

Chapter 2

Macroscopic Quantum Electrodynamics

Photon, photon, shining bright! Diffracting through the lab, at night. Can even God, with all his might, measure thy position right?

The author

2.1 Field Quantisation in Vacuum

2.1.1 Quantising the Light Field

We now learn from an early age that light is a form of radiation and a wave phenomenon that admits a continuum of frequencies. The visible light that the human eye perceives unaided is but a part of a larger electromagnetic spectrum that includes high frequency ultra-violet light at one end and low-frequency radiowaves at the other, involving an interplay of oscillating electric and magnetic fields that is well-described by Maxwell's equations. It is also an increasingly familiar thought, since the early 20th century, that light is in some sense particulate; it deposits its energy in discrete packets called photons, as postulated by Max Planck in his explanation of blackbody radiation, and hypostatized by Einstein in his treatment of the photo-electric effect. In fact, the reverberating dualism between light *qua* 'wave' and light *qua* 'atoms' has been traced back as far as the philosophers of classical Greece and the Hindu schools of ancient India.

In the quantum theory of light, first formulated by the British scientist Paul Dirac, something of a reconciliation of these seemingly incompatible conceptions is achieved—at least, at a formal level. Dirac proposed a quantisation of the electromagnetic field, as described by Maxwell's equations, involving an ensemble of harmonic oscillators with discrete energy levels. These 'quanta' are called photons. We will briefly propound a simple version of this *quantum electrodynamics* here.

2.1.2 Maxwell's Equations

In the vacuum, the phenomenon of light is characterised by the electric field \mathbf{E} and the magnetic induction \mathbf{B} . The classical electrodynamic field obeys Maxwell's equations,

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.1.1)$$

$$\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (2.1.2)$$

with the added constraint that the fields vanish at infinity. A useful representation of the fields involves the vector potential \mathbf{A} ,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}. \quad (2.1.3)$$

Expressed in this form, the first two of Maxwell's equations (2.1.1) are automatically satisfied. By imposing the Coulomb gauge,

$$\nabla \cdot \mathbf{A} = 0 \quad (2.1.4)$$

the third of Maxwell's equations (2.1.2) is also satisfied. Using the only remaining non-trivial equation (2.1.2) we can derive the wave equation of light:

$$\nabla \times \nabla \times \mathbf{A} + \frac{1}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0. \quad (2.1.5)$$

In quantum field theory the classical amplitudes must be replaced by quantum observables. The postulated connection between the classical and the quantum fields is elegantly simple, however: the classical fields are the ensemble averages of the quantum fields, e.g.

$$\mathbf{E} = \langle \psi | \hat{\mathbf{E}} | \psi \rangle, \quad (2.1.6)$$

where $|\psi\rangle$ is the state vector of the system, and $\hat{\mathbf{E}}$ is the 'operator-valued' quantum electric field. From the linearity of quantum mechanics and of Maxwell's equations, it follows that the quantum operators $\hat{\mathbf{E}}$ and $\hat{\mathbf{B}}$ also obey both Maxwell's equations (2.1.1, 2.1.2) and the electromagnetic wave equation (2.1.5). The specifically quantum-mechanical character of the field operators, however, resides in a fundamental commutation relation. It can be shown that [1]

$$[\hat{\mathbf{E}}(\mathbf{r}, t), \hat{\mathbf{E}}(\mathbf{r}', t)] = [\hat{\mathbf{B}}(\mathbf{r}, t), \hat{\mathbf{B}}(\mathbf{r}', t)] = \mathbf{0}, \quad (2.1.7)$$

$$[\hat{\mathbf{E}}(\mathbf{r}, t), \hat{\mathbf{B}}(\mathbf{r}', t)] = \frac{i\hbar}{\epsilon_0} \nabla \times \delta^T(\mathbf{r} - \mathbf{r}'), \quad (2.1.8)$$

where δ^T is the transversal delta function. Embedded in this result is a second assumption concerning the Hamiltonian of light, which governs the time evolution of the field and describes the total energy of the system: the *quantum* Hamiltonian has the same structure as the classical energy of the electromagnetic field [2]:

$$\hat{H} = \frac{1}{2} \int (\hat{\mathbf{E}} \cdot \hat{\mathbf{E}} + \hat{\mathbf{B}} \cdot \hat{\mathbf{B}}) dV, \quad (2.1.9)$$

where the volume integration is over all space.

2.1.3 The Quantum Light Mode

It is useful for our purposes to focus primarily on the vector potential \mathbf{A} , which determines the electromagnetic field (2.1.4), and to expand this quantum operator in terms of an appropriate set of light modes:

$$\hat{\mathbf{A}}(\mathbf{r}, t) = \sum_k \left(\mathbf{A}_k(\mathbf{r}, t) \hat{a}_k + \mathbf{A}_k^*(\mathbf{r}, t) \hat{a}_k^\dagger \right). \quad (2.1.10)$$

In this formalism the modes, which are a set of classical waves $\{\mathbf{A}_k(\mathbf{r}, t)\}$ obeying the laws of electromagnetism, are conjoined with quantum amplitude operators $\{\hat{a}_k, \hat{a}_k^\dagger\}$, which are associable with discrete excitations of the field. In the case of normal modes, the quantum amplitudes behave as creation and annihilation operators, satisfying Bose commutation relations

$$[\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}, \quad [\hat{a}_k, \hat{a}_{k'}] = 0. \quad (2.1.11)$$

For convenience, we consider the special case of monochromatic modes¹ that oscillate at single frequencies,

$$\mathbf{A}_k(\mathbf{r}, t) = \mathbf{A}_k(\mathbf{r}) \exp(-i\omega_k t). \quad (2.1.12)$$

Under this representation, the Hamiltonian (2.1.9) can be rendered very simply:

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right). \quad (2.1.13)$$

The total energy of the field is thus the sum of the energies of the modes, where each mode k carries an energy of $\hbar\omega_k (n + 1/2)$, and n is the number of photons present in that mode.

¹ Monochromatic modes are stationary modes that conserve energy. We therefore expect the Hamiltonian to be the sum of the Hamiltonians of the individual modes.

2.1.4 Zero-Point Energy

The form of the Hamiltonian in Eq. (2.1.13) is ubiquitous in quantum field theory; it embodies the expression for the energy of a simple quantum harmonic oscillator with discrete energy levels,

$$E = \hbar\omega (n + 1/2), \quad n \in \mathbb{Z}, \quad (2.1.14)$$

for which even the unexcited state of the system ($n = 0$) has a non-zero energy of $\hbar\omega/2$. This is the so-called ‘zero-point energy’ associated with the ground-state of the field. We see that each mode of the field is like a distinct oscillator; each represents a degree of freedom. But we also see that there are an infinite number of them in the electromagnetic field. It follows that even in the ground state of each mode—*ergo*, the absence of any photons in the field—the minimal value of the energy is

$$E_0 = \sum_k \frac{\hbar\omega_k}{2} = \infty. \quad (2.1.15)$$

It seems obvious that infinities of this sort are unfortunate artifacts. It is not so obvious how to ‘fix’ quantum field theory to prevent it from producing them. Still, as we have seen in Chap. 1, the zero point energy leads to experimentally confirmed results. One of these results is the Casimir force (1.1.23).

2.1.5 External Boundaries on a Quantum Field

As it stands, however, Eq. (2.1.15) fails to be physically meaningful on at least two counts: it is both infinite and without empirical reference. There is no reason to think that infinite plane waves exist in nature. One way to register these constraints is through the imposition of ‘external’ boundary conditions, in which the field strength and its derivatives are fixed at certain locations. In our calculation in Chap. 1, we required that the transversal components of the electric field and the normal component of the magnetic field should vanish at the locations of the mirrors. The frequencies ω_k , in this case, refer to a set of standing waves associated with the cavity, and with an energy that varies with the size of the cavity. This energy is finite, consequent upon an appropriate regularisation in which we effectively acknowledge that such a cavity cannot support an infinite number of frequencies.²

However, this method of representing the presence of material bodies within the system is acutely limited: real materials are dispersive, responding differently to different frequencies; they are dissipative, transforming electromagnetic energy into currents; and their presence is felt over spatially extended regions. Moreover, electromagnetic fields are present within material bodies, as well as the vacuum, where

² See Sect. 1.1.2.

Casimir forces are also to be expected. To facilitate a more sophisticated approach to the interaction of light with materials, we must learn to quantise the electromagnetic field in media.

2.2 Field Quantisation in Media

Our concern is with material bodies interacting with each other through the quantum electrodynamic field. Since such bodies involve large numbers of bound, charged particles, a microscopic description of their interaction is impractical, and we must instead consider the effective influence of these particles on the electromagnetic field. Contrary to conventional wisdom, a canonical quantisation of the electromagnetic field in the presence of media can in fact be performed [3, 4]. However, the process is somewhat involved, and the problems we wish to discuss formally do not depend upon the details. We will content ourselves, then, with a more phenomenological approach,³ for the purpose of recovering the critical results we require with minimal effort and familiarising the reader with some essential ideas.

2.2.1 The Macroscopic Maxwell Equations

In differential form, using *SI* units, the classical electrodynamic field obeys the macroscopic Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (2.2.1)$$

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \times \mathbf{H} = \frac{1}{c^2} \frac{\partial \mathbf{D}}{\partial t}, \quad (2.2.2)$$

where the fields \mathbf{D} and \mathbf{H} are effective electric and magnetic fields respectively, associated with the polarisation and magnetisation of the media. We do not concern ourselves with external free charges, in the Casimir Effect. However, in extending quantum electrodynamics from light in empty space to light in media, we must lay down a set of constitutive equations coupling the fields $\hat{\mathbf{E}}$ to $\hat{\mathbf{D}}$ and $\hat{\mathbf{H}}$ to $\hat{\mathbf{B}}$. In the linear response regime

$$\mathbf{D} = \epsilon_0 \epsilon \mathbf{E}, \quad \mathbf{B} = \mu_0 \mu \mathbf{H}, \quad \epsilon_0 \mu_0 = 1/c^2, \quad (2.2.3)$$

³ The basic approach adopted in this section is developed in detail in [8–10]. However, we will not use the Lorentz force expression, which does not recover the standard results for forces in media, but an analogue of the Minkowski stress tensor. Pitaevskii's comments here are relevant [13, 14].

where $(\epsilon_0) \epsilon$ is the *permittivity* of (free) space and $(\mu_0) \mu$ is the *permeability* of (free) space. For inhomogeneous media, ϵ and μ will vary in space. In the quantisation of the light field, all classical fields must be replaced by operator-valued quantum observables. It is convenient to introduce frequency components of the form

$$\hat{\mathbf{f}} = \int_0^\infty d\omega \hat{\mathbf{f}}(\omega) + \text{H.c.}, \quad (2.2.4)$$

for some quantum operator $\hat{\mathbf{f}}$, where H.c. denotes the Hermitian conjugate of the preceding term. The quantised Maxwell equations can now be written:

$$\nabla \cdot \hat{\mathbf{B}} = 0, \quad \nabla \times \hat{\mathbf{E}} - i\omega \hat{\mathbf{B}} = \mathbf{0}. \quad (2.2.5)$$

$$\nabla \cdot \hat{\mathbf{D}} = 0, \quad \nabla \times \hat{\mathbf{H}} + i\omega \hat{\mathbf{D}} = \mathbf{0}. \quad (2.2.6)$$

2.2.2 Quantum Noise

The Heisenberg uncertainty principle entails the presence of quantum fluctuations that must be felt by the properties of the medium. At this level of description, without invoking the additional apparatus involved in the canonical theory of macro-QED [4], we must follow Lifshitz' somewhat ad hoc procedure [5] of introducing noise polarisation and magnetisation terms into the descriptions of the fields:

$$\hat{\mathbf{D}} = \epsilon_0 \epsilon \hat{\mathbf{E}} + \hat{\mathbf{P}}_N, \quad (2.2.7)$$

$$\hat{\mathbf{H}} = \frac{1}{\mu_0 \mu} \hat{\mathbf{B}} - \hat{\mathbf{M}}_N. \quad (2.2.8)$$

Situated in the effective fields associated with the media, these sources generate internal noise currents and noise charge densities within the material:

$$\hat{\mathbf{J}}_N = -i\omega \hat{\mathbf{P}}_N + \nabla \times \hat{\mathbf{M}}_N, \quad (2.2.9)$$

$$\hat{\rho}_N = -\nabla \cdot \hat{\mathbf{P}}_N. \quad (2.2.10)$$

These source fields satisfy the continuity equation

$$-i\omega \hat{\rho}_N + \nabla \cdot \hat{\mathbf{J}}_N = 0. \quad (2.2.11)$$

In Lifshitz theory, the electromagnetic fields remain formally unquantised and classical, with the behavior of the noise fields being governed by the results of Rytov theory [6] to produce the appropriate stochastic behaviour [7]. Here, we will follow the more conspicuously quantum-mechanical approach developed in [8–10].

2.2.3 Bosonic Field Operators

Cast in the form of quantum operators, our noise fields must be made to satisfy quantum commutation relations. We choose to adopt relations such that the fluctuation spectrum obeys the fluctuation-dissipation theorem [11], entailing (among other things) that the noise should vanish *on average*. This can be achieved by appropriately relating the polarisation and magnetisation to bosonic creation and annihilation operators $\hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega)$ and $\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega)$ for the electric and magnetic contributions to the field $\lambda \in \{e, m\}$,

$$\hat{\mathbf{P}}_N(\mathbf{r}, \omega) = i \sqrt{\frac{\hbar \epsilon_0}{\pi}} \operatorname{Im} \epsilon(\mathbf{r}, \omega) \hat{\mathbf{f}}_e(\mathbf{r}, \omega), \quad (2.2.12)$$

$$\hat{\mathbf{M}}_N(\mathbf{r}, \omega) = i \sqrt{\frac{\hbar}{\pi \mu_0}} \frac{\operatorname{Im} \mu(\mathbf{r}, \omega)}{|\mu(\mathbf{r}, \omega)|^2} \hat{\mathbf{f}}_m(\mathbf{r}, \omega), \quad (2.2.13)$$

which themselves obey bosonic commutation relations:

$$\left[\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega), \hat{\mathbf{f}}_{\lambda'}(\mathbf{r}', \omega') \right] = \left[\hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega), \hat{\mathbf{f}}_{\lambda'}^\dagger(\mathbf{r}', \omega') \right] = \mathbf{0}, \quad (2.2.14)$$

$$\left[\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega), \hat{\mathbf{f}}_{\lambda'}^\dagger(\mathbf{r}', \omega') \right] = \delta_{\lambda\lambda'} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'). \quad (2.2.15)$$

In fact, these operators will serve to describe the collective, polariton-like, bosonic excitations of the body-field system, for which we may define a system ground-state:

$$\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega) |\{0\}\rangle = \mathbf{0} \quad \forall \lambda, \mathbf{r}, \omega. \quad (2.2.16)$$

When the system is in its ground state, the electromagnetic field is likewise ('the quantum vacuum'). A complete Hilbert-space spanned by Fock states can be obtained in the usual way by repeated application of creation operators $\hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega)$ to the ground state:

$$|\{n_{\lambda_1}(\mathbf{r}_1, \omega_1), n_{\lambda_2}(\mathbf{r}_2, \omega_2), \dots\}\rangle = \prod_{k, n_{\lambda_k}} \frac{1}{\sqrt{n_{\lambda_k}(\mathbf{r}_k, \omega_k)}!} \hat{\mathbf{f}}_{\lambda_k}^\dagger(\mathbf{r}_k, \omega_k) |\{0\}\rangle. \quad (2.2.17)$$

2.2.4 Field Fluctuations

For a system prepared in a state $|\phi\rangle$, where $|\phi\rangle$ is represented as a vector of a Hilbert space, the quantum average of an observable \hat{q} is given by

$$\langle \hat{q} \rangle = \langle \phi | \hat{q} | \phi \rangle. \quad (2.2.18)$$

The fluctuations associated with a quantum operator are

$$\langle (\Delta \hat{q})^2 \rangle = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2. \quad (2.2.19)$$

As noted, these fluctuations necessarily occur for two non-commuting operators as a direct consequence of the Heisenberg uncertainty principle:

$$\langle (\Delta \hat{f})^2 (\Delta \hat{g})^2 \rangle \geq \frac{1}{4} \left| \langle [\hat{f}, \hat{g}] \rangle \right|^2. \quad (2.2.20)$$

Our field operators \hat{f} clearly have a vanishing ground-state average:

$$\begin{aligned} \langle \{0\} | \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega) | \{0\} \rangle &= \mathbf{0}, \\ \langle \{0\} | \hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega) | \{0\} \rangle &= \mathbf{0}. \end{aligned} \quad (2.2.21)$$

It follows that the noise polarisation and magnetisation operators, defined in Eqs. (2.2.12) and (2.2.13), consequently have zero-averages as well:

$$\begin{aligned} \langle \hat{\mathbf{P}}_N(\mathbf{r}, \omega) \rangle &= \mathbf{0}, \\ \langle \hat{\mathbf{M}}_N(\mathbf{r}, \omega) \rangle &= \mathbf{0}. \end{aligned} \quad (2.2.22)$$

For paired field operators, we obtain the following results:

$$\begin{aligned} \langle \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega) \hat{\mathbf{f}}_{\lambda'}(\mathbf{r}', \omega') \rangle &= \mathbf{0}, \\ \langle \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega) \hat{\mathbf{f}}_{\lambda'}^\dagger(\mathbf{r}', \omega') \rangle &= \delta_{\lambda\lambda'} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'), \\ \langle \hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega) \hat{\mathbf{f}}_{\lambda'}(\mathbf{r}', \omega') \rangle &= \mathbf{0}, \\ \langle \hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega) \hat{\mathbf{f}}_{\lambda'}^\dagger(\mathbf{r}', \omega') \rangle &= \mathbf{0}. \end{aligned} \quad (2.2.23)$$

From these, it follows that the noise polarisation and magnetisation have non-zero fluctuations, leading to the expressions [10]

$$\langle S [\Delta \hat{\mathbf{P}}_N(\mathbf{r}, \omega) \Delta \hat{\mathbf{P}}_N^\dagger(\mathbf{r}', \omega')] \rangle = \frac{\hbar}{2\pi} \epsilon_0 \text{Im } \chi(\mathbf{r}, \omega) \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'), \quad (2.2.24)$$

$$\langle S [\Delta \hat{\mathbf{M}}_N(\mathbf{r}, \omega) \Delta \hat{\mathbf{M}}_N^\dagger(\mathbf{r}', \omega')] \rangle = \frac{\hbar}{2\pi} \frac{\text{Im } \zeta(\mathbf{r}, \omega)}{\mu_0} \delta(\mathbf{r} - \mathbf{r}') \delta(\omega - \omega'), \quad (2.2.25)$$

where the electric susceptibility $\chi(\mathbf{r}, \omega)$ is related to the electric permittivity via

$$\epsilon(\mathbf{r}, \omega) = 1 + \chi(\mathbf{r}, \omega), \quad (2.2.26)$$

the magnetic susceptibility $\zeta(\mathbf{r}, \omega)$ is related to the magnetic permeability via

$$\mu(\mathbf{r}, \omega) = \frac{1}{1 - \zeta(\mathbf{r}, \omega)}, \quad (2.2.27)$$

and S denotes a symmetrised operator product

$$S[\hat{a}\hat{b}] = \frac{1}{2}(\hat{a}\hat{b} + \hat{b}\hat{a}). \quad (2.2.28)$$

This result concurs with the fluctuation-dissipation theorem [11], which relates the fluctuations of a quantity with the rate of absorption of energy by the system when an external force is applied. In fact, we intentionally defined (2.2.12) and (2.2.13) in order for them to do so. This is the same spectrum recovered by Rytov theory, and its application to the electromagnetic field lies at the heart of the Lifshitz calculation of the Casimir force.

2.2.5 The Fundamental Fields

It is now possible to express the other fields in terms of the fundamental field operators. Of course, we achieve this by solving Maxwell's equations. Noting that the noise operators we have introduced have effectively created internal currents (2.2.9) and charges (2.2.10) in the system, we can use the inhomogeneous Helmholtz equation for the electric field

$$\left[\nabla \times \frac{1}{\mu} \nabla \times - \frac{\omega^2}{c^2} \epsilon \right] \hat{\mathbf{E}} = i\mu_0\omega \hat{\mathbf{j}}_N. \quad (2.2.29)$$

This equation is formally solveable by means of a classical Green tensor characterising the linear response of the field to the current sources:

$$\hat{\mathbf{E}}(\mathbf{r}, \omega) = i\mu_0\omega \int d^3r' \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \cdot \hat{\mathbf{j}}_N(\mathbf{r}', \omega). \quad (2.2.30)$$

The noise current (2.2.9) was earlier expressed in terms of the noise polarisation (2.2.12) and magnetisation (2.2.13), and these have been rewritten in terms of the fundamental field operators $\hat{\mathbf{f}}_e(\mathbf{r}, \omega)$ and $\hat{\mathbf{f}}_m(\mathbf{r}, \omega)$. With reference to Eqs. (2.2.9), (2.2.12) and (2.2.13), an expression for the field can be straightforwardly written as

$$\hat{\mathbf{E}}(\mathbf{r}) = \int_0^\infty d\omega \sum_{\lambda=e,m} \int d^3r' \mathbf{G}_\lambda(\mathbf{r}, \mathbf{r}', \omega) \cdot \hat{\mathbf{f}}_\lambda(\mathbf{r}', \omega) + \text{H.c.}, \quad (2.2.31)$$

where

$$\mathbf{G}_e(\mathbf{r}, \mathbf{r}', \omega) = i \frac{\omega^2}{c^2} \left(\frac{\hbar}{\pi \epsilon_0} \text{Im} \epsilon(\mathbf{r}', \omega) \right)^{1/2} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega), \quad (2.2.32)$$

$$\mathbf{G}_m(\mathbf{r}, \mathbf{r}', \omega) = i \frac{\omega}{c} \left(\frac{\hbar}{\pi \epsilon_0} \frac{\text{Im} \mu(\mathbf{r}', \omega)}{|\mu(\mathbf{r}, \omega)|^2} \right)^{1/2} [\nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega)]^T. \quad (2.2.33)$$

The other fields may also be expressed similarly. Using Eq. (2.2.31) and the Maxwell equation (2.2.5), we obtain

$$\hat{\mathbf{B}}(\mathbf{r}) = \int_0^\infty \frac{d\omega}{i\omega} \sum_{\lambda=e,m} \int d^3 r' \nabla \times \mathbf{G}_\lambda(\mathbf{r}, \mathbf{r}', \omega) \cdot \hat{\mathbf{f}}_\lambda(\mathbf{r}', \omega) + \text{H.c.} \quad (2.2.34)$$

From (2.2.7) and (2.2.12), conjoined with (2.2.31) we obtain

$$\begin{aligned} \hat{\mathbf{D}}(\mathbf{r}) = \int_0^\infty d\omega \left[\epsilon_0 \epsilon(\mathbf{r}, \omega) \sum_{\lambda=e,m} \int d^3 r' \mathbf{G}_\lambda(\mathbf{r}, \mathbf{r}') \cdot \mathbf{f}_\lambda(\mathbf{r}', \omega) \right. \\ \left. + i \left(\frac{\hbar \epsilon_0}{\pi} \text{Im} \epsilon(\mathbf{r}, \omega) \right)^{1/2} \mathbf{f}_e(\mathbf{r}, \omega) \right] + \text{H.c.} \end{aligned} \quad (2.2.35)$$

And from (2.2.8) and (2.2.13), conjoined with (2.2.34) we obtain

$$\begin{aligned} \mathbf{H}(\mathbf{r}) = \int_0^\infty d\omega \left[\frac{1}{i\omega \mu_0 \mu(\mathbf{r}, \omega)} \sum_{e,m} \int d^3 r' \nabla \times \mathbf{G}_\lambda(\mathbf{r}, \mathbf{r}') \cdot \mathbf{f}_\lambda(\mathbf{r}', \omega) \right. \\ \left. - \left(\frac{\hbar \kappa_0}{\pi} \frac{\text{Im} \mu(\mathbf{r}, \omega)}{\mu(\mathbf{r}, \omega)} \right)^{1/2} \mathbf{f}_m(\mathbf{r}, \omega) \right] + \text{H.c.} \end{aligned} \quad (2.2.36)$$

With a little further work [8], we can recover the commutation relations

$$[\mathbf{E}(\mathbf{r}), \mathbf{E}(\mathbf{r}')] = [\mathbf{B}(\mathbf{r}), \mathbf{B}(\mathbf{r}')] = \mathbf{0}, \quad (2.2.37)$$

$$[\mathbf{E}(\mathbf{r}), \mathbf{B}(\mathbf{r}')] = \frac{i\hbar}{\epsilon_0} \nabla \times \delta(\mathbf{r} - \mathbf{r}'), \quad (2.2.38)$$

which agree with those in free-space (2.1.8), and deduce the ground-state fluctuation spectrum of the field [10]

$$\left\langle S \left[\Delta \hat{\mathbf{E}}(\mathbf{r}, \omega) \Delta \hat{\mathbf{E}}^\dagger(\mathbf{r}', \omega') \right] \right\rangle = \frac{\hbar}{2\pi} \omega^2 \mu_0 \text{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \delta(\omega - \omega'), \quad (2.2.39)$$

which is in accordance with the fluctuation-dissipation theorem [11].

2.2.6 The Hamiltonian

The Hamiltonian governs the dynamics of a system. Since the electromagnetic field operators we introduced are linear combinations of the fundamental field operators, our Hamiltonian must therefore generate the correct time-dependence of the field operators, so that Maxwell's equations and the constitutive equations hold. This behaviour is implemented using the Hamiltonian

$$\hat{H} = \sum_{\lambda=e,m} \int d^3r \int_0^\infty d\omega \hbar\omega \hat{\mathbf{f}}_\lambda^\dagger(\mathbf{r}, \omega) \cdot \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega). \quad (2.2.40)$$

From the Heisenberg equation of motion, we find

$$\frac{d}{dt} \hat{\mathbf{f}}_\lambda = \frac{1}{i\hbar} [\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega), \hat{H}] = -i\omega \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega). \quad (2.2.41)$$

This differential equation is solved by

$$\hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega, t) = \hat{\mathbf{f}}_\lambda(\mathbf{r}, \omega) e^{-i\omega t}. \quad (2.2.42)$$

This gives the correct behaviour for recovering the Maxwell equations: the time-dependent frequency components of the electromagnetic fields are ordinary Fourier components. The ground state defined earlier is clearly an eigenstate of this Hamiltonian,

$$\hat{H} | \{0\} \rangle = 0. \quad (2.2.43)$$

Using the bosonic commutation relations (2.2.15), we find that

$$\hat{H} | n_\lambda(\mathbf{r}, \omega) \rangle = \hbar\omega n_\lambda(\mathbf{r}, \omega) | n_\lambda(\mathbf{r}, \omega) \rangle, \quad (2.2.44)$$

and more generally that

$$\begin{aligned} \hat{H} | n_{\lambda_1}(\mathbf{r}_1, \omega_1), n_{\lambda_2}(\mathbf{r}_2, \omega_2), \dots \rangle \\ = \hbar (n_{\lambda_1}\omega_1 + n_{\lambda_2}\omega_2 + \dots) | n_{\lambda_1}(\mathbf{r}_1, \omega_1), n_{\lambda_2}(\mathbf{r}_2, \omega_2), \dots \rangle, \end{aligned} \quad (2.2.45)$$

so a multi-mode quantum Fock state is an energy eigenstate, where the energy is the sum of the energies associated with each individual excitation.

2.2.7 Photons and Polaritons

Significantly, the quantum of the interacting system of dielectric material and electromagnetic fields is no longer that of the photon; the Hamiltonian has been

cast in the form of a summation over field operators in which the material and electromagnetic components of the system have been *mixed*. This is a kind of polariton, and the quantisation of the coupled system in terms of polaritons has implications for how we understand the *nature* of the Casimir force. This question will be explored further in Chap. 3. At this point, we will content ourselves simply with observing that the Hamiltonian of (2.1.9) can be related to the Hamiltonian of (2.2.40) as the limiting case in which the refractive index of the material (or its resistance to the field, if you will) is so high that only a negligible fraction of the radiation can be absorbed. In such idealised cases, the polariton is almost entirely ‘photonic’.

2.3 The Casimir Force Density

2.3.1 The Stress Tensor

As we have seen, the ground-state of the coupled system of electromagnetic field and dielectric is one with non-zero current density within the medium, consistent with the fluctuation-dissipation theorem. The Casimir force arises from the interaction of these currents. In Lifshitz’ original theory, the force was obtained by averaging Maxwell’s stress tensor in a vacuum with respect to electromagnetic fluctuations [5],

$$\sigma_F = (\mathbf{E} \otimes \mathbf{E}) + (\mathbf{B} \otimes \mathbf{B}) - \frac{1}{2} (\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) \mathbf{1}_3, \quad (2.3.1)$$

which governs the flow of momentum associated with the electromagnetic field.⁴ In the extension of Lifshitz theory to the general case of Casimir forces between bodies embedded in media [12], the Casimir (or ‘Casimir-Lifshitz’) forces in a system are ultimately determined by an analogue of the Minkowski stress tensor

$$\sigma_M = (\mathbf{D} \otimes \mathbf{E}) + (\mathbf{B} \otimes \mathbf{H}) - \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \mathbf{1}_3, \quad (2.3.2)$$

which concerns the momentum associated with the body-assisted field, reducing to the Maxwell stress tensor in the vacuum. Here we will derive a general expression for the Casimir stress tensor in an inhomogeneous medium, relating the Minkowski stress tensor to the Casimir force density. The derived expression also applies to piece-wise homogeneous systems and an infinite homogeneous medium as special cases. The subject of the Casimir stress tensor has been a matter of some controversy.⁵ However, we offer here a fresh and simple argument that coincides with the results

⁴ See Appendix B and [21].

⁵ The accepted use of the ‘Minkowski-like’ stress tensor for computing Casimir forces in the Lifshitz theory [22] was challenged in [23], resulting in some debate [13, 24, 25]. A new argument for the

of Lifshitz theory⁶; the standard predictions may be recovered after quantising the designated stress tensor at the end of what is essentially a classical demonstration.

It must be emphasised from the outset that this is intended more as an illustrative argument than a ‘water-tight’ proof. Once the effects of temperature are considered and absorption is included, the derivation becomes substantially more involved, but the final result for the stress tensor is the same [3, 12].

2.3.1.1 Prior Conditions on the Stress Tensor

The stress tensor describing the momentum flow for an object embedded in a fluid medium experiencing the fluctuations of the electromagnetic field must satisfy certain properties, as discussed at length by Pitaevskii [13, 14]. It must incorporate both the electromagnetic and fluid-mechanical aspects of the problem, and be decomposable into the form

$$\sigma = -P_0(\rho)1_3 + \sigma_E, \quad (2.3.3)$$

where P_0 is defined as the pressure of a uniform infinite liquid of density ρ in the absence of electromagnetic fluctuations, and σ_E is the contribution to the stress in the system arising specifically from the fluctuations of the field. For our purposes, temperature is not a variable; whilst the purely quantum-mechanical Casimir force requires thermodynamic equilibrium, it occurs at zero temperature. In addition, we require that the stress tensor must be symmetric: $\sigma = \sigma^T$.

The physical picture is this: electromagnetic fluctuations in the fluid exert a radiation pressure on the molecules that compose it. Mechanical equilibrium is ensured, however, by the presence of a counteracting pressure term in the stress tensor (2.3.3) preventing a permanent flow in the liquid (which would lead to a *perpetuum mobile*). The total stress tensor must therefore be derived in the circumstances of both thermodynamic and mechanical equilibrium, and consequently we require that

$$\nabla \cdot \sigma = 0. \quad (2.3.4)$$

2.3.1.2 The Pressure Force

We consider a small deformation at the surface of a body with a displacement vector field $\delta \mathbf{r}(\mathbf{r})$. The change in the free energy is

$$\delta F = - \int dV \mathbf{f} \cdot \delta \mathbf{r}, \quad (2.3.5)$$

(Footnote 5 continued)

disputed result can be found in [3], where the Casimir stress was derived in the context of the canonical theory of macroscopic quantum electrodynamics [4].

⁶ Lifshitz theory, in this case, refers to the more general results obtained in [12].

where \mathbf{f} is the force per unit volume on the body during the deformation. From the continuity equation of fluid dynamics, we infer that

$$\delta\rho + \nabla \cdot (\rho \delta\mathbf{r}) = 0 \implies \delta\rho = -\nabla \cdot (\rho \delta\mathbf{r}). \quad (2.3.6)$$

The variation of the permittivity ϵ is related to the material deformation via

$$\delta\epsilon = \frac{\partial\epsilon}{\partial\rho}\delta\rho = -\frac{\partial\epsilon}{\partial\rho}\nabla \cdot (\rho \delta\mathbf{r}). \quad (2.3.7)$$

Expanding the divergence, and observing that

$$\frac{\partial\epsilon}{\partial\rho}\delta\mathbf{r} \cdot \nabla\rho = \nabla\epsilon \cdot \delta\mathbf{r}, \quad (2.3.8)$$

we obtain

$$\delta\epsilon = -\delta\mathbf{r} \cdot \nabla\epsilon - \rho \frac{\partial\epsilon}{\partial\rho} \nabla \cdot \delta\mathbf{r}, \quad (2.3.9)$$

where $\epsilon = \epsilon(\rho)$ and $\rho = \rho(\mathbf{r})$. The variation of the permeability μ is governed similarly. The energy associated with the electromagnetic field in media is of the form [2]

$$E_f = \frac{1}{2} \int \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right) dV. \quad (2.3.10)$$

We postulate that the variation in the free energy⁷ is of the form [15]

$$\delta F = \delta F_0 - \frac{1}{2} \int \left(\delta\epsilon E^2 + \delta \left(\mu^{-1} \right) B^2 \right) dV. \quad (2.3.11)$$

Inserting (2.3.9) and its analogue for μ into (2.3.11), we obtain

$$\begin{aligned} \delta F = \delta F_0 &+ \frac{1}{2} \int \left[(\nabla\epsilon) E^2 + \left(\nabla \frac{1}{\mu} \right) B^2 \right] \cdot \delta\mathbf{r} dV \\ &+ \frac{1}{2} \int \left(E^2 \rho \frac{\partial\epsilon}{\partial\rho} \nabla \cdot \delta\mathbf{r} + B^2 \rho \frac{\partial}{\partial\rho} \left(\frac{1}{\mu} \right) \nabla \cdot \delta\mathbf{r} \right) dV. \end{aligned} \quad (2.3.12)$$

Using integration by parts, we find that

⁷ This result is not valid when there is absorption. Note that the expression for the variation of the free energy in Lifshitz theory takes a similar form [12]:

$$\delta F = \delta F_0 - \frac{T}{4\pi} \sum_{n=0}^{\infty} \int D_{ii}(\mathbf{r}, \mathbf{r}, \xi_n) \delta\epsilon(\mathbf{r}, i\xi_n) d^3\mathbf{r}.$$

$$\int \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \nabla \cdot \delta \mathbf{r} \right) \right] dV = \int_{\partial V} E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \delta \mathbf{r} \right) dA - \int \nabla \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) \right] \cdot \delta \mathbf{r} dV. \quad (2.3.13)$$

For a sufficiently large volume, the field terms vanish on the surface. We deduce that

$$\int \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \nabla \cdot \delta \mathbf{r} \right) \right] dV = - \int \nabla \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) \right] \cdot \delta \mathbf{r} dV. \quad (2.3.14)$$

The terms in μ may be treated similarly. Hence Eq. (2.3.12) may be recast in the form

$$\delta F = \delta F_0 + \frac{1}{2} \int dV \left\{ (\nabla \epsilon) E^2 + \left(\nabla \frac{1}{\mu} \right) B^2 - \nabla \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) \right] - \nabla \left[B^2 \left(\rho \frac{\partial}{\partial \rho} \frac{1}{\mu} \right) \right] \right\} \cdot \delta \mathbf{r}. \quad (2.3.15)$$

The free energy is clearly associated, via Eq. (2.3.5), with a force of the form

$$\mathbf{f} = -\nabla P_0 - \frac{1}{2} (\nabla \epsilon) E^2 - \frac{1}{2} \left(\nabla \frac{1}{\mu} \right) B^2 + \frac{1}{2} \nabla \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) + B^2 \left(\rho \frac{\partial}{\partial \rho} \frac{1}{\mu} \right) \right], \quad (2.3.16)$$

acting per unit volume of the body. We consider the case of mechanical equilibrium, in which the forces on the body are balanced, i.e.

$$\mathbf{f} = 0 \implies \mathbf{f}_P + \mathbf{f}_E = \mathbf{0}, \quad (2.3.17)$$

where we decompose the force into two contributions:

$$\begin{aligned} \mathbf{f}_P &= -\nabla P_0 + \frac{1}{2} \nabla \left[E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) + B^2 \left(\rho \frac{\partial}{\partial \rho} \frac{1}{\mu} \right) \right], \\ \mathbf{f}_E &= -\frac{1}{2} \left[(\nabla \epsilon) E^2 + \nabla \left(\mu^{-1} \right) B^2 \right]. \end{aligned} \quad (2.3.18)$$

It will become clear that these balancing contributions can be assigned distinct physical meanings.⁸ Our expression for the total force (2.3.17) may be recast in the form of a stress tensor σ satisfying

$$\mathbf{f} = \nabla \cdot \sigma = 0. \quad (2.3.19)$$

⁸ We note that the first contribution \mathbf{f}_P contains a pressure term which is present in the absence of electromagnetic fluctuations, and a contribution due to the deformation of the medium which vanishes in the limit of an incompressible medium.

It follows that the complete stress tensor takes the form

$$\begin{aligned} \sigma = & -P_0 \mathbf{1}_3 + \frac{1}{2} \left\{ E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) + B^2 \left(\rho \frac{\partial}{\partial \rho} \frac{1}{\mu} \right) \right\} \mathbf{1}_3 \\ & - \frac{1}{2} \left\{ \int \nabla \epsilon(\mathbf{r}) E^2(\mathbf{r}) \cdot d\mathbf{r} + \int \nabla \left(\frac{1}{\mu(\mathbf{r})} \right) B^2(\mathbf{r}) \cdot d\mathbf{r} \right\} \mathbf{1}_3, \end{aligned} \quad (2.3.20)$$

which we similarly decompose into two contributions

$$\sigma = \sigma_P + \sigma_{\mathcal{E}}, \quad (2.3.21)$$

where

$$\sigma_P = -P_0 \mathbf{1}_3 + \frac{1}{2} \left\{ E^2 \left(\rho \frac{\partial \epsilon}{\partial \rho} \right) + B^2 \left(\rho \frac{\partial}{\partial \rho} \frac{1}{\mu} \right) \right\} \mathbf{1}_3, \quad (2.3.22)$$

$$\sigma_{\mathcal{E}} = -\frac{1}{2} \left\{ \int \nabla \epsilon(\mathbf{r}) E^2(\mathbf{r}) \cdot d\mathbf{r} + \int \nabla \left(\frac{1}{\mu(\mathbf{r})} \right) B^2(\mathbf{r}) \cdot d\mathbf{r} \right\} \mathbf{1}_3. \quad (2.3.23)$$

The two force densities may be defined with respect to the stress components:

$$\mathbf{f}_P = \nabla \cdot \sigma_P, \quad \mathbf{f}_{\mathcal{E}} = \nabla \cdot \sigma_{\mathcal{E}}. \quad (2.3.24)$$

2.3.1.3 The Minkowski Contribution

The term we have labelled $\sigma_{\mathcal{E}}$ can be related to the Minkowski stress tensor via Maxwell's equations, in the absence of external charges and under the condition of equilibrium. We begin with the general form of the classical Maxwell equations:

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \cdot \mathbf{D} = \rho, \quad \nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}. \quad (2.3.25)$$

From the first equation, it follows that

$$\nabla (\mathbf{B} \otimes \mathbf{H}) = \mathbf{H} (\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla) \mathbf{H} = (\mathbf{B} \cdot \nabla) \mathbf{H}. \quad (2.3.26)$$

From the third, we deduce

$$\nabla (\mathbf{D} \otimes \mathbf{E}) = \mathbf{E} (\nabla \cdot \mathbf{D}) + (\mathbf{D} \cdot \nabla) \mathbf{E} = \rho \mathbf{E} + (\mathbf{D} \cdot \nabla) \mathbf{E}. \quad (2.3.27)$$

Combining (2.3.26) and (2.3.27), we obtain

$$\nabla (\mathbf{D} \otimes \mathbf{E}) + \nabla (\mathbf{B} \otimes \mathbf{H}) = \rho \mathbf{E} + (\mathbf{D} \cdot \nabla) \mathbf{E} + (\mathbf{B} \cdot \nabla) \mathbf{H}. \quad (2.3.28)$$

From the second and fourth equations, we find

$$\frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) = (\nabla \times \mathbf{E}) \times \mathbf{D} + (\nabla \times \mathbf{H}) \times \mathbf{B} - \mathbf{J} \times \mathbf{B}. \quad (2.3.29)$$

It follows from (2.3.28) and (2.3.29) that

$$\mathbf{F}_L + \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) = \mathbf{T}, \quad (2.3.30)$$

where \mathbf{F}_L is the Lorentz force that acts on external charges and currents in the system,

$$\mathbf{F}_L = \rho \mathbf{E} + \mathbf{J} \times \mathbf{B}, \quad (2.3.31)$$

and \mathbf{T} can be decomposed into the sum of two terms:

$$\mathbf{T}_1 = (\nabla \times \mathbf{E}) \times \mathbf{D} - (\mathbf{D} \cdot \nabla) \mathbf{E} + (\nabla \times \mathbf{H}) \times \mathbf{B} - (\mathbf{B} \cdot \nabla) \mathbf{H}, \quad (2.3.32)$$

$$\mathbf{T}_2 = \nabla (\mathbf{D} \otimes \mathbf{E}) + \nabla (\mathbf{B} \otimes \mathbf{H}). \quad (2.3.33)$$

Both sides of Eq. (2.3.30) must have units of force density. However, in macroscopic electromagnetism applied solely to dielectrics (including metals) there are no external currents or charges: $\rho = 0$ and $\mathbf{J} = 0$. It follows that the Lorentz force is identically zero:

$$\mathbf{F}_L = 0. \quad (2.3.34)$$

Whilst a microscopic Lorentz force could be defined with respect to the *internal* source terms introduced in the Lifshitz theory, these terms are not the referents of (2.3.25), and this approach does not recover the accepted expression for the forces in media [13, 14]. Also, we will restrict ourselves to considering equilibrium states, so the time derivative does not contribute either. Under these conditions, Eq. (2.3.30) becomes simply

$$\mathbf{T} = \mathbf{T}_1 + \mathbf{T}_2 = \mathbf{0}. \quad (2.3.35)$$

In the argument that follows, we recall the vector identity

$$\nabla (\mathbf{a} \cdot \mathbf{b}) = (\mathbf{a} \cdot \nabla) \mathbf{b} + (\mathbf{b} \cdot \nabla) \mathbf{a} + \mathbf{a} \times (\nabla \times \mathbf{b}) + \mathbf{b} \times (\nabla \times \mathbf{a}), \quad (2.3.36)$$

and note that the constitutive relations are given by

$$\mathbf{D}(\mathbf{r}, \omega) = \epsilon(\mathbf{r}, \omega) \mathbf{E}(\mathbf{r}, \omega), \quad \mathbf{H}(\mathbf{r}, \omega) = \frac{1}{\mu(\mathbf{r}, \omega)} \mathbf{B}(\mathbf{r}, \omega), \quad (2.3.37)$$

where we have chosen to absorb the constants ϵ_0 and μ_0 into the definitions of the respective fields. Under these conditions, and for a fixed frequency ω ,

$$\begin{aligned}
(\nabla \times \mathbf{D}(\mathbf{r})) \times \mathbf{E}(\mathbf{r}) &= (\nabla \times \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})) \times \mathbf{E}(\mathbf{r}) \\
&= \nabla \epsilon(\mathbf{r}) (\mathbf{E}(\mathbf{r}) \times \mathbf{E}(\mathbf{r})) + \epsilon(\mathbf{r}) (\nabla \times \mathbf{E}(\mathbf{r})) \times \mathbf{E}(\mathbf{r}). \quad (2.3.38)
\end{aligned}$$

Clearly, the first term, involving the cross product of \mathbf{E} with itself, is identically zero. Evaluated at a fixed point \mathbf{r}_0 ,

$$\begin{aligned}
[(\nabla \times \mathbf{D}(\mathbf{r})) \times \mathbf{E}(\mathbf{r})]_{\mathbf{r}_0} &= \epsilon(\mathbf{r}_0) [\nabla \times \mathbf{E}(\mathbf{r})]_{\mathbf{r}_0} \times \mathbf{E}(\mathbf{r}_0) \\
&= [\nabla \times \mathbf{E}(\mathbf{r})]_{\mathbf{r}_0} \times \epsilon(\mathbf{r}_0)\mathbf{E}(\mathbf{r}_0). \quad (2.3.39)
\end{aligned}$$

Hence we may write

$$[(\nabla \times \mathbf{D}(\mathbf{r})) \times \mathbf{E}(\mathbf{r})]_{\mathbf{r}_0} = [(\nabla \times \mathbf{E}(\mathbf{r})) \times \mathbf{D}(\mathbf{r})]_{\mathbf{r}_0}, \quad (2.3.40)$$

in which the field terms have been exchanged, noting that

$$\mathbf{D}(\mathbf{r}) = \epsilon(\mathbf{r})\mathbf{E}(\mathbf{r}). \quad (2.3.41)$$

We may perform the same trick for the magnetic fields:

$$[(\nabla \times \mathbf{H}(\mathbf{r})) \times \mathbf{B}(\mathbf{r})]_{\mathbf{r}_0} = [(\nabla \times \mathbf{B}(\mathbf{r})) \times \mathbf{H}(\mathbf{r})]_{\mathbf{r}_0}, \quad (2.3.42)$$

noting that

$$\mathbf{H}(\mathbf{r}) = \frac{1}{\mu(\mathbf{r})}\mathbf{B}(\mathbf{r}). \quad (2.3.43)$$

In addition, we find that

$$\begin{aligned}
(\mathbf{E}(\mathbf{r}) \cdot \nabla) \mathbf{D}(\mathbf{r}) &= \mathbf{E}(\mathbf{r}) \cdot \nabla (\epsilon(\mathbf{r})\mathbf{E}(\mathbf{r})) \\
&= \epsilon(\mathbf{r}) (\mathbf{E}(\mathbf{r}) \cdot \nabla) \mathbf{E}(\mathbf{r}) + \nabla \epsilon(\mathbf{r}) (\mathbf{E}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})). \quad (2.3.44)
\end{aligned}$$

Evaluated at a point \mathbf{r}_0 :

$$[(\mathbf{E}(\mathbf{r}) \cdot \nabla) \mathbf{D}]_{\mathbf{r}_0} = \left[(\mathbf{D}(\mathbf{r}) \cdot \nabla) \mathbf{E}(\mathbf{r}) + \nabla \epsilon(\mathbf{r}) E^2(\mathbf{r}) \right]_{\mathbf{r}=\mathbf{r}_0}. \quad (2.3.45)$$

We note this time that, in exchanging the order of the fields, we acquire an additional term that depends upon the gradient of the permittivity. Similarly, we find

$$[(\mathbf{B}(\mathbf{r}) \cdot \nabla) \mathbf{H}]_{\mathbf{r}_0} = \left[(\mathbf{H}(\mathbf{r}) \cdot \nabla) \mathbf{B}(\mathbf{r}) + \nabla \left(\frac{1}{\mu(\mathbf{r})} \right) B^2(\mathbf{r}) \right]_{\mathbf{r}=\mathbf{r}_0}, \quad (2.3.46)$$

where we have again acquired an additional term that depends upon the gradient of the permeability. Using Eq. (2.3.40) and (2.3.45) and the vector identity (2.3.36), the first term of \mathbf{T}_1 , evaluated at \mathbf{r}_0 , can be rewritten

$$(\nabla \times \mathbf{E}) \times \mathbf{D} = (\mathbf{D} \cdot \nabla) \mathbf{E} - \frac{1}{2} \nabla (\mathbf{D} \cdot \mathbf{E}) + \frac{1}{2} \nabla [\epsilon(\mathbf{r})] E^2(\mathbf{r}). \quad (2.3.47)$$

Similarly, for the third term, using Eqs. (2.3.35), (2.3.42) and (2.3.46), we obtain

$$(\nabla \times \mathbf{H}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{H} - \frac{1}{2} \nabla (\mathbf{B} \cdot \mathbf{H}) + \frac{1}{2} \nabla \left[\frac{1}{\mu(\mathbf{r})} \right] B^2(\mathbf{r}). \quad (2.3.48)$$

Reexpressing (2.3.33) using (2.3.42–2.3.48), it follows from (2.3.35) that

$$\nabla \cdot \sigma_M = -\frac{1}{2} \left(\nabla [\epsilon(\mathbf{r})] E^2(\mathbf{r}) + \nabla \left[\mu(\mathbf{r})^{-1} \right] B^2(\mathbf{r}) \right), \quad (2.3.49)$$

where σ_M defines the familiar Minkowski stress tensor for the electromagnetic field

$$\sigma_M = (\mathbf{D} \otimes \mathbf{E}) + (\mathbf{B} \otimes \mathbf{H}) - \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \mathbf{1}_3. \quad (2.3.50)$$

In fact, expression (2.3.49) is equal to $\mathbf{f}_{\mathcal{E}}$, defined in Eq. (2.3.18). It follows from the right-hand side of (2.3.49) that, in an equilibrium state, in the absence of external currents or charges, σ_M may be alternatively cast in the form

$$\sigma_M = -\frac{1}{2} \left\{ \int \nabla \epsilon(\mathbf{r}) E^2(\mathbf{r}) \cdot d\mathbf{r} + \int \nabla \left(\frac{1}{\mu(\mathbf{r})} \right) B^2(\mathbf{r}) \cdot d\mathbf{r} \right\} \mathbf{1}_3. \quad (2.3.51)$$

2.3.1.4 The Abraham Stress Tensor

Exchanging $\sigma_{\mathcal{E}}$ for σ_M in the stress tensor (2.3.21), and rewriting σ_M in the form

$$\sigma_M = (\mathbf{D} \otimes \mathbf{E}) + (\mathbf{B} \otimes \mathbf{H}) - \frac{1}{2} \left(\epsilon E^2 + \frac{1}{\mu} B^2 \right) \mathbf{1}_3, \quad (2.3.52)$$

we recover the celebrated Abraham tensor of the stationary electromagnetic field in a dielectric fluid:

$$\sigma = -P_0 \mathbf{1}_3 + (\mathbf{D} \otimes \mathbf{E}) + (\mathbf{B} \otimes \mathbf{H}) - \frac{1}{2} \left\{ E^2 \left(\epsilon - \rho \frac{\partial \epsilon}{\partial \rho} \right) + B^2 \left(\frac{1}{\mu} - \rho \frac{\partial}{\partial \rho} \frac{1}{\mu} \right) \right\} \mathbf{1}_3. \quad (2.3.53)$$

This is the total stress tensor applied in the general form of Lifshitz theory [12] for the prediction of Casimir phenomena in media.

2.3.1.5 The Casimir-Lifshitz Force and the Fluid Pressure

The Casimir-Lifshitz force in a liquid medium may be understood as arising in the equilibrium balance between the radiation pressure and fluid pressure components of the stress tensor. Its stress tensor is given by the quantum analogue of

$$\sigma_M = (\mathbf{D} \otimes \mathbf{E}) + (\mathbf{B} \otimes \mathbf{H}) - \frac{1}{2} (\mathbf{D} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}) \mathbf{1}_3, \quad (2.3.54)$$

which may be associated with an electromagnetic force density

$$\mathbf{f}_C = \mathbf{f}_M = \nabla \cdot \sigma_M. \quad (2.3.55)$$

It may alternatively be expressed in the form

$$\mathbf{f}_C = \mathbf{f}_E = \nabla \cdot \sigma_E \equiv -\frac{1}{2} \left(\nabla [\epsilon(\mathbf{r})] E^2(\mathbf{r}) + \nabla \left[\mu(\mathbf{r})^{-1} \right] B^2(\mathbf{r}) \right), \quad (2.3.56)$$

where it is specified in terms of the gradients of the dielectric functions. It is exactly opposed by a compensating stress containing the pressure gradient due to the fields,

$$\sigma_P = \left\{ -P_0 + \frac{1}{2} \left(\rho \frac{\partial \epsilon}{\partial \rho} E^2 + \rho \frac{\partial}{\partial \rho} \left(\frac{1}{\mu} \right) B^2 \right) \right\} \mathbf{1}_3. \quad (2.3.57)$$

Consequently,

$$\nabla \cdot \sigma = \nabla \cdot \sigma_M + \nabla \cdot \sigma_P = \mathbf{f}_C + \mathbf{f}_P = 0, \quad (2.3.58)$$

and mechanical equilibrium is maintained. The Casimir force density may thus be inferred either from the quantum analogues of the pressure or Minkowski contributions to the stress:

$$\mathbf{f}_C = -\mathbf{f}_P. \quad (2.3.59)$$

The total pressure in the fluid (a scalar quantity) includes the contribution P_0 defined in the absence of electromagnetic fluctuations.

2.3.1.6 The Casimir Force on Interacting Bodies in a Medium

The Casimir-Lifshitz force is the quantum analogue of the force determined by the volume integral of the force density, which is equivalent to the integration of the Minkowski stress over an enclosing surface:

$$\mathbf{F}_C = \int_V d\mathbf{r} \nabla \cdot \sigma_M = \int_{\partial V} d\mathbf{A} \cdot \sigma_M. \quad (2.3.60)$$

From Eq. (2.3.56), we observe that $\nabla \cdot \sigma_M = 0$ and hence $\mathbf{F}_C = 0$ for any surface enclosing a volume of uniform fluid. However, for any surface surrounding a solid body, the gradients of the dielectric functions are non-zero, and integration over the opposing electromagnetic or pressure components of the stress produces the force acting on the body.

2.3.1.7 Piece-wise Homogeneous Case

Let us consider now a piece-wise planar homogeneous case with a boundary at $x = x_0$, as a special case of an inhomogeneous system. In fact, most theoretical Casimir problems are concerned with the interactions between homogeneous bodies; consequently, their global dielectric functions are piece-wise homogeneous. The constitutive relations are as stated earlier, but here ϵ is sharply discontinuous at the boundary:

$$\epsilon(x) = \begin{cases} \epsilon_1 & x < x_0 \\ \epsilon_2 & x \geq x_0 \end{cases}. \quad (2.3.61)$$

The permittivity $\epsilon(x)$ can be rewritten in terms of the Heaviside function H :

$$\epsilon(x) = \epsilon_1 + H(x - x_0)\Delta\epsilon, \quad (2.3.62)$$

where $\Delta\epsilon = \epsilon_2 - \epsilon_1$. The magnetic response of a material can often be ignored ($\mu \rightarrow 1$). Consider Eq. (2.3.56). For the piece-wise homogeneous system of our example, the first term involves

$$\frac{\partial}{\partial x} [\epsilon(x)] E^2(\mathbf{r}) = \delta(x - x_0) \Delta\epsilon E^2(\mathbf{r}). \quad (2.3.63)$$

The second term also gives a non-zero contribution only at $x = x_0$, i.e. on the boundary. For piece-wise homogeneous media, then, the stress is discontinuous at the interface between two different media, and the force-density is concentrated in a delta-function (in this case, at $x = x_0$):

$$\nabla \cdot \sigma_M = -\frac{1}{2}\delta(x - x_0) \left[\Delta\epsilon E^2(\mathbf{r}) \right]. \quad (2.3.64)$$

Notice that inside an infinite homogeneous medium, where the gradients of the dielectric functions are zero, Eq. (2.3.56) implies that the Casimir force must be identically zero.

2.3.1.8 General Remarks

We find then that the force density can be written solely in terms containing spatial derivatives of the dielectric functions, for both piecewise homogeneous systems and inhomogeneous media. Rewritten as a quantum correlation function, averaged over the vacuum state, σ_M ultimately affords the Casimir force density for a system, including Casimir's original case involving a cavity formed by two perfect mirrors. The Casimir force therefore depends upon material inhomogeneities in a system that scatter the (virtual) photons of the electromagnetic field, and it is this scattered part of the field that is relevant to computing the Casimir stresses in a system. This is a thought we will repeatedly come back to.

2.3.2 Averaging over the Quantum Stress Tensor

We have established that the Casimir force density is equal to the divergence of the Minkowski stress tensor. Of course, this tensor must first be quantised. The work has essentially been done, however: it is sufficient to replace the classical field components of the stress tensor with the quantum operators we have defined in this chapter:

$$\hat{\sigma} = \hat{\mathbf{E}} \otimes \hat{\mathbf{D}} + \hat{\mathbf{B}} \otimes \hat{\mathbf{H}} - \frac{1}{2} (\hat{\mathbf{E}} \cdot \hat{\mathbf{D}} + \hat{\mathbf{B}} \cdot \hat{\mathbf{H}}) \mathbf{1}_3. \quad (2.3.65)$$

This effectively rewrites the stress in terms of the fundamental field operators. Using the Gauss theorem, the total force operator can then be written in the form

$$\hat{\mathbf{F}} = \int_{\partial V} d\mathbf{A} \cdot \hat{\sigma} - \int_V d^3r \frac{\partial}{\partial t} (\hat{\mathbf{D}} \times \hat{\mathbf{B}}). \quad (2.3.66)$$

To obtain the measured Casimir force, we must average the stress tensor in the ground state of the polariton field. The force is then

$$\mathbf{F} = \int_{\partial V} d\mathbf{A} \cdot \langle \{0\} | \hat{\sigma} | \{0\} \rangle - \int_V d^3r \frac{\partial}{\partial t} \langle \{0\} | \hat{\mathbf{D}} \times \hat{\mathbf{B}} | \{0\} \rangle. \quad (2.3.67)$$

According to the Schrödinger equation, the ground-state is constant in time. The second term therefore vanishes, and we can state the force as

$$\mathbf{F} = \int_{\partial V} d\mathbf{A} \cdot \sigma, \quad (2.3.68)$$

where σ is the expectation value of the stress tensor in the ground-state. The ground-state expectation values can be computed by expanding the fields in terms of the

fundamental bosonic field operators (2.2.31–2.2.36), exploiting their properties (2.2.23), but neglecting absorption.⁹ In doing so, we find that [1, 16]

$$\langle \hat{\mathbf{E}}(\mathbf{r}) \otimes \hat{\mathbf{D}}(\mathbf{r}') \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \mu_0 \omega^2 \epsilon_0 \epsilon(\mathbf{r}, \omega) \operatorname{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega), \quad (2.3.69)$$

$$\langle \hat{\mathbf{B}}(\mathbf{r}) \otimes \hat{\mathbf{H}}(\mathbf{r}') \rangle = \frac{\hbar}{\pi} \int_0^\infty d\omega \frac{1}{\mu(\mathbf{r}, \omega)} \nabla \times \operatorname{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \times \overleftarrow{\nabla}'. \quad (2.3.70)$$

The stress tensor, evaluated at position \mathbf{r} , is then readily cast into the form [1]

$$\sigma(\mathbf{r}) = \frac{\hbar}{\pi} \int_0^\infty d\omega \left(\tau(\mathbf{r}) - \frac{1}{2} \operatorname{Tr} [\tau(\mathbf{r})] \mathbf{1}_3 \right), \quad (2.3.71)$$

where Tr implements the trace function. This involves integrating over all frequencies the function

$$\tau(\mathbf{r}) = \frac{\omega^2}{c^2} \epsilon(\mathbf{r}, \omega) \operatorname{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}, \omega) + \frac{1}{\mu(\mathbf{r}, \omega)} \left[\nabla \times \operatorname{Im} \mathbf{G}(\mathbf{r}, \mathbf{r}', \omega) \times \overleftarrow{\nabla}' \right]_{\mathbf{r}'=\mathbf{r}}. \quad (2.3.72)$$

The stress is therefore determined by the classical Green function of the field.

2.3.3 Regularising the Stress Tensor

However, the stress tensor (2.3.71), like the zero-point energy, turns out to be *infinite*. To facilitate the removal of the divergence, we introduce an additional positional argument \mathbf{r}' and define the regularised stress tensor

$$\sigma = \lim_{\mathbf{r}' \rightarrow \mathbf{r}} \langle \hat{\sigma}(\mathbf{r}, \mathbf{r}') - \hat{\sigma}_0(\mathbf{r}, \mathbf{r}') \rangle. \quad (2.3.73)$$

The first term of (2.3.73) is a quantum correlation function defined by

$$\hat{\sigma}(\mathbf{r}, \mathbf{r}') = \hat{\mathbf{E}}(\mathbf{r}) \otimes \hat{\mathbf{D}}(\mathbf{r}') + \hat{\mathbf{B}}(\mathbf{r}) \otimes \hat{\mathbf{H}}(\mathbf{r}') - \frac{1}{2} \left(\hat{\mathbf{E}}(\mathbf{r}) \cdot \hat{\mathbf{D}}(\mathbf{r}') + \hat{\mathbf{B}}(\mathbf{r}) \cdot \hat{\mathbf{H}}(\mathbf{r}') \right) \mathbf{1}_3, \quad (2.3.74)$$

⁹ We retain only the *real* parts. To take proper account of absorption in media requires a more sophisticated formulation of macro-QED [4]. However, the stress tensor is the same [3].

which is identical in form to the quantum stress in the limit as $\mathbf{r}' \rightarrow \mathbf{r}$. The subtraction of the second term $\hat{\sigma}_0$ in (2.3.73) is typically associated with the removal of illicit ‘self-forces’. Both of these terms in the expression are finite for $\mathbf{r}' \neq \mathbf{r}$, so the subtraction is well-defined. Applying once again results (2.3.69) and (2.3.70), the stress correlation function (2.3.74) can be expressed in the form [1]

$$\sigma(\mathbf{r}, \mathbf{r}') = -\frac{\hbar}{\pi} \int_0^\infty d\xi \left(\tau(\mathbf{r}, \mathbf{r}') - \frac{1}{2} \text{Tr} [\tau(\mathbf{r}, \mathbf{r}')] \mathbf{1}_3 \right), \quad (2.3.75)$$

which involves integrating over the imaginary axis the associated correlation function¹⁰

$$\tau(\mathbf{r}, \mathbf{r}') = \frac{\xi^2}{c^2} \epsilon(\mathbf{r}, i\xi) \mathbf{G}(\mathbf{r}, \mathbf{r}', i\xi) - \frac{1}{\mu(\mathbf{r}, i\xi)} \nabla \times \mathbf{G}(\mathbf{r}, \mathbf{r}', i\xi) \times \overleftarrow{\nabla}'. \quad (2.3.76)$$

The additional step of rotating to the imaginary axis is a mathematical trick to produce a more rapidly converging integral, necessitating the evaluation of the dielectric functions at imaginary frequencies. These quantities can be obtained from the original functions by Hilbert transforms. The correlation function is expressed in terms of the classical Green function of the electromagnetic field, which may be associated in this case with the magnitude of a field at \mathbf{r} produced by a dipole situated at \mathbf{r}' oscillating with ‘imaginary frequency’ ξ , which probes the properties of the system.

2.3.3.1 The Meaning of the Regularisation

Concerning the regularisation procedure, it must be emphasised that, in attempting to isolate and remove the divergences in the stress tensor, we are not in general at liberty to subtract anything that depends upon the inhomogeneity of the medium. This could arbitrarily modify the Casimir force, which itself arises as a result of scattering due to material inhomogeneities in the system (see Eq. (2.3.56)), as we have discussed. This fairly obvious stricture has frequently been violated without adequate justification in the literature. However, the purchase of finitude at the cost of *meaning* is too high a price to pay. In Lifshitz theory, the regularisation is assigned a fairly precise semantics [14] best expressed at the level of the Green function, which we may write as

$$\mathbf{G}(\mathbf{r}, \mathbf{r}') = (\mathbf{G}(\mathbf{r}, \mathbf{r}') - \mathbf{G}_0(\mathbf{r}, \mathbf{r}')) + \mathbf{G}_0(\mathbf{r}, \mathbf{r}'), \quad (2.3.77)$$

where \mathbf{G}_0 is an auxiliary Green function for an infinite homogeneous medium whose dielectric properties are the same as that of the actual medium at the point \mathbf{r}' . As we have argued, there is no Casimir force in an infinite homogeneous medium; the third term in (2.3.77) can effectively be absorbed into a bulk stress term that is not associ-

¹⁰ For a discussion of Wick rotation to the imaginary axis, see Appendix A.

ated with the Casimir force. This results in a renormalisation of the stress in which both the form and meaning of σ_0 in Eq. (2.3.73) are now apparent: to determine the *physical* part of the stress tensor relevant to the measurable Casimir force, we subtract a bulk stress σ_0 corresponding to an infinite homogeneous medium, effectively renormalising the Casimir stress to zero in the absence of any inhomogeneity in the system. We may also define the renormalised Green function

$$\mathbf{G}_S(\mathbf{r}, \mathbf{r}') = \mathbf{G}(\mathbf{r}, \mathbf{r}') - \mathbf{G}_0(\mathbf{r}, \mathbf{r}'), \quad (2.3.78)$$

corresponding to the *scattered* part of the electromagnetic field. Simply computing the stress using the scattered Green function (2.3.78) is equivalent to computing the stress using Eq. (2.3.73).

2.4 The Lifshitz Result for Two Half-spaces

Lifshitz' original paper considered the case of the force produced due to fluctuations between two parallel dielectric half-spaces, separated by empty space [5]. This result was subsequently generalised to allow for the case of a liquid medium between the moving bodies, as well as vacuum, producing a general expression for the force that recovers Casimir's own expression in appropriate limits [12]. By solving the wave equation to determine the correct Green function, we can recover this well-known result from our more general expressions (2.3.73–2.3.76). The calculation is somewhat involved, however, and we present here only an outline.

2.4.1 The Force in a Dielectric Sandwich

2.4.1.1 The Planar Green Function

In planar dielectrics, we can arrange our coordinate system so that ε and μ depend on x , but not on y and z . In view of the symmetry in y and z , we introduce a spatial Fourier transformation to simplify the problem:

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \tilde{G}(x, x', k_y, k_z) e^{ik_y(y-y') + ik_z(z-z')} dk_y dk_z, \quad (2.4.1)$$

noting that

$$\nabla = (\partial_x, ik_y, ik_z), \quad \nabla' = (\partial_{x'}, -ik_y, -ik_z). \quad (2.4.2)$$

The Green function obeys the Fourier transformed wave equation

$$\nabla \times \frac{1}{\mu} \nabla \times \tilde{G} + \varepsilon \kappa^2 \tilde{G} = \mathbf{1}_3 \delta(x - x'), \quad (2.4.3)$$

where $\kappa = \xi/c$. Solving the wave equation is slightly involved, even for this simple system, and we will not repeat all the details here.¹¹ The Fourier-transformed Green function may be expressed in the form

$$\tilde{G} = \tilde{G}_e + \tilde{G}_m, \quad \text{for } x \neq x', \quad (2.4.4)$$

consisting of two separate contributions [1],

$$\tilde{G}_e = \mathbf{n}_E \tilde{g}_e \otimes \mathbf{n}'_E, \quad \tilde{G}_m = - \frac{\nabla \times \mathbf{n}_E \tilde{g}_m \otimes \mathbf{n}'_E \times \overleftarrow{\nabla'}}{\varepsilon \varepsilon' \kappa^2}, \quad (2.4.5)$$

where

$$\mathbf{n}_E = \frac{1}{\sqrt{k_y^2 + k_z^2}} \begin{pmatrix} 0 \\ -k_z \\ k_y \end{pmatrix}, \quad \mathbf{n}'_E = -\mathbf{n}_E. \quad (2.4.6)$$

Primed terms are evaluated at position r' , and the operator $\overleftarrow{\nabla'}$ acts to the left with respect to r' . For this geometry, the problem may be essentially rewritten in terms of electric and magnetic scalar Green functions that solve the wave equations

$$\nabla \cdot \frac{1}{\mu(x)} \nabla \tilde{g}_e(x) - \varepsilon(x) \kappa^2 \tilde{g}_e(x) = \delta(x - x'), \quad (2.4.7)$$

$$\nabla \cdot \frac{1}{\varepsilon(x)} \nabla \tilde{g}_m(x) - \mu(x) \kappa^2 \tilde{g}_m(x) = \delta(x - x'). \quad (2.4.8)$$

2.4.1.2 The Planar Stress

We can write the stress correlation functions (2.3.75, 2.3.76) in terms of the Fourier-transformed Green function (2.4.4), shifting the prefactor and the integrations to the second correlation function:

$$\sigma = \sum_{\lambda=e,m} \left(\tau_\lambda - \frac{1}{2} \text{Tr } \tau_\lambda \mathbf{1}_3 \right), \quad (2.4.9)$$

where we have separated the stress into components that depend on the electric and magnetic Green functions, and

¹¹ The full details of this approach can be found in [1].

$$\tau_\lambda = \frac{\hbar c}{\pi} \int_0^\infty d\kappa \int_{-\infty}^{+\infty} \frac{d\mathbf{k}_\parallel}{(2\pi)^2} \left(-\kappa^2 \varepsilon \tilde{G}_\lambda + \frac{1}{\mu} \nabla \times \tilde{G}_\lambda \times \overleftarrow{\nabla'} \right), \quad (2.4.10)$$

with $\mathbf{k}_\parallel = (0, k_y, k_z)$. The inverse Fourier-transform nullifies any terms that are odd in k_y or k_z and both the \tilde{G}_e and the \tilde{G}_m contributions have ‘off-diagonal’ components that then vanish in the integral. It follows that the stress correlation function is isotropic, i.e.

$$\sigma = \text{diag}(\sigma_{xx}, \sigma_{yy}, \sigma_{zz}). \quad (2.4.11)$$

Furthermore, the Green function does not depend on y and z , which is evident from the symmetry of this problem. The only component that is relevant to the Casimir force, in this case, is σ_{xx} , where

$$\sigma_{xx} = \sum_{\lambda=e,m} \left(\tau_{\lambda,xx} - \frac{1}{2} \text{Tr } \tau_\lambda \right). \quad (2.4.12)$$

Consider the contribution of the electric component of the correlation function. We find that

$$\tau_{e,xx} = -\frac{\hbar c}{\pi} \int_0^\infty d\kappa \int_{-\infty}^{+\infty} \frac{d\mathbf{k}_\parallel}{(2\pi)^2} \frac{1}{\mu} (k_y^2 + k_z^2) \tilde{g}_e, \quad (2.4.13)$$

$$\text{Tr } \tau_e = \frac{\hbar c}{\pi} \int_0^\infty d\kappa \int_{-\infty}^{+\infty} \frac{d\mathbf{k}_\parallel}{(2\pi)^2} \left(\kappa^2 \varepsilon - \frac{1}{\mu} (k_y^2 + k_z^2 + \partial_x \partial_{x'}) \right) \tilde{g}_e. \quad (2.4.14)$$

It follows that the electric component of the stress is of the form [1]

$$\sigma_{e,xx} = -\frac{\hbar c}{2} \int_0^\infty d\kappa \int_{-\infty}^{+\infty} \frac{d\mathbf{k}_\parallel}{(2\pi)^2} \frac{1}{\mu} (w^2 - \partial_x \partial_{x'}) \tilde{g}_e, \quad (2.4.15)$$

where

$$w = \sqrt{k_y^2 + k_z^2 + n^2 \kappa^2}, \quad n = \sqrt{\varepsilon \mu}. \quad (2.4.16)$$

Similarly, for the magnetic contribution, we find that

$$\sigma_{m,xx} = -\frac{\hbar c}{2} \int_0^\infty d\kappa \int_{-\infty}^{+\infty} \frac{d\mathbf{k}_\parallel}{(2\pi)^2} \frac{1}{\varepsilon} (w^2 - \partial_x \partial_{x'}) \tilde{g}_m. \quad (2.4.17)$$

Both of these contributing terms, however, are infinite. To obtain a finite and physical result, we need to renormalise the stress tensor. This may be achieved at the level of the scalar Green functions by isolating the scattered components of \tilde{g}_e and \tilde{g}_m .

2.4.1.3 The Scattered Green Function

We are considering a dielectric sandwich consisting of three uniform regions characterised by dielectric constants $\{\epsilon_1, \mu_1\}$, $\{\epsilon_2, \mu_2\}$, and $\{\epsilon_3, \mu_3\}$. The outer dielectrics are idealised to be infinitely thick, the left outer plate extending from $x \in (-\infty, 0)$, the inner region from $x \in [0, a]$, and the right outer plate from $x \in (a, \infty)$. We could proceed by solving the scalar wave equations, with the corresponding boundary conditions, to obtain \tilde{g}_e and \tilde{g}_m . However, it is mathematically simpler (and physically more engaging) to determine the solution by thinking about the structure of the scalar waves and the multiple propagations and reflections that will occur at the boundaries.

The scalar Green functions should describe waves emitted at x' that are multiply reflected at the dielectric interfaces. Let r_L and r_R be the left and right reflection coefficients of the plates. These factors are determined by the relevant permittivities and permeabilities. Multiple reflections and propagations add up to

$$\sum_{m=0}^{\infty} \left(e^{-2aw} r_L r_R \right)^m = \frac{1}{1 - e^{-2aw} r_L r_R}. \quad (2.4.18)$$

Now consider all possible multiple reflections and propagations from x' with the strength $-1/2w$ of the bare Green function

$$\tilde{g}_0 = -\frac{1}{2w} e^{(-w|x-x'|)}. \quad (2.4.19)$$

This collection consists of forward and backwards-directed waves emitted at x' and measured at x , after multiple reflections. For the forwards-directed waves, there are two cases:

1. when $x' < x$, the wave proceeds directly to x on its first cycle, and the distance between the source and the point of measurement is simply $x - x'$;
2. when $x' > x$, the wave picks up an extra reflection r_R on the right on its first cycle; the distance between source and measurement is $(a - x') + (a - x) = 2a - x - x'$.

And for the backwards-directed waves, there are also two cases:

1. when $x' > x$, the wave proceeds directly to x on its first cycle; the distance between source and measurement is $x' - x$.
2. when $x' < x$, the wave picks up an extra reflection r_L on the left before it is measured at x ; the distance between source and measurement is $x' + x' + (x - x') = x + x'$.

In total, this gives

$$\tilde{g}_1 = -\frac{1}{(1 - e^{-2aw} r_L r_R)} \cdot \frac{1}{2w} \left(e^{-w(x-x')} + e^{-w(x+x')} r_L + e^{w(x-x')} + e^{w(x+x'-2a)} r_R \right). \quad (2.4.20)$$

Notice, however, that if $x' < x$, the direct propagation from x' to x for the backwards-directed wave,

$$\tilde{g} = -\frac{1}{2w} e^{w(x-x')}, \quad (2.4.21)$$

is not possible; only indirect propagations that involve multiple reflections are possible. Likewise, if $x < x'$, then direct forward-propagating waves are not possible. We must therefore subtract these contributions¹²; *ergo*, we must add the scalar Green function

$$\tilde{g}_2 = \frac{1}{2w} \left(e^{w(x-x')} + e^{-w(x-x')} \right). \quad (2.4.22)$$

We may therefore express the scalar Green function as

$$\tilde{g} = \tilde{g}_0 + \tilde{g}_1 + \tilde{g}_2. \quad (2.4.23)$$

The renormalised scalar Green function, which is associated with scattering, is then

$$\tilde{g}_S = \tilde{g} - \tilde{g}_0 = \tilde{g}_1 + \tilde{g}_2. \quad (2.4.24)$$

Consider that, in the stress function, the terms $e^{-w(x+x')} r_L$ and $e^{w(x+x'-2a)} r_R$ are eliminated due to the term $(w^2 - \partial_x \partial_{x'}) \tilde{g}_S$. The only terms that are retained are those that contain $e^{\pm w(x-x')}$, and these are effectively doubled. Incorporating this factor of two into the stress, and taking the limit $x' \rightarrow x$, we can express the scalar Green function as [1]

$$\tilde{g}_S = -\frac{1}{w(r_L^{-1} r_R^{-1} e^{2aw} - 1)}. \quad (2.4.25)$$

2.4.1.4 The Regularised Stress

It is now possible to state the relevant, renormalised component of the stress exactly:

$$\sigma_{xx} = 2\hbar c \sum_{\lambda=e,m} \int_0^\infty \frac{d\kappa}{2\pi} \int_{-\infty}^{+\infty} \frac{d^2 \mathbf{k}_\parallel}{(2\pi^2)} w \frac{r_{\lambda L} r_{\lambda R} e^{-2aw}}{1 - r_{\lambda L} r_{\lambda R} e^{-2aw}}. \quad (2.4.26)$$

¹² In the limit as $x' \rightarrow x$, this amounts to subtracting the self-interacting terms. Only waves which have explored the environment contribute to the Casimir force.

This is Lifshitz' celebrated result [5] in a more general form.¹³ When we consider the Green functions in the outer plates, we note that the only terms that depend on $e^{\pm w(x-x')}$ are those describing direct propagation, and that these are subtracted from \tilde{g} in \tilde{g}_S . It follows that the Casimir force vanishes in the outer plates. The force density is concentrated in the interface as a delta function, pointing inwards (attractive) if σ_{xx} is positive and outward (repulsive) if σ_{xx} is negative. For the case of uniform plates, the electromagnetic reflection coefficients are given by [2]:

$$r_{eL} = \frac{\mu_1 w_2 - \mu_2 w_1}{\mu_1 w_2 + \mu_2 w_1}, \quad r_{mL} = \frac{\varepsilon_1 w_2 - \varepsilon_2 w_1}{\varepsilon_1 w_2 + \varepsilon_2 w_1}, \quad (2.4.27)$$

$$r_{eR} = \frac{\mu_3 w_2 - \mu_2 w_3}{\mu_3 w_2 + \mu_2 w_3}, \quad r_{mR} = \frac{\varepsilon_3 w_2 - \varepsilon_2 w_3}{\varepsilon_3 w_2 + \varepsilon_2 w_3}, \quad (2.4.28)$$

where the physical quantities are evaluated in the subscripted regions.

2.4.2 The Casimir Force in the Limit

To recover Casimir's original result, we assume vacuum between the plates $\varepsilon_2 = \mu_2 = 1$, and we take the limit $\varepsilon_1, \varepsilon_3 \rightarrow \infty$, with $\mu_l, \mu_r = 1$. The second condition corresponds to the case of a perfect electric mirror. We see from (2.4.27) that $r_{eL} = -1$. Similarly, $r_{eR} = r_{mL} = r_{mR} = -1$. This is the phase jump of π at the mirror. We obtain

$$\sigma_{xx} = \frac{\hbar c}{2\pi^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{w}{e^{2aw} - 1} dk_y dk_z d\kappa. \quad (2.4.29)$$

This integral is best expressed in spherical coordinates, with w as the radius:

$$\sigma_{xx} = \frac{\hbar c}{2\pi^3} \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi/2} \frac{w^3 \sin \phi}{e^{2aw} - 1} d\phi d\theta d\kappa = \frac{\hbar c}{\pi^2} \int_0^{\infty} \frac{w^3}{e^{2aw} - 1}. \quad (2.4.30)$$

On calculating the integral, we find that

$$\sigma_{xx} = \frac{\hbar c \pi^2}{240 a^4} = \sigma_{\text{Casimir}}. \quad (2.4.31)$$

We have recovered Casimir's original result (1.1.23) from Lifshitz theory, expressed in terms of the Minkowski stress.

¹³ The Lifshitz result was not originally cast in terms of reflection coefficients, until Kats observed that it was natural to do so [26]. Indeed, expressed in this form [1] it remains valid even when the optical response of the mirrors cannot be described by a local dielectric response function [27].

2.4.3 Boyer's Repulsive Mirrors

As a passing curiosity that has prompted much speculation, there is a second and similar case that has been calculated [17] in which the Casimir force turns out to be repulsive. This time, whilst retaining a perfect electric mirror on the left, with $\varepsilon_1 \rightarrow \infty$, $\mu_1 = 1$, we substitute a perfect magnetic mirror on the right, with $\varepsilon_3 = 1$, $\mu_3 \rightarrow \infty$. In this case, the reflection coefficients are $r_{eL} = r_{mL} = -1$, but $r_{eR} = r_{mR} = 1$. It follows that

$$\sigma_{xx} = -\frac{\hbar c}{2\pi^3} \int_0^\infty \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{w}{e^{2aw} + 1} dk_y dk_z dk = -\frac{\hbar c}{\pi^2} \int_0^\infty \frac{w^3}{e^{2aw} + 1}. \quad (2.4.32)$$

On computing the integral, we find

$$\sigma_{xx} = -\frac{7}{8} \sigma_{\text{Casimir}}. \quad (2.4.33)$$

Here, the vacuum force causes an electric and a magnetic mirror to *repel* each other. However, like Casimir's result, this holds only for the highly idealised case of perfect mirrors. Repulsive Casimir forces in real materials remain the debated exception, with prohibitive theorems extending both to metamaterials¹⁴ and chiral materials [18–20].

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¹⁴ Metamaterials incorporate arrays of micro-engineered circuitry, and can be engineered to produce a strong magnetic response at certain frequencies.

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