Chapter 2
Basic Concepts About Manifolds and Fibre Bundles

Mathematics, the Queen of Sciences...
Carl Friedrich Gauss

2.1 Introduction

General Relativity is founded on the concept of *differentiable manifolds*. The mathematical model of *space-time* that we adopt is given by a pair $(\mathscr{M}, g)$ where $\mathscr{M}$ is a differentiable manifold of dimension $D = 4$ and g is a *metric*, that is a rule to calculate the length of curves connecting points of $\mathscr{M}$. In physical terms the points of $\mathscr{M}$ take the name of *events* while every physical process is a continuous succession of events. In particular the motion of a *point-like particle* is represented by a *world-line*, namely a curve in $\mathscr{M}$ while the motion of an *extended object* of dimension p is given by a $d = p + 1$ dimensional *world-volume* obtained as a continuous succession of p-dimensional hypersurfaces $\Sigma_p \subset \mathscr{M}$.

Therefore, the discussion of such *physical concepts* is necessarily based on a collection of *geometrical concepts* that constitute the backbone of differential geometry. The latter is at the basis not only of General Relativity but of all Gauge Theories by means of which XX century Physics obtained a consistent and experimentally verified description of all Fundamental Interactions.

The central notions are those which fix the geometric environment:

- Differentiable Manifolds
- Fibre-Bundles

and those which endow such environment with structures accounting for the measure of lengths and for the rules of parallel transport, namely:

- Metrics
- Connections

Once the geometric environments are properly mathematically defined, the metrics and connections one can introduce over them turn out to be the structures which encode the Fundamental Forces of Nature.

The present chapter introduces Differentiable Manifolds and Fibre-Bundles while the next one is devoted to a thorough discussion of Metrics and Connections.

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2.2 Differentiable Manifolds

First and most fundamental in the list of geometrical concepts we need to introduce is that of a *manifold* which corresponds, as we already explained, to our intuitive idea of a *continuous space*. In mathematical terms this is, to begin with, a *topological space*, namely a set of elements where one can define the notion of *neighborhood* and *limit*. This is the correct mathematical description of our intuitive ideas of vicinity and close-by points. Secondly the characterizing feature that distinguishes a manifold from a simple topological space is the possibility of labeling its points with a set of coordinates. Coordinates are a set of real numbers $x_1(p), \dots, x_D(p) \in \mathbb{R}$ associated with each point $p \in \mathscr{M}$ that tell us *where* we are. Actually in General Relativity each point is an event so that coordinates specify not only its *where* but also its *when*. In other applications the coordinates of a point can be the most disparate parameters specifying the state of some complex system of the most general kind (dynamical, biological, economical or whatever).

In classical physics the laws of motion are formulated as a set of differential equations of the second order where the unknown functions are the three Cartesian coordinates x, y, z of a particle and the variable t is time. Solving the dynamical problem amounts to determine the continuous functions $x(t)$, $y(t)$, $z(t)$, that yield a parametric description of a curve in $\mathbb{R}^3$ or better define a curve in $\mathbb{R}^4$, having included the time t in the list of coordinates of each event. Coordinates, however, are not uniquely defined. Each observer has its own way of labeling space points and the laws of motion take a different form if expressed in the coordinate frame of different observers. There is however a privileged class of observers in whose frames the laws of motion have always the same form: these are the inertial frames, that are in rectilinear relative motion with constant velocity. The existence of a privileged class of inertial frames is common to classical Newtonian physics and to Special Relativity: the only difference is the form of coordinate transformations connecting them, Galileo transformations in the first case and Lorentz transformations in the second. This goes hand in hand with the fact that the space-time manifold is the *flat affine*[1] *manifold* $\mathbb{R}^4$ in both cases. By definition all points of $\mathbb{R}^N$ can be covered by one coordinate frame $\{x^i\}$ and all frames with such a property are related to each other by general linear transformations, that is by the elements of the general linear group $\mathrm{GL}(N, \mathbb{R})$:

$$x^{i'} = A^i{}_j \, x^j; \quad A^i{}_j \in \mathrm{GL}(N, \mathbb{R}) \tag{2.2.1}$$

The restriction to the Galilei or Lorentz subgroups of $\mathrm{GL}(4, \mathbb{R})$ is a consequence of the different *scalar product* on $\mathbb{R}^4$ vectors one wants to preserve in the two cases, but the relevant common feature is the fact that the space-time manifold has a vector-space structure. The privileged coordinate frames are those that use the corresponding vectors as labels of each point.

A different situation arises when the space-time manifold is not flat, like, for instance, the surface of a hypersphere $\mathbb{S}^N$. As chartographers know very well there

[1] A manifold (defined in this section) is named *affine* when it is also a vector space.

is no way of representing all points of a curved surface in a single coordinate frame, namely in a single *chart*. However we can succeed in representing all points of a curved surface by means of an *atlas*, namely by a collection of charts, each of which maps one open region of the surface and such that the union of all these regions covers the entire surface. Knowing the transition rule from one chart to the next one, in the regions where they overlap, we obtain a complete coordinate description of the curved surface by means of our atlas.

The intuitive idea of an *atlas* of *open charts*, suitably reformulated in mathematical terms, provides the very definition of a differentiable manifold, the geometrical concept that generalizes our notion of space-time, from $\mathbb{R}^N$ to more complicated non-flat situations.

There are many possible *atlases* that describe the same manifold $\mathscr{M}$, related to each other by more or less complicated transformations. For a generic $\mathscr{M}$ no privileged choice of the atlas is available differently from the case of $\mathbb{R}^N$: here the inertial frames are singled out by the additional *vector space* structure of the manifold, which allows to label each point with the corresponding vector. Therefore if the laws of physics have to be universal and have to accommodate non-flat space-times, then they must be formulated in such a way that they have the same form in whatsoever atlas. This is the principle of *general covariance* at the basis of General Relativity: all observers see the same laws of physics.

Similarly, in a wider perspective, the choice of a particular set of parameters to describe the state of a complex system should not be privileged with respect to any other choice. The laws that govern the dynamics of a system should be intrinsic and should not depend on the set of variables chosen to describe it.

2.2.1 Homeomorphisms and the Definition of Manifolds

A fundamental ingredient in formulating the notion of differential manifolds is that of homeomorphism.[2]

Definition 2.2.1 Let X and Y be two topological spaces and let h be a map:

$$h : X \to Y \tag{2.2.2}$$

If h is one-to-one and if both h and its inverse h^{-1} are continuous, then we say that h is a *homeomorphism*.

As a consequence of the theorems proved in all textbooks about elementary topology and calculus, homeomorphisms preserve all topological properties. Indeed let h be a homeomorphism mapping X onto Y and let $A \subset X$ be an open subset: its

[2] We assume that the reader possesses the basic notions of general topology concerning the notions of bases of neighborhoods, open and close subsets, boundary and limit.

image through h, namely $h(A) \subset Y$ is also an open subset in the topology of Y. Similarly the image $h(C) \subset Y$ of a closed subset $C \subset X$ is a closed subset. Furthermore for all $A \subset X$ we have:

$$h(\overline{A}) = \overline{h(A)} \tag{2.2.3}$$

namely the closure of the image of a set A coincides with the image of the closure.

Definition 2.2.2 Let X and Y be two topological spaces. If there exists a homeomorphism $h : X \to Y$ then we say that X and Y are homeomorphic.

It is easy to see that given a topological space X, the set of all homeomorphisms $h : X \to X$ constitutes a group, usually denoted $\mathrm{Hom}(X)$. Indeed if $h \in \mathrm{Hom}(X)$ is a homeomorphism, then also $h^{-1} \in \mathrm{Hom}(X)$ is a homeomorphism. Furthermore if $h \in \mathrm{Hom}(X)$ and $h' \in \mathrm{Hom}(X)$ then also $h \circ h' \in \mathrm{Hom}(X)$. Finally the identity map:

$$\mathbf{1} : X \to X \tag{2.2.4}$$

is certainly one-to-one and continuous and it coincides with its own inverse. Hence $\mathbf{1} \in \mathrm{Hom}(X)$. As we discuss later on, for any manifold X the group $\mathrm{Hom}(X)$ is an example of an infinite and continuous group.

Let now $\mathscr{M}$ be a topological Hausdorff space. An *open chart* of $\mathscr{M}$ is a pair (U, φ) where $U \subset \mathscr{M}$ is an open subset of $\mathscr{M}$ and φ is a homeomorphism of U on an open subset $\mathbb{R}^m$ (m being a positive integer). The concept of open chart allows to introduce the notion of coordinates for all points $p \in U$. Indeed the coordinates of p are the m real numbers that identify the point $\varphi(p) \in \varphi(U) \subset \mathbb{R}^m$.

Using the notion of open chart we can finally introduce the notion of differentiable structure.

Definition 2.2.3 Let $\mathscr{M}$ be a topological Hausdorff space. A differentiable structure of dimension m on $\mathscr{M}$ is an *atlas* $\mathscr{A} = \bigcup_{i \in A} (U_i, \varphi_i)$ of open charts (U_i, φ_i) where $\forall i \in A$, $U_i \subset \mathscr{M}$ is an open subset and

$$\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{R}^m \tag{2.2.5}$$

is a homeomorphism of U_i in $\mathbb{R}^m$, namely a continuous, invertible map onto an open subset of $\mathbb{R}^m$ such that the inverse map

$$\varphi_i^{-1} : \varphi_i(U_i) \to U_i \subset \mathscr{M} \tag{2.2.6}$$

is also continuous (see Fig. 2.1). The atlas must fulfill the following axioms:

M_1 It covers $\mathscr{M}$, namely

$$\bigcup_i U_i = \mathscr{M} \tag{2.2.7}$$

so that each point of $\mathscr{M}$ is contained at least in one chart and generically in more than one: $\forall p \in \mathscr{M} \mapsto \exists (U_i, \varphi_i) / p \in U_i$.

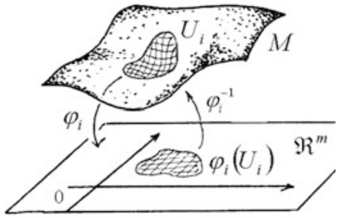


Fig. 2.1 An open chart is a homeomorphism of an open subset U_i of the manifold $\mathscr{M}$ onto an open subset of $\mathbb{R}^m$

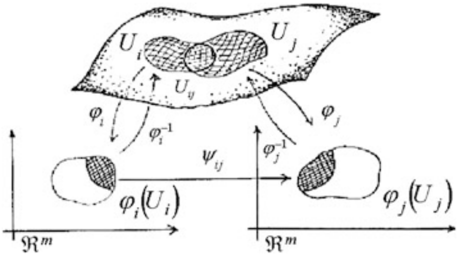


Fig. 2.2 A transition function between two open charts is a differentiable map from an open subset of $\mathbb{R}^m$ to another open subset of the same

M_2 Chosen any two charts (U_i, φ_i), (U_j, φ_j) such that $U_i \bigcap U_j \neq \emptyset$, on the intersection

$$U_{ij} \stackrel{\text{def}}{=} U_i \bigcap U_j \tag{2.2.8}$$

there exist two homeomorphisms:

$$\begin{aligned} \varphi_i|_{U_{ij}} &: U_{ij} \to \varphi_i(U_{ij}) \subset \mathbb{R}^m \\ \varphi_j|_{U_{ij}} &: U_{ij} \to \varphi_j(U_{ij}) \subset \mathbb{R}^m \end{aligned} \tag{2.2.9}$$

and the composite map:

$$\begin{aligned} \psi_{ij} &\stackrel{\text{def}}{=} \varphi_j \circ \varphi_i^{-1} \\ \psi_{ij} &: \ \varphi_i(U_{ij}) \subset \mathbb{R}^m \to \varphi_j(U_{ij}) \subset \mathbb{R}^m \end{aligned} \tag{2.2.10}$$

named the *transition function* which is actually an m-tuplet of m real functions of m real variables is requested to be *differentiable* (see Fig. 2.2).

M_3 The collection $(U_i, \varphi_i)_{i \in A}$ is the maximal family of open charts for which both M_1 and M_2 hold true.

Next we can finally introduce the definition of differentiable manifold.

Definition 2.2.4 A differentiable manifold of dimension m is a topological space $\mathscr{M}$ that admits at least one differentiable structure $(U_i, \varphi_i)_{i \in A}$ of dimension m.

The definition of a differentiable manifold is constructive in the sense that it provides a way to construct it explicitly. What one has to do is to give an atlas of open charts (U_i, φ_i) and the corresponding transition functions ψ_{ij} which should satisfy the necessary consistency conditions:

$$\forall i, j \qquad \psi_{ij} = \psi_{ji}^{-1} \tag{2.2.11}$$

$$\forall i, j, k \qquad \psi_{ij} \circ \psi_{jk} \circ \psi_{ki} = \mathbf{1} \tag{2.2.12}$$

In other words a general recipe to construct a manifold is to specify the open charts and how they are *glued* together. The properties assigned to a manifold are the properties fulfilled by its transition functions. In particular we have:

Definition 2.2.5 A differentiable manifold $\mathscr{M}$ is said to be *smooth* if the transition functions (2.2.10) are *infinitely differentiable*

$$\mathscr{M} \text{ is smooth} \quad \Leftrightarrow \quad \psi_{ij} \in \mathbb{C}^{\infty}\big(\mathbb{R}^m\big) \tag{2.2.13}$$

Similarly one has the definition of a complex manifold.

Definition 2.2.6 A real manifold of even dimension $m = 2\nu$ is *complex* of dimension ν if the 2ν real coordinates in each open chart U_i can be arranged into ν complex numbers so that (2.2.5) can be replaced by

$$\varphi_i : U_i \to \varphi_i(U_i) \subset \mathbb{C}^{\nu} \tag{2.2.14}$$

and the transition functions ψ_{ij} are *holomorphic maps*:

$$\psi_{ij} : \varphi_i(U_{ij}) \subset \mathbb{C}^{\nu} \to \varphi_j(U_{ij}) \subset \mathbb{C}^{\nu} \tag{2.2.15}$$

Although the constructive definition of a differentiable manifold is always in terms of an atlas, in many occurrences we can have other intrinsic global definitions of what $\mathscr{M}$ is and the construction of an atlas of coordinate patches is an a posteriori operation. Typically this happens when the manifold admits a description as an algebraic locus. The prototype example is provided by the $\mathbb{S}^N$ sphere which can be defined as the locus in $\mathbb{R}^{N+1}$ of points with distance r from the origin:

$$\{X_i\} \in \mathbb{S}^N \quad \Leftrightarrow \quad \sum_{i=1}^{N+1} X_i^2 = r^2 \tag{2.2.16}$$

In particular for $N = 2$ we have the familiar $\mathbb{S}^2$ which is diffeomorphic to the compactified complex plane $\mathbb{C} \bigcup \{\infty\}$. Indeed we can easily verify that $\mathbb{S}^2$ is a one-dimensional complex manifold considering the atlas of holomorphic open charts suggested by the geometrical construction named *the stereographic projection*. To this effect consider the picture in Fig. 2.3 where we have drawn the two-sphere $\mathbb{S}^2$ of radius $r = 1$ centered in the origin of $\mathbb{R}^3$. Given a generic point $P \in \mathbb{S}^2$ we can construct its image on the equatorial plane $\mathbb{R}^2 \sim \mathbb{C}$ drawing the straight line in $\mathbb{R}^3$ that goes through P and through the *North Pole* of the sphere N. Such a line will intersect the equatorial plane in the point P_N whose value z_N, regarded as a complex

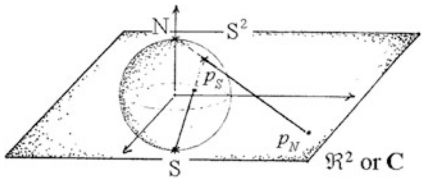


Fig. 2.3 Stereographic projection of the two sphere

number, we can identify with the complex coordinate of P in the open chart under consideration:

$$\varphi_N(P) = z_N \in \mathbb{C} \tag{2.2.17}$$

Alternatively we can draw the straight line through P and the *South Pole* S. This intersects the equatorial plane in another point P_S whose value as a complex number, named z_S, is just the reciprocal of z_N: $z_S = 1/z_N$. We can take z_S as the complex coordinate of the same point P. In other words we have another open chart:

$$\varphi_S(P) = z_S \in \mathbb{C} \tag{2.2.18}$$

What is the domain of these two charts, namely what are the open subsets U_N and U_S? This is rather easily established considering that the North Pole projection yields a finite result $z_N < \infty$ for all points P except the North Pole itself. Hence $U_N \subset \mathbb{S}^2$ is the open set obtained by subtracting one point (the North Pole) to the sphere. Similarly the South Pole projection yields a finite result for all points P except the South Pole itself and U_S is $\mathbb{S}^2$ minus the south pole. More definitely we can choose for U_N and U_S any two open neighborhoods of the South and North Pole respectively with non-vanishing intersection (see Fig. 2.4). In this case the intersection $U_N \bigcap U_S$ is a band wrapped around the equator of the sphere and its image in the complex equatorial plane is a circular corona that excludes both a circular neighborhood of the origin and a circular neighborhood of infinity. On such an intersection we have the transition function:

$$\psi_{NS} : z_N = \frac{1}{z_S} \tag{2.2.19}$$

which is clearly holomorphic and satisfies the consistency conditions in (2.2.11), (2.2.12). Hence we see that $\mathbb{S}^2$ is a complex 1-manifold that can be constructed with an atlas composed of two open charts related by the transition function (2.2.19). Obviously a complex 1-manifold is *a fortiori* a *smooth real* 2-*manifold*. Manifolds with infinitely differentiable transition functions are named smooth not without a reason. Indeed they correspond to our intuitive notion of smooth hypersurfaces without conical points or edges. The presence of such defects manifests itself through the lack of differentiability in some regions.

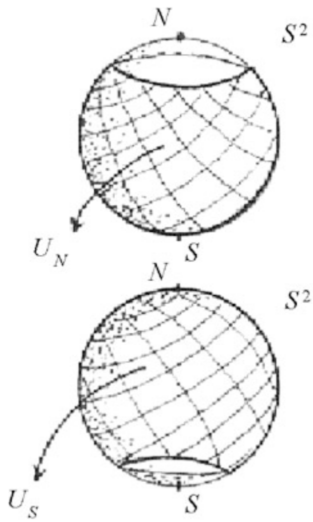


Fig. 2.4 The open charts of the North and South Pole

2.2.2 *Functions on Manifolds*

Being the mathematical model of possible space-times, manifolds are the geometrical support of physics. They are the arenas where physical processes take place and where physical quantities take values. Mathematically, this implies that calculus, originally introduced on $\mathbb{R}^N$ must be extended to manifolds. The physical entities defined over manifolds with which we have to deal are mathematically characterized as *scalar functions, vector fields, tensor fields, differential forms, sections of more general fibre-bundles*. We introduce such basic geometrical notions slowly, beginning with the simplest concept of a *scalar function*.

Definition 2.2.7 A real scalar function on a differentiable manifold $\mathscr{M}$ is a map:

$$f : \mathscr{M} \to \mathbb{R} \tag{2.2.20}$$

that assigns a real number $f(p)$ to every point $p \in \mathscr{M}$ of the manifold.

The properties of a scalar function, for instance its differentiability, are the properties characterizing its local description in the various open charts of an atlas. For each open chart (U_i, φ_i) let us define:

$$f_i \stackrel{\text{def}}{=} f \circ \varphi_i^{-1} \tag{2.2.21}$$

By construction

$$f_i : \mathbb{R}^m \supset \varphi_i(U_i) \to \mathbb{R} \tag{2.2.22}$$

is a map of an open subset of $\mathbb{R}^m$ into the real line $\mathbb{R}$, namely a real function of m real variables (see Fig. 2.5). The collection of the real functions $f_i(x_1^{(i)}, \ldots, x_m^{(i)})$

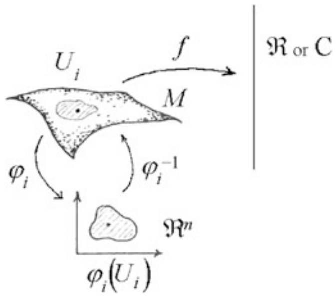


Fig. 2.5 Local description of a scalar function on a manifold

constitute the local description of the scalar function f. The function is said to be *continuous, differentiable, infinitely differentiable* if the real functions f_i have such properties. From Definition (2.2.21) of the local description and from Definition (2.2.10) of the transition functions it follows that we must have:

$$\forall U_i, U_j : \quad f_j|_{U_i \bigcap U_j} = f_i|_{U_i \bigcap U_j} \circ \psi_{ij} \tag{2.2.23}$$

Let $x^{\mu}_{(i)}$ be the coordinates in the patch U_i and $x^{\mu}_{(j)}$ be the coordinates in the patch U_j. For points p that belong to the intersection $U_i \bigcap U_j$ we have:

$$x^{\mu}_{(j)}(p) = \psi^{(ji)}_{\mu}\left(x^1_{(j)}(p), \dots x^m_{(j)}(p)\right) \tag{2.2.24}$$

and the gluing rule (2.2.23) takes the form:

$$f(p) = f_j(x_{(j)}) = f_j\left(\psi_{ji}(x_{(i)})\right) = f_i(x_{(i)}) \tag{2.2.25}$$

The practical way of assigning a function on a manifold is therefore that of writing its local description in the open charts of an atlas, taking care that the various f_i glue together correctly, namely through (2.2.23). Although the number of continuous and differentiable functions one can write on any open region of $\mathbb{R}^m$ is infinite, the smooth functions globally defined on a non-trivial manifold can be very few. Indeed it is only occasionally that we can consistently glue together various local functions $f_i \in \mathbb{C}^\infty(U_i)$ into a global f. When this happens we say that $f \in \mathbb{C}^\infty(\mathscr{M})$.

All what we said about real functions can be trivially repeated for complex functions. It suffices to replace $\mathbb{R}$ by $\mathbb{C}$ in (2.2.20).

2.2.3 Germs of Smooth Functions

The local geometry of a manifold is studied by considering operations not on the space of smooth functions $\mathbb{C}^\infty(\mathscr{M})$ which, as just explained, can be very small, but on the space of germs of functions defined at each point $p \in \mathscr{M}$ that is always an infinite dimensional space.

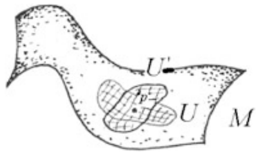


Fig. 2.6 A germ of a smooth function is the equivalence class of all locally defined function that coincide in some neighborhood of a point p

Definition 2.2.8 Given a point $p \in \mathscr{M}$, the space of germs of smooth functions at p, denoted $\mathbb{C}_p^\infty$ is defined as follows. Consider all the open neighborhoods of p, namely all the open subsets $U_p \subset \mathscr{M}$ such that $p \in U_p$. Consider the space of smooth functions $\mathbb{C}^\infty(U_p)$ on each U_p. Two functions $f \in \mathbb{C}^\infty(U_p)$ and $g \in \mathbb{C}^\infty(U'_p)$ are said to be equivalent if they coincide on the intersection $U_p \bigcap U'_p$ (see Fig. 2.6):

$$f \sim g \quad \Leftrightarrow \quad f|_{U_p \bigcap U'_p} = g|_{U_p \bigcap U'_p} \tag{2.2.26}$$

The union of all the spaces $\mathbb{C}^\infty(U_p)$ modded by the equivalence relation (2.2.26) is the space of germs of smooth functions at p:

$$\mathbb{C}_p^\infty \equiv \frac{\bigcup_{U_p} \mathbb{C}^\infty(U_p)}{\sim} \tag{2.2.27}$$

What underlies the above definition of germs is the familiar principle of analytic continuation. Of the same function we can have different definitions that have different domains of validity: apparently we have different functions but if they coincide on some open region than we consider them just as different representations of a single function. Given any germ in some open neighborhood U_p we try to extend it to a larger domain by suitably changing its representation. In general there is a limit to such extension and only very special germs extend to globally defined functions on the whole manifold $\mathscr{M}$. For instance the power series $\sum_{k\in\mathbb{N}} z^k$ defines a holomorphic function within its radius of convergence $|z| < 1$. As everybody knows, within the convergence radius the sum of this series coincides with $1/(1-z)$ which is a holomorphic function defined on a much larger neighborhood of $z = 0$. According to our definition the two functions are equivalent and correspond to two different representatives of the same *germ*. The germ, however, does not extend to a holomorphic function on the whole Riemann sphere $\mathbb{C} \bigcup \infty$ since it has a singularity in $z = 1$. Indeed, as stated by Liouville theorem, the space of global holomorphic functions on the Riemann sphere contains only the constant function.

2.3 Tangent and Cotangent Spaces

In elementary geometry the notion of a *tangent line* is associated with the notion of a curve. Hence to introduce tangent vectors we have to begin with the notion of *curves in a manifold*.

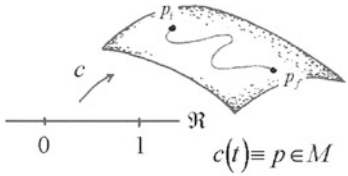


Fig. 2.7 A curve in a manifold is a continuous map of an interval of the real line into the manifold itself

Definition 2.3.1 A curve $\mathscr{C}$ in a manifold $\mathscr{M}$ is a continuous and differentiable map of an interval of the real line (say $[0, 1] \subset \mathbb{R}$) into $\mathscr{M}$:

$$\mathscr{C} : [0, 1] \to \mathscr{M} \tag{2.3.1}$$

In other words a curve is one-dimensional submanifold $\mathscr{C} \subset \mathscr{M}$ (see Fig. 2.7).

There are curves with a *boundary*, namely $\mathscr{C}(0) \bigcup \mathscr{C}(1)$ and open curves that do not contain their boundary. This happens if in (2.3.1) we replace the closed interval $[0, 1]$ with the open interval $]0, 1[$. *Closed curves* or *loops* correspond to the case where the initial and final point coincide, that is when $p_i \equiv \mathscr{C}(0) = \mathscr{C}(1) \equiv p_f$. Differently said

Definition 2.3.2 A closed curved is a continuous differentiable map of a circle into the manifold:

$$\mathscr{C} : \mathbb{S}^1 \to \mathscr{M} \tag{2.3.2}$$

Indeed, identifying the initial and final point means to consider the points of the curve as being in one-to-one correspondence with the equivalence classes

$$\mathbb{R}/\mathbb{Z} \equiv \mathbb{S}^1 \tag{2.3.3}$$

which constitute the mathematical definition of the circle. Explicitly (2.3.3) means that two real numbers r and r' are declared to be equivalent if their difference $r' - r = n$ is an integer number $n \in \mathbb{Z}$. As representatives of these equivalence classes we have the real numbers contained in the interval $[0, 1]$ with the proviso that $0 \sim 1$.

We can also consider *semiopen curves* corresponding to maps of the semiopen interval $[0, 1[$ into $\mathscr{M}$. In particular, in order to define tangent vectors we are interested in open branches of curves defined in the neighborhood of a point.

2.3.1 Tangent Vectors in a Point $p \in \mathscr{M}$

For each point $p \in \mathscr{M}$ let us fix an open neighborhood $U_p \subset \mathscr{M}$ and let us consider the semiopen curves of the following type:

$$\begin{cases} \mathscr{C}_p : [0, 1[\to U_p \\ \mathscr{C}_p(0) = p \end{cases} \tag{2.3.4}$$

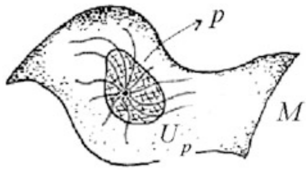


Fig. 2.8 In a neighborhood U_p of each point $p \in \mathscr{M}$ we consider the curves that go through p

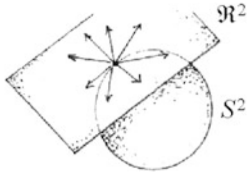


Fig. 2.9 The tangent space in a generic point of an $\mathbb{S}^2$ sphere

In other words for each point p let us consider all possible curves $\mathscr{C}_p(t)$ that go trough p (see Fig. 2.8).

Intuitively the tangent in p to a curve that starts from p is the vector that specifies the curve's *initial* direction. The basic idea is that in an m-dimensional manifold there are as many directions in which the curve can depart as there are vectors in $\mathbb{R}^m$: furthermore for sufficiently small neighborhoods of p we cannot tell the difference between the manifold $\mathscr{M}$ and the flat vector space $\mathbb{R}^m$. Hence to each point $p \in \mathscr{M}$ of a manifold we can attach an m-dimensional real vector space

$$\forall p \in \mathscr{M}: \quad p \mapsto T_p\mathscr{M} \quad \dim T_p\mathscr{M} = m \tag{2.3.5}$$

which parameterizes the possible directions in which a curve starting at p can depart. This vector space is named the tangent space to $\mathscr{M}$ at the point p and is, by definition, isomorphic to $\mathbb{R}^m$, namely $T_p\mathscr{M} \sim \mathbb{R}^m$. For instance to each point of an $\mathbb{S}^2$ sphere we attach a tangent plane $\mathbb{R}^2$ (see Fig. 2.9).

Let us now make this intuitive notion mathematically precise. Consider a point $p \in \mathscr{M}$ and a germ of smooth function $f_p \in C_p^\infty(\mathscr{M})$. In any open chart $(U_\alpha, \varphi_\alpha)$ that contains the point p, the germ f_p is represented by an infinitely differentiable function of m-variables:

$$f_p\big(x^1_{(\alpha)}, \ldots, x^m_{(\alpha)}\big) \tag{2.3.6}$$

Let us now choose an open curve $\mathscr{C}_p(t)$ that lies in U_α and starts at p:

$$\mathscr{C}_p(t): \begin{cases} \mathscr{C}_p : [0,1[\to U_\alpha \\ \mathscr{C}_p(0) = p \end{cases} \tag{2.3.7}$$

and consider the composite map:

$$f_p \circ \mathscr{C}_p : \quad [0,1[\subset \mathbb{R} \to \mathbb{R} \tag{2.3.8}$$

which is a real function

$$f_p\big(\mathscr{C}_p(t)\big) \equiv g_p(t) \tag{2.3.9}$$

of one real variable (see Fig. 2.10).

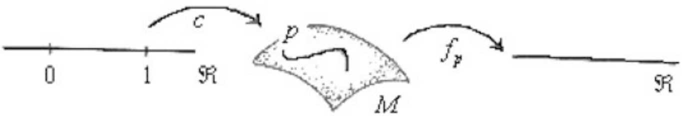


Fig. 2.10 The composite map $f_p \circ \mathscr{C}_p$ where f_p is a germ of smooth function in p and $\mathscr{C}_p$ is a curve departing from $p \in \mathscr{M}$

We can calculate its derivative with respect to t in $t = 0$ which, in the open chart $(U_\alpha, \varphi_\alpha)$, reads as follows:

$$\frac{d}{dt} g_p(t)\bigg|_{t=0} = \frac{\partial f_p}{\partial x^\mu} \cdot \frac{dx^\mu}{dt}\bigg|_{t=0} \tag{2.3.10}$$

We see from the above formula that the increment of any germ $f_p \in \mathbb{C}_p^\infty(\mathscr{M})$ along a curve $\mathscr{C}_p(t)$ is defined by means of the following m real coefficients:

$$c^\mu \equiv \frac{dx^\mu}{dt}\bigg|_{t=0} \in \mathbb{R} \tag{2.3.11}$$

which can be calculated whenever the parametric form of the curve is given: $x^\mu = x^\mu(t)$. Explicitly we have:

$$\frac{df_p}{dt} = c^\mu \frac{\partial f_p}{\partial x^\mu} \tag{2.3.12}$$

Equation (2.3.12) can be interpreted as the action of a differential operator on the space of germs of smooth functions, namely:

$$\mathbf{t}_p \equiv c^\mu \frac{\partial}{\partial x^\mu} \quad \Rightarrow \quad \mathbf{t}_p : \mathbb{C}_p^\infty(\mathscr{M}) \mapsto \mathbb{C}_p^\infty(\mathscr{M}) \tag{2.3.13}$$

Indeed for any germ f and for any curve

$$\mathbf{t}_p f = \frac{dx^\mu}{dt}\bigg|_{t=0} \frac{\partial f}{\partial x^\mu} \in \mathbb{C}_p^\infty(\mathscr{M}) \tag{2.3.14}$$

is a new germ of a smooth function in the point p. This discussion justifies the mathematical definition of the tangent space:

Definition 2.3.3 The tangent space $T_p\mathscr{M}$ to the manifold $\mathscr{M}$ in the point p is the vector space of *first order differential operators* on the germs of smooth functions $\mathbb{C}_p^\infty(\mathscr{M})$.

Next let us observe that the space of germs $\mathbb{C}_p^\infty(\mathscr{M})$ is an *algebra* with respect to linear combinations with real coefficients $(\alpha f + \beta g)(p) = \alpha f(p) + \beta g(p)$ and pointwise multiplication $f \cdot g(p) \equiv f(p)g(p)$:

$$\begin{aligned} \forall \alpha, \beta \in \mathbb{R} \quad \forall f, g \in \mathbb{C}^\infty_p(\mathscr{M}) \qquad & \alpha f + \beta g \in \mathbb{C}^\infty_p(\mathscr{M}) \\ \forall f, g \in \mathbb{C}^\infty_p(\mathscr{M}) \qquad & f \cdot g \in \mathbb{C}^\infty_p(\mathscr{M}) \\ & (\alpha f + \beta g) \cdot h = \alpha f \cdot h + \beta g \cdot h \end{aligned} \tag{2.3.15}$$

and a tangent vector $\mathbf{t}_p$ is a *derivation* of this algebra.

Definition 2.3.4 A *derivation* $\mathscr{D}$ of an algebra $\mathscr{A}$ is a map:

$$\mathscr{D} : \mathscr{A} \to \mathscr{A} \tag{2.3.16}$$

that

1. is linear

$$\forall \alpha, \beta \in \mathbb{R} \quad \forall f, g \in \mathscr{A} : \quad \mathscr{D}(\alpha f + \beta g) = \alpha \mathscr{D} f + \beta \mathscr{D} g \tag{2.3.17}$$

2. obeys Leibnitz rule

$$\forall f, g \in \mathscr{A} : \quad \mathscr{D}(f \cdot g) = \mathscr{D} f \cdot g + f \cdot \mathscr{D} g \tag{2.3.18}$$

That tangent vectors fit into Definition 2.3.4 is clear from their explicit realization as differential operators (2.3.13), (2.3.14). It is also clear that the set of *derivations* $D[\mathscr{A}]$ of an algebra constitutes a real vector space. Indeed a linear combination of derivations is still a derivation, having set:

$$\forall \alpha, \beta \in \mathbb{R}, \ \forall \mathscr{D}_1, \mathscr{D}_2 \in D[\mathscr{A}], \ \forall f \in \mathscr{A} : \quad (\alpha \mathscr{D}_1 + \beta \mathscr{D}_2) f = \alpha \mathscr{D}_1 f + \beta \mathscr{D}_2 f \tag{2.3.19}$$

Hence an equivalent and more abstract definition of the tangent space is the following:

Definition 2.3.5 The tangent space to a manifold $\mathscr{M}$ at the point p is the vector space of derivations of the algebra of germs of smooth functions in p:

$$T_p \mathscr{M} \equiv D\big[\mathbb{C}^\infty_p(\mathscr{M})\big] \tag{2.3.20}$$

Indeed for any tangent vector (2.3.13) and for any pair of germs $f, g \in \mathbb{C}^\infty_p(\mathscr{M})$ we have:

$$\begin{aligned} \mathbf{t}_p(\alpha f + \beta g) &= \alpha \mathbf{t}_p(f) + \beta \mathbf{t}_p(g) \\ \mathbf{t}_p(f \cdot g) &= \mathbf{t}_p(f) \cdot g + f \cdot \mathbf{t}_p(g) \end{aligned} \tag{2.3.21}$$

In each coordinate patch a tangent vector is, as we have seen, a first order differential operator singled out by its *components*, namely by the coefficients c^μ. In the language of tensor calculus the tangent vector *is identified* with the m-tuplet of real numbers c^μ. The relevant point, however, is that such m-tuplet representing the

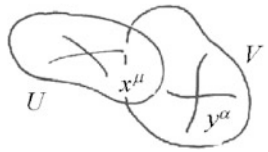


Fig. 2.11 Two coordinate patches

same tangent vector is different in different coordinate patches. Consider two coordinate patches (U, φ) and (V, ψ) with non-vanishing intersection. Name x^μ the coordinate of a point $p \in U \bigcap V$ in the patch (U, φ) and y^α the coordinate of the same point in the patch (V, ψ). The transition function and its inverse are expressed by setting:

$$x^\mu = x^\mu(y); \qquad y^\nu = y^\nu(x) \tag{2.3.22}$$

Then the same first order differential operator can be alternatively written as:

$$\mathbf{t}_p = c^\mu \frac{\partial}{\partial x^\mu} \quad \text{or} \quad \mathbf{t}_p = c^\mu \left(\frac{\partial y^\nu}{\partial x^\mu} \right) \frac{\partial}{\partial y^\nu} = c^\nu \frac{\partial}{\partial y^\nu} \tag{2.3.23}$$

having defined:

$$c^\nu \equiv c^\mu \left(\frac{\partial y^\nu}{\partial x^\mu} \right) \tag{2.3.24}$$

Equation (2.3.24) expresses the transformation rule for the components of a tangent vector from one coordinate patch to another one (see Fig. 2.11).

Such a transformation is *linear* and the matrix that realizes it is the *inverse of the Jacobian matrix* $(\partial y/\partial x) = (\partial x/\partial y)^{-1}$. For this reason we say that the components of a tangent vector constitute a *controvariant world vector*. By definition a *covariant world vector* transforms instead with the *Jacobian matrix*. We will see that covariant world vectors are the components of a differential form.

2.3.2 Differential Forms in a Point $p \in \mathscr{M}$

Let us now consider the total differential of a function (better of a germ of a smooth function) when we evaluate it along a curve. $\forall f \in \mathbb{C}^\infty_p(\mathscr{M})$ and for each curve $c(t)$ starting at p we have:

$$\frac{d}{dt} f\big(c(t)\big)\bigg|_{t=0} = c^\mu \frac{\partial}{\partial x^\mu} f \equiv \mathbf{t}_p f \tag{2.3.25}$$

where we have named $\mathbf{t}_p = \frac{dc^\mu}{dt}|_{t=0} \frac{\partial}{\partial x^\mu}$ the tangent vector to the curve in its initial point p. So, fixing a tangent vector means that for any germ f we know its total differential along the curve that admits such a vector as tangent in p. Let us now reverse our viewpoint. Rather than keeping the tangent vector fixed and letting the germ f vary let us keep the germ f fixed and let us consider all possible curves that depart from the point p. We would like to evaluate the total derivative of the germ $\frac{df}{dt}$ along each curve. The solution of such a problem is easily obtained: given the tangent vector $\mathbf{t}_p$ to the curve in p we have $df/dt = \mathbf{t}_p f$. The moral of this tale is the following: the concept of *total differential of a germ* is the *dual* of the concept of *tangent vector*. Indeed we recall from linear algebra that the dual of a vector space is the space of linear functionals on that vector space and our discussion shows that the total differential of a germ is precisely a linear functional on the tangent space $T_p\mathscr{M}$.

Definition 2.3.6 The total differential df_p of a smooth germ $f \in \mathbb{C}_p^\infty(\mathscr{M})$ is a *linear functional* on $T_p\mathscr{M}$ such that

$$\begin{array}{ll} \forall \mathbf{t}_p \in T_p\mathscr{M} & df_p(\mathbf{t}_p) = \mathbf{t}_p f \\ \forall \mathbf{t}_p, \mathbf{k}_p \in T_p\mathscr{M}, \forall \alpha, \beta \in \mathbb{R} & df_p(\alpha \mathbf{t}_p + \beta \mathbf{k}_p) = \alpha df_p(\mathbf{t}_p) + \beta df_p(\mathbf{k}_p) \end{array} \tag{2.3.26}$$

The linear functionals on a finite dimensional vector space $\mathscr{V}$ constitute a vector space $\mathscr{V}^\star$ (the dual) with the same dimension. This justifies the following

Definition 2.3.7 We name *cotangent space* to the manifold $\mathscr{M}$ in the point p the vector space $T_p^*\mathscr{M}$ of linear functionals (or 1-forms in p) on the tangent space $T_p\mathscr{M}$:

$$T_p^*\mathscr{M} \equiv \mathrm{Hom}(T_p\mathscr{M}, \mathbb{R}) = (T_p\mathscr{M})^\star \tag{2.3.27}$$

So we name differential 1-forms in p the elements of the cotangent space and $\forall \omega_p \in T_p^*\mathscr{M}$ we have:

$$\begin{array}{ll} 1)\ \forall \mathbf{t}_p \in T_p\mathscr{M}: & \omega_p(\mathbf{t}_p) \in \mathbb{R} \\ 2)\ \forall \alpha, \beta \in \mathbb{R}, \forall \mathbf{t}_p, \mathbf{k}_p \in T_p\mathscr{M}: & \omega_p(\alpha \mathbf{t}_p + \beta \mathbf{k}_p) = \alpha \omega_p(\mathbf{t}_p) + \beta \omega_p(\mathbf{k}_p) \end{array} \tag{2.3.28}$$

The reason why the above linear functionals are named differential 1-forms is that in every coordinate patch $\{x^\mu\}$ they can be expressed as linear combinations of the coordinate differentials:

$$\omega_p = \omega_\mu \, dx^\mu \tag{2.3.29}$$

and their action on the tangent vectors is expressed as follows:

$$\mathbf{t}_p = c^\mu \frac{\partial}{\partial x^\mu} \quad \Rightarrow \quad \omega_p(\mathbf{t}_p) = \omega_\mu c^\mu \in \mathbb{R} \tag{2.3.30}$$

Indeed in the particular case where the 1-form is exact (namely it is the differential of a germ) $\omega_p = df_p$ we can write $\omega_p = \partial f/\partial x^\mu \, dx^\mu$ and we have $df_p(\mathbf{t}_p) \equiv \mathbf{t}_p f = c^\mu \partial f/\partial x^\mu$. Hence when we extend our definition to differential forms that are not exact we continue to state the same statement, namely that the value of the 1-form on a tangent vector is given by (2.3.30).

Summarizing, in each coordinate patch, a differential 1-form in a point $p \in \mathscr{M}$ has the representation (2.3.29) and its coefficients ω_μ constitute a *controvariant vector*. Indeed, in complete analogy to (2.3.23), we have

$$\omega_p = \omega_\mu \, dx^\mu \quad \text{or} \quad \omega_p = \omega_\mu \left(\frac{\partial x^\mu}{\partial y^\nu} \right) dy^\nu = \omega_\nu \, dy^\nu \tag{2.3.31}$$

having defined:

$$\omega_\nu \equiv \omega_\mu \left(\frac{\partial x^\mu}{\partial y^\nu} \right) \tag{2.3.32}$$

Finally the duality relation between 1-forms and tangent vectors can be summarized writing the rule:

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu \tag{2.3.33}$$

2.4 Fibre Bundles

The next step we have to take is *gluing together* all the tangent $T_p\mathscr{M}$ and cotangent spaces $T^*_p\mathscr{M}$ we have discussed in the previous sections. The result of such a gluing procedure is not a vector space, rather it is a vector bundle. Vector bundles are specific instances of the more general notion of *fibre bundles*.

The concept of *fibre bundle* is absolutely central in contemporary physics and provides the appropriate mathematical framework to formulate modern field theory since all the fields one can consider are either *sections* of *associated bundles* or *connections* on *principal bundles*. There are two kinds of fibre-bundles:

1. principal bundles,
2. associated bundles.

The notion of a principal fibre-bundle is the appropriate mathematical concept underlying the formulation of *gauge theories* that provide the general framework to describe the dynamics of all non-gravitational interactions. The concept of a connection on such principal bundles codifies the physical notion of the bosonic particles mediating the interaction, namely the gauge bosons, like the photon, the gluon or the graviton. Indeed, gravity itself is a gauge theory although of a very special type. On the other hand the notion of associated fibre-bundles is the appropriate mathematical framework to describe *matter fields* that interact through the exchange of the *gauge bosons*.

Also from a more general viewpoint and in relation with all sort of applications the notion of fibre-bundles is absolutely fundamental. As we already emphasized, the points of a manifold can be identified with the possible states of a complex system specified by an m-tuplet of parameters $x_1, \ldots, x_m$. Real or complex functions of such parameters are the natural objects one expects to deal with in any scientific theory that explains the phenomena observed in such a system. Yet, as we already anticipated, calculus on manifolds that are not trivial as the flat $\mathbb{R}^m$ cannot be confined to functions, which is a too restrictive notion. The appropriate generalization of functions is provided by the *sections* of fibre-bundles. Locally, namely in each coordinate patch, functions and sections are just the same thing. Globally, however, there are essential differences. A section is obtained by gluing together many local functions by means of non-trivial transition functions that reflect the geometric structure of the fibre-bundle.

To introduce the mathematical definition of a fibre-bundle we need to recall the definition of a Lie group which the reader should have met in other basic courses.

Definition 2.4.1 A Lie group G is:

- A group from the algebraic point of view, namely a set with an internal composition law, the product

$$\forall g_1 g_2 \in G \quad g_1 \cdot g_2 \in G \tag{2.4.1}$$

  which is associative, admits a unique neutral element e and yields an inverse for each group element.
- A smooth manifold of finite dimension $\dim G = n < \infty$ whose transition function are not only infinitely differentiable but also real analytic, namely they admit an expansion in power series.
- In the topology defined by the manifold structure the two algebraic operations of taking the inverse of an element and performing the product of two elements are real analytic (admit a power series expansion).

The last point in Definition (2.4.1) deserves a more extended explanation. To each group element the product operation associates two maps of the group into itself:

$$\begin{aligned} \forall g \in G: &\quad L_g : G \to G : g' \to L_g(g') \equiv g' \cdot g \\ \forall g \in G: &\quad R_g : G \to G : g' \to R_g(g') \equiv g \cdot g' \end{aligned} \tag{2.4.2}$$

respectively named the *left translation* and the *right translation*. Both maps are required to be real analytic for each choice of $g \in G$. Similarly the group structure induces a map:

$$(\cdot)^{-1} : G \to G : g \to g^{-1} \tag{2.4.3}$$

which is also required to be real analytic.

Coming now to fibre-bundles let us begin by recalling that a pedagogical and pictorial example of such spaces is provided by the celebrated picture by Escher of an ant crawling on a Mobius strip (see Fig. 2.12).

The basic idea is that if we consider a piece of the bundle this cannot be distinguished from a trivial direct product of two spaces, an open subset of the base manifold and the fibre. In Fig. 2.12 the base manifold is the strip and the fibre is the space containing all possible positions of the ant. However, the relevant point

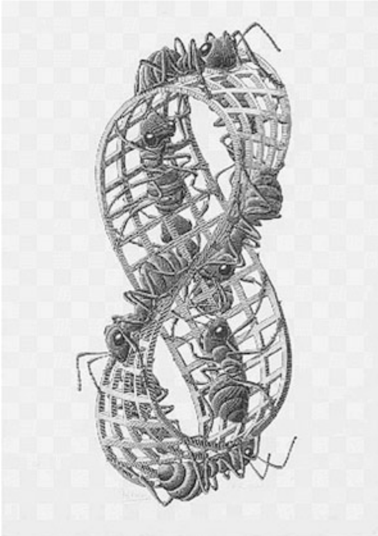


Fig. 2.12 Escher's ant crawling on a Mobius strip provides a pedagogical example of a fibre-bundle

is that, *globally*, the bundle *is not a direct product of spaces*. If the ant is placed in some orientation at a certain point on the strip, taking her around the strip she will be necessarily reversed at the end of her trip.

Hence the notion of fibre-bundle corresponds to that of a differentiable manifold P with dimension $\dim P = m + n$ that locally *looks like* the direct product $U \times F$ of an open manifold U of dimension $\dim U = m$ with another manifold F (the standard fibre) of dimension $\dim F = n$. Essential in the definition is the existence of a map:

$$\pi : P \to \mathscr{M} \tag{2.4.4}$$

named *the projection* from the *total manifold* P of dimension $m + n$ to a manifold $\mathscr{M}$ of dimension m, named the *base manifold*. Such a map is required to be continuous. Due to the difference in dimensions the projection cannot be invertible. Indeed to every point $\forall p \in \mathscr{M}$ of the base manifold the projection associates a submanifold $\pi^{-1}(p) \subset P$ of dimension $\dim \pi^{-1}(p) = n$ composed by those points of $x \in P$ whose projection on $\mathscr{M}$ is the chosen point p: $\pi(x) = p$. The submanifold $\pi^{-1}(p)$ is named the *fibre over* p and the basic idea is that each fibre is homeomorphic to the *standard fibre* F. More precisely for each open subset $U_\alpha \subset \mathscr{M}$ of the base manifold we must have that the submanifold

$$\pi^{-1}(U_\alpha)$$

is homeomorphic to the direct product

$$U_\alpha \times F$$

This is the precise meaning of the statement that, locally, the bundle looks like a direct product (see Fig. 2.13). Explicitly what we require is the following: there

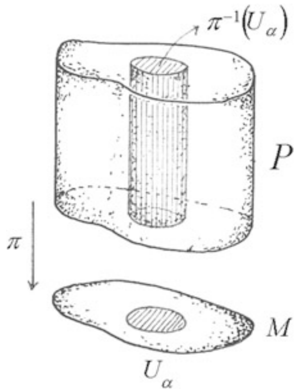


Fig. 2.13 A fibre-bundle is locally trivial

should be a family of pairs (U_α, ϕ_α) where U_α are open charts covering the base manifold $\bigcup_\alpha U_\alpha = \mathscr{M}$ and ϕ_α are maps:

$$\phi_\alpha : \pi^{-1}(U_\alpha) \subset P \to U_\alpha \otimes F \tag{2.4.5}$$

that are required to be one-to-one, bicontinuous (= continuous, together with its inverse) and to satisfy the property that:

$$\pi \circ \phi_\alpha^{-1}(p, f) = p \tag{2.4.6}$$

Namely the projection of the image in P of a base manifold point p *times* some fibre point f is p itself.

Each pair (U_α, ϕ_α) is named a *local trivialization*. As for the case of manifolds, the interesting question is what happens in the intersection of two different local trivializations. Indeed if $U_\alpha \bigcap U_\beta \neq \emptyset$, then we also have $\pi^{-1}(U_\alpha) \bigcap \pi^{-1}(U_\beta) \neq \emptyset$. Hence each point $x \in \pi^{-1}(U_\alpha \bigcap U_\beta)$ is mapped by ϕ_α and ϕ_β in two different pairs $(p, f_\alpha) \in U_\alpha \otimes F$ and $(p, f_\beta) \in U_\alpha \otimes F$ with the property, however, that the first entry p is the same in both pairs. This follows from property (2.4.6). It implies that there must exist a map:

$$t_{\alpha\beta} \equiv \phi_\beta^{-1} \circ \phi_\alpha : \quad \left(U_\alpha \bigcap U_\beta\right) \otimes F \to \left(U_\alpha \bigcap U_\beta\right) \otimes F \tag{2.4.7}$$

named *transition function*, which acts exclusively on the fibre points in the sense that:

$$\forall p \in U_\alpha \bigcap U_\beta, \ \forall f \in F \quad t_{\alpha\beta}(p, f) = \big(p, t_{\alpha\beta}(p).f\big) \tag{2.4.8}$$

where for each choice of the point $p \in U_\alpha \bigcap U_\beta$,

$$t_{\alpha\beta}(p) : F \mapsto F \tag{2.4.9}$$

is a continuous and invertible map of the standard fibre F into itself (see Fig. 2.14).

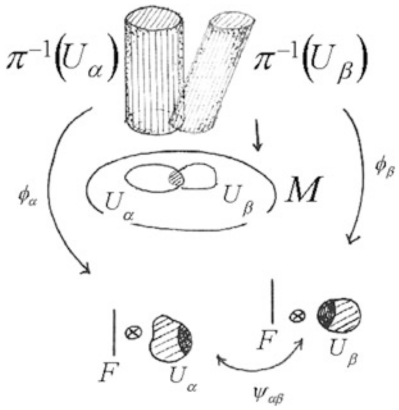


Fig. 2.14 Transition function between two local trivializations of a fibre-bundle

The last bit of information contained in the notion of fibre-bundle is related with the *structural group*. This has to do with answering the following question: where are the transition functions chosen from? Indeed the set of all possible continuous invertible maps of the standard fibre F into itself constitute a group, so that it is no restriction to say that the transition functions $t_{\alpha\beta}(p)$ are group elements. Yet the group of all homeomorphisms $\mathrm{Hom}(F, F)$ is very very large and it makes sense to include into the definition of fibre bundle the request that the transition functions should be chosen within a smaller hunting ground, namely inside some finite dimensional Lie group G that has a well defined action on the standard fibre F.

The above discussion can be summarized into the following technical definition of fibre bundles.

Definition 2.4.2 A fibre bundle $(P, \pi, \mathscr{M}, F, G)$ is a geometrical structure that consists of the following list of elements:

1. A differentiable manifold P named the *total space*.
2. A differentiable manifold $\mathscr{M}$ named the *base space*.
3. A differentiable manifold F named the *standard fibre*.
4. A Lie group G, named the *structure group*, which acts as a transformation group on the standard fibre:

$$\forall g \in G; \quad g : F \longrightarrow F \quad \{i.e. \, \forall f \in F \quad g.f \in F\} \tag{2.4.10}$$

5. A surjection map $\pi : P \longrightarrow \mathscr{M}$, named the *projection*. If $n = \dim \mathscr{M}$, $m = \dim F$, then we have $\dim P = n + m$ and $\forall p \in \mathscr{M}$, $F_p = \pi^{-1}(p)$ is an m-dimensional manifold diffeomorphic to the standard fibre F. The manifold F_p is named the *fibre at the point* p.
6. A covering of the base space $\cup_{(\alpha \in A)} U_\alpha = \mathscr{M}$, realized by a collection $\{U_\alpha\}$ of open subsets ($\forall \alpha \in A \; U_\alpha \subset \mathscr{M}$), equipped with a homeomorphism:

$$\phi_\alpha^{-1} : U_\alpha \times F \longrightarrow \pi^{-1}(U_\alpha) \tag{2.4.11}$$

such that

$$\forall p \in U_\alpha, \forall f \in F: \quad \pi \cdot \phi_\alpha^{-1}(p, f) = p \tag{2.4.12}$$

The map ϕ_α^{-1} is named a *local trivialization* of the bundle, since its inverse ϕ_α maps the open subset $\pi^{-1}(U_\alpha) \subset P$ of the total space into the direct product $U_\alpha \times F$.

7. If we write $\phi_\alpha^{-1}(p, f) = \phi_{\alpha,p}^{-1}(f)$, the map $\phi_{\alpha,p}^{-1} : F \longrightarrow F_p$ is the homeomorphism required by point (6) of the present definition. For all points $p \in U_\alpha \cap U_\beta$ in the intersection of two different local trivialization domains, the composite map $t_{\alpha\beta}(p) = \phi_{\alpha,p} \cdot \phi_{\beta,p}^{-1} F \longrightarrow F$ is an element of the structure group $t_{\alpha\beta} \in G$, named the *transition function*. Furthermore the transition function realizes a smooth map $t_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G$. We have

$$\phi_\beta^{-1}(p, f) = \phi_\alpha^{-1}\big(p, t_{\alpha\beta}(p).f\big) \tag{2.4.13}$$

Just as manifolds can be constructed by gluing together open charts, fibre-bundles can be obtained by gluing together local trivializations. Explicitly one proceeds as follows.

1. First choose a base manifold $\mathscr{M}$, a typical fibre F and a structural Lie Group G whose action on F must be well-defined.
2. Then choose an atlas of open neighborhoods $U_\alpha \subset \mathscr{M}$ covering the base manifold $\mathscr{M}$.
3. Next to each non-vanishing intersection $U_\alpha \bigcap U_\beta \neq \emptyset$ assign a transition function, namely a smooth map:

$$\psi_{\alpha\beta} : U_\alpha \bigcap U_\beta \mapsto G \tag{2.4.14}$$

from the open subset $U_\alpha \bigcap U_\beta \subset \mathscr{M}$ of the base manifold to the structural Lie group. For consistency the transition functions must satisfy the two conditions:

$$\begin{aligned} &\forall U_\alpha, U_\beta / U_\alpha \bigcap U_\beta \neq \emptyset: && \psi_{\beta\alpha} = \psi_{\alpha\beta}^{-1} \\ &\forall U_\alpha, U_\beta, U_\gamma / U_\alpha \bigcap U_\beta \bigcap U_\gamma \neq \emptyset: && \psi_{\alpha\beta} \cdot \psi_{\beta\gamma} \cdot \psi_{\gamma\alpha} = \mathbf{1}_G \end{aligned} \tag{2.4.15}$$

Whenever a set of local trivializations with consistent transition functions satisfying (2.4.15) has been given a fibre-bundle is defined. A different and much more difficult question to answer is to decide whether two sets of local trivializations define the same fibre-bundle or not. We do not address such a problem whose proper treatment is beyond the scope of this course. We just point out that the classification of inequivalent fibre-bundles one can construct on a given base manifold $\mathscr{M}$ is a problem of global geometry which can also be addressed with the techniques of algebraic topology and algebraic geometry.

Typically inequivalent bundles are characterized by topological invariants that receive the name of *characteristic classes*.

In physical language the transition functions (2.4.14) from one local trivialization to another one are the *gauge transformations*, namely group transformations depending on the position in space-time (*i.e.* the point on the base manifold).

Definition 2.4.3 A principal bundle $P(\mathscr{M}, G)$ is a fibre-bundle where the standard fibre coincides with the structural Lie group $F = G$ and the action of G on the fibre is the left (or right) multiplication (see (2.4.2)):

$$\forall g \in G \quad \Rightarrow \quad L_g : G \mapsto G \tag{2.4.16}$$

The name principal is given to the fibre-bundle in Definition 2.4.3 since it is a "*father*" bundle which, once given, generates an infinity of *associated vector bundles*, one for each linear representation of the Lie group G.

Let us recall the notion of linear representations of a Lie group.

Definition 2.4.4 Let V be a vector space of finite dimension $\dim V = m$ and let $\mathrm{Hom}(V, V)$ be the group of all linear homomorphisms of the vector space into itself:

$$\begin{array}{ll} f \in \mathrm{Hom(V, V)}/ & f : V \to V \\ \forall \alpha, \beta \in \mathbb{R} \ \ \forall v_1, v_2 \in V : & f(\alpha v_1 + \beta v_2) = \alpha f(v_1) + \beta f(v_2) \end{array} \tag{2.4.17}$$

A linear representation of the Lie group G of dimension n is a *group homomorphism*:

$$\begin{cases} \forall g \in G & g \mapsto D(g) \in \mathrm{Hom}(V, V) \\ \forall g_1 g_2 \in G & D(g_1 \cdot g_2) = D(g_1) \cdot D(g_2) \\ & D(e) = \mathbf{1} \\ \forall g \in G & D(g^{-1}) = [D(g)]^{-1} \end{cases} \tag{2.4.18}$$

Whenever we choose a basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of the vector space V every element $f \in \mathrm{Hom}(V, V)$ is represented by a matrix $f_i^{\ j}$ defined by:

$$f(\mathbf{e}_i) = f_i^{\ j} \, \mathbf{e}_j \tag{2.4.19}$$

Therefore a linear representation of a Lie group associates to each abstract group element g an $n \times n$ matrix $D(g)_i^{\ j}$. As it should be known to the student, linear representations are said to be *irreducible* if the vector space V admits *no* non-trivial vector subspace $W \subset V$ that is *invariant* with respect to the action of the group: $\forall g \in G/D(g)W \subset W$. For simple Lie groups reducible representations can always be decomposed into a direct sum of irreducible representations, namely $V = V_1 \oplus V_2 \oplus \cdots \oplus V_r$ (with V_i irreducible) and irreducible representations are completely defined by the structure of the group. These notions that we have recalled from group theory motivate the definition:

Definition 2.4.5 An *associated vector bundle* is a fibre-bundle where the standard fibre $F = V$ is a vector space and the action of the structural group on the standard fibre is a linear representation of G on V.

The reason why the bundles in Definition 2.4.5 are named associated is almost obvious. Given a principal bundle and a linear representation of G we can immedi-

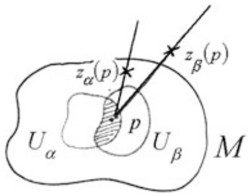


Fig. 2.15 The intersection of two local trivializations of a line bundle

ately construct a corresponding vector bundle. It suffices to use as transition functions the linear representation of the transition functions of the principal bundle:

$$\psi^{(V)}_{\alpha\beta} \equiv D\bigl(\psi^{(G)}_{\alpha\beta}\bigr) \in \mathrm{Hom}(V, V) \tag{2.4.20}$$

For any vector bundle the dimension of the standard fibre is named the *rank* of the bundle.

Whenever the base-manifold of a fibre-bundle is complex and the transition functions are holomorphic maps, we say that the bundle is *holomorphic*.

A very important and simple class of holomorphic bundles are the *line bundles*. By definition these are principal bundles on a complex base manifold $\mathscr{M}$ with structural group $\mathbb{C}^\star \equiv \mathbb{C}\backslash 0$, namely the multiplicative group of non-zero complex numbers.

Let $z_\alpha(p) \in \mathbb{C}^\star$ be an element of the standard fibre above the point $p \in U_\alpha \bigcap U_\beta \subset \mathscr{M}$ in the local trivialization α and let $z_\beta(p) \in \mathbb{C}^\star$ be the corresponding fibre point in the local trivialization β. The transition function between the two trivialization is expressed by (see Fig. 2.15):

$$z_\alpha(p) = \underbrace{f_{\alpha\beta}(p)}_{\in\mathbb{C}^\star} \cdot z_\beta(p) \quad \Rightarrow \quad f_{\alpha\beta}(p) = \frac{z_\alpha(p)}{z_\beta(p)}, \neq 0 \tag{2.4.21}$$

2.5 Tangent and Cotangent Bundles

Let $\mathscr{M}$ be a differentiable manifold of dimension $\dim \mathscr{M} = m$: in Sect. 2.3 we have seen how to construct the tangent spaces $T_p\mathscr{M}$ associated with each point $p \in \mathscr{M}$ of the manifold. We have also seen that each $T_p\mathscr{M}$ is a real vector space isomorphic to $\mathbb{R}^m$. Considering the definition of fibre-bundles discussed in the previous section we now realize that what we actually did in Sect. 2.3 was to construct a vector-bundle, the *tangent bundle* $T\mathscr{M}$ (see Fig. 2.16).

In the tangent bundle $T\mathscr{M}$ the *base manifold* is the differentiable manifold $\mathscr{M}$, the *standard fibre* is $F = \mathbb{R}^m$ and the structural group is $\mathrm{GL}(m, \mathbb{R})$ namely the group of real $m \times m$ matrices. The main point is that the transition functions are not newly introduced to construct the bundle rather they are completely determined from the transition functions relating open charts of the base manifold. In other words, whenever we define a manifold $\mathscr{M}$, associated with it there is a unique vector bundle $T\mathscr{M} \to \mathscr{M}$ which encodes many intrinsic properties of $\mathscr{M}$. Let us see how.

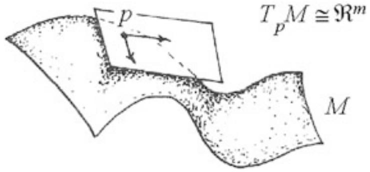


Fig. 2.16 The tangent bundle is obtained by gluing together all the tangent spaces

Consider two intersecting local charts (U_α, ϕ_α) and (U_β, ϕ_β) of our manifold. A tangent vector, in a point $p \in \mathscr{M}$ was written as:

$$\mathbf{t}_p = c^\mu(p) \frac{\partial}{\partial x^\mu}\bigg|_p \tag{2.5.1}$$

Now we can consider choosing smoothly a tangent vector for each point $p \in \mathscr{M}$, namely introducing a map:

$$p \in \mathscr{M} \mapsto \mathbf{t}_p \in T_p\mathscr{M} \tag{2.5.2}$$

Mathematically what we have obtained is a *section of the tangent bundle*, namely a smooth choice of a point in the fibre for each point of the base. Explicitly this just means that the components $c^\mu(p)$ of the tangent vector are smooth functions of the base point coordinates x^μ. Since we use coordinates, we need an extra label denoting in which local patch the vector components are given:

$$\begin{cases} \mathbf{t} = c^\mu_{(\alpha)}(x) \frac{\partial}{\partial x^\mu}|_p \Rightarrow \text{in chart } \alpha \\ \mathbf{t} = c^\nu_{(\beta)}(y) \frac{\partial}{\partial y^\nu}|_p \Rightarrow \text{in chart } \beta \end{cases} \tag{2.5.3}$$

having denoted x^μ and y^ν the local coordinates in patches α and β, respectively. Since the tangent vector is the same, irrespectively of the coordinates used to describe it, we have:

$$c^\nu_{(\beta)}(y) \frac{\partial}{\partial y^\nu} = c^\mu_{(\alpha)}(x) \frac{\partial y^\nu}{\partial x^\mu} \frac{\partial}{\partial y^\nu} \tag{2.5.4}$$

namely:

$$c^\nu_{(\beta)}(p) = c^\mu_{(\alpha)}(p) \left(\frac{\partial y^\nu}{\partial x^\mu} \right)(p) \tag{2.5.5}$$

In formula (2.5.5) we see the explicit form of the transition function between two local trivializations of the tangent bundle: it is simply the *inverse Jacobian matrix* associated with the transition functions between two local charts of the base manifold $\mathscr{M}$. On the intersection $U_\alpha \bigcap U_\beta$ we have:

$$\forall p \in U_\alpha \bigcap U_\beta : \quad p \to \psi_{\beta\alpha}(p) = \left(\frac{\partial y}{\partial x} \right)(p) \in \mathrm{GL}(m, \mathbb{R}) \tag{2.5.6}$$

as it is pictorially described in Fig. 2.17.

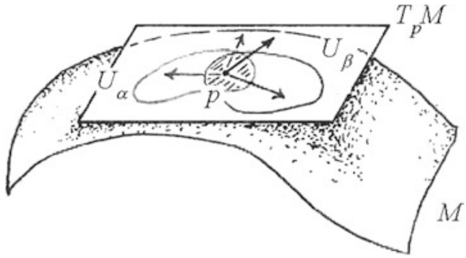


Fig. 2.17 Two local charts of the base manifold $\mathcal{M}$ yield two local trivializations of the tangent bundle $T\mathcal{M}$

2.5.1 Sections of a Bundle

It is now the appropriate time to associate a precise definition to the notion of bundle section that we have implicitly advocated in (2.5.2).

Definition 2.5.1 Consider a generic fibre-bundle $E \xrightarrow{\pi} \mathcal{M}$ with generic fibre F. We name *section of the bundle* a rule s that to each point $p \in \mathcal{M}$ of the base manifold associates a point $s(p) \in F_p$ in the fibre above p, namely a map

$$s : \mathcal{M} \mapsto E \tag{2.5.7}$$

such that:

$$\forall p \in \mathcal{M} : \quad s(p) \in \pi^{-1}(p) \tag{2.5.8}$$

The above definition is illustrated in Fig. 2.18 which also clarifies the intuitive idea standing behind the chosen name for such a concept.

It is clear that sections of the bundle can be chosen to be *continuous, differentiable, smooth* or, in the case of complex manifolds, even *holomorphic*, depending on the properties of the map s in each local trivialization of the bundle. Indeed given a local trivialization and given open charts for both the base manifold $\mathcal{M}$ and for the fibre F, the local description of the section reduces to a map:

$$\mathbb{R}^m \supset U \mapsto F_U \subset \mathbb{R}^n \tag{2.5.9}$$

where m and n are the dimensions of the base manifold and of the fibre respectively.

We are specifically interested in smooth sections, namely in section that are infinitely differentiable. Given a bundle $E \xrightarrow{\pi} \mathcal{M}$, the set of all such sections is denoted by:

$$\Gamma(E, \mathcal{M}) \tag{2.5.10}$$

Of particular relevance are the smooth sections of vector bundles. In this case to each point of the base manifold p we associate a vector $\mathbf{v}(p)$ in the vector space above the point p. In particular we can consider sections of the tangent bundle $T\mathcal{M}$ associated with a smooth manifold $\mathcal{M}$. Such sections correspond to the notion of *vector fields*.

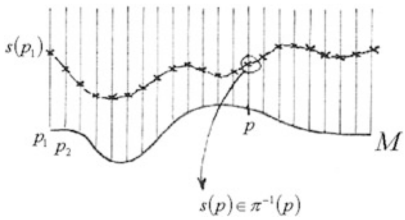


Fig. 2.18 A section of a fibre bundle

Definition 2.5.2 Given a smooth manifold $\mathcal{M}$, we name *vector field* on $\mathcal{M}$ a smooth section $\mathbf{t} \in \Gamma(T\mathcal{M}, \mathcal{M})$ of the tangent bundle. The local expression of such vector field in any open chart (U, ϕ) is

$$\mathbf{t} = t^{\mu}(x)\frac{\partial}{\partial x^{\mu}} \quad \forall x \in U \subset \mathcal{M} \tag{2.5.11}$$

2.5.1.1 Example: Holomorphic Vector Fields on $\mathbb{S}^2$

As we have seen above, the 2-sphere $\mathbb{S}^2$ is a complex manifold of complex dimension one covered by an atlas composed by two charts, that of the North Pole and that of the South Pole (see Fig. 2.19) and the transition function between the local complex coordinate in the two patches is the following one:

$$z_N = \frac{1}{z_S} \tag{2.5.12}$$

Correspondingly, in the two patches, the local description of a holomorphic vector field $\mathbf{t}$ is given by:

$$\begin{aligned} \mathbf{t} &= v_N(z_N)\frac{d}{dz_N} \\ \mathbf{t} &= v_S(z_S)\frac{d}{dz_S} \end{aligned} \tag{2.5.13}$$

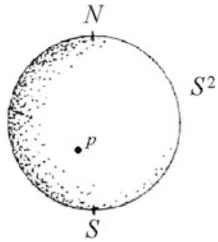
where the two functions $v_N(z_N)$ and $v_S(z_S)$ are supposed to be holomorphic functions of their argument, namely to admit a Taylor power series expansion:

$$\begin{aligned} v_N(z_N) &= \sum_{k=0}^{\infty} c_k z_N^k \\ v_S(z_S) = v_S(z_S) &\sum_{k=0}^{\infty} d_k z_S^k \end{aligned} \tag{2.5.14}$$

However, from the transition function (2.5.12) we obtain the relations:

$$\frac{d}{dz_N} = -z_S^2\frac{d}{dz_S}; \qquad \frac{d}{dz_S} = -z_N^2\frac{d}{dz_N} \tag{2.5.15}$$

Fig. 2.19 The 2-sphere



and hence:

$$\mathbf{t} = -\sum_{k=0}^{\infty} c_k z_S^{2-k} \frac{d}{dz_S} = \sum_{k=0}^{\infty} d_k z_S^k \frac{d}{dz_S} = -\sum_{k=0}^{\infty} d_k z_N^{2-k} \frac{d}{dz_N} = \sum_{k=0}^{\infty} c_k z_N^k \frac{d}{dz_N} \tag{2.5.16}$$

The only way for (2.5.16) to be self consistent is to have:

$$\forall k > 2 \quad c_k = d_k = 0; \qquad c_0 = -d_2, \quad c_1 = -d_1, \quad c_2 = -d_0 \tag{2.5.17}$$

This shows that the space of holomorphic sections of the tangent bundle $T\mathbb{S}^2$ is a *finite dimensional* vector space of dimension *three* spanned by the three differential operators:

$$\begin{aligned} \mathbf{L}_0 &= -z\frac{d}{dz} \\ \mathbf{L}_1 &= -\frac{d}{dz} \\ \mathbf{L}_{-1} &= -z^2\frac{d}{dz} \end{aligned} \tag{2.5.18}$$

We will have more to say about these operators in the sequel.

What we have so far discussed can be summarized by stating the transformation rule of vector field components when we change coordinate patch form x^μ to $x^{\mu'}$:

$$t^{\mu'}(x') = t^\nu(x)\frac{\partial x^{\mu'}}{\partial x^\nu} \tag{2.5.19}$$

Indeed a convenient way of defining a fibre-bundle is provided by specifying the way its sections transform from one local trivialization to another one which amounts to giving all the transition functions. This method can be used to discuss the construction of the cotangent bundle.

2.5.2 The Lie Algebra of Vector Fields

In Sect. 2.3 we saw that the tangent space $T_p\mathscr{M}$ at point $p \in \mathscr{M}$ of a manifold can be identified with the vector space of derivations of the algebra of germs (see

Definition 2.3.5). After gluing together all tangent spaces into the tangent bundle $T\mathcal{M}$ such an identification of tangent vectors with the derivations of an algebra can be extended from the local to the global level. The crucial observation is that the set of smooth functions on a manifold $\mathbb{C}^\infty(\mathcal{M})$ constitutes an algebra with respect to point-wise multiplication just as the set of germs at point p. The vector fields, namely the sections of the tangent bundle, are derivations of this algebra. Indeed each vector field $\mathbf{X} \in \Gamma(T\mathcal{M}, \mathcal{M})$ is a linear map of the algebra $\mathbb{C}^\infty(\mathcal{M})$ into itself:

$$\mathbf{X} : \mathbb{C}^\infty(\mathcal{M}) \to \mathbb{C}^\infty(\mathcal{M}) \tag{2.5.20}$$

that satisfies the analogue properties of those mentioned in (2.3.21) for tangent vectors, namely:

$$\begin{aligned} \mathbf{X}(\alpha f + \beta g) &= \alpha \mathbf{X}(f) + \beta \mathbf{X}(g) \\ \mathbf{X}(f \cdot g) &= \mathbf{X}(f) \cdot g + f \cdot \mathbf{X}(g) \\ &\qquad \big[\forall \alpha, \beta \in \mathbb{R} \text{ (or } \mathbb{C}); \ \forall f, g \in \mathbb{C}^\infty(\mathcal{M})\big] \end{aligned} \tag{2.5.21}$$

On the other hand the set of vector fields, renamed for this reason:

$$\mathbb{D}\mathrm{iff}(\mathcal{M}) \equiv \Gamma(T\mathcal{M}, \mathcal{M}) \tag{2.5.22}$$

forms a Lie algebra with respect to the following Lie bracket operation:

$$[\mathbf{X}, \mathbf{Y}] f = \mathbf{X}\big(\mathbf{Y}(f)\big) - \mathbf{Y}\big(\mathbf{X}(f)\big) \tag{2.5.23}$$

Indeed the set of vector fields is a vector space with respect the scalar numbers ($\mathbb{R}$ or $\mathbb{C}$, depending on the type of manifold, real or complex), namely we can take linear combinations of the following form:

$$\forall \lambda, \mu \in \mathbb{R} \text{ or } \mathbb{C} \ \ \forall \mathbf{X}, \mathbf{Y} \in \mathbb{D}\mathrm{iff}(\mathcal{M}) : \quad \lambda \mathbf{X} + \mu \mathbf{Y} \in \mathbb{D}\mathrm{iff}(\mathcal{M}) \tag{2.5.24}$$

having defined:

$$[\lambda \mathbf{X} + \mu \mathbf{Y}](f) = \lambda\big[\mathbf{X}(f)\big] + \mu\big[\mathbf{Y}(f)\big], \quad \forall f \in \mathbb{C}^\infty(\mathcal{M}) \tag{2.5.25}$$

Furthermore the operation (2.5.23) is the commutator of two maps and as such it is antisymmetric and satisfies the Jacobi identity.

The Lie algebra of vector fields is named $\mathbb{D}\mathrm{iff}(\mathcal{M})$ since each of its elements can be interpreted as the generator of an infinitesimal diffeomorphism of the manifold onto itself. As we are going to see $\mathbb{D}\mathrm{iff}(\mathcal{M})$ is a Lie algebra of *infinite dimension*, but it can contain finite dimensional subalgebras generated by particular vector fields. The typical example will be the case of the Lie algebra of a Lie group: this is the finite dimensional subalgebra $\mathbb{G} \subset \mathbb{D}\mathrm{iff}(G)$ spanned by those vector fields defined on the Lie group manifold that have an additional property of invariance with respect to either left or right translations (see Chap. 3).

2.5.3 The Cotangent Bundle and Differential Forms

Let us recall that a differential 1-form in the point $p \in \mathscr{M}$ of a manifold $\mathscr{M}$, namely an element $\omega_p \in T^*_p\mathscr{M}$ of the cotangent space over such a point was defined as a real valued linear functional over the tangent space at p, namely

$$\omega_p \in \mathrm{Hom}(T_p\mathscr{M}, \mathbb{R}) \tag{2.5.26}$$

which implies:

$$\forall \mathbf{t}_p \in T_p\mathscr{M} \quad \omega_p : \mathbf{t}_p \mapsto \omega_p(\mathbf{t}_p) \in \mathbb{R} \tag{2.5.27}$$

The expression of ω_p in a coordinate patch around p is:

$$\omega_p = \omega_\mu(p)\, dx^\mu \tag{2.5.28}$$

where $dx^\mu(p)$ are the differentials of the coordinates and $\omega_\mu(p)$ are real numbers. We can glue together all the cotangent spaces and construct the cotangent bundles by stating that a *generic smooth section* of such a bundle is of the form (2.5.28) where $\omega_\mu(p)$ are now smooth functions of the base manifold point p. Clearly if we change coordinate system, an argument completely similar to that employed in the case of the tangent bundle tells us that the coefficients $\omega_\mu(x)$ transform as follows:

$$\omega_\mu\big(x'\big)' = \omega^\nu(x)\frac{\partial x^\nu}{\partial x^{\mu'}} \tag{2.5.29}$$

and (2.5.29) can be taken as a definition of the *cotangent bundle* $T^*\mathscr{M}$, whose sections transform with the Jacobian matrix rather than with the inverse Jacobian matrix as the sections of the tangent bundle do (see (2.5.19)). So we can write the

Definition 2.5.3 A differential 1-form ω on a manifold $\mathscr{M}$ is a section of the cotangent bundle, namely $\omega \in \Gamma(T^*\mathscr{M}, \mathscr{M})$.

This means that a differential 1-form is a map:

$$\omega : \Gamma(T\mathscr{M}, \mathscr{M}) \mapsto \mathbb{C}^\infty(\mathscr{M}) \tag{2.5.30}$$

from the space of vector fields (*i.e.* the sections of the tangent bundle) to smooth functions. Locally we can write:

$$\begin{aligned} \Gamma(T\mathscr{M}, \mathscr{M}) &\ni \omega = \omega_\mu(x)\, dx^\mu \\ \Gamma\big(T^*\mathscr{M}, \mathscr{M}\big) &\ni \mathbf{t} = t^\mu(x)\frac{\partial}{\partial x^\mu} \end{aligned} \tag{2.5.31}$$

and we obtain

$$\omega(\mathbf{t}) = \omega_\mu(x)t^\nu(x)\, dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \omega_\mu(x)t^\mu(x) \tag{2.5.32}$$

using

$$dx^\mu \left(\frac{\partial}{\partial x^\nu} \right) = \delta^\mu_\nu \tag{2.5.33}$$

which is the statement that coordinate differentials and partial derivatives are dual bases for 1-forms and tangent vectors respectively.

Since TM is a vector bundle it is meaningful to consider the addition of its sections, namely the addition of vector fields and also their pointwise multiplication by smooth functions. Taking this into account we see that the map (2.5.30) used to define sections of the cotangent bundle, namely 1-forms is actually an F-linear map. This means the following. Considering any F-linear combination of two vector fields, namely:

$$f_1\mathbf{t}_1 + f_2\mathbf{t}_2, \quad f_1, f_2 \in \mathbb{C}^\infty(\mathscr{M}) \;\; t_1, t_2 \in \Gamma(T\mathscr{M}, \mathscr{M}) \tag{2.5.34}$$

for any 1-form $\omega \in \Gamma(T^*\mathscr{M}, \mathscr{M})$ we have:

$$\omega(f_1\mathbf{t}_1 + f_2\mathbf{t}_2) = f_1(p)\omega(\mathbf{t}_1)(p) + f_2(p)\omega(\mathbf{t}_2)(p) \tag{2.5.35}$$

where $p \in \mathscr{M}$ is any point of the manifold $\mathscr{M}$.

It is now clear that the definition of differential 1-form generalizes the concept of *total differential* of the germ of a smooth function. Indeed in an open neighborhood $U \subset \mathscr{M}$ of a point p we have:

$$\forall f \in \mathbb{C}^\infty_p(\mathscr{M}) \quad df = \partial_\mu f\, dx^\mu \tag{2.5.36}$$

and the value of df at p on any tangent vector $\mathbf{t}_p \in T_p\mathscr{M}$ is defined to be:

$$df_p(\mathbf{t}_p) \equiv \mathbf{t}_p(f) = t^\mu \partial_\mu f \tag{2.5.37}$$

which is the directional derivative of the local function f along $\mathbf{t}_p$ in the point p. If rather than the germ of a function we take a global function $f \in \mathbb{C}^\infty(\mathscr{M})$ we realize that the concept of 1-form generalizes the concept of total differential of such a function. Indeed the total differential df fits into the definition of a 1-form, since for any vector field $\mathbf{t} \in \Gamma(T\mathscr{M}, \mathscr{M})$ we have:

$$df(\mathbf{t}) = t^\mu(x)\partial_\mu f(x) \equiv \mathbf{t}f \in \mathbb{C}^\infty(\mathscr{M}) \tag{2.5.38}$$

A first obvious question is the following. Is any 1-form $\omega = \omega_\mu(x)\, dx^\mu$ the differential of some function? The answer is clearly no and in any coordinate patch there is a simple test to see whether this is the case or not. Indeed, if $\omega^{(1)}_\mu = \partial_\mu f$ for some germ $f \in \mathbb{C}^\infty_p(\mathscr{M})$ then we must have:

$$\frac{1}{2}\left(\partial_\mu \omega^{(1)}_\nu - \partial_\nu \omega^{(1)}_\mu\right) = \frac{1}{2}[\partial_\mu, \partial_\nu] f = 0 \tag{2.5.39}$$

The left hand side of (2.5.39) are the components of what we will name a differential 2-form

$$\omega^{(2)} = \omega^{(2)}_{\mu\nu}\, dx^\mu \wedge dx^\nu \tag{2.5.40}$$

and in particular the 2-form of (2.5.39) will be identified with the exterior differential of the 1-form $\omega^{(1)}$, namely $\omega^{(2)} = d\omega^{(1)}$. In simple words the exterior differential operator d is the generalization on any manifold and to differential forms of any degree of the concept of *curl*, familiar from ordinary tensor calculus in $\mathbb{R}^3$. Forms whose exterior differential vanishes will be named *closed forms*. All these concepts need appropriate explanations that will be provided shortly from now. Yet, already at this intuitive level, we can formulate the next basic question. We saw that, in order to be the total differential of a function, a 1-form must be necessarily closed. Is such a condition also sufficient? In other words are all closed forms the differential of something? Locally the correct answer is yes, but globally it may be no. Indeed in any open neighborhood a closed form can be represented as the differential of another differential form, but the forms that do the job in the various open patches may not glue together nicely into a globally defined one. This problem and its solution constitute an important chapter of geometry, named cohomology. Actually cohomology is a central issue in algebraic topology, the art of characterizing the topological properties of manifolds through appropriate algebraic structures.

2.5.4 *Differential k-Forms*

Next we introduce differential forms of degree k and the exterior differential d. In a later section, after the discussion of homology we show how this relates to the important construction of cohomology. For the time being our approach is simpler and down to earth.

We have seen that the 1-forms at a point $p \in \mathscr{M}$ of a manifold are linear functionals on the tangent space $T_p\mathscr{M}$. First of all we discuss the construction of exterior k-forms on any vector space W defined to be the kth linear antisymmetric functionals on such a space.

2.5.4.1 Exterior Forms

Let W a vector space of finite dimension over the field F (F can either be $\mathbb{R}$ or $\mathbb{C}$ depending on the case). In this section we show how we can construct a sequence of vector spaces $\Lambda_k(W)$ with $k = 0, 1, 2, \dots, n = \dim W$ defined in the following way:

$$\begin{aligned}
\Lambda_0(W) &= F \\
\Lambda_1(W) &= W^\star \\
&\ \vdots \\
\Lambda_k(W) &= \text{vector space of } k\text{-linear antisymmetric functionals over } W
\end{aligned} \tag{2.5.41}$$

The spaces $\Lambda_k(W)$ contain the linear functionals on the kth exterior powers of the vector space W. Such functionals are denoted *exterior forms* of degree k on W.

Let $\phi^{(k)} \in \Lambda_k(W)$ be a k-form. It describes a map:

$$\phi^{(k)} : W \otimes W \otimes \cdots \otimes W \to F \tag{2.5.42}$$

with the following properties:

$$\begin{aligned}
&\text{(i)}\ \ \phi^{(k)}(w_1, w_2, \ldots, w_i, \ldots, w_j, \ldots, w_k) \\
&\qquad = -\phi^{(k)}(w_1, w_2, \ldots, w_j, \ldots, w_i, \ldots, w_k) \\
&\text{(ii)}\ \phi^{(k)}(w_1, w_2, \ldots, \alpha x + \beta y, \ldots, w_k) \\
&\qquad = \alpha\phi^{(k)}(w_1, w_2, \ldots, x, \ldots, w_k) + \beta\phi^{(k)}(w_1, w_2, \ldots, y, \ldots, w_k)
\end{aligned} \tag{2.5.43}$$

where $\alpha, \beta \in F$ and $w_i, x, y \in W$.

The first of properties (2.5.43) guarantees that the map $\phi^{(k)}$ is antisymmetric in any two arguments. The second property states that $\phi^{(k)}$ is linear in each argument.

The sequence of vector spaces $\Lambda_k(W)$:

$$\Lambda(W) \equiv \bigcup_{k=0}^{n} \Lambda_k(W) \tag{2.5.44}$$

can be equipped with an additional operation, named exterior product that to each pair of a k_1 and a k_2 form $(\phi^{(k_1)}, \phi^{(k_2)})$ associates a new $(k_1 + k_2)$-form. Namely we have:

$$\wedge : \Lambda_{k_1} \otimes \Lambda_{k_2} \to \Lambda_{k_1+k_2} \tag{2.5.45}$$

More precisely we set:

$$\phi^{(k_1)} \wedge \phi^{(k_2)} \in \Lambda_{k_1+k_2}(W) \tag{2.5.46}$$

and we write:

$$\begin{aligned}
\phi^{(k_1)} \wedge \phi^{(k_2)}(w_1, w_2, \ldots, w_{k_1+k_2}) = \sum_P (-)^{\delta_P} \frac{1}{(k_1+k_2)!} \big(\phi^{(k_1)}(w_{P(1)}, \ldots, w_{P(k)}) \\
\times \phi^{(k_2)}(w_{P(k_1+1)}, \ldots, w_{P(k_1+k_2)})\big)
\end{aligned} \tag{2.5.47}$$

where P are the permutations of $k_1 + k_2$ objects, namely the elements of the symmetric group $\mathscr{S}_{k_1+k_2}$ and δ_P is the parity of the permutation P ($\delta_P = 0$ if P contains an even number of exchanges with respect to the identity permutation, while $\delta_P = 1$ if such a number is odd).

In order to make this definition clear, consider the explicit example where $k_1 = 2$ and $k_2 = 1$. We have:

$$\phi^{(2)} \wedge \phi^{(1)} = \phi^{(3)} \tag{2.5.48}$$

and we find

$$\begin{aligned}\phi^{(3)}(w_1,w_2,w_3) &= \frac{1}{3!}\big(\phi^{(2)}(w_1,w_2)\phi^{(1)}(w_3) - \phi^{(2)}(w_2,w_1)\phi^{(1)}(w_3)\\ &\quad - \phi^{(2)}(w_1,w_3)\phi^{(1)}(w_2) - \phi^{(2)}(w_31,w_2)\phi^{(1)}(w_1)\\ &\quad + \phi^{(2)}(w_2,w_3)\phi^{(1)}(w_1) + \phi^{(2)}(w_3,w_1)\phi^{(1)}(w_2)\big)\\ &= \frac{1}{3}\big(\phi^{(2)}(w_1,w_2)\phi^{(1)}(w_3) + \phi^{(2)}(w_2,w_3)\phi^{(1)}(w_1)\\ &\quad + \phi^{(2)}(w_3,w_1)\phi^{(1)}(w_2)\big)\end{aligned} \tag{2.5.49}$$

The exterior product we have just defined has the following formal property:

$$\phi^{(k)} \wedge \phi_{k'} = (-)^{kk'}\phi_{k'} \wedge \phi_k \quad \big[\forall\phi^{(k)} \in \Lambda_k(W);\ \forall\phi_{k'} \in \Lambda_{k'}(W)\big] \tag{2.5.50}$$

which can be immediately verified starting from Definition (2.5.47). Indeed, assuming for instance that $k_2 > k_1$, it is sufficient to consider the parity of the permutation:

$$\Pi = \begin{pmatrix} 1, & 2, & \ldots, & k_1, & k_1+1, & \ldots, & k_2, & k_2+1, & \ldots, & k_1+k_2 \\ k_1, & k_1+2, & \ldots, & k_1+k_1, & 2k_1+1, & \ldots, & k_1+k_2, & 1, & \ldots, & k_1 \end{pmatrix} \tag{2.5.51}$$

which is immediately seen to be:

$$\delta_\Pi = k_1 k_2 \bmod 2 \tag{2.5.52}$$

Setting $P = P'\Pi$ (which implies $\delta_P = \delta_{P'} + \delta_\Pi$) we obtain:

$$\begin{aligned}\phi^{(k_2)} \wedge \phi^{(k_1)}(w_1,\ldots,w_{k_1+k_2}) &= \sum_P (-)^{\delta_P}\phi^{(k_2)}(w_{P(1)},\ldots,w_{P(k_2)})\\ &\quad \times \phi^{(k_1)}(w_{P(k_2+1)},\ldots,w_{P(k_1+k_2)})\\ &= \sum_{P'} (-)^{\delta_{P'}+\delta_\Pi}\phi^{(k_2)}(w_{P'\Pi(1)},\ldots,w_{P'\Pi(k_2)})\\ &\quad \times \phi^{(k_1)}(w_{P'\Pi(k_2+1)},\ldots,w_{P'\Pi(k_2+k_1)})\\ &= (-)^{\delta_\Pi}\sum_{P'}(-)^{\delta_{Pi'}}\phi^{(k_2)}(w_{P'(k_1+1)},\ldots,w_{P'(k_1+k_2)})\\ &\quad \times \phi^{(k_1)}(w_{P'(1)},\ldots,w_{P'(k_1)})\\ &= (-)^{\delta_\Pi}\phi^{(k_1)} \wedge \phi^{(k_2)}(w_1,\ldots,w_{k_1+k_2})\end{aligned} \tag{2.5.53}$$

2.5.4.2 Exterior Differential Forms

It follows that on $T_p\mathscr{M}$ we can construct not only the 1-forms but also all the higher degree k-forms. They span the vector space $\Lambda_k(T_p\mathscr{M})$. By gluing together all such vector spaces, as we did in the case of 1-forms, we obtain the vector-bundles of k-forms. More explicitly we can set:

Definition 2.5.4 A differential k-form $\omega^{(k)}$ is a smooth assignment:

$$\omega^{(k)} : p \mapsto \omega_p^{(k)} \in \Lambda_k(T_p\mathscr{M}) \tag{2.5.54}$$

of an exterior k-form on the tangent space at p for each point $p \in \mathscr{M}$ of a manifold.

Let now (U, φ) be a local chart and let $\{dx_p^1, \dots, dx_p^m\}$ be the usual natural basis of the cotangent space $T_p^*\mathscr{M}$. Then in the same local chart the differential form $\omega^{(k)}$ is written as:

$$\omega^{(k)} = \omega_{i_1,\dots,i_k}(x_1, \dots, x_m)\, dx^{i_1} \wedge \cdots \wedge dx^{i_k} \tag{2.5.55}$$

where $\omega_{i_1,\dots,i_k}(x_1, \dots, x_m) \in \mathbb{C}^\infty(U)$ are smooth functions on the open neighborhood U, completely antisymmetric in the indices $i_1, \dots, i_k$.

At this point it is obvious that the operation of exterior product, defined on exterior forms, can be extended to *exterior differential forms*. In particular, if $\omega^{(k)}$ and $\omega^{(k')}$ are a k-form and a k'-form, respectively, then $\omega^{(k)} \wedge \omega^{(k')}$ is a $(k + k')$-form. As a consequence of (2.5.50) we have:

$$\omega^{(k)} \wedge \omega^{(k')} = (-)^{kk'} \omega^{(k')} \wedge \omega^{(k)} \tag{2.5.56}$$

and in local coordinates we find:

$$\omega^{(k)} \wedge \omega^{(k')} = \omega^{(k)}_{[i_1\dots i_k}(x_1, \dots, x_m)\omega^{(k)}_{i_{k+1}\dots i_{k+k'}]}\, dx^1 \wedge \cdots \wedge dx^{k+k'} \tag{2.5.57}$$

where $[\dots]$ denotes the complete antisymmetrization on the indices.

Let $\mathscr{A}_0(\mathscr{M}) = \mathbb{C}^\infty(\mathscr{M})$ and let $\mathscr{A}_k(\mathscr{M}) = \mathbb{C}^\infty(\mathscr{M})$ be the $\mathbb{C}^\infty(\mathscr{M})$-module of differential k-forms. To justify the naming module, observe that we can construct the product of a smooth function $f \in \mathbb{C}^\infty(\mathscr{M})$ with a differential form $\omega^{(k)}$ setting:

$$\left[f\omega^{(k)}\right](\mathbf{Z}_1, \dots, \mathbf{Z}_k) = f \cdot \omega^{(k)}(\mathbf{Z}_1, \dots, \mathbf{Z}_k) \tag{2.5.58}$$

for each k-tuplet of vector fields $\mathbf{Z}_1, \dots, \mathbf{Z}_k \in \Gamma(T\mathscr{M}, \mathscr{M})$

Furthermore let

$$\mathscr{A}(\mathscr{M}) = \bigoplus_{k=0}^{m} \mathscr{A}_k(\mathscr{M}) \quad \text{where } m = \dim \mathscr{M} \tag{2.5.59}$$

Then $\mathscr{A}$ is an algebra over $\mathbb{C}^\infty(\mathscr{M})$ with respect to the exterior wedge product $\bigwedge$.

To introduce the exterior differential d we proceed as follows. Let $f \in \mathbb{C}^\infty(\mathscr{M})$ be a smooth function: for each vector field $\mathbf{Z} \in \mathbb{D}\mathrm{iff}(\mathscr{M})$, we have $\mathbf{Z}(f) \in \mathbb{C}^\infty(\mathscr{M})$ and therefore there is a unique differential 1-form, noted df such that $df(\mathbf{Z}) =$

$\mathbf{Z}(f)$. This differential form is named the total differential of the function f. In a local chart U with local coordinates $x^1, \ldots, x^m$ we have:

$$df = \frac{\partial f}{\partial x^j} dx^j \tag{2.5.60}$$

More generally we can see that there exists an endomorphism d, $(\omega \mapsto d\omega)$ of $\mathscr{A}(\mathscr{M})$ onto itself with the following properties:

$$\begin{aligned}
&\text{(i) } \forall \omega \in \mathscr{A}_k(\mathscr{M}) \qquad d\omega \in \mathscr{A}_{k+1}(\mathscr{M}) \\
&\text{(ii) } \forall \omega \in \mathscr{A}(\mathscr{M}) \qquad d\, d\omega = 0 \\
&\text{(iii) } \forall \omega^k \in \mathscr{A}_k(\mathscr{M}) \qquad \forall \omega^{k'} \in \mathscr{A}_{k'}(\mathscr{M}) \\
&\quad d(\omega^{(k)} \wedge \omega^{(k')}) = d\omega^{(k)} \wedge \omega^{(k')} + (-1)^k \omega^{(k)} \wedge d\omega^{(k')} \\
&\text{(iv) if } f \in \mathscr{A}_0(\mathscr{M}) \qquad df = \text{total differential}
\end{aligned} \tag{2.5.61}$$

In each local coordinate patch the above intrinsic definition of the exterior differential leads to the following explicit representation:

$$d\omega^{(k)} = \partial_{[i_1} \omega_{i_2 \ldots i_{k+1}]} dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} \tag{2.5.62}$$

As already stressed the exterior differential is the generalization of the concept of curl, well known in elementary vector calculus.

In the next section we introduce the notions of homotopy, homology and cohomology that are crucial to understand the global properties of manifolds and Lie groups and will also play an important role in formulating supergravity.

2.6 Homotopy, Homology and Cohomology

Differential 1-forms can be integrated along differentiable paths on manifolds. The higher differential p-forms, to be introduced shortly from now, can be integrated on p-dimensional submanifolds. An appropriate discussion of such integrals and of their properties requires the fundamental concepts of algebraic topology, namely *homotopy* and *homology*. Also the global properties of Lie groups and their many-to-one relation with Lie algebras can be understood only in terms of homotopy. For this reason we devote the present section to an introductory discussion of homotopy, homology and of its dual, cohomology.

The kind of problems we are going to consider can be intuitively grasped if we consider Fig. 2.20, displaying a closed two-dimensional surface with two handles (actually an oriented, closed Riemann surface of genus $g = 2$) on which we have drawn several different closed 1-dimensional paths $\gamma_1, \ldots, \gamma_6$.

Consider first the path γ_5. It is an intuitive fact that γ_5 can be continuously deformed to just a point on the surface. Paths with such a property are named *homotopically trivial* or *homotopic to zero*. It is also an intuitive fact that neither γ_2, nor γ_3, nor γ_1, nor γ_4 are homotopically trivial. Paths of such a type are *homotopically*

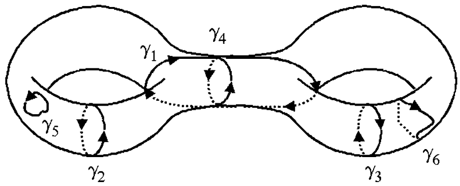


Fig. 2.20 A closed surface with two handles marked by several different closed 1-dimensional paths

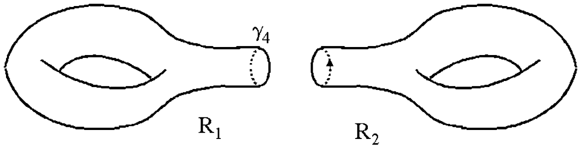


Fig. 2.21 When we cut a surface along a path that is a boundary, namely it is homologically trivial, the surface splits into two separate parts

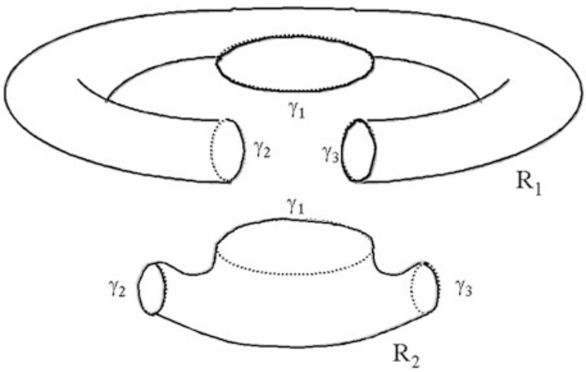


Fig. 2.22 The sum of the three paths $\gamma_1, \gamma_2, \gamma_3$ is homologically trivial, namely $\gamma_2 + \gamma_3$ is homologous to $-\gamma_1$

non-trivial. Furthermore we say that two paths are homotopic if one can be continuously deformed into the other. This is for instance the case of γ_6 which is clearly homotopic to γ_3.

Let us now consider the difference between path γ_4 and path γ_1 from another viewpoint. Imagine the result of cutting the surface along the path γ_4. After the cut the surface splits into two separate parts, R_1 and R_2 as shown in Fig. 2.21. Such a splitting does not occur if we cut the original surface along the path γ_1. The reason for this different behavior resides in this. The path γ_4 is the boundary of a region on the surface (the region R_1 or, equivalently its complement R_2) while γ_1 is not the boundary of any region. A similar statement is true for the paths γ_2 or γ_3. We say that γ_4 is *homologically trivial* while γ_1, γ_2, γ_3 are *homologically non-trivial*.

Next let us observe that if we simultaneously cut the original surface along γ_1, γ_2, γ_3 the surface splits once again into two separate parts as shown in Fig. 2.22.

This is due to the fact that the sum of the three paths is the boundary of a region: either R_1 or R_2 of Fig. 2.22. In this case we say that $\gamma_2 + \gamma_3$ is *homologous* to $-\gamma_1$, since the difference $\gamma_2 + \gamma_3 - (-\gamma_3)$ is a boundary.

In order to give a rigorous formulation to these intuitive concepts,which can be extended also to higher dimensional submanifolds of any manifold we proceed as follows.

2.6.1 Homotopy

Let us come back to Definition 2.3.1 of a curve (or path) in a manifold and slightly generalize it.

Definition 2.6.1 Let $[a, b]$ be a closed interval of the real line $\mathbb{R}$ parameterized by the parameter t and subdivide it into a finite number of closed, partial intervals:

$$[a, t_1], [t_1, t_2], \ldots, [t_{n-1}, t_n], [t_n, b] \tag{2.6.1}$$

We name *piece-wise differentiable path* a continuous map:

$$\gamma : [a, b] \to \mathcal{M} \tag{2.6.2}$$

of the interval $[a, b]$ into a differentiable manifold $\mathcal{M}$ such that there exists a splitting of $[a, b]$ into a finite set of closed subintervals as in (2.6.1) with the property that on each of these intervals the map γ is not only continuous but also infinitely differentiable.

Since we have parametric invariance we can always rescale the interval $[a, b]$ and reduce it to be

$$[0, 1] \equiv I \tag{2.6.3}$$

Let

$$\begin{aligned} \sigma &: I \to \mathcal{M} \\ \tau &: I \to \mathcal{M} \end{aligned} \tag{2.6.4}$$

be two piece-wise differentiable paths with coinciding extrema, namely such that (see Fig. 2.23):

$$\begin{aligned} \sigma(0) &= \tau(0) = x_0 \in \mathcal{M} \\ \sigma(1) &= \tau(1) = x_1 \in \mathcal{M} \end{aligned} \tag{2.6.5}$$

Definition 2.6.2 We say that σ is homotopic to τ and we write $\sigma \simeq \tau$ if there exists a continuous map:

$$F : I \times I \to \mathcal{M} \tag{2.6.6}$$

such that:

$$\begin{aligned} F(s, 0) &= \sigma(s) \quad && \forall s \in I \\ F(s, 1) &= \tau(s) \quad && \forall s \in I \\ F(0, t) &= \ \ x_0 \quad && \forall t \in I \\ F(1, t) &= \ \ x_1 \quad && \forall t \in I \end{aligned} \tag{2.6.7}$$

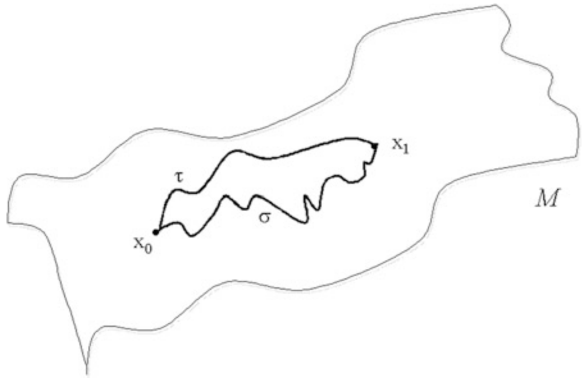


Fig. 2.23 Two paths with coinciding extrema

In particular if σ is a closed path, namely a loop at x_0, i.e. if $x_0 = x_1$ and if τ homotopic to σ is the *constant loop* that is

$$\forall s \in I : \quad \tau(s) = x_0 \tag{2.6.8}$$

then we say that σ is *homotopically trivial* and that it can be contracted to a point.

It is quite obvious that the homotopy relation $\sigma \simeq \tau$ is an equivalence relation. Hence we shall consider the homotopy classes $[\sigma]$ of paths from x_0 to x_1.

Next we can define a binary product operation on the space of paths in the following way. If σ is a path from x_0 to x_1 and τ is a path from x_1 to x_2 we can define a path from x_0 to x_2 traveling first along σ and then along τ. More precisely we set:

$$\sigma\tau(t) = \begin{cases} \sigma(2t) & 0 \le t \le \frac{1}{2} \\ \tau(2t-1) & \frac{1}{2} \le t \le 1 \end{cases} \tag{2.6.9}$$

What we can immediately verify from this definition is that if $\sigma \simeq \sigma'$ and $\tau \simeq \tau'$ then $\sigma\tau \simeq \sigma'\tau'$. The proof is immediate and it is left to the reader. Hence without any ambiguity we can multiply the equivalence class of σ with the equivalence class of τ always assuming that the final point of σ coincides with the initial point of τ. Relying on these definitions we have a theorem which is very easy to prove but has an outstanding relevance:

Theorem 2.6.1 *Let $\pi_1(\mathscr{M}, x_0)$ be the set of homotopy classes of loops in the manifold $\mathscr{M}$ with base in the point $x_0 \in \mathscr{M}$. If the product law of paths is defined as we just explained above, then with respect to this operation $\pi_1(\mathscr{M}, x_0)$ is a group whose identity element is provided by the homotopy class of the constant loop at x_0 and the inverse of the homotopy class $[\sigma]$ is the homotopy class of the loop σ^{-1} defined by*:

$$\sigma^{-1}(t) = \sigma(1-t) \quad 0 \le t \le 1 \tag{2.6.10}$$

(*In other words σ^{-1} is the same path followed backward.*)

Proof Clearly the composition of a loop σ with the constant loop (from now on denoted as x_0) yields σ. Hence x_0 is effectively the identity element of the group. We still have to show that $\sigma\sigma^{-1} \simeq x_0$. The explicit realization of the required homotopy is provided by the following function:

$$F(s,t) = \begin{cases} \sigma(2s) & 0 \leq 2s \leq t \\ \sigma(t) & t \leq 2s \leq 2-t \\ \sigma^{-1}(2s-1) & 2-t \leq 2s \leq 2 \end{cases} \tag{2.6.11}$$

Let us observe that having defined F as above we have:

$$\begin{aligned} F(s,0) &= \{\sigma(0) = x_0 \quad \forall s \in I \\ F(s,1) &= \begin{cases} \sigma(2s) & 0 \leq s\frac{1}{2} \\ \sigma^{-1}(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} \end{aligned} \tag{2.6.12}$$

and furthermore:

$$\begin{aligned} F(0,t) &= \{\sigma(0) = x_0 \quad \forall t \in I \\ F(1,t) &= \left\{\sigma^{-1}(1) = x_0 \quad \forall t \in I\right. \end{aligned} \tag{2.6.13}$$

Therefore it is sufficient to check that $F(s,t)$ is continuous. Dividing the square $[0, 1] \times [0, 1]$ into three triangles as in Fig. 2.24 we see that $F(s,t)$ is continuous in each of the triangles and that is consistently glued on the sides of the triangles. Hence F as defined in (2.6.11) is continuous. This concludes the proof of the theorem. □

Theorem 2.6.2 *Let α be a path from x_0 to x_1. Then*

$$[\sigma] \xrightarrow{\alpha} \left[\alpha^{-1}\sigma\alpha\right] \tag{2.6.14}$$

is an isomorphism of $\pi_1(\mathscr{M}, x_0)$ into $\pi_1(\mathscr{M}, x_1)$.

Proof Indeed, since

$$[\sigma\tau] \longrightarrow \left[\alpha^{-1}\sigma\alpha\right]\left[\alpha^{-1}\tau\alpha\right] = \left[\alpha^{-1}\sigma\tau\alpha\right] \tag{2.6.15}$$

we see that $\xrightarrow{\alpha}$ is a homomorphism. Since also the inverse $\xrightarrow{\alpha^{-1}}$ does exist, then the homomorphism is actually an isomorphism. □

From this theorem it follows that in a arc-wise connected manifold, namely in a manifold where every point is connected to any other by at least one piece-wise differentiable path, the group $\pi_1(\mathscr{M}, x_0)$ is independent from the choice of the base point x_0 and we can call it simply $\pi_1(\mathscr{M})$. The group $\pi_1(\mathscr{M})$ is named the first homotopy group of the manifold or simply the *fundamental group* of $\mathscr{M}$.

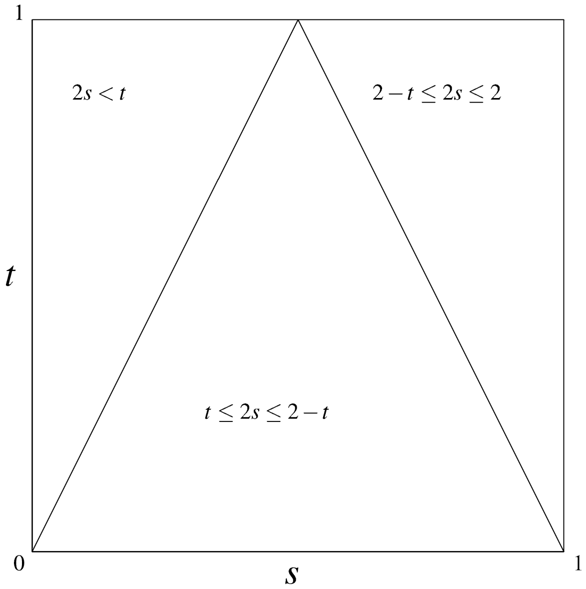


Fig. 2.24 The continuous map the realizes the homotopy between the constant loop and the product of any loop with its own inverse

Definition 2.6.3 A differentiable manifold $\mathscr{M}$ which is arc-wise connected is named *simply connected* if its fundamental group $\pi_1(\mathscr{M})$ is the trivial group composed only by the identity element.

$$\pi_1(\mathscr{M}) = \mathrm{id} \quad \Leftrightarrow \quad \mathscr{M} = \text{simply connected} \tag{2.6.16}$$

2.6.2 Homology

The notion of homotopy led us to introduce an internal composition group for paths, the fundamental group $\pi_1(\mathscr{M})$, whose structure is a topological invariant of the manifold $\mathscr{M}$, since it does not change under continuous deformations of the latter. For this group we have used a multiplicative notations since nothing guarantees a priori that it should be Abelian. Generically the fundamental homotopy group of a manifold is non-Abelian. As mentioned above there are higher homotopy groups $\pi_n(\mathscr{M})$ whose elements are the homotopy classes of $\mathbb{S}^n$ spheres drawn on the manifold.

In this section we turn our attention to another series of groups that also codify topological properties of the manifold and are on the contrary all Abelian. These are the homology groups:

$$H_k(\mathscr{M}); \quad k = 0, 1, 2, \ldots, \dim(\mathscr{M}) \tag{2.6.17}$$

We can grasp the notion of *homology* if we persuade ourselves that it makes sense to consider linear combinations of submanifolds or regions of dimension p of

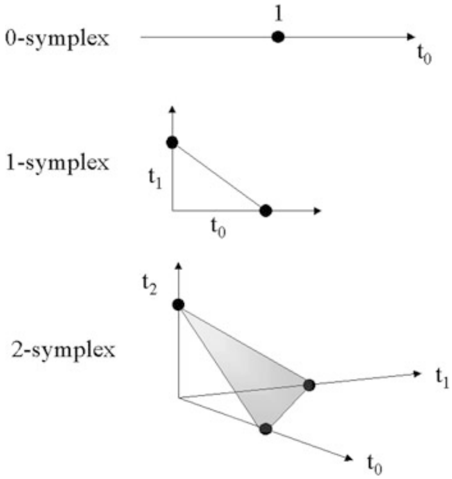


Fig. 2.25 The standard p-simplexes for $p = 0, 1, 2$

a manifold $\mathscr{M}$, with coefficients in a ring $\mathscr{R}$ that can be either $\mathbb{Z}$, or $\mathbb{R}$ or, sometimes $\mathbb{Z}_n$. The reason is that the submanifolds of dimension p are just fit to integrate p-differential forms over them. This fact allows to give a meaning to an expression of the following form:

$$\mathscr{C}^{(p)} = m_1 S_1^{(p)} + m_2 S_2^{(p)} + \cdots + m_k S_k^{(p)} \tag{2.6.18}$$

where $S_i^{(p)} \subset \mathscr{M}$ are suitable p-dimensional submanifolds of the manifold $\mathscr{M}$, later on called *simplexes*, and $m_i \in \mathscr{R}$ are elements of the chosen ring of coefficients. What we systematically do is the following. For each differential p-form $\omega^{(p)} \in \Lambda_p(\mathscr{M})$ we set:

$$\int_{\mathscr{C}^{(p)}} \omega^{(p)} = \int_{m_1 S_1^{(p)} + m_2 S_2^{(p)} + \cdots + m_k S_k^{(p)} \mathscr{C}^{(p)}} \omega^{(p)} = \sum_{i=1}^{k} m_i \int_{S_i^{(p)}} \omega^{(p)} \tag{2.6.19}$$

and in this we define the integral of $\omega^{(p)}$ on the region $\mathscr{C}^{(p)}$. Next let us give the precise definition of the p-simplexes of which we want to take linear combinations.

Definition 2.6.4 Let us consider the Euclidian space $\mathbb{R}^{p+1}$. The standard p-simplex Δ^p is the set of all points $\{t_0, t_1, \dots, t_p\} \in \mathbb{R}^{p+1}$ such that the following conditions are satisfied:

$$t_i \geq 0; \qquad t_0 + t_1 + \cdots + t_p = 1 \tag{2.6.20}$$

It is just easy to see that the standard 0-simplex is a point, namely $t_0 = 1$, the standard 1-simplex is a segment of line, the standard 2-simplex is a triangle, the standard 3-simplex is a tetrahedron and so on (see Fig. 2.25).

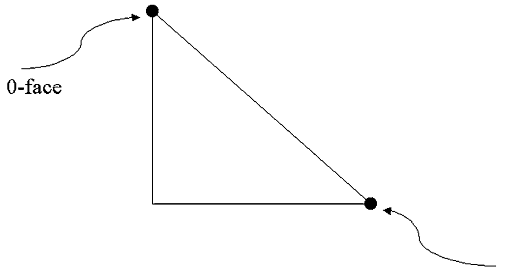


Fig. 2.26 The faces of the standard 1-simplex

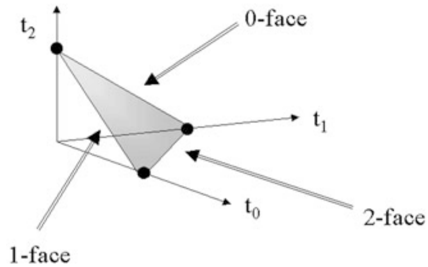


Fig. 2.27 The faces of the standard 2-simplex

Let us now consider the standard $(p-1)$-simplex $\Delta^{(p-1)}$ and let us observe that there are $(p+1)$ canonical maps ϕ_i that map $\Delta^{(p-1)}$ into Δ^p:

$$\phi_i : \Delta^{(p-1)} \mapsto \Delta^p \tag{2.6.21}$$

These maps are defined as follows:

$$\phi_i^{(p)}(t_0, \ldots, t_{i-1}, t_{i+1}, \ldots, t_p) = (t_0, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_p) \tag{2.6.22}$$

Definition 2.6.5 The $p+1$ standard simplexes Δ^{p-1} immersed in the standard p-simplex Δ^p by means of the $p+1$ maps of (2.6.22) are named the *faces* of Δ^p and the index i enumerates them. Hence the map $\phi_i^{(p)}$ yields, as a result, the *ith face* of the standard p-simplex.

For instance the two faces of the standard 1-simplex are the two points $(t_0 = 0, t_1 = 1)$ and $(t_0 = 1, t_1 = 0)$ as shown in Fig. 2.26.

Similarly the three segments $(t_0 = 0, t_1 = t, t_2 = 1-t)$, $(t_0 = t, t_1 = 0, t_2 = 1-t)$ and $(t_0 = t, t_1 = 1-t, t_2 = 0)$ are the three faces of the standard 2-simplex (see Fig. 2.27).

Definition 2.6.6 Let $\mathscr{M}$ be a differentiable manifold of dimension m. A continuous map:

$$\sigma^{(p)} : \Delta^{(p)} \mapsto \mathscr{M} \tag{2.6.23}$$

of the standard p-simplex into the manifold is named a *singular p-simplex* or simply a *simplex* of $\mathscr{M}$.

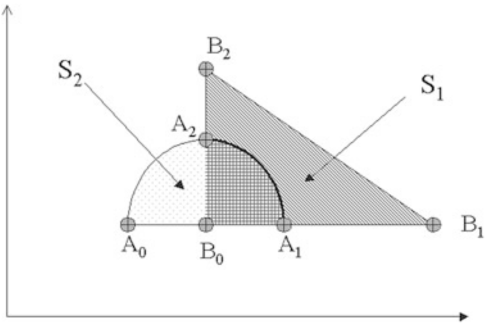


Fig. 2.28 $S_1^{(2)}$ and $S_2^{(2)}$ are two distinct 2-simplexes, namely two triangles with vertices respectively given by (A_0, A_1, A_2) and B_0, B_1, B_2. The 2-simplex $S_3^{(2)}$ with vertices B_0, A_1, A_2 is the intersection of the other two $S_3^{(2)} = S_1^{(2)} \bigcap S_2^{(2)}$

Clearly a 1-simplex is a continuous path in $\mathscr{M}$, a 2-simplex is a portion of surface immersed M and so on. The ith face of the simplex $\sigma^{(p)}$ is given by the $(p-1)$-simplex obtained by composing $\sigma^{(p)}$ with ϕ_i:

$$\sigma^{(p)} \circ \phi_i : \Delta^{(p-1)} \mapsto \mathscr{M} \tag{2.6.24}$$

Let $\mathscr{R}$ be a commutative ring.

Definition 2.6.7 Let $\mathscr{M}$ be a manifold of dimension m. For each $0 \leq n \leq m$ the group of n-chains with coefficients in $\mathscr{R}$, named $C(\mathscr{M}, \mathscr{R})$, is defined as the *free* $\mathscr{R}$*-module* having a generator for each n-simplex in $\mathscr{M}$.

In simple words Definition 2.6.7 states that $C_p(\mathscr{M}, \mathscr{R})$ is the set of all possible linear combination of p-simplexes with coefficients in $\mathscr{R}$:

$$\mathscr{C}^{(p)} = m_1 S_1^{(p)} + m_2 S_2^{(p)} + \cdots + m_k S_k^{(p)} \tag{2.6.25}$$

where $m_i \in \mathscr{R}$. The elements of $C_p(\mathscr{M}, \mathscr{R})$ are named *p-chains*.

The concept of p-chains gives a rigorous meaning to the intuitive idea that any p-dimensional region of a manifold can be constructed by gluing together a certain number of simplexes. For instance a path γ can be constructed gluing together a finite number of segments (better their homeomorphic images). In the case $p = 2$, the construction of a two-dimensional region by means of 2-simplexes corresponds to a triangulation of a surface.

As an example consider the case where the manifold we deal with is just the complex plane $\mathscr{M} = \mathbb{C}$ and let us focus on the 2-simplexes drawn in Fig. 2.28.

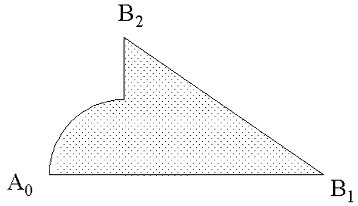
The chain:

$$\mathscr{C}^{(2)} = S_1^{(2)} + S_2^{(2)} \tag{2.6.26}$$

denotes the region of the complex plane depicted in Fig. 2.29, with the proviso that when we compute the integral of any 2-form on $\mathscr{C}^{(2)}$ the contribution from the simplex $S_3^{(2)} = S_1^{(2)} \bigcap S_2^{(2)}$ (the shadowed area in Fig. 2.29) has to be counted twice since it belongs both to $S_1^{(2)}$ and to $S_2^{(2)}$.

Relying on these notions we can introduce the boundary operator.

Fig. 2.29 Geometrically the chain $S_1^{(2)} + S_2^{(2)}$ is the union of the two simplexes $S_1^{(2)} \bigcup S_2^{(2)}$



Definition 2.6.8 The boundary operator ∂ is the map:

$$\partial : C_n(\mathscr{M}, \mathscr{R}) \to C_{n-1}(\mathscr{M}, \mathscr{R}) \tag{2.6.27}$$

defined by the following properties:

1. *$\mathscr{R}$-linearity*

$$\begin{aligned} &\forall \mathscr{C}_1^{(p)}, \mathscr{C}_2^{(p)} \in C_p(\mathscr{M}, \mathscr{R}), \quad \forall m_1, m_2 \in \mathscr{R} \\ &\partial\left(m_1 \mathscr{C}_1^{(p)} + m_2 \mathscr{C}_2^{(p)}\right) = m_1 \partial \mathscr{C}_1^{(p)} + m_2 \partial \mathscr{C}_2^{(p)} \end{aligned} \tag{2.6.28}$$

2. *Action on the simplexes*

$$\begin{aligned} \partial\sigma &\equiv \sigma \circ \phi_0 - \sigma \circ \phi_1 + \sigma \circ \phi_1 - \cdots \\ &= \sum_{i=1}^{p} (-)^i \sigma \circ \phi_i \end{aligned} \tag{2.6.29}$$

The image of a chain $\mathscr{C}$ through ∂, namely $\partial\mathscr{C}$, is called the *boundary* of the chain.

As an exercise we can compute the boundary of the 2-chain $\mathscr{C}^{(2)} = \mathscr{S}_1^{(2)} + \mathscr{S}_2^{(2)}$ of Fig. 2.28, with the understanding that the relevant ring is, in this case $\mathbb{Z}$. We have:

$$\begin{aligned} \partial C^{(2)} &= \partial S_1^{(2)} + \partial S_2^{(2)} \\ &= \overrightarrow{A_1 A_2} - \overrightarrow{A_0 A_2} + \overrightarrow{A_0 A_1} + \overrightarrow{B_1 B_2} - \overrightarrow{B_0 B_2} + \overrightarrow{B_1 B_2} \end{aligned} \tag{2.6.30}$$

where $\overrightarrow{A_1 A_2}, \ldots$ denote the oriented segments from A_1 to A_2 and so on. As one sees the change in sign is interpreted as the change of orientation (which is the correct interpretation if one thinks of the chain and of its boundary as the support of an integral). With this convention the 1-chain:

$$\overrightarrow{A_1 A_2} - \overrightarrow{A_0 A_2} + \overrightarrow{A_0 A_1} = \overrightarrow{A_1 A_2} + \overrightarrow{A_2 A_0} + \overrightarrow{A_0 A_1} \tag{2.6.31}$$

is just the oriented boundary of the $S_1^{(2)}$-simplex as shown in Fig. 2.30.

Theorem 2.6.3 *The boundary operator ∂ is nilpotent, namely it is true that*:

$$\partial^2 \equiv \partial \circ \partial = 0 \tag{2.6.32}$$

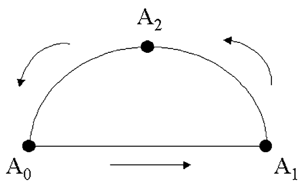


Fig. 2.30 The oriented boundary of the $S^{(2)}$ symplex

Proof It is sufficient to observe that, as a consequence of their own definition, the maps ϕ_i defined in (2.6.22) have the following property:

$$\phi_i^{(p)} \circ \phi_j^{(p-1)} = \phi_j^{(p)} \circ \phi_{i-1}^{(p-1)} \tag{2.6.33}$$

Then, for the p-simplex σ we have:

$$\begin{aligned}
\partial\partial\sigma &= \sum_{i=0}^{p} (-)^i \delta[\sigma \circ \phi_i] \\
&= \sum_{i=0}^{p} \sum_{j=0}^{p-1} (-)^i (-)^j \sigma \circ \left(\phi_i^{(p)} \circ \phi_j^{(p-1)}\right) \\
&= \sum_{j<i=1}^{p} (-)^{i+j} \sigma \circ \left(\phi_j^{(p)} \circ \phi_{i-1}^{(p-1)}\right) + \sum_{0=i\le j}^{p-1} \sigma\left(\phi_i^{(p)} \circ \phi_j^{(p-1)}\right)
\end{aligned} \tag{2.6.34}$$

We can verify that everything in the last line of (2.6.34) cancels identically and this proves the theorem. □

As an illustration we can calculate $\partial\partial S_1^{(2)}$ for the 2-simplex $S_1^{(2)}$ described in Fig. 2.28. We obtain:

$$\partial\partial S_1^{(2)} = A_2 - A_1 - A_2 + A_0 + A_1 - A_0 = 0 \tag{2.6.35}$$

The nilpotency of the boundary operator ∂ that acts on the *chains* is the counterpart of the nilpotency of the exterior derivative d that acts on differential forms as explained in Sect. 2.5.4. Consider Fig. 2.31. As one sees the sequence of the vector spaces C_m of m-chains can be put into correspondence with the sequence of vector spaces Λ_m of differential m-forms.

The operator:

$$\partial : C_k \to C_{k-1} \tag{2.6.36}$$

makes you to travel on the sequence from left to right, while the exterior derivative operator:

$$d : \Lambda_k \to \Lambda_{k+1} \tag{2.6.37}$$

causes you to travel along the same sequence in the opposite direction from right to left. Both ∂ and d are nilpotent maps.

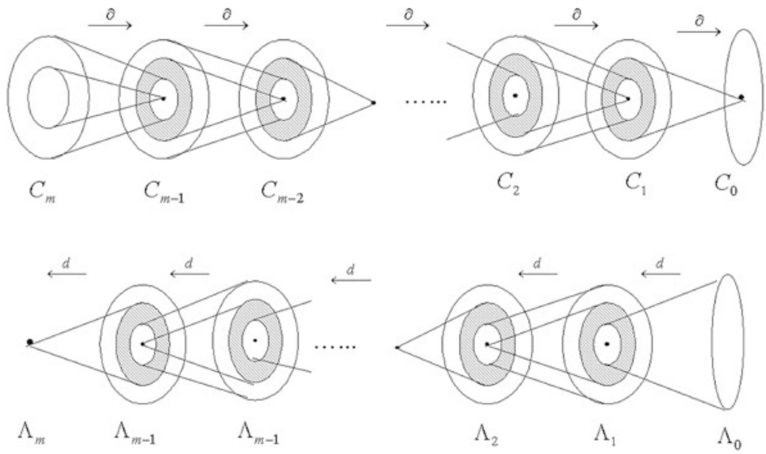


Fig. 2.31 Homology versus cohomology groups

2.6.3 Homology and Cohomology Groups: General Construction

Let $\pi : X \to Y$ be a linear map between vector spaces. We define *kernel* of π and we denote $\ker \pi$ the subspace of X whose elements have the property of being mapped into $0 \in Y$ by π:

$$\ker \pi = \left\{ x \in X / \pi(x) = 0 \in Y \right\} \tag{2.6.38}$$

We call image of π and we denote $\operatorname{Im} \pi$ the subspace of Y whose elements have the property that they are the image through π of some element of X:

$$\operatorname{Im} \pi = \left\{ y \in Y / \exists x \in X / \pi(x) = y \right\} \tag{2.6.39}$$

A nilpotent operator that acts on a sequence of vector spaces X_i defines a sequence of linear maps π_i:

$$X_1 \xrightarrow{\pi_1} X_2 \xrightarrow{\pi_2} X_3 \longrightarrow \cdots \longrightarrow X_i \xrightarrow{\pi_i} X_{i+1} \tag{2.6.40}$$

that have the following property:

$$\operatorname{Im} \pi_i \subset \ker \pi_{i+1} \tag{2.6.41}$$

The inclusion of $\operatorname{Im} \pi_i$ in $\ker \pi_{i+1}$ is what has been pictorially described in Fig. 2.31 and applies both to the boundary and to the exterior derivative operator. This situation suggests the following terminology:

Definition 2.6.9 In every space $C_k(\mathscr{M}, \mathbb{R})$ we name *cycles* the elements of $\ker \partial$, namely the chains $\mathscr{C}$, whose boundary vanishes $\partial \mathscr{C} = 0$. Similarly in every space $\Lambda_k(\mathscr{M})$ we name closed forms or *cocycles* the elements of $\ker d$, namely the differential forms ω such that $d\omega = 0$.

At the same time:

Definition 2.6.10 In every space $C_k(\mathscr{M}, R)$ we name *boundaries* all k-chains that are the boundary of a $k+1$-chain:

$$\mathscr{C}^{(k)} = \text{boundary} \quad \Leftrightarrow \quad \exists \mathscr{C}^{(k)} = \partial \mathscr{C}^{(k+1)} \tag{2.6.42}$$

Similarly in every space $\Lambda_k(\mathscr{M})$ we name exact forms or *coboundaries* all differential forms $\omega^{(k)}$ such that they can be written as the exterior derivative of a $(k-1)$-form: $\omega^{(k)} = d\omega^{(k-1)}$.

Clearly (2.6.41) can be translated by saying that every boundary is a cycle and every coboundary is a cocycle. The reverse statement, however, is not true in general. There are cycles that are not boundaries and there are cocycles that are not coboundaries.

The concept of homology (or cohomology) previously discussed in an intuitive way can be formalized in the following way.

Definition 2.6.11 Consider the k-cycles. We say that two cycles $C_1^{(k)}$ and $C_2^{(k)}$ are *homologous* and we write $C_1^{(k)} \sim C_2^{(k)}$ if their difference is a boundary:

$$C_1^{(k)} \sim C_2^{(k)} \quad \Rightarrow \quad \exists C_3^{(k+1)} / C_1^{(k)} - C_2^{(k)} = \partial C_3^{(k+1)} \tag{2.6.43}$$

Clearly homology is an equivalence relation since:

$$\left.\begin{array}{l} C_1^{(k)} - C_2^{(k)} = \partial C_a^{(k+1)} \\ C_2^{(k)} - C_3^{(k)} = \partial C_b^{(k+1)} \end{array}\right\} \quad \Rightarrow \quad C_1^{(k)} - C_3^{(k)} = \partial \big[C_a^{(k+1)} + C_b^{(k+1)} \big] \tag{2.6.44}$$

Definition 2.6.12 We name *kth homology group* and we denote $H_k(\mathscr{M}, \mathbb{R})$ the group of equivalence classes of the kth cycles with respect to the k-boundaries.

Similarly we define *kth cohomology group* and we denote $H^k(\mathscr{M}, \mathbb{R})$ the group of equivalence classes of the k-cocycles with respect to the kth coboundaries. Indeed we say that two closed forms ω and ω' are cohomologous if their difference is an exact form: $\omega \sim \omega' \Rightarrow \exists \phi / \omega - \omega' = d\phi$.

More generally when we have a sequence of vector spaces X_i as in (2.6.40) and a sequence of linear maps π_i satisfying (2.6.41) we define the *cohomology groups* relative to the operator π as:

$$H^i_{(\pi)} \equiv \frac{\ker \pi_i}{\operatorname{Im} \pi_{i-1}} \tag{2.6.45}$$

The relation existing between homology and cohomology is fully contained in the following formula which generalizes to an arbitrary smooth manifold and to differential forms of any degree the familiar Gauss lemma or Stokes lemma:

$$\int_{\partial \mathscr{C}^{(k+1)}} \omega^{(k)} = \int_{\mathscr{C}^{(k+1)}} d\omega^{(k)} \tag{2.6.46}$$

Equation (2.6.46), whose general proof we omit, implies that in the case $\mathscr{C}^{(k)}$ is a cycle we have:

$$\int_{\mathscr{C}^{(k)}} \left[\omega^{(k)} + d\phi^{(k-1)}\right] = \int_{\mathscr{C}^{(k)}} \omega^{(k)} \tag{2.6.47}$$

namely the integral of a closed differential form along a cycle depends only on the cohomology class and not on the choice of the representative. Similarly if $\omega^{(k)}$ is a closed form:

$$\int_{\mathscr{C}^{(k)}+\partial \mathscr{C}^{(k+1)}} \omega^{(k)} = \int_{\mathscr{C}^{(k)}} \omega^{(k)} \tag{2.6.48}$$

namely the integral of a cocycle along a cycle depends on the homology class of the class and not on the choice of the representative inside the class.

2.6.4 Relation Between Homotopy and Homology

The relation between homotopy and homology groups of a manifold is provided by a fundamental theorem of algebraic geometry that we state without proof:

Theorem 2.6.4 *Let $\mathscr{M}$ be a smooth manifold. Then there exists a homomorphism*:

$$\chi : \pi_1(\mathscr{M}) \to H_1(\mathscr{M}, \mathbb{Z}) \tag{2.6.49}$$

that sends the homotopy class of each loop γ into the 1-simplex γ. If $\mathscr{M}$ is arc-wise connected, then the map χ is surjective and the kernel of χ is the subgroup of commutators in $\pi_1(\mathscr{M})$.

We recall that the subgroup of commutators of a discrete group G is the group G' generated by all elements of the form $x^{-1}y^{-1}xy$ for some $x, y \in G$.

From this theorem we have two consequences:

Corollary 2.6.1 *If $\pi_1(\mathscr{M})$ is Abelian, then $H_1(\mathscr{M}) \simeq \pi_1(\mathscr{M})$, namely the homotopy and cohomology groups coincide.*

Corollary 2.6.2 *If a manifold $\mathscr{M}$ is simply connected ($\pi_1(\mathscr{M}) = 1$) then also the first homology group is trivial $H_1(\mathscr{M}) = 0$.*

The second of the above corollaries implies that in a simply connected manifold every closed loop is homologous to zero, namely it is the boundary of some region.

On the foundations of Differential Geometry, the notions of Manifolds and Fibre Bundles and all the basic concepts introduced in the present chapter there exist many classical textbooks. A short list reflecting just the preferences of the authors is the following one [1–3].

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