Preface

In 1955, Chevalley [67] published constructions of simple groups of Lie type from Lie algebras. At that time, Dieudonné [119] had already published constructions of the classical groups and had started to focus on their geometric interpretation [120]. The universal geometric counterpart though was provided by Tits, whose approach and way of thinking by then began to transpire through lectures and papers such as [280]. However, the full extent of his constructions as well as the geometric classification of groups of Lie type (at least for rank at least 3) became available through [286]. This is his first comprehensive work on buildings, and deals primarily with the classification of spherical buildings of rank at least three. In later work, using joint work with Bruhat, Tits [290] took care of the classification of buildings of affine type and rank at least four (see [306] for an excellent account). Their automorphism groups are infinite groups, and we will only provide a simple example of the kind of geometry involved. One of the surprising aspects that came forward from Tits' work on buildings is the notion of a diagram. Diagrams prescribe what the geometries underlying the groups of Lie type look like locally. Here, a geometry is an abstract object, little more than a multipartite graph, and the local information alluded to concerns residues, that is, subgeometries induced on the set of vertices adjacent to a set of mutually adjacent vertices of that graph. In the case of bipartite graphs, the geometry is a so-called generalized polygon. As the word indicates, this is a very natural generalization of polygons. Generalized 3-gons, for instance, are projective planes. The ordinary 3-gon, a combinatorial triangle, is a generalized 3-gon and can be viewed as the smallest projective plane. Moreover, projective planes contain hoards of triangles.

In the classification of (non-Abelian) finite simple groups, the groups of Lie type play a crucial role, simply because, apart from these and the alternating groups, there are only 26 more—the so-called sporadic groups. Buekenhout launched the idea of employing the diagrammatic description of the geometries for groups of Lie type to a wider class of groups, preferably one that would lead to a classification of the finite simple groups by diagram geometry. By judiciously extending the classes of bipartite graphs allowed as residues, the right diagrams might occur that would fit all simple groups rather than just those of Lie type. The idea led to a flurry of activities, ranging from the construction of diagram geometries on the basis of known groups and a system of their subgroups to the classification of all geometries pertaining to a given diagram. Although quite a few diagrams have been found for finite simple groups and quite a few interesting classifications of geometries have seen the light, the classification of finite simple groups has been completed without a satisfactory framework offered by diagram geometry. Nevertheless diagram geometry structured several characterizations of individual sporadic groups, and provided tools that are useful for geometric alternatives to certain existing parts of the classification. Besides, a lot of finite group theory is of a very geometric nature, although the proofs are not always formulated in the associated terminology.

Incidence geometry, though, has beauty in its own right. This is not only reflected by quite intriguing diagrams for several simple groups, but also by striking axioms characterizing spaces related to classical geometries. Most impressive is the Buekenhout-Shult description of a polar space by means of the single condition that, for each line and each point off that line, either one or all points of the line are collinear with the point. From this axiom, together with some light nondegeneracy conditions, the full building belonging to any classical group distinct from a special linear group and of rank at least two (that is, having a subspace of dimension at least two in the natural representation space of the group on which the invariant defining form completely vanishes) can be reconstructed. Besides, whereas diagram geometry functions best in cases where ranks are finite, much of the polar space approach remains valid for spaces of arbitrary rank. The construction of geometries from spaces with few axioms is a major theme of diagram geometry. In this respect, the root filtration spaces form the counterpart of polar spaces. Their axioms are more involved, but examples exist for all finite groups of Lie type of rank at least two and distinct from ${}^2\mathrm{F}_4$. In this book, these spaces are introduced and enough properties are derived so as to be able to characterize spaces related to projective spaces and to line Grassmannians of polar spaces.

This book provides a self-contained introduction to diagram geometry. The first three chapters are spent on the basic theory. The fourth chapter shows the tight connection with group theory; it deals with thin geometries, which are very close to quotients of Cayley graphs of Coxeter groups. These geometries are abundant in buildings, like the triangles in projective planes. We then treat projective and affine geometry in two chapters. These are geometries with a linear diagram and linear shadow spaces, which implies that they are matroids. This opens the door to variations of the geometries connected with buildings. We restrict ourselves to a limited number of variations, just enough to give the flavor of combinatorial structures like Steiner systems in the context of diagram geometry. The next four chapters are devoted to polar spaces. Their complete classification is found in Tits [286]. Here we use a different approach, starting with Veldkamp's method [297] of embedding the polar space in a projective space. There are exceptions, such as the polar spaces whose projective planes are not Desarguesian, to which we devote little attention. This reflects the idea that the book is primarily an introduction to diagram geometry and the associated synthetic treatment of fascinating geometric spaces. The intention is to give a flavor of the topic rather than an exhaustive treatment. The final chapter gives a brief introduction to the theory of buildings and shows that every spherical building of rank at least three is connected with a root filtration space. The references to the literature in the Notes sections are meant to enable the interested reader to further knowledge in several directions.

The switching of viewpoints within a single geometry by use of the diagram leads to axiom systems of various kinds for the classical geometries. As mentioned above, the root filtration spaces are special among these as every possible finite group of Lie type distinct from ${}^2F_4$ acts acts almost faithfully on such a space. A representative collection of these spaces is directly related to the Lie algebras introduced by Chevalley. It is the purpose of the second author to complete a second volume dedicated to these spaces and the non-classical geometries of spherical Coxeter type.

Preliminary versions of this book, including the intended chapters of the second volume, have been available on the Internet for over fifteen years. We are very grateful to comments received by enthusiastic readers and acknowledgments in the form of references to such a volatile site as the place where the individual chapters were to be found. We hope the readers will be pleased that—at least the first part of—the Internet version has finally been turned into a book.

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