Remarks on Gaussian Noise Stability, Brascamp-Lieb and Slepian Inequalities

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Abstract E. Mossel and J. Neeman recently provided a heat flow monotonicity proof of Borell's noise stability theorem. In this note, we develop the argument to include in a common framework noise stability, Brascamp-Lieb inequalities (including hypercontractivity), and even a weak form of Slepian inequalities. The scheme applies furthermore to families of measures with are more log-concave than the Gaussian measure.

1 Introduction

Borell's noise stability theorem [12] expresses that if γ is the standard Gaussian measure $d\gamma(x) = d\gamma^n(x) = \mathrm{e}^{-|x|^2/2} \frac{dx}{(2\pi)^{n/2}}$ on $\mathbb{R}^n$, and if A, B are Borel measurable sets in $\mathbb{R}^n$ and H, K parallel half-spaces

$$H = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n ; x_1 \leq a\big\}, \quad K = \big\{(x_1, \ldots, x_n) \in \mathbb{R}^n ; x_1 \leq b\big\}$$

with respectively the same Gaussian measures $\gamma(H) = \gamma(A)$, $\gamma(K) = \gamma(B)$, then, for every $t \geq 0$,

$$\int_{\mathbb{R}^n} 1_A \, Q_t(1_B) d\gamma \leq \int_{\mathbb{R}^n} 1_H \, Q_t(1_K) d\gamma. \tag{1}$$

Here $(Q_t)_{t \geq 0} = (Q_t^n)_{t \geq 0}$ is the Ornstein-Uhlenbeck semigroup defined, on suitable functions $f : \mathbb{R}^n \to \mathbb{R}$, by

$$Q_t f(x) = \int_{\mathbb{R}^n} f\big(\mathrm{e}^{-t} x + \sqrt{1 - \mathrm{e}^{-2t}}\, y\big) d\gamma(y), \quad t \geq 0, \ x \in \mathbb{R}^n. \tag{2}$$

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According to this representation, setting $\rho = \mathrm{e}^{-t}$, if $X = X^n$ and $Y = Y^n$ are independent with distribution $\gamma = \gamma^n$,

$$\int_{\mathbb{R}^n} 1_A \, Q_t(1_B) d\gamma = \mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in B\big)$$

so that the conclusion (1) equivalently reads as

$$\mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in B\big) \leq \mathbb{P}\big(X \in H, \rho X + \sqrt{1-\rho^2}\, Y \in K\big). \tag{3}$$

The result then extends to any $\rho \in [-1, +1]$, with however the inequality in (3) reversed when $\rho \in [-1, 0]$. For simplicity in the exposition, we mostly only consider $\rho \in [0, 1]$ below (actually $\rho \in (0, 1)$ since the cases $\rho = 0$ and $\rho = 1$ are straightforward). The content of (3) is that the (Gaussian) noise stability (of a set A)

$$\mathbb{P}\big(X \in A, \rho X + \sqrt{1-\rho^2}\, Y \in A\big)$$

is maximal for half-spaces.

Towards the proof of (1), C. Borell [12] developed symmetrization arguments with respect to the Gaussian measure introduced by A. Ehrhard in [20] (see also [10,15]). Recently, E. Mossel and J. Neeman [28] proposed an alternative semigroup proof. The purpose of this note is to somewhat broaden their argument to cover in the same mould various related inequalities such as hypercontractivity, Brascamp-Lieb and Slepian inequalities. Heat flow arguments towards Brascamp-Lieb inequalities [13] have been investigated in the recent years by E. Carlen et al. [16] and J. Bennett et al. [8] (see also [5, 7, 17]). Section 2 describes the main theorem of [28] as an equivalent concavity property covering at the same time hypercontractivity and Borell's noise stability theorem. In Sect. 3, we consider multidimensional versions which were recently emphasized in [30], and discuss their applications to various families of concave functions towards Brascamp-Lieb and (a weak form of) Slepian-type inequalities. In the next section, we address extensions from the Gaussian model to families of measures $d\mu = \mathrm{e}^{-V} dx$ with a lower bound on the Hessian of V following the basic semigroup interpolation argument. Section 5 comments on some analogous issues on the discrete cube which raise questions on a family of concave functions in connection with the recent discrete proof by A. De et al. [18] of the "Majority is Stablest" theorem of [29].

2 Hypercontractivity and Gaussian Noise Stability

The main result of E. Mossel and J. Neeman [28] expresses an integral concavity property for correlated Gaussian vectors for a specific family of functions on $\mathbb{R}^2$. Say that a C^2 function J on $\mathbb{R}^2$, or some open rectangle $\mathcal{R} = I_1 \times I_2 \subset \mathbb{R}^2$, where I_1 and I_2 are open intervals, is ρ-concave for some $\rho \in \mathbb{R}$ if the matrix

$$\begin{pmatrix} \partial_{11} J & \rho\, \partial_{12} J \\ \rho\, \partial_{12} J & \partial_{22} J \end{pmatrix}$$

is (uniformly) semi-negative definite. $\rho = 1$ amounts to standard concavity while $\rho = 0$ amounts to concavity along each coordinate.

Theorem 1. *Let $\rho \in (0,1)$ and let J on $\mathcal{R} = I_1 \times I_2 \subset \mathbb{R}^2$ be of class C^2. Then,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(f(x), g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big) d\gamma(x) d\gamma(y) \\ \leq J\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma\bigg) \tag{4}$$

for every suitably integrable functions $f : \mathbb{R}^n \to I_1$, $g : \mathbb{R}^n \to I_2$ if and only if J is ρ-concave.

Let us sketch at this stage the heat flow proof of Theorem 1 following [28], the detailed argument being developed in the more general context of Sect. 4. Consider, for t (> 0) fixed and (smooth) functions $f : \mathbb{R}^n \to I_1$, $g : \mathbb{R}^n \to I_2$,

$$\psi(s) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(Q_s f(x), Q_s g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big) d\gamma(x) d\gamma(y), \quad s \geq 0.$$

By ergodicity, $Q_s f \to \int_{\mathbb{R}^n} f d\gamma$ and $Q_s g \to \int_{\mathbb{R}^n} g d\gamma$ as $s \to \infty$ so that it is enough to show that ψ is non-decreasing in order that $\psi(0) \leq \psi(\infty)$. Differentiating ψ and integrating by parts with respect to the infinitesimal generator $\mathrm{L} = \Delta - x \cdot \nabla$ of the Ornstein-Uhlenbeck semigroup $(Q_s)_{s \geq 0}$ yields

$$\begin{aligned} \psi'(s) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big[\partial_1 J\, \mathrm{L} Q_s f + \partial_2 J\, \mathrm{L} Q_s g\big] d\gamma\, d\gamma \\ &= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \big[\partial_{11} J\, |\nabla Q_s f|^2 + \rho \partial_{12} J\, \nabla Q_s f \cdot \nabla Q_s g\big] d\gamma\, d\gamma \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_2 J\, \mathrm{L} Q_s g\, d\gamma\, d\gamma. \end{aligned}$$

By Gaussian rotational invariance, setting $(x,y)_\rho = \rho x + \sqrt{1-\rho^2}\, y$,

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_2 J\, \mathrm{L} Q_s g\, d\gamma\, d\gamma \\ &\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_2 J\big(Q_s f(x), Q_s g\big((x,y)_\rho\big)\big) \mathrm{L} Q_s g\big((x,y)_\rho\big) d\gamma(x) d\gamma(y) \\ &\quad = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_2 J\big(Q_s f\big((x,-y)_\rho\big), Q_s g(x)\big) \mathrm{L} Q_s g(x) d\gamma(x) d\gamma(y) \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[\rho \partial_{12} J\big(Q_s f\big((x,-y)_\rho\big), Q_s g(x)\big) \nabla Q_s f\big((x,-y)_\rho\big) \cdot \nabla Q_s g(x) \\
&\qquad\qquad + \partial_{22} J\big(Q_s f\big((x,-y)_\rho\big), Q_s g(x)\big) \left|\nabla Q_s g(x)\right|^2\Big] d\gamma(x) d\gamma(y) \\
&= -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \Big[\rho \partial_{12} J\big(Q_s f(x), Q_s g\big((x,y)_\rho\big)\big) \nabla Q_s f(x) \cdot \nabla Q_s g\big((x,y)_\rho\big) \\
&\qquad\qquad + \partial_{22} J\big(Q_s f(x), Q_s g\big((x,y)_\rho\big)\big) \left|\nabla Q_s g\big((x,y)_\rho\big)\right|^2\Big] d\gamma\, d\gamma.
\end{aligned}
$$

Finally

$$
\begin{aligned}
\psi'(s) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \; &\Big[(-\partial_{11} J)|\nabla Q_s f|^2 + (-\partial_{22} J)|\nabla Q_s g|^2 \\
&\quad -2\,\rho\, \partial_{12} J\, \nabla Q_s f \cdot \nabla Q_s g\Big] d\gamma\, d\gamma
\end{aligned}
$$

From the hypothesis of ρ-concavity on J, it follows that $\psi' \geq 0$ which is the result.

The converse was observed by R. O'Donnell, and communicated to us by J. Neeman. Indeed, applying (4) to $f(x) = a + \varepsilon x$ and $g(y) = b + \varepsilon y$ (in dimension one) and letting $\varepsilon \to 0$ shows that J is ρ-concave.

It may be mentioned that due to the product structure of the Gaussian measure γ^n, the inequality of Theorem 1 immediately tensorizes so that it is actually enough to establish it in dimension one.

Let us now illustrate the application of Theorem 1 to two main examples of ρ-concave function J, covering hypercontractivity and noise stability at the same time.

Let first

$$
J^{\mathrm{H}}(u,v) = u^\alpha v^\beta, \quad (u,v) \in [0,\infty)^2.
$$

Since

$$
\partial_{11} J^{\mathrm{H}} = \alpha(\alpha-1)u^{\alpha-2}v^\beta, \quad \partial_{22} J^{\mathrm{H}} = \beta(\beta-1)u^\alpha v^{\beta-2}, \quad \partial_{12} J^{\mathrm{H}} = \alpha\beta u^{\alpha-1}v^{\beta-1},
$$

J^{H} is ρ-concave on $(0,\infty)^2$ as soon as $\alpha, \beta \in [0,1]$ and

$$
\rho^2 \alpha\beta \leq (\alpha-1)(\beta-1). \tag{5}
$$

The function J^{H} will be called the hypercontractive function in this context.

Indeed, let $1 < p < q < \infty$ and $\rho = \mathrm{e}^{-t} \in (0,1)$ be such that

$$
\frac{1}{\rho^2} = \frac{q-1}{p-1}\,.
$$

Denote by q' the conjugate of q, $\frac{1}{q} + \frac{1}{q'} = 1$. Then, according to (5), the function J^{H} with $\alpha = \frac{1}{q'}$ and $\beta = \frac{1}{p}$ is ρ-concave on $(0,\infty)^2$. By Theorem 1, for strictly positive functions $f, g : \mathbb{R}^n \to \mathbb{R}$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^{1/q'}(x) g^{1/p}\big(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}\, y\big)\, d\gamma(x) d\gamma(y) \leq \bigg(\int_{\mathbb{R}^n} f\, d\gamma\bigg)^{1/q'} \bigg(\int_{\mathbb{R}^n} g\, d\gamma\bigg)^{1/p}.$$

In other words, changing f into $f^{q'}$ and g into g^p,

$$\int_{\mathbb{R}^n} f\, Q_t g\, d\gamma \leq \|f\|_{q'} \|g\|_p.$$

By duality

$$\|Q_t g\|_q \leq \|g\|_p$$

which amounts to hypercontractivity of the Ornstein-Uhlenbeck semigroup [23, 31] (cf. e.g. [3]). Clearly, the conclusion of Theorem 1 for J^{H} is actually equivalent to hypercontractivity. Note that a prior to the proof of hypercontractivity along these lines may be found in [24].

The second example involves a new function introduced in [28] defined for $(u, v) \in [0,1]^2$ by

$$J^{\mathrm{B}}(u,v) = J^{\mathrm{B}}_\rho(u,v) = \mathbb{P}\big(X^1 \leq \Phi^{-1}(u), \rho X^1 + \sqrt{1-\rho^2}\, Y^1 \leq \Phi^{-1}(v)\big)$$

where $\Phi(a) = \gamma^1((-\infty, a])$, $a \in \mathbb{R}$, is the distribution of the standard normal on $\mathbb{R}$ and $\rho \in [-1, +1]$. For the connection with Borell's theorem, observe that if H and K are the (parallel) half-spaces $H = \{x_1 \leq a\}$ and $K = \{x_1 \leq b\}$ for some $a, b \in \mathbb{R}$, with $\rho = e^{-t}$ and the integral representation (2) of Q_t,

$$\begin{aligned} J^{\mathrm{B}}\big(\gamma(H), \gamma(K)\big) &= \mathbb{P}\big(X^1 \leq a, \rho X^1 + \sqrt{1-\rho^2}\, Y^1 \leq b\big) \\ &= \int_{\mathbb{R}^n} 1_H\, Q_t(1_K) d\gamma. \end{aligned} \tag{6}$$

Apply now Theorem 1 to the function J^{B}. Since $J^{\mathrm{B}}(u, 0) = J^{\mathrm{B}}(0, v) = 0$ and $J^{\mathrm{B}}(1,1) = 1$, for f and g approaching 1_A and 1_B respectively,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(f(x), g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big) d\gamma(x) d\gamma(y) = \int_{\mathbb{R}^n} 1_A\, Q_t(1_B) d\gamma.$$

We then recover Borell's noise stability theorem (1) since by (6),

$$\begin{aligned} J^{\mathrm{B}}\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma\bigg) = J^{\mathrm{B}}\big(\gamma(A), \gamma(B)\big) &= J^{\mathrm{B}}\big(\gamma(H), \gamma(K)\big) \\ &= \int_{\mathbb{R}^n} 1_H\, Q_t(1_K) d\gamma \end{aligned}$$

for parallel half-spaces H and K such that respectively $\gamma(A) = \gamma(H)$ and $\gamma(B) = \gamma(K)$. When $\rho \in [-1, 0]$, observe that

$$J^{\mathrm{B}}_\rho(u, v) = u - J^{\mathrm{B}}_{-\rho}(u, 1 - v)$$

so that J^{B} is ρ-convex in this case, and the conclusion of Theorem 1 for the function J^{B}_ρ is thus reversed. As pointed out in [28], (1) on sets may actually be turned to Theorem 1 (for J^{B}) through epigraphs of functions on $\mathbb{R}^{n-1}$.

It remains to check the ρ-concavity of J^{B}. To this task, it is convenient to recall that $Q_t f(x)$ may be given alternatively by the Mehler kernel

$$Q_t f(x) = \int_{\mathbb{R}^n} f(y) q_t(x, y) d\gamma(y) \tag{7}$$

where, for $t > 0$, $(x, y) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$\begin{aligned} q_t(x, y) &= q_t^n(x, y) \\ &= \frac{1}{\sqrt{1 - \mathrm{e}^{-2t}}} \exp\bigg(- \frac{\mathrm{e}^{-2t}}{2(1 - \mathrm{e}^{-2t})} \big[|x|^2 + |y|^2 - 2\,\mathrm{e}^t x \cdot y \big] \bigg). \end{aligned} \tag{8}$$

In particular, if $\rho = \mathrm{e}^{-t}$,

$$J^{\mathrm{B}}(u, v) = \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} q_t^1(x, y) d\gamma^1(x) d\gamma^1(y).$$

Observe also that $(\Phi^{-1})' = \frac{1}{\varphi \circ \Phi^{-1}}$ where $\varphi = \Phi'$ is the density of γ^1 on $\mathbb{R}$. Hence,

$$\partial_1 J^{\mathrm{B}}(u, v) = \int_{-\infty}^{\Phi^{-1}(v)} q_t^1 \big(\Phi^{-1}(u), y \big) d\gamma^1(y)$$

and

$$\partial_{12} J^{\mathrm{B}}(u, v) = q_t^1 \big(\Phi^{-1}(u), \Phi^{-1}(v) \big).$$

On the other hand, by the integral representations (2) and (7), for h smooth enough,

$$\partial_x \int_{\mathbb{R}} h(y) q_t^1(x, y) d\gamma^1(y) = \partial_x Q_t^1 h(x) = \rho\, Q_t^1 h'(x) = \rho \int_{\mathbb{R}} h'(y) q_t^1(x, y) d\gamma^1(y).$$

With h a smooth approximation of $1_{(-\infty, b]}$,

$$\partial_x \int_{-\infty}^b q_t^1(x, y) d\gamma^1(y) = -\rho\, q_t^1(x, b) \varphi(b).$$

Therefore,

$$\partial_{11} J^{\mathrm{B}}(u, v) = -\rho\, q_t^1\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)\, \frac{\varphi \circ \Phi^{-1}(v)}{\varphi \circ \Phi^{-1}(u)}\,.$$

Similarly,

$$\partial_{22} J^{\mathrm{B}}(u, v) = -\rho\, q_t^1\big(\Phi^{-1}(u), \Phi^{-1}(v)\big)\, \frac{\varphi \circ \Phi^{-1}(u)}{\varphi \circ \Phi^{-1}(v)}\,.$$

Hence, on $(0, 1)^2$,

$$\partial_{11} J^{\mathrm{B}}\, \partial_{22} J^{\mathrm{B}} - \rho^2 (\partial_{12} J^{\mathrm{B}})^2 = 0$$

and $\partial_{11} J^{\mathrm{B}} \le 0$, $\partial_{22} J^{\mathrm{B}} \le 0$ so that J^{B} is indeed ρ-concave.

It would be of interest to find other relevant examples of function J. It is also of interest to directly compare the conclusion of Theorem 1 for the hypercontractive function J^{H} and for the Borell noise stability function J^{B}, and namely to show that noise stability is a stronger statement implying hypercontractivity. One way towards this end, however along a rather long detour, is to observe, as emphasized in [26], that Borell's noise stability theorem may be used to reach the Gaussian isoperimetric inequality. Now, the latter implies in turn the standard logarithmic Sobolev inequality for the Gaussian measure, equivalent to hypercontractivity (cf. [3, 26]).

There is an alternative direct argument towards this relationship, applying Theorem 1 for J^{B} to εf and δg and letting $\varepsilon, \delta \to 0$. To this task, it is necessary to investigate the asymptotics of $J^{\mathrm{B}}(\varepsilon u, \delta v)$ as $\varepsilon, \delta \to 0$. Similar asymptotics are investigated in [19].

Set $\rho = \mathrm{e}^{-t} \in (0, 1)$ and fix $0 < u, v < 1$. Let furthermore $0 < \varepsilon < 1$, $\delta = \varepsilon^{\kappa^2}$ where $\rho < \kappa < \frac{1}{\rho}$, and

$$Z = \sqrt{2 \log \frac{1}{\varepsilon}}\,, \quad U = \log \frac{1}{u}\,, \quad V = \log \frac{1}{v}\,.$$

In this notation, after a change of variables,

$$J^{\mathrm{B}}(\varepsilon u, \delta v) = \frac{UV}{\kappa Z^2} \int_c^\infty \int_d^\infty \tilde{q}_t^1\Big(- Z - \frac{Ux}{Z}, -\kappa Z - \frac{Vy}{\kappa Z}\Big) dxdy$$

where

$$c = -\frac{Z}{U}\big[Z + \Phi^{-1}(\varepsilon u)\big] \quad \text{and} \quad d = -\frac{Z}{\kappa V}\big[\kappa Z + \Phi^{-1}(\delta v)\big],$$

and

$$\tilde{q}_t^1(x,y) = (2\pi)^{-1} q_t^1(x,y)\, \mathrm{e}^{-(x^2+y^2)/2}, \quad (x,y) \in \mathbb{R} \times \mathbb{R}.$$

After some algebra,

$$J^{\mathrm{B}}(\varepsilon u, \delta v) = \frac{UV\, \mathrm{e}^{\sigma Z^2}}{2\pi \sqrt{1-\rho^2}\kappa Z^2} \int_c^\infty \int_d^\infty \mathrm{e}^{-\alpha U x - \beta V y - R(x,y)} dxdy$$

where

$$\sigma = -\frac{1 - 2\kappa\rho + \kappa^2}{2(1-\rho^2)}, \quad \alpha = \frac{1-\kappa\rho}{1-\rho^2}, \quad \beta = \frac{1-\kappa^{-1}\rho}{1-\rho^2}$$

and

$$R(x,y) = -\frac{1}{2(1-\rho^2)} \bigg(\frac{U^2 x^2}{Z^2} + \frac{V^2 x^2}{\kappa^2 Z^2} - 2\rho\, \frac{UVxy}{\kappa Z^2} \bigg).$$

It is classical that

$$\Phi^{-1}(\varepsilon) = -\sqrt{2 \log \frac{1}{\varepsilon}} + o\bigg(\sqrt{2 \log \frac{1}{\varepsilon}} \bigg)$$

as $\varepsilon \to 0$, so that

$$\Phi^{-1}(\varepsilon u) = -Z - \frac{U}{Z} + o(Z)$$

as $Z \to \infty$. Moreover, $o(Z)$ can be made uniform over $\eta \le u \le 1-\eta$ for $\eta > 0$ fixed. As a consequence, as $\varepsilon \to 0$, $c, d \to 1$ and

$$2\pi \sqrt{1-\rho^2}\, \kappa Z^2 \mathrm{e}^{-\sigma Z^2} J^{\mathrm{B}}(\varepsilon u, \delta v) \to UV \int_1^\infty \int_1^\infty \mathrm{e}^{-\alpha U x - \beta V y} dxdy = \frac{1}{\alpha\beta}\, \mathrm{e}^{-\alpha U - \beta V}.$$

By definition of U and V, the right-hand side is $\frac{1}{\alpha\beta}\, u^\alpha v^\beta$.

Let now f, g on $\mathbb{R}^n$ such that $\eta \le f, g \le 1-\eta$ for some fixed $\eta > 0$. Translating the preceding asymptotics into the inequality

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J^{\mathrm{B}}\big(\varepsilon f(x), \delta g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big)\, d\gamma(x) d\gamma(y) \\ \le J^{\mathrm{B}}\bigg(\varepsilon \int_{\mathbb{R}^n} f\, d\gamma, \delta \int_{\mathbb{R}^n} g\, d\gamma \bigg)$$

yields

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f^\alpha(x)g^\beta\big(\rho x+\sqrt{1-\rho^2}\,y\big)d\gamma(x)d\gamma(y)\le\bigg(\int_{\mathbb{R}^n} f\,d\gamma\bigg)^\alpha\bigg(\int_{\mathbb{R}^n} g\,d\gamma\bigg)^\beta.$$

This inequality extends to all positive measurable functions $f, g : \mathbb{R}^n \to \mathbb{R}$ by homogeneity. Now, as is immediately checked, for the values of α, β defined above,

$$(\alpha-1)(\beta-1)=\rho^2\alpha\beta,$$

that is condition (5) of hypercontractivity holds. Given therefore any $\alpha, \beta \in (0, 1)$ satisfying this relation, one may choose $\rho < \kappa < \frac{1}{\rho}$ such that $\alpha = \frac{1-\kappa\rho}{1-\rho^2}$ and $\beta = \frac{1-\kappa^{-1}\rho}{1-\rho^2}$ as above. The announced claim follows.

It would be worthwhile to examine similarly noise stability for the Lebesgue measure λ with respect to the standard heat kernel expressing that for Borel sets A, B in $\mathbb{R}^n$ with finite volume,

$$\int_{\mathbb{R}^n} 1_A\,H_t(1_B)dx\le\int_{\mathbb{R}^n} 1_C\,H_t(1_D)dx$$

where

$$H_t f(x)=\int_{\mathbb{R}^n} f(y)\,\mathrm{e}^{-|x-y|^2/4t}\,\frac{dy}{(4\pi t)^{n/2}},\qquad t>0,\ x\in\mathbb{R}^n,$$

and C and D are centered balls in $\mathbb{R}^n$ such that $\lambda(A) = \lambda(C)$ and $\lambda(B) = \lambda(D)$. This classical result is going back to the Riesz rearrangement inequality [33] (see also [14,27,34]), and one might wonder for a heat flow proof. A similar question may be formulated on the sphere (cf. [1, 10, 15]).

3 Multidimensional Extensions

On the basis of the heat flow proof of Theorem 1, we address in this section multidimensional extensions and develop connections to Brascamp-Lieb and Slepian-type inequalities. The multidimensional version of noise stability was already put forward by J. Neeman in [30]. The Brascamp-Lieb applications are essentially contained with the same approach in [8, 16] (see also [17]). At the same time, the investigation provides a somewhat different analytical treatment of the conclusions of Sect. 2.

Let J be a (smooth) real-valued function on some rectangle subset $\mathcal{R}$ of $\mathbb{R}^m$. It will implicitly be assumed below that a composition like $J \circ f$ is meant for functions f with values in $\mathcal{R}$.

Let $f_1, \dots, f_m$ be (smooth) functions on $\mathbb{R}^n$ and consider, for $f = (f_1, \dots, f_m)$,

$$\psi(s) = \int_{\mathbb{R}^n} J \circ Q_s f \, d\gamma, \quad s \geq 0,$$

where $(Q_s)_{s \geq 0}$ is the Ornstein-Uhlenbeck semigroup on $\mathbb{R}^n$ (extended to functions with values in $\mathbb{R}^m$). Arguing as in Sect. 2, by integration by parts with respect to the Ornstein-Uhlenbeck generator,

$$\begin{aligned} \psi'(s) &= \sum_{k=1}^{n} \int_{\mathbb{R}^n} \partial_k J \circ Q_s f \, \mathrm{L} Q_s f_k \, d\gamma \\ &= -\sum_{k,\ell=1}^{m} \int_{\mathbb{R}^n} \partial_{k\ell} J \circ Q_s f \, \nabla Q_s f_k \cdot \nabla Q_s f_\ell \, d\gamma. \end{aligned} \tag{9}$$

Definition. Given a smooth function J on an open subset of $\mathbb{R}^m$ and $\Gamma = (\Gamma_{k\ell})_{1 \leq k,\ell \leq m}$ where $\Gamma_{k\ell}$ are $p \times p$ matrices ($p \geq 1$), say that J is Γ-concave if

$$\sum_{k,\ell=1}^{m} \partial_{k\ell} J \, \Gamma_{k\ell} \, v_k \cdot v_\ell \leq 0 \tag{10}$$

for all vectors v_k, $k = 1, \dots, m$, in $\mathbb{R}^p$. If $p = 1$, the meaning of this condition is that the point-wise (Hadamard) product $\mathrm{Hess}(J) \circ \Gamma$ of the Hessian of J and of Γ is (semi-) negative definite.

When $m = 2$, $p = n$ and Γ is the $2n \times 2n$ matrix

$$\begin{pmatrix} \mathrm{Id}_n & \rho \, \mathrm{Id}_n \\ \rho \, \mathrm{Id}_n & \mathrm{Id}_n \end{pmatrix} \tag{11}$$

where $\rho \in \mathbb{R}$, the Γ-concavity of J (on $\mathbb{R}^2$) amounts to its ρ-concavity.

In (9), replace now n by qn, $q \geq 1$ integer, and assume that for every $k = 1, \dots, m$,

$$f_k = g_k \circ A_k$$

where $g_k : \mathbb{R}^p \to \mathbb{R}$ and A_k is a (constant) $p \times qn$ matrix such that $A_k \, {}^t A_k$ is the identity matrix (of $\mathbb{R}^p$). By the integral representation (2) of Q_s,

$$\nabla Q_s f_k = \mathrm{e}^{-s} \, {}^t A_k (\nabla Q_s g_k) \circ A_k$$

where on the left-hand side the semigroup Q_s is acting on $\mathbb{R}^{qn}$ and on the right-hand side, it is acting on $\mathbb{R}^p$. Hence

$$\psi'(s) = -\,\mathrm{e}^{-2s} \sum_{k,\ell=1}^{m} \int_{\mathbb{R}^{qn}} \partial_{k\ell} J \circ Q_s f \; \Gamma_{k\ell} (\nabla Q_s g_k) \circ A_k \cdot (\nabla Q_s g_\ell) \circ A_\ell \, d\gamma$$

where $\Gamma_{k\ell} = A_\ell{}^t A_k$ (which is a $p \times p$ matrix).

With this choice of $\Gamma = (\Gamma_{k\ell})_{1\leq k,\ell\leq m}$, the following proposition summarizes the conclusion at this level of generality.

Proposition 2. *In the preceding setting, assume that J is Γ-concave. Then $\int_{\mathbb{R}^{qn}} J \circ f \, d\gamma \leq J(\int_{\mathbb{R}^{qn}} f \, d\gamma)$, that is*

$$\int_{\mathbb{R}^{qn}} J(g_1 \circ A_1, \dots, g_m \circ A_m) d\gamma \leq J\bigg(\int_{\mathbb{R}^{qn}} g_1 \circ A_1 d\gamma, \dots, \int_{\mathbb{R}^{qn}} g_m \circ A_m d\gamma \bigg).$$

To connect with Sect. 2, take for example $p = n$ and $q = m = 2$ and let A_1 and A_2 be the $n \times 2n$ matrices $A_1 = (\mathrm{Id}_n; 0_n)$ and $A_2 = (\rho\,\mathrm{Id}_n; \sqrt{1-\rho^2}\,\mathrm{Id}_n)$ so that

$$f_1(x,y) = g_1(x) \quad \text{and} \quad f_2(x,y) = g_2\big(\rho x + \sqrt{1-\rho^2}\, y\big), \quad (x,y) \in \mathbb{R}^n \times \mathbb{R}^n.$$

Since Γ is given by (11), the monotonicity property follows from the ρ-concavity of J.

We next systematically investigate illustrations of Proposition 2 for some main examples of interest. For simplicity, we only consider $p = q = 1$, the multidimensional cases being often obtained by tensor products with the identity matrix (as in the preceding example). The Γ-concavity thus amounts to $\mathrm{Hess}(J) \circ \Gamma \leq 0$ in the following.

(i) The first illustration examines Brascamp-Lieb inequalities under geometric conditions. Consider unit vectors $A_1, \dots, A_m$ which decompose the identity in $\mathbb{R}^n$ in the sense that for $0 \leq c_k \leq 1$, $k = 1, \dots, m$,

$$\sum_{k=1}^m c_k A_k \otimes A_k = \mathrm{Id}_n. \tag{12}$$

Then, for

$$J(u_1, \dots, u_m) = u_1^{c_1} \cdots u_m^{c_m}$$

on $(0,\infty)^m$ and $f_k(x) = g_k(A_k \cdot x)$, $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, the Γ-concavity with respect to $\Gamma_{k\ell} = A_k \cdot A_\ell$, $k, \ell = 1, \dots, m$, is expressed by

$$\sum_{k,\ell=1}^m c_k c_\ell A_k \cdot A_\ell \, v_k v_\ell \leq \sum_{k=1}^m c_k v_k^2 \tag{13}$$

for all $v_1, \dots, v_m \in \mathbb{R}$. Now, if $x = \sum_{k=1}^m c_k A_k v_k$,

$$|x|^2 = \sum_{k=1}^m c_k A_k v_k \cdot x \leq \bigg(\sum_{k=1}^m c_k v_k^2 \bigg)^{1/2} \bigg(\sum_{k=1}^m c_k (A_k \cdot x)^2 \bigg)^{1/2}$$

Since, by the decomposition (12), $|x|^2 = \sum_{k=1}^m c_k(A_k \cdot x)^2$, it follows that

$$|x|^2 = \bigg| \sum_{k=1}^m c_k A_k v_k \bigg|^2 \leq \sum_{k=1}^m c_k v_k^2$$

which is precisely the requested condition (13). We therefore conclude to the following result.

Corollary 3. *Under the decomposition (12), for non-negative functions g_k on $\mathbb{R}$, $k = 1, \ldots, m$,*

$$\int_{\mathbb{R}^n} \prod_{k=1}^m g_k^{c_k}(A_k \cdot x) d\gamma \leq \prod_{k=1}^m \bigg(\int_{\mathbb{R}} g_k d\gamma \bigg)^{c_k}.$$

This inequality is part of the Brascamp-Lieb inequalities under the geometric Ball condition (12) [4] (cf. e.g. [7, 8]). It is more classically stated with respect to the Lebesgue measure as

$$\int_{\mathbb{R}^n} \prod_{k=1}^m f_k^{c_k}(A_k \cdot x) dx \leq \prod_{k=1}^m \bigg(\int_{\mathbb{R}} f_k dx \bigg)^{c_k}$$

which is immediately obtained after the change $f_k(x) = g_k(x)\mathrm{e}^{-x^2/2}$ (using that $\sum_{k=1}^m c_k = n$).

The heat flow proof of Corollary 3 is thus going back to [16] and [8] in which more general statements are considered and achieved in this way. One of the motivations of [16] was actually to investigate similar inequalities for coordinates on the sphere. Let $\mathbb{S}^{n-1}$ be the standard n-sphere in $\mathbb{R}^n$ and denote by σ the uniform (normalized) measure on it. In this framework, one result then reads as follows. If g_k, $k = 1, \ldots, n$, are, say bounded measurable, functions on $\mathbb{R}$, then

$$\int_{\mathbb{S}^{n-1}} J\big(g_1(x_1), \ldots, g_n(x_n)\big) d\sigma \leq J\bigg(\int_{\mathbb{S}^{n-1}} g_1(x_1) d\sigma, \ldots, \int_{\mathbb{S}^{n-1}} g_n(x_n) d\sigma \bigg)$$

as soon as J on $\mathbb{R}^n$, or some open (convex) set in $\mathbb{R}^n$, is separately concave in any two variables. The proof proceeds as the one of Proposition 2 along now the heat flow of the Laplace operator

$$\Delta = \frac{1}{2} \sum_{k,\ell=1}^n (x_k \partial_\ell - x_\ell \partial_k)^2$$

on $\mathbb{S}^{n-1}$. The monotonicity condition on J then takes the form

$$\sum_{k,\ell=1}^n \partial_{k\ell} J \, (\delta_{k\ell} - x_k x_\ell) v_k v_\ell \leq 0$$

which is easily seen to be satisfied under concavity of J in any two variables. The case considered in [16] simply corresponds to

$$J(u_1, \dots, u_n) = (u_1 \cdots u_n)^{1/2}$$

on $\mathbb{R}^n_+$. More general forms under decompositions (12) of the identity have been considered in [6, 7].

In the further illustrations, consider $X = (X_1, \dots, X_m)$ a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix $\Gamma = A^t A$ such that $\Gamma_{kk} = 1$ for every $k = 1, \dots, m$. The vector X has the distribution of Ax, $x \in \mathbb{R}^n$, under the standard normal distribution γ on $\mathbb{R}^n$. Applying the general Proposition 2 to the unit vectors ($1 \times n$ matrices) A_k, $k = 1, \dots, m$, which are the lines of the matrix A, and to $f_k(x) = g_k(A_k \cdot x)$, $x \in \mathbb{R}^n$, where $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$, with respect to γ, yields that whenever $\mathrm{Hess}(J) \circ \Gamma \leq 0$, under suitable integrability properties on the g_k's,

$$\mathbb{E}\Big(J\big(g_1(X_1), \dots, g_m(X_m)\big)\Big) \leq J\Big(\mathbb{E}\big(g_1(X_1)\big), \dots, \mathbb{E}\big(g_m(X_m)\big)\Big). \tag{14}$$

Note that, as in Sect. 1, the condition $\mathrm{Hess}(J) \circ \Gamma \leq 0$ is actually necessary and sufficient for (14) to hold.

(ii) This illustration deals with a correlation inequality for Gaussian vectors which covers in particular the classical hypercontractivity property. For a Gaussian vector X as above, let as in the first illustration,

$$J(u_1, \dots, u_m) = u_1^{c_1} \cdots u_m^{c_m}$$

on $(0, \infty)^m$, with $0 \leq c_k \leq 1$, $k = 1, \dots, m$. This function J is the suitable multidimensional analogue of the hypercontractive function J^{H}. For this choice of J, the condition $\mathrm{Hess}(J) \circ \Gamma \leq 0$ (where Γ is the covariance matrix of X) amounts to

$$\sum_{k,\ell=1}^{m} c_k c_\ell \, \Gamma_{k\ell} \, v_k v_\ell \leq \sum_{k=1}^{m} c_k v_k^2 \tag{15}$$

for all $v_k \in \mathbb{R}$, $k = 1, \dots, m$. Note that this condition expresses equivalently that $\Gamma \leq \Delta_c$ in the sense of symmetric matrices where Δ_c is the diagonal matrix $(\frac{1}{c_k})_{1 \leq k \leq m}$. While the next Corollary 4 is somewhat part of the folklore (implicit for example in [7]), it has been emphasized recently in [17] together with reverse and multidimensional versions (in particular, if $\Gamma \geq \Delta_c$, the conclusion is reversed in (16)).

Corollary 4. *Under (15), for all non-negative functions $g_k : \mathbb{R} \to \mathbb{R}$, $k = 1, \dots, m$,*

$$\mathbb{E}\bigg(\prod_{k=1}^{m} g_k^{c_k}(X_k)\bigg) \le \prod_{k=1}^{m} \mathbb{E}\big(g_k(X_k)\big)^{c_k}. \tag{16}$$

One application concerns the Ornstein-Uhlenbeck process $Z = (Z_t)_{t\ge 0}$ (in dimension one) with stationary measure $\gamma = \gamma^1$ and associated Markov semigroup $(Q_t)_{t\ge 0} = (Q_t^1)_{t\ge 0}$. If X is the vector $(Z_{t_1}, \dots, Z_{t_m})$ with $0 \le t_1 \le \dots \le t_m$, the covariance matrix Γ has entries $\Gamma_{k\ell} = \mathrm{e}^{-|t_k - t_\ell|}$, $k, \ell = 1, \dots, m$. In particular, for $t_1 = 0$ and $t_2 = t > 0$, (15) reads

$$2\,\mathrm{e}^{-t} c_1 c_2 v_1 v_2 \le c_1(1-c_1)v_1^2 + c_2(1-c_2)v_2^2$$

for all $v_1, v_2 \in \mathbb{R}$ which amounts to (5)

$$\mathrm{e}^{-2t} c_1 c_2 \le (c_1 - 1)(c_2 - 1)$$

and the conclusion of Corollary 4 leads to hypercontractivity. The condition

$$\sum_{k,\ell=1}^{m} c_k c_\ell\, \mathrm{e}^{-|t_k - t_\ell|} v_k v_\ell \le \sum_{k=1}^{m} c_k v_k^2$$

yields a multidimensional form of hypercontractivity

$$\mathbb{E}\bigg(\prod_{k=1}^{m} g_k^{c_k}(Z_{s_k})\bigg) \le \prod_{k=1}^{m} \mathbb{E}\big(g_k(Z_{s_k})\big)^{c_k}.$$

In terms of the Mehler kernel (8),

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \prod_{k=1}^{m} g_k^{c_k}(x_k)\, q_{t_2 - t_1}(x_1, x_2) \cdots q_{t_m - t_{m-1}}(x_{m-1}, x_m)\, d\gamma(x_1) \cdots d\gamma(x_m)$$
$$\le \prod_{k=1}^{m} \bigg(\int_{\mathbb{R}} g_k\, d\gamma\bigg)^{c_k}.$$

(iii) We next turn to the multidimensional versions of Gaussian noise stability following [30]. As above, let $X = (X_1, \dots, X_m)$ be a centered Gaussian vector on $\mathbb{R}^m$ with (non-degenerate) covariance matrix Γ. Define, for $u_1, \dots, u_m$ in $(0, 1)$,

$$J(u_1, \dots, u_m) = \mathbb{P}\big(X_1 \le \alpha_1(u_1), \dots, X_m \le \alpha_m(u_m)\big) \tag{17}$$

where $\alpha_1, \dots, \alpha_m$ are smooth functions on $(0,1)$. For specific choices of α_k, this function will turn as the multidimensional analogue of the noise stability function J^{B}. Denoting by p the density of the distribution of X with respect to the Lebesgue measure, elementary (although a bit tedious, see [30]) differential calculus leads to

$$\partial_{k\ell} J = \alpha_k'(u_k)\alpha_\ell'(u_\ell) \int_{-\infty}^{\alpha_1(u_1)} \cdots \int_{-\infty}^{\alpha_m(u_m)} p_{k\ell}\, dx$$

for $k \neq \ell$ and

$$\begin{aligned}\partial_{kk} J = &\left(\alpha_k''(u_k) - \frac{\alpha_k(u_k)\alpha_k'(u_k)^2}{\Gamma_{kk}}\right) \int_{-\infty}^{\alpha_1(u_1)} \cdots \int_{-\infty}^{\alpha_m(u_m)} p_k\, dx \\ &- \alpha_k'(u_k)^2 \sum_{\ell \neq k} \frac{\Gamma_{k\ell}}{\Gamma_{kk}} \int_{-\infty}^{\alpha_1(u_1)} \cdots \int_{-\infty}^{\alpha_m(u_m)} p_{k\ell}\, dx\end{aligned}$$

where

$$\begin{aligned} p_k &= p\big(x_1, \dots, \alpha_k(u_k), \dots, x_m\big), \\ p_{k\ell} &= p\big(x_1, \dots, \alpha_k(u_k), \dots, \alpha_\ell(u_\ell), \dots, x_m\big). \end{aligned}$$

Choose now $\alpha_k = \Phi^{-1}$, $k = 1, \dots, m$, where we recall the distribution function Φ of the standard normal, and φ its derivative. Since

$$\alpha_k' = \frac{1}{\varphi \circ \Phi^{-1}} \quad \text{and} \quad \alpha_k'' = \frac{\Phi^{-1}}{(\varphi \circ \Phi^{-1})^2},$$

in order for the condition $\mathrm{Hess}(J) \circ \Gamma \leq 0$ to hold it is thus sufficient that $\Gamma_{kk} = 1$ for every $k = 1, \dots, m$ and

$$\sum_{k=1}^m \sum_{\ell \neq k} \Gamma_{k\ell}\, p_{k\ell}\, v_k^2 - \sum_{k \neq \ell} p_{k\ell}\, \Gamma_{k\ell}\, v_k v_\ell \geq 0$$

for all $v_1, \dots, v_m \in \mathbb{R}$. This holds as soon as $\Gamma_{k\ell} \geq 0$ for all k, ℓ.

For the application to the following corollary, recall that for the choice of $\alpha_k = \Phi^{-1}$, the function J of (17) is equal to 0 if one of the u_k's is (approaches) 0, and is equal to (approach) 1 if all the u_k's are equal to 1. The following corollary, thus due to J. Neeman [30], is then a consequence of (14) applied to $g_k = 1_{B_k}$, $k = 1, \dots, m$. The restriction $\Gamma_{kk} = 1$, $k = 1, \dots, m$, is lifted after a simple scaling of the Gaussian vector and the Borel sets.

Corollary 5. *Let $X = (X_1, \dots, X_m)$ be a centered Gaussian vector in $\mathbb{R}^m$ with (non-degenerate) covariance matrix Γ such that $\Gamma_{k\ell} \geq 0$ for all $k, \ell = 1, \dots, m$. Then, for any Borel sets $B_1, \dots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(X_1 \in B_1, \dots, X_m \in B_m) \leq \mathbb{P}(X_1 \leq b_1, \dots, X_m \leq b_m)$$

where $\mathbb{P}(X_k \in B_k) = \Phi(b_k/\sigma_k)$, $\sigma_k = \sqrt{\Gamma_{kk}}$, $k = 1, \dots, m$.

When $\Gamma_{k\ell} \leq 0$ whenever $k \neq \ell$, the inequality in the conclusion of Corollary 5 is reversed. As developed in [30], the result applies similarly to Gaussian vectors $X_1, \dots, X_m$ with covariance identity matrix. A related work by M. Isaksson and E. Mossel [25] establishes the conclusion of Corollary 5 under the (stronger) hypothesis that the off-diagonal elements of the inverse of Γ are non-positive. Their approach relies on a rearrangement inequality for kernels on the sphere. Corollary 5 (as well as actually, after some work, the result of [25]—see [30]) covers the example of the Ornstein-Uhlenbeck process, and thus of C. Borell's result [12] in the form of the following corollary.

Corollary 6. *Let $(Z_t)_{t \geq 0}$ be the Ornstein-Uhlenbeck process on the line, and let $0 \leq t_1 \leq \dots \leq t_m$. For any Borel sets $B_1, \dots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(Z_{t_1} \in B_1, \dots, Z_{t_m} \in B_m) \leq \mathbb{P}(Z_{t_1} \leq b_1, \dots, Z_{t_m} \leq b_m)$$

where $\mathbb{P}(Z_{t_k} \in B_k) = \gamma(B_k) = \Phi(b_k)$, $k = 1, \dots, m$.

(iv) This illustration is a variation on the previous multidimensional noise stability result which actually leads to a weak form of the classical Slepian inequalities. Let as above $X = (X_1, \dots, X_m)$ be a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix $\Gamma = \Gamma^X$ such that $\Gamma^X_{kk} = 1$ for every $k = 1, \dots, m$. Consider furthermore $Y = (Y_1, \dots, Y_m)$ a centered Gaussian vector on $\mathbb{R}^m$ with covariance matrix Γ^Y also such that $\Gamma^Y_{kk} = 1$ for every $k = 1, \dots, m$, yielding a J function (17)

$$J(u_1, \dots, u_m) = \mathbb{P}\big(Y_1 \leq \alpha_1(u_1), \dots, Y_m \leq \alpha_m(u_m)\big), \quad u_1, \dots, u_m \in (0, 1).$$

Choose now again $\alpha_k = \Phi^{-1}$. Arguing as in *(iii)* towards $\mathrm{Hess}(J) \circ \Gamma^X \leq 0$, the condition is now that

$$\sum_{k=1}^m \sum_{\ell \neq k} \Gamma^Y_{k\ell}\, p_{k\ell}\, v_k^2 - \sum_{k \neq \ell} p_{k\ell}\, \Gamma^X_{k\ell}\, v_k v_\ell \geq 0$$

for all $v_1, \dots, v_m \in \mathbb{R}$ (where p is here the density of the law of Y). This holds as soon as $\Gamma^Y_{k\ell} \geq 0$ and

$$\big(\Gamma^X_{k\ell}\big)^2 \leq \big(\Gamma^Y_{k\ell}\big)^2$$

for all $k \neq \ell$. As a conclusion

Corollary 7. *Let $X = (X_1, \dots, X_m)$ and $Y = (Y_1, \dots, Y_m)$ be centered Gaussian vectors on $\mathbb{R}^m$ with respective (non-degenerate) covariance matrices Γ^X and Γ^Y. Assume that $\Gamma^X_{kk} = \Gamma^Y_{kk} = 1$ and*

$$\left|\Gamma^X_{k\ell}\right| \leq \Gamma^Y_{k\ell}$$

for all $k, \ell = 1, \dots, m$. Then, for any Borel sets $B_1, \dots, B_m$ in $\mathbb{R}$,

$$\mathbb{P}(X_1 \in B_1, \dots, X_m \in B_m) \leq \mathbb{P}(Y_1 \leq b_1, \dots, Y_m \leq b_m)$$

where $\mathbb{P}(X_k \in B_k) = \Phi(b_k)$, $k = 1, \dots, m$. In particular, for every $r_1, \dots, r_m$ in $\mathbb{R}$,

$$\mathbb{P}(X_1 \leq r_1, \dots, X_m \leq r_m) \leq \mathbb{P}(Y_1 \leq r_1, \dots, Y_m \leq r_m).$$

This result is of course a weak form, in particular due to the constraint $\Gamma^Y_{k\ell} \geq 0$ (with however a somewhat stronger conclusion), of the classical Slepian lemma which indicates that for Gaussian vectors X and Y in $\mathbb{R}^m$, the conclusion of Corollary 7 holds whenever $\Gamma^X_{kk} = \Gamma^Y_{kk}$ and $\Gamma^X_{k\ell} \leq \Gamma^Y_{k\ell}$ for all $k, \ell = 1, \dots, m$. Note that the traditional proof of Slepian's lemma [21, 22, 32, 36] is an interpolation between the covariances Γ^X and Γ^Y which is not exactly the same as the one at the root of Corollary 7.

4 Log-Concave Measures

In this section, we develop the heat flow proof of Theorem 1 of E. Mossel and J. Neeman in the somewhat extended context of probability measures $d\mu = \mathrm{e}^{-V}dx$ on $\mathbb{R}^n$ such that V is a smooth potential with a uniform lower bound on its Hessian. The typical application actually concerns potentials V which are more convex than the quadratic one, corresponding to Gaussian measures. The argument may be extended to the more general context of Markov diffusion semigroups and the Γ-calculus as exposed in [3] although for the simplicity of this note, we stay in the familiar Euclidean setting.

Consider therefore a probability measure $d\mu = \mathrm{e}^{-V}dx$ on the Borel sets of $\mathbb{R}^n$, invariant and symmetric measure of the second order differential operator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ where V is a smooth potential on $\mathbb{R}^n$. The (symmetric) semigroup $(P_t)_{t\geq 0}$ with generator L may be represented by (smooth) probability kernels

$$P_t h(x) = \int_{\mathbb{R}^n} h(y)\, p_t(x, dy). \tag{18}$$

It will be assumed that $V - c\,\frac{|x|^2}{2}$ is convex for some $c \in \mathbb{R}$, in other words the Hessian of V is bounded from below by $c\,\mathrm{Id}_n$ as symmetric matrices. It is by now classical (cf. [3]) that this convexity assumption ensures that for all (smooth) $h : \mathbb{R}^n \to \mathbb{R}$,

$$|\nabla P_t h| \leq \mathrm{e}^{-ct} P_t\big(|\nabla h|\big). \tag{19}$$

The Gaussian example of the Ornstein-Uhlenbeck semigroup $(Q_t)_{t\geq 0}$ with invariant measure γ is included with $c = 1$. In this case, due to the representation (2), the gradient bound (19) actually turns into the identity $\nabla Q_t h = \mathrm{e}^{-t} Q_t(\nabla h)$.

We start with the analogue of Theorem 1 in this context following therefore the argument of [28].

Theorem 8. *Let J be ρ-concave, $\rho > 0$, on $\mathcal{R} = I_1 \times I_2 \subset \mathbb{R}^2$ where I_1 and I_2 are open intervals. Then, for every $f : \mathbb{R}^n \to I_1$, $g : \mathbb{R}^n \to I_2$ suitably integrable, and with $\rho = \mathrm{e}^{-ct}$, $t > 0$,*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(f(x), g(y)\big)\, p_t(x,dy)d\mu(x) \leq J\bigg(\int_{\mathbb{R}^n} f\,d\mu, \int_{\mathbb{R}^n} g\,d\mu\bigg).$$

Proof. It is enough to assume that f and g are taking values in respective compact sub-intervals of I_1 and I_2. Set

$$\psi(s) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J\big(P_s f(x), P_s g(y)\big)\, p_t(x,dy)d\mu(x), \quad s \geq 0.$$

The task is to show that ψ is non-decreasing. Taking derivative in time s,

$$\begin{aligned}\psi'(s) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \mathrm{L} P_s f(x)\, p_t(x,dy)d\mu(x) \\ &\quad + \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_2 J\big(P_s f(x), P_s g(y)\big) \mathrm{L} P_s g(y)\, p_t(x,dy)d\mu(x).\end{aligned}$$

By integration by parts in space with respect to the operator L, expressed (for smooth functions $\xi, \zeta : \mathbb{R}^n \to \mathbb{R}$) by

$$\int_{\mathbb{R}^n} \xi(-\mathrm{L}\zeta)d\mu = \int_{\mathbb{R}^n} \nabla\xi \cdot \nabla\zeta\, d\mu,$$

it holds

$$\begin{aligned}&\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \mathrm{L} P_s f(x)\, p_t(x,dy)d\mu(x) \\ &\qquad = -\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_{11} J\big(P_s f(x), P_s g(y)\big) \big|\nabla P_s f(x)\big|^2 p_t(x,dy)d\mu(x) \\ &\qquad\quad - \int_{\mathbb{R}^n \times \mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \nabla P_s f(x) \cdot \nabla_x p_t(x,dy)d\mu(x).\end{aligned}$$

For $x \in \mathbb{R}^n$ fixed, consider $h(y) = \partial_1 J(P_s f(x), P_s g(y))$, $y \in \mathbb{R}^n$. Since

$$\nabla P_t h(z) = \int_{\mathbb{R}^n} h(y) \nabla_z p_t(z, dy), \quad z \in \mathbb{R}^n,$$

at $z = x$,

$$\int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \nabla P_s f(x) \cdot \nabla_x p_t(x, dy) = \nabla P_t h(x) \cdot \nabla P_s f(x).$$

Now, by (19),

$$\big|\nabla P_t h(x)\big| \le \mathrm{e}^{-ct} P_t\big(|\nabla h|\big)(x) = \mathrm{e}^{-ct} \int_{\mathbb{R}^n} \big|\nabla h(y)\big| p_t(x, dy).$$

Since

$$\nabla h(y) = \partial_{12} J\big(P_s f(x), P_s g(y)\big) \nabla P_s g(y),$$

it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \partial_1 J\big(P_s f(x), P_s g(y)\big) \nabla_x p_t(x, dy) \cdot \nabla P_s f(x) d\mu(x) \\
&\quad \le \mathrm{e}^{-ct} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\partial_{12} J|\big(P_s f(x), P_s g(y)\big) \big|\nabla P_s g(y)\big| \big|\nabla P_s f(x)\big| p_t(x, dy) d\mu(x).
\end{aligned}$$

Summarizing, and by the symmetric conclusion in the y variable, $\psi'(s)$ is bounded from below by

$$\begin{aligned}
\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \Big[(-\partial_{11} J)|\nabla P_s f|^2 + (-\partial_{22} J)|\nabla P_s g|^2 - 2\,\mathrm{e}^{-cs} |\partial_{12} J| |\nabla P_s f| |\nabla P_s g|\Big]& \\
\times\, p_t(x, dy) d\mu(x).&
\end{aligned}$$

From the hypothesis on the Hessian of J, it follows that $\psi' \ge 0$ which is the result. □

As in the Gaussian case, the examples of illustration of Theorem 8 cover both hypercontractivity and noise stability for the choices of $J = J^{\mathrm{H}}$ or $J = J^{\mathrm{B}}$. Under $c > 0$, the choice of J^{H} yields hypercontractivity of the semigroup associated to this family of invariant measures, and thus the equivalent logarithmic Sobolev inequality for μ (cf. [3]). On the other hand, the noise stability part actually turns into a comparison theorem.

Corollary 9. *Let $(P_t)_{t \ge 0}$ be the Markov semigroup with invariant reversible measure $d\mu = \mathrm{e}^{-V} dx$ where V is a smooth potential on $\mathbb{R}^n$ such that* $\mathrm{Hess}(V) \ge c\, \mathrm{Id}_n$

with $c > 0$*. Then, whenever* A, B *are Borel sets in* $\mathbb{R}^n$ *and* H, K *are respective parallel half-spaces such that* $\mu(A) = \gamma(H)$*,* $\mu(B) = \gamma(K)$*, then*

$$\int_{\mathbb{R}^n} 1_A P_t(1_B) d\mu \le \int_{\mathbb{R}^n} 1_H \, Q_{ct}(1_K) d\gamma.$$

Again, as in the Gaussian setting (cf. [26]), the comparison property of Corollary 9 may be shown to imply the isoperimetric comparison theorem of [2] (see [3]) comparing the isoperimetric profile of measures $d\mu = \mathrm{e}^{-V} dx$ to the Gaussian one.

Next, we turn to the multidimensional version of the preceding result, with therefore in the following $c > 0$. Let $X = (X_t)_{t \ge 0}$ be the Markov process with generator $\mathrm{L} = \Delta - \nabla V \cdot \nabla$ and initial invariant distribution $d\mu = \mathrm{e}^{-V} dx$. We are interested in the distribution of $(X_{t_1}, \dots, X_{t_m})$ where $0 \le t_1 \le \dots \le t_m$. Consider the covariance matrix Γ the Ornstein-Uhlenbeck process at speed ct, that is $\Gamma_{k\ell} = \mathrm{e}^{-c|t_k - t_\ell|}$, $k, \ell = 1, \dots, m$. In the Gaussian case, this extension (for thus the Ornstein-Uhlenbeck process) was achieved by the study of general Gaussian vectors. In the present case, we deal with the kernels as given by (18), for simplicity one-dimensional.

Theorem 10. *In the preceding notation, assume that the Hadamard product of* $(|\partial_{k\ell} J|)_{1 \le k, \ell \le m}$ *and* Γ *is (semi-) negative-definite. Then, for every* $f_i : \mathbb{R} \to I_i$*,* $i = 1, \dots, m$*, suitably integrable,*

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} J\big(f_1(x_1), \dots, f_m(x_m)\big) p_{t_m - t_{m-1}}(x_{m-1}, dx_m) \cdots p_{t_2 - t_1}(x_1, dx_2) d\mu(x_1)$$
$$\le J\bigg(\int_{\mathbb{R}} f_1 \, d\mu, \dots, \int_{\mathbb{R}} f_m \, d\mu\bigg).$$

We outline the argument when $m = 3$. Consider

$$\psi(s) = \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} J\big(P_s f(x), P_s g(y), P_s h(z)\big) p_{t-u}(y, dz) p_t(x, dy) d\mu(x), \quad s \ge 0,$$

for $t > u > 0$ and three functions f, g, h. Differentiating ψ and integrating by parts in space leads to consider expressions such as

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \partial_1 J \; p_{t-u}(y, dz) \partial_x p_u(x, dy) \partial_x P_s f \; d\mu(x).$$

Arguing as in the proof of Theorem 8, this expression is equal to

$$\int_{\mathbb{R}} \partial_x P_s k \, \partial_x P_s f \; d\mu(x)$$

where $k = k(y) = \int_{\mathbb{R}} \partial_1 J \; p_{t-u}(y, dz)$. Now by (19)

$$|\partial_x P_s k| \leq \mathrm{e}^{-cs} P_s\big(|\partial_y k|\big).$$

Since

$$\partial_y k = \int_{\mathbb{R}} \partial_{12} J \; p_{t-u}(y, dz) + \int_{\mathbb{R}} \partial_1 J \; \partial_y p_{t-u}(y, dz),$$

similarly

$$|\partial_y k| \leq \int_{\mathbb{R}} |\partial_{12} J| p_{t-u}(y, dz) + \mathrm{e}^{-c(t-u)} \int_{\mathbb{R}} |\partial_{13} J| \partial_y p_{t-u}(y, dz).$$

The proof is then completed in the same way.

With the J function (17) associated to a finite-dimensional distribution of the Ornstein-Uhlenbeck process, the following consequence holds true.

Corollary 11. *Let $c > 0$ and $0 \leq t_1 \leq \cdots \leq t_m$. For any Borel sets $B_1, \ldots, B_m$ in $\mathbb{R}$,*

$$\mathbb{P}(X_{t_1} \in B_1, \ldots, X_{t_m} \in B_m) \leq \mathbb{P}(Z_{ct_1} \leq b_1, \ldots, Z_{ct_m} \leq b_m)$$

where $\mathbb{P}(X_{t_k} \in B_k) = \mu(B_k) = \Phi(b_k)$, $k = 1, \ldots, m$ and where $(Z_{ct})_{t \geq 0}$ is the Ornstein-Uhlenbeck process with speed ct.

As suggested by J. Neeman following his arguments developed in [30], Corollary 11 may be used towards a comparison property between hitting times. For a Borel set B in $\mathbb{R}$, let $e_B^X = \inf\{t \geq 0\,; X_t \notin B\}$ be the exit time of the Markov process $X = (X_t)_{t \geq 0}$ from the set B.

Corollary 12. *Under the preceding notation, for any $s \geq 0$,*

$$\mathbb{P}\big(e_B^X \geq s\big) \leq \mathbb{P}\big(e_H^Z \geq s\big)$$

where H is a half-line in $\mathbb{R}$ such that $\gamma(H) = \mu(B)$ and $Z = (Z_{ct})_{t \geq 0}$ the Ornstein-Uhlenbeck process at speed ct.

5 The Discrete Cube

To conclude this note, we briefly address in this last section the corresponding noise stability issue on the discrete cube, and collect a few remarks in connection with the recent development [18] on the "Majority is Stablest" theorem.

By Theorem 1, a function J is ρ-concave in the sense that

$$\begin{pmatrix} \partial_{11} J & \rho\, \partial_{12} J \\ \rho\, \partial_{12} J & \partial_{22} J \end{pmatrix} \leq 0$$

if and only if for all suitable functions f and g on $\mathbb{R}^n$,

$$\begin{aligned} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} J\big(f(x), g\big(\rho x + \sqrt{1-\rho^2}\, y\big)\big) d\gamma(x) d\gamma(y) & \\ \leq J\bigg(\int_{\mathbb{R}^n} f\, d\gamma, \int_{\mathbb{R}^n} g\, d\gamma\bigg). & \end{aligned} \tag{20}$$

The analogue of the Gaussian couple $(X, \rho X + \sqrt{1-\rho^2}\, Y)$ with correlation $\rho\,\mathrm{Id}_n$, $\rho \in [-1, 1]$, on the discrete cube $\Sigma^n = \{-1, +1\}^n$, with $n = 1$ to start with, leads to consider a couple with distribution

$$(1 + \rho xy) d\mu(x) d\mu(y)$$

on Σ^2, where μ is the uniform probability measure on $\Sigma = \{-1, +1\}$. The latter inequality (20) on the two-point space $\Sigma = \{-1, +1\}$ therefore amounts to

$$\int_\Sigma \int_\Sigma J\big(f(x), g(y)\big) K_\rho(x, y) d\mu(x) d\mu(y) \leq J\bigg(\int_\Sigma f\, d\mu, \int_\Sigma g\, d\mu\bigg) \tag{21}$$

for every functions $f, g : \Sigma \to \mathbb{R}$, where $K_\rho(x, y) = 1 + \rho xy$. This inequality (21) is stable under product. On $\Sigma^n = \{-1, +1\}^n$ equipped with the uniform product measure μ^n, let for $\rho \in \mathbb{R}$ and $x = (x_1, \ldots, x_n) \in \Sigma^n$, $y = (y_1, \ldots, y_n) \in \Sigma^n$,

$$K_\rho(x, y) = \prod_{i=1}^n (1 + \rho\, x_i\, y_i).$$

If (21) holds, for every f, g on Σ^n,

$$\int_{\Sigma^n} \int_{\Sigma^n} J\big(f(x), g(y)\big) K_\rho(x, y) d\mu^n(x) d\mu^n(y) \leq J\bigg(\int_{\Sigma^n} f d\mu^n, \int_{\Sigma^n} g d\mu^n\bigg).$$

One may of course wonder for the equivalence of (21) with the ρ-concavity of J. Actually, (21) expresses equivalently a 4-point inequality similar to the standard characterization of concavity. Say namely that a function J on some open convex set $\mathcal{O}$ of $\mathbb{R}^2$ is strongly ρ-concave for some $\rho \in \mathbb{R}$ if for all $(u, v) \in \mathcal{O}$, $(u', v') \in \mathcal{O}$,

$$\begin{aligned} \tfrac{1+\rho}{4}\, J(u, v) + \tfrac{1-\rho}{4}\, J(u', v) + \tfrac{1-\rho}{4}\, J(u, v') + \tfrac{1+\rho}{4}\, J(u', v') & \\ \leq J\Big(\tfrac{u+u'}{2}, \tfrac{v+v'}{2}\Big). & \end{aligned} \tag{22}$$

Lemma 13. *Strong ρ-concavity implies ρ-concavity (for smooth functions).*

Proof. By a Taylor expansion, at any $(a,b) \in \mathcal{O}$, $(h,k) \in \mathbb{R}^2$, such that $(a \pm h, b \pm k) \in \mathcal{O}$,

$$\begin{aligned}
&(1+\rho)\big[J(a+h,b+k) + J(a-h,b-k) - 2J(a,b)\big] \\
&\quad + (1-\rho)\big[J(a+h,b-k) + J(a-h,b+k) - 2J(a,b)\big] \\
&\qquad = 2h^2\partial_{11}J(a,b) + 4\rho hk\partial_{12}J(a,b) + 2k^2\partial_{22}J(a,b) + o(h^2+k^2).
\end{aligned}$$

With $u = a+h$, $v = b+k$, $u' = a-h$, $v' = b-k$, (22) implies the ρ-concavity of J as $h,k \to 0$. □

It is a main result, namely the Bonami-Beckner hypercontractivity theorem [9,11], that the hypercontractive function J^{H} is strongly ρ-concave under (5) (along the equivalence between hypercontractivity and Theorem 1 described in Sect. 2 for the Ornstein-Uhlenbeck semigroup). However, we could not establish directly the strong ρ-concavity of J^{H} in this case. Such a proof could give a better understanding of the strong ρ-concavity property.

On the other hand, it is not true in general that ρ-concavity implies back strong ρ-concavity and one example, taken from [18], is simply Borell's noise stability function J^{B} (with parameter $\rho \in (0,1)$). Indeed, for $u = v = 1$ and $u' = v' = 0$ (in the Boolean analysis terminology, this choice corresponds to the dictator functions $f(x) = \frac{1}{2} + \frac{x}{2}$, $g(y) = \frac{1}{2} + \frac{y}{2}$), (22) would imply that

$$1 + \rho \le 4\, J^{\mathrm{B}}\Big(\frac{1}{2}, \frac{1}{2}\Big) \tag{23}$$

since $J^{\mathrm{B}}(1,1) = 1$ and $J^{\mathrm{B}}(1,0) = J^{\mathrm{B}}(0,1) = J^{\mathrm{B}}(0,0) = 0$. But

$$J^{\mathrm{B}}\Big(\frac{1}{2}, \frac{1}{2}\Big) = \int_{-\infty}^{0}\int_{-\infty}^{0} q_t^1(x,y)d\gamma^1(x)d\gamma^1(y) = \int_0^\infty \Phi(\alpha x)d\gamma^1(x)$$

where $\alpha = \frac{\rho}{\sqrt{1-\rho^2}}$ and $\rho = \mathrm{e}^{-t}$. Taking the derivative in α easily shows that

$$J^{\mathrm{B}}\Big(\frac{1}{2}, \frac{1}{2}\Big) = \frac{1}{4} + \frac{1}{2\pi}\arctan(\alpha) = \frac{1}{2} - \frac{1}{2\pi}\arccos(\rho)$$

so that (23) indeed fails as $\rho \to 0$. This value of $J^{\mathrm{B}}\big(\frac{1}{2}, \frac{1}{2}\big)$ appears in Sheppard's formula put forward in [35] as early as 1899 as the asymptotic noise stability of the Majority function (see [18,29]).

It would be of interest to understand which additional property to ρ-concavity ensures strong ρ-concavity. A. De et al. [18] recently observed by a suitable Taylor expansion that there exists, for any $\rho \in (0,1)$, $C(\rho) > 0$ such that

$$\left|\frac{\partial^3 J_\rho^{\mathrm{B}}(u,v)}{\partial^i u\, \partial^j v}\right| \le C(\rho)\big[uv(1-u)(1-v)\big]^{-C(\rho)}$$

for all $i, j \geq 0$ with $i + j = 3$. This property then implies the approximate validity of (22) in the sense that for every $u, u', v, v' \in [\varepsilon, 1-\varepsilon]$ for some $\varepsilon > 0$,

$$\begin{aligned} &\tfrac{1+\rho}{4} J_\rho^{\mathrm{B}}(u,v) + \tfrac{1-\rho}{4} J_\rho^{\mathrm{B}}(u',v) + \tfrac{1-\rho}{4} J_\rho^{\mathrm{B}}(u,v') + \tfrac{1+\rho}{4} J_\rho^{\mathrm{B}}(u',v') \\ &\quad \leq J_\rho^{\mathrm{B}}\Big(\tfrac{u+u'}{2}, \tfrac{v+v'}{2}\Big) + C'(\rho)\, \varepsilon^{-C'(\rho)}\big(|u-u'|^3 + |v-v'|^3\big). \end{aligned} \tag{24}$$

As a main achievement, the authors of [18] develop from this conclusion and tensorization a fully discrete proof of the "Majority is Stablest" theorem of [29] (with Sheppard's constant $J^{\mathrm{B}}\big(\frac{1}{2}, \frac{1}{2}\big)$ as stability value) by suitably controlling the error term via the influences of the Boolean functions under investigation.

One further observation of [18] is that the preceding two-point inequality (24) is still good enough to reach, after tensorization and the central limit theorem, Borell's noise stability theorem for the Ornstein-Uhlenbeck semigroup.

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