Chapter 2
Diagrams

A diagram is a structure defined on a set of types I. This structure generally is close to a labelled graph and provides information on the isomorphism class of residues of rank two of geometries over I. This way diagrams lead naturally to classification questions like all residually connected geometries pertaining to a given diagram.

In Sect. 2.1, we start with one of the most elementary kinds of diagrams, the digon diagram. In Sect. 2.2, we explore some parameters of bipartite graphs that help distinguish relevant isomorphism classes of rank two geometries. Projective and affine planes can be described in terms of these parameters, but we also discuss some other remarkable examples, such as generalized m-gons; for $m = 3$, these are projective planes. The full abstract definition of a diagram appears in Sect. 2.3. The core interest is in the case where all rank 2 geometries are generalized m-gons, in which case the diagrams involved are called Coxeter diagrams, the topic of Sect. 2.4.

The significance of the axioms for geometries introduced via these diagrams becomes visible when we return to elements of a single kind, or, more generally to flags of a single type. The structure inherited from the geometry becomes visible through so-called shadows, studied in Sect. 2.5. Here, the key notion is that of a line space, where the lines are particular kinds of shadows. In order to construct flag-transitive geometries from groups with a given diagram, we need a special approach to diagrams for groups. This is carried out in Sect. 2.6. Finally in this chapter, a series of examples of a flag-transitive geometry belonging to a non-linear Coxeter diagram is given, all of whose proper residues are projective geometries.

2.1 The Digon Diagram of a Geometry

The digon diagram is a slightly simpler structure than a diagram and is useful in view of the main result, Theorem 2.1.6, which allows us to conclude that certain elements belonging to a given residue are incident. Let I be a set of types.

Definition 2.1.1 Suppose that $i, j \in I$ are distinct. A geometry over $\{i, j\}$ is called a *generalized digon* if each element of type i is incident with each element of type j.

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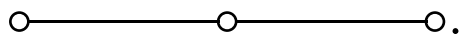
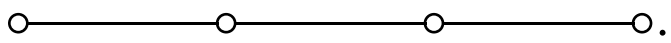
Let Γ be a geometry over I. The *digon diagram* $\mathcal{I}(\Gamma)$ of Γ is the graph whose vertex set is I and whose edges are the pairs $\{i, j\}$ from I for which there is a residue of type $\{i, j\}$ that is not a generalized digon.

In other words, a generalized digon has a complete bipartite incidence graph. The choice of the name digon will become clear in Sect. 2.2, where the notion of generalized polygon is introduced.

Example 2.1.2 In the examples of Sect. 1.1, the cube, the icosahedron, a polyhedron, a tessellation of $\mathbb{E}^2$, and the Euclidean space $\mathbb{E}^3$ all have digon diagram



The digon diagram of Example 1.1.5 (tessellation of $\mathbb{E}^3$ by polyhedra) is



The digon diagram of a geometry Γ is a concise way of capturing some of the structure of Γ. This information is called *local* as it involves rank two residues only.

The usefulness of the digon diagram can be illustrated as follows. Assume that we are looking for all auto-correlations of Γ. It is obvious that each auto-correlation α of Γ permutes the types of Γ, inducing a permutation on I that is an automorphism of $\mathcal{I}(\Gamma)$. If $\mathcal{I}(\Gamma)$ is *linear*, that is, has the shape of single path, we see at once that $\mathcal{I}(\Gamma)$ has exactly two automorphisms: the identity and an involution permuting the endpoints of $\mathcal{I}(\Gamma)$. This implies that Γ has at most two families of auto-correlations: automorphisms and dualities (interchanging elements whose types are at the extreme ends of $\mathcal{I}(\Gamma)$, such as points and hyperplanes in projective geometries). The latter need not exist, as can be seen from the cube geometry of rank three; its dual geometry is the octahedron, which is not isomorphic to the original cube.

The digon diagram of a geometry is not necessarily linear. The digon diagram of Example 1.3.10, for instance, is a triangle. Actually every graph is the digon diagram of some geometry. Nevertheless, most of the geometries studied in this book have a digon diagram that is close to being linear. We now give some examples with non-linear digon diagrams.

Example 2.1.3 Consider the geometry constructed from a tessellation of $\mathbb{E}^3$ by cubes discussed in Example 1.1.5. Up to Euclidean isometry, there is a unique way to color the vertices with two colors $\mathtt{b}$ (for black) and $\mathtt{w}$ (for white), and the cubes (the cells) with two other colors $\mathtt{g}$ (for green) and $\mathtt{r}$ (for red) in such a way that two adjacent vertices (respectively, cubes) do not bear the same color. Let Γ be the geometry over $\{\mathtt{b}, \mathtt{w}, \mathtt{g}, \mathtt{r}\}$ on the bicolored vertices and cubes; incidence is symmetrized inclusion for vertices and cubes, and adjacency for two vertices, as well as for two cubes. The digon diagram of Γ is a quadrangle in which $\mathtt{b}$ and $\mathtt{w}$ represent opposite vertices and $\mathtt{g}$ and $\mathtt{r}$ likewise. Here, and elsewhere, a *quadrangle* is a circuit of length four without further adjacencies; in other words a complete bipartite graph with two parts of size two each.

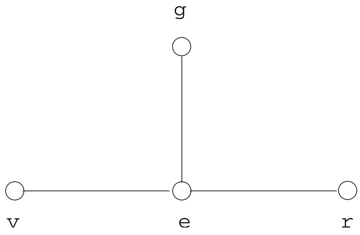


Fig. 2.1 The digon diagram of Δ from Example 2.1.3

For a variation, consider the geometry Δ over $\{\mathtt{v}, \mathtt{e}, \mathtt{g}, \mathtt{r}\}$ whose elements of type $\mathtt{g}$ and $\mathtt{r}$ are as above, and whose elements of type $\mathtt{v}$ and $\mathtt{e}$ are the vertices and edges of the tessellation, respectively. Define incidence by the Principle of Maximal Intersection (cf. Exercise 1.9.20). Then the digon diagram of Δ is as indicated in Fig. 2.1.

There is a relation between the digon diagram of Γ and those of its residues. We will use the notion partial subgraph introduced in Definition 1.2.1.

Proposition 2.1.4 *Let Γ be a geometry over I and let F be a flag of Γ. The digon diagram $\mathcal{I}(\Gamma_F)$ is a partial subgraph of the digon diagram $\mathcal{I}(\Gamma)$.*

Proof It suffices to apply Proposition 1.5.3 and to see that every residue of type $\{i, j\}$ in Γ_F is also a residue of type $\{i, j\}$ in Γ. $\square$

Remark 2.1.5 It may happen that two vertices of $\mathcal{I}(\Gamma_F)$ are not joined while they are joined in $\mathcal{I}(\Gamma)$. Of course, the digon diagram $\mathcal{I}(\Gamma_F)$ is a subgraph of $\mathcal{I}(\Gamma)$ for all flags F if and only if, for any two distinct types i, j, either all or none of the residues in Γ of type $\{i, j\}$ are generalized digons. This property holds for flag-transitive geometries and for most of the geometries we study later. It gives rise to a useful heuristic trick. Given such a 'pure' geometry Γ, its digon diagram $\mathcal{I}(\Gamma)$, and a flag F, it suffices to remove the vertices of $\tau(F)$ from $\mathcal{I}(\Gamma)$ as well as the edges having a vertex in $\tau(F)$ in order to obtain the digon diagram of Γ_F.

Theorem 2.1.6 (Direct Sum) *Let Γ be a residually connected geometry of finite rank and let i and j be types in distinct connected components of the digon diagram $\mathcal{I}(\Gamma)$. Then every element of type i in Γ is incident with every element of type j in Γ.*

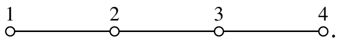
Proof Write $\Gamma = (X, *, \tau)$ and $r = \mathrm{rk}(\Gamma)$. We proceed by induction on r. For $r = 2$, the graph $\mathcal{I}(\Gamma)$ consists of the non-adjacent vertices i and j, so Γ is a generalized digon, for which the theorem obviously holds. Let $r \geq 3$, and let k be an element of $I = \tau(X)$ distinct from i, j. As i and j are in distinct connected components of $\mathcal{I}(\Gamma)$, we may assume that k and i are not in the same connected component of $\mathcal{I}(\Gamma)$. Let x_i (respectively, x_j) be an element of type i (respectively, j). By Lemma 1.6.3, there is an $\{i, j\}$-chain from x_i to x_j in Γ. We must show that x_i, x_j is such a path.

Suppose that we have $x_i * y_j * y_i * x_j$ with $y_j \in X_j$, $y_i \in X_i$. By Corollary 1.6.6 applied to the flag $\{y_i\}$, there is a $\{j, k\}$-chain from y_j to x_j in Γ_{y_i}, say $y_j = y_j^1, z_k^1, y_j^2, z_k^2, \ldots, z_k^s, x_j$, with $z_k^m \in X_k$ and $y_j^m \in X_j$ for all $m \in [s]$. By Proposition 2.1.4, the types i and k are in distinct connected components of $\mathcal{I}(\Gamma_{y_j})$ and by the induction hypothesis we obtain $x_i * z_k^1$. Similarly, in $\Gamma_{z_k^1}$, we find $x_i * y_j^2$. Repeated use of this argument eventually leads to $x_i * x_j$.

The preceding paragraph shows that an $\{i, j\}$-chain of length $m \geq 3$ between an element of X_i and an element of X_j can be shortened to an $\{i, j\}$-chain between the same endpoints of length $m - 2$. Hence the minimal length of such a path, being odd, must be 1. $\square$

In view of Exercise 2.8.20, the condition that I has finite cardinality is needed in Theorem 2.1.6.

Remark 2.1.7 Interesting geometries tend to have a connected digon diagram. However such a geometry may have residues whose digon diagrams are no longer connected. Take for instance a residually connected geometry Γ over [3] whose digon diagram is



If x is an element of type 2, then any element of type 1 and any element of type 3, both incident with x, are necessarily incident with each other. This is the kind of use we will make of the Direct Sum Theorem 2.1.6.

A consequence of the theorem is that any residually connected geometry of finite rank with a disconnected digon diagram can be seen as a direct sum of geometries whose digon diagrams are connected.

Definition 2.1.8 Let J be a set of indices and $(I_j)_{j \in J}$ a system of pairwise disjoint sets. Let $\Gamma_j = (X_j, *_j, \tau_j)$ be a geometry over I_j where $(X_j)_{j \in J}$ is a system of pairwise disjoint sets. The *direct sum* of the geometries Γ_j $(j \in J)$ is the triple $\Gamma = (X, *, \tau)$ where $X = \bigcup_{j \in J} X_j$, $*|_{X_j} = *_j$, $x * y$ for $x \in X_j$, $y \in X_k$, $j \neq k$, and $\tau|_{X_j} = \tau_j$.

Example 2.1.9 A direct sum of two rank one geometries is a generalized digon. Any generalized digon is obtained in this way.

For the direct sum Γ as in Definition 2.1.8, it is clear that Γ is a geometry over $I = \bigcup_{j \in J} I_j$ whose digon diagram $\mathcal{I}(\Gamma)$ is the disjoint union of the digon diagrams $\mathcal{I}(\Gamma_j)$ for $j \in J$. Here is a converse of this statement.

Corollary 2.1.10 *Let Γ be a residually connected geometry of finite rank. Let $(I_j)_{j \in J}$ be the collection of all connected components of the digon diagram $\mathcal{I}(\Gamma)$. Then Γ is isomorphic to the direct sum of the I_j-truncations ${}_{I_j}\Gamma$ $(j \in J)$ of Γ.*

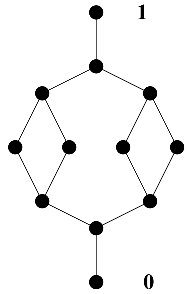


Fig. 2.2 A Jordan-Dedekind poset that is neither firm nor residually connected

Proof Set $\Gamma = (X, *, \tau)$ and ${}_{I_j}\Gamma = (X_j, *_j, \tau_j)$. The sets X_j $(j \in J)$ are pairwise disjoint and $X = \bigcup_{j \in J} X_j$. Also, $*|_{X_j} = *_j$ and $\tau|_{X_j} = \tau_j$. Finally, Theorem 2.1.6 forces $x * y$ for any $x \in X_j$ and $y \in X_k$ with $j \neq k$. □

Example 2.1.11 Suppose that $(P, \leq)$ is a partially ordered set (sometimes abbreviated to *poset*) with a maximal element $\mathbf{1}$ and a minimal element $\mathbf{0}$. It is said to have the *Jordan-Dedekind property* if, for every pair $x, y \in P$ with $x \leq y$, all maximal totally ordered subsets of P with x and y as their minimal, respectively, maximal element, have the same finite cardinality. If $(P, \leq)$ satisfies the Jordan-Dedekind property, we denote by $\Gamma(P, \leq)$ the triple $(P \setminus \{\mathbf{0}, \mathbf{1}\}, *, \tau)$, where $*$ is symmetrized $\leq$, and, for any $x \in P$, the value $\tau(x) + 1$ is the cardinality of any maximal totally ordered subset of P with minimal element $\mathbf{0}$ and maximal element x. If $n = \tau(\mathbf{1}) - 1$, this is a geometry over $[n]$ with a digon diagram that is a partial subgraph of the graph on $[n]$ consisting of the single path $1, 2, \ldots, n$.

The geometry $\Gamma(P, \leq)$ need neither be firm nor residually connected; see Fig. 2.2, where the poset on the vertex set drawn is obtained from the figure by letting $a \leq b$ if and only if a appears at the end of a path downward from b.

Now, $\Gamma(P, \leq)$ is firm if and only if for all a, b, c in P with $a < b < c$ there exists $x \neq b$ in P such that $a < x < c$. We can characterize residual connectedness similarly. Define an interval (a, b) with $a < b$ in P, as the set of all x in P such that $a < x < b$. The elements of (a, b) constitute the vertices of a graph whose edges are the pairs $\{x, y\}$ of distinct vertices x, y such that either $x < y$ or $y < x$. Then, $\Gamma(P, \leq)$ is residually connected if and only if the graph of each interval in P containing at least two elements x, y with $x < y$, is connected. This follows readily from the Direct Sum Theorem 2.1.6.

Conversely, suppose that $\Gamma = (X, *, \tau)$ is a geometry over $[n]$ with a linear digon diagram. For $x, y \in X$, put $x \leq y$ if $\tau(x) \leq \tau(y)$ and $x * y$. Then $\leq$ is a transitive relation by the Direct Sum Theorem 2.1.6. Also, by the definition of incidence system, $x \leq y$ and $y \leq x$ imply $x = y$. Hence $(X \cup \{\mathbf{0}, \mathbf{1}\}, \leq)$, where $\leq$ is extended by the rule $\mathbf{0} \leq x \leq \mathbf{1}$ for all $x \in X$, is a partially ordered set with the Jordan-Dedekind property. Observe that $\Gamma(X \cup \{\mathbf{0}, \mathbf{1}\}, \leq)$ coincides with Γ. These examples show that the local structure (read off from the digon diagram) may be equivalent to the global structure (the ordering on X).

2.2 Some Parameters for Rank Two Geometries

In Sect. 2.1 we began with the idea that a diagram for a geometry Γ over a set of types I would assign to any pair $\{i, j\}$ of elements of I information on the residues of type $\{i, j\}$ of Γ, and went on to distinguish rank two residues that are generalized digons from arbitrary rank two geometries. We now introduce some refinements of this information, capturing more characteristics of rank two geometries. In the remainder of this section, I will be a finite index set and Γ a geometry over I. Often, I will be the set $\{\mathtt{p}, \mathtt{l}\}$ of cardinality two.

Definition 2.2.1 Let Δ be a graph. In Definition 1.6.1, we introduced the distance function d (or d_Δ). For a vertex x of Δ, the *diameter of Δ at x* is the largest distance from x to any other vertex of Δ. The *diameter* δ_Δ of Δ is the largest distance between two vertices of Δ. If Δ does not have finite diameter, then we put $\delta_\Delta = \infty$.

Often, for two vertices x, y of Δ, we write $x \perp y$ (or $x \perp_\Delta y$ if confusion is imminent) to denote $d(x, y) \leq 1$, that is, x and y are equal ($x = y$) or adjacent ($x \sim y$). For a vertex x and a set Y of vertices of Δ, we write $x^\perp$ for the set of all vertices y with $y \perp x$ and $Y^\perp$ for $\bigcap_{y\in Y} y^\perp$.

Let $\Gamma = (X, *, \tau)$ be an incidence system over I. If Δ is the incidence graph of Γ, then $x * y$ is equivalent to $x \perp y$, and $x^\perp = x^*$, etc. *Distance* in Γ is usually understood to be distance in Δ, and similarly for the diameter. For $j \in I$, the *j-diameter* d_j of Γ is the largest number occurring as a diameter of the incidence graph of Γ at some element of type j.

Now take $I = \{\mathtt{p}, \mathtt{l}\}$. The *collinearity graph* of Γ on $X_\mathtt{p} = \tau^{-1}(\mathtt{p})$ is the graph $(X_\mathtt{p}, \sim)$ with vertex set $X_\mathtt{p}$ in which x and y are adjacent (equivalently, $x \sim y$ holds) whenever they have distance two in Γ; equivalently, whenever they are distinct and there is a line $L \in X_\mathtt{l}$ such that $x * L * y$.

The *shadow* of $L \in X_\mathtt{l}$ on $\{\mathtt{p}\}$ is the set $L^* \cap X_\mathtt{p}$. It is a clique in the collinearity graph of Γ on $X_\mathtt{p}$.

By the symmetry of the roles of $\mathtt{p}$ and $\mathtt{l}$, we also have the notion of the collinearity graph on $X_\mathtt{l}$. In this graph, two lines are adjacent whenever they are distinct and there is a point to which both are incident.

If Γ as above is connected (cf. Definition 1.6.1), then there are exactly two connected components in the graph on X in which adjacency is defined as having mutual distance two: the parts $X_\mathtt{p}$ and $X_\mathtt{l}$. The subgraphs induced on these parts are the two collinearity graphs of Γ.

If $I = \{\mathtt{p}, \mathtt{l}\}$, the difference between $d_\mathtt{p}$ and $d_\mathtt{l}$ is at most one and the larger one is equal to the diameter δ of Γ. If δ is odd, then $d_\mathtt{p} = d_\mathtt{l} = \delta$, since both a point and a line are involved in a pair of elements at maximal distance.

Example 2.2.2 If Γ is a polygon with n vertices, considered as a geometry over $\{\mathtt{p}, \mathtt{l}\}$, then $d_\mathtt{p} = d_\mathtt{l} = n$. In a generalized digon, we have $d_\mathtt{p} = d_\mathtt{l} = 2$. In the real affine plane $\mathbb{E}^2$ (cf. Example 1.1.2), we have $d_\mathtt{p} = 3$, $d_\mathtt{l} = 4$, and in the real projective plane $\mathrm{PG}(\mathbb{R}^3)$ (cf. Example 1.4.9), we have $d_\mathtt{p} = d_\mathtt{l} = 3$. This means that the collinearity graph of the projective plane on the point set (and on the line set) is a clique. The collinearity graph of an affine plane on the line set has diameter two, while the collinearity graph on the point set is again a clique.

We introduce yet another parameter.

Definition 2.2.3 A *circuit* in the geometry Γ over $I = \{\mathtt{p}, \mathtt{l}\}$ is a chain $x = x_0, x_1, x_2, \ldots, x_{2n} = x$ from x to x, with $x_i \neq x_{i+1}, x_{i+2}$ for $i = 0, \ldots, 2n$ (all indices taken modulo $2n$ and $n > 0$). Its length $2n$ is necessarily even. The minimal number $g > 0$ such that Γ has a circuit of length $2g$ is called the *girth* of Γ. If Γ has no circuits, we put $g = \infty$.

Lemma 2.2.4 *The girth g of a firm geometry Γ over $I = \{\mathtt{p}, \mathtt{l}\}$ satisfies*

$$\begin{array}{ll} \textit{either} & 2 \leq g \leq d_{\mathtt{p}} \leq d_{\mathtt{l}} \leq d_{\mathtt{p}} + 1 \\ \textit{or} & 2 \leq g \leq d_{\mathtt{l}} \leq d_{\mathtt{p}} \leq d_{\mathtt{l}} + 1. \end{array}$$

Proof This is a direct consequence of the observations preceding Example 2.2.2. The assertions also hold for $g = \infty$. □

Definition 2.2.5 The *dual geometry* $\Gamma^\vee$ of a geometry Γ over $I = \{\mathtt{p}, \mathtt{l}\}$ is the triple $(X, *, \tau^\vee)$ where $\tau^\vee(x) = \mathtt{l}$ if and only if $\tau(x) = \mathtt{p}$.

The girths of Γ and $\Gamma^\vee$ are equal while $d^\vee_{\mathtt{p}} = d_{\mathtt{l}}$ and $d^\vee_{\mathtt{l}} = d_{\mathtt{p}}$ (with the obvious interpretations of $d^\vee_{\mathtt{p}}$ and $d^\vee_{\mathtt{l}}$).

Definition 2.2.6 If Γ is a $\{\mathtt{p}, \mathtt{l}\}$-geometry with finite diameter d and with girth g having the same diameter d_i at all elements of type i (for $i = \mathtt{p}, \mathtt{l}$), then Γ is called a $(g, d_{\mathtt{p}}, d_{\mathtt{l}})$*-gon* over $(\mathtt{p}, \mathtt{l})$. If, in addition, $g = d_{\mathtt{p}} = d_{\mathtt{l}}$, then Γ is called a *generalized g-gon*.

Allowing $g = \infty$ in the above definition of a generalized g-gon, we see that generalized ∞-gons have no circuits.

If Γ is a $(g, d_{\mathtt{p}}, d_{\mathtt{l}})$-gon over $(\mathtt{p}, \mathtt{l})$, then it is a $(g, d_{\mathtt{l}}, d_{\mathtt{p}})$-gon over $(\mathtt{l}, \mathtt{p})$.

The definition of a generalized 2-gon coincides with that of a generalized digon in Definition 2.1.1. In view of the terminology below, the name digon fits a 2-gon.

Definition 2.2.7 Generalized 3-gons are also called *projective planes*, generalized 4-gons are called *generalized quadrangles*. Likewise, generalized 6-gons are called *generalized hexagons* and generalized 8-gons are called *generalized octagons*. *Generalized polygons* is the name used for all generalized g-gons ($g \geq 2$).

For $g \in \mathbb{N} \cup \{\infty\}$, the *(ordinary) g-gon* is defined as the thin generalized g-gon.

It is easy to see that the ordinary g-gon is indeed unique: it is isomorphic to the geometry consisting of the set $\mathbb{Z}/2g\mathbb{Z}$ with types `even` and `odd` and incidence

$x * y \iff x - y \in \{0, 1, -1\}$ for $x, y \in \mathbb{Z}/2g\mathbb{Z}$. Here, in case $g = \infty$, we interpret $\mathbb{Z}/2\infty\mathbb{Z}$ as $\mathbb{Z}$.

Generalized polygons are the building blocks of the classical geometries that we encounter in later chapters. Although there is no hope of describing all graphs that are $(g, d_{\mathtt{p}}, d_{\mathtt{l}})$-gons for arbitrary g, $d_{\mathtt{p}}$, $d_{\mathtt{l}}$, a lot of structure can be pinned down if $g = d_{\mathtt{p}} = d_{\mathtt{l}}$. A classification of all finite generalized g-gons with $g = 3$ or $g = 4$ can hardly be expected in the presence of so many wild examples. The examples in the remainder of this section provide some evidence for this. The occurrence of projective planes and generalized quadrangles as residues in geometries of higher rank usually imposes conditions which enable us to classify them.

Example 2.2.8 The Petersen graph (cf. Example 1.3.3) is a $(5, 5, 6)$-gon over $(\mathtt{vertex}, \mathtt{edge})$ and the cube is a $(4, 6, 6)$-gon over $(\mathtt{vertex}, \mathtt{edge})$. The Fano plane (cf. Exercise 1.9.7) is a generalized 3-gon.

Theorem 2.2.9 *Let Γ be a $\{\mathtt{p}, \mathtt{l}\}$-geometry. Then Γ is a projective plane if and only if*

(i) *every pair of points is incident with a unique line*;
(ii) *every pair of lines is incident with a unique point*;
(iii) *there exists a non-incident point-line pair.*

Proof Set $\Gamma = (X, *, \tau)$ and write $P = X_{\mathtt{p}}$ and $L = X_{\mathtt{l}}$. First, suppose that Γ is a projective plane. Let $x, y \in P$ be distinct. As the point-diameter of Γ is 3, there is a path of length at most 3 from x to y. But this length must be even and strictly greater than 0, hence equal to 2. In other words, there is $h \in L$ incident with both x and y. If $m \in L$ is a line distinct from h also incident with x and y, then x, h, y, m, x is a 4-circuit in Γ, which contradicts that the girth is 6. Hence (i). Statement (ii) follows likewise. Finally (iii) is a consequence of the existence of a path of length three in the incidence graph of Γ.

Next suppose that Γ satisfies (i), (ii), and (iii). Then (i) and (ii) force $d_{\mathtt{p}} \leq 3$ and $d_{\mathtt{l}} \leq 3$, respectively, while (iii) implies that equality holds for both. If $g = 2$, then there are two points incident with two distinct lines, contradicting both (i) and (ii). Hence $g \geq 3$. But Γ is firm (it is a $\{\mathtt{p}, \mathtt{l}\}$-geometry) so, by the inequalities in Lemma 2.2.4, $g = 3$. □

The geometries $\mathrm{PG}(\mathbb{D}^3)$ of Example 1.4.9 for any division ring $\mathbb{D}$ are examples. But there are more, as will be clear from Example 2.3.4.

Example 2.2.10 Let $X_{\mathtt{p}}$ be the set of all pairs from $[6]$ and let $X_{\mathtt{l}}$ be the set of all partitions of $[6]$ into three pairs. The geometry $\Gamma = (X_{\mathtt{p}}, X_{\mathtt{l}}, *)$, where $*$ is symmetrized containment, is a generalized quadrangle with 15 points and 15 lines. It is drawn in Fig. 2.3, where lines are represented by arcs and line segments.

Theorem 2.2.11 *Every generalized quadrangle with three points on each line and three lines through each point is isomorphic to the one of Example 2.2.10.*

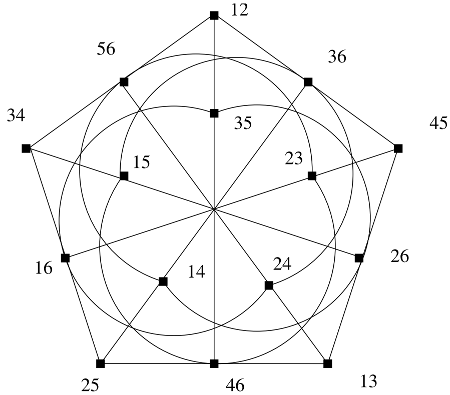


Fig. 2.3 A generalized quadrangle with 15 points and 15 lines

Proof Let $(X_{\mathtt{p}}, X_{\mathtt{l}}, *)$ be such a generalized quadrangle. It must have 15 points: if we fix one, say ∞, then there are $3 \cdot 2 = 6$ neighbors (two on each line through ∞) in the collinearity graph on the points; as each neighbor of ∞ is collinear with only one point that is also a neighbor of ∞, a count of edges from points at distance one to points at distance two, using $d_{\mathtt{p}} = d_{\mathtt{l}} = 4$, shows that there are $6 \cdot 4/3 = 8$ points at distance two from ∞; finally, as $d_{\mathtt{p}} = 4$, there are no points at distance greater than two from ∞.

The generalized quadrangle is determined by the collinearity graph on the point set $X_{\mathtt{p}}$, as the lines correspond bijectively to the maximal cliques (of size three).

If a and b are distinct non-collinear points in $X_{\mathtt{p}}$, then $\{a, b\}^{\perp}$ has exactly three points (for b is collinear with one point on each line incident with a). We claim that $(\{a, b\}^{\perp})^{\perp}$ also has exactly three points. To see this, let x, y, z be the three points of $\{a, b\}^{\perp}$, and let c be the third point of $\{x, y\}^{\perp}$ distinct from a and b. It suffices to show $c \sim z$. If not, there would be four distinct lines on z: beside the lines incident with a and b, there would be lines incident with the third points on the line incident with c and y and on the line incident with c and x. Indeed, coincidence of any two of these four lines on z would lead to a contradiction with the girth being 4. But there are precisely three lines on z, so we must have $c \sim z$, as required for the claim.

We finish by identifying the collinearity graph on P with the graph on the pairs from [6] (see Example 2.2.10) in which two vertices are adjacent whenever they are disjoint. To this end, start with the configuration on the six points of the previous paragraph and label the points as follows: $a = 12$, $b = 23$, $c = 13$, $x = 45$, $y = 56$, $z = 46$. The subgraph of the collinearity graph of P induced on these points is as it should be. Each of the nine lines on two collinear points from the sextet has a third point as yet unaccounted for. Label these nine points by the pair that complements the two pairs from the points already assigned on that line. For instance, the third point of the line on a and x receives label 36. This way we have labelled all 15 points and the remaining lines are forced as indicated by Example 2.2.10. □

Remark 2.2.12 Putting together the above uniqueness result and Example 2.2.10, we see that the symmetric group Sym_6 acts as a group of automorphisms on the generalized quadrangle Γ of Example 2.2.10. The uniqueness proof can also be used to show that $\mathrm{Aut}(\Gamma)$ is isomorphic to Sym_6. Besides, the flag transitivity of this group on Γ shows that the geometry can alternatively be described as $\Gamma(G, (C_G((1,2)), C_G((1,2)(3,4)(5,6))))$, where $G = \mathrm{Sym}_6$ and $C_G(x)$ denotes the centralizer in G of the element x of G.

Adding a formal point 0 to X_{p}, we can define addition on X_{p} as follows: for distinct $u, v \in X_{\mathrm{p}}$ we set $0 + u = u + 0 = u$, $u + u = 0$ and $u + v = w$, where w is the unique point other than u, v in $\{u, v\}^{\perp\perp} = (\{u, v\}^{\perp})^{\perp}$. In view of the abundance of automorphisms, we only need check the existence and uniqueness of w for $u = 12$ and $v = 34$ or $v = 13$; in these respective cases, w is 56 or 23. The addition leads to an $\mathbb{F}_2$-vector space structure on $V := \{0\} \cup X_{\mathrm{p}}$, turning it into $\mathbb{F}_2^4$ (there being $1 + 15 = 2^4$ points). So the generalized quadrangle is a subgeometry of the $\{1, 2\}$-truncation of $\mathrm{PG}(\mathbb{F}_2^4)$ (see Examples 1.4.9 and 1.5.6).

There is a bilinear form $f : V \times V \to \mathbb{F}_2$ given by $f(x, y) = 1$ if $x, y \in X_{\mathrm{p}}$ are non-collinear points, and 0 otherwise. This form is nondegenerate and antisymmetric. The generalized quadrangle can now be described completely in terms of the vector space and the bilinear form: its points and its lines are the one, respectively, two dimensional singular subspaces of V (as defined in Example 1.4.13). It follows that the generalized quadrangle is an absolute geometry of the projective geometry $\mathrm{PG}(V)$.

Example 2.2.13 The uniqueness of the generalized quadrangle Γ of Example 2.2.10 implies the existence of a duality: the dual geometry $\Gamma^{\vee}$ also satisfies the properties of Theorem 2.2.11 and so must be isomorphic to Γ. This means that $\mathrm{Aut}(\Gamma)$ has index two in $\mathrm{Cor}(\Gamma)$. The bigger group contains 'outer' (as opposed to inner) involutions of Sym_6. Here, we will construct polarities of Γ in a geometric way, using ovoids and spreads.

An *ovoid* of Γ is a set O of points with the property that each line in X_{l} is incident with exactly one point in O. A simple count shows that an ovoid is a set of five pairwise non-collinear points (and conversely). Dually, a *spread* of Γ is an ovoid of $\Gamma^{\vee}$, that is, a subset S of X_{l} with the property that every point is on exactly one line in S. Denote by $\mathcal{O}$ and $\mathcal{S}$ the collection of ovoids and spreads, respectively, of Γ. It is easy to see that the ovoids are of the form $O_i = \{\{i, j\} \mid j \in [6] \setminus \{i\}\}$ and that $i \mapsto O_i$ is a bijection $[6] \to \mathcal{O}$. Any two members O_i and O_j meet in exactly one point, *viz.* $\{i, j\}$. Using the duality, corresponding statements for spreads can be derived.

The diagonal action of Sym_6 on $\mathcal{O} \times \mathcal{S}$ is transitive. For, in view of duality, every spread is left invariant by a subgroup of Sym_6 isomorphic to Sym_5, and, by taking a specific spread, it is easily seen that the stabilizer acts transitively on [6] whence on $\mathcal{O}$.

Now, let (O, S) be a pair in $\mathcal{O} \times \mathcal{S}$. We claim that there is a unique polarity π of Γ mapping $x \in O$ to the unique line $\pi(x) \in S$ to which it belongs (thus $x \in \pi(x)$). By transitivity of Sym_6 on $\mathcal{O} \times \mathcal{S}$, it suffices to check this for a single pair (O, S). To determine the image of $y \in X_{\mathrm{p}} \setminus O$, consider the line $h \in S$ containing y, and the lines $m, n \in S$ distinct from h containing a point of O collinear with y. Then $\pi(y)$

Fig. 2.4 Distribution diagram of the double six

must be the line through the point $\pi(h) \in O$ (on h) and meeting both m and n. This, and the fact that π has order two, uniquely determines the polarity π on Γ.

As $\mathrm{Aut}(\Gamma)$ is isomorphic to Sym_6, we find that $\mathrm{Cor}(\Gamma)$ is a group isomorphic to $\mathrm{Sym}_6 \rtimes \mathrm{C}_2$. The transposition $(1,2)$ acts naturally on $[6]$, and, as $i \mapsto O_i$ is an equivalence of Sym_6-representations between $[6]$ and $\mathcal{O}$, also on $\mathcal{O}$. But this element has no fixed points on $\mathcal{S}$ (for, if $S \in \mathcal{S}$ would be fixed by $(1,2)$, then the line in S on $\{1,3\}$ will have a point $\{k,l\}$ with $k,l \neq 1,2$, that is, a fixed point, so the line must be fixed by $(1,2)$, contradicting that its point $\{1,3\}$ is mapped to $\{2,3\}$). Therefore, its image under the action on $\mathcal{S}$ is a product of three transpositions. This shows that the map $g \mapsto \pi g \pi$ $(g \in G)$ is an outer automorphism of G.

Example 2.2.14 On $\mathcal{O} \times \mathcal{S}$ of Example 2.2.13 as a vertex set, an interesting graph Δ arises by letting (O,S) and (O',S') be adjacent whenever $O \neq O'$ and $S \neq S'$ but $O \cup S$ and $O' \cup S'$ have a flag in common. This graph is known as the *double six*. By the above, the intersection of $O \cup S$ and $O' \cup S'$ always contains a point and a line. A fixed pair (O,S) has 5 neighbors (given the choice of a flag in $O \cup S$, the adjacent vertex meeting in that flag is uniquely determined). It requires a little elaboration to see that a vertex (O',S') for which the intersection consists of a non-incident point-line pair, is adjacent to a unique neighbor of (O,S), and that there are 20 of them. Finally, the 10 vertices (O,S') and (O',S) with $S' \neq S$ and $O' \neq O$ are at distance three from (O,S). Schematically, this information is conveyed in the distribution diagram (cf. Example 1.7.16) depicted in Fig. 2.4.

Another generalized quadrangle arises from the double six Δ. Consider the graph obtained from Δ on the same vertex set by letting two vertices be adjacent whenever they are at distance three in Δ. This graph is the 6×6-*grid*, that is, the Cartesian product of two cliques of size six. Its vertex set and its set of maximal cliques form a generalized quadrangle with six points per line but two lines per point.

Example 2.2.15 We construct a generalized hexagon with three points on each line and three lines on each point. Let $\mathbb{F}$ be $\mathbb{F}_9$, the field of 9 elements. Equip the vector space $\mathbb{F}^3$ with the standard unitary inner product

$$f(x,y) = \sum_{i=1}^{3} x_i^3 y_i \quad \left(x, y \in \mathbb{F}^3\right).$$

Let P be the set of nonsingular points of the underlying projective space (cf. Example 1.4.9), that is, $P = \{x\mathbb{F} \mid f(x,x) \neq 0\}$, and write $x\mathbb{F} \sim y\mathbb{F}$ whenever $f(x,y) = 0$. Maximal cliques in the graph $(P, \sim)$ have size three, and correspond to orthonormal bases of $\mathbb{F}^3$ (up to scalar multiples for the basis vectors). Let L be the collection of all these maximal cliques. We claim that $(P, L, *)$, where $*$ is symmetrized containment, is a generalized hexagon. It has 63 points and 63 lines.

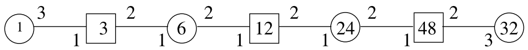


Fig. 2.5 The distribution diagram of the generalized hexagon of Example 2.2.15

To verify this, observe that any pair $a\mathbb{F}$, $b\mathbb{F}$ of points with $f(a,b)=0$ lies on a line. Suppose that $\{a\mathbb{F}, b\mathbb{F}, c\mathbb{F}\}$ is a line. Without loss of generality, we normalize $f(a,a)=f(b,b)=f(c,c)=1$. Now $f(b+c,b+c)=f(b-c,b-c)=2\neq 0$, and $f(b\pm c,a)=f(b+c,b-c)=0$, so $\{a\mathbb{F}, (b+c)\mathbb{F}, (b-c)\mathbb{F}\}$ is a line on a. It is readily seen that there is only one more line on a, *viz.* $\{a\mathbb{F}, (b+ic)\mathbb{F}, (b-ic)\mathbb{F}\}$, where i is a square root of -1 in $\mathbb{F}$. Thus, each point is on exactly 3 lines. Interchanging the roles of a, b, and c, we obtain the lines on $b\mathbb{F}$ and $c\mathbb{F}$ as well, and see that no point collinear with a neighbor of $a\mathbb{F}$ is collinear with a neighbor of $b\mathbb{F}$. In particular, the girth of $(P, L, *)$ is at least six.

Suppose $d\mathbb{F}\in P$ is not collinear with any neighbor of $a\mathbb{F}$, $b\mathbb{F}$, or $c\mathbb{F}$. Write $d=\alpha a+\beta b+\gamma c$, so $\alpha\beta\gamma\neq 0$. There is a common neighbor in P of $c\mathbb{F}$ and $d\mathbb{F}$ if and only if $\alpha\in\{\pm 1,\pm i\}\beta$, that is, $\alpha^4=\beta^4$. Changing the roles of a, b, c again, and using that $\alpha^4+\beta^4+\gamma^4\neq 0$, we see that there exists a unique point in $\{a,b,c\}$ that is at distance two to $d\mathbb{F}$. The conclusion is that the point diameter and line diameter are 6 and that the girth is at least 6. But then the girth must be precisely 6 by Lemma 2.2.4. This establishes the claim that $(P, L, *)$ is a generalized hexagon.

Figure 2.5 gives a pictorial description of this generalized hexagon, from which it is immediate that there are precisely 63 points and just as many lines.

There is a nice way to visualize automorphisms. To each $a\mathbb{F}\in P$, we associate the following linear transformation r_a of $\mathbb{F}^3$.

$$r_a(x)=x+af(a,a)^{-1}f(a,x)\quad \big(x\in\mathbb{F}^3\big).$$

Such a linear transformation is the special case of the unitary reflection $r_{a,\phi}$ with $\phi(x)=2f(a,a)^{-1}f(a,x)$, presented in Exercise 1.9.31. Observe that r_a does not depend on the choice of vector in $\mathbb{F}a$. Being a member of the unitary group $\mathrm{U}(\mathbb{F}^3, f)$, it preserves the unitary inner product, so induces an automorphism of $(P, L, *)$. The reflection r_a fixes every vector in the hyperplane $\{x\in\mathbb{F}^3 \mid f(x,a)=0\}$, whence all points of $(P, L, *)$ collinear with $a\mathbb{F}$. It moves each point at distance two from $a\mathbb{F}$ in the collinearity graph to its unique neighbor at distance two from $a\mathbb{F}$.

By use of these reflections, the transitivity of $G=\mathrm{U}(\mathbb{F}_9^3, f)$ on each of the sets $\{(a,b)\in P\times P \mid d(a,b)=i\}$ for $i=2,4,6$ is readily established. The map $P\to G$ given by $a\mathbb{F}\mapsto r_a$ sends P to a conjugacy class of reflections in G. Moreover, three straightforward checks show that r_ar_b is a transformation of order 2, 4, or 3 according to whether $a\mathbb{F}$ and $b\mathbb{F}$ have mutual distance 2, 4, or 6 in $(P\cup L, *)$. This gives an alternative description of $(P\cup L, *)$, entirely in terms of the group G: a line can be seen as the set of three nontrivial reflections in a subgroup S isomorphic to C_2^3 generated by reflections. As a consequence, the generalized hexagon can be described as $\Gamma(G, (C_G(r), N_G(S)))$, where $C_G(r)$ is the centralizer of a reflection r in S and $N_G(S)=\{g\in G \mid gSg^{-1}=S\}$, the *normalizer* of S in G.

For geometries with finite rank 1 residues, some more parameters are useful.

Definition 2.2.16 Let Γ be a geometry over I and fix $i \in I$. If each residue of Γ of type $\{i\}$ has the same finite size $s_i + 1$, then s_i is called the *i-order* of Γ. If Γ is a geometry over I having i-order s_i for each $i \in J$ for some subset J of I, then $(s_i)_{i\in J}$ is called the *J-order* of Γ. In case $J = I$ we just speak of the *order* of Γ. In particular, for a $\{\mathtt{p}, \mathtt{l}\}$-geometry in which both line and point order exist, we speak of the order $(s_{\mathtt{p}}, s_{\mathtt{l}})$ over $(\mathtt{p}, \mathtt{l})$.

Thus, Example 2.2.10 gives a generalized quadrangle of order (2, 2) and Example 2.2.15 a generalized hexagon of order (2, 2).

Remark 2.2.17 The parameters g, $d_{\mathtt{p}}$, $d_{\mathtt{l}}$, $s_{\mathtt{p}}$, $s_{\mathtt{l}}$ can be used to enrich digon diagrams. We summarize the information they provide by a diagram such as

$$\overset{\mathtt{p}}{\underset{s_{\mathtt{p}}}{\circ}}\overset{d_{\mathtt{p}}\qquad g\qquad d_{\mathtt{l}}}{\rule[0.5ex]{8em}{0.4pt}}\overset{\mathtt{l}}{\underset{s_{\mathtt{l}}}{\circ}}.$$

The diagram stands for the class of all $(g, d_{\mathtt{p}}, d_{\mathtt{l}})$-gons of order $(s_{\mathtt{p}}, s_{\mathtt{l}})$ over $(\mathtt{p}, \mathtt{l})$. A member of this class is said to *belong to the diagram*. If the subscripts $s_{\mathtt{p}}$, $s_{\mathtt{l}}$ are dropped, a geometry belongs to the diagram if and only if it is a $(g, d_{\mathtt{p}}, d_{\mathtt{l}})$-gon over $(\mathtt{p}, \mathtt{l})$. For example, the Petersen graph belongs to $\overset{\mathtt{p}}{\underset{1}{\circ}}\overset{5\qquad 5\qquad 6}{\rule[0.5ex]{6em}{0.4pt}}\overset{\mathtt{l}}{\underset{2}{\circ}}$ and the real affine plane to $\overset{\mathtt{p}}{\circ}\overset{3\qquad 3\qquad 4}{\rule[0.5ex]{6em}{0.4pt}}\overset{\mathtt{l}}{\circ}$.

The Petersen graph is easily seen to be the unique geometry with the given diagram (including the specified orders). Replacing the order 2 by 6, we obtain a more complicated example.

Example 2.2.18 We construct a graph with a lot of symmetry known as the Hoffman-Singleton graph. Start with the set $X_{\mathtt{p}} = \binom{[7]}{3}$ of all triples from [7]. It has cardinality 35. There are 30 collections of 7 elements of $X_{\mathtt{p}}$ that form the lines of a Fano plane with point set [7]. The group Sym_7 acts transitively on this set of 30 Fano plane structures, but the alternating group Alt_7 has two orbits, of cardinality 15 each. Select one and call it $X_{\mathtt{l}}$. Now consider the following graph HoSi constructed from the incidence graph $(X_{\mathtt{p}} \cup X_{\mathtt{l}}, *)$ by additionally joining two elements of $X_{\mathtt{p}}$ whenever they have empty intersections. The resulting graph HoSi has 50 vertices, is regular of valency 7, and has a group of automorphisms isomorphic to Alt_7. But there are additional automorphisms, fusing the orbits of Alt_7 on the vertex set of sizes 35 and 15, as we will see in Theorem 2.2.19 below. This graph HoSi is the *Hoffman-Singleton graph*.

We give an alternative description of HoSi. At first sight it is not clear at all that we are dealing with the same graph (up to isomorphism), so we will call this graph HoSi′. Recall the double six Δ of Example 2.2.14 and the related collections $\mathcal{O}$ and $\mathcal{S}$ of ovoids and spreads. Select two types, $\mathtt{o}$ and $\mathtt{s}$ and build the tree T on 14 vertices with a central edge $\{\mathtt{o}, \mathtt{s}\}$ both of whose vertices are adjacent to 6 end nodes (that is, vertices of valency 1). Label the end nodes of T adjacent to $\mathtt{o}$ by the members of $\mathcal{O}$, and those adjacent to $\mathtt{s}$ by the members of $\mathcal{S}$. Now let HoSi′ be the graph whose vertex set is the disjoint union of the vertex sets of T and Δ and in which T and Δ are subgraphs; join the vertex $S \in \mathcal{S}$ of T to every (O', S) in Δ for $O' \in \mathcal{O}$, and, likewise, join the vertex $O \in \mathcal{O}$ of T to (O, S') for every $S' \in \mathcal{S}$. From this description, we see that $\mathrm{Aut}(\mathrm{Sym}_6)$ acts on HoSi'.

It is readily derived that the geometries of vertices and edges of both HoSi and HoSi' are $\{\mathtt{p}, \mathtt{l}\}$-geometries with parameters

$$\overset{\mathtt{p}}{\underset{1}{\circ}}\!\!\overset{5 \qquad 5 \qquad 6}{\rule[0.5ex]{8em}{0.4pt}}\!\!\overset{\mathtt{l}}{\underset{6}{\circ}}.$$

Here is a characterization of the Hoffman-Singleton graph.

Theorem 2.2.19 *There is a unique* $(5, 5, 6)$*-gon of order* $(1, 6)$ *up to isomorphism. It has a flag-transitive group of automorphisms.*

Proof Let $\Gamma = (X_{\mathtt{p}}, X_{\mathtt{l}}, *)$ be a $(5, 5, 6)$-gon of order $(1, 6)$. As the girth is greater than two and $s_{\mathtt{p}} = 1$, we may view Γ as the geometry of points and edges of the collinearity graph $X = (X_{\mathtt{p}}, \perp)$. The line order is 6, so this graph has valency 7. Moreover, any point $x \in X_{\mathtt{p}}$ has 7 neighbors and each neighbor is adjacent to exactly 6 points at distance two from x in X, whereas no point at distance two has more than one neighbor in common with x. Hence, X has $1 + 7 + 7 \times 6 = 50$ vertices.

If x and y are nonadjacent vertices of X, then there is a unique path in the collinearity graph of length two from x to y, and each of the 6 edges on y not on this path of length two is on a unique path of length three from x to y. Thus every path of length two lies on exactly 6 pentagons. Let P be a pentagon in Γ and write $X_1(P)$ for the set of points outside P collinear with a point of P. The set $X_1(P)$ has $5 \times 5 = 25$ vertices (no vertex outside P can be adjacent to two members of P), and each of these lies on exactly two paths of length three between two nonadjacent members of P. Consequently, $X_1(P)$ is a regular subgraph of Γ of valency two. Set $X_2(P) = X_{\mathtt{p}} \setminus (P \cup X_1(P))$. Each vertex in this set has distance two to every member of P, so lies on precisely five edges having a vertex in $X_1(P)$. Consequently, $X_2(P)$ is also regular of valency two. Suppose that a, b, c is a path in $X_2(P)$. There are at least four paths from a to c of length three having both non-end points in $X_1(P)$; let a, d, e, c be one of them. Denote by f, g the unique neighbor of e, d, respectively, in $X_1(P) \setminus \{e, d\}$. Now f and b cannot be adjacent, so there must be a common neighbor x say. As x cannot be one of a or c, we must have $x \in X_1(P)$, so x is the unique neighbor of f in $X_1(P) \setminus \{e\}$. Tracing adjacencies with vertices of P, we see that x is also adjacent to the unique point z of P at distance two to both e and d, so that $\{x\} = \{z, b\}^{\perp}$. But then, arguing with g instead of f, we find that x is also adjacent to g, leading to the 5-circuit d, e, f, x, g in $X_1(P)$. Varying over the 4 edges in $X_1(P)$ on paths of length 3 from a to c, we see that $X_1(P)$ has at least four 5-circuits, whence consists entirely of five disjoint pentagons. Repeating the argument with P' a pentagon in $X_1(P)$, we find that $X_1(P') = P \cup X_2(P)$ also consists of five disjoint pentagons. Thus, there exists a partition Π of $X_{\mathtt{p}}$ into 10 pentagons.

Clearly, we get much more than that partition Π. Every pentagon A in Π determines the partition uniquely. Moreover, A is 'adjacent' to five other members of

Π that are pairwise non-adjacent. Here, adjacency of two pentagons from Π means that each vertex of the first is adjacent to one and only one from the second. On each pair of adjacent pentagons, the induced subgraph of X is a Petersen graph.

This forces a unique partition of Π in two sets of five pentagons say, Π_1, Π_2, with the property that each member of Π_i is adjacent to each member of Π_j for $j \neq i$. Recalling that the full graph has no circuits of length smaller than five and making a picture of the five pentagons in Π_1 next to those of Π_2, we find that, up to symmetry, there is indeed at most one graph with the required properties.

The statement on flag transitivity follows by comparing the two constructions given above. From HoSi, we derive the existence of a group of automorphisms isomorphic to Alt_7 with vertex orbits of lengths 15 and 35, and from HoSi′ a group of automorphisms isomorphic to $\mathrm{Aut}(\mathrm{Sym}_6)$ with orbits of lengths 2, 12 (in T), and 36 (in Δ). Thus, the automorphism group must be transitive on the vertex set. Let x be a vertex of X. By the first description, the stabilizer of x contains an element of order 7 permuting the 7 neighbors transitively. By the second description, the stabilizer of x and a neighbor y induces Sym_6 on the six remaining neighbors of x. Thus, the automorphism group is flag transitive on the corresponding geometry Γ. □

Remark 2.2.20 The automorphism group G of the Hoffman-Singleton graph can be determined further. If p is a vertex of HoSi, the stabilizer in Aut(HoSi) of the set of 7 vertices adjacent to p contains a copy of Sym_7 and cannot be larger. Hence, G has order $50 \cdot 7! = 252000$. It can be shown that G has a simple subgroup H of index 2 which is isomorphic to $\mathrm{PSU}(\mathbb{F}_{25}^3, f)$, the quotient by the center of the subgroup $\mathrm{SU}(\mathbb{F}_{25}^3, f)$ of the unitary group $\mathrm{U}(\mathbb{F}_{25}^3, f)$ with respect to the standard unitary form f on $\mathbb{F}_{25}^3$ consisting of all transformations of determinant 1. This group is simple and G is obtained from it by adjoining the field automorphism (sending each matrix entry to its fifth power).

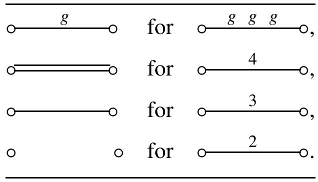
There are exactly 100 *cocliques* (that is, subgraphs without edges) in HoSi of size 15. A typical example is the Alt_7-orbit of length 15 in the vertex set of HoSi. The group Aut(HoSi) is transitive on this set of cocliques. Its index two subgroup H has two orbits, of size 50 each.

Notation 2.2.21 As we have mentioned before, the extreme cases where $g = d_\mathrm{p} = d_\mathrm{l}$ will be studied most intensively. To economize on notation, the abbreviations indicated in Table 2.1 will be frequently used (with $g \in \mathbb{N} \cup \{\infty\}$).

2.3 Diagrams for Higher Rank Geometries

Theorem 2.2.9 shows how to capture the classical definition of a projective plane in terms of the parameters recorded in diagrams for rank two geometries. There is an analogue of this result for affine planes. We will discuss it as well as their connection with projective planes and show that not all affine planes are of the form $\mathrm{AG}(\mathbb{D}^2)$ for a division ring $\mathbb{D}$. The class of these rank two geometries serves as an example

Table 2.1 Pictorial abbreviations



for describing geometries of higher rank by means of diagrams. The crux of the diagram information for a geometry Γ is that it gives us an isomorphism class to which specified rank two residues of Γ belong.

After the introduction of a diagram for geometries of arbitrary rank, we discuss variations on the notion of a graph that is locally isomorphic to a given graph. These variations are governed by a diagram for geometries of rank three.

Recall from Definition 1.2.3 that $x^* = \{x\}^*$ is the set of all elements of Γ incident with x. Throughout the section, we let I stand for a set of types.

Definition 2.3.1 A $\{\mathtt{p}, \mathtt{l}\}$-geometry $\Gamma = (X_{\mathtt{p}}, X_{\mathtt{l}}, *)$ is called an *affine plane* if it satisfies the following three axioms.

(1) Any pair of distinct points is on a unique line.
(2) If $x \in X_{\mathtt{p}}$ and $l \in X_{\mathtt{l}}$ are non-incident, then there is a unique line in $X_{\mathtt{l}} \cap x^*$ at distance strictly greater than two to l. It is called the line through x *parallel* to l.
(3) There exists a non-incident pair in $X_{\mathtt{p}} \times X_{\mathtt{l}}$.

Example 2.3.2 The affine geometries $\mathrm{AG}(V)$, for V a 2-dimensional vector space, are examples. The *real affine plane* (of points and affine lines in $\mathbb{R}^2$) is an example, and so is the analogue over any field other than $\mathbb{R}$.

For q a prime power, the affine plane $\mathrm{AG}(\mathbb{F}_q^2)$ has q^2 points, $q^2 + q$ lines, and $q + 1$ classes of parallel lines. See Fig. 2.6 for $q = 4$. This affine plane has order $(q - 1, q)$. We often abbreviate this statement to saying that its order is q.

Exercise 2.8.9(a) shows a more general construction of an affine plane from a projective plane Π and a line h of it: the subgeometry induced on the set of elements of Π not incident with h. Conversely, Exercise 2.8.9(b) shows that the relation $\parallel$ on the line set of an affine plane $(X_{\mathtt{p}}, X_{\mathtt{l}}, *)$, defined by $l \parallel m$ if and only if $l = m$ or $d(l, m) = 4$, is an equivalence relation on $X_{\mathtt{l}}$ and that the addition of a line (at 'infinity') built up from $\parallel$-equivalence classes leads to a projective plane.

Remark 2.3.3 Every affine plane is a $(3, 3, 4)$-gon, but not every $(3, 3, 4)$-gon is an affine plane. An example of a non-affine plane that is a $(3, 3, 4)$-gon is obtained by removing a point from the Fano plane. A geometry results having six points and seven lines, three of which have only two points.

Example 2.3.4 The most general construction of an affine plane, in the sense that each affine plane can be constructed in this way, makes use of a *ternary ring*. This is a set R with two distinguished elements 0 and 1 and a ternary operation $T : R^3 \to R$ satisfying the following properties for all $a, b, c, d \in R$.

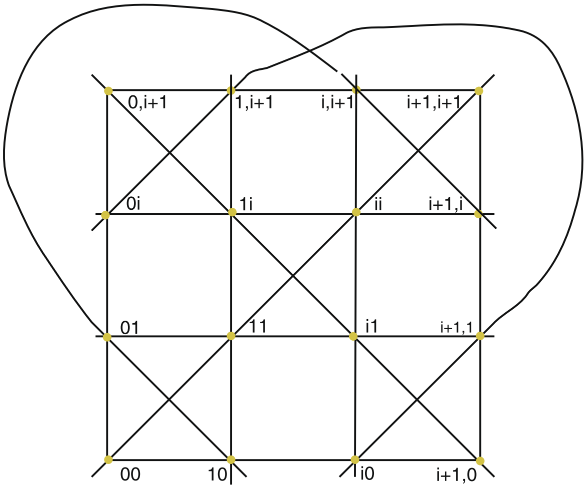


Fig. 2.6 The affine plane of order four with three of the five classes of parallel lines drawn

(1) $T(a, 0, c) = T(0, b, c) = c$.
(2) $T(a, 1, 0) = T(1, a, 0) = a$.
(3) If $a \neq c$, then there is a unique solution $x \in R$ to the equation $T(x, a, b) = T(x, c, d)$.
(4) There is a unique solution $x \in R$ to the equation $T(a, b, x) = c$.
(5) If $a \neq c$, then there is a unique solution $(x, y) \in R^2$ to the two equations $T(a, x, y) = b$ and $T(c, x, y) = d$.

Given a ternary ring (R, T), we construct an affine plane by taking as point set $P = R^2$ and letting the line set L consist of all sets of the forms $\{(x, y) \in P \mid y = T(x, a, b)\}$ and $\{(a, y) \in P \mid y \in R\}$ for $a, b \in R$. The triple $(P, L, *)$, where $*$ denotes symmetrized inclusion, is an affine plane. Any affine plane is isomorphic to one constructed in this way, and any two affine planes are isomorphic if and only if their ternary rings are isomorphic.

We will discuss three examples, where

$$T(a, b, c) = ab + c \tag{2.1}$$

for a multiplication $(a, b) \mapsto ab$ and addition $(a, b) \mapsto a + b$ on R. Such examples are called *linear*.

First, if $R = \mathbb{D}$ is a division ring, then the construction of an affine plane using (2.1) copies the usual one for $\mathrm{AG}(\mathbb{D}^2)$.

For the second example, we let C be the 8-dimensional vector space over $\mathbb{R}$, with basis $e_0, e_1, \ldots, e_7$ and define multiplication as the bilinear operation on C determined by $e_0 = 1$, the identity element, $e_i^2 = -1$ for $i \in [7]$, and $e_i e_j = -e_j e_i = e_k$ whenever the 3-cycle (i, j, k) (in Sym_7) can be transformed to the 3-cycle $(1, 2, 4)$ under a power of the permutation $(1, 2, 3, 4, 5, 6, 7)$. For instance, $e_5 e_7 = e_4$ as $(5, 7, 4) = (4, 5, 7) = (i, 1 + i, 3 + i)$ for $i = 4$. The unordered triples arising in this way are the lines of the Fano plane as pictured in Fig. 1.21. The multiplication on C is not associative; cf. Exercise 2.8.11. The algebra C is known as the *Cayley* *division ring*. The ternary operation defined on it by means of (2.1) gives a ternary ring and hence an affine plane.

The third example is a finite one: the set $J = \{0, \pm 1, \pm i, \pm j, \pm k\}$ of size 9 with multiplication as in the quaternion group of order 8, with 0 added; so $ij = -ji = k$, and $i^2 = j^2 = k^2 = -1$ whereas $0x = x0 = 0$ for all $x \in J$. An Abelian group structure on J is determined by $1 + 1 = -1$, $1 + i = j$, $1 - i = k$, and $x + x + x = 0$ for all $x \in J$. The ternary ring on J determined by (2.1) leads to an affine plane on 81 points. It is not isomorphic to $\mathrm{AG}(\mathbb{F}_9^2)$.

If we specify point and line orders, we can characterize finite affine planes by means of a diagram.

Proposition 2.3.5 *Let $q \in \mathbb{N}$, $q \geq 2$. Every $\{\mathtt{p}, \mathtt{l}\}$-geometry belonging to the diagram* $\overset{\mathtt{p}}{\underset{q-1}{\circ}}\overset{3\ \ 3\ \ 4}{\text{———}}\overset{\mathtt{l}}{\underset{q}{\circ}}$ *is a finite affine plane. Conversely, every finite affine plane belongs to such a diagram.*

Proof Axioms (1) and (3) of Definition 2.3.1 are obviously satisfied. For Axiom (2) let x be a point, and m a line such that x and m are not incident. There are $q + 1$ lines on x, each containing $q - 1$ points distinct from x. Since any two points are collinear and no two points can be on more than one line (the girth is 3), the total number of points amounts to $1 + (q + 1)(q - 1) = q^2$. Moreover, each point of m is incident with a unique line on x. This accounts for q of the $q + 1$ lines on x, leaving a single line on x parallel to m. Hence Axiom (2).

The second assertion is immediate from the definition of affine plane, except for the order information. Let Π be a finite affine plane and suppose that m is a line of Π of size q. We need to show that each line of Π has size q and each point is on exactly $q + 1$ lines. If p is a point of Π not on m, there are exactly q lines on p meeting m in a point. Adding to this number the line n on p parallel to m, we find that there are exactly $q + 1$ lines on p. By a similar argument, each line disjoint from p has exactly q points. Now, by firmness, n must have a second point. Arguing for this point as for p, we see that each line, except possibly n has exactly q points. Hence, each point on m is on exactly $q + 1$ lines, and so, taking a point on m and arguing as before, we obtain that n has exactly q points. Hence the second assertion. □

Notation 2.3.6 Since the class of affine planes is important, it is denoted by the separate diagram $\overset{\mathtt{p}}{\circ}\overset{\mathrm{Af}}{\text{————}}\overset{\mathtt{l}}{\circ}$.

We have come to the point where any class $\mathcal{K}$ of geometries over $\{\mathtt{p}, \mathtt{l}\}$ can be depicted by a labelled edge between vertices $\mathtt{p}$ and $\mathtt{l}$, as in $\overset{\mathtt{p}}{\circ}\overset{\mathcal{K}}{\text{————}}\overset{\mathtt{l}}{\circ}$. This will be a useful convention for specifying rank two residues when working with geometries of rank greater than 2. A specific example of which we will make use later is $\overset{\mathtt{p}}{\circ}\overset{\mathrm{C}}{\text{————}}\overset{\mathtt{l}}{\circ}$ for the class C of all geometries over $\{\mathtt{p}, \mathtt{l}\}$ whose point collinearity graphs are complete and whose line order is one (so any two points are on a line of size two). A subtlety to note here is that the label depends on the directed edge $(\mathtt{p}, \mathtt{l})$, whereas the class of rank two geometries is defined over the unordered set $\{\mathtt{p}, \mathtt{l}\}$. In particular, $\overset{\mathtt{l}}{\circ}\!\overset{\mathrm{C}}{\rule[0.5ex]{3em}{0.4pt}}\!\overset{\mathtt{p}}{\circ}$ represents a (different) class of geometries over $\{\mathtt{p}, \mathtt{l}\}$, namely those that are dual to members of C.

Here is the general notion of a diagram.

Definition 2.3.7 A *diagram* over I is a map D defined on $\binom{I}{2}$, the set of unordered pairs from I, that assigns to every pair $\{i, j\}$ some class $D(i, j) = D(j, i)$ of rank two geometries over $\{i, j\}$.

A geometry Γ *belongs to the diagram* D over I if, for all distinct types $i, j \in I$ and every flag F of Γ such that Γ_F is of type $\{i, j\}$, the residue Γ_F is isomorphic to a geometry in $D(i, j)$. In this case, Γ is also said to be of *type* D.

If Γ belongs to a diagram D over I and has order $(s_j)_{j \in J}$ for some subset J of I, then the parameters s_j are often given as subscripts in a pictorial description of D as in Remark 2.2.17.

Remark 2.3.8 We can view the diagram D over I as a complete labelled graph on the vertex set I, in which the label depends on the directed edge. The label of the directed edge (j, i) is determined by that of (i, j) by the prescription that the latter is assigned the dual geometry of the former (cf. Definition 2.2.5). Thus, given $D(j, i) = D(i, j)$, we need only draw one of them. For instance, if, for some i, $j \in I$, the class $D(i, j)$ consists of all (5, 5, 6)-gons over (i, j) (or, equivalently, all (5, 6, 5)-gons over (j, i)), we adorn the edge with the numbers 5, 5, and 6 in such a way that the node j is closest to 6 (cf. Remark 2.2.17). The dual geometry of a member of $D(i, j)$ is then a (5, 6, 5)-gon over (i, j) and does not belong to $D(j, i)$.

Example 2.3.9 Let Δ be a graph. We say that a graph Σ is *locally* Δ if, for every $x \in \Sigma$, the subgraph induced on the neighbors of x is isomorphic to Δ. For instance, the complete graph on n vertices is locally the complete graph on $n-1$ vertices (and is the unique connected graph with that property). A graph that is locally Petersen (cf. Example 1.3.3) can be obtained on the vertex set of all transpositions (i.e., conjugates of $(1, 2)$) of Sym_7 by demanding that two vertices are adjacent whenever they commute. The fact that Σ is locally Δ can be expressed in diagram language. Consider the following diagram

$$D_\Delta := \underset{1}{\overset{\mathtt{p}}{\circ}}\!\rule[0.5ex]{5em}{0.4pt}\!\underset{1}{\overset{\mathtt{l}}{\circ}}\!\overset{\Delta}{\rule[0.5ex]{5em}{0.4pt}}\!\overset{\mathtt{c}}{\circ}.$$

Here, we abbreviated $\{\Delta\}$ to Δ. Suppose that Σ is a connected graph such that each vertex lies in at least two cliques of size three. Take $X_{\mathtt{p}}$, $X_{\mathtt{l}}$, and $X_{\mathtt{c}}$ to be the sets of all vertices, all edges, and all cliques of size 3 of Σ, respectively, and define incidence $*$ by symmetrized containment to obtain a $\{\mathtt{p}, \mathtt{l}, \mathtt{c}\}$-geometry $(X_{\mathtt{p}}, X_{\mathtt{l}}, X_{\mathtt{c}}, *)$. This geometry belongs to the above diagram of rank three if and only if Σ is locally Δ.

Example 2.3.10 Suppose that Γ is a $\{\mathtt{p}, \mathtt{l}, \mathtt{c}\}$-geometry belonging to the diagram D_Δ of Example 2.3.9. Consider the graph Σ whose vertex set is $X_\mathtt{p}$ and in which two vertices are adjacent if and only if there is a member of $X_\mathtt{l}$ incident with both. The map $\phi : X_\mathtt{p} \cup X_\mathtt{l} \to X_\mathtt{p} \cup \binom{X_\mathtt{p}}{2}$ which is the identity on $X_\mathtt{p}$ and maps any line in $X_\mathtt{l}$ to the pair of points to which it is incident, is a homomorphism from the $\{\mathtt{p}, \mathtt{l}\}$-truncation of Γ to Σ (viewed as a geometry whose vertices have type $\mathtt{p}$ and whose edge have $\mathtt{l}$). This homomorphism is surjective as a map between point sets.

However, Σ need not be locally Δ. An easy example that is not firm shows what may go wrong: take Γ to be the rank three geometry of the octahedron from which half the faces are removed; those that are white in a black and white coloring (such that no two adjacent faces have the same color). Then Σ is the octahedron graph, which is locally a quadrangle, but the graph Δ of the diagram D_Δ for Γ is the disjoint union of two cliques of size two rather than a quadrangle.

Example 2.3.11 We give a flag-transitive geometry Γ with diagram D_{Pet}, whose associated graph Σ is locally the complete graph on 10 points (instead of Pet). We construct it by use of a transitive extension of the alternating group Alt_5 in its action on Pet.

Suppose that G_o is a group of automorphisms on a graph Δ with vertex set Ω_o. A *transitive extension* of $(G_\mathrm{o}, \Omega_\mathrm{o})$ is a transitive permutation group G on the set $\Omega = \Omega_\mathrm{o} \cup \{\mathrm{o}\}$ (the extension of Ω_o by the single point o outside Ω_o) such that the stabilizer of o in G coincides with G_o (in accordance with what the notation suggests). In general, such an extension need not exist. Suppose that (G, Ω) is a transitive extension of $(G_\mathrm{o}, \Omega_\mathrm{o})$. Denote by $\mathcal{B}$ the G-orbit of the triple $\{\mathrm{o}, a, b\}$, where $\{a, b\}$ is an edge of Δ. Suppose that Δ is connected. If G_o is edge transitive on Δ, the geometry $\Gamma = (\Omega, \binom{\Omega}{2}, \mathcal{B}, *)$, where $*$ is symmetrized inclusion, is residually connected and belongs to D_Δ. Furthermore, the collinearity graph Σ of Ω is the complete graph on Ω, and so it is impossible to reconstruct $\mathcal{B}$ or Γ from Σ.

Now take $G_\mathrm{o} = \mathrm{Alt}_5$ acting on the Petersen graph $\Delta = \mathrm{Pet}$ (cf. Example 1.3.3) with the vertex set $\Omega_\mathrm{o} = \binom{[5]}{2}$. We know that G_o is edge transitive on Pet. In order to construct a transitive extension, we pin down some necessary conditions. Consider the triple $B = \{\mathrm{o}, 12, 34\} \in \mathcal{B}$. Its point-wise stabilizer in G lies in G_o, where it is readily seen to be $\{\mathrm{id}, (1,2)(3,4)\}$, of order two.

The set stabilizer in G_o of the triple B contains the involution $(1,3)(2,4)$ inducing a transposition on B. As G_o is transitive on the edge set of Pet, it is transitive on the set of triples in $\mathcal{B}$ containing o, so G is transitive on the collection of incident pairs from $\Omega \times \mathcal{B}$. The (set-wise) stabilizer G_B of B in G induces $\mathrm{Sym}(B)$ on B. Thus $G_B \cong \mathrm{C}_2.\mathrm{Sym}_3$ (that means, there is a normal subgroup of order 2 in G_B with quotient group Sym_3). By Lagrange's Theorem, $|\mathcal{B}| = |G|/|G_B| = 11|G_\mathrm{o}|/|2.\mathrm{Sym}_3| = 660/12 = 55$. The triples in $\mathcal{B}$ containing o correspond to the edges of Pet; there are precisely 15 of them. So there are 40 triples in $\mathcal{B}$ lying entirely in Ω_o; they form a union of G_o-orbits. On the other hand, the $\binom{10}{3} = 120$ triples in Ω_o fall into G_o-orbits of lengths 10 (the neighbors of a vertex in Pet), 20 (cocliques of size 3), 30, 30 (for two orbits on triples carrying a single edge), and 30 (paths of length 3). The only way to make 40 is via $10 + 30$;

this indicates that the triple $\{34, 45, 35\}$ (consisting of the neighbors of 12 in Pet) must belong to $\mathcal{B}$.

Now switch the viewpoint to the residue of 12. The triples in $\mathcal{B}$ containing 12 determine the edges of a Petersen graph on $\Omega \setminus \{12\}$, say Pet_{12}. The neighbors of o are known from the triples in $\mathcal{B}$ containing o and 12; these are $\{\mathrm{o}, 12, a\}$ for $a \in \{34, 45, 35\}$. The neighbors of 45 in Pet form the triple $\{12, 13, 23\}$, so 13 and 23 are adjacent in Pet_{12}. If the paths of length 3 in Pet were to be triples in $\mathcal{B}$, then both $\{45, 13\}$ and $\{45, 23\}$ are edges of Pet_{12}, leading to a clique on $\{45, 13, 23\}$, a contradiction as there are no cliques of size 3 in a Petersen graph. Hence the missing triples of $\mathcal{B}$ must come from one of the G_{o}-orbits of triples in Pet inducing a subgraph with a single edge. Since they are interchanged by Sym_5, it does not matter which one we take. Pick the one containing $\{12, 15, 35\}$. Inspection shows that the new edges coming from the triples containing 12 in this G_{o}-orbit indeed make Pet_{12} into a Petersen graph. Identification of the two Petersen graphs Pet and Pet_{12} can now be achieved via the permutation

$$g = (\mathrm{o}, 12)(13, 25)(14, 23)(15, 24) \in \mathrm{Sym}(\Omega).$$

As g preserves $\mathcal{B}$, it is readily checked that the group $G = \langle g, G_{\mathrm{o}} \rangle$ generated by g and G_{o} is a transitive extension of G_{o} on Ω_{o}.

In fact, $G \cong \mathrm{PSL}(\mathbb{F}_{11}^2)$ (as introduced at the end of Example 1.8.16). As G acts flag transitively on the geometry, a direct existence proof of Γ is the following description

$$\Gamma\big(G, \big(G_{\mathrm{o}}, \langle d, f, g \rangle, \langle d, e, g \rangle\big)\big)$$

in terms of subgroups of Sym_{11}, where $G = \langle d, e, f, g \rangle$, the permutation g is as above, and the permutations d, e, f correspond to the following elements of Alt_5 and $\mathrm{Sym}(\Omega)$.

Element	In Alt_5	Action on Ω
d	$(1, 2)(3, 4)$	$(13, 24)(14, 23)(15, 25)(35, 45)$
e	$(1,3)(2,4)$	$(12, 34)(14, 23)(15, 35)(25, 45)$
f	$(3,4,5)$	$(12, 13, 14)(25, 26, 27)(34, 45, 35)$

Flag transitivity also implies the existence of a $\subset$-order of Γ; it is 2.

2.4 Coxeter Diagrams

Geometries belonging to a so-called Coxeter diagram are a central theme in this book. Fix a set I.

Definition 2.4.1 A diagram D over I will be called a *Coxeter diagram* (over I) if, for each pair i, j of types, there is a number m_{ij} such that $D(i, j)$ is the class of all generalized m_{ij}-gons.

Table 2.2 Coxeter diagrams of some geometries

Geometry name	Coxeter diagram	Diagram name
Cube	$\overset{1}{\circ}=\!=\!=\overset{2}{\circ}\text{———}\overset{3}{\circ}$	
Octahedron	$\overset{1}{\circ}\text{———}\overset{2}{\circ}=\!=\!=\overset{3}{\circ}$	B_3
Icosahedron	$\overset{1}{\circ}\text{———}\overset{2}{\circ}\overset{5}{\text{———}}\overset{3}{\circ}$	H_3
Dodecahedron	$\overset{1}{\circ}\overset{5}{\text{———}}\overset{2}{\circ}\text{———}\overset{3}{\circ}$	
$\mathbb{E}^2$ tiling by quadrangles	$\overset{1}{\circ}=\!=\!=\overset{2}{\circ}=\!=\!=\overset{3}{\circ}$	$\widetilde{\mathrm{B}}_2$
$\mathbb{E}^2$ tiling by hexagons	$\overset{1}{\circ}\overset{6}{\text{———}}\overset{2}{\circ}\text{———}\overset{3}{\circ}$	$\widetilde{\mathrm{G}}_2$
$\mathbb{E}^2$ tiling by triangles	$\overset{1}{\circ}\text{———}\overset{2}{\circ}\overset{6}{\text{———}}\overset{3}{\circ}$	

In view of Table 2.1 it is no surprise that the Coxeter diagram is fully determined by a matrix satisfying the following definition.

Definition 2.4.2 A *Coxeter matrix* over I is a matrix $M = (m_{i,j})_{i,j\in I}$ where $m_{i,j} \in \mathbb{N} \cup \{\infty\}$ with $m_{i,i} = 1$ for $i \in I$, and $m_{i,j} = m_{j,i} > 1$ for distinct $i, j \in I$.

The Coxeter matrix $M = (m_{ij})_{i,j\in I}$ and the Coxeter diagram D for which $D(i, j)$ is the class of $m_{i,j}$-gons, determine each other. If Γ is a geometry of type D, we also say that Γ is of *Coxeter type* M.

Example 2.4.3 Table 2.2 contains a list of some geometries of Coxeter type that we have met before and a Coxeter diagram to which they belong. All orders are equal to 1. The labels near the nodes indicate types of the elements. The Coxeter matrix of the dodecahedron, for instance, is

$$\begin{pmatrix} 1 & 5 & 2 \\ 5 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Notice that dual (weakly isomorphic) geometries, like the icosahedron and the dodecahedron, have 'dual' diagrams. The hemi-dodecahedron of Example 1.3.4 also belongs to the dodecahedron diagram.

Example 2.4.4 The Coxeter diagram on the tricolored vertices of the $\mathbb{E}^2$ tiling by triangles of Example 1.3.10 is a triangle. The Coxeter diagram of the $\mathbb{E}^3$ tiling by bicolored cubes and bicolored vertices is a quadrangle, in accordance with its digon diagram discussed in Example 2.1.3. The Coxeter diagrams of the $\mathbb{E}^3$ tilings by cubes and bicolored cubes, respectively, are given in Fig. 2.7.

The 12 vertices, 30 edges, and 12 pentagons of an icosahedron, with incidence being symmetrized containment, lead to the thin geometry of the *great dodecahedron* with diagram

$$\circ\overset{5}{\text{———}}\circ\overset{5}{\text{———}}\circ.$$

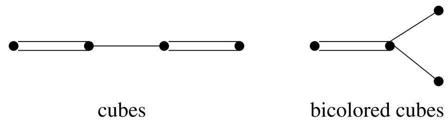


Fig. 2.7 $\mathbb{E}^3$ tilings and their Coxeter diagrams

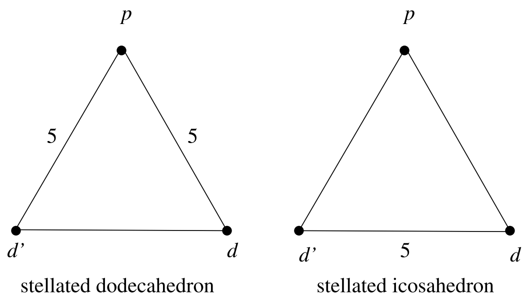


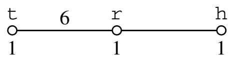
Fig. 2.8 Diagrams of the stellated dodecahedron and icosahedron

It has a flag-transitive group of automorphisms, and a quotient geometry by a group of order two having the same diagram (see Exercise 2.8.13).

Example 2.4.5 Let D be the dodecahedron and D' the associated great stellated dodecahedron (see Example 1.3.4 and Fig. 1.12). Let $X_{\mathtt{p}}$ be the vertex set of D, and $X_{\mathtt{d}}$, $X_{\mathtt{d}'}$ the set of all pentagons of D and D', respectively. Then the incidence system $\Gamma(X_{\mathtt{p}}, X_{\mathtt{d}} \cup X_{\mathtt{d}'})$ over $\{\mathtt{p}, \mathtt{d}, \mathtt{d}'\}$ built by use of the Principle of Maximal Intersection (cf. Exercise 1.9.20) is a geometry. It belongs to the diagram at the left hand side of Fig. 2.8.

The icosahedron can also be stellated. Taking $X_{\mathtt{p}}$, $X_{\mathtt{d}}$, and $X_{\mathtt{d}'}$ as for the dodecahedron, we find a geometry $\Gamma(X_{\mathtt{p}}, X_{\mathtt{d}} \cup X_{\mathtt{d}'})$ over $\{\mathtt{p}, \mathtt{d}, \mathtt{d}'\}$ of type the diagram at the right hand side of Fig. 2.8.

Example 2.4.6 Consider the tiling of $\mathbb{E}^2$ in Fig. 1.3. We derive another tiling T' from it. The elements of T' are the hexagons of T (type $\mathtt{h}$), the rectangles obtained as the union of two adjacent squares of T (type $\mathtt{r}$) and the triangles obtained as the union of four triangles of T (type $\mathtt{t}$). Define incidence on the elements of T' by the Principle of Maximal Intersection (Exercise 1.9.20). We find a geometry of the following Coxeter type.



Another geometry of the same Coxeter type and same orders can be constructed as follows. Let Δ be the graph on six vertices obtained from the complete graph by deletion of all edges of a single hexagon. There are nine edges and three hexagons in Δ. The vertices, edges, and hexagons give a rank three geometry whose automorphism group has order 12. This geometry is not flag transitive.

Recall from Example 1.4.9 the definition of the *projective geometry* $\mathrm{PG}(V)$.

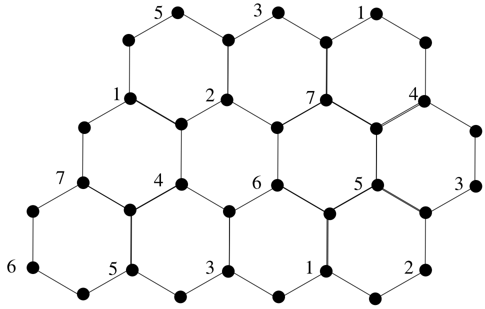
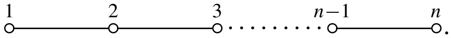


Fig. 2.9 The incidence graph of the Fano plane. Vertices that are not labelled represent lines. Labelled vertices represent points. Vertices with the same label need to be identified

Proposition 2.4.7 *Let $n > 0$. If V is a vector space over a division ring of finite dimension $n + 1$, then* $\mathrm{PG}(V)$ *is a thick* $[n]$*-geometry with Coxeter diagram*

$\mathrm{A}_n =$ 

If the division ring has size q, then all i-orders are equal to q.

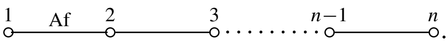
Proof Let V be defined over the division ring $\mathbb{D}$, so $V \cong \mathbb{D}^{n+1}$. Residual connectedness of $\mathrm{PG}(V)$ follows from the fact that every residue is a direct sum of geometries of the form $\mathrm{PG}(W)$ for W a vector space over $\mathbb{D}$. A projective plane $\mathrm{PG}(\mathbb{D}^3)$ is a generalized 3-gon as defined in Definition 2.2.6 and so belongs to the Coxeter diagram A_2. In $\mathrm{PG}(V)$, each rank two residue of type $\{i, i+1\}$, for some $i \in [n]$, is also a projective plane isomorphic to $\mathrm{PG}(\mathbb{D}^3)$. The points on a line are in bijective correspondence with the disjoint union of $\mathbb{D}$ and a point 'at infinity'. Therefore, the 1-order, and, by dualization and induction, all other i-orders of $\mathrm{PG}(V)$ are one less than $|\mathbb{D}|$. This implies that the geometry is thick. □

Example 2.4.8 The smallest thick projective geometry is $\mathrm{PG}(\mathbb{F}_2^3)$. Figure 1.21 provides a picture of it as a classical plane with points and lines, but Fig. 2.9 depicts its incidence graph.

Remark 2.4.9 The diagram A_n of $\mathrm{PG}(V)$ has only one nontrivial symmetry and so the order of the quotient of the correlation group of $\mathrm{PG}(V)$ by its automorphism group $\mathrm{Aut}(\mathrm{PG}(V))$ is at most two. If such a duality exists, then $\mathbb{D}$ must be isomorphic to $\mathbb{D}^{\mathrm{op}}$; cf. Exercise 1.9.13.

Affine geometries $\mathrm{AG}(V)$ as defined in Example 1.4.10 do not belong to a Coxeter diagram because affine subplanes are not generalized polygons. The rank two diagram Af for affine planes appearing in the diagram of Proposition 2.4.10 below was introduced in Notation 2.3.6.

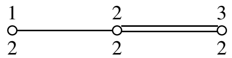
Proposition 2.4.10 *The geometry* $\mathrm{AG}(\mathbb{D}^n)$ *is firm and residually connected and belongs to the diagram*

Af_n: 

Proof The point residues of $\mathrm{AG}(\mathbb{D}^n)$ are projective geometries isomorphic to $\mathrm{PG}(\mathbb{D}^n)$. The i-orders for $i > 1$ are $|\mathbb{D}|$ as for $\mathrm{PG}(\mathbb{D}^n)$, discussed in Proposition 2.4.7. The number of points on a line equals $\mathbb{D}$, and so $\mathrm{AG}(\mathbb{D}^n)$ is firm with 1-order $|\mathbb{D}| - 1$. □

The converses of Propositions 2.4.7 and 2.4.10, which recognize $\mathrm{PG}(V)$ and $\mathrm{AG}(V)$ as geometries of the indicated types, will appear in Chap. 6.

Example 2.4.11 We construct a remarkable geometry of Coxeter type B_3 (see Table 2.2). Let X_1 be the set $[7]$, and let X_2 be the collection of all triples from X_1. Each Fano plane with X_1 as point set can be viewed as a collection of 7 triples from X_1, and hence as a set of 7 elements of X_2. As discussed in Example 2.2.18, the alternating group on X_1 has two orbits on the collection of all Fano planes, interchanged by an odd permutation of X_1; each orbit has 15 members. Let X_3 be one of these orbits. Then $\Gamma := (X_1, X_2, X_3, *)$, where $*$ is symmetrized containment, is a geometry belonging to the diagram



on which $G := \mathrm{Alt}_7$ acts flag transitively. This accounts for the full group $\mathrm{Aut}(\Gamma)$ as restriction to X_1 is a faithful homomorphism, and $\mathrm{Sym}(X_1)$ is the only permutation group on X_1 properly containing G and does not preserve the selected class of 15 Fano planes. This rank three geometry is known as the *Neumaier geometry*. Observe that each 1-element is incident with each 3-element. By Theorem 2.2.11 or, by identification of the description in terms of pairs and partitions of $[6]$, the residue of a 1-element is the generalized quadrangle of Example 2.2.10. In terms of subgroups, Γ is the geometry $\Gamma(G, (\mathrm{Alt}(\{2, \dots, 6\}), G_l, L))$, where $l = \{1, 2, 4\}$, a line of the Fano plane described in Exercise 1.9.7, and L is the automorphism group of that plane. Corollary 1.8.13 applied to this description of Γ readily gives that the Neumaier geometry is residually connected.

There is an extraordinary relation with the projective space of $\mathbb{F}_2^4$ (cf. Examples 1.4.9 and 1.5.6). Consider the truncated geometry ${}_{\{2,3\}}\Gamma$. Any two distinct members of X_3 meet in a unique element of X_2 (to see this, use the fact that Alt_7 acts transitively on the collection of ordered pairs of distinct elements from X_3). Put $V = \{0\} \cup X_3$, and define addition of $u, v \in V$ on V by $v + v = 0$, $u + v = v + u = v$ if $u = 0$, and $u + v = w$ if $u, v \in X_3$ are incident with $l \in X_2$ and $\{u, v, w\} = X_3 \cap l^*$. This turns V into an additive group isomorphic to $\mathbb{F}_2^4$, so it can be viewed as a vector space. Moreover, X_3 and X_2 can be identified with the point and line set of this space. Since $\mathrm{Aut}(\Gamma)$ acts faithfully on V, we have obtained an embedding of Alt_7 in the general linear group $\mathrm{GL}(V) = \mathrm{GL}(\mathbb{F}_2^4)$. From this it is easy to derive the sporadic isomorphism $\mathrm{GL}(\mathbb{F}_2^4) \cong \mathrm{Alt}_8$. For, $\mathrm{GL}(V)$ has order $|\mathrm{Alt}_8|$, so G maps to a subgroup, H say, of $\mathrm{GL}(V)$ isomorphic to Alt_7 and of index eight in $\mathrm{GL}(V)$. By Theorem 1.7.5, $\mathrm{GL}(V)$ has a transitive representation on $\mathrm{GL}(V)/H$ of degree eight. Since $\mathrm{GL}(V)$ is a simple group (a fact that is not hard to prove, but assumed for now), the representation is faithful, and it is an embedding in Alt_8. By comparison of orders, the embedding must be an isomorphism. The stabilizer in G of a triple

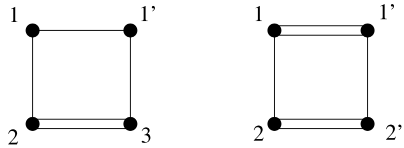


Fig. 2.10 Coxeter diagrams of geometries related to HoSi

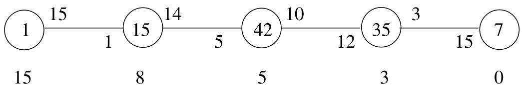


Fig. 2.11 Distribution diagram of the hundred 15-cocliques of HoSi

l of [7] (that is, an element of X_2, or a projective line of V) has orbits of lengths 1, 12, 18, 4 on X_2. There are precisely $3 \times 6 = 18$ projective lines meeting l in a point. Apparently, in terms of projective lines, the G_l-orbits of lengths 12 and 4 fuse to the single $\mathrm{GL}(V)_l$-orbit of all lines disjoint from l.

Example 2.4.12 Consider once more the Hoffman-Singleton graph HoSi of Example 2.2.18. There are two geometries related to this graph with diagrams as depicted in Fig. 2.10.

Their constructions are based on a partition of the maximal cocliques (i.e., complements of cliques) of HoSi that are of size 15. Consider the graph on these cocliques in which two cocliques are adjacent if their intersection has size eight. This graph has the following distribution diagram. For each $i \in \{0, \dots, 4\}$, the size of the intersection of two 15-cocliques at mutual distance i is given underneath the circle at distance i from the left hand node (which represents a fixed 15-coclique of HoSi).

In order to construct the two rank four geometries, take X_1 and $X_{1'}$ to be two copies of the vertex set of HoSi, take X_2 and X_3 to be the H-orbits of cocliques of size 15 in HoSi, where H is the index two subgroup of Aut(HoSi) described in Remark 2.2.20. It can be read off from the distribution diagram of Fig. 2.11 that the graph has no odd cycles, so it is bipartite. The sets X_2 and X_3 are the two classes of the unique partition. Finally, take $X_{2'}$ to be a copy of X_2. Define incidence $*$ for $a \in X_1$, $a' \in X_{1'}$, $b \in X_2$, $b' \in X_{2'}$, and $c \in X_3$, by

$$\begin{array}{rcl@{\qquad}rcl}
a * a' & \Longleftrightarrow & a \sim a' \text{ in HoSi}, & a * b & \Longleftrightarrow & a \in b,\\
a * b' & \Longleftrightarrow & a \notin b', & a * c & \Longleftrightarrow & a \notin c,\\
a' * b & \Longleftrightarrow & a' \notin b, & a' * b' & \Longleftrightarrow & a' \in b',\\
a' * c & \Longleftrightarrow & a' \in c, & b * b' & \Longleftrightarrow & b \cap b' = \emptyset,\\
b * c & \Longleftrightarrow & |b \cap c| = 8. & & &
\end{array}$$

The *Wester HoSi geometry* is $(X_1, X_{1'}, X_2, X_3, *)$, and the *Neumaier HoSi geometry* is $(X_1, X_{1'}, X_2, X_{2'}, *)$. The residues of type B_3 are isomorphic to the Neumaier geometry of Example 2.4.11.

Remark 2.4.13 The graph obtained from the graph, say Δ, on the 15-cocliques of HoSi discussed in Example 2.4.12 can be used to construct another graph on the same vertex set with an interesting automorphism group. In the graph Δ' meant here, two vertices are adjacent whenever they are at distance one or four in Δ. This graph Δ' has 100 vertices and is regular of valency 22. Moreover, two adjacent vertices have no common neighbors and two non-adjacent vertices have six common neighbors. The automorphism group of Δ' has a unique index two subgroup; it is isomorphic to the sporadic simple group HS, named after Higman and Sims.

2.5 Shadows

We started Chap. 1 by saying that we want to abandon the usual physical viewpoint according to which each element of a geometry is a point or a set of points, and we have described a more abstract viewpoint. However, now that the latter has been developed, we want to point out how to recover the physical viewpoint, because the latter has an important role for instance in the construction of examples and in characterizations. We fix a finite set of types I, let $\Gamma = (X, *, \tau)$ be an I-geometry, and let J be a non-empty subset of I. Often, but not always, J will just be a singleton.

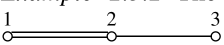
Definition 2.5.1 For every flag F of Γ, the *J-shadow* (of F) or *shadow* of F on J is the set $\mathrm{Sh}_J(F)$ of all flags of type J that are incident with F. If we need to emphasize the dependence on Γ of the shadow on J, we write $\mathrm{Sh}_J(F, \Gamma)$ instead of $\mathrm{Sh}_J(F)$. If $J = \{j\}$, we will often write $\mathrm{Sh}_j(F)$ instead of $\mathrm{Sh}_J(F)$.

For $j \in J$, the shadow of a flag of type $I \setminus \{j\}$ on J is said to be a *j-line* or a *line of type* j. Let T be a subset of J. The *shadow space* $\mathrm{ShSp}(\Gamma, J, T)$ of type (J, T) is the pair $(\mathrm{Sh}_J(\emptyset), L)$ where L is the collection of all j-lines for $j \in T$. The members of $\mathrm{Sh}_J(\emptyset)$ are called the *points* and the members of L are called the *lines* of $\mathrm{ShSp}(\Gamma, J, T)$. We write $\mathrm{ShSp}(\Gamma, J) = \mathrm{ShSp}(\Gamma, J, J)$ and call it the *shadow space* of Γ on J. If $J = \{j\}$, we also write $\mathrm{ShSp}(\Gamma, j)$ instead of $\mathrm{ShSp}(\Gamma, J)$, and speak of the shadow space on j instead of $\{j\}$.

In order to distinguish between shadow spaces viewed as line spaces and shadow spaces equipped with all shadows, we will refer to the latter as *full shadow spaces*.

If Γ is a firm geometry, then every shadow on J containing more than one point and not containing any other shadow satisfying this requirement, is a line of $\mathrm{ShSp}(\Gamma, J)$.

If Γ belongs to a diagram D over I, then we depict $\mathrm{ShSp}(\Gamma, J)$ in D by drawing a circuit around the set of vertices of I belonging to J.

Example 2.5.2 The geometry Γ of a cube belongs to the Coxeter diagram  with eight elements of type 1, twelve of type 2, and six of type 3. There are seven non-empty subsets J in $I = [3]$. Each of these gives rise to a J-space which can be represented by one of the seven semi-regular convex polyhedra having the same group of isometries as the cube in $\mathbb{E}^3$. The edges of the polyhedron representing $\mathrm{ShSp}(\Gamma, J)$ are precisely the lines while its faces are the other non-trivial shadows.

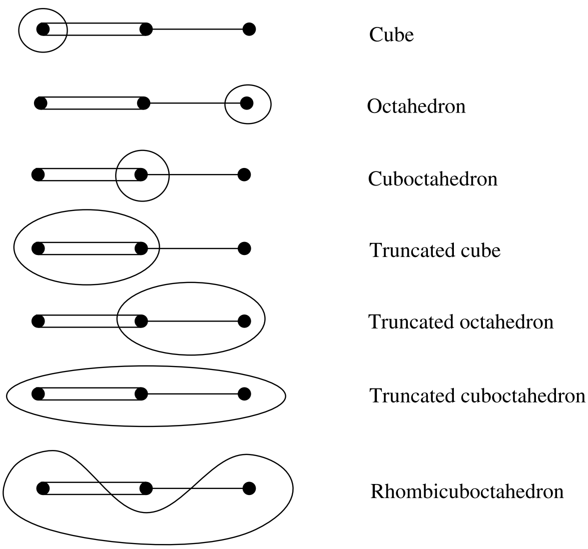


Fig. 2.12 The truncations of the cube

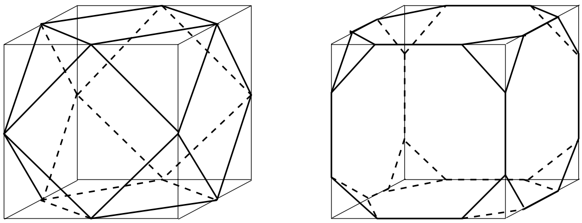


Fig. 2.13 The cuboctahedron and the truncated cube

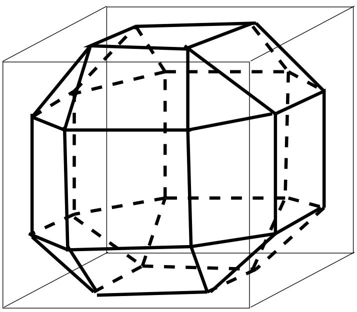


Fig. 2.14 The rhombicuboctahedron

Figure 2.12 contains a list of these spaces with diagrams and names. Two of these truncations are drawn in Fig. 2.13, and another in Fig. 2.14.

The truncated cuboctahedron appeared in Fig. 1.1 and is redrawn in Fig. 3.1. The lines of various shadow spaces in the geometry of the cube are the edges drawn in Figs. 2.13 and 2.14. In the real affine geometry $\mathrm{AG}(\mathbb{R}^3)$, with $J = \{1\}$, the physical

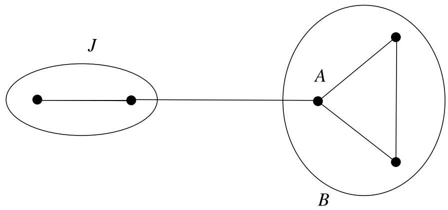


Fig. 2.15 The vertex set A separates J from B

viewpoint emerges from $\mathrm{ShSp}(\mathbb{E}^3, J)$, where points and lines are the 'usual' objects in $\mathbb{E}^3$.

Example 2.5.3 Shadow spaces $\mathrm{ShSp}(\Gamma, j)$ for a single type j are studied most intensively. Nevertheless, in projective geometry we need a case where $|J| = 2$, which appears in the projective geometry $\mathrm{PG}(V)$ of the vector space V of dimension $n + 1$ over a division ring (cf. Example 1.4.9). We take $J = \{1, n\}$, so the points of $\mathrm{ShSp}(\mathrm{PG}(V), J)$ correspond to incident point-hyperplane pairs. Two pairs (x, H) and (y, K), where x, y are 1-dimensional and H, K are n-dimensional, are 1-collinear (i.e., both incident with a 1-element) if and only if $H = K$ and n-collinear if and only if $x = y$. Verify that, in the former case, a 1-line on (x, H) and (y, H) is represented by all pairs (z, H) with z contained in the linear subspace spanned by x and y. Similarly, in the latter case, an n-line on (x, H) and (x, K) consists of all pairs (x, L) with L an n-dimensional subspace of V containing $H \cap K$.

In Definition 2.5.1, we did not allow $J = \emptyset$ because the resulting space is trivial and useless. In the examples where J is a singleton, i.e., consists of a single element of I, we see that the lines of $\mathrm{ShSp}(\Gamma, J)$ are shadows of flags of type $\bar{J}$ where $\bar{J}$ is the *neighborhood* of J in the digon diagram of Γ, i.e., the set of all $i \in I \setminus J$ such that i is on some edge of $\mathcal{I}(\Gamma)$ having a vertex in J. These considerations lead to the following development.

Definition 2.5.4 Let $\mathcal{I} = (I, \sim)$ be a graph. If J, A, B are subsets of I, then A *separates* J from B if no connected component of the subgraph of $\mathcal{I}$ induced on $I \setminus A$ meets both J and B. The J-*reduction* of B is the smallest subset of B separating J from B; it is readily seen to exist for all J and B.

An example is given in Fig. 2.15.

In the result below, the notion of separation will be applied to the digon diagram of a geometry.

Lemma 2.5.5 *If* $F_1 \cup F_2$ *is a flag of* Γ *such that* $\tau(F_1)$ *separates* J *from* $\tau(F_2)$ *in* $\mathcal{I}(\Gamma)$, *then* $\mathrm{Sh}_J(F_1) \subseteq \mathrm{Sh}_J(F_2)$, *so* $\mathrm{Sh}_J(F_1 \cup F_2) = \mathrm{Sh}_J(F_1)$.

Proof By the Direct Sum Theorem 2.1.6, each flag of type J contained in F_1^* is also contained in F_2^*. $\square$

Theorem 2.5.6 *Let Γ be an I-geometry of finite rank and let J be a non-empty subset of I. For $j \in J$, every j-line is the shadow on J of some flag of Γ of type J_j, where J_j is the J-reduction of $I \setminus \{j\}$. Conversely, the J-shadow of every flag of Γ of type J_j is a j-line.*

Proof Let $j \in J$ and suppose that l is a j-line. Then $l = \mathrm{Sh}_J(F)$ for some flag F of cotype $\{j\}$. If $H \subseteq F$ is the subflag of F of cotype J_j, then J_j separates J from the subset $I \setminus (\{j\} \cup J_j)$ of $J \setminus \{j\}$, so $\mathrm{Sh}_J(H) \subseteq \mathrm{Sh}_J(F \setminus H)$ by Lemma 2.5.5. This implies $\mathrm{Sh}_J(H) = \mathrm{Sh}_J(F) = l$, so H is a flag of type J_j such that $l = \mathrm{Sh}_j(H)$, as required.

In order to prove the converse, suppose that Y is a flag of type J_j. We show that its J-shadow is a j-line. Extend Y to a flag F of Γ of cotype $\{j\}$. By definition of J_j, the type $\tau(Y)$ separates J from $I \setminus \{j\}$ and hence also from $\tau(F \setminus Y)$, so Lemma 2.5.5 gives $\mathrm{Sh}_J(Y) = \mathrm{Sh}_J(F)$, which is a j-line. □

Remark 2.5.7 Consideration of the (ordinary) digon shows that in an I-geometry with $j \in J \subseteq I$, the map $F \mapsto \mathrm{Sh}_J(F)$ from the set of flags of type J_j onto the set of j-lines need not be bijective.

In conclusion, from an abstract geometry, we have indicated a method to obtain various 'physical interpretations' in terms of points and lines. We refer to these interpretations as spaces. We would like to stress that our spaces are quite combinatorial in that they refer to collinearity, but not to a metric, topological, or differential structure. Starting from a space of points and lines, we can now construct diagram geometries, and go back to other spaces. We will do so in later chapters. The rest of this section is devoted to the basics of line spaces.

Definition 2.5.8 A *line space* is a pair (P, L) consisting of a set P, whose members are called *points*, and a collection L of subsets of P of size at least two, whose members are called *lines*. A line in L is called *thin* if it has exactly two points, and *thick* otherwise.

Let X be a subset of P. Then $L(X)$ denotes the collection of subsets $l \cap X$ of X of size at least 2 where $l \in L$. The resulting line space $(X, L(X))$ will be called the *restriction* of the line space (P, L) to X, or the line space *induced* on X.

A *subspace* X of (P, L) is a subset of P such that every line of L containing two distinct points of X is entirely contained in X (in other words, $L(X) \subseteq L$). Thus, $X = \emptyset$ and $X = P$ are trivial specimens of subspaces.

A *homomorphism* $\alpha : (P, L) \to (P', L')$ of line spaces is a map $\alpha : P \to P'$ such that the image under α of every line in L is contained in a line of L'. If it is injective and the image of every line in L is a line in L', then the homomorphism is also called an *embedding*. The notions *isomorphism* and *automorphism* are defined in the obvious way.

Often we will denote the lines space (P, L) by a single symbol, such as Z. We will then also abuse Z to indicate its point set as well, for instance, when writing

$p \in Z$ and $Z \setminus \{p\}$ to indicate that p is a point of Z and the set of points of Z distinct from p, respectively.

Finally, the *dual line space* of a line space (P, L) is the point shadow space of the dual geometry $\Gamma^\vee$ (cf. Definition 2.2.5) of the geometry $\Gamma = (P, L, *)$, where $*$ is symmetrized containment. It is a line space only if each point in P is on at least two members of L.

Example 2.5.9 Shadow spaces (cf. Definition 2.5.1) are examples of line spaces. In Chap. 5 we will study the line spaces that are shadow spaces on 1 of the geometries $\mathrm{AG}(V)$ and $\mathrm{PG}(V)$ for a vector space V.

Graphs are examples of line spaces all of whose lines have size two. Each subset of the vertex set is a subspace. The collinearity graph of the dual line space of a graph Δ is known as the line graph of Δ.

Lemma 2.5.10 *In a line space, the intersection of any set of subspaces is again a subspace.*

Proof Straightforward. □

Definition 2.5.11 If X is a set of points in a line space Z, the subspace $\langle X \rangle$ of Z *generated* by X is the intersection of all subspaces of Z containing X.

It is possible to construct $\langle X \rangle$ from X by 'linear combination', see Exercise 2.8.21.

Definition 2.5.12 A line space (P, L) can be viewed as a rank two geometry $(P, L, *)$ by letting $*$ be symmetrized containment. This means that $*$ is determined by the rule that for $x \in P$, $y \in L$ we have $x * y$ (and $y * x$) if and only if $x \in y$. We will refer to this geometry as the *geometry of the space* (P, L). The line space (P, L) is called *connected* (*firm*, *thin*, *thick*) whenever the corresponding geometry $(P, L, *)$ is connected (firm, thin, thick, respectively).

The *collinearity graph* of a line space is the collinearity graph on the point set of the geometry of the space (introduced in Definition 2.2.1).

The geometry of the line space induced on a subset X of points of a line space (P, L) is a subgeometry of $(P, L, *)$ in the sense of Definition 1.4.1.

Definition 2.5.13 A subspace of a line space is said to be *singular* if any two of its points are on a line and *linear* if any two of its points are on a unique line. It is called *partial linear* if any two of its points are on at most one line. The unique line containing two collinear points p and q of a partial linear space is usually denoted by pq.

The definition of singularity is in accordance with Example 1.4.13, where the subspaces on which the form f defined there vanishes completely are indeed singular in the current sense.

Remark 2.5.14 The shadow spaces of affine and projective geometries are linear, but there are many other examples.

All the subspaces of a linear space are linear. All the subspaces of a partial linear space are partial linear. All singular subspaces of a partial linear space are linear. If X is a subset of a linear space Z and L is the set of lines of Z entirely contained in X, then (X, L) is a partial linear space.

The notions of point diameter and girth from Definitions 2.2.1 and 2.2.3 will be used to introduce a diagram for firm linear spaces on the basis of Definition 2.5.12.

Theorem 2.5.15 *Linear spaces relate to rank two geometries as follows.*

(i) *If* (P, L) *is a firm linear space, then the geometry* $(P, L, *)$ *of this space is a* $[2]$*-geometry with girth* 3 *and* 1*-diameter* 3.

(ii) *If* Γ *is a* $\{\mathtt{p}, \mathtt{l}\}$*-geometry with girth* 3 *and* $\mathtt{p}$*-diameter* 3, *then the shadow space of* Γ *on* $\mathtt{p}$ *is a firm linear space.*

Proof (i) Suppose that (P, L) is a firm linear space. By definition of firmness for linear spaces, $\Gamma = (P, L, *)$ is firm. Since two distinct points cannot be incident with two distinct lines in (P, L), there are no circuits of length four in the incidence graph of Γ. There are circuits of length six as any two points are collinear and three non-collinear points can be found. Thus, the girth of Γ is three.

The point diameter of Γ is also three: given a point a, an argument similar to the above gives a line l not containing a, so $d(a, l) \geq 3$. As any point x distinct from a is on the line ax, it follows that $d(a, x) \leq 2$, so $d(a, l) = 3$. Therefore, the point diameter $\delta_{\mathtt{p}}$ is equal to three.

(ii) Suppose that $\Gamma = (X_{\mathtt{p}}, X_{\mathtt{l}}, *)$ is a firm geometry over $\{\mathtt{p}, \mathtt{l}\}$ with girth and point diameter both equal to three. Here, as usual, $\mathtt{p}$ stands for points and $\mathtt{l}$ for lines. An element (a line) in $X_{\mathtt{l}}$ may be identified with its shadow on $X_{\mathtt{p}}$: lines have at least two points and if two distinct lines l, m would both be incident with two distinct points $a, b \in X_{\mathtt{p}}$, then $a * l * b * m * a$ would be a 4-circuit, a contradiction with the girth being three. This argument also shows that two distinct points are on at most one line. Again, let a, b be distinct points. Then $d(a, b)$ (distance in Γ) is an even number; it is at most three as $\delta_{\mathtt{p}} = 3$. Hence $d(a, b) = 2$. Therefore there is a line containing a and b, which proves that the shadow space of Γ on $\mathtt{p}$ is linear. As it is clearly firm, this ends the proof of the theorem. □

Notation 2.5.16 We denote by $\underset{\mathtt{p}}{\circ}\overset{\mathrm{L}}{\rule{2em}{0.4pt}}\underset{\mathtt{l}}{\circ}$ the collection of firm rank two geometries of the theorem. The subclass of all geometries $(P, L, *)$ where (P, L) is the complete graph will be denoted by $\underset{\mathtt{p}}{\circ}\overset{\mathrm{C}}{\rule{2em}{0.4pt}}\underset{\mathtt{l}}{\circ}$ (for clique or complete graph).

In Chap. 5, we will apply Theorem 2.5.15 to affine and projective geometries. In order to separate these nice examples from the other linear spaces, we look for conditions enabling us to build higher rank geometries by use of subspaces.

2.6 Group Diagrams

When a geometry is being constructed from a system of subgroups of a given group, the isomorphism types of the residues are not easy to describe beforehand. To remedy this, we adhere a diagram to a system of groups in such a way that the resulting coset geometry (if any) has the corresponding diagram in the sense of Definition 2.3.7. Fix a set I.

Definition 2.6.1 Let G be a group with a system of subgroups $(G_i)_{i\in I}$, and let D be a diagram over I. We say that G has *diagram* D over $(G_i)_{i\in I}$ if, for each pair $\{i, j\} \subseteq I$, the coset geometry $\Gamma(G_{I\setminus\{i,j\}}, (G_{I\setminus\{j\}}, G_{I\setminus\{i\}}))$, with $G_{I\setminus\{j\}}$ of type i and $G_{I\setminus\{i\}}$ of type j, belongs to $D(i, j)$.

Theorem 1.8.10 gives that $\Gamma(G_{I\setminus\{i,j\}}, (G_{I\setminus\{j\}}, G_{I\setminus\{i\}}))$ is indeed a geometry, for $G_{I\setminus\{i,j\}}$ acts flag transitively on it by Lemma 1.8.6.

If G has diagram D over $(G_i)_{i\in I}$, then G_J has diagram $D|_{(I\setminus J)}$ over $(G_{J\cup\{i\}})_{i\in I\setminus J}$. Here, $D|_{(I\setminus J)}$ stands for the restriction of D to the collection of pairs from $I \setminus J$. In the flag-transitive case, the above definition coincides with the ordinary diagram.

Proposition 2.6.2 *Suppose that I is finite and that G is a group having diagram D over a system of subgroups $(G_i)_{i\in I}$. If G is flag transitive on the coset incidence system $\Gamma = \Gamma(G, (G_i)_{i\in I})$, then Γ is a geometry belonging to D.*

Proof This is a direct consequence of Theorem 1.8.10(ii). □

We analyze a case similar to the Direct Sum Theorem 2.1.6, in which generalized digons (cf. Definition 2.1.1) play a role.

Lemma 2.6.3 *Let I be finite and let G be a group with diagram D over a system of subgroups $(G_i)_{\in I}$. Suppose that the following two conditions hold.*

(i) $G_J = \langle G_{J\cup\{i\}} \mid i \in I \setminus J\rangle$ *for each* $J \subseteq I$ *with* $|I \setminus J| \geq 2$.
(ii) *I is partitioned into R and L in such a way that $D(r, l)$ consists of generalized digons for every $r \in R$ and $l \in L$.*

Then $G = G_L G_R$. In particular $G = G_l G_r$ for all $l \in L$, $r \in R$.

Proof If $|I| = 2$, then $R = \{r\}$ and $L = \{l\}$, so $\Gamma(G, (G_r, G_l))$ is a generalized digon. This means $gG_r \cap hG_l \neq \emptyset$ for each $g, h \in G$. In particular, if $x \in G$, then $xG_r \cap G_l \neq \emptyset$, so $x \in G_l G_r$. This settles $G = G_l G_r$ and establishes the rank two case. As G is a group, we can derive $G = G_r G_l$ by taking inverses.

In the case of arbitrary rank, we have

$$G = \langle G_i \mid i \in I\rangle = \langle G_J \mid |J| = 2, J \subseteq I\rangle = \cdots = \langle G_{I\setminus\{i\}} \mid i \in I\rangle.$$

But, for $r \in R$ and $l \in L$, by Definition 2.6.1 and what we have seen in the rank two case, $G_{I\setminus\{r,l\}} = G_{I\setminus\{l\}}G_{I\setminus\{r\}} = G_{I\setminus\{r\}}G_{I\setminus\{l\}}$, so

$$G = \langle G_{I\setminus\{i\}} \mid i \in I\rangle = \langle G_{I\setminus\{r\}} \mid r \in R\rangle\langle G_{I\setminus\{l\}} \mid l \in L\rangle \subseteq G_L G_R.$$

The last statement of the lemma follows as $G_L \subseteq G_l$ and $G_R \subseteq G_r$. □

The case of a *linear diagram* D, which means that, as a graph, D is a path, lends itself to an easy criterion for flag transitivity.

Theorem 2.6.4 *Let I be finite and let G be a group with a system of subgroups $(G_i)_{i\in I}$ such that*

(i) $G_J = \langle G_{J\cup\{i\}} \mid i \in I \setminus J\rangle$ *for each* $J \subseteq I$ *with* $|I \setminus J| \geq 2$;
(ii) *G has a linear diagram over* $(G_i)_{i\in I}$.

Then $\Gamma(G, (G_i)_{i\in I})$ is a residually connected geometry on which G acts flag transitively.

Proof In view of Corollary 1.8.13, the incidence system $\Gamma(G, (G_i)_{i\in I})$ is a residually connected geometry if G acts flag transitively on it. The latter is immediate if $|I| \leq 2$. Suppose, therefore, $|I| \geq 3$.

In view of Theorem 1.8.10(iii) and induction on $|I|$, it suffices to show that, for every subset J of I of size three, G is transitive on the set of all flags of type J. Denote the linear diagram by D and write $J = \{i, j, k\}$ where i, j, k are chosen so that $D(j, k)$ consists of generalized digons (note that this is always possible as D is linear). Now, by Lemma 2.6.3 applied to G_i, we have $G_i = G_{\{i,j\}}G_{\{i,k\}}$, so $G_iG_k \cap G_jG_k = G_{\{i,j\}}G_k \cap G_jG_k$. But this is equal to $G_{\{i,j\}}G_k$ as $G_{\{i,j\}} \subseteq G_j$, so $G_iG_k \cap G_jG_k = (G_i \cap G_j)G_k$. By Lemma 1.8.9(ii), this implies the required transitivity. □

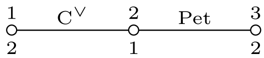
Example 2.6.5 Let G be the group given by the following presentation.

$$\begin{aligned} G = \big\langle a, d, e, z, t \mid {} & a^2 = d^3 = e^3 = z^2 = t^2 = 1, \\ & [e, z] = [a, z] = [z, t] = [d, e] = [a, t]z = 1, \\ & (dz)^2 = (et)^2 = (da)^5 = (dza)^5 = (ate)^3 = 1\big\rangle. \end{aligned}$$

A coset enumeration with respect to the subgroup $N = \langle a, d, z, eae^{-1}\rangle$ of G shows that we can also view G as the subgroup of Sym_{12} with generators

$$\begin{aligned} a &= (3, 4)(5, 6)(7, 10)(8, 11), \\ d &= (4, 5, 7)(6, 8, 9)(10, 12, 11), \\ e &= (1, 3, 2)(6, 9, 8)(10, 12, 11), \\ z &= (5, 7)(6, 10)(8, 11)(9, 12), \\ t &= (1, 2)(6, 10)(8, 12)(9, 11). \end{aligned}$$

Fig. 2.16 The diagram of the geometry associated with M_{11}



In fact, further computations show that G is 3-transitive on [12] (cf. Exercise 2.8.12) and has order 7920. From this it is not hard to derive that G is isomorphic to the Mathieu group M_{11} (see Definition 5.6.4). But we will not need this. We distinguish the following subgroups.

$$G_1 = \langle a, d, z, t\rangle, \qquad G_2 = \langle d, e, z, t\rangle, \qquad G_3 = \langle a, e, z, t\rangle.$$

Observe that G_1 stabilizes the set [2], that G_2 stabilizes [3], and that G_3 stabilizes the set [4]. In fact, they are the full stabilizers; G_1 has index 66 in G, while G_2 has index 220, and G_3 has index 165.

Now $G_{12} = \langle d, z, t\rangle$ is a group of order 12, and $G_{13} = \langle a, z, t\rangle$ is a group of order 8, and $G_{23} = \langle e, z, t\rangle$ is a group of order 12. Consequently, $G_1 = \langle a, d, z\rangle = \langle G_{12}, G_{13}\rangle$, $G_2 = \langle d, e, z\rangle = G_{12}G_{23}$, $G_3 = \langle a, e, z\rangle = \langle G_{13}, G_{23}\rangle$. Since G_2 is factored into G_{12} and G_{23}, the group G has a linear diagram over (G_1, G_2, G_3). By Theorem 2.6.4, G is flag transitive on $\Gamma := \Gamma(G, (G_1, G_2, G_3))$ and Γ is a residually connected geometry. It is firm, as G_{123} has order four, whence index 3, 2, 3, in G_{12}, G_{13}, G_{23}, respectively.

The geometry belongs to the diagram depicted in Fig. 2.16. We already know that the diagram is linear. A residue of type $\{1, 2\}$ can be viewed as the geometry whose 1-elements are 2-sets and whose 2-elements are 3-sets in a given 4-set of [12], whence the dual C diagram $\mathrm{C}^\vee$. A residue of type $\{2, 3\}$ can be viewed as a graph on $\{3, \dots, 12\}$, with the 1-elements being its vertices and the 2-elements being edges $\{x, y\}$ such that $\{1, 2, x, y\}$ belongs to the G_1-orbit of $\{1, 2, 3, 4\}$. There are precisely three such edges on $\{1, 2, 3\}$ and the group acting on this residue is $G_1 \cong \mathrm{Sym}_5$; this implies that the graph must be the Petersen graph.

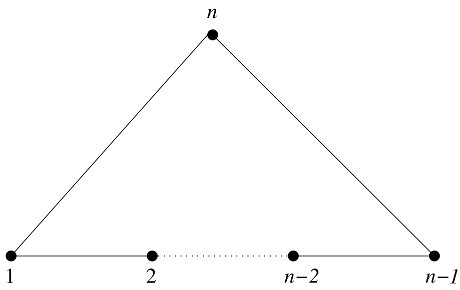
2.7 A Geometry of Type $\widetilde{\mathrm{A}}_{n-1}$

So far, we have dealt with various constructions of geometries with linear diagrams from groups. In this section we provide an example having the non-linear Coxeter diagram $\widetilde{\mathrm{A}}_{n-1}$ depicted in Fig. 2.17, where $n \in \mathbb{N}$, $n > 2$. For $n = 2$, the construction is also valid; the resulting diagram appears in Remark 2.7.15. Example 2.4.4 showed thin geometries of this type, the triangle ($n = 3$) and the quadrangle ($n = 4$). The geometries constructed in this section are thick and all of their rank $n - 1$ residues are isomorphic to $\mathrm{PG}(V)$ for some vector space V of dimension n.

We first review some local ring theory from commutative algebra.

Definition 2.7.1 A commutative ring R is called a *discrete valuation ring* if it is a *principal ideal domain* (i.e., an associative ring with 1 and without zero divisors in which each ideal is generated by a single element), with a unique nonzero max-

Fig. 2.17 Coxeter type $\widetilde{\mathrm{A}}_{n-1}$



imal ideal. A generator of the maximal ideal is called a *local parameter*. The field obtained from R by modding out its maximal ideal is called its *residue field*.

Example 2.7.2 Let p be a prime.

(i) The ring of *p-local integers* consists of all rational numbers a/b with $\gcd(b, p) = 1$. The subset $\{pa/b \mid a, b \in \mathbb{Z}, \gcd(b, p) = 1\}$ is its unique maximal ideal and p is a local parameter (and so are $-p$ and $p(1 + p)$). The residue field of this ring is isomorphic to $\mathbb{F}_p$.
(ii) Let $\mathbb{F}$ be a field and set $R = \{f/g \mid f, g \in \mathbb{F}[X], \gcd(g, X) = 1\}$. Then $\{Xf/g \mid f, g \in \mathbb{F}[X], \gcd(g, X) = 1\}$ is the unique maximal ideal in R and X is a local parameter. The residue field of R is isomorphic to $\mathbb{F}$.

Lemma 2.7.3 *Let R be a discrete valuation ring with local parameter π. Then, for each nonzero element x of R there is a unique integer $i \in \mathbb{N}$ such that $x = \pi^i y$ with y an invertible element of R.*

Proof If $x \notin \pi R$, then the ideal generated by x coincides with R, so there is $z \in R$ with $xz = 1$; in other words, x is invertible in R. Since, obviously, invertible elements do not belong to πR, we find that $R \setminus \pi R$ is the set of invertible elements of R. In particular, for $x \notin \pi R$, the lemma holds with $i = 0$.

Observe that $(\pi^i R)_{i\in\mathbb{N}}$ is a strictly descending chain of ideals. For if $\pi^i R = \pi^{i+1} R$ for some $i \in \mathbb{N}$, then $\pi^i = \pi^{i+1} r$ for some $r \in R$ and so, as R is a domain, $1 = \pi r$, contradicting that π is not invertible.

We derive $\bigcap_{i\in\mathbb{N}} \pi^i R = \{0\}$. The left hand side is an ideal of R. As R is a principal ideal domain, there is $a \in R$ such that $\bigcap_{i\in\mathbb{N}} \pi^i R = aR$. We claim that $\pi aR = aR$. Let $x \in aR$. As $aR \subseteq \pi R$, there is an element y in R with $x = \pi y$. Let $j \in \mathbb{N}$ be arbitrary. As $x \in aR$, there is $z_j \in R$ such that $x = \pi^{j+1} z_j$. Now $\pi y = \pi^{j+1} z_j$, so $y = \pi^j z_j$. We find $y \in \bigcap_{i\in\mathbb{N}} \pi^i R = aR$ and $x \in \pi aR$. In particular, $\pi aR = aR$. This means that there is $b \in R$ with $b\pi a = a$. But $(b\pi - 1)$ is invertible as it is not in πR, so $a = 0$. Therefore, indeed, $\bigcap_{i\in\mathbb{N}} \pi^i R = aR = \{0\}$.

So, if $x \in R$, then $x \in \pi^i R \setminus \pi^{i+1} R$ for some $i \in \mathbb{N}$. Then $x = \pi^i y$ for some $y \in R \setminus \pi R$, and, by the above, y is invertible. □

As a consequence, an element of a discrete valuation ring is invertible if and only if it does not belong to the maximal ideal. Being an integral domain, a discrete valuation ring has a field of fractions into which it embeds.

Corollary 2.7.4 *The elements of the field of fractions of a discrete valuation ring R are all of the form $\pi^i y$ with $y \in R \setminus \pi R$ and $i \in \mathbb{Z}$.*

Proof Suppose $v, w \in R$ with $w \neq 0$. By Lemma 2.7.3, there are $i, j \in \mathbb{N}$ and invertible elements $x, y \in R$ with $v = \pi^i x$ and $w = \pi^j y$, so $vw^{-1} = \pi^{i-j} xy^{-1}$, with $i - j \in \mathbb{Z}$ and xy^{-1} an invertible element of R, as required. $\square$

Definition 2.7.5 Let R be a discrete valuation ring with local parameter π and field of fractions $\mathbb{F}$. Consider $G = \mathrm{SL}(\mathbb{F}^n)$, with $n \geq 3$. For $i \in [n]$, take G_i to be the set of matrices in G, of the form

$$\begin{pmatrix} A_{i,i} & \pi^{-1} B_{i,n-i} \\ \pi C_{n-i,i} & D_{n-i,n-i} \end{pmatrix},$$

where $X_{k,l}$, for X one of A, B, C, D, denotes a $k \times l$-matrix with entries in R. In particular, $G_n = \mathrm{SL}(R^n)$. It is easily checked that each G_i is a subgroup of G. The corresponding coset incidence system $\Gamma = \Gamma(G, (G_i)_{i \in [n]})$ over $[n]$ is called the *π-adic geometry* of $\mathbb{F}^n$.

The word geometry in the name will be justified by Theorem 2.7.14 below. This group-theoretic definition of the incidence system is quite succinct. But, in order to derive properties of it, a geometric description will be used as well. The necessary objects stem from a generalization of lattices over the ring of integers $\mathbb{Z}$ to discrete valuation rings. For the introduction of lattices, we use the notion of a module over a ring, which is the well-known generalization of a vector space over a field.

Definition 2.7.6 Let V be a vector space over the field $\mathbb{F}$. For a subring R of $\mathbb{F}$, an *R-lattice* in V is an R-submodule of V generated by a vector space basis of V.

The notion of a lattice in the context of a poset, as used in Exercise 3.7.7, is an entirely different one.

Remark 2.7.7 A basis $b_1, \ldots, b_n$ of $\mathbb{F}^n$ determines the R-lattice

$$L = Rb_1 \oplus Rb_2 \oplus \cdots \oplus Rb_n$$

in V. We collect the vectors b_i as columns in a matrix B. These can be used to describe L as the image $B(R^n)$ of the map $R^n \to \mathbb{F}^n$, $x \mapsto Bx$. The matrix B belongs to $\mathrm{GL}(\mathbb{F}^n)$ as its columns constitute a basis of $\mathbb{F}^n$.

Lemma 2.7.8 *Suppose that the R-lattice L is generated by the columns of the matrix $B \in \mathrm{GL}(\mathbb{F}^n)$ and that $L' = B'(R^n)$ is the sublattice of L generated by the columns of the matrix $B' \in \mathrm{GL}(\mathbb{F}^n)$. The determinant* $\det(B)$ *is a divisor of* $\det(B')$

in R*. Moreover,* $L = L'$ *if and only if the quotient* $\det(B')/\det(B)$ *is invertible in* R*. In particular,* $\det(B)$ *is determined by* L *up to multiplication by an invertible element of* R*.*

Proof Denote the i-th column of B by b_i, and the i-th column of B' by b'_i. Since $\{B'x \mid x \in R^n\} \subseteq \{Bx \mid x \in R^n\}$, there are elements $x_{ij} \in R$ $(1 \le i, j \le n)$ such that $b'_i = \sum_{j=1}^n x_{ji} b_j$. This means that $X = (x_{ij})_{i,j\in[n]}$ is a matrix such that $B' = BX$, and so $\det(B') = \det(B)\det(X)$. This proves that $\det(B)$ divides $\det(B')$.

If $L = L'$, then there is also a matrix Y with entries in R such that $B = B'Y$. But then YX is the identity matrix and so X and Y are invertible, whence the last two statements of the lemma. □

Definition 2.7.9 Two R-lattices L, M of the same vector space are called *homothetic* if there is a scalar $\lambda \in \mathbb{F}$ such that $L = \lambda M$.

Being homothetic is an equivalence relation on the set of all R-lattices in $\mathbb{F}^n$.

Notation 2.7.10 For an R-lattice in $\mathbb{F}^n$, we denote by $[L]$ its *homothety class*, that is, its equivalence class with respect to the relation of being homothetic. Furthermore, we write $[L] * [M]$ to denote that, for some $L' \in [L]$ and $M' \in [M]$, we have $\pi L' \subseteq M' \subseteq L'$.

We use $*$ for a geometric definition of the π-adic incidence system. Let $\mathcal{L}$ be the collection of homothety classes of R-lattices in $\mathbb{F}^n$ and, for $[L] \in \mathcal{L}$, denote by $\tau([L])$ the integer $i \in [n]$ for which there is a matrix B with entries in $\mathbb{F}$ such that $B(R^n) \in [L]$ and $\det(B) = \pi^i$.

Lemma 2.7.11 *For each* $n \in \mathbb{N}$, $n \ge 2$, *and discrete valuation ring* R, *the triple* $(\mathcal{L}, *, \tau)$ *is an incidence system over* $[n]$.

Proof We first verify that τ is well defined. Suppose, to this end, that $B(R^n) = \lambda B'(R^n)$ for two matrices B, B' in $\mathrm{GL}(\mathbb{F}^n)$ and $\lambda \in \mathbb{F}$. It follows that $\det(B) = \lambda^n \det(B')$ and $\lambda \neq 0$. Write $j = \tau([B(R^n)])$ and $k = \tau([B'(R^n)])$. If D is a diagonal matrix with all diagonal entries but one equal to 1 and the remaining entry equal to an invertible element z of R, then $B(R^n) = BD(R^n)$ and $\det(BD) = \pi^j z$. So, replacing B and B' by suitable matrices, we may assume $\det(B) = \pi^{j+pn}$ and $\det(B') = \pi^{k+qn}$ for certain $p, q \in \mathbb{Z}$. As $\lambda \neq 0$, by Lemma 2.7.3, there are $i \in \mathbb{Z}$ and $y \in R \setminus \pi R$ such that $\lambda = \pi^i y$. Consequently, $\pi^{j+pn} = \det(B) = \lambda^n \det(B') = \pi^{ni+k+qn} y^n$. By uniqueness of the exponent of π (see Lemma 2.7.3), we find $j - k = n(i + q - p)$. As both $j, k \in [n]$, it follows that $j = k$, which proves that τ does not depend on the choice of B.

Next, we show that $*$ is a symmetric relation. If $[L] * [M]$, then, by definition, there are $L' \in [L]$ and $M' \in [M]$ with $\pi L' \subseteq M' \subseteq L'$. Now $\pi(\pi^{-1}M') \subseteq L' \subseteq \pi^{-1}M'$, so $[\pi^{-1}M'] * [L]$. Since $[\pi^{-1}M'] = [M'] = [M]$, this proves symmetry of $*$.

Suppose, once more, $[L] * [M]$ but, in addition $\tau([L]) = \tau([M]) = i$. Choose L' and M' as before. Multiplying both lattices L and M by the same suitable power of π, we may assume $M = M'$. Let A and B be $n \times n$ matrices over $\mathbb{F}$ with $L' = A(R^n)$ and $M = B(R^n)$. Without loss of generality, we take $\det(B) = \pi^i$. By Lemma 2.7.8, there are $a, b \in R$ such that $\det(\pi A) = a \det(B)$ and $\det(B) = b \det(A)$. This gives $\pi^n \det(A) = \det(\pi A) = a \det(B) = ab \det(A)$, so $\pi^n = ab$. Now $\tau([L]) = i$ means that $\det(A) = \pi^{i+kn} z$ for some $k \in \mathbb{Z}$ and z is an invertible element of R. Furthermore, $\det(B) = b \det(A)$ leads to $\pi^i = b\pi^{i+kn} z$, and so $b = \pi^{-kn} z^{-1}$. Now $b \in R$ implies $k \leq 0$ and the fact that b divides π^n means that only $k = 0$ and $k = -1$ are possible. If $k = 0$, then $b \in R \setminus \pi R$ and Lemma 2.7.8 gives $M = L'$; if $k = -1$, then $a = z \in R \setminus \pi R$, so $\pi L' = M$. This proves $[L] = [M]$. We conclude that the restriction of $*$ to $\tau^{-1}(i)$ is the identity, so all conditions of Definition 1.2.2 are satisfied for $(\mathcal{L}, *, \tau)$. □

A special chamber of $(\mathcal{L}, *, \tau)$ is the set $\{[L_i] \mid i \in [n]\}$, where

$$L_i = R\varepsilon_1 + \cdots + R\varepsilon_{n-i} + \pi R\varepsilon_{n-i+1} + \cdots + \pi R\varepsilon_n. \tag{2.2}$$

Here, $\varepsilon_1, \ldots, \varepsilon_n$ denotes the standard basis of $\mathbb{F}^n$. So $L_i = D_{n-i}(R^n)$, where D_i is the diagonal matrix with ones in the first i diagonal entries and π in the remaining $n - i$ diagonal entries. As $\det(D_{n-i}) = \pi^i$, we have $\tau([L_i]) = i$.

The group $\mathrm{GL}(\mathbb{F}^n)$ acts on the set of R-lattices via left multiplication: $A(R^n) \mapsto BA(R^n)$ for $A, B \in \mathrm{GL}(\mathbb{F}^n)$.

Lemma 2.7.12 *Let $\Gamma = (\mathcal{L}, *, \tau)$. The group* $\mathrm{Cor}(\Gamma)$ *of auto-correlations of Γ is transitive on $\mathcal{L}$. More specifically, the following statements hold.*

(i) *The group* $\mathrm{SL}(\mathbb{F}^n)$ *acts on Γ via* $B([L]) = [BL]$ $(B \in \mathrm{SL}(\mathbb{F}^n))$.
(ii) *The matrix*

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & \cdot & \cdot & 0 \\ 0 & 0 & 1 & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \ddots & \cdot & \vdots \\ 0 & 0 & 0 & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ \pi & 0 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

belongs to $\mathrm{GL}(\mathbb{F}^n)$ *and induces a correlation of Γ permuting the type set $[n]$ cyclically, according to $(1, 2, \ldots, n)$.*

Proof The group $\mathrm{GL}(\mathbb{F}^n)$ preserves homothety classes, so its action on $\mathcal{L}$ is well defined. Also, the action is easily seen to preserve incidence.

If $B \in \mathrm{GL}(\mathbb{F}^n)$ and $\det(B) = \pi^i x$ for some $i \in \mathbb{Z}$ and some invertible $x \in R$, then $\det(BA) = \det(B)\det(A)$, so $\tau(B[A(R^n)]) \equiv i + \tau([A(R^n)]) \pmod n$ for each $A \in \mathrm{GL}(\mathbb{F}^n)$. This shows that B maps elements of $\mathcal{L}$ of type j to elements of type $i + j$ modulo n.

(i) Suppose now $B \in \mathrm{SL}(\mathbb{F}^n)$. Then $\det(B) = 1$, so B preserves types.

An arbitrary representative of a homothety class in $\mathcal{L}$ is given by a matrix A whose columns are a basis of $\mathbb{F}^n$. Write $\det(A) = \pi^k x$ with $k \in \mathbb{Z}$ and invertible $x \in R$. After applying a suitable homothety, we may assume $k \in [n]$ without changing $[A(R^n)]$. But then $B = AD_{n-k}^{-1} \in \mathrm{SL}(\mathbb{F}^n)$ and $A(R^n) = BD_{n-k}(R^n) = B(L_k)$ in view of (2.2), so $\mathrm{SL}(\mathbb{F}^n)$ is transitive on $\tau^{-1}(k)$.

(ii) As $\det(\alpha) = \pm\pi$, we have $\alpha \in \mathrm{GL}(\mathbb{F}^n)$. Moreover, $\alpha L_i = L_{i+1}$ (indices taken mod n and in $[n]$), so the induced permutation of α on the type set $[n]$ is as stated.

Together with (i), it follows that the subgroup of $\mathrm{GL}(\mathbb{F}^n)$ generated by α and $\mathrm{SL}(\mathbb{F}^n)$ is transitive on $\mathcal{L}$. □

The natural quotient map $R \to R/\pi R$ is a ring homomorphism onto the residue field $\mathbb{K} = R/\pi R$ of R. It leads to a group homomorphism $\phi : \mathrm{SL}(R^n) \to \mathrm{SL}(\mathbb{K}^n)$ given by reduction modulo πR in each entry. In turn, this homomorphism provides an action of $\mathrm{SL}(R^n)$ on $\mathbb{K}^n$.

Lemma 2.7.13 *The residue of each element of $\Gamma = (\mathcal{L}, *, \tau)$ is isomorphic to* $\mathrm{PG}(\mathbb{K}^n)$. *Let Γ_n denote the residue of $[R^n]$ in Γ. The map $\beta : \Gamma_n \to \mathrm{PG}(\mathbb{K}^n)$ given by $\beta([M]) = M/\pi R^n$, where M is chosen in such a way that $\pi R^n \subseteq M \subseteq R^n$, is an isomorphism establishing an equivalence of the natural action of the group* $\mathrm{SL}(R^n)$ *on Γ_n to the action via ϕ on* $\mathrm{PG}(\mathbb{K}^n)$.

Proof By Lemma 2.7.12 it suffices to prove the first statement for the element $[R^n]$. Suppose that $[M] \in \mathcal{L}$, of type $i \in [n-1]$, is in the residue Γ_n. We can choose the R-lattice M in such a way that $\pi R^n \subseteq M \subseteq R^n$. Now $M/\pi R^n$ is a subspace of $R^n/\pi R^n = \mathbb{K}^n$ of dimension $n - i$ (see Exercise 2.8.31).

We claim that the map β given by $\beta([M]) = M/\pi R^n$ is an isomorphism $\Gamma_n \to \mathrm{PG}(\mathbb{K}^n)$. First of all, β is injective because each homothety class has at most one representative M such that $\pi R^n \subseteq M \subseteq R^n$. Moreover, β is surjective because each subspace of $\mathbb{K}^n$ of dimension i is the image of R^n under an $n \times n$-matrix U over $\mathbb{K}$ of rank i and so each $n \times n$-matrix B over R whose entries map onto those of U defines a class $[M]$, where $M = B(R^n)$, that is, an element of Γ_n with $\beta([M]) = U(\mathbb{K}^n)$.

If $[M] * [M']$ for some element $[M']$ of Γ_n with $\pi R^n \subseteq M' \subseteq R^n$, then there is an integer j such that $M \subseteq \pi^j M' \subseteq \pi^{-1} M$. Now $\pi R^n \subseteq M, M' \subseteq R^n$ forces $j = 0, -1$. Interchanging the roles of M and M' if needed, we may take $j = 0$, which means $M \subseteq M'$ and implies $\beta(M) \subseteq \beta(M')$, proving $\beta([M]) * \beta([M'])$ in $\mathrm{PG}(\mathbb{K}^n)$. The reverse implication is proved similarly.

As for the group actions, let $B \in \mathrm{SL}(R^n)$ and let M be an R-lattice in $\mathbb{F}^n$ with $\pi R^n \subseteq M \subseteq R^n$. Then $\beta(B([M])) = \beta([BM]) = BM/\pi R^n = \phi(B)\beta(M)$. This establishes that β is an equivalence between the actions of $\mathrm{SL}(R^n)$ on Γ_n and $\mathrm{PG}(\mathbb{K}^n)$. □

We now relate the incidence system $(\mathcal{L}, *, \tau)$ of Lemma 2.7.11 to the coset incidence system $\Gamma(G, (G_i)_{i\in[n]})$ of Definition 2.7.5. The type $\widetilde{\mathrm{A}}_{n-1}$ appearing in the theorem below is depicted in Fig. 2.17.

Theorem 2.7.14 *Let $n \in \mathbb{N}$, $n \geq 2$, and let R be a discrete valuation ring with local parameter π and field of fractions $\mathbb{F}$. The incidence system $(\mathcal{L}, *, \tau)$ is an $[n]$-geometry of type $\widetilde{\mathrm{A}}_{n-1}$. Moreover, the action of $G = \mathrm{SL}(\mathbb{F}^n)$ on $(\mathcal{L}, *, \tau)$ is flag transitive with stabilizers G_i, so it is equivalent to the geometric representation on the π-adic geometry $\Gamma(G, (G_i)_{i\in[n]})$ over $\mathbb{F}^n$.*

Proof We show that G acts flag transitively on $\Gamma = (\mathcal{L}, *, \tau)$. Let J be a non-empty subset of $[n]$ and let $(M_i)_{i\in J}$ be R-lattices such that $([M_i])_{i\in J}$ is a flag of type J in Γ. We need to show that this flag is in the same G-orbit as $([L_i])_{i\in J}$ of (2.2). In view of Lemma 2.7.12, we may assume $n \in J$. But then $([M_i])_{i\in J\setminus\{n\}}$ is a flag of type $J \setminus \{n\}$ in Γ_n and so $(\beta([M_i]))_{i\in J\setminus\{n\}}$, where β is as in Lemma 2.7.13, is a flag of $\mathrm{PG}(\mathbb{K}^n)$. By flag transitivity of $\mathrm{SL}(\mathbb{K}^n)$ on this geometry (cf. Example 1.8.16), there is $A \in \mathrm{SL}(\mathbb{K}^n)$ such that $A(\beta([M_i]))_{i\in J\setminus\{n\}} = (\beta([L_i]))_{i\in J\setminus\{n\}}$. The equivalence of the two $\mathrm{SL}(R^n)$-actions of Lemma 2.7.13 gives that $([M_i])_{i\in J\setminus\{n\}}$ and $([L_i])_{i\in J\setminus\{n\}}$ are in the same $\mathrm{SL}(R^n)$-orbit. It follows that G acts flag transitively on Γ.

Fix $i \in [n]$. We verify that G_i is the stabilizer in G of the element $[L_i] \in \mathcal{L}$. Clearly, G_i stabilizes L_i and hence $[L_i]$. Let H_i be the stabilizer of $[L_i]$ in $\mathrm{SL}(\mathbb{F}^n)$ and take $A \in H_i$. Then H_i must stabilize L_i itself. If $j \leq n-i$, then $A\varepsilon_j \in L_i$ means that the j-th column of A has the first $n-i$ entries in R and the last i entries in πR. If $j > n-i$, then $A\varepsilon_j \in L_i$ means that the j-th column of A has the first $n-i$ entries in $\pi^{-1}R$ and the last i entries in R. Therefore, $H_i \subseteq G_i$ and so $H_i = G_i$.

Proposition 1.8.7 gives that the geometric representation of G over $(G_i)_{i\in[n]}$ is equivalent to the action on Γ. It follows from Theorem 1.8.10 that Γ is a geometry. By now it is easy to see that Γ has Coxeter diagram $\widetilde{\mathrm{A}}_{n-1}$. For, the residue of each element is isomorphic to $\mathrm{PG}(\mathbb{K}^n)$ and so belongs to the diagram A_{n-1}.

It remains to show that Γ is residually connected. As all residues of non-empty flags are residues inside a geometry isomorphic to $\mathrm{PG}(\mathbb{K}^n)$, Lemma 1.8.9 shows that we only need to verify that G is generated by all G_i $(i \in [n])$. But this readily follows from the relation

$$\begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi^{-1} & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \begin{pmatrix} 0 & \pi^{-1} & 0 \\ -\pi & 0 & 0 \\ 0 & 0 & I_{n-2} \end{pmatrix} \in G_n G_1. \qquad \square$$

Remark 2.7.15 (i) For $n = 2$, the above construction of $(\mathcal{L}, *, \tau)$ is still valid and leads to a flag-transitive [2]-geometry of type $\overset{1}{\circ}\!\!\overset{\infty}{\text{———}}\!\!\overset{2}{\circ}$. This geometry is also known as a generalized ∞-gon (cf. Definition 2.2.6). The incidence graph of this geometry is a connected graph without circuits, also known as a *tree*.

(ii) If R is as in Example 2.7.2(i), then, due to Proposition 2.4.7, all i-orders of $(\mathcal{L}, *, \tau)$ are equal to p. Similarly, if $R = \mathbb{F}_q[[X]]$ (see Exercise 2.8.27), then all i-orders are equal to q.

2.8 Exercises

Section 2.1

Exercise 2.8.1 For the geometry Γ of Example 2.1.3 construct an auto-correlation that induces a permutation of order 4 on the set of types. Does the geometry Δ of Example 2.1.3 possess trialities?

Exercise 2.8.2 Let J be a finite index set. Prove that a direct sum of geometries Γ_j $(j \in J)$ is residually connected if and only if each Γ_j is residually connected.

Section 2.2

Exercise 2.8.3 Show that the (ordinary) m-gon, that is, with vertices as points and edges as lines, is the only generalized m-gon of order $(1, 1)$. Verify that there are no $(m, d_{\mathtt{p}}, d_{\mathtt{l}})$-gons of order $(1, 1)$ unless $m = d_{\mathtt{p}} = d_{\mathtt{l}}$.

Exercise 2.8.4 Let Γ be a generalized m-gon. Take Δ to be the rank two geometry over $\{\mathtt{c}, \mathtt{o}\}$ whose $\mathtt{o}$-elements are the elements (points and lines) of Γ and whose $\mathtt{c}$-elements are the chambers of Γ; incidence is symmetrized containment. Show that Δ is a generalized $2m$-gon. Is it thick?

Exercise 2.8.5 Let Γ be a geometry of rank two satisfying the following conditions for some $m \in \mathbb{N}$ with $m \geq 2$.

(a) Every pair of elements is joined by a chain of length at most m and by at most one chain of length smaller than m.
(b) There exists an m-gon in Γ (that is, a subgeometry isomorphic to the ordinary m-gon).

Prove that Γ is a generalized m-gon.
(*Hint:* Let M be an m-gon in Γ and x an element of Γ. Then there is an element $y \in M$ such that x is at distance m from y in the incidence graph. Construct a bijection from Γ_x to Γ_y.)

Exercise 2.8.6 Show that the generalized hexagon of Example 2.2.15 admits no dualities.
(*Hint:* The part of the collinearity graph of the dual line space induced on the set of points at distance three from a given point differs from the original.)

Exercise 2.8.7 Consider the vector space $V = \mathbb{F}_9^3$ with nondegenerate bilinear form f given by $f(x, y) = x_1y_1 + x_2y_2 + x_3y_3$ for $x = \varepsilon_1x_1 + \varepsilon_2x_2 + \varepsilon_3x_3$ and $y = \varepsilon_1y_1 + \varepsilon_2y_2 + \varepsilon_3y_3$ in $\mathbb{F}_9^3$. Put $G = \mathrm{O}(V, f)$, the orthogonal group introduced in Exercise 1.9.31. Write $\mathbb{F} = \mathbb{F}_9$ and prove the following statements.

(a) The set P of projective points $a\mathbb{F} \in \mathbb{P}(V)$ for which $f(a,a)$ is a nonzero square in $\mathbb{F}$, is a single G-orbit of size 45.
(b) The set Q of orthogonal frames in P, that is, subsets $\{a\mathbb{F}, b\mathbb{F}, c\mathbb{F}\}$ of P with a, b, c mutually orthogonal, is a single G-orbit of size 30. The kernel of the action of G on Q is $\mathrm{Z}(G)$, the center of G, which is a group of size two.
(c) Turn Q into a graph by letting $x \sim y$ for $x, y \in Q$ if and only $x \neq y$ and $x \cap y \neq \emptyset$. This graph has diameter four with 3, 6, 12, and 8 vertices at distance 1, 2, 3, and 4 from a given vertex, respectively. The group G acts *distance transitively* on Q (this means that G acts transitively on the sets of pairs of points at mutual distance i for each $i \in \mathbb{N}$). The distance distribution diagram coincides with the incidence graph of the rank two geometry of a generalized quadrangle of order $(2, 2)$. So the graph on Q is bipartite with two parts of size 15.
(d) The group G is isomorphic to $\mathrm{C}_2 \times (\mathrm{Alt}_6 \rtimes \langle\sigma\rangle)$, where σ acts on Alt_6 as one of the outer automorphisms of Sym_6 given in Example 2.2.13.
(e) For $a\mathbb{F} \in P$, the orthogonal reflection $r_{\alpha,\phi} : V \to V$ from Exercise 1.9.31, with ϕ given by $\phi(x) = 2f(a,a)^{-1}f(a,x)$, has the form $r_{a,\phi}(x) = x + af(a,a)^{-1}f(a,x)$ $(x \in V)$ and belongs to G. The subgroup H of G generated by all such orthogonal reflections has order 720 and preserves the partition of Q into two cocliques. In particular, $\mathrm{Z}(H) = \mathrm{Z}(G)$ and $H/\mathrm{Z}(H) \cong \mathrm{Alt}_6$.

Exercise 2.8.8 Let $G = \mathrm{Alt}_6$. This group has two conjugacy classes, say C_1 and C_2, of subgroups isomorphic to Alt_5. Show that the following graph is isomorphic to HoSi.

(1) Its vertex set consists of C_1, C_2, the twelve subgroups of G isomorphic to Alt_5 and the 36 Sylow 5-subgroups of G.
(2) The unordered pair $\{C_1, C_2\}$ is an edge. For $i \in [2]$, a subgroup of G isomorphic to Alt_5 is adjacent to C_i if it is a member of C_i. A subgroup of G isomorphic to Alt_5 and a Sylow 5-subgroup of G are adjacent if the latter is contained in the former. Two Sylow 5-subgroups of G are adjacent if together they generate G and there is an involution of G normalizing each. There are no further adjacencies.

Section 2.3

Exercise 2.8.9 (Cited in Example 2.3.2) Projective planes and affine planes are closely related.

(a) Let $\Pi = (P, L, *)$ be a thick projective plane. Fix $h \in L$. Show that the subgeometry of Π induced on $(P \setminus h^*) \cup (L \setminus \{h\})$ is an affine plane.
(b) Suppose that $A = (Q, M, *)$ is an affine plane. Prove that being parallel is an equivalence relation on M. For $m \in M$ we denote by $\overline{m}$ the parallel class (i.e., equivalence class) of m. Let P be the disjoint union of Q and the set h of equivalence classes of the parallelism relation on M. Let L be the disjoint union

of M and the singleton $\{h\}$. Extend the relation $*$ to a symmetric and reflexive relation on all of $P \cup L$ by demanding that, for $l, m \in M$, we have
(1) $\overline{m} * h$ if and only if $\overline{m} \in h$;
(2) $l * \overline{m}$ if and only if $l \in \overline{m}$;
(3) $x * h$ for no $x \in Q$.
Prove that $(P, L, *)$ is a projective plane, whose subgeometry induced on $Q \cup M$ coincides with A.

Exercise 2.8.10 Show that every projective plane of order four is isomorphic to $\mathbb{P}(\mathbb{F}_4^2)$.
(*Hint:* Use Exercise 2.8.9 and prove that there is a unique affine plane of order four.)

Exercise 2.8.11 Let C be the Cayley division ring of Example 2.3.4.

(a) Prove that C is not associative, but satisfies the laws $x^2y = x(xy)$ and $xy^2 = (xy)y$ for all $x, y \in C$ (here, of course, x^2 stands for xx). Algebras satisfying this law are called *alternative*.
(b) Define $\overline{x} = 2x_0e_0 - x$ for $x = \sum_i x_ie_i \in C$ and prove that $N(x) = x\overline{x}$ is multiplicative (i.e., $N(xy) = N(x)N(y)$) and positive definite (i.e., $N(x) > 0$ whenever $x \neq 0$). (Algebras with such a map N are called composition algebras.)
(c) Derive from (b) that each nonzero element x of C has an inverse x^{-1}.
(d) Show that, if $x, y \in C$ with x nonzero, then $(yx)x^{-1} = y$. (This property helps to verify (3) and (5) of the definition of a ternary ring by means of (2.1).)

Exercise 2.8.12 (This exercise is used in Examples 2.6.5 and 6.2.7) Let $G \to \mathrm{Sym}(X)$ be a permutation representation of a group G on a set X. Let $t \in \mathbb{N}$ with $0 < t \leq |X|$. The action of G on X is said to be t*-transitive* on X if G acts transitively on X and, whenever $t > 1$, the stabilizer G_x in G of a point $x \in X$ acts $(t-1)$-transitively on $X \setminus \{x\}$. Prove the following assertions.

(a) The group G acts 2-transitively on X if and only if there is $g \in G$ with $G = H \cup HgH$, where $H = G_x$.
(b) The group G (isomorphic to $\mathrm{PSL}(\mathbb{F}_{11}^2)$) of Example 2.3.11 acts 2-transitively on the collection of cosets of its subgroup G_o isomorphic to Alt_5.
(c) For every division ring $\mathbb{D}$, the group $\mathrm{PGL}(\mathbb{D}^2)$ is 3-transitive on the set of points of $\mathbb{P}(\mathbb{D}^2)$.
(d) Let $\mathcal{B}$ be a G-orbit of subsets of X. Assume that G acts t-transitively on X. There is a positive integer λ such that each set of t elements of X lies in exactly λ members of $\mathcal{B}$.

Section 2.4

Exercise 2.8.13 Consider the Petersen graph Pet and its two Alt_5-orbits, B and R, say, of pentagons discussed in Example 1.3.4. Define a rank three geometry $\Gamma =$

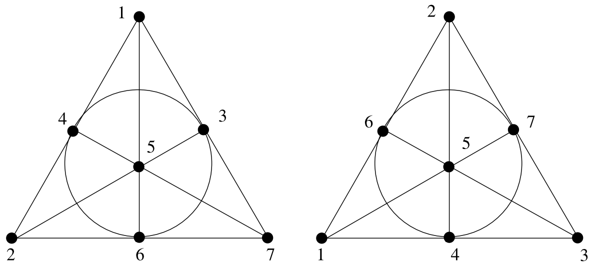


Fig. 2.18 Two disjoint Fano planes

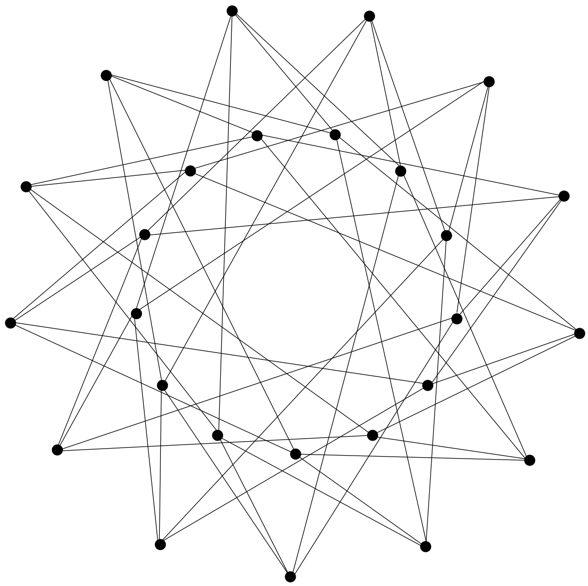


Fig. 2.19 The incidence graph of the projective plane $\mathbb{P}(\mathbb{F}_3^3)$. Rotations give a cyclic group of collineations of order 13

$(E, B, R, *)$ where E is the edge set of Pet, in such a way that it is isomorphic to the quotient of the great dodecahedron of Example 2.4.4 by a group of order two.

Exercise 2.8.14 (Cited in Remark 4.1.7) Let $\Omega = [7]$ and let G be the permutation group of order 42 consisting of all maps $x \mapsto ax + b \ \ (x \in \Omega)$ for $a \not\equiv 0 \bmod 7$, where addition and multiplication are taken modulo 7. Consider the incidence system Γ over [3] whose 1-elements are all points of Ω, whose 2-elements are all unordered pairs of points, and whose 3-elements are all orbits of size three of subgroups of G of order three. Incidence is symmetrized inclusion. Prove the following three statements.

(a) The set of 3-elements of Γ consists of the 14 subsets of Ω of size three depicted as lines in the two disjoint Fano planes of Fig. 2.18.

(b) Γ is a thin [3]-geometry with diagram $\overset{1}{\underset{1}{\circ}}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{2}{\underset{1}{\circ}}\!\!\overset{6}{-\!\!\!-\!\!\!-\!\!\!-\!\!\!-}\!\!\overset{3}{\underset{1}{\circ}}$.

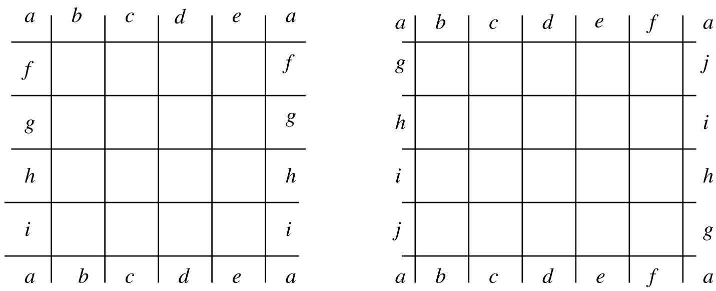


Fig. 2.20 *Left*: A toroidal geometry. *Right*: A Klein bottle geometry

(c) The group of automorphisms of Γ coincides with G. It is incidence transitive on Γ and has exactly two orbits on the set of chambers.

Exercise 2.8.15 Prove that the incidence graph of a projective plane of order 3 is necessarily as given in Fig. 2.19. Conclude that $\mathbb{P}(\mathbb{F}_3^3)$ is the unique projective plane of order three.

Exercise 2.8.16 Prove that $\mathbb{P}(\mathbb{F}_4^3)$ is the unique projective plane of order four.

Exercise 2.8.17 Consider a tiling T of the Euclidean plane by squares of unit side length. Let $u, v \in \mathbb{N}$, $u, v \geq 2$.

(a) Cut out a $u \times v$ rectangle from T that is covered by precisely uv tiles, and identify the border patches in the usual way to obtain a torus, as visualized in the left-hand side of Fig. 2.20 for $u = v = 5$. The result is $\mathbb{R}^2/(u\mathbb{Z} \times v\mathbb{Z})$, the quotient of $\mathbb{R}^2$ by the equivalence relation $\sim$ given by $(x, y) \sim (x', y')$ if and only if $x - x' \in u\mathbb{Z}$ and $y - y' \in v\mathbb{Z}$. Show that this quotient with vertices, edges, and tiles, leads to a geometry with diagram $\underset{1}{\circ}\!=\!=\!=\!\underset{1}{\circ}\!=\!=\!=\!\underset{1}{\circ}$ and that Γ is flag transitive if and only if $u = v$.

(b) Next, assume $v > 2$, start again with the $u \times v$ rectangle cut out from T and proceed to identify border patches as suggested by the right-hand side of Fig. 2.20 for $u = 6$ and $v = 5$. The result is the quotient of $\mathbb{R}^2$ by the equivalence relation $\sim$ given by $(x, y) \sim (x', y')$ if and only if either $x - x' \in u\mathbb{Z} \setminus 2u\mathbb{Z}$ and $y' + y \in v\mathbb{Z}$ or $x - x' \in 2u\mathbb{Z}$ and $y' - y \in v\mathbb{Z}$. It is the so-called *Klein bottle*. Show that the Klein bottle with vertices, edges, and tiles, leads to a geometry over $\underset{1}{\circ}\!=\!=\!=\!\underset{1}{\circ}\!=\!=\!=\!\underset{1}{\circ}$ whose automorphism group of order $\gcd(2, v)u$ is not transitive on the set of points.

Exercise 2.8.18 Consider the Petersen graph Pet. Its complement occurs in Example 1.7.16 for $\varepsilon = 1$ and $n = 5$. It is also the collinearity graph of the classical *Desargues configuration*; cf. Fig. 2.21. The depicted geometry Des has 10 lines of three points each. Observe that not all cliques of size three of the complement of Pet are represented by lines.

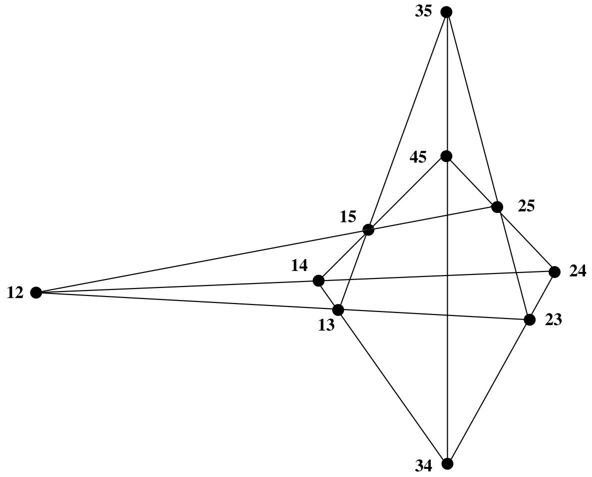


Fig. 2.21 Desargues configuration Des

(a) Show that the geometry Des belongs to $\underset{2}{\overset{\mathrm{p}}{\circ}}\overset{5 \quad 3 \quad 5}{\text{———}}\underset{2}{\overset{1}{\circ}}$.
(b) Let Δ be the geometry over $[3]$ whose 1-elements are the 10 points, whose 2-elements are the 30 pairs of collinear points, and whose 3-elements are the 15 induced quadrangles in the collinearity graph of Des (here, a quadrangle is a set of four points such that the collinearity induced on them is a 4-circuit). Incidence is symmetrized containment. Show that Δ belongs to $\underset{1}{\overset{1}{\circ}}=\!=\!=\underset{1}{\overset{2}{\circ}}\overset{6}{\text{———}}\underset{1}{\overset{3}{\circ}}$.
(c) Prove that $\mathrm{Aut}(\Delta)$ is isomorphic to Sym_5 and that this group acts flag transitively on Δ.

Section 2.5

Exercise 2.8.19 For $n, j \in \mathbb{N}$ with $1 \leq j < n$, the *Johnson graph* with parameters (n, j) has vertex set $\binom{[n]}{j}$ and adjacency $x \sim y$ given by $|x \cap y| = j - 1$. Show that the Johnson graph is the shadow space on j of the geometry of rank $n - 1$ of Example 1.2.6.

Exercise 2.8.20 Consider the geometry Γ of Exercise 1.9.18, which has infinite rank.

(a) Show that Γ does not satisfy the conclusion of Theorem 2.1.6.
(b) Verify that, for each $n \in \mathbb{N}$ with $n > 1$, the space $\mathrm{ShSp}(\Gamma, \{1/n\})$ is the space whose point set is $X_{1/n}$ (the set of elements of type $1/n$), and whose lines are of the form L_b for $b \in \mathbb{R}$, where L_b is the set of all members of $X_{1/n}$ containing b.

Exercise 2.8.21 (This exercise is used in Lemma 9.2.4) Let Z be a linear space and $X \subseteq P$. Define the derived set $X^{(1)}$ of X as the union of X and of all lines of Z

having at least two points in X. Put $X^{(0)} = X$, and for $n \in \mathbb{N}$ with $n > 0$, define the n-th *derived set* $X^{(n)}$ as $(X^{(n-1)})^{(1)}$. Show that $\langle X \rangle = \bigcup_{n\in\mathbb{N}} X^{(n)}$.

Exercise 2.8.22 Give an example to demonstrate that a homomorphism $\alpha : (P, L) \to (P', L')$ of line spaces does not necessarily correspond to a homomorphism $(P, L, *) \to (P', L', *')$ of geometries. Show that if (P', L') is linear and $\alpha(l)$ has at least two points for every $l \in L$, there is a homomorphism $(P, L, *) \to (P', L', *')$ inducing α on the point shadow space.

Exercise 2.8.23 Let (P, L) and (P', L') be linear spaces having at least three points each. Suppose that $\alpha : P \to P'$ is a bijection. Show that α is an isomorphism if and only if both α and α^{-1} map each set of three collinear points to a set of three collinear points.

Exercise 2.8.24 Let $Z = (P, L)$ be a line space and $\equiv$ an equivalence relation on P. By $d(x, y)$ we denote the distance between x and y in the collinearity graph of Z and by $\overline{x}$ the equivalence class of x in P. The *quotient space* $Z/\equiv$ of Z by $\equiv$ is the pair (P', L') where P' consists of the equivalence classes in P and L' consists of the sets $\{\overline{x} \in P' \mid x \in l\}$ for some $l \in L$. We say that $\equiv$ is a *standard equivalence* if, whenever $x \equiv x'$ and $x \perp y$ for $x, x', y \in P$, there is $y' \in P$ such that $x' \perp y'$ and $y' \equiv y$. Prove the following two assertions for a standard equivalence $\equiv$.

(a) If $d(x, x') \geq 2$ for any two points x and x' in P with $x \equiv x'$, then $Z/\equiv$ is a line space.
(b) Let A be a group of automorphisms of Z. Show that the equivalence relation of being in the same A-orbit is standard.
(c) The *quotient space* Z/A of Z by A is defined as the quotient space $Z/\equiv$, where $\equiv$ is as in (b). As in Definition 1.6.1, we denote by $d(x, y)$ the distance between x and y in the collinearity graph of Z. Prove that, if Z is a partial linear space and $d(x, \sigma(x)) \geq 3$ for each $x \in P$ and $\sigma \in A \setminus \{1\}$, then Z/A is a partial linear space.
(d) Give a connected line space with a standard equivalence satisfying $d(x, y) \geq 3$ for all distinct $x, y \in P$ with $x \equiv y$, for which (c) is not valid.
(*Hint:* Take $P = \{a_i, b_i, c_i, d_i \mid i \in [3]\}$; the line set L having $\{a_1, b_1, c_1\}$ and $\{b_2, c_2, d_2\}$ of size three and the following eight lines of size two: $\{b_1, d_3\}$, $\{c_1, d_1\}$, $\{b_3, d_1\}$, $\{a_2, c_2\}$, $\{a_2, b_3\}$, $\{a_3, b_2\}$, $\{a_3, c_3\}$, $\{c_3, d_3\}$; and take the equivalence of having the same letter in the name of the point.)

$\sharp$ **Exercise 2.8.25** (The tilde geometry) Put $\tau := \cos(\pi/5) = \frac{1}{4}(1 + \sqrt{5})$, $\rho := \cos(3\pi/5) = \frac{1}{4}(1 - \sqrt{5})$, and $\omega = \mathrm{e}^{2\pi i/3}$, so $\omega^2 = -\omega - 1$. In unitary space $\mathbb{C}^3$ with standard hermitian inner product f and standard orthonormal basis ε_1, ε_2, ε_3, consider the vectors $\alpha_1 = \varepsilon_1$, $\alpha_2 = \omega^2\tau\varepsilon_1 - \rho\varepsilon_2 + \omega/2\varepsilon_3$, and $\alpha_3 = 1/2\varepsilon_1 + \rho\varepsilon_2 + \tau\varepsilon_3$. For $\alpha \in \{\alpha_1, \alpha_2, \alpha_3\}$, the unitary reflection $r_{\alpha,\phi} : \mathbb{C}^3 \to \mathbb{C}^3$ from Exercise 1.9.31, with ϕ given by $\phi(x) = 2f(\alpha, x)$, belongs to $\mathrm{U}(\mathbb{C}^3, f)$. Denote by G the group generated by these three unitary reflections of order two.

(a) Prove that G leaves invariant the set

$$\begin{aligned}\Phi = \mu_6 \Big\{ &\varepsilon_i, \frac{1}{2}\varepsilon_i \pm \rho\varepsilon_j \pm \tau\varepsilon_k, \omega^2\tau\varepsilon_i \pm \rho\varepsilon_j \pm \frac{1}{2}\omega\varepsilon_k, \\ &\frac{1}{2}\varepsilon_i \pm \frac{1}{2}\omega^2\varepsilon_j \pm \left(\frac{1}{2} + \omega\tau\right)\varepsilon_k, \left(\frac{1}{2} + \omega^2\tau\right)(\varepsilon_i \pm \omega^2\varepsilon_j) \\ &\mid (i,j,k) \in \{(1,2,3),(3,1,2),(2,3,1)\} \Big\}\end{aligned}$$

containing $\alpha_1, \alpha_2, \alpha_3$. Here, $\mu_6 = \{\pm 1, \pm\omega, \pm\omega^2\}$.

(b) Establish that Φ is a single G-orbit of size 6×45.

(c) Observe that $p = \{\pm\varepsilon_1, \pm\omega^2\varepsilon_2, \pm\omega\varepsilon_3\}$ is the only orthonormal basis up to signs of $\mathbb{C}^3$ inside $M = \mu_6\varepsilon_1 \cup \mu_6\varepsilon_2 \cup \mu_6\varepsilon_3$ containing ε_1 and invariant under the normalizer in G of M. Let P be the G-orbit of p and show it has size 45.

(d) Set

$$\begin{aligned} d &= \varepsilon_2(-\omega^2\tau - 1/2) + \varepsilon_3(\omega\tau + 1/2\omega^2), \\ e &= \varepsilon_2(\omega^2\tau + 1/2) + \varepsilon_3(\omega\tau + 1/2\omega^2). \end{aligned}$$

Prove that every orthonormal basis of $\mathbb{C}^3$ containing ε_1, contained in Φ, and not contained in M lies in $N = \{\mu_6\varepsilon_1, \mu_6 d, \mu_6 e\}$ and that $l = \{\pm\varepsilon_1, \pm d, \pm e\}$ is the only orthonormal triple up to signs from N invariant under the normalizer in G of N. Let L be the G-orbit of l and show it has size 45.

(e) Show that the graph $(P \cup L, \sim)$, where $x \sim y$ if and only if $|x \cap y| = 1$, is a bipartite graph with parts P and L and with 3, 6, 12, 24, 24, 12, 6, 2 vertices at distance 1, 2, 3, 4, 5, 6, 7, 8, respectively, from a given vertex.

(f) Let Til be the incidence system $(P, L, *)$ over $\{\mathtt{p}, \mathtt{l}\}$, where $x * y$ if and only if $x \sim y$ or $x = y$. It is called the *tilde geometry*. Prove that Til is a connected and flag transitive geometry with parameters $d_{\mathtt{p}} = d_{\mathtt{l}} = 8$ and $g = 5$.

(g) Let A be the subgroup of G consisting of homotheties by a scalar from $\{1, \omega, \omega^2\}$. Verify that the elements of Til at distance eight from an element x are of the form ax for $a \in A \setminus \{1\}$. Conclude that being at distance eight is an equivalence relation on $P \cup L$.

(h) Consider the subring $\mathbb{Z}[\omega, \sqrt{5}, \frac{1}{2}]$ of algebraic numbers in $\mathbb{C}$ and the finite field $\mathbb{F}_9$ of order 9 with square root i of -1. Verify that there is a unique homomorphism $\sigma : \mathbb{Z}[\omega, \sqrt{5}, \frac{1}{2}] \to \mathbb{F}_9$ of rings determined by $\sigma(\frac{1}{2}) = -1$, $\sigma(\sqrt{5}) = i$, and $\sigma(\omega) = 1$.

(i) Observe that all components of vectors in Φ are actually in the ring $\mathbb{Z}[\omega, \sqrt{5}, \frac{1}{2}]$. By applying σ coordinatewise we obtain a map $\mathbb{Z}[\omega, \sqrt{5}, \frac{1}{2}]^3 \to \mathbb{F}_9^3$. The image of Φ under this map is a set $\overline{\Phi}$ of 45 vertices up to sign changes. Use this map to derive that the quotient incidence system Til$/A$ (cf. Definition 1.3.5) of Til by the group A is the generalized quadrangle of order $(2, 2)$ identified in Exercise 2.8.7.

Section 2.6

Exercise 2.8.26 Fix a finite-dimensional vector space V over the field $\mathbb{F}$, a natural number $k \leq \dim(V)/2$, and consider the graph Γ whose vertices are the k-dimensional vector spaces of V and in which two vertices X and Y are adjacent whenever $\dim(X \cap Y) = k - 1$.

(a) Prove that two vertices X, Y of Γ are at distance h in Γ (i.e., $d_\Gamma(X, Y) = h$) if and only if $\dim(X \cap Y) = k - h$.
(b) Let $h \in [k]$. Show that $\mathrm{GL}(V)$ is transitive on the set of ordered pairs of vertices $\{(X, Y) \mid d_\Gamma(X, Y) = h\}$.
(c) Describe the distribution diagram of Γ for $\mathbb{F} = \mathbb{F}_q$, the finite field of order q.

A graph with a group of automorphisms satisfying (b) is called distance transitive.

Section 2.7

Exercise 2.8.27 Let $\mathbb{F}$ be a field and $\mathbb{F}[[X]]$ be the ring of formal power series in X with coefficients in $\mathbb{F}$. Prove that $\mathbb{F}[[X]]$ is a discrete valuation ring with local parameter X and residue field isomorphic to $\mathbb{F}$. Compare the result with Example 2.7.2(ii) and conclude that the residue field does not uniquely determine the discrete valuation ring.

Exercise 2.8.28 Let V and W be right vector spaces over a division ring $\mathbb{D}$. A map $g : V \to W$ is called *semi-linear* if, for each $x \in V$, there is an automorphism σ_x of $\mathbb{D}$ such that

$$g(u\lambda + v\mu) = (gu)\sigma_u(\lambda) + (gv)\sigma_v(\mu) \quad (\lambda, \mu \in \mathbb{D}; v, w \in V).$$

(a) Prove that σ_v is the same for all $v \in V$ with $gv \neq 0$; it is called the automorphism of $\mathbb{D}$ *induced* by g. Such a map g is called σ*-linear*, where $\sigma = \sigma_v$ whenever $gv \neq 0$.
(b) Show that the set of all invertible semi-linear maps $V \to V$, denoted $\Gamma\mathrm{L}(V)$, is a subgroup of $\mathrm{Sym}(V)$ containing $\mathrm{GL}(V)$ as a normal subgroup.
(c) For $\lambda \in \mathbb{D} \setminus \{0\}$, the *homothety* with respect to λ on V is the map $h_\lambda : V \to V$ given by $h_\lambda(v) = v\lambda$. Prove that the set of homotheties is a subgroup H of $\Gamma\mathrm{L}(V)$, and that $H \subseteq \mathrm{GL}(V)$ if and only if $\mathbb{D}$ is a field.
(d) Verify that the homothety class of an R-lattice as in Definition 2.7.9 is the same as its H-orbit.

Exercise 2.8.29 Suppose that R is a discrete valuation ring with local parameter π. Let $\mathrm{ord} : R \to \mathbb{N}$ be the map such that $x \in \pi^{\mathrm{ord}(x)}R \setminus \pi^{\mathrm{ord}(x)+1}R$ for all $x \in R$. Prove that ord satisfies the following rules for $x, y \in R$.

(1) $\mathrm{ord}(xy) = \mathrm{ord}(x)\mathrm{ord}(y)$,
(2) $\mathrm{ord}(x + y) \geq \min(\mathrm{ord}(x), \mathrm{ord}(y))$,
(3) $\mathrm{ord}(x) = 0$ if and only if x is invertible.

Exercise 2.8.30 (Cited in proof of Theorem 2.7.14) Prove that $[L] = \{\pi^j L \mid j \in \mathbb{Z}\}$. Derive from this that the stabilizers in $\mathrm{SL}(\mathbb{F}^n)$ of $[L]$ and of L coincide.

Exercise 2.8.31 (This exercise is used in Lemma 2.7.13) Let R be a discrete valuation ring, $n \in \mathbb{N}$, $n > 1$, and take $i \in [n-1]$. Show that the map $M \mapsto M/\pi R^n$ is a bijective correspondence between the R-lattices M such that $\pi R^n \subseteq M \subseteq R^n$ and the $(n-i)$-dimensional subspaces of the vector space $(R/\pi R)^n$.

2.9 Notes

The concept of a diagram can be traced back to Schläfli in 1853 who exploited it in order to classify the regular convex polytopes. The discovery of the generalized polygons and of important classes of flag-transitive geometries over Coxeter diagrams is due to Tits; see [280, 281].

Section 2.1

The partially ordered set with the Jordan-Dedekind property, described in Example 2.1.11, are known as ranked posets in [267].

Section 2.2

The general concept of a diagram was coined by Buekenhout [44, 45] in order to encompass examples related to sporadic groups. For $(g, d_{\mathtt{p}}, d_{\mathtt{l}})$-gons, see [46].

A wealth of information on generalized polygons can be found in [236, 294] (the terminology differs slightly in that lines in their generalized polygons are not allowed to have exactly two points (and dually); such generalized polygons are called weak).

In Definition 2.2.7, generalized g-gons were given a special name only for $g \in \{2, 3, 4, 6, 8\}$. The reason is that these are the only values for which thick finite generalized g-gons exist; see [36] for an overview. Besides, these are also the only values for which thick generalized g-gons exist that are Moufang. In [291] all generalized Moufang polygons are classified.

Good introductory texts to projective planes are [167, 240]. The usual definition of a projective plane in these books does not allow for lines to be thin.

Free constructions show that thick infinite generalized g-gons for all values of g occur; see for instance [275].

The construction of the generalized hexagon of order (2, 2) in Example 2.2.15 is from [281]; see also [71]. In [83] it is proved that it is not isomorphic to its dual and that, up to isomorphism and duality, it is the unique one of its order (a fact already announced in [281]). A characterization of the known generalized octagons is given in [295].

The Hoffman-Singleton graph HoSi first appeared with uniqueness proof in [163]. More papers with details on HoSi are [17] and [92]. In the proof of Theorem 2.2.19, we follow [174]. The graph is a Moore graph (referring to a different person from his namesake mentioned in Sect. 1.10), whose definition and theory is summarized in [36].

Section 2.3

Proofs of most of the statements regarding affine planes can be found in [167]. The idea of a ternary ring goes back to M. Hall [146]. As a consequence of the strict correspondence, various geometric properties are reflected by properties of the ternary ring. The ternary ring is a division ring if and only if the corresponding projective plane is Desarguesian. The ternary ring is an alternative division ring if and only if the projective plane is Moufang, etc. Good introductions to alternative rings are [246, 265].

All locally Petersen graphs, of which only one has been mentioned in Example 2.3.10, have been determined in [148]. The notion of graphs being locally Δ was brought forward in 1980 [24] but appeared as early as 1963 in a question by Zykov. Several studies have been conducted since for special classes of graphs; see [25, 52, 81, 85, 121, 140, 150] for results related to geometry.

If the girth of the local graph is larger than five, free examples exist, whereas full characterizations seem often feasible in cases of smaller girth. See [300, 301], which show that the case of a local hexagon, the tiling of the plane by honeycombs, and the locally Petersen graphs, characterized in [148], are at the division line. Group-theoretical results in the same vein can be found in [176, Theorem D]. Some characterizations using extra hypotheses approach the verge of what is possible; examples are found in [78, 140].

Section 2.4

The Neumaier geometry of Example 2.4.11 can be found in [230]; for characterizations as a flag-transitive geometry with diagram B_3, see [6, 315]. The geometries in Example 2.4.12 are from [309] and [230], respectively; see also [214].

Heiss [158] established their uniqueness assuming that all residues with diagram are Neumaier geometries.

The graph Δ' of Remark 2.4.13 was constructed in [160] on the basis of a known Steiner system, which is described in Exercise 5.7.34. The sporadic simple group HS appears in Table 5.2.

More details on most of the sporadic geometries constructed in this chapter, as well as additional references, can be found in [36].

Section 2.5

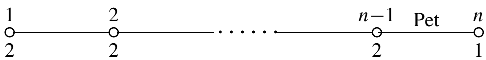
The concepts of linear space and linear subspace have been used in many places. Around 1960 they received explicit recognition, thanks to the efforts of Libois [206]. Theorem 2.5.15 is a slight generalization of Tits' characterization of projective planes as generalized 3-gons [281].

In [286], the original definition of a shadow space can be found.

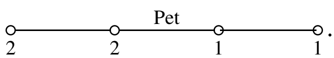
Section 2.6

The special case of linear diagrams for group geometries, as appearing in Theorem 2.6.4 is treated in [217].

The coset enumeration mentioned in Example 2.6.5 is a systematic method of producing the representation of a group G on a subgroup H when G is given by finite sets of generators and relations, H by a finite set of generators expressed as words in the generators of G, provided G/H is finite. See [84] for details. Example 2.6.5 is taken from [215]. It is only one example of a vast number of flag-transitive geometries in which the Petersen graph occurs as a rank two residue; see for example [171, 173], where the geometries are not only constructed but also classified as those with a given diagram and a flag-transitive automorphism group. For instance, there are exactly eight flag-transitive Pet_n-geometries; these are geometries of type

$\mathrm{Pet}_n =$ 

where n is the rank. As in Example 2.3.9, the label Pet indicates a rank two geometry isomorphic to the geometry of vertices and edges of the Petersen graph treated in Example 2.2.8. The automorphism group of the unique example up to isomorphism for $n = 2$ is isomorphic to Sym_5. For $n = 3$, they correspond to the Mathieu group M_{22}, and its central 3-cover $3 \cdot \mathrm{M}_{22}$, for $n = 4$ to the Mathieu group M_{23}, the second Conway group Co_2, an extension of this group $3^{23}\mathrm{Co}_2$, and the fourth Janko group J4 and for $n = 5$ to the Baby Monster group B and an extension of type $3^{4371}\mathrm{B}$. Also, the McLaughlin group McL acts flag transitively on a geometry of type



This accounts for six sporadic groups.

Section 2.7

The geometries of type $\widetilde{\mathrm{A}}_{n-1}$ are the easiest nontrivial examples of buildings of affine type. Their analysis and classification (for rank at least four) is a major piece of work by Bruhat and Tits [39]. An excellent treatment of this material, together with new details on the classification, is given by Weiss [306]. Some examples of finite geometries of type $\widetilde{\mathrm{A}}_2$ are given in [196, 242]. For other affine Coxeter diagrams, some finite examples of this kind are known as well; see for instance [183] for type $\widetilde{\mathrm{D}}_4$. A survey of such constructions is given in [184], and a classification of finite flag-transitive quotient geometries of affine buildings, obtained by means of a discrete flag-transitive subgroup of the full automorphism group, is discussed in [185].

These geometries have 2-coverings of the above type, that is, which are buildings. Some examples of finite geometries of rank three having affine type $\widetilde{\mathrm{C}}_2$ that are not covered by buildings are given in [182, 186].

Section 2.8

Part (a) of the definition of a generalized polygon in Exercise 2.8.5 refers to Tits' first definition of the notion in [281]. Part (b) was added later to avoid anomalies.

Exercise 2.8.8 was suggested to us by Ernest Shult.

The notion of standard equivalence in Exercise 2.8.24 is a slight variation of the one introduced in [107]. We owe the counterexample at the end of this exercise to Pasini.

The rank two geometry Til of Exercise 2.8.25 is discussed under the name Foster graph in [36, Sect. 13.2A]. See also [171, Sect. 6.2] and [234]. The automorphism group of Til is isomorphic to $3 \cdot \mathrm{Sym}_6.2$. The group G of Exercise 2.8.25 is isomorphic to $\mathrm{C}_2 \times (3 \cdot \mathrm{Alt}_6)$ and is identified in [69]. There are several interesting flag-transitive Til_n-geometries (see [171, 173]); these are geometries of type

$\mathrm{Til}_n =$ 

where n is the rank. Among the groups admitting flag-transitive Til_n-geometries there are an infinite series with group $3^{s(n)}\mathrm{Sp}(2n,2)$, where $s(n) = (2^n - 1) \times (2^n - 2)/6$; furthermore, for rank $n = 3$, the Mathieu group M_{24} and the Held group He, for rank 4, the first Conway group Co_1, and for rank 5, the biggest sporadic simple group, called the Monster. Together with those from the diagram related to the Petersen graphs, this accounts for ten sporadic groups.

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