g_k(x) \chi_{E_k}(x)\,dx \right| \le C\|h_j\|_{p(\cdot)},$$

and so the sequence $\{\int h_j g_k \chi_{E_k}\,dx\}_k$ is bounded. Hence, it has a convergent subsequence, and so by a Cantor diagonalization argument we can find a subsequence of functions $\{g_{k_i}\chi_{E_{k_i}}\}_i$ such that for every j, the sequence $\{\int h_j g_{k_i}\chi_{E_{k_i}}\,dx\}_i$ converges and so is Cauchy.

From this fact we will see that for any $F \subset \Omega$ the sequence

$$\left\{\int_\Omega f(x)\chi_F(x)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx\right\}_i \tag{2.12}$$

is Cauchy. Fix $\epsilon > 0$ and let h_j be such that $\|h_j - f\chi_F\|_{p(\cdot)} < \epsilon$. Then for all i and l,

$$\begin{aligned}
&\left|\int_\Omega f(x)\chi_F(x)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx - \int_\Omega f(x)\chi_F(x)g_{k_l}(x)\chi_{E_{k_l}}(x)\,dx\right| \\
&\qquad \le \left|\int_\Omega \big(f(x)\chi_F(x) - h_j(x)\big)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx\right| \\
&\qquad\qquad + \left|\int_\Omega \big(f(x)\chi_F(x) - h_j(x)\big)g_{k_l}(x)\chi_{E_{k_l}}(x)\,dx\right| \\
&\qquad\qquad + \left|\int_\Omega h_j(x)\big(g_{k_i}(x)\chi_{E_{k_i}}(x) - g_{k_l}(x)\chi_{E_{k_l}}(x)\big)\,dx\right|.
\end{aligned}$$

By Hölder's inequality the first two terms are bounded by $C\epsilon$ and the last term is less than ϵ for all i and l sufficiently large. Thus the sequence (2.12) is Cauchy and so converges.

Since the sets E_{k_i} are nested, we can define a sequence of measures on E_1 by

$$\mu_i(F) = \int_{E_1} f(x)\chi_F(x)g_{k_i}(x)\chi_{E_{k_i}}(x)\,dx, \qquad F \subset E_1.$$

Since (2.12) converges, there exists a set function μ such that

$$\mu(F) = \lim_{i\to\infty} \mu_i(F).$$

Since $|E_1| < \infty$, by the Hahn-Saks theorem μ is an absolutely continuous measure on E_1. (See Hewitt and Stromberg [169, ex. 19.68, p. 339].) Therefore, there exists $g \in L^1_{\mathrm{loc}}(E_1)$ such that

$$\mu(F) = \int_F g(x)\,dx.$$

We claim that $g \equiv 0$. To see this, note that since the sets E_k are nested and $|E_k| \to 0$, $|\cap_i\ E_{k_i}| = 0$. Now fix any i and let F be such that $|F \cap E_{k_i}| = 0$. Then

$$\mu(F) = \lim_{i\to\infty} \mu_i(F) = 0.$$

This is true for all such sets F; in particular we can take F to be the set where $g\chi_{E_1\setminus E_{k_i}}$ is positive or negative. Hence, we must have that $g \equiv 0$ on $E_1 \setminus E_{k_i}$. Since this is true for all i, $g \equiv 0$ on E_1. But then

$$0 = \mu(E_1) = \lim_{i\to\infty} \mu_i(E_1) = \lim_{i\to\infty} \int_\Omega f(x)\chi_{E_{k_i}}(x)g_{k_i}(x)\,dx \geq \epsilon,$$

which is a contradiction. Hence, $L^{p(\cdot)}(\Omega)$ is not separable. □

2.8 The Dual Space of a Variable Lebesgue Space

In this section we consider the dual space of $L^{p(\cdot)}(\Omega)$: that is, the Banach space $L^{p(\cdot)}(\Omega)^*$ of continuous linear functionals $\Phi : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ with norm

$$\|\Phi\| = \sup_{\|f\|_{p(\cdot)}\leq 1} |\Phi(f)|.$$

In the classical Lebesgue spaces, $L^{p'} \subset (L^p)^*$ (up to isomorphism), and equality holds if $p < \infty$. The behavior of the variable Lebesgue spaces is analogous if $p_+ < \infty$.

We will begin by constructing a large family of continuous linear functionals and showing that they are induced by elements of $L^{p'(\cdot)}(\Omega)$. Given a measurable function g, define the linear functional Φ_g on $L^{p(\cdot)}(\Omega)$ by

$$\Phi_g(f) = \int_\Omega f(x)g(x)\,dx.$$

Proposition 2.79. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, and a measurable function g, then Φ_g is a continuous linear functional on $L^{p(\cdot)}(\Omega)$ if and only if $g \in L^{p'(\cdot)}(\Omega)$. Furthermore, $\|\Phi_g\| = \|g\|'_{p'(\cdot)}$, and so*

$$k_{p'(\cdot)}\|g\|_{p'(\cdot)} \leq \|\Phi_g\| \leq K_{p'(\cdot)}\|g\|_{p'(\cdot)}. \tag{2.13}$$

Proof. Given any measurable function g, it follows from the definitions that $\|\Phi_g\| = \|g\|'_{p'(\cdot)}$, and so by Theorem 2.34 (with the roles of f and g exchanged in the statement and $p(\cdot)$ replaced by $p'(\cdot)$), Φ_g is continuous if and only if $g \in L^{p'(\cdot)}(\Omega)$ and we get inequality (2.13). □

The linear mapping $g \mapsto \Phi_g$ provides a natural identification between $L^{p'(\cdot)}(\Omega)$ and a subspace of $L^{p(\cdot)}(\Omega)^*$. When $p(\cdot)$ is bounded, we get every element of the dual space in this way.

Theorem 2.80. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, the following are equivalent:*

1. $p_+ < \infty$;
2. *The map $g \mapsto \Phi_g$ is an isomorphism: given any $g \in L^{p'(\cdot)}(\Omega)$, the functional Φ_g is continuous and linear; conversely, given any continuous linear functional $\Phi \in L^{p(\cdot)}(\Omega)^*$ there exists a unique $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$ and $\|g\|_{p'(\cdot)} \approx \|\Phi\|$.*

It follows from Theorem 2.80 that when $p_+ < \infty$ the dual space and the associate space of $L^{p(\cdot)}(\Omega)$ (see Proposition 2.37) coincide. In this case we will simply write $L^{p(\cdot)}(\Omega)^* = L^{p'(\cdot)}(\Omega)$; the isomorphism will be implicit.

As an immediate corollary to Theorem 2.80 we can characterize when $L^{p(\cdot)}(\Omega)$ is reflexive. (Recall that a Banach space X is reflexive if $X^{**} = X$, with equality in the sense of isomorphism.)

Corollary 2.81. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, $L^{p(\cdot)}(\Omega)$ is reflexive if and only if $1 < p_- \leq p_+ < \infty$.*

Proof of Theorem 2.80. Suppose first that $p_+ < \infty$. Fix $\Phi \in L^{p(\cdot)}(\Omega)^*$; we will find $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$. Note that by (2.13) we immediately get that $\|g\|_{p'(\cdot)} \approx \|\Phi\|$.

We initially consider the case when $|\Omega| < \infty$. Define the set function μ by $\mu(E) = \Phi(\chi_E)$ for all measurable $E \subset \Omega$. Since Φ is linear and $\chi_{E\cup F} = \chi_E + \chi_F$ if $E \cap F = \emptyset$, μ is additive. To see that it is countably additive, let

$$E = \bigcup_{j=1}^{\infty} E_j,$$

where the sets $E_j \subset \Omega$ are pairwise disjoint, and let

$$F_k = \bigcup_{j=1}^{k} E_j.$$

Then by Corollary 2.48,

$$\begin{aligned} \|\chi_E - \chi_{F_k}\|_{p(\cdot)} &\leq (1 + |\Omega|)\|\chi_E - \chi_{F_k}\|_{p_+} \\ &= (1 + |\Omega|)|E \setminus F_k|^{1/p_+}. \end{aligned}$$

Since $|E| < \infty$, $|E \setminus F_k|$ tends to 0 as $k \to \infty$; thus $\chi_{F_k} \to \chi_E$ in norm. Therefore, by the continuity of Φ, $\Phi(\chi_{F_k}) \to \Phi(\chi_E)$; equivalently,

$$\sum_{j=1}^{\infty} \mu(E_j) = \mu(E),$$

and so μ is countably additive.

In other words μ is a measure on Ω. Further, it is absolutely continuous: if $E \subset \Omega$, $|E| = 0$, then $\chi_E \equiv 0$, and so

$$\mu(E) = \Phi(\chi_E) = 0.$$

By the Radon-Nikodym theorem (see Royden [301]), absolutely continuous measures are gotten from L^1 functions. More precisely, there exists $g \in L^1(\Omega)$ such that

$$\Phi(\chi_E) = \mu(E) = \int_\Omega \chi_E(x) g(x)\, dx.$$

By the linearity of Φ, for every simple function $f = \sum a_j \chi_{E_j}$, $E_j \subset \Omega$,

$$\Phi(f) = \int_\Omega f(x) g(x)\, dx.$$

By Corollary 2.73, the simple functions are dense in $L^{p(\cdot)}(\Omega)$, and so Φ and Φ_g agree on a dense subset. Thus, by continuity $\Phi = \Phi_g$, and so by Proposition 2.79, $g \in L^{p'(\cdot)}(\Omega)$.

Finally, to see that g is unique, it is enough to note that if $g, \tilde{g} \in L^{p'(\cdot)}(\Omega)$ are such that $\Phi_g = \Phi_{\tilde{g}}$, then for all $f \in L^{p(\cdot)}(\Omega)$,

$$\int_\Omega f(x)(g(x) - \tilde{g}(x))\, dx = 0. \tag{2.14}$$

Since $|\Omega| < \infty$, by Corollary 2.50, $g - \tilde{g} \in L^{p'(\cdot)}(\Omega) \subset L^{p'(\cdot)_-}(\Omega) = L^{(p_+)'}(\Omega)$, and since (2.14) holds for all $f \in L^{p_+}(\Omega) \subset L^{p(\cdot)}(\Omega)$, by the duality theorem for the classical Lebesgue spaces, $g - \tilde{g} = 0$ almost everywhere.

We now consider the case when $|\Omega| = \infty$. Write

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k,$$

where for each k, $|\Omega_k| < \infty$ and $\overline{\Omega}_k \subset \Omega_{k+1}$. Given $\Phi \in L^{p(\cdot)}(\Omega)^*$, by restriction Φ induces a bounded linear functional on $L^{p(\cdot)}(\Omega_k)$ for each k. Therefore, by the above argument, there exists $g_k \in L^{p'(\cdot)}(\Omega_k)$ such that for all $f \in L^{p(\cdot)}(\Omega)$, $\operatorname{supp}(f) \subset \overline{\Omega}_k$,

$$\Phi(f) = \int_{\Omega_k} f(x) g_k(x)\, dx.$$

Further, $\|g_k\|_{p'(\cdot)} \le k^{-1}_{p'(\cdot),\Omega_k} \|\Phi\| \le 3\|\Phi\|$. Since the sets Ω_k are nested, we must have that for all f with support in Ω_k,

$$\int_{\Omega_k} f(x) g_k(x)\, dx = \int_{\Omega_{k+1}} f(x) g_{k+1}(x)\, dx.$$

Since the functions g_k are unique, we must have that $g_k = g_{k+1}\chi_{\Omega_k}$. Therefore, we can define g by $g(x) = g_k(x)$ for all $x \in \Omega_k$. Since $\operatorname{supp}(g_k) \subset \overline{\Omega}_k$, the sequence $|g_k|$ increases to $|g|$; hence, by the monotone convergence theorem for variable Lebesgue spaces (Theorem 2.59),

$$\|g\|_{p'(\cdot)} = \lim_{k\to\infty} \|g_k\|_{p'(\cdot)} \leq 3\|\Phi\| < \infty.$$

Thus $g \in L^{p'(\cdot)}(\Omega)$.

Now fix $f \in L^{p(\cdot)}(\Omega)$ and let $f_k = f\chi_{\Omega_k}$. Then $f_k \to f$ pointwise almost everywhere and $|f - f_k| \leq |f|$, so by the dominated convergence theorem in variable Lebesgue spaces (Theorem 2.62), $f_k \to f$ in norm. Further, $f_k g \to fg$ pointwise, and by Hölder's inequality for variable Lebesgue spaces (Theorem 2.26), $|f_k g| \leq |fg| \in L^1(\Omega)$. Therefore, by the classical dominated convergence theorem and the continuity of Φ,

$$\begin{aligned}\int_\Omega f(x)g(x)\,dx &= \lim_{k\to\infty}\int_{\Omega_k} f_k(x)g(x)\,dx \\ &= \lim_{k\to\infty}\int_{\Omega_k} f_k(x)g_k(x)\,dx = \lim_{k\to\infty}\Phi(f_k) = \Phi(f).\end{aligned}$$

Finally, since the restriction of g to each Ω_k is uniquely determined, g itself is the unique element of $L^{p'(\cdot)}(\Omega)$ with this property. This completes the proof of the first half of the theorem.

Now suppose that $p_+ = \infty$; we will show that there exists $\Phi \in L^{p(\cdot)}(\Omega)^*$ such that $\Phi \neq \Phi_g$ for any $g \in L^{p'(\cdot)}(\Omega)$.

If $|\Omega_\infty| > 0$, then we use the fact that $L^\infty(\Omega_\infty)^*$ contains (the isomorphic image of) $L^1(\Omega_\infty) = L^{p'(\cdot)}(\Omega_\infty)$ as a proper subset (see, for example, Brezis [37] or Dunford and Schwartz [95]); in other words there exists $\Phi \in L^\infty(\Omega_\infty)^*$ that is not induced by any element of $L^1(\Omega_\infty)$. By the Hahn-Banach theorem we can extend Φ to an element of $L^{p(\cdot)}(\Omega)^*$. This is clearly the desired element: if it were equal to Φ_g for some $g \in L^{p'(\cdot)}(\Omega)$, then its restriction to $L^{p(\cdot)}(\Omega_\infty)$ would be induced by $g\chi_{\Omega_\infty}$, contradicting our choice of Φ.

Now assume that $|\Omega_\infty| = 0$ but $p_+(\Omega \setminus \Omega_\infty) = \infty$. We will prove that the desired Φ exists by contradiction. The proof starts as in the proof of Theorem 2.78. Suppose to the contrary that every $\Phi \in L^{p(\cdot)}(\Omega)^*$ is of the form Φ_g, $g \in L^{p'(\cdot)}(\Omega)$. Fix sets E_k and the function f as constructed in the proof of Theorem 2.75. Then f is non-negative, $\|f\|_{p(\cdot)} \leq 1$, $\|\chi_{E_k}\|_{p(\cdot)} \to 0$, and for every k, $\|f\chi_{E_k}\|_{p(\cdot)} \geq 1/2$. Therefore, by Theorem 2.34 there exist non-negative functions $g_k \in L^{p'(\cdot)}(\Omega)$, $\|g_k\|_{p'(\cdot)} \leq 1$, and $\epsilon > 0$ such that

$$\int_\Omega f(x)\chi_{E_k}(x)g_k(x)\,dx \geq \epsilon. \tag{2.15}$$

Without loss of generality we may assume that for all k, $g_k = g_k\chi_{E_k}$.

Define the sets

$$G_k = \left\{ \Phi \in L^{p(\cdot)}(\Omega)^* : |\Phi(f\chi_{E_k})| < \epsilon/2 \right\}.$$

Then we have that $L^{p(\cdot)}(\Omega)^* = \bigcup_k G_k$. To see this, fix $\Phi \in L^{p(\cdot)}(\Omega)^*$; by our original assumption there exists $g \in L^{p'(\cdot)}(\Omega)$ such that $\Phi = \Phi_g$. By Hölder's inequality (Theorem 2.26), $fg \in L^1(\Omega)$, and so by the classical dominated convergence theorem,

$$\lim_{k\to\infty} \Phi_g(f\chi_{E_k}) = \lim_{k\to\infty} \int_\Omega f(x)\chi_{E_k}(x)g(x)\,dx = 0.$$

Hence, for k sufficiently large, $\Phi \in G_k$.

By definition, the sets G_k are open in the weak* topology on $L^{p(\cdot)}(\Omega)^*$. Therefore, the collection $\{G_k\}$ is an open cover of the ball $B = \{\Phi \in L^{p(\cdot)}(\Omega)^* : \|\Phi\| \leq 4\}$. By the Banach-Alaoglu Theorem (see Brezis [37] or Conway [51]), B is weak* compact, and so there exists $N > 0$ and a collection of indices $1 \leq k_1 < k_2 < \cdots < k_N$ such that $\{G_{k_i}\}_{i=1}^N$ is a finite subcover of B.

Define $\Phi_k \in L^{p(\cdot)}(\Omega)^*$ by

$$\Phi_k(h) = \Phi_{g_k}(h) = \int_\Omega h(x)\chi_{E_k} g_k(x)\,dx.$$

Since $\|g_k\|_{p'(\cdot)} \leq 1$, by Theorem 2.34, $\|\Phi_k\| \leq 4$ and so $\Phi_k \in B$. Let k_i be such that $\Phi_k \in G_{k_i}$; then we have that $\Phi_k(f\chi_{E_{k_i}}) = |\Phi_k(f\chi_{E_{k_i}})| < \epsilon/2$. Since the sets E_k are nested, for all $k \geq k_N$,

$$\begin{aligned}\int_{E_k} f(x)g_k(x)\,dx &= \int_\Omega f(x)\chi_{E_k}(x)g_k(x)\,dx \\ &\leq \int_\Omega f(x)\chi_{E_{k_i}} g_k(x)\,dx = \Phi_k(f\chi_{E_{k_i}}) < \epsilon/2.\end{aligned}$$

But this contradicts inequality (2.15). Therefore, our original supposition is false, and there exists $\Phi \in L^{p(\cdot)}(\Omega)^*$ not induced by any $g \in L^{p'(\cdot)}(\Omega)$. This completes our proof. □

2.9 The Lebesgue Differentiation Theorem

We conclude this chapter with a generalization of the Lebesgue differentiation theorem to variable Lebesgue spaces. In the classical case (see Grafakos [143]) if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \fint_{B_r(x)} f(y)\,dy = f(x).$$

Such points x are referred to as Lebesgue points of the function f. This limit also holds if balls are replaced by cubes centered at x or more generally by a nested sequence of balls or cubes whose intersection contains x. In particular, it holds for the sequence of dyadic cubes containing x. (See Sect. 3.2 below.) If $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, then by Proposition 2.41 f is locally integrable, so the Lebesgue differentiation theorem holds for such f.

However, if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$, $1 \le p < \infty$, then a stronger result holds (again see [143]): for almost every $x \in \mathbb{R}^n$,

$$\lim_{r\to 0} \fint_{B_r(x)} |f(y) - f(x)|^p\,dy = 0.$$

An analog of this is true in the variable Lebesgue spaces.

Proposition 2.82. *Given $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $|\Omega_\infty| = 0$, and $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, then for almost every $x \in \mathbb{R}^n$ there exists $\alpha > 0$ such that*

$$\lim_{r\to 0} \fint_{B_r(x)} \big|\alpha\big(f(y) - f(x)\big)\big|^{p(y)}\,dy = 0. \tag{2.16}$$

If $p_+ < \infty$, then we can take $\alpha = 1$.

Proof. Since this is a local result, it will suffice to fix a ball B and prove (2.16) for almost every $x \in B$. Since $f \in L^{p(\cdot)}_{\mathrm{loc}}(\mathbb{R}^n)$, there exists $\lambda > 1$ such that

$$\int_B \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy < \infty.$$

Enumerate the rationals as $\{q_i\}$ and define $\beta_i = \big(2\lambda(|q_i| + 1)\big)^{-1}$. Then

$$\begin{aligned}\int_B \big|\beta_i\big(f(y) - q_i\big)\big|^{p(y)}\,dy &\le \int_B 2^{p(y)-1}\big(\big|\beta f(y)\big|^{p(y)} + |\beta q_i|^{p(y)}\big)\,dy \\ &\le \frac{1}{2}\int_B \left(\frac{|f(y)|}{\lambda}\right)^{p(y)} dy + \int_B \left(\frac{|q_i|}{|q_i| + 1}\right)^{p(y)} dy < \infty.\end{aligned}$$

Therefore, by the classical Lebesgue differentiation theorem, for each i and for almost every $x \in B$,

$$\lim_{r\to 0} \fint_{B_r(x)} \big|\beta_i\big(f(y) - q_i\big)\big|^{p(y)}\,dy = \big|\beta_i\big(f(x) - q_i\big)\big|^{p(x)}.$$

Since the countable union of sets of measure 0 again has measure 0, this limit holds for all i and almost every $x \in B$. Fix such an x and fix ϵ, $0 < \epsilon < 1$. Then there exists i such that

$$\big|\beta_i\big(f(x) - q_i\big)\big| < \epsilon.$$

Define $\alpha = \beta_i/2$. Then by Remark 2.8 we have that

$$\begin{aligned}
&\limsup_{r\to 0} \fint_{B_r(x)} \big|\alpha\big(f(y) - f(x)\big)\big|^{p(y)}\,dy \\
&\quad\le \limsup_{r\to 0}\bigg(\fint_{B_r(x)} 2^{p(y)-1}\Big|\frac{\beta_i}{2}\big(f(y) - q_i\big)\Big|^{p(y)}\,dy \\
&\qquad\qquad + \fint_{B_r(x)} 2^{p(y)-1}\Big|\frac{\beta_i}{2}\big(f(x) - q_i\big)\Big|^{p(y)}\,dy\bigg) \\
&\quad\le \frac{1}{2}\limsup_{r\to 0}\bigg(\fint_{B_r(x)} \big|\beta_i\big(f(y) - q_i\big)\big|^{p(y)}\,dy \\
&\qquad\qquad + \fint_{B_r(x)} \big|\beta_i\big(f(x) - q_i\big)\big|\,dy\bigg) \\
&\quad= \frac{1}{2}\Big(\big|\beta_i\big(f(x) - q_i\big)\big|^{p(x)} + \big|\beta_i\big(f(x) - q_i\big)\big|\Big) \\
&\quad< \epsilon.
\end{aligned}$$

The limit (2.16) follows at once.

Finally if $p_+ < \infty$, then the above proof can be readily modified to take $\alpha = \beta_i = 1$. □

Remark 2.83. When $p_+ < \infty$, by Theorem 2.58 the modular limit implies a limit of norms:

$$\lim_{r\to 0} \big\||B_r(x)|^{-1/p(\cdot)}|f(\cdot) - f(x)|\big\|_{p(\cdot)} = 0.$$

2.10 Notes and Further Results

2.10.1 References

As we discussed in Chap. 1, the variable Lebesgue spaces were considered by a number of authors independently and so many of the results in this chapter were probably discovered several times. In our treatment, we have primarily followed the work of Kováčik and Rákosník [219] and Diening [80]. (This work, Diening's habilitation thesis, has recently been expanded into a book, written jointly with Harjulehto, Hästö and Růžička [82].) The structure of variable Lebesgue spaces is also treated by Samko [313, 314] and Fan and Zhao [122]. A briefer overview, combined with an extensive bibliography, is given by Harjulehto and Hästö [150]. The structural parallels between the classical and variable Lebesgue spaces are clearest when $p_+ < \infty$, and this is the case frequently considered in the literature. Our approach has been to provide a unified treatment of bounded and unbounded exponents.

The local log-Hölder continuity condition LH_0 (Definition 2.2) first appeared in Sharapudinov [331] and later in Zhikov [358, 359, 361], Karapetyants and Ginzburg [189, 190], Ross and Samko [300], Samko [313], and Diening [77]. Since these papers this condition has become ubiquitous. The log-Hölder condition at infinity was introduced in [62]. Both log-Hölder conditions play a central role in harmonic analysis on variable Lebesgue spaces, as we will make clear in subsequent chapters.

The modular in Definition 2.6 is taken from [219]; for alternative definitions see Sect. 2.10.2 below. The variable Lebesgue space norm in Definition 2.16 is usually referred to as the Luxemburg norm, because it is analogous to the norm on Orlicz spaces (cf. [25]). However, it appeared in Musielak and Orlicz [275] in the more general context of modular spaces, and earlier in Nakano [280]. Independently it was defined by Sharapudinov [329], who based it on a more general result of Kolmogorov [210] about Minkowski functionals. For this reason, some authors refer to this norm as the Kolmogorov-Minkowski norm (e.g., [313]).

The extension theorem in Lemma 2.4 was first proved in [61]. A weaker version for functions in LH_0 appeared in [80] and for Lipschitz functions in [106]. The construction in the second half of the proof of Proposition 2.12 is due to Kováčik and Rákosník [219]; this construction and the variant of it in Theorem 2.75 play a major role in understanding the properties of variable Lebesgue spaces with unbounded exponents. A somewhat different and more general version of Proposition 2.18 (including the case $|\Omega_\infty| > 0$ and replacing the constant s by a bounded function) is due independently to Samko [314] and Edmunds and Rakósník [106]; the simpler version given here was proved independently in [61]. Corollary 2.23 for $p_+ < \infty$ appeared in [122]; our version is adapted from Diening *et al.* [81]. Variants of this estimate have appeared elsewhere in the literature: see, for example, de Cicco *et al.* [73]. The proof of Proposition 2.25 is taken from Samko [313]. In the more general setting of modular spaces this was proved by Nakano [280] (who attributed this definition of the norm to Amemiya). See also Musielak [274] and Maligranda [244]. Independently, and both working in the more general setting of Musielak-Orlicz spaces, Fan [114] and Šragin [335] proved that the Amemiya norm is equal to the associate norm when $|\Omega_\infty| = 0$. (Šragin assumed that $|\Omega_\infty| = 0$. This result was also noted for modular spaces without proof by Hudzik and Maligranda [180, Remark 4].) For an application of the Amemiya norm, see [131].

Our proof of Hölder's inequality (Theorem 2.26) is taken from [219]. The generalized Hölder's inequality (Corollary 2.28) was proved by Diening [80] and earlier by Samko [313, 314] with the additional hypothesis that $r_+(\Omega \setminus \Omega_\infty^{r(\cdot)}) < \infty$. In the same papers, Samko also proved Corollary 2.30 and Minkowski's integral inequality (Corollary 2.38). His proof of Corollary 2.30 shows that the constant can be taken to be $\sum [p_i(\cdot)_-]^{-1}$.

The L^∞ embedding in Proposition 2.43 was shown to us by Diening. Theorem 2.45 is due to Diening [77]; when $|\Omega| < \infty$ (i.e., Corollary 2.48) it was proved by Kováčik and Rákosník [219] and Samko [314]. A quantitative version when $p(\cdot)$ and $q(\cdot)$ are close was proved by Edmunds, Lang and Nekvinda [102]. The embedding in Theorem 2.51 was implicit in [67] and is explicit in Diening [80]. Proposition 2.53 and other, related embedding theorems were proved by Diening and Samko [92].

Our definition of modular convergence, Definition 2.54, is classical in the study of modular spaces; see Maligranda [244] or Musielak [274]. Diening [80] also uses this definition; both [219] and [122] assume $\beta = 1$ in the definition. The monotone convergence theorem for variable Lebesgue spaces (Theorem 2.59) was first stated without proof in [101]; a proof in the case $p_+ < \infty$ appeared in [58] and the full result was proved in [56]. Fatou's lemma and the dominated convergence theorem for variable Lebesgue spaces (Theorems 2.61 and 2.62) are new. The weak converse of the dominated convergence theorem, Proposition 2.67 is also new. For the converse in the case of the classical Lebesgue spaces see Brezis [37] or Lieb and Loss [238]. Theorem 2.68 for $p_+ < \infty$ is in [219] and implicit in [122]; our version is new. Theorem 2.69 is stated by Fan and Zhao [122] but the proof is only sketched. The complete proof was given in [60]; also see below.

The completeness of the variable Lebesgue spaces was proved by Kováčik and Rákosník [219] and Diening [80]; our proof is different and follows the proof in Bennett and Sharpley [25] for abstract Banach function spaces. Our approach also yields the Riesz-Fischer property (Theorem 2.70). Theorem 2.72 and Corollary 2.73 are in [219]. Theorem 2.75 is due to Kalyabin [187] and also to Edmunds, Lang and Nekvinda [101]. Theorem 2.77 is new; Harjulehto [149] gave a specific example of a space in which functions of compact support were not dense. Theorem 2.78 in the case $p_+ = \infty$ is new, but it depends critically on the construction from [219] and adapts an argument in [25].

Theorem 2.80 is proved in [219], but their proof depends on deeper results on Orlicz-Musielak spaces due to Hudzik [179] and Kozek [220]. Our proof is direct: when $p_+ < \infty$ we followed the proof for classical Lebesgue spaces in Royden [301], and for $p_+ = \infty$ we adapted an argument in Bennett and Sharpley [25]. A different proof of the characterization of reflexivity (Corollary 2.81) is due to Lukeš, Pick and Pokorný [242]: see Sect. 2.10.3 below.

The generalization of the Lebesgue differentiation theorem to the variable setting (Proposition 2.82) was proved by Harjulehto and Hästö [152] when $p_+ < \infty$. Our proof is a simple modification of theirs.

2.10.2 Musielak-Orlicz Spaces and Modular Spaces

The variable Lebesgue spaces are a particular example of a larger class of function spaces that also includes the classical and weighted Lebesgue spaces and Orlicz spaces as special cases. Given a set Ω, let $\Phi : \Omega \times \mathbb{R}^+ \to [0,\infty]$ be such that for each $x \in \Omega$, the function $\Phi(x, \cdot)$ is non-decreasing, continuous and convex on the set where it is finite. Assume that $\Phi(x, 0) = 0$, $\Phi(x, t) > 0$ if $t > 0$, and $\Phi(x, t) \to \infty$ as $t \to \infty$. We also assume that for each $t \geq 0$, the function $\Phi(\cdot, t)$ is a measurable function.

Define the Musielak-Orlicz space $L^{\Phi(,)}(\Omega)$ to be the set of all functions f such that for some $\lambda > 0$,

$$\rho_{\Phi(,)}(f) = \int_\Omega \Phi(x, |f(x)|/\lambda)\, dx < +\infty. \tag{2.17}$$

Then by arguments analogous to those above one can show that $L^{\Phi(,)}(\Omega)$ is a Banach function space with the norm

$$\|f\|_{L^{\Phi(,)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \Phi(x, |f(x)|\lambda^{-1})\, dx \leq 1\right\}.$$

In this setting the norm is referred to as the Luxemburg norm. It is possible to define a so-called complementary function Ψ which also generates a Musielak-Orlicz space. This space can be used to define the associate norm, which is also called the Orlicz norm. See [244, 274] for further details. Because the spaces $L^{\Phi(,)}$ generalize Orlicz spaces in the same way that $L^{p(\cdot)}$ generalizes the classical Lebesgue spaces, it makes sense to refer to $L^{\Phi(,)}$ as a variable Orlicz space, but this terminology has not been widely adopted.

Musielak-Orlicz spaces are themselves a special case of abstract Banach spaces called modular spaces. Given a set X that is a real vector space, a convex modular is a function $\rho : X \to [0, \infty]$ such that:

1. $\rho(x) = 0$ if and only if $x = 0$;
2. $\rho(-x) = \rho(x)$ for all $x \in X$;
3. ρ is convex;
4. The map $\lambda \mapsto \rho(\lambda x)$ is left-continuous.

If we let X_ρ be the set of all $x \in X$ such that $\rho(\lambda^{-1}x) < \infty$ for some $\lambda > 0$, then this becomes a normed vector space with norm

$$\|x\|_{X_\rho} = \inf\{\lambda > 0 : \rho(\lambda^{-1}x) \leq 1\}.$$

For more further details, see [82, 244, 274].

The function ρ_Φ defined by (2.17) is a convex modular in this sense and $L^{\Phi(,)}$ is a modular space. In particular, if $p(\cdot) \in \mathcal{P}(\Omega)$, then (by Proposition 2.7) $\rho_{p(\cdot)}$ is a convex modular. Many of the classical Banach function spaces can also be viewed as Musielak-Orlicz spaces or as modular spaces. If let $\Phi(x, t) = t^p$, $1 \leq p < \infty$, we get the classical Lebesgue space $L^p(\Omega)$. If we let $\Phi(x, t) = t^p w(x)$, where w is a positive, locally integrable function, then we get the weighted Lebesgue space $L^p(\Omega, w)$. If $\Phi(x, t) = \Phi(t)$, then we get the Orlicz spaces. For example, we can take $\Phi(t) = t^p \log(e+t)^a$, in which case L^Φ becomes the Zygmund space $L^p(\log L)^a$. (See Bennett and Sharpley [25].)

We can weaken the definition of modular by replacing (1) by

(1a) $\rho(0) = 0$;
(1b) If $\rho(\lambda x) = 0$ for all $\lambda > 0$, then $x = 0$.

Such functions ρ are referred to as semi-modulars, and the theory of modular spaces readily extends to this setting. For example, if we let $\Phi(x,t) = \infty \cdot \chi_{(1,\infty)}(t)$ (letting $0 \cdot \infty = 0$), then (2.17) defines a semi-modular and we get $L^\infty(\Omega)$. We can extend this approach to get a very elegant definition of the variable Lebesgue spaces. Given $p(\cdot) \in \mathcal{P}(\Omega)$, define

$$\tilde{\rho}_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx,$$

with the convention that $t^\infty = \infty \cdot \chi_{(1,\infty)}(t)$. Then $\tilde{\rho}_{p(\cdot)}$ is a semi-modular. It is not equivalent to $\rho_{p(\cdot)}$: for example, if we let $\Omega = \mathbb{R}$, $p(x) = \infty$, and $f(x) = c > 0$, then $\rho_{p(\cdot)}(f) = c$, but $\tilde{\rho}_{p(\cdot)}(f) = 0$ if $0 < c \leq 1$ and $\tilde{\rho}_{p(\cdot)}(f) = \infty$ if $c > 1$. Nevertheless, the norm $\|\cdot\|_{X_{\tilde{\rho}}}$ is equivalent to $\|\cdot\|_{p(\cdot)}$: for all f,

$$\|f\|_{X_{\tilde{\rho}}} \leq \|f\|_{p(\cdot)} \leq 2\|f\|_{X_{\tilde{\rho}}}. \tag{2.18}$$

The whole theory of variable Lebesgue spaces can be developed from this perspective; it is done this way, for example, in [80, 82]. (A proof of (2.18) can be found in both.) This approach is extremely elegant and is also advantageous in some applications, since in certain limiting cases the space that appears naturally is a Musielak-Orlicz space. For instance, in Sect. 3.7.3 below, the behavior of the Hardy-Littlewood maximal operator is considered for functions $f \in L^{p(\cdot)}(\log L)^{q(\cdot)}$, the Musielak-Orlicz space generated by $\Phi(x,t) = t^{p(x)}\log(e+t)^{q(x)}$. These are generalizations of the Zygmund spaces and were first considered in [59] and later by Mizuta and various co-authors [138, 166, 167, 243, 265, 267]. For another example generalizing the space $\exp L$, see Harjulehto and Hästö [153].

2.10.3 Banach Function Spaces

Another abstract approach to the variable Lebesgue spaces is that of Banach function spaces as defined by Bennett and Sharpley [25]. Let $\Omega \subset \mathbb{R}^n$ and let $\mathcal{M}$ be the set of all measurable functions with respect to Lebesgue measure. Given a mapping $\|\cdot\|_X : \mathcal{M} \to [0,\infty]$, the set

$$X = \{f \in \mathcal{M} : \|f\|_X < \infty\},$$

is a Banach function space if the pair $(X, \|\cdot\|_X)$ satisfies the following properties for all $f, g \in \mathcal{M}$:

1. $\|f\|_X = \||f|\|_X$ and $\|f\|_X = 0$ if and only if $f \equiv 0$;
2. $\|f + g\|_X \le \|f\|_X + \|g\|_X$;
3. For all $a \in \mathbb{R}$, $\|af\|_X = |a|\|f\|_X$;
4. X is a complete normed vector space with respect to $\|\cdot\|_X$;
5. If $|f| \le |g|$ almost everywhere, then $\|f\|_X \le \|g\|_X$;
6. If $\{f_n\} \subset \mathcal{M}$ is a sequence such that $|f_n|$ increases to $|f|$ almost everywhere., then $\|f_n\|_X$ increases to $\|f\|_X$;
7. If $E \subset \Omega$ is a measurable set and $|E| < \infty$, then $\|\chi_E\|_X < \infty$;
8. $\int_E |f(x)|\,dx \le C_E\|f\|_X$ if $|E| < \infty$, where $C_E < \infty$ depends on E and X, but not on f.

It follows at once from the results in this chapter that $\|\cdot\|_{p(\cdot)}$ is a Banach function space. This was first observed by Edmunds, Lang and Nekvinda [101] (see also Lukeš, Pick and Pokorný [242]). Many of the results proved in this chapter—especially the functional analytic ones on duality, separability, etc.—can be proved in this more general setting.

Here we give one such general result. We say that a function $f \in X$ has absolutely continuous norm if given any nested sequence of sets $\{E_k\}$ such that $|E_k| \to 0$, $\|f\chi_{E_k}\|_X \to 0$. The norm $\|\cdot\|_X$ is absolutely continuous if every function in X has absolutely continuous norm. We define the associate space of X to be the space X' of functions g such that

$$\|g\|_{X'} = \sup\left\{\int_\Omega |f(x)g(x)|\,dx : \|f\|_X \le 1\right\} < \infty.$$

Denoting by X^* the dual space of X, then the following are equivalent [25]:

1. $\|\cdot\|_X$ is absolutely continuous;
2. X is separable;
3. $X^* = X'$ (up to isomorphism).

As a corollary to Theorems 2.58 and 2.62 we have that the norm $\|\cdot\|_{p(\cdot)}$ is absolutely continuous if and only if $p_+ < \infty$. In proving this fact, as well as in proving separability and duality (Theorems 2.78 and 2.80) the construction from Proposition 2.12 played a central role.

The Banach space properties of the variable Lebesgue spaces have been considered by several authors. The subspace of functions in $L^{p(\cdot)}$, $p_+ = \infty$, that have absolutely continuous norm was examined by Edmunds, Lang and Nekvinda [101]. A Banach space X is uniformly convex if for every $\epsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$, $\|x\|_X = \|y\|_X = 1$ and $\|x - y\|_X \ge \epsilon$, then $\|x + y\|_X \le 2 - \delta$. Lukeš, Pick and Pokorný [242] showed that the following are equivalent:

1. $1 < p_- \le p_+ < \infty$;
2. $L^{p(\cdot)}(\Omega)$ is reflexive;
3. $L^{p(\cdot)}(\Omega)$ and $L^{p'(\cdot)}(\Omega)$ have absolutely continuous norms;
4. $L^{p(\cdot)}(\Omega)$ is uniformly convex.

Earlier, the uniform convexity of $L^{p(\cdot)}(\Omega)$ was proved by Nakano [280] (when $\Omega = [0,1]$, see also [245]), Diening [80] and also by Fan and Zhao [122]; the uniform convexity of modular spaces was considered by Musielak [274]. In the same paper, Lukeš *et al*. characterized the exponents such that $L^{p(\cdot)}(\Omega)$ has the Radon-Nikodym and Daugavet properties. Dinca and Matei [93, 94] have considered the Gateaux derivative of the norm of $L^{p(\cdot)}(\Omega)$ and have also considered uniform convexity and the derivative of the norm for variable Sobolev spaces (see Chap. 6).

2.10.4 Alternative Definitions of the Modular

In the framework we have adopted there are several equivalent definitions of the modular. One alternative is

$$\rho'_{p(\cdot)}(f) = \max\left(\int_{\Omega\setminus\Omega_\infty} |f(x)|^{p(x)}\,dx, \|f\|_{L^\infty(\Omega_\infty)}\right);$$

then $\rho'_{p(\cdot)}(f)$ is equivalent to $\rho_{p(\cdot)}(f)$ for all f, and the same results hold with minor modifications of the proof. This definition was used by Edmunds and Rákosník [106].

Another, more interesting alternative was considered by Samko [313] and developed systematically by Diening *et al*. [80, 82]. Modify the definition of the modular

$$\rho^*_{p(\cdot)}(f) = \int_{\Omega_*} \frac{1}{p(x)}|f(x)|^{p(x)}\,dx + \|f\|_{L^\infty(\Omega_\infty)},$$

and use this to define the norm

$$\|f\|^*_{p(\cdot)} = \inf\{\lambda > 0 : \rho^*_{p(\cdot)}(f/\lambda) \le 1\}. \tag{2.19}$$

If $p_+ < \infty$, then it is immediate that

$$(p_+)^{-1}\rho^*_{p(\cdot)}(f) \le \rho_{p(\cdot)}(f) \le (p_-)^{-1}\rho^*_{p(\cdot)}(f),$$

and it follows that $\|\cdot\|_{p(\cdot)}$ and $\|\cdot\|^*_{p(\cdot)}$ are equivalent norms. However, it can be shown that this is the case even when $p_+ = \infty$.

One advantage of this definition is that Hölder's inequality follows with a universal constant. Indeed, the proof of Theorem 2.26 can be modified to show that

$$\int_\Omega |f(x)g(x)|\,dx \le 2\|f\|^*_{p(\cdot)}\|g\|^*_{p'(\cdot)}. \tag{2.20}$$

Furthermore, as Samko [313] pointed out, if in the definition of $\|\cdot\|^*_{p(\cdot)}$ we replace the constant 1 by $1/2$ on the right-hand side of (2.19), then the constant in (2.20) becomes 1. This phenomenon is exactly parallel to the behavior of the norm on Orlicz spaces and follows from the structure of the Luxemburg norm. See Miranda [264] or Greco, Iwaniec and Moscariello [145].

2.10.5 Variable Lebesgue Spaces and Orlicz Spaces

In certain applications where $p_- = 1$ and $|\Omega| < \infty$ (see, for instance Sect. 3.7.3 below) it is natural to ask if there is an embedding of $L^{p(\cdot)}(\Omega)$ into the Zygmund space $L \log L(\Omega)$: more precisely, when

$$\|f\|_{L\log L(\Omega)} \leq C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{2.21}$$

These embeddings were first studied by Hästö [163], and then in Futamura and Mizuta [136], Mizuta, Ohno and Shimomura [266], and also in [59]. They hold if $p(\cdot)$ satisfies a decay condition when $p(\cdot)$ is close to 1 in value. More precisely, let

$$\lambda(s) = 1 + \frac{\log\log(1/s)}{\log(1/s)}.$$

If for all $s > 0$ sufficiently small,

$$|\{x \in \Omega : p(x) \leq \lambda(s)\}| \leq Ks,$$

then (2.21) holds.

Necessary and sufficient conditions for the embeddings between Orlicz spaces and variable Lebesgue spaces can be gotten as special cases of a general theorem for Orlicz-Musielak spaces. Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, and given a Young function Φ and the corresponding Orlicz space $L^\Phi(\Omega)$, then $L^{p(\cdot)}(\Omega) \subset L^\Phi(\Omega)$ if and only if there exists $K > 1$ and $h \in L^1(\Omega)$ such that for all $x \in \Omega$ and $t > 0$,

$$\Phi(t) \leq Kt^{p(x)} + h(x).$$

Conversely, $L^\Phi(\Omega) \subset L^{p(\cdot)}(\Omega)$ if and only if there exists $K > 1$ and $g \in L^1(\Omega)$ such that

$$t^{p(x)} \leq K\Phi(t) + g(x).$$

This theorem is due to Ishii [182]; see also Hudzik [177], Kozek [220], or Musielak [274]. This result was used by Diening [77] to prove Theorem 2.45.

2.10.6 More on Convergence

Theorem 2.69 shows that convergence in norm, modular and measure are equivalent if $p_+ < \infty$. The relationship between these three kinds of convergence is more complicated when $p_+ = \infty$. As we showed in Proposition 2.56 and Theorem 2.66, convergence in norm always implies convergence in modular and convergence in measure. Conversely, convergence in modular implies convergence in norm exactly when $p_- = \infty$ or $p_+(\Omega\setminus\Omega_\infty) < \infty$ (Theorem 2.58), and the sequence of functions constructed in Theorem 2.66 also shows that convergence in measure never implies convergence in norm.

The relationship between convergence in modular and convergence in measure is more complicated. The proof of Theorem 2.69 can be generalized to prove the following results.

Theorem 2.84. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$, for each $M \geq 1$ let*

$$E_M = \{x \in \Omega \setminus \Omega_\infty : p(x) > M\}.$$

Then the following are equivalent:

1. *For any sequence $\{f_k\} \in L^{p(\cdot)}(\Omega)$ and $f \in L^{p(\cdot)}(\Omega)$, if $f_k \to f$ in modular, then $f_k \to f$ in measure and for every $\gamma > 0$ sufficiently small, $\rho(\gamma f_k) \to \rho(\gamma f)$;*
2. *$|E_M| \to 0$ as $M \to \infty$.*

Theorem 2.85. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $|\Omega_\infty| = 0$, if $f \in L^{p(\cdot)}(\Omega)$ and $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ are such that $f_k \to f$ in measure and for some γ, $0 < \gamma < 1$, $\rho(\gamma f) < \infty$ and $\rho(\gamma f_k/3) \to \rho(\gamma f/3)$, then $f_k \to f$ in modular.*

For proofs and a complete discussion of the relationship between these three notions of convergence, see [60].

Beyond the three types of convergence, we can also consider weak convergence. A sequence $\{f_n\} \subset L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega)$ if for every $\Phi \in L^{p(\cdot)}(\Omega)^*$, $\Phi(f_n) \to \Phi(f)$. When $p_+ < \infty$, by Theorem 2.80, we have that $f_k \to f$ weakly in $L^{p(\cdot)}(\Omega)$ if for every $g \in L^{p'(\cdot)}(\Omega) = L^{p(\cdot)}(\Omega)^*$,

$$\int_\Omega f_k(x)g(x)\,dx \to \int_\Omega f(x)g(x)\,dx.$$

In the classical Lebesgue spaces, by the Radon-Riesz theorem, if $1 < p < \infty$, $f_k \to f$ weakly, and $\|f_k\|_p \to \|f\|_p$, then $f_k \to f$ in norm. This is also true in the variable Lebesgue spaces.

Proposition 2.86. *Given Ω and $p(\cdot) \in \mathcal{P}(\Omega)$ such that $1 < p_- \leq p_+ < \infty$, if the sequence $\{f_k\} \subset L^{p(\cdot)}(\Omega)$ converges weakly to $f \in L^{p(\cdot)}(\Omega)$, and if $\|f_k\|_{p(\cdot)} \to \|f\|_{p(\cdot)}$, then $f_k \to f$ in norm.*

The proof is the same as in the classical case (see Hewitt and Stromberg [169]): it follows from the fact that with these hypotheses, $L^{p(\cdot)}(\Omega)$ is uniformly convex. (See Sect. 2.10.3.) For an example of the application of weak convergence in variable Lebesgue spaces, see Zecca [352] (which generalizes [146]).

2.10.7 Variable Sequence Spaces

The sequence spaces $\ell^p,\ 1 \le p < \infty$, can be generalized to get a discrete version of the variable Lebesgue spaces. Given a function $p(\cdot) : \mathbb{N} \to [1,\infty)$, define $\ell^{p(\cdot)}$ to be the space of sequences $\alpha = \{a_k\}$ such that

$$\|\alpha\|_{\ell^{p(\cdot)}} = \inf\left\{\lambda > 0 : \sum_{k=1}^{\infty}\left(\frac{|a_k|}{\lambda}\right)^{p(k)} \le 1.\right\}.$$

Arguing as above we can prove that $\ell^{p(\cdot)}$ is Banach space. These spaces were first considered by Orlicz [290] and Nakano [279] (see also [245]), and more recently by Edmunds and Nekvinda [104] and by Nekvinda [281, 283]. Diening [80] treats variable sequence spaces as a special case of the modular spaces, since the above definition of the norm is gotten from the definition of the norm on $L^{p(\cdot)}$ if we replace the underlying space by $\mathbb{N}$ and Lebesgue measure by counting measure.

Recently, Hästö has shown that the variable sequence spaces have applications to the study of operators on variable Lebesgue spaces. See [165] and Sect. 5.6.6 below.

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