What's Location Got to Do with It? Place, Space, and the Infinite in Classical Greek Mathematics

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Here is a basic question: how much philosophy of mathematics can one pull out of Greek mathematical texts? Obviously, this depends on what we are taking as philosophy of mathematics. We can describe easily enough what goes on in a typical Greek mathematical treatise, but even here, 'typical' is a loaded word. We define the scope to our liking and to some 7 or 8 extant authors, or maybe to more, but the more we extend our list, the less does the result fit, for example, the tidy and austere world that Netz (1999) portrays in his ground breaking study. Even within this group, we should expect difference and variation. Austerity aside, unless they tell us, we have not a clue how Greek mathematicians thought about their work. In discussions of the philosophy of Greek mathematics, it is very common, following in the path of Proclus, for moderns to find a philosophy of Greek mathematics and then to trace the philosophy to Aristotle or Plato. If we distinguish, however, issues that are intrinsic to a mathematical exposition from external questions such as the ontology of mathematics, we can observe that there is very little evidence of any views about ontology expressed in any Hellenistic mathematicians, while later mathematicians tend to come out of neo-Platonism. Did Autolycus, Euclid, Archimedes, or Apollonius believe that mathematical objects were intermediates, ideal impressions in the imagination, physical objects qua mathematical, or something else, or did they not think about the issue? We have not a clue, except that some of these are unlikely in the Hellenistic Age. If anything, a Hellenistic mathematician was more likely to be a Stoic than an Aristotelian or Platonist, to say nothing of an Epicurean. But why any school at all? Even if we did find a fragment of philosophy, it would be a strange induction to infer that the view expressed was

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anything other than the view of that author. It would be merely less peculiar to infer that the author's view told us how we should understand the ontology expressed in the texts. Many a very good mathematician has made a not so good philosopher of mathematics. When we look at issues internal to the structure of mathematical treatises, we are on better ground, but here too we should resist many temptations, e.g., to connect practices to Aristotle, to over interpret, to generalize about conceptual issues and so forth. With this banal warning, I am going to look at one case where such ontology and mathematics might be thought to cross paths, namely in ancient treatments of place, space, infinite, and related notions: how they play out in ancient philosophical treatments of mathematics and how they show up in mathematical texts. It seems obvious that the notion of the infinite in Greek mathematics can have no other origin than the tradition of Anaximander, Zeno, Melissus, but most significantly Anaxagoras and Democritus. The ordinary treatment of place probably emerges from ordinary geometrical thinking, whether in architecture, surveying, or any other activity that might involve measuring and representation. Speculations on these lie outside my present concern.

I begin with a discussion how certain key words that dominate philosophical discussions of location (taken broadly) are used in mathematical texts. I then look at philosophical discussions that indicate how Aristotle and subsequent Hellenistic philosophers apply locational notions to geometry. If any of these discussions impinged on their treatments in mathematical texts, we would expect there to be a notion of a geometrical space or a conception of spatial relations between geometrical objects in mathematical texts that restricts some mathematical practice or that provides a robust avenue for a mathematical technique. Does either occur? My answer will be mostly negative. Whatever the origins of 'infinite in extent' in Greek discussions of the nature of the world, it becomes a small part of the language of mathematics. Since the infinite is an important spot for trying to see if someone has attitudes about spatial limitations on the treatment of mathematical objects, I shall then turn to Aristotle's claim that the geometers do not use the infinite. In this, I shall look at Euclid, Archimedes, Apollonius, Alexander of Aphrodisias, the third century CE professor of Aristotelian philosophy, and finally at Heron of Alexandria. Here, I think the results will have a certain irony.

A fundamental issue in any philosophy of science or mathematics is that of descriptivism (non-revisionism) versus revisionism, i.e., whether the philosophy aims to describe practice and fundamental conceptions or to change them or even to abandon the science altogether (e.g., because they inhibit one from the good life or because they are irredeemably false). In so far as the philosophy is revisionist, if philosophers can convince practitioners to change their ways, then the revision will become practice and the followers of the philosophers will be espousing a new descriptivism. If they fail, then the conflict will remain unresolved, with the two attitudes remaining incompatible, even if ignored by the practitioners. So I shall examine extents to which Aristotle, Stoics or Epicureans are revisionists and the extent to which their views express an incompatibilism with practice.

We need to distinguish three issues in Greek mathematics. First is the question of how Greek mathematicians treat ordinary spatial relations in their practice.

This is largely a philosophical and interpretative question. Of course, if we can extend lines, draw lines between disjoint and separate figures, move a disjoint line or figure to another, there is a sense in which spatial relations play an important role. This involves much more than drawing straight lines or circles. It involves mechanisms of construction such as sliding rotating lines through a loop (conchoids), coordinated motions (spirals and quadratices), even pointwise constructions; adoptions of simple plane techniques to solid geometry, extending cones or planes, slicing solids, etc.; and maybe even infinite lines. Since the mathematics we are primarily looking at is only metrical in using given magnitudes as measures, these extensions will involve either connecting points, arbitrary but adequate extensions, or multiplying lengths (as in the diagrams for numbers and magnitudes in the proportion theory). To do any of this requires a region where this can be done. Let's call this naïve space.

Much Greek geometry involves merely planar relations (say, where two lines intersect, any third line will intersect at least one of the others if they are extended), and the concept of a plane as extended in two directions is fairly clear. One can ask, however, whether having the concept of objects in the same planar relations requires having the concept of a planar space, that is a plane large enough for all plane objects to be placed, or, more minimally (and plausibly), whether any construction takes place in a plane large enough to place the objects under construction, even if in some cases the plane might be infinite. And one can raise a similar question about solids. This is a cognitive and a philosophical question which I admit I do not know how to address. But it is also a historical question whether certain geometers made any such inference about planar relations to having a conscious theory about planar spaces. Let's say that naïve planar space is that the concept that any new objects introduced will have the same planar relations as other objects already introduced. If we are dealing with solid geometry, then objects in the same plane will have the same planar relations and if objects are being sliced by planes, well those planes are objects. One could go on to define a planar relation more carefully (and a linear relation, etc.), but I prefer here to leave the notion as intuitive. If more is needed for cognitive reasons, then let it be added, but keeping things minimal. The historical question will be whether some Greek mathematicians required more and expressed it in their mathematical discourse.

I take it that naïve space is necessary for doing much of Greek mathematics, although it is not clear to me whether it is also necessary for doing Babylonian or Egyptian mathematics where constructions do not explicitly occur.[1] This feature arises from the constructional nature Greek mathematics. By itself, it tells us no more about Greek mathematicians' conceptions of space than the activity of a pâtissière producing an elaborately layered cake would tell us about her conception of space.

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[1] Or at least one could say that the constructed buildings are given, e.g. in BM 85194 (Sippur) Problem (v), one is given a wall of certain dimensions and a rate of construction and asked what is the length of the wall that one worker builds. I suppose one could say that the author treats space being filled, but only very weirdly.

1 Ordinary and Technical Location in Hellenistic Greek Mathematics

It is very easy in engaging the treatment of space in Greek mathematics to turn the study into a rich philosophical study of the foundations of ancient mathematics, to produce a rival to ancient philosophical discussion, where the modern result can be no better than the underlying philosophical assumptions, that is, to fill in a cognitive story of linear, planar, and corporeal space. My goal will be more modest (but not naïve—philosophical attitudes will always come in, even in grouping together certain notions), to investigate some ways in which spatial notions play out explicitly in Hellenistic Greek mathematics and to see any relation with ancient philosophical discussions. By space here, I merely mean regions to be filled in with figures, ordinary space (fill in your favourite theory).[2] Of course, figures may be extended, unconnected figures joined, regions identified. Up and down are normally oriented in relation to a part of the diagram related to the construction ('up' indicating away, 'down' indicating towards), and only incidentally to the orientation of the viewer.[3] Direction is one way, the reader is to scan the positional relations between figures in the diagram,[4] so that diagrams are oriented to the user, as Aristotle notes (see § 2 below). Space is not measured, although figures constructed within a space may be.

Construction itself plays oddly different roles even within Euclid's *Elements*. In the Definitions of Book I, only parallel lines involve a construction in their definition (23: when extended they do not meet). Yet, in the definitions of Book XI, the sphere (Definition 14), cone (Definition 18), and cylinder (Definition 21) are

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[2] So I resist the temptation to do a philosophy of ancient geometry, such as the a priori construction of the history of geometry that one finds in Husserl (1989). It is not that such a philosophy is completely wrong. For example, it is obvious that Hellenistic treatments of geometrical objects are extremely different from Hilbert's conception of a space, and obviously so, in the difference between treating the geometrical space as a set of points and creating a mathematics of geometrical objects (and where points are just one sort of object), whose spatial relations are determined by geometrical objects. My observation concerns what geometers commonly did, and not some ontology or epistemology of some special object, Greek Mathematics.

[3] A small survey of Euclid should make this clear. The direction is indicated by the verbal prefix, ἀνα- (commonly: up-, back-, re-) or by κατα- (sometimes: downward). For 'lead up (ἀνάγειν)' and 'lead down (κατἀγειν)', cf. Mendell (1984, p. 362) and pace Makin (2006, pp. 233–234) on the same issue. For 'draw up (ἀναγρἀφειν)', see Euclid, *Elements* II 11, where one square is drawn up on a line that is at right angles to a line on which another square is drawn (presumably, up for one and either to the right or left for the other). Cf. also I 47. The apparent contrast between 'draw up (ἀναγρἀφειν)' and 'draw down (καταγρἀφειν)' in II 7 is not at all about directions, but is between the square drawn on (from) the line and the figure *being completed* (diagonals and rectangles filled in), where 'let it be drawn down (καταγεγρἀφθω)' is related to the common word for the diagram (καταγραφή), e.g., II 8, VI 27–29, X 91–96, XIII 1–5; *Data* 58, 59. For 'be stood up (ἀνίστασθαι)', cf. *Elements* XII 10, where the common base of the cone and cylinder is drawn, but the erected figures rise, as it were, from the diagram towards the reader. For all three, cf. Mugler (1958, ad verbum). The common verb 'to construct (κατασκευζειν)' indicates nothing about direction.

[4] Netz (1999, Chap. 1).

defined by rotations. In the Postulates 1–3, someone may draw (ἀγαγεῖν: aorist active) or extend (ἐκβαλεῖν: aorist active) lines, and draw (γράφεσθαι: middle?) a circle, and in problems one is normally required (δεῖ) to do something, to construct or to find (only points and lines and magnitudes in a given ratio[5]). Furthermore, the rare uses of superposition or fitting one figure on another seems to be a change of place, i.e., unless the superimposed figure is never in between.[6] Such actions require a 'there' to do the actions. But there is no account of the 'there'.

To us, it seems a remarkable feature of Greek mathematics that one creates a figure by delimiting its limits, and what is bounded becomes the object. How the stuff between the limits comes to be, whether as space or out of nothing or by stipulation or something else is just not a part of the story (something similar happens with units appearing in Euclid's treatment of multiplication). In any case, we should resist seeing Greek geometry as about space or spatial relations. That is incidental to the objects studied. We can see this through features common to much of Hellenistic geometry. Mathematical texts do not come with any such an ontology.

Archimedes never treats of a distance between figures that is not either within a figure or a constructed line. For example, his famous first assumption in *De sphaera et cylindro*, "a straight-line is the smallest of those having the same ends," concerns lines between two points and not distances between them. Euclid only uses 'distance (διάστημα)' in the *Elements* and *Data* when referring to Postulate 3, the construction of a circle.[7]

The distinction between the figure and the space outside the figure is conceptually, but not visually marked. All we see in a diagram are the borders of the figure. It is not typically filled in, even in the fragments of ancient mathematical texts.[8] We know that there is a conceptual difference. We do not need to appeal to Aristotle, who makes this clear in his discussions of the matter of magnitude.[9] The evidence comes from the definitions of figures that we find in Archimedes, Apollonius, Euclid, and Theodosius that the figure is enclosed (περιεχόμενον) or surrounded (περιληφέν) by a boundary, lines or planes. Thus, it is not the enclosing lines or planes.[10] Unless challenged, one would not have thought otherwise. This is not to say that the language sometimes suggests that a circle is a circumference, as when a

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[5] A point is found, namely the center of a circle at *El*. III 1, but lines in a given ratio in *El*. VI and magnitudes of a certain kind that implies a relation to other magnitudes (commensurable, incommensurable, etc.) are found in *Elements* X. Numbers are also found in VII–X. Outside Euclid, I do not find any such caution.

[6] *Elements* I 4, 8, III 24. But see also Archimedes, *Stomacheion* 2.416.15–18.4 (as filled in by recent editions of the Archimedes Palimpsest), where the figure is transposed to another place and gets another position (εἰς ἕτερον τόπον … μετατιθεμένου … καὶ ἕτεραν θέσιν λαμβάνοντος). Obviously, the more applied the work is, the less this would gently raise an eyebrow. See also Apollonius, *Conics* VI Definition 1, where conic sections, which are lines, are said to be 'equal' by superposition.

[7] In fact, διάστημα only occurs in the dative in the formula given by Postulate 3.

[8] Some of the astronomical diagrams in the *Ars Eudoxi* (Paris Gr. 1) are partially filled in, e.g., to distinguish dark parts and lit parts of a body.

[9] Cf. Aristotle, *Physics* IV 3.209b2–9, *Met*. V 17.1022a4–6

circle has been drawn about a triangle (just as the circle is enclosed by its circumference).[11] But one can just easily understand the triangle becoming a part of the circle as the circle qua circumference enclosing an area of which the triangle is a part, with its vertices on the 'circle'. It is also possible that the distinction gets loosened in discussions about great circles, after the author gets underway. How would Euclid reply to Heath's point (*Euclid* I 185), that III 10 (circles do not cut circles at more than two points) treats circles as circumferences? We would have to know more about the intentional states of each author.[12]

We would expect the issue of explicit distance (often διάστημα) between figures to arise only in issues of convergence of lines[13] and in applied mathematics, e.g., mechanics, optics or astronomy.[14] In the case of convergence, the distance is crucial but will be handled, naturally, by comparing drawn lines (is there another way in ancient mathematics?). For optics especially, one wants to have theorems about objects nearer and further from the eye. Again, this is studied through the drawn light-ray.

We shall see four central 'spatial' terms playing out in philosophical texts, 'position (θέσις)', 'place (τόπος)', 'room (χώρα)', and 'void or vacuum (κενόν)'.[15] It is useful to begin with a survey of how they are used in Greek mathematical texts, where only 'position' and 'place' are at all common terms, 'position' being mostly

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[10] E.g. *Elements* I Definitions 14, 15, 18–20 and XII Definitions 9, 10, 12–14, 18, 25–28; Archimedes, *De conoid. et sphaer*. p. 246.16–19, 246.22–248.6, 248.28–250.1, etc., *De lin. spir*. p. 6.22–6, 8.23–5; Apollonius, *Conica* I Definition 1, 11.1–16, etc.; Theodosius, *De sphaera* post. 1. Contrast with Plato, *Meno* 76A, "figure is the end of a solid."

[11] Things will also get messy in later discourse. So, from late antiquity, ps.-Heron, Definition 7 (helix) says that the line that results from a line rotated about a fixed point is a circle, while Definition 27 (circle) includes an awkward definition of 'circle' that is the line that makes distances equal to all its parts.

[12] Mugler (1958, 260–1) cites no example where κύκλος means 'circumference'. For Archimedes a section (τομή) is a line, the segment (τηῆμα) an area. So one assumes something similar in Apollonius, e.g., from his description of the contents of Book IV (proem 4.17–19), in how many ways a section (τομή) of a cone or a circumference of a circle meet (cf. esp. IV 25, but also throughout the treatise). Clearly, 'circle' denotes a disk, so that the section should be a line. However, at I 3, the vertical section is a triangle, which should then be a figure, unless he uses 'triangle' loosely. We should be cautious about imposing much semantic order here.

[13] So Apollonius, *Conics* II 14: When extended infinitely the asymptotes and the [hyperbola] section draw nearer to themselves and come to a distance smaller than any given distance. The proof takes the distance smaller than the given to be determined by a line drawn parallel to the tangent of the hyperbola between the asymptotes, namely, the segment between one asymptote and the intersection of the parallel line and the neighboring part of the curve. So the distance derived is actually larger than the nearest distance between the point and the asymptote, which goes unremarked.

[14] Of course, διάστημα has a different meaning in harmonics, which need not concern us here.

[15] In the following word counts, taken from the TLG_E, I take the following authors without distinguishing works, Autolycus, Euclid, Archimedes, Apollonius, Hypsicles, Serenus. For Pappus, I search only the *Collectio*. For Theodosius, I separate out *De sphaera*, but mention counts for *De habit*. and *De dieb. et noct*. For Heron, I restrict searches to *Pneumatica, De automatis, Frag. de horoscopiis, Mechanicorum frag., Catoptrica, Metrica, Dioptra,* and *Belopoeica*.

either 'in position (θέσει)' or, less commonly, 'having a position (θέσιν ἔχειν)', and 'place' having one of two meanings, a region and a line, plane or solid region where every object of a certain kind has a given property.

In looking at mathematical terminology, a good place to start is Mugler (1958). Mugler finds no uses of 'room (χώρα)' in geometry between Plato and Proclus, where, he says, it is a synonym for 'area (χωρίον)'; indeed, the word is very rare in geometers other than Heron.[16] However, Heron uses 'room (χώρα)' mostly in the more applied, mechanical works *Pneumatica* and De *automatis*, where the areas are less classified and rely more on the drawing. In any case, χώρα has little to do with questions about space and location in geometry.[17]

With regard to the related word 'area (χωρίον),' I note one small, perhaps Aristotelian, feature about its very common use in all geometers. If 'figure (σχῆμα)' and words for particular figures denote an object *qua* having a certain quality, namely shape, 'area (χωρίον)' denotes the object *qua* so-much or being so much (quantum). Here the primary referent may be a rectangle (e.g. *Elements* X) because that is the most common sort of area considered (as, in English, 'eggs' normally and in a general context refers to chicken eggs, even though the word never means 'chicken eggs'). In other words, area is not an abstract entity. That said, it is often the case that 'area (χωρίον)' is just a general word for any two dimensional object,[18] ἐμβαδόν being the more common word for an area qua measured (compare στερέον as 'solid' and as 'volume').

'Void (κενόν)' does not appear in Mugler, nor, with one or two unimportant exceptions, in the core mathematical texts.[19] A method for measuring disorderly figures by measuring an absence of body appears in Heron. In *Metrica* II 12–13, he introduces a technique for measuring shells determined by one volume separated from a similar, smaller volume, where the result is the difference between the two volumes, and recommends it as a method for measuring washtubs, conch shells (with an insignificant error), arches, and vaults. He then says, "Given that the surface within is hollow/concave (κοιλῆς), that is, empty/void (κενῆς), again each of them will be the excess of two similar segments." At the end of *Metrica* II (20), he commends Archimedes' method of measuring a disorderly body by placing it in a full rectangular tank and measuring the emptied place (τὸν ἐκκεκενωμένον τόπον)

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[16] χώρα occurs: Autolycus (0), Euclid (1), Archimedes (1 ordinary use in *Sand Reckoner*), Apollonius (0), Theodosius (0), Hypsicles (0), Serenus (0), Heron (69), Pappus (1).

[17] We can supplement Mugler's remarks by noting that in ps.-Heron (e.g. *Mensurae* and *Liber geeponicus*), the word is used to mark an area treated as a figure, e.g., *De Mensuris* 54.1.1: isosceles triangular χώρα (χώρα τρίγωνος ἰσοσκελής). It is not quite a synonym for χωρίον (area), except in its pre-Euclidean uses (cf. Mugler 1958, p. 451, Plato, *Meno* 82B, square area [τετράγωνον χωρίον]).

[18] See the citations in Mugler (1958, ad verbum).

[19] κενόν occurs: Autolycus (0), Euclid (0), Archimedes (1, 'empty speech'), Apollonius (0), Theodosius (0), Hypsicles (0), Serenus (0), Heron (59: *Pneumatica* (56), *De automatis* (2), *Metrica* (1)), Pappus (1, 'with empty hands'). In Heron, the verb 'to empy (κενόω)', the noun 'emptying (κένωσις)', and two compounds, 'to empty together (συνκενόω)' and 'to empy out (ἐκκενόω)', occur 71 times.

in the tank when the object is removed, and a second method where the body cannot be moved, of covering the body with wax or clay to form a rectangular figure that can be measured and then measuring the wax or clay, now molded into a rectangular figure. It's fairly obvious that, in general, Heron has a more physical and environmental treatment of geometrical objects anyway. Or rather, none of the extant works is a work of geometry. All are applied.[20] But one might wonder whether the technique brings along a tiny shift in how to conceive of volumes in a practical context.[21] I am somewhat sceptical, since, I think, it at most reflects Heron's own views about applied geometry. First, the use of 'void (κενόν)' looks like the ordinary use, indeed, as does the use of 'hollow/concave (κοιλῆς)', for Heron does not say that the region is empty of everything. One might be led to think that something unusual is going on by the fact that the inner figure in II 12–13 and

[20] Heron's division of methodologies in the *Metrica* is somewhat subtle. There are three levels of discussion, the purely abstract, the metrical (arbitrary numbers in pure units (μονάδες), not in terms of standard measures), and loose applications where physical objects are mentioned. Through most of the treatise and depending on the complexity of the problem, he often gives an analysis abstractly followed by a synthesis with numbers. Furthermore, metrica is throughout an application of geometry. So at III 10, p. 160.14–7 (cf. also I 7, 8, esp. p. 20.6), he contrasts the numerical presentation with a geometrical demonstration (i.e., the abstract presentation), where numbers are not used. Accordingly, he seems to mean in I 6 (p. 16.11–14), "We acted up to now taking into account (or: calculating on/by the) geometrical demonstrations; in what follows we will make measurements according to the analysis through the synthesis of the numbers (Μέχρι μὲν οὖν τούτου ἐπιλογιζόμενοι τὰς γεωμετρικὰς ἀποδείξεις ἐποιησάμεθα, ἑξῆς δὲ κατὰ ἀνάλυσιν διὰ τῆς τῶν ἀριθμῶν συνθέσεως τὰς μετρήσεις ποιησόμεθα.)," that the employment of numbers follows the geometrical demonstration of the theorem, while later he will provide a geometrical analysis followed by a synthesis that uses numbers. For this is somewhat what he has done and somewhat what he will do, although not until I 10. In this regard, the distinction between orderly and disorderly figures is just that between those where the figures have been studied and have been given an abstract treatment for finding area or volume and those where no clear geometrical reduction is known so that one needs to use metrical or even physical means, whether approximations (*Metrica* I 39) or the methods discussed above (II 20), i.e., those to which geometry can be applied directly and those to which it cannot, for whatever reasons. I would be very surprised if Heron, or any ancient mathematician, thought there was a distinction between geometrical figures per se and non-geometrical figures or even whether a construction employing a mechanism is thereby a part of mechanics, an error that famously goes back to Descartes. Hence, given that the context is never purely geometrical, we expect that geometrical methods will be used but blended with more metrical methods, however these are to be distinguished, although Heron does some work on this in the *Metrica*. Nonetheless, we do not expect a distinction between geometrical plane figures and non-geometrical plane figures (pace Tybjerg 2004, esp. pp. 39–43). Furthermore, just because geometrical shapes get their names from physical objects they represent, it does not follow that they are those objects, and just because a mathematician uses vivid language (e.g., 'pierce') it does not follow that he is thinking physically and not geometrically (again, pace Tybjerg). That said, this might become irrelevant if we can discern anything of Heron's views on geometry (see § 6).

[21] The method appears in late antiquity in works which are probably based on Heron. Here, 'vacuum (κένωμα)' occurs in *De mensuris* (5) and *Stereomtrica* (24), sometimes to measure the vacuum and solid and then the vacuum in order to determine the volume of the solid. The word might have shown up in other works of Heron in this context, but he uses it only once, quite differently (*De automatis* 26.2).

the emptied bath are not figures. After all, this sort of determining areas or volumes by subtraction is as old as geometry, but here they are regarded as devoid of at least the figure in question, but with emphasis. Much more significant is the water, wax, and clay that keep their volume in changing shape, revealing that metrica can be more concrete than the mere adding of measures would suggest. In any case, it is very optimistic to read ontological commitments about geometry from a treatment of applied mathematics, as is reading the end of *Metrica* II as providing the level of abstraction for the entire work. Nonetheless, we shall see in § 6 that Heron takes into account practical considerations in apparently purely geometrical constructions.

Thirdly, let us turn briefly to 'position (θέσις)'.[22] 'In position' occurs in the *Data* and the *Porisms*, a lost work of Euclid, as reported by Pappus.[23] Here is how Euclid defines 'given in position' in the *Data* (Definition 4):

> Points and lines and angles that always keep the same place are said *to be given in position*.
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> δʹ. *Τῇ θέσει δεδόσθαι* λέγονται σημεῖά τε καὶ γραμμαὶ καὶ γωνίαι, ἃ τὸν αὐτὸν ἀεὶ τόπον ἐπέχει.

'Place' is not mentioned again in the *Data*. In other words, Euclid here treats 'place' as an intuitive notion from which one may define the technical notion. Objects do not necessarily keep their place in the *Elements*, as already noted, but this does not seem to be the point here. As a practical matter, one may use the position of something that is given in position in showing that other things are given in position. So it establishes the spatial relation between different things so given. This seems to be all Euclid really needs to mean and all that other geometers need in using the expression. So things have a position (θέσιν ἔχει) when they can be located in a configuration, when their position is given. A circle may be given in position without any point on it being given in position. A point on a circle may be given in position without the circle itself being given in position.

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[22] θέσις occurs: Nominative: Autolycus (0), Euclid (10), Archimedes (1), Apollonius (0), Theodosius (0), Heron (7), Pappus (6). Genitive: Autolycus (1, in *De ort. et occ.*), Euclid (2), Archimedes (0), Apollonius (0), Theodosius (0), Heron (3), Pappus (28). Dative singular: Autolycus (0), Euclid (210), Archimedes (1), Apollonius (50), Theodosius (0), Heron (26), Pappus (185). Dative plural: Autolycus (0), Euclid (0), Archimedes (0), Apollonius (0), Theodosius (0), Heron (0), Pappus (0). Accusative singular with 'having': Autolycus (18: 4 in *De sphaera*, 14 in *De ort. et occ.*), Euclid (41), Archimedes (0), Apollonius (0), Theodosius (0), Heron (84), Pappus (20). Other accusatives: Autolycus (0), Euclid (3), Archimedes (1), Apollonius (0), Theodosius (0), Heron (10: 5 with 'find (εὑρίσκειν)', 1 with 'get (λαμβάνειν)', 1 with 'I have (ἔχω), and 3 others), Pappus (8: 7 with 'get (λαμβάνειν)' and 1 with 'preserve (φυλάττειν)'). The distribution of datives largely reflects the language of analysis and 'given' (esp. Euclid, Apollonius, Heron, and Pappus), while the accusative with 'have' or 'get' is largely astronomical or at least involves moving figures. For example, if we add the two astronomical works of Theodosius, there would be 29 occurrences of the accusative, 28 with 'have'. So too all occurrences in Autolycus. The Euclid passages are all from the *Phenomena*, and the three accusatives without 'having' from the *Optics*. Additionally, Serenus and Hypsicles do not use the word.

[23] Pappus, 636.18–30, 648.19–20 and following. Presumably, the *Places* (*Loci*) *on a Plane* (*Τόποι ο πρὸς ἐπιφανεῖᾳ*), also mentioned by Pappus (first passage), used 'given in position'.

One might presume (perhaps from thinking about basic properties of atoms in Democritus) that whereas position is a relative concept, place is absolute so that this will mark the difference between them. But, while important in astronomy and any science where things are absolutely located, it is hard to see what role absolute place could play in pure geometry, or in any science where things do not need absolute position for the theorems to be stated. Even in Archimedes' statics, all that is needed is up and down. So we should expect, and do find, a different distinction.

Furthermore, different authors might have different tastes. In Autolycus, Euclid's *Phenomena*, and Theodosius, only circles (or circular-arcs in Theodosius) have a position (θέσις). However, in Autolycus the sun may have a place (τόπος), in Euclid places are rising and setting points on the horizon, while Theodosius doesn't use the term at all. Euclid, the only one of these to use 'room (χώρα)', uses it for the region where the North Star turns. So he keeps place to the horizon. Quite differently, in the commentary of Hipparchus on Aratus and Eudoxus, stars and constellations have positions, but, besides places on the earth (e.g., the regions of Greece), 'place' is used only five times, but in ways different from the others here. From Hipparchus' comments, however, it would seem that in his astronomy Eudoxus, like Euclid in his astronomy, treated place as absolute.[24]

Finally, 'place (τόπος)', is also important in mathematics.[25] Here, Mugler identifies two principal uses. The first is a region defined by lines or surfaces, while the second, technical sense is the familiar, locus use, although the two uses can be merged. In the technical sense, in a 'locus (τόπος)' theorem, a place or locus is a given point, line, area or region, solid or region, such that every object of a determined sort within the locus has certain additional properties (συμπτώματα). For example, in Aristotle, *Meteorologica* III 5, every point in a given ratio (not 1:1)

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[24] Two of these are quotations from Eudoxus. Hipparchus, *In Arati et Eudoxi phaen*. I 4.1, where the issue is whether "there is a star always remaining at the same place (ἔστι δέ τις ἀστὴρ μένων ἀεὶ κατὰ τὸν αὐτὸν τόπον)," and Eudoxus' claim (I 9.2), "The sun appears as making a difference according to the places of its turnings (τῶν κατὰ τὰς τροπὰς τόπων)." The places in the last are relative to the observer on earth, as in Euclid's *Phaenomena*, while Euclid uses 'room (χωρα)', where Eudoxus used 'place (τόπος)' for the position of the North Star, perhaps with a different theory. Observe that this amounts to treating the fixed stars as changing place but not position (relative to each other), which would encourage one to use 'position' for the stars and not 'place'. The pole (quoting Eudoxus) has place. If this is right, Eudoxus treats place as absolute. Two more passages concern the fact that the constellations of the zodiac do not extend over their proper places, i.e., are more or less than their twelfth-part (II 1.8, 4.4). Finally, it may indicate a region (I 8.6), namely the region between the River (Eridanus) and the rudder of the Argo that is not large.

[25] τόπος occurs, where I restrict Theodosius to *De sphaera,* and Pappus to the *Collectio*: Autolycus (4, in *De ort. et occ.*), Euclid (64), Archimedes (14), Apollonius (27), Theodosius (0), Hypsicles (1, also a degree arc (μοῖρα) is called τοπική (not mentioned again since it is the primary notion, as opposed to a time degree, χρονική (frequent)), Serenus (1 as 'topic'), Heron (134), Pappus (95). Proclus (In Eucl. 194.25–195.5, on Com. Notion 1) quotes Apollonius as holding that figures are equal that occupy the same τόπος to show the transitivity of equality. Additionally, τόπος occurs 31 times in the astronomical works of Theodosius.

of distances from two given points lies on a given circle, where the ratio is given iff the circle is given. The locus here would be the circumference (Aristotle does not use the term in this way).

Proclus defines a 'local-theorem (*In pr. Eucl. el.* 394.16–395.1, trans. Morrow (1992), with changes):

> I call 'local-[theorem]s'[26] those in which the same property occurs throughout the whole of a certain locus, and I call 'locus' a position of a line or surface producing one and the same property of a line or a surface producing one and the same property. Some local-[theorem]s refer to lines, others to surfaces; and since some lines are plane and others solid—plane lines being those which, like the straight-line, lie in a plane and whose generation is simple, and solid line those which are produced by some sectioning of a solid figure, like the cylindrical helix and the conic lines—I should say further that of local-[theorem]s referring to lines some have a plane and others a solid locus.

καλῶ δὲ τοπικὰ μέν, ὅσοις ταὐτὸν σύμ-
πτωμα πρὸς ὅλῳ τινὶ τόπῳ συμβέβηκεν, τόπον δὲ
γραμμῆς ἢ ἐπιφανείας θέσιν ποιοῦσαν ἓν καὶ ταὐ-
τὸν σύμπτωμα. τῶν γὰρ τοπικῶν τὰ μέν ἐστι πρὸς
γραμμαῖς συνιστάμενα, τὰ δὲ πρὸς ἐπιφανείαις. καὶ
ἐπειδὴ τῶν γραμμῶν αἱ μέν εἰσιν ἐπίπεδοι, αἱ δὲ στε-
ρεαί, – ἐπίπεδοι μέν, ὧν ἐν ἐπιπέδῳ ἁπλῆ ἡ γένεσις,
ὡς τῆς εὐθείας, στερεαὶ δέ, ὧν ἡ γένεσις ἔκ τινος
τομῆς ἀναφαίνεται στερεοῦ σχήματος, ὡς τῆς κυλιν-
δρικῆς ἕλικος καὶ τῶν κωνικῶν γραμμῶν – φαίην
395.1 ἂν καὶ τῶν πρὸς γραμμαῖς τοπικῶν τὰ μὲν ἐπίπεδον
ἔχειν τόπον, τὰ δὲ στερεόν.

22 γένεσις ver Eecke Morrow νόησις Friedlein mss

Proclus own classification of local-theorems is probably derived from Apollonius, or at least one might infer that from the more detailed discussion in Pappus.[27]

Proclus (395.3–12, 396.2–9) then goes on to identify three theorems in the *Elements* as local-theorems: *Elements* I 35 (parallelograms on the same base and on the same parallels are equal), where the locus is the region between the parallel lines;[28] as well as III 21 (angles on the same segment of a circle are equal) and 31 (angles in a semicircle are right, in a larger segment smaller, and in a smaller

[26] 'Theorem' is understood from context; however, I prefer to translate τοπικά as 'local' than 'locus', reserving that for the region. However, the usual description is not quite an idiosyncracy of Proclus—Pappus uses it once (*Collectio*, VII 652.2), but normally just calls a local theorem a 'locus (τόπος)˜, which at least obviates the issue that some are problems.

[27] Cf. Jones (1986, pp. 539–46).

[28] Proclus' description of *Elements* I 35 as a local-theorem is somewhat awkward, since he takes the theorem as concerning parallel lines drawn from the base, which is only implicit in the theorem. For our purposes, Proclus' example is fine, namely as showing that within the τόπος all parallelogram areas on the same base have the same salient property.

larger), where the locus is the circle or disk,[29] as well as a theorem on hyperbolae (parallelograms inscribed in the asymptotes and hyperbola are equal), where the locus is the hyperbola.[30]

It is enough for our purposes that a locus need not be a finite figure or line, but can be an infinite line, surface, or solid, so long as it is partially bounded so that there are points, lines, etc. that are not included in the locus, as this marks the distinction between a local theorem and a general theorem.

Proclus reports local-theorems (cf. 67.20–23), or at least this sort of problem as going back at least to Hermotimus of Colophon, who would have been younger than Eudoxus (say, mid-4th cent. BCE), but does not attribute to him the actual discovery of local theorems. So far as I can tell, the terminology is not in Aristotle, at least not in the two places where we find local theorems, *Meteorologica* III 3 and 5. It is important to keep in mind that Euclid's definition of circle (*Elements* I Definition 15) is a local definition that does not use the word 'locus (τόπος)'.

> A circle is a plane figure enclosed by one line [which is called circular arc], such that all the straight lines falling on it from one point among those lying within the figure [to the circular arc of the circle] are equal to one another, …

Κύκλος ἐστὶ σχῆμα ἐπίπεδον ὑπὸ μιᾶς γραμμῆς περιεχόμενον [ἣ καλεῖται περιφέρεια], πρὸς ἣν ἀφ' ἑνὸς σημείου τῶν ἐντὸς τοῦ σχήματος κειμένων πᾶσαι αἱ προσπίπτουσαι εὐθεῖαι [πρὸς τὴν τοῦ κύκλου περιφέρειαν] ἴσαι ἀλλήλαις εἰσίν.

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29 Proclus does not actually say this, but we may infer it from his drawing an analogy with I 35.

30 Morrow (1992, p. 311 n. 70 ad 395.20) follows ver Eecke in identifying the theorem with Apollonius, *Conics* II 12. However, the theorems do not seem to be the same in one way that is important for this discussion. Proclus seems to be saying that if one draws a line from the hyperbola parallel to the asymptote to the opposite asymptote and completes the parallelogram, it will be equal to any other drawn similarly. Apollonius states that if from a point on the hyperbola lines are drawn to the two asymptotes and from another point on the hyperbola parallel lines are drawn to the asymptotes, the rectangles formed by the lines from the one point are equal to those from the others. Except for special cases, the rectangles are not inscribed in the asymptotes at all. The two basic properties of hyperbolae are trivially related, however. Suppose the lines in Apollonius are drawn as in Proclus, and let the respective lines be a_1, a_2, b_1, b_2, so that by *Conics* II 12, Rectangle(a_1, a_2) = Rectangle(b_1, b_2). Since the parallelograms are drawn in equal angles (the angle between the asymptotes), let it be α, the parallelograms will also be equal. In modern terminology, the area of each parallelogram will be: Rectangle (a_1, a_2 Sin(α)) = Rectangle (b_1, b_2 Sin(α)). Nonetheless, the theorem in Apollonius is more apposite to Proclus' point. For it is unclear why Proclus takes the τόπος of the hyperbola theorem to be the line and not the region between the asymptotes and the hyperbola. For *Elements* I 35 and the hyperbola theorem concern parallelograms and not lines. Perhaps this is what he intended, to draw the parallel, but then digressed to say that the line is a solid line (constructed by a section). One could read *Conics* II 12, however, as concerned with points on a hyperbola or as concerned with lines drawn between the hyperbola and the asymptotes, which would be more akin to Proclus' examples from the *Elements*. Clearly, the terminology is a little loose.

Additionally, Knorr argued that the conics were probably also originally defined in this way.[31] For early texts, Euclid wrote a lost treatise, books on τόποι in planes,[32] while local-propositions also occur in Euclid, *Optics* (37, 38, 44–49), but with a slightly different terminology.[33] Is there any philosophical significance in all this for understanding Greek philosophy of mathematics, concepts of location or space? I do not know. Obviously, a place is defined relative to some other object, distinct from it, given points or lines, etc.

The first use of τόπος mentioned by Mugler is more immediately relevant to our purpose. This is a region determined by some lines or surfaces. We find this use in the discussion of the elements in Plato's *Timaeus*, and one could argue that Aristotle's own discussion of the places of the elements is an attempt to fit this in as well. In the *Elements*, it only occurs in III 16 for the region between a circumference and a tangent, but it also occurs in Apollonius' *Conics*, in propositions about hyperbolas, as the region within an angle or between an angle and the hyperbola;[34] in Archimedes, mostly in less purely mathematical contexts, as length;[35] and in Pappus, with various senses perhaps representing the variety of his sources, but sometimes for the area bounded by different figures and once for the region about a point.[36] I have already noted that in Heron it can be a volume.[37] So let's think of this as an ordinary use. Does it have philosophical or ontological commitments?

Consider Aristotle's category of 'Where?' in the *Categories*, where 'in the Lyceum' gives a region within a boundary. Honing the region to the location of the object is also the motivation behind the technical notion of primary place that we find developed in *Physics* Δ 2. So far as anyone in the Academy or Lyceum is concerned, the notion of a bounded region within which an object may be found must be compatible with any reasonable treatment of place. The dialectical issue would be whether it is compatible with one's opponent's theory.

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[31] Cf. Knorr (1986, pp. 62–6), Jones' response (1986, pp. 572–599). Knorr apparently intended to reply to Jones, but I am only aware of a sketchy draft.

[32] Pappus, *Collectio* VII 636.21. I am assuming that Euclid precedes Aristaeus. The list in Pappus is not chronological. Also, cf. Apollonius, *Conica* I prol. 30–7.

[33] A place is a position of the eye (a point) or of the seen object (a line), e.g. the places (τόποι) in *Optica* 47 are each points on a semicircle where, with the eye placed at any of them, two adjacent lines will appear equal.

[34] Apollonius, *Conica* I 32, 35, 36, II 13, 32, 33, 42, 49, III 24, IV 42, 51.

[35] Cf. Archimedes, *Sand Reckoner* 236.11–12 (length of 25 poppy seeds as a τόπος), where the other occurrences concern the place of the eye (222.22, 224.4, 12, 29, 226.8), and *Floating Bodies* I 4 328.4 (closed region), II 10 386.3, 392.15, 408.13, 412.14 (area: the base will be cut off at a larger place, etc., by the liquid), but also Stomacheion referred to in note 6.

[36] Pappus, *Collectio* IV 224.15, 252.18 (closed areas bounded by different figures), 242.15 (region under cochloid), 244.26 (region between cochloid and base line); V 306.8, 9, 15 (ὁ περὶ τὸν αὐτὸν τόπος, i.e., the region about the same point that can be filled up with identical equilateral figures). On this last, cf. also ps.-Heron, Definition 71.3–9 and 136.45.

[37] Cf. also the late, ps.-Heron, *Geometrica* 4.15.40–57, cf. 4.8, *Stereometrica* 42a12–13.

I note, however, that in the one quasi-mathematical place where you might expect Aristotle to use τόπος in this way, he doesn't. At *De caelo* A 5.271b28–33, he is discussing an infinite ray rotating about a point. From Aristotle's perspective, the argument is physical and not per se mathematical—the infinite ray is a physical line of the rotating heaven from the center of the earth:

> For if the circularly rotating body is infinite, the lines extended from the middle will be infinite. But the interval [or extension] of infinite lines is infinite. I means by 'interval of lines', that where it is not possible to take a magnitude outside touching the lines [that is part of the interval]. And so, this must be infinite, since for finite-lines, [the interval] will always be finite.

271b28 Εἰ γὰρ ἄπειρον τὸ κύκλῳ φερόμε-
νον σῶμα, ἄπειροι ἔσονται αἱ ἀπὸ τοῦ μέσου ἐκβαλλόμε-
271b30 ναι. Τῶν δ' ἀπείρων τὸ διάστημα ἄπειρον· διάστημα δὲ λέγω
τῶν γραμμῶν, οὗ μηδὲν ἔστιν ἔξω λαβεῖν μέγεθος ἁπτόμε-
νον τῶν γραμμῶν. Τοῦτ' οὖν ἀνάγκη ἄπειρον εἶναι· τῶν γὰρ
πεπερασμένων ἀεὶ ἔσται πεπερασμένον.

In effect, the distance is the region within an angle. So a Greek mathematician might well have used the word 'place (τόπος)' instead of 'interval (διάστημα)'. If so, then the word 'interval' might well be doing the work that 'place' conceived as interval would be doing. Given his definition of place as the inner limit of a containing body and his resulting view that the finite universe has no place, is Aristotle queesy here? If so, is he queesy about calling an infinite region a τόπος, especially if it is unbounded in one direction, even in a reductio ad absurdum (cf. his criticism of Anaxagoras at *Physics* III 205b1–b24). Or is he queesy about something else?[38] Or is the terminology just not there? We cannot know.

The technical use of 'place (τόπος)', as presented by Proclus, is really just this first use in a particular technical context, where certain objects in the determined region have certain properties. Of course, the technical use expands the ordinary use, since in the ordinary use, lines are not places.

Besides these uses of 'place (τόπος)' mentioned by Mugler, I might point out that there are other uses in Greek mathematical texts not mentioned by him, but occurring in definitions or in Heron and later texts, some of which have already been indicated.

[38] Cf. Heron, *Dioptra* 6.5–7: "Nevertheless, let us examine the places given in the interval between the points, how they are related to one another and the initial given points (οὐ μὴν ἀλλὰ καὶ τοὺς δοθέντας τόπους ἐν τῷ μεταξὺ διαστήματι τῶν σημείων ἐπισκεψώμεθα, πῶς ἔχουσι πρὸς ἀλλήλους καὶ τὰ ἐξ ἀρχῆς δοθέντα σημεῖα)".

1. a τόπος can be the ordinary location of objects in a configuration, as Euclid, *Data* Definition 9.[39]
2. τόπος is a region (dare I say space) where a construction is to take place, e.g. the figures that can fill up a place (τόπος).
3. A τόπος can be a physical length or an area (in Archimedes), even a volume (Heron).
4. The word τοπικός is used both as a metamathematical term (as in Proclus' definition above) and an adjective for something having a place.

To summarize, of the four terms that become important in philosophical discussion in the Hellenistic Age, only 'position (θέσις)' and 'place (τόπος)' are important notions. A partially determined region outside object is a place, but not the entire region or the entire plane or three dimensional world. Place is not space. But in its many uses, it is not the same as position either. For to be given in position might well be to be given in some place, and certainly lines that are loci should be given in position (whether as initially given or as the goal of a problem). But in local theorems, the place or locus is where things get positioned. So there seems to be a general functional difference between the two terms (no doubt with many exceptions). A place in both primary senses is a region within which things have positions. This holds whether the place is open ended (e.g., the region closed by a line) or closed (e.g., a quadrant, within which the loci that make up the quadratrix will find position). Usually, but far from always, things are just given in position when they are given.

Although such notions take on mathematical significance, regions where objects may be located, objects given in positions, it is unclear that one can say much about any relation to philosophical discussions. This is where we shall now turn.

2 Aristotle on the Location of Mathematical Objects

Let's begin with a quick survey of views of place in Classical and Hellenistic physics. Here our task is threefold, to present views about place, views about mathematical objects, and the sense in which mathematical objects can have location. Since it encompasses all views on place through the Hellenistic Age, I like to start with a tableau that one can derive from Aristotle (Fig. 1).[40]

In the *Physics*, Aristotle argues that the place of a body is the inner limit of a container of the body. He then argues that points do not have place because they cannot be distinct from the inner limit of a container. By the same argument lines

[39] Cf. ps.-Heron, Definition 4 on the definition of a straight line as that which when rotated about its end points always keeps the same place (τὸν αὐτὸν ἀεὶ τόπον ἔχουσα) and 8 on the definition of surface as the limit of body and of place, and 9 on congruence. Definition 11 (solid and place as having three dimensions) obviously is incompatible with Aristotle's *Physics*, but not with his *Categories* or almost any post-Theophrastus view of place.

[40] Cf. Mendell (1987, p. 219) and (2005, p. 355). the diagram is slightly different here, to emphasize the two possible ways an extension might be separate from the contained.

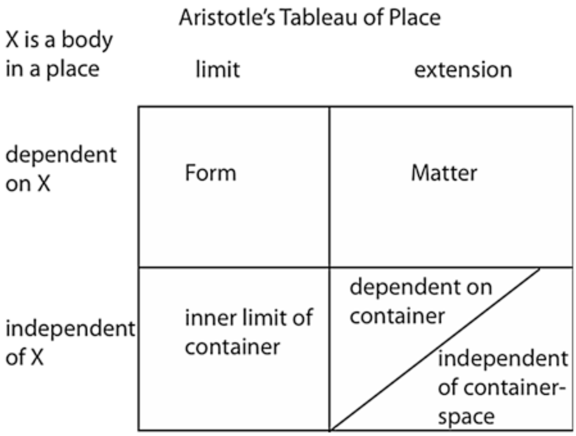

Fig. 1 Aristotle's tableau of place

and surfaces should not have place, but that would be our inference. Later, Aristotle also defines two bodies as being in contact if they have their extremities together, that is in one primary place (*Phys*. E 3.227b21–23). So when two bodies are in contact, the extremities may be distinct but are together. So if two bodies are in contact at a point, it follows that points have primary place. This contradiction, which bothered G.E.L. Owen,[41] for example, requires us to distinguish two approaches, one very strict and one loose. However, the points in question, are associated with physical and not purely mathematical bodies.

Nonetheless, one might think that the official Aristotelian position should be: physical bodies smaller than the universe and below the outer shell of the heaven have place; mathematical objects have position, but not place. Aristotle seems to confirm this in several discussions, some less decisive than others, e.g., at the beginning of his discussion of place, where he is surveying puzzles (he need not be asserting his own beliefs); or in an argument against Academic, intermediate mathematicals, which should not have place on the view of the Academics (*Met*. N 5.1092a17–21):

> But at the same time providing a place to mathematical solids is also absurd (for place is a proprium of particulars, whence they are separate in place, but the mathematicals are not anywhere), that is, saying that they will be somewhere, but not what place is.

> ἄτοπον δὲ καὶ τὸ τόπον ἅμα τοῖς στερεοῖς τοῖς μαθηματικοῖς ποιῆσαι (ὁ μὲν γὰρ τόπος τῶν καθ' ἕκαστον ἴδιος, διὸ χωριστὰ τόπῳ, τὰ δὲ μαθηματικὰ οὐ πού), καὶ τὸ εἰπεῖν μὲν ὅτι ποὺ ἔσται, τί δέ ἐστιν ὁ τόπος μή.

[41] It is remarkable how often Owen returned to the issue, cf. Owen (1961), (1970), (1976), and (1986a).

In *De caelo* Γ 6.305a22–31, Aristotle argues what is unmoveable and mathematical, as lacking weight, cannot be in place.

> In fact, the elements cannot be composed from some body. For it will follow that another body will be prior to the elements. If this will have heaviness or lightness it will be one of the elements, but without having downwards-inclination it will be unmovable and mathematical. Being of this sort, it will not be in a place. For [in the place] in which it rests, it will also be able to move. And if it [can] by force, it will do so contrary to nature; and if not by force, according to nature. And so if it will be in place and somewhere, it will be one of the elements. And if it is not in a place, nothing will be [composed] from it. For what comes to be and that from which it comes to be must be together.

> Ἀλλὰ μὴν οὐδ' ἐκ σώματός τινος ἐγχωρεῖ γίνεσθαι τὰ στοιχεῖα· συμβήσεται γὰρ ἄλλο σῶμα πρότερον εἶναι τῶν στοιχείων. Τοῦτο δ' εἰ μὲν ἕξει βάρος ἢ κουφότητα, τῶν στοιχείων ἔσται τι, μηδεμίαν δ' ἔχον ῥοπὴν ἀκίνητον ἔσται καὶ μαθηματικόν· τοιοῦτον δὲ ὂν οὐκ ἔσται ἐν τόπῳ. Ἐν ᾧ γὰρ ἠρεμεῖ, ἐν τούτῳ καὶ κινεῖσθαι δυνατόν. Καὶ εἰ μὲν βίᾳ, παρὰ φύσιν, εἰ δὲ μὴ βίᾳ, κατὰ φύσιν. Εἰ μὲν οὖν ἔσται ἐν τόπῳ καί που, ἔσται τι τῶν στοιχείων· εἰ δὲ μὴ ἐν τόπῳ, οὐδὲν ἐξ αὐτοῦ ἔσται· τὸ γὰρ γινόμενον, καὶ ἐξ οὗ γίγνεται, ἀνάγκη ἅμα εἶναι.

The argument here is against a physics such as atomism that makes natural bodies have only mathematical properties. However, the argument seems general: mathematical objects cannot be in place in so far as this requires that they be movable.

Thus it may be surprising to some that, in discussing how efficient causes work in *De gen. et corr.* I 6.322b9–3a12, Aristotle denies this:

> And so, just as nearly every other noun is also said in many ways, some homonymously, others from other (uses) and prior (uses), so it holds too in the case of contact. Nevertheless, what is said principally belongs to things having position, and position to just those things that also have place. For one must attribute similarly contact and place also to mathematicals, whether each of them is separate or exists in some other way. And so, if, just as was defined earlier (*Physics* E 3), being in contact is having the limits together, these would be in contact with one another which, by being determinate magnitudes and having position, have their limits together. And since position belongs to those things to which place also belongs, and the first differentia of place is up and down, and these sorts are opposites, all things touching one another would have weight or lightness, either both or one. And these sorts are affective and effective. Thus it is obvious that those things are of a nature to touch one another which, by their being divided magnitudes, have their limits together, as they are causing motion and are moved by one another.

Σχεδὸν μὲν οὖν, ὥσπερ καὶ τῶν ἄλλων ὀνομάτων ἕκαστον λέγεται πολλαχῶς, καὶ τὰ μὲν ὁμωνύμως τὰ δὲ θάτερα ἀπὸ τῶν ἑτέρων καὶ τῶν προτέρων, οὕτως ἔχει καὶ περὶ ἁφῆς. Ὅμως δὲ τὸ κυρίως λεγόμενον ὑπάρχει τοῖς ἔχουσι θέσιν, θέσις δ' οἷσπερ καὶ τόπος· καὶ γὰρ τοῖς μαθηματικοῖς ὁμοίως ἀποδοτέον ἁφὴν καὶ τόπον, εἴτ' ἐστὶ κεχωρισμένον ἕκαστον αὐτῶν εἴτ' ἄλλον τρόπον. Εἰ οὖν ἐστίν, ὥσπερ διωρίσθη πρότερον, τὸ ἅπτεσθαι τὸ τὰ ἔσχατα ἔχειν ἅμα, ταῦτα ἂν ἅπτοιτο ἀλλήλων ὅσα διωρισμένα μεγέθη καὶ θέσιν ἔχοντα ἅμα ἔχει τὰ ἔσχατα. Ἐπεὶ δὲ θέσις μὲν ὅσοις καὶ τόπος ὑπάρχει, τόπου δὲ διαφορὰ πρώτη τὸ ἄνω καὶ τὸ κάτω καὶ τὰ τοιαῦτα τῶν ἀντικειμένων, ἅπαντα τὰ ἀλλήλων ἁπτόμενα βάρος ἂν ἔχοι ἢ κουφότητα, ἢ ἄμφω ἢ θάτερον. Τὰ δὲ τοιαῦτα παθητικὰ καὶ ποιητικά· ὥστε φανερὸν ὅτι ταῦτα ἅπτεσθαι πέφυκεν ἀλλήλων, ὧν διῃρημένων μεγεθῶν ἅμα τὰ ἔσχατά ἐστιν, ὄντων κινητικῶν καὶ κινητῶν ὑπ' ἀλλήλων.

Here is a summation of the argument that we are interested in.

1. Things that have their limits together are in contact (Phys. E 3).
2. The primary sense of 'contact' belongs to things that have position.
3. Things that have position have place.
4. The first differentia of place is up and down.
5. [Things that are up or down have heaviness or lightness or both].
6. ∴ Things in contact have heaviness or lightness or both.
7. Things that have heaviness or lightness are affective or effective (with respect to motion, see step 9).
8. Divided magnitudes have their limits together.
9. ∴ Divided magnitudes are effective and affective with respect to motion.

I assume that the divided magnitudes at the end of the argument are perceptible or physical magnitudes. Yet in the middle of this argument, Aristotle says (a1–3), "For one must give similarly contact and place also to mathematicals, whether each of them is separate or exists in some other way." Clearly, one must block the inference that mathematical magnitudes are effective and affective. As most commentators agree, this must occur at step 4.[42] At *Physics* Δ 1.208b22–26, in setting out the worthy opinions (endoxa) that place exists, Aristotle argues that we must

[42] Similar analyses may be found in Joachim (1922, ad loc.), de Haas et al. (2004, ad loc.), etc. Buchheim (2010) avoids the interference by switching (2) to its converse (everything in place has position) and altering (1), but at the sacrifice of removing 1–2 as part of an argument. As for our concern, it will still be the case that mathematical objects have place, so that one will still need either to read the rest of the argument very differently (per Buchheim) or to block the inference that mathematical objects are efficiently causal.

distinguish between right/left, up/down relative to us, but that there is also an up/down, left/right, front/back in nature.

> The mathematicals make this clear (that there is a distinction between natural up/down, etc. and relative), since, although they are not in place, nevertheless they have in position right and left relative to us as things merely spoken of due to position, without having each of these by nature.
>
> δηλοῖ δὲ καὶ τὰ μαθηματικά· οὐκ ὄντα γὰρ ἐν τόπῳ ὅμως κατὰ τὴν θέσιν τὴν πρὸς ἡμᾶς ἔχει δεξιὰ καὶ ἀριστερὰ ὡς τὰ μόνον λεγόμενα διὰ θέσιν, οὐκ ἔχοντα φύσει τούτων ἕκαστον.

So, on the reasonable assumption that Aristotle would endorse this endoxon (he might not), mathematicals do have relative but not natural up and down. Presumably, he is referring to the orientation of the diagram to its user, keeping in mind that this issue is distinct from the use of the prefixes 'up- (ἀνα-)' and 'down- (κατα-)' in the verbs discussed in § 1, and where the diagram, whether in sand or on papyrus, will be flat.

We can block the inference then by one of two Aristotelian claims:

1. The sense of 'contact' is clearly different for mathematical magnitudes and physical magnitudes, so that the sense of 'position' and 'place' must also be different.
2. The sense of 'up' and 'down' is clearly different in mathematics and in physics. In mathematics, it is merely relative to the observer of the diagram (or the adjacent text); in physics it is absolute.

To say that natural up and down are the first differentia of place does not entail that there might be a way in which something is in place where neither differentia applies, namely place that is neither up nor down.

In favor of (1), there is a clear difference between physical magnitudes being in contact, where their limits are different, i.e., the end of the pen in contact with the paper is not the same as the point on the paper; and mathematical ends, where it makes no sense to distinguish the end of a line and the point on a plane that it touches. The line is continuous with the plane, or, as Aristotle might say, the point on the plane and the point on the line are one in number but two in being or definition.[43] Putting emphasis on the word 'similarly', one might understand, "To the same extent, mathematicals are in position and in place." Against, Aristotle does not imply that mathematicals are defectively in position, and hence in place. The argument in favor of (1) is basically a rational reconstruction, which could easily be wrong.

[43] Cf. *Physics* Δ 13.222a10–17 (cf. 11.220a11–13), where a point is both beginning of one line and end of another.

In favor of (2), 'place' is a word that can appear in Greek mathematics. So when Aristotle says that up and down are the first differentia of 'place' he might not mean that everything in place is so only in a derivative way. This is particularly so in the case of the heaven and the sublunary world. What is up in the case of the heaven is neither heavy nor light and is effective and affective only in respect of locomotion. So Aristotle's argument is restricted to natural sublunary bodies.

This may answer the question how general Aristotle's argument is, but it does not answer our question how he can allow geometrical figures to be in place. For according our standard modern interpretations of Aristotle on mathematical objects, the geometrical objects just are physical objects with physical properties removed from their logical structure.[44] The physical universe seems to be one of the things substracted, and with it place. This reasoning may be too quick and easy. It is perfectly possible that Aristotle thinks of the geometrical universe as the entire physical magnitude of the universe qua magnitude. In this way, mathematical solids might have place. However, I think there is no evidence for this view in Aristotle, although we shall return to the suggestion later.

On the other hand, if we wish to distinguish place and position in Aristotle, we might think that whereas position concerns the parts of a single figure, place concerns the relations between distinct geometrical objects, best represented by *Elements* I 2 (to position a line equal to a given line at a given point). This too is a rational reconstruction, which could easily be wrong. So we have an interpretative puzzle that we can dance around with Aristotelian solutions, but it is not at all clear whether any of them capture Aristotle's view. At least, it is reasonable to say that in so far as there is a sense of place that does not presuppose motion and rest, Aristotle does not seem to object to mathematical objects as he understands them (as opposed to the views of his Academic opponents) having place, but the question remains what that sense of place is. In any case, his view seems to have little to do with mathematical practice or to affect mathematical terminology.

3 Basic Spatial Notions in the Hellenistic Age

Eudemus and Theophrastus are probably the last people in the Hellenistic age to have endorsed an Aristotelian view of place and his rejection of the void. Starting with Epicurus, the dominant view would be something like the following:[45]

τόπος	place, extension occupied by a body (in Heron, also an extension being emptied)
κενόν	void or vacuum, extension that a body does not occupy or extension that a body does not occupy but that a body could occupy

[44] This is a common point of most modern accounts of Aristotle on mathematical object. Cf. Mendell (2004).

[45] Cf. Sedley (1982), Long and Sedley (1987, pp. 294–7), and Algra (1995).

χώρα	room, extension that a body is moving into or that is partially occupied (as such, it may blend into the next use)
[ἀναφὴς φύσις, or unamed]	intangible nature, the disjunction of the three (Epicureans) or χώρα—room that is indifferently occupied or not (possibly some Stoics[46])

None of the first three terms quite picks out a notion of space, although, as terminology slackens, any of them can. The intangible nature or room as indifferently occupied or not is space.

An issue for the Stoics is whether void is infinite, as Chrysippus and probably most Stoics, or does it just extend to where the universe expands when it burns up, as Posidonius. I take the varying testimonia as evidence of alternative views.[47] Additionally, I should mention the Peripatetic view of Strato of Lampsacus.[48] He too thinks of place as extension where an object is, but has a finite Aristotelian kosmos, with no void outside. However, within the kosmos are small interstices of void. This view influences Heron in his *Pneumatics*.[49]

Now, since our concern is with geometry, we can well ask whether any of this has anything to do with Greek geometry, and my suspicion is, 'not very much'. Although the issue arises in several Hellenistic naturalist contexts, it is hard to see Greek mathematicians thinking of lines and planes as demarcations of space. Of course, reasonable question arises about professional demarcations between philosophers and mathematicians. However, this question is misguided both for the ancient world and ours. The contrast in the Hellenistic Age should be between the study of nature and mathematics. This is, after all, the division that Posidonius attempts to make, between subjects that seek a cause and those that prove things from assumptions.[50] We can use this to mark professional identities, but I am not sure this is necessary. Some we would put down in both ranks, Ptolemy (cf. the

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46 The evidence for 'room (χώρα)' being for the Stoics partly occupied and partly unoccupied is primarily Sextus Empiricus, *Adv. math*. 10.3–4. (SVF 2.505). Since void and place are two of the four Stoic incorporeals, one might think that they would not form another sort that is neutral between the two. However, 'room' might well have done that work. Galen, *Qual. inc*. 19.464.10–4 (SVF 2.502) suggests that the Stoics were compelled to agree to extension in three dimensions (τὸ τριχῇ διαστατόν) as something common. Cf. Long and Sedley (1987, p. 296). De Harven (2012, p. 29) takes extension to be a mode of incorporeals and so not something distinct.

47 For Chrysippus, cf. inter alia Arius Didymus, fr. 25 in *Dox. Graec*. 460.460.18–361.3 (SVF 2.503). For Stoics and Posidonius in contrast on this point, cf. Aetius in *Dox. Graec*. 338a16–19, b16–21. If Cleomedes is relying on Posidonius for the argument of *Caelestia* I 1, that void is unlimited, this report might be incorrect.

48 Cf. Fr. 54–67 (Werhli) and Algra (1995, pp. 58–70).

49 One might expect, as a result, some terminological care. Yet, although 'empty (κενόν)' does mean 'empty', Heron is, I suspect, casual about 'place (τόπος)' and 'room (χώρα)'. Cf. '(the air) goes into the emptying place (εἰς τὸν κενούμενον τόπον χωρεῖ)' at I 40.29. However, this needs more careful examination.

50 From an epitome of Posidonius' *Meteorology* by Geminus and quoted by Alexander, whence Simplicius, *In Phys.* 291.20–292.31.

introduction to the *Tetrabiblos*), Heron; and some we would not, but possibly because we just do not know enough.

I am now also violating my principle of not trying to read ontology/physics into geometrical texts. So let me violate my principle a little more. Aristotle's friend, Eudemus represents no one but himself. Like his mentor, he rejects void, but, in building on Aristotle's argument (*Physics* Z 1) that two points cannot be in succession, he does consider void (fr. 99 = Simplicius, *In Phys*. 10.928.28–9.3):

> In the handbook, Eudemus made use of this [argument]: "For if," he says, "the partless are in succession, it is altogether necessary that there be something between them that is not of the same kind, so that it would not be a point, but a *line or void in a length between points*. And so, the line will not be [composed] from points, since successive points will not be in it. But if there is void, the void will be more in the [class of] continua than [in the class of] things from which it is [composed], that is, [more] than from the points said to be in succession, or there won't be any magnitude at all. For just as two points do not make a length, so too nor does a point and void."

Ὁ δὲ Εὔδημος τῷ ἐπιχειρήματι οὕτως ἐχρήσατο "εἰ γάρ ἐστι, φησίν,
ἐφεξῆς τὰ ἀμερῆ, δεῖ πάντως εἶναί τι αὐτῶν μεταξὺ μὴ ὁμογενές, ὥστε
στιγμὴ μὲν οὐκ ἂν εἴη, γραμμὴ δὲ ἢ κενὸν μεταξὺ στιγμῶν ἐν μήκει. οὐκ
ἔσται οὖν ἐκ τῶν στιγμῶν ἡ γραμμή· οὐ γὰρ ἐν αὐτῇ αἱ ἐφεξῆς στιγμαί·
929.1 εἰ δὲ κενόν, πλέον ἔσται τὸ κενὸν ἐν τοῖς συνεχέσι τῶν ἐξ ὧν, τουτέστι
τῶν ἐφεξῆς λεγομένων στιγμῶν, ἢ οὐδὲ ἔσται μέγεθος ὅλως. ὥσπερ γὰρ
ἁπτόμεναι δύο στιγμαὶ μῆκος οὐδὲν ποιοῦσιν, οὕτως οὐδὲ στιγμὴ καὶ κενόν."

Now, the quotation from Eudemus establishes that one could argue that there is no difficulty in thinking of distances between points of the void as lengths. Or rather one might think that lines could represent distances between void-points. However, Eudemus clearly makes a conceptual distinction between void-lengths and lines, where the lines are presumably physical lines. Someone else might make a different move, e.g., Epicureans.[51] So let us turn to the Stoics, who, after all, are the most likely ancient philosophers to have had an influence on Greek mathematicians.

4 Incompatibilism? Compatibalism? Revisionism? Descriptivism?

Certainly, there are many Stoics and Epicureans who were credited with great knowledge in mathematics. What the standard for great knowledge is is another matter. Nonetheless, it is possible that some Epicureans were up in the field,[52]

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[51] Cf. Verde (2010, esp. pp. 256–61) on minima in time and space among Epicureans. However, even if the Epicureans regard void has having minima, etc., it would not show that they made the additional move of treating distances in the void as lines, as geometrical objects.

[52] Cf. most recently Verde (2010, pp. 213–256). Certainly, there were many Epicureans who are reported to have had some proficiency with geometry: Polienus of Lampsacus, Zeno of Sidon,

maybe Philonides, IF he is the Philonides whom Apollonius commends in the introductory letter to Eudemus that begins *Conics* II.[53] Much less compelling is the suggestion that the skillful correspondent of Hypsicles (the presumed the author of *Elements* XIV) is also the Epicurean Protarchus.[54] In any case, no extant Epicurean text shows a mathematical concern beyond the contents of *Elements* I. Epicureans interested in mathematics, at least according to our sources, were primarily concerned with minima and division and are unabashed revisionists, who seem to have had little influence on extant mathematical practice. Unrestricted division is taken for granted in every standard mathematical text, and without comment. The principle that given two unequal comparable magnitudes one may continuously slice the larger so that the remainder is smaller than the other, the basis of the so-called Eudoxan method of exhaustion, implies that there is no smallest magnitude. Of course, the principle is different from one that says that a magnitude may be infinitely or completely divided, as, no doubt, Anaxagoras had already seen in the 5th cent. (cf. fr. B3). Indeed, the distinction may even be the primary point of the Eudoxan method. The principle is not an isolatable feature of the work of Eudoxus, Euclid, Archimedes, and Apollonius, any more than is the assumption that any line

(Footnote 52 continued)
Demetrius of Laconia, Philonides of Laodicea by the sea. It is another question how proficient they were.

[53] The argument goes back to Crönert (1900). Apollonius recommends that his correspondent, Eudemus in Pergamum, make known Book II to Philonides, whom he had introduced to Eudemus in Ephesus, should he show up in town. PHerc. 1044, a life of Philonides, praises his geometrical skills and says that his first teacher was Eudemus and his second Dionysodorus son of Dionysodorus of Caunus. There is at least one known, post-Archimedean mathematician named Dionysodorus, so that the coincidence of names might seem somewhat compelling, even though Apollonius never suggests that Philonides was a student of Eudemus and indeed may suggest that he was already a geometer when the introduction took place. Nor is there any evidence that the two Dionysodori are the same person. Furthermore, we do not know why Apollonius mentions Philonides; if he is the Epicurean, maybe because he is well connected at the Seleucid court. Hence, the evidence gets thinner the more it is examined. In any case, none of this suggests that Apollonius or any working mathematician, which the Epicurean Philonides seems not to have been, had Epicurean connections, even less that they were influenced. It would at best attest to geometrical skill with one member of the Garden, who, like so many ex-scientists, used his knowledge in a very limited way.

[54] Cf. Crönert (1900) and Verde (2010, pp. 233–4). Hypsicles tells a story about Basileides of Tyre and his father in Alexandria and their common interest in a treatise of Apollonius to explain how he got interested in the topic of dodecahedra and icosohedra, and mentions the friendship of Protarchus and his father, as well as his geometrical prowess. Nowhere does he suggest that Basileides and Protarchus are familiars. The coincidence here is that a Basileides was the fourth head of the Epicurean Garden, and a Protarchus of Bargalia was the teacher of Demetrius of Laconia, whom the Epicureans considered skilled in mathematics. Basileides has no known connection to mathematics, while Protarchus' only connection is that he taught Demetrius, although we do not know what he taught. It is now standard to cite the Epicurean as 'Basilides of Tyre'. I would suggest more caution. Not everyone with the same name is the same person or even from the same town. Netz has informed me of even more compelling reasons to doubt the identifications of the two Philonidai and the two Protarchi.

may be divided at any point. No ancient mathematician asks with which segment the point of a line division goes.

Nor is it typical to think of lines as composed of points. However, Archimedes' *Method* famously treats *n*-dimensional objects as composed of $n - 1$ dimensional objects, figures of lines, solids of planes. Props. 1–13 also treat each of these as having weight. It has been pointed out that nowhere does Archimedes actually say that the multitude of composing lines is finite or infinite. So "one cannot rule out the possibility that he thought of them as indivisible minimal bodies."[55] I think, on the contrary, that one can rule out the issue as playing any role in the argument, and it is very easy to see why. In 1–13 he uses dimensional-reduction (commonly mis-called 'indivisibiles') and the balance, while in 14 he just uses dimensional-reduction, but in 15 he gives a Eudoxus/Archimedes proof of the previous theorem, which will require unlimited division. So, he has not abandoned the condition of unrestricted divisibility in the treatise and so has not endorsed surreptitiously an Epicurean doctrine as the ideology of the treatise.

Archimedes identifies Props. 1–13 as heuristic because, like the argument of Q*uadrature of the Parabola* 6–17, they use mechanics. Whether he also regards 14 to be heuristic may never be known.[56] In the introduction, Archimedes cites *Conoids and Spheroids* 1,[57] which he will use explicitly in Proposition 14. Anachronistically, if we have two series of magnitudes equal in multitude (τῷ πλήθει ἴσα), $a_1, \ldots$, and $b_1, \ldots$, and pairwise for any *i* and *j*, $a_i : a_j = b_i : b_j$, and another two series $c_1, \ldots$ and $d_1, \ldots$, where $a_i : c_i = b_i : d_i$, then Totality($a_i, \ldots$) : Totality($c_i, \ldots$) = Totality($b_i, \ldots$) : Totality($d_i, \ldots$). The proof of the lemma in the earlier treatise only deals with the finite case. Indeed, it is hard to see how Archimedes could prove it for the infinite case. Yet all he says in the application is that each of the four groups of magnitudes are 'equal in multitude (ἴσα τὸ πλῆθος)' to each other, a formula one readily recognizes from Euclid and Archimedes as a variant on the other formula. Yet, 'equal in multitude', here without anachronism, in effect means that the groups can be paired up (dare I say, "1:1"). So in a sense, the suggestion is right that Archimedes does not mention either that he is making a leap from the finite case to the infinite nor that he is treating the figure as composed of finite figures. Since, in the proof, all the a's are

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[55] Cambiano (2008, p. 588).

[56] That said, it is very plausible that Archimedes also regards *Method* 14 as a heuristic. My point is just that we cannot know this. See most recently Christianidis and Demis (2010), who argue for this plausible thesis, but, much more controversially, that Archimedes believes that one can have mechanical demonstrations of geometrical propositions, so that the heuristic feature of the *Method* is the method of indivisibles. The difficulty is that Archimedes only contrasts: his heuristic method, which he does not describe; theorems investigated through mechanics, which is the topic of the first part of the book; and geometrical proofs which will come at the end. Everything beyond is our guesswork, except that Archimedes makes the same point in the introduction to the *Quadrature of the Parabola*, that the quadrature was discovered through mechanics and that he is sending to Dositheus how through mechanics it was observed and how through geometry it was demonstrated.

[57] Whether the actual mention of the book is an interpolation is unimportant, but see Netz et al. (2001, p. 20).

equal (triangles) and all the b's are equal (lines), at least we do not have to worry about the ratios of the adjacent triangles and lines. We know that without a lot of fairly sophisticated mathematics the proof won't work either way. Without restrictions, we know the method will be vitiated by Cavalieri paradoxes. Yet, it works.[58] The theorem also requires that we have the ratio of the diameter of a parabolic segment to a parallel to the diameter and that of a triangle in a prism to a triangle in a cut cylinder. Will we get this if there are minima? There is, however, another paradox, much more ancient, that calls into question the method. Nevertheless, if Archimedes thinks of it merely as an effective heuristic, who's to damn it, either then or in the 17th century?

It is some time since Luria[59] tried to connect Archimedes' *Method* to atomism. So far the only certain connection remains Archimedes' correction of his earlier praise of Eudoxus in *De sphaer. et cyl.*, namely that Democritus first stated the theorem of the ratio of cone to cylinder, and that could have come from anywhere, but most likely from some learned Alexandrian, such as Eratosthenes. If so, it contributes little to our understanding of Archimedes. Whatever Archimedes may have thought, it is clear that either he just assumes that we will read his method one way and not the other, or he doesn't care since the issue of how many lines compose a figure doesn't enter into the argument. It would be just like him to tease the reader with something hanging, but how would we know? After all, he was that kind of guy! Nonetheless, the *Method* presumes propositions whose truth and proofs are incompatible with atomism.

The Stoics seem better candidates for having an influence on mathematical treatments. They can even claim a notable or two, in particular Eratosthenes, who studied with Ariston and even met Zeno once.[60] Well, having studied with Stoics doesn't make one a Stoic any more than having studied geometry makes one a geometer.[61] One would have to find evidence in Eratosthenes of Stoicism at work. There is some evidence that mathematicians at least shared some vocabulary with the Stoa.[62] I shall consider here four issues in the evidence for Stoic views:

58 These issues are fully explored in Netz et al. (2001). They rightly do not consider the possibility that the method is finitist.

59 Cf. Luria (1933).

60 Cf. Strabo, I 2.2, Athenaeus, *Deip*. 7.14–20 (SVF 1.341). Strabo suggests more familiarity (γνώριμος) between the pre-teen and the septuagenarian, Zeno, which we can put down to the ordinary exaggeration that accrues to such stories over time.

61 ˜I fail see any significance in Chrysippus knowing that great circles bisect one another, the most basic theorem in spherical geometry (a few generations later, it will become Theodosius, *De sphaera* I 1), assuming that the evidence in Cicero, *De fato* 15 does indicate this (so Mansfeld 1983, p. 66). Whether Chrysippus accepts or rejects geometry, he can know this as a theorem. This is not to impugn Chrysippus' knowledge of mathematics, which may well have been extensive, just the way scholars embroider evidence.

62 'Property (σύμπτωμα)' is the obvious example, although it already has a logical meaning in Aristotle's *Organon, Topics*. Δ 6.126b35–7a2, less clearly *Cat*. 8.9b19–10a10. However, it is not a central term in Aristotle, nor, for that matter, anywhere else in 4th century BCE discourse. This says nothing about the direction of influence.

treatments of the basic definitions in geometry and whether some Stoics, especially Posidonius, were committed to a conceptualism or to a modified physicalism; Chrysippus on locus theorems; Chrysippus on infinite divisibility (including of void) and the cone paradox; Posidonius on the nature of figures. Whatever they may indicate about Stoic views on Greek mathematics, they will say very little about Greek mathematics itself. Regardless of whether some were Stoics, Greek mathematicians do not wear Stoic badges on their tunics.

This is not the place to sort out the woefully inadequate evidence on the status of mathematical objects in the Stoa and the degree to which views may have differed.[63] Given the Stoic distinction between existing (ὑπάρχον), the status of corporeals, and subsisting (ὑφεστηκός), the status of incorporeals, the standard list being place, void, time, sayables,[64] where do planes, lines, and points go? In addition to the standard list of incorporeals, Cleomedes and probably Posidonius, took them to include the limits of bodies,[65] while Plutarch explicitly takes the Stoics as treating points and planes as incorporeals.[66] Hence, solid figures (σχήματα) are qualia and hence corporeal,[67] while it would follow that lower dimensional figures are incorporeals. Unless the Stoics generally are physicalists, this will not tell us how the objects of mathematics are to be treated. That is, are they incorporeal subsistents dependent on bodies? Or are they conceptual and distinct in some sense from incorporeal limits?

Because Geminus (early 1st. cent. CE) preserved much of Posidonius and was available to Proclus, when combined with a few other fragments, we can get a slightly clearer picture at least of his views on the status of mathematical objects. Diogenes Laertius (*Vitae* VII 134) says that most of the major Stoics (Zeno, Cleanthes, Chrysippus, Archedemus, and Posidonius) distinguish between principles that are eternal, incorporeal, and lacking shape and the elements that have shape but get destroyed in the conflagration of the world.

Diogenes goes on to say (VII 135):

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[63] Cf. especially Long and Sedley (1987, pp. 297–303), Robertson (2004), Ju (2009), and de Harven (2012).

[64] Cf. Long and Sedley (1987, pp. 162, 164–5).

[65] Cf. Ju (2009, pp. 381 and 385–6), citing Cleomedes, *Caelestia* I 1.139–44 and 3.34–5. I take it that her argument (pp. 381–6) effectively refutes the view that they are corporeals for any Stoic. This is not to say that there might not be some serious concerns, as Paparazzo (2005) raises, but ultimately the issue here concerns Pliny's treatment of patinas, and not Posidonius. Nonetheless, it is of some concern that the evidence for Posidonius, Diogenes Laert., *Vitae* VII 135 (discussed below), comes from book 5 of *De meteora*. One really wants to know the context.

[66] *De comm. not*. 1081B5 for plane, B11–2 for point, 1080EF for the contacts in general of a body with an incorporeal.

[67] So Simplicius, *In cat.* 271.20–22 (SVF 2.383), and compare with 217.32–218.1 (SVF 2.389), that if what's qualified is body the quale is body, and if bodiless bodiless. Cf. Long and Sedley (1987, pp. 169, 172). The context of Simplicius in the first passage is why Aristotle puts figure with qualities. So the Stoics agree with Aristotle in making them qualities but not in making them bodies, where Aristotle classifies bodies as quantities. It is enough for Simplicius' argument that some figures are bodies, namely bodies so qualified.

> A body, as Apollodorus says in the *Physics*, is what’s extended in three ways, in length, in width, in depth. This is also called ‘solid body’. A surface is a limit of a body or what has length and width alone and not depth. Posidonius in the fifth [book] on *Things Seen Above* [meteorology] admits this both in attentive-thought and in substance. And line is a limit of a surface or widthless length or what has length alone. A point (στιγμή) is a limit of a line or what is a smallest mark (σημεῖον).[68]

7.135.1 Σῶμα δ′ ἐστίν, ὥς φησιν Ἀπολλόδωρος ἐν τῇ Φυσικῇ, τὸ
τριχῇ διαστατόν, εἰς μῆκος, εἰς πλάτος, εἰς βάθος· τοῦτο δὲ καὶ
στερεὸν σῶμα καλεῖται. ἐπιφάνεια δ′ ἐστὶ σώματος πέρας ἢ τὸ
μῆκος καὶ πλάτος μόνον ἔχον βάθος δ′ οὔ· ταύτην δὲ Ποσειδώνιος
7.135.5 ἐν πέμπτῳ Περὶ μετεώρων καὶ κατ′ ἐπίνοιαν καὶ καθ′ ὑπόστασιν
ἀπολείπει. γραμμὴ δ′ ἐστὶν ἐπιφανείας πέρας ἢ μῆκος ἀπλατὲς
ἢ τὸ μῆκος μόνον ἔχον. στιγμὴ δ′ ἐστὶ γραμμῆς πέρας, ἥτις ἐστὶ
σημεῖον ἐλάχιστον.

First, the definitions would all seem to be from Apollodorus, but endorsed by Posidonius. What Posidonius adds is that surface is both in thought and in substance. Apollodorus’ definition of body as three dimensional is notable in two ways. First, many Stoics defined body by affective and effective capacities. So we might suspect a more geometrical definition is given in order to define a geometrical body, where the definiendum would be solid body (στερεὸν σῶμα), perhaps as opposed to natural body. It is not clear why Diogenes proceeds from talking about eternal principles to specific principles, the definitions of basic mathematical objects. As to the other definitions, the only thing notable is that Stoics should prefer the definitions that provide an $n-1$-dimensional object as the limit of an n-dimensional body. For them, as for Aristotle, this is the ontologically correct version. Yet, we have a bevy of triples of definitions, but mostly unremarkable, as is clear from a comparison with Euclid, *Elements* I Definitions 1–3, 5, 6 (or any perusal of definitions in Aristotle). So the order in Diogenes is from three to no-dimensions and with each (except for the point): the ontologically correct definition, the privative definition, the definition by a list of dimensions. Euclid gives a selection of these in reverse order:

point	1.	different definition	2.	ontologically correct
line	3.	privative definition	4.	ontologically correct
surface	5.	list of dimensions	6.	ontologically correct

There is no definition of solid in Euclid, but Posidonius’ is the sort of definition we would expect.[69] So the only thing of note here is that Diogenes gives both the privative and list forms, which Euclid seems not to distinguish. The definition of

[68] Taking στιγμή as the definiendum is a little strange. This common word for ‘point’ in Aristotle is never used in an any mathematical text between Eudemus and Theon of Smyrna, while σημεῖον is the common word.

[69] Cf. the citing of Euclid’s definitions in Philo, *De congressu eruditionis gratia* 147, ending with “what has three dimensions, length, width, depth (ὃ τὰς τρεῖς ἔχει διαστάσεις, μῆκος, πλάτος, βάθος).”

'point' is very strange, but may be an attempt at a substitute for Euclid's definition, "of which there is no part (οὗ οὐ μέρος οὐθέν)," to meet some atomist objection. If the order is significant, it is a reasonable guess that Apollodorus and Posidonius want the ontologically correct version to come first.

This is confirmed by how Posidonius treats the surface, as according to attentive-thought[70] and subsistence, where attentive-thought involves, presumably forming an object of thought where none exists. A similar claim occurs in Proclus, where limits subsist according to attentive-thought.[71] When he finishes his argument that limits can be efficacious, he refers back to this claim and seems to distinguish two Stoic views, one that involves conceptually separating limits from bodies (a physicalist abstractionism) and the other treating them as purely conceptual (a fictionalism).[72] In either case, they would be somethings in a presentational/representational faculty (φαντασία) that is distinct from the surfaces, etc. that do subsist as incorporeals. If so, is this because they are concerned with the idealization problem, that physical objects are never so round as the geometrical ones?

(Footnote 69 continued)
This definition somewhat appears at Aristotle, *Physics* Δ 1.209a4–6, "And so, [place] has three distances, length, width, depth, by which every body is defined (διαστήματα μὲν οὖν ἔχει τρία, μῆκος καὶ πλάτος καὶ βάθος, οἷς ὁρίζεται σῶμα πᾶν)." The third class of definition is not in Aristotle, but cf. *Met.* Δ 13.1020a11–13, where each kind is given by the number of dimension and the last dimension as the definiendum, e.g., depth is what's continuous in three (τὸ δ' ἐπὶ τρίαβάθος, with συνεχές understood from earlier).

70 Chrysippus treats ἐπίνοια as the faculty that is able to conceive things that are logically possible, e.g., to divide mentally fire into two bodies. Cf. Galen, *In Hippocr. de nat. hom.* p. 30 (SVF 2.409).

71 Proclus, *In Eucl. El.* 89.15–7 (SVF 2.488, part), "One should not believe that such limits, I mean 'of bodies', subsist according to a fine attentive-thought, just as the Stoics suppose (ὅτι δὲ οὐ δεῖ νομίζειν κατ' ἐπίνοιαν ψιλὴν ὑφεστάναι τὰ τοιαῦτα πέρατα, λέγω τῶν σωμάτων, ὥσπερ οἱ ἀπὸ τῆς Στοᾶς ὑπέλαβον)."

72 The full context is important (*In Eucl. El.* 91.19–24), where, in refutation of the Stoic view, Proclus has just argued for the causal efficacy of poles and axes, "In observing things imperfectly subsisting within what are themselves limited the many believe their subsistence to be obscure, that is, some say that these are separated according to attentive-thought alone from perceptibles, while others, I suppose, that they have no being other than in our attentive-thought (οἱ δὲ πολλοὶ τὰ ἐν τοῖς περατουμένοις αὐτοῖς ἀτελῶς ὑφεστηκότα θεωροῦντες ἀμυδρὰν αὐτῶν οἴονται τὴν ὑπόστασιν εἶναι καὶ οἱ μὲν κατ' ἐπίνοιαν μόνην χωρίζεσθαί φασιν αὐτὰ τῶν αἰσθητῶν, οἱ δὲ μηδὲ ἀλλαχοῦ που τὴν οὐσίαν ἔχειν ἢ ἐν ταῖς ἡμετέραις ἐπινοίαις.)." Given the switch from 'subsistence' to 'being', one might think that the Stoics are the first group, but the switch need not be significant given the use of 'subsistence' in the main clause. Cf. Plutarch, *De animae procreatione in Timaeo* 1023B7 on Posidonius, "the existence of the limits about the body (τὴν τῶν περάτων οὐσίαν περὶ τὰ σώματα)." Instead, I think 'the many' just is 'most Stoics'. They are the only group in question here, if Proclus meant most people, even most people who think about these things, the comment would be strange indeed, as most people in his time are his people. And so, we have two Stoic views, one abstractionist and the other fictionalist (putting geometrical objects with centaurs and the like, as neither corporeal nor incorporeal). Since, on either view, there can still be two sorts of limits, as Robertson (2004) and De Harven (2012).

Well, there is paltry evidence that any Hellenistic Stoic ruminated on this.[73] It is, however, at least a nice story that the common interpretation of Aristotle, that the mind contemplates mathematical objects that are derived from sensation but are presented in the imagination, ultimately derives from the Stoa.[74]

The two positions might also reveal different attitudes towards spatial relations and what is acceptable in mathematics. The fictionalist may have more freedom in allowing conflicts with the structure of the physical world, e.g., the decomposition of figures and the extension of lines. It is to these that we now turn. I shall discuss in this section some mathematical issues that arise for local-theorems, Eudoxan division of a figure, Archimedean dimensional reduction, the nature of a figure. In the next section, I shall also look at a small issue on infinite extension in the Stoa.

In his discussion of local-theorems, discussed in § 1, Proclus (*In prim. Eucl. el.* 395.13–21) says that Geminus reports that Chrysippus compares local-theorems to the Ideas (presumably Stoic conceptions[75] and not Platonic Forms). For the simile to be meaningful, it must concern local-theorems and not theorems in general. Just as the Ideas encompass the generation of unlimited entities within determinate limits (ἐν πέρασιν ὡρισμένοις), so the theorems encompass unlimited entities within determinate places (ἐν ὡρισμένοις τόποις). To know more, we have to know more about Chrysippus on universals. On a strong nominalist reading of the remark, it might mean that just as the conception determines the limits within which each thing is so grouped, the locus theorem determines limits, i.e. the locus (τόπος), within which certain geometrical individuals are equal. If Chrysippus had said that ideas are like general theorems, he would merely have indicated that general notions are like general notions in mathematics, a not very profound remark. So a general theorem can easily be nominalized to a statement about individuals ("Every triangle …," to "given a triangle, ABC, …"), just as can any other general claim. By making the simile with locus theorems, he points to the spatial boundary that the locus theorem sets. So the nominalization, "given parallelograms ABDE and ABGC between the parallel lines AB, ECDG," does not dispose of the locus between parallel lines, since that too is an individual as is the region between the parallel lines. If this interpretation is right, the simile says little about mathematics, but much about how Chrysippus wants us to understand general propositions and general notions. Nothing here is revisionist, as it involves merely treating a locus by its boundary. Yet it is important that a locus is not necessarily a figure, indeed, just as is required.

How then would Stoics broach the question raised in the previous section about whether the void is subject to mathematical treatment? Cleomedes, *De motu* 8.10–14, argues that void is incorporeal, intangible, and lacking shape and not being shaped, that it is merely capable of receiving body. Whether or not anyone

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[73] Basically, the only evidence is Proclus' discussion and claim that they are imperfectly subsisting (previous note), but this is Proclus' issue. How do we know whether it also bothered the Stoa?

[74] Cf. Mueller (1990), who traces the view to Alexander.

[75] Cf. Long and Sedley (1987, pp. 179–183).

ever reasoned in this way, someone holding this view should probably hold that void is neither divisible nor indivisible and that its infinity outside the cosmos is to be explained counter-factually, by the conceptual possibility of the world expanding infinitely. On the other hand, according to Stobaeus, Chrysippus held that void is infinitely divisible.[76] Does this imply that it is possible to do a geometry of the extra-cosmic void, at least on the abstractionist view?

Yet this testimonium itself is mathematically curious. Stobaeus goes on to say that given a body divided up ad infinitum, a body is not composed of infinite bodies, nor a surface, nor a line nor place. We may infer that he would also apply the principle to the other divisibles that have just been mentioned, time and void. But when would one think of a surface as composed of infinite surfaces (as opposed to, say, lines in the Archimedes' *Method*). If Chrysippus is as knowledgeable about mathematics as his fans believe, then the most natural place would be in a continuation of a typical Eudoxan decomposition, e.g., of a circle into successive inscribed approximating figures. Yet the claim is suspiciously like Aristotle's account of potential infinity. In any case, with the availability of Eudoxan techniques for avoiding exhaustion, the issue is moot.

Nonetheless, Chrysippus pyramid paradox and his discussion of Democritus' cone paradox, as hostilely reported by Plutarch, *De comm. not*. 1078E-1080E, might pose a challenge to Stobaeus' report. The cone paradox of Democritus goes: Suppose we cut a cone into a top cone and frustum. Is the bottom surface of the top cone equal or unequal to the top surface of the frustum. If unequal, the cone will be jagged; if equal, then all such surfaces will be equal and the cone will have been a cylinder.[77] The pyramid paradox is less clear but may have been similar. It is tempting to see the paradox as related not just to Epicureanism but also to any technique of dimensional reduction. If Chrysippus objects to a body composed from an infinite decomposition of a body into bodies, and so forth, then a fortiori he would object to a body composed from an infinite decomposition of a body into planes. Given Stobaeus, the second horn of the dilemma will have to involve decomposing the cone into planes, inferring by transitivity of identity that they are all equal, and then inferring that the original cone was somehow not composed of the planes but that the equality still holds. This is not what Plutarch reports.

Now, if we want to see a dialogue between the philosopher and the mathematician, Archimedes' *Method* may well reflect an older mathematical method that would prompt Chrysippus' looking at the paradox.[78] Alternatively, it is very

[76] Stobaeus, *Eclogae* I 14.1e (SVF 2.482). "Chrysippus said that bodies are cut ad infinitum as well as things that are like bodies, e.g., plane, line, place, void, time. And when they are cut ad infinitum, neither are bodies composed from infinite bodies, nor a plane nor a line nor a place nor <a void nor a time> . (Χρύσιππος ἔφασκε τὰ σώματα εἰς ἄπειρον τέμνεσθαι καὶ τὰ τοῖς σώμασι προσεοικότα, οἷον ἐπιφάνειαν, γραμμήν, τόπον, κενόν, χρόνον· εἰς ἄπειρόν τε τούτων τεμνομένων οὔτε σῶμα ἐξ ἀπείρων σωμάτων συνέστηκεν οὔτ' ἐπιφάνεια οὔτε γραμμὴ οὔτε τόπος < οὔτε κενὸν οὔτε χρόνος > .)".

[77] Cf. Hahm (1972), Robertson (2004), inter alios, for a sample of many rival interpretations. My goal here is not to given an interpretation of the text.

[78] Cf. Knorr (1996).

tempting to suppose that just as Hipparchus was to do half a century later,[79] Archimedes is tweaking his nose at his contemporary Chrysippus.[80] Alternatively, Chrysippus might be attacking the *Method*. Of course, we do not know if either is aware of the other.

The problem is not merely that any story we come up with here is at best a nice historical fiction. Other than that it is not the one just suggested, it is not even clear what Chrysippus' solution was and what it says about his views on mathematics. His solution contains four claims:

1. the surfaces are neither equal nor unequal.
2. the bodies are unequal, due to (1).
3. there is a sense in which something can be larger (μεῖζον) without exceeding (ὑπερέχον).
4. bodies are in contact at a limit and not at a part (i.e., we might note, in agreement with Aristotle)

With skill, one can find in these non-revisionist readings of these three claims, and one of these readings might be right.[81] Whatever non-revisionist readings one takes, it must allow one to say that when one runs a plane through two equal cones parallel to the base, the volumes are determined by the diameters of the surfaces and the heights, so that the bottom of the top cone equals the top surface of the frustum. Otherwise, the mathematics will become bizarre, an orphan from a different millennium.[82]

Let us turn now to the fourth issue. In commenting on Euclid's definition of figure (El. I 14):

> A figure is what's contained by some boundary or boundaries (Σχῆμά ἐστι τὸ ὑπό τινος ἤ τινων ὅρων περιεχόμενον.)

Proclus (*In Eucl.* 143.5–21) holds that there is a conceptual difference between the way Euclid and Posidonius conceive 'figure (σχῆμα)':

> And so, in calling what's enfigured 'figure', he reasonably named in addition what's enmattered and existent in quantity 'contained'. But, separating the account of 'figure' from quantity and positing that (the definiens) is a cause of its being determined and limited and of the containing, Posidonius defines 'figure' as an enclosing limit. For the closing is different from what's enclosed and the limit from what's limited. And one (Posidonius) seems perhaps to look towards the boundary that surrounds from without, while the other (Euclid) to a whole, the substrate, so that one will say that a circle is wholly the plane figure

[79] Cf. Plutarch, *De repug. Stoic.* 1047DE and Acerbi (2003).

[80] Chrysippus outlived Archimedes, but the *Method* was probably fairly late in his career.

[81] It may even be that Chrysippus takes the paradox to be about physical bodies and not mathematical objects, Long and Sedley (1987, p. 302). However, the solution is then easily refuted. One constructs a cone equal to the top cone and a frustum equal to the bottom frustum. By transitivity of identity the bottom surface of the cone is equal to the top surface of the frustum. Now repeat ad infinitum.

[82] This is at least how I understand several proposals such as that of White (1992, pp. 284–313).

and the outside containing, while the other according to the circumference. And one indicates that the enfigured as contemplated with the substrate gets defined, while the other that he wants the account itself of ‘figure’ to make clear what limits and encloses the quantity.

ὁ μὲν οὖν Εὐκλείδης τὸ ἐσχηματισμένον σχῆμα καλῶν καὶ τὸ ἔνυλον καὶ τῷ ποσῷ συνυπάρχον περιεχόμενον εἰκότως αὐτὸ προσείρηκεν, ὁ δὲ Ποσειδώνιος πέρας συγκλεῖον ἀφορίζεται τὸ σχῆμα τὸν λόγον τοῦ σχήματος χωρίζων τῆς ποσότητος καὶ αἴτιον αὐτὸν εἶναι τιθέμενος τοῦ ὡρίσθαι καὶ πεπεράσθαι καὶ τῆς περιοχῆς. τὸ γὰρ κλεῖον ἕτερόν ἐστι τοῦ συγκλειομένου καὶ τὸ πέρας τοῦ πεπερασμένου, καὶ δοκεῖ πως ὁ μὲν εἰς τὸν ἔξωθεν περικείμενον ὅρον ἀποβλέπειν, ὁ δὲ εἰς ὅλον τὸ ὑποκείμενον, ὥστε τὸν κύκλον ὁ μὲν ἐρεῖ καθ′ ὅλον τὸ ἐπίπεδον εἶναι σχῆμα καὶ τὴν ἔξω περιοχήν, ὁ δὲ κατὰ τὴν περιφέρειαν. ἐνδείκνυται δὲ ὁ μὲν ὅτι τὸ ἐσχηματισμένον ἀφορίζεται καὶ σὺν τῷ ὑποκειμένῳ θεωρούμενον, ὁ δὲ ὅτι τὸν λόγον τοῦ σχήματος αὐτὸν τὸν περατοῦντα καὶ συγκλείοντα τὸ ποσὸν ἐμφανίζειν ἐθέλει.

The issue relates to a perplexing question for a philosophy of geometry: how does drawing the perimeter of a figure ‘create’ the matter in between (see § 1). As a result one might think that geometry is either a study of demarcated spatial extension or demarcations of a material plenum. Whether Greek geometers are committed to anything beyond the method of constructing a figure by constructing its perimeter is an issue that I have eschewed in my discussion, in as much as there is no evidence from Greek mathematical texts. Posidonius seems to hold the view that describing a figure is just drawing the perimeter. Without the need to add Proclus’ ontology, we have seen that Euclid does treat ‘circle (κύκλος)’ as disk, albeit with some difficulties (§ 1), while Proclus is not claiming that Posidonius treated plane figures merely as perimeters. He does say that the enclosing limit is the cause of the figure being determined and contained. One might well smell a little Aristotelianism here (a geometrical figure as its form or shape enclosing its matter or extension), while worrying about the sense of ‘cause’. Given that such figures are incorporeals anyway, if not mere conceptions, it does not seem to affect how one is to understand geometrical objects or their relations. However, we also see here that the definition of ‘figure’, the quale, of an extension, reverses the ontological ordering we saw earlier in Posidonius. To know better, we would have to have much more evidence, including a sense of how Posidonius defined figures such as circles.[83]

[83] The other principal evidence for Posidonius’ view of geometrical objects is Plutarch, *De an. procr. in Timaeo* 1023BD.

Has the naturalist and metaphysical discourse of the Hellenistic Age that relates to volume and spatial relations been at all reflected in mathematical discourse? Where Epicureans or Stoics were odd or non-standard, they seem to be ignored, or at best teased. Where they were standard, the standards would seem already to have been set by the time of Euclid. Where mathematicians went non-standard, namely Archimedes' *Method*, we can discover many *ben trovata*, but the ones that look to Epicureans don't look very well. Where mathematics becomes more physical, e.g., in the Heronian tradition, the influence will be greater. Where the content is more natural, standard views about nature, such as the sphericity of the (liquid) world in Archimedes' *Floating Bodies* I, will be part of the mathematics. To conclude this part of my discussion, I have found little reason to see any particular view about place or space in mathematical texts, except perhaps in the method of measuring area of a solid by measuring vacuums.

5 Infinite Extensions and Other Infinite Activities

The previous discussions were warm up. If all philosophers except for Epicureans and Skeptics thought that the body of the universe is finite, then what do we do with long lines, large planes, and big spheres. These are all a part of mathematics. In this, it is normal to begin with a notorious claim of Aristotle. He has just argued that in the case of physical bodies the actual infinite in extension does not exist except for time, which does not exist 'together'. The sense in which the infinite exists is that some divisibles may be always divided more and that conversely, an ever smaller amount may be added to a magnitude, so long as it does not exceed every given magnitude. For, the world is finite. It is important to realize that in the entire corpus of Aristotle's work there appears only one argument against actual infinities in mathematics (*Phys*. Γ 5.204b4–10), a 'logical' argument that bodies, by definition, are bounded. However, *Met*. Δ 13.1020a11–14 completely undercuts this argument by defining 'body' as finite magnitude in three dimensions, whence infinite three-dimensional magnitudes could be a distinct species. So, without any argument that infinite extension is impossible in mathematics, Aristotle then says (*Physics* Γ 7.207b27–34):

> The argument does not take contemplation from the mathematicans, although it does take away there being an infinite in such a way that it is in actuality, in increment, and untraversible. For they don't need the infinite, since they don't use it, but merely [need] there to be as much finite [line? increase?] as they want. It is possible for any other sized magnitude to be cut in the same ratio as the largest magnitude, so that the fact of its existence [any size?] in existent magnitudes will make no difference to them for [the purposes] of proving.[84]

[84] It is natural to take the feminine participle at b31 as referring to a line. However, the use of the neuter 'magnitude' in the next line undercuts this. The only feminine noun that could make sense is 'increase'. It is also unclear what exists in or among the existent magnitudes at b34. Whatever it is should to be unneeded in the proof. So Hussey (1983) and the Oxford translation take it that the (potential?) infinite exists in the existent magnitudes. Surely, what must exist in or among the

οὐκ ἀφαιρεῖται δ' ὁ λόγος οὐδὲ τοὺς μαθηματικοὺς τὴν θεωρίαν, ἀναιρῶν οὕτως εἶναι ἄπειρον ὥστε ἐνεργείᾳ εἶναι ἐπὶ τὴν αὔξησιν ἀδιεξίτητον· οὐδὲ γὰρ νῦν δέονται τοῦ ἀπείρου (οὐ γὰρ χρῶνται), ἀλλὰ μόνον εἶναι ὅσην ἂν βούλωνται πεπερασμένην· τῷ δὲ μεγίστῳ μεγέθει τὸν αὐτὸν ἔστι τετμῆσθαι λόγον ὁπηλικονοῦν μέγεθος ἕτερον. ὥστε πρὸς μὲν τὸ δεῖξαι ἐκείνοις οὐδὲν διοίσει τὸ [δ'] εἶναι ἐν τοῖς οὖσιν μεγέθεσιν.

This passage has perplexed readers. In addition to the few problems with ellipsis, it is unclear what Aristotle has in mind. There are two issues about infinity in the text, both relevant.

1. The actual infinite does not exist, but the potential infinite exists. Mathematicians only need the potential infinite.
2. The universe is finite, so that there is in fact a largest line. So how could one prove things, e.g., about convergent lines that converge outside the universe.

Additionally, there are two general ways of understanding Aristotle's solution, perhaps not very different.

(a) The lines used by the mathematician are the lines in the diagram. These are finite, indeed. So the mathematician does not ever actually use infinite lines. The line cut in the diagram is 'proportional' to any line. This might be seen as a solution to (1) as well as (2).
(b) Since geometrical proofs are about similar configurations, there is a general principle in geometry that the metrical size of the figures is unimportant to the universalization from the particular case.

Of course, Aristotle gives us no hint whether he intends (a), (b), or something else. I do not find (a) very satisfactory, in as much as the diagram is a representation and not an instance of the large figure, especially in reductio arguments. So the convergence of parallel lines in *Elements* I appears as crooked lines, as here from Paris Gr2466(p) folio 9v. Obviously, in a reductio something has to be wrong. My point is merely that the finite depiction of two long lines is not a small representation of long-lines (Fig. 2).

Or take a representation of an infinite line from the same manuscript (4v, Proposition I 12). AB is infinite (Fig. 3).

(Footnote 84 continued)
existent magnitudes should be something that makes it unimportant that one does not have the larger magnitude. My suggestion is that the fact that the smaller figure is used makes no difference to the proof. So what exists in magnitudes is probably their being of any particular size, loosely from line b32.

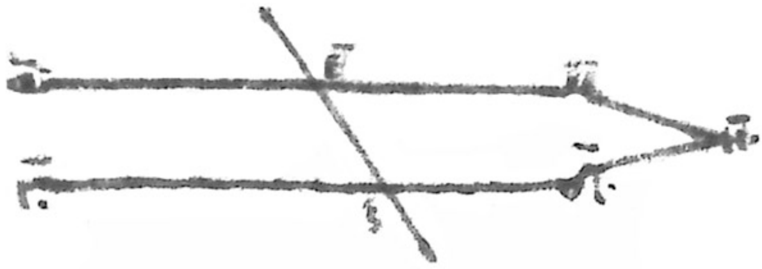

Fig. 2 Reductio arguments

Fig. 3 Representation of an infinite line

So much more is going on than merely reducing the size. I don't wish here to involve us in questions about diagrams. It is enough that we need some account that deals with (1) and (2). Here, Hussey's account of how to modify Euclidean mathematics to resolve (1) and (2) is probably fine.[85] Yet, I would like to make a few points about each.

If you adopt a different version of the parallel postulate, e.g., where you substitute for it a principle that interior alternate angles of a line intersecting two lines are equal iff the lines are parallel, you will be able to prove the other theorems needed for parallels. But this is not what Aristotle suggests. His principle concerns ratios. And as we shall see, the parallel theorems in Euclid are not the only problem with the geometry for a small world.

My second point concerns (1). If one allows an arbitrarily large but not actually infinite universe, it doesn't really matter that one defines parallel lines as lines that will never meet no matter how they are extended. That is, the universe may be

[85] Hussey (1983, ad loc.).

potentially infinite. Proofs that lines are parallel can readily be accomplished by reductios, as is the norm in Euclid.

There is, however, another sort of infinitary proof that employs infinitary sequences. Such proofs normally involve issues distinct from my present concerns. However, Aristotle uses such a technique; so I don't see how he could object to them.[86] Indeed, Aristotle allows himself the luxury of setting up an infinitary, recursive sequence to argue that time is continuous iff motion is, concluding (*Phys*. Z 2.233a7–10):

> For the faster will divide the time, and the slower the length. And so if it is true that they always reciprocate, a division always comes about in their reciprocating.

> διαιρήσει γὰρ τὸ μὲν θᾶττον τὸν χρόνον, τὸ δὲ βραδύτερον τὸ μῆκος. εἰ οὖν αἰεὶ μὲν ἀντιστρέφειν ἀληθές, ἀντιστρεφομένου δὲ αἰεὶ γίγνεται διαίρεσις,

Aristotle does not give us the reductio that we expect. So he can have no serious objection to this type of argument, except that he can object to its being 'completed'. Nonetheless, this sort of argument does not impinge on questions about the infinite in extent.

I suspect it is common, however, at least to say of the standard texts of Greek mathematics that 'infinite' means 'potentially infinite' in Aristotle's sense, so that one can readily eliminate actual infinities from such texts. That is an interesting claim about the logic of Euclid's *Elements*. Since Aristotle probably invented the notion of the potential infinite, although it is implicit in Anaxagoras, if Euclid intends us to understand 'potential infinite' in a few places in *Elements* I, that would show a philosophical dependence of Euclid on Aristotle. There are four groups of texts that mention infinity in Euclid's corpus:

Texts relating to parallel lines (I def 23, post. 5 (cf. 2), props. 29, 44 (both quoting post. 5, but 29 for a reductio);
I 12: to construct a perpendicular from given point to an infinite line;
I 22, *Data* 39 (related to *Elements* I 22): construction of triangle from three given lines;
Text on number (VII 31: the fundamental theorem of arithmetic, but notably not IX 20, the prime number theorem);
Texts on the infinitude of irrationals (X Definition 3, Proposition 115 that there are infinitely different irrationals).

I shall not be concerned here with the parallel postulate, which states the familiar condition under which two lines infinitely extended will intersect. If lines intersect, they intersect at a finite length from a given point. So the fact that they are infinitely extended is convenient overkill. We can think of the parallel postulate as having a

[86] This is a very brief reply to Hussey on this point in a generally excellent discussion (ibid.).

logical form (with P(x,y) = x is parallel to y, and C(x,y) = x and y meet if extended infinitely):

Postulate 5: $F(x,y) \rightarrow C(x,y)$
Definition 23: $P(x,y) \leftrightarrow \sim C(x,y)$
So, the argument for the first proposition on parallels shows of two lines:
Proof of alternate angles (26):$A(a, b)$ & $C(a, b) \rightarrow \bot; \therefore A(a, b) \rightarrow \sim C(a, b)$

So even if in principle the lines could be extended infinitely, it really makes no difference whether we understand the lines as infinitely extended or as extended as much as is needed.

Nor will I be concerned with the fundamental theorem of arithmetic, that every composite number is measured by a prime number, since the infinite appears in a reductio, that if the number is not measured, then it will be measured by infinitely many, ever smaller numbers.

The texts that there are infinite irrationals are more interesting, as they involve infinitely many distinct constructions. Yet, there is some general agreement today that they are late additions to respond to the claim of X Definition 3 (it will be shown that to a given line there are infinitely many commensurable and incommensurable lines). The procedure is to start with a medial A and a rational B and to take the side of O(A, B) = Γ. Then take the side of O(Γ, B) = Δ, and so forth. So, the author gives us a recursive procedure for generating infinitely many irrationals. Is this potential or actual infinity? Well, this has nothing to do with space and place. So we can drop the unanswerable question about a possibly unknown mathematician.

My principal concern then is with I 12, and I 22.

I 12 To draw a straight perpendicular line to a given infinite line from a given point (where 'altitude' is a line drawn to a given line at right angles, and a perpendicular is a line drawn at right angles from a given line. Direction is important!) (Ἐπὶ τὴν δοθεῖσαν εὐθεῖαν ἄπειρον ἀπὸ τοῦ δοθέντος σημείου, ὃ μή ἐστιν ἐπ' αὐτῆς, κάθετον εὐθεῖαν γραμμὴν ἀγαγεῖν.)

I 22 To construct a triangle from three straight lines, which are equal to given straight-lines. … The proof constructs the base of the triangle on a line that's infinite in one direction: Let a line be displayed DE which is finite along D, but infinite along E, … (Ἐκκείσθω τις εὐθεῖα ἡ ΔΕ πεπερασμένη μὲν κατὰ τὸ Δ ἄπειρος δὲ κατὰ τὸ Ε, …)

I 12 is in Book I to complement the drawing of a perpendicular from a given line, but, as Mueller informs us,[87] it is not used until the squaring problem, II 14. Proclus is probably right that Euclid decides to construct the figure on an infinite line, because otherwise there is no guarantee that there will be a altitude to the line, which may fall short. A change in the nature of the problem might deal with the

[87] Mueller (1981, p. 20).

problem. Draw a line from the given point to the given line. Double the length of this line. Draw a circle with the double line as radius and the given point as center, and extend the given line, if necessary, to meet the circle. The altitude will meet the extended line. A similar issue is at work in I 22. One needs a line that is at least as long as the sum of the three lines. So, for convenience, take a partially infinite line. However, this can be very trivially avoided. Instead of cutting off three segments of the partially infinite line, just draw a line with one length and, using circles, extend it twice, for the other two lengths. So why does Euclid ask for infinite lines?

Should we assume that Euclid understands these infinite lines to be really lines that are long enough, or should we just take him to be saying that he wants actual infinite lines? Here the issue is not what Euclid could have done. That is trivial enough. It is what he chose to do. If he had the slightest queesiness about infinite lines, he easily could have avoided them, as he does, in effect, in the case of parallel lines. I should also point out that this rather cavalier introduction of merely convenient infinite lines is somewhat unusual in Greek mathematics.[88]

We can contrast the ambiguity in Euclid with Apollonius. He really does use infinite to mean potential infinite but also has no difficulty in speaking of actual infinities as well. For example, in stating the asymptote theorem for hyperbolas (II 14), he says, "As they are extended ad infinitum, the asymptotes and the section move nearer to themselves and arrive at a distance smaller than any/every given distance (Αἱ ἀσύμπτωτοι καὶ ἡ τομὴ εἰς ἄπειρον ἐκβαλλόμεναι ἔγγιόν τε προσάγουσιν ἑαυταῖς καὶ παντὸς τοῦ δοθέντος διαστήματος εἰς ἔλαττον ἀφικνοῦνται διάστημα)." One might worry about the second clause, but the quantifier is correctly placed and there is nothing more infinitary than the Anaxagorean/Eudoxan principle that it is possible to get a magnitude smaller than any given magnitude, as is clear from the proof, where we are given a magnitude K to get a smaller distance.[89]

One might also, with much caution, see an infinite, spatial plane in Apollonius' marking out four regions (τόποι) by the asymptotes to opposite hyperbolas of the curve at II 33, that is the two regions determined by the 'angle containing the section', i.e., by the asymptotes to the two hyperbolas, and the two regions outside the two angles formed by the asymptotes to the two hyperbolas. In post-Cartesian mathematics, these evolve into, but should not be confused with the regions determined by x and y axes. These regions, as determined by the asymptotes, are infinite if the asymptotes are infinite. It would be anachronistic to see here an

[88] The Heronian Definitions 119 clearly understands actual infinities in talking about infinite magnitudes as, "A magnitude is what increases and is cut ad infinitum. There are 3 species of it: line, surface, solid. A magnitude is infinite than which nothing larger is conceived in existence (ὑπόστασιν) of whatever size so that there is no limit of it." There is no suggestion that such magnitudes are impossible.

[89] So too I Definition 1, Proposition 8 (the infinite extension of a section whose diameter is parallel to the axis of the cone). Here, one might note that Apollonius says that as the surface of the cone and the cutting plane increase ad infinitum, the section also increases. One might wonder about the claim at II 44 that we will find infinite diameters to a section. After all, he could have said 'however many'. Basically, it is as if one said, "it is possible to find infinite parallels to a given line." Is Apollonius speaking hyperbolically?

infinite space divided up into four sections within which the asymptotes lie, as the spatial regions are determined by the asymptotes, and, as sector of the cone, the plane is just another, perhaps infinite geometrical object, that is, if the question what it is ever gets asked. Yet there is no evidence anyone put these two together into a notion of geometrical space (as opposed to geometrical object).

In fact, in Book VI, Apollonius clearly treats some conic lines as infinite.[90] Props. 1–10 concern equal conic sections, those that can be superimposed (Definition 1). Consider Proposition 1, "If the upright sides of parabolic sections (the parameter or *latus rectus*), applied to which the perpendiculars drawn down to the axes equal in power (the ordinates), are equal, then the sections are equal, and if the sections are equal the upright sides are equal (my trans.)." This is only true if the parabolic sections are infinite; otherwise, two parabolic sections with the same parameter but axes of different lengths will not superimpose and be equal. Recall that the square on the ordinate equals the rectangle of the parameter and the cut off part of the axis (the abscissa). And the proof of the proposition bears this out. So too, in proving Proposition 3, that none of the three sorts of sections, ellipse, hyperbola, parabola, is equal to any section of the other sorts, Apollonius argues that it is obvious that an ellipse is not equal to a parabola or hyperbola, because it is finite while a parabola and hyperbola is infinite (Toomer 1985, p. 275.8; Rashed 2009, p. 101.16).

When it comes to infinite lines, I conclude then that Euclid and Apollonius were not bothered.[91]

6 Alexander Was Bothered; Was Anyone Else? Well, Maybe by the Practical?

In his commentary on *Physics* III 8, Simplicius reports an objection of Alexander of Aphrodisias to Euclid, *Elements* I 1. What if we want to construct an equilateral triangle on the diameter of the universe? The construction requires that we go beyond the rim of heaven, so that Euclid's construction is not universal (Simplicius, *In Arist. phys*. 511.30–2.9)

> Alexander inquired how the first theorem of the *Elements* of Euclid is not destroyed, if it is not possible also to extend a straight line outside the universe or to draw a circle (for [this would be the case] if the given finite straight-line on which it is required to construct the equilateral triangle can be the diameter of the kosmos, and it is impossible to construct an equilateral triangle on this, if there be nothing outside the kosmos, since the diameter of the universe becomes the radius of the circles where the [lines] joining their common section to the end-points of the given [line] with it produce the equilateral triangle). After inquiring, he solves it, saying, "Since [that magnitude] is infinite [i.e., potentially] where it is always possible for those who take it in quantity to take something outside, as was shown, it is

[90] I am very thankful to Vincenzo De Risi for pointing this out in a workshop on Apollonius at the Humboldt University, July, 2014. Some comments of Sabetai Unguru on Book III at the same workshop were also particularly helpful for my discussion of Apollonius.

[91] The British reader is permitted to see an allusion to Catherine Tate.

clear that the mathematicians also suppose such lines which they suppose as infinite, so that it is possible that they increase. For these [lines] are infinite where there is something outside. But it is not possible for the diameter of the kosmos to grow. And so they suppose a [line] smaller than the diameter, if they suppose a finite-line, since those [lines] are also infinite which they can add to and which they can extend."

511.30 Ζητήσας δὲ ὁ Ἀλέξανδρος, πῶς οὐκ ἀναιρεῖται τὸ πρῶτον θεώρημα
τῶν Εὐκλείδου Στοιχείων, εἴπερ μὴ καὶ ἔξω τοῦ παντὸς δυνατὸν εὐθεῖαν
ἐκβάλλειν ἢ κύκλον γράφειν (εἰ γὰρ δύναται μὲν ἡ δοθεῖσα εὐθεῖα πεπε-
ρασμένη, ἐφ' ἧς δεῖ τὸ ἰσόπλευρον τρίγωνον συστήσασθαι, ἡ διάμετρος εἶναι
τοῦ κόσμου, ἀδύνατον δὲ ἐπὶ ταύτης τρίγωνον ἰσόπλευρον συστήσασθαι, εἰ
511.35 μηδὲν εἴη τοῦ κόσμου ἐκτός· ἡ γὰρ διάμετρος τοῦ παντὸς ἐκ τοῦ κέντρου
512.1 γίνεται τῶν κύκλων, ὧν ἀπὸ τῆς κοινῆς τομῆς αἱ ἐπὶ τὰ πέρατα τῆς δοθεί-
σης ἐπιζευγνύμεναι τὸ ἰσόπλευρον τρίγωνον μετ' αὐτῆς ποιοῦσι), τοῦτο οὖν
ζητήσας λύει λέγων· "ἐπειδὴ ἄπειρόν ἐστιν, οὗ κατὰ ποσὸν λαμβάνουσιν
ἀεί τι λαβεῖν ἔστιν ἔξω, ὡς δέδεικται, δῆλον ὅτι οἱ μαθηματικοὶ καὶ ἃς
512.5 ἀπείρους γραμμὰς ὑποτίθενται, τοιαύτας ὑποτίθενται, ὥστε δυνατὸν αὐτὰς
αὐξῆσαι. ὧν γάρ ἐστιν ἐκτός τι, αὗται ἄπειροι· τὴν δὲ διάμετρον τοῦ
κόσμου οὐκ ἔστιν αὐξῆσαι· ἐλάττονα οὖν ὑποτίθενται τῆς διαμέτρου, εἴπερ
πεπερασμένην ὑποτίθενται, ὁπότε καὶ ἄπειροι αὗταί εἰσιν, αἷς προσθεῖναι
δύνανται καὶ ἃς ἐκβάλλειν."

The professor of Aristotelian studies needs to assume that mathematicians just keep their lines smaller. The neo-Platonic exile has no such difficulty, since he holds that the objects of mathematics reside in the imagination, where the potential infinite is unencumbered by physical limitations, and this, presumably, is why we do not find Alexander's puzzle in Proclus' commentary on Euclid, although he is concerned about actual infinities. Yet it is difficult to know whether it would have been, in fact, a serious problem even for Aristotle, since the maximal line is still the diameter of the physical universe qua diameter of the heaven qua sphere, so that the size does not enter in. Does the limitations on the size of the line qua perceptible line have any impact on constructions on the same line qua line?

If such a puzzle appeared among philosophers in the Hellenistic Age, we would expect, however, it to be a response to Stoic or late Peripatetic and not Aristotle's own views of the heaven and to Stoic or late Peripatetic and not Aristotle's own views of geometrical objects, at least not before the first century BCE Aristotelian revival. Now our Stoic fictionalists need not trouble about the issue. Yet an abstractionist might be concerned whether lines represented in the imagination need to be no larger than some size. There would seem to be at most one Stoic who might have been so concerned, Posidonius, who thought that the void is only large enough to accommodate the exploded cosmos during the cyclical, great conflagration.[92]

[92] Cf. note 47.

Now, according to al-Nairīzī[93] and Proclus (*In Eucl.* 176.5–17),[94] Posidonius attempts a variation on the definition of parallel lines (Proclus, *In Eucl.* 176.5–17):

> And Euclid defines parallel lines in this way, but Posidonius says that parallels are those in one plane that neither converge nor diverge but have all their perpendiculars equal that are drawn from points on one to the other. Those that always make their perpendiculars smaller converge with one another. For the perpendicular is able to determine the heights of areas and the distances between the lines. Hence, when the perpendiculars are equal, the distances between the lines are equal, when larger or smaller the distances too will be made smaller and they will converge on the side where the perpendiculars are smaller.

> Καὶ ὁ μὲν Εὐκλείδης τοῦτον ὁρίζεται τὸν τρόπον τὰς παραλλήλους εὐθείας, ὁ δὲ Ποσειδώνιος, παράλληλοι, φησίν, εἰσὶν αἱ μήτε συνεύουσαι μήτε ἀπονεύουσαι ἐν ἑνὶ ἐπιπέδῳ, ἀλλ' ἴσας ἔχουσαι πάσας τὰς καθέτους τὰς ἀγομένας ἀπὸ τῶν τῆς ἑτέρας σημείων ἐπὶ τὴν λοιπήν. ὅσαι δ' ἂν ἐλάττους ἀεὶ ποιῶσι τὰς καθέτους συνεύουσιν ἀλλήλαις· ἡ γὰρ κάθετος τά τε ὕψη τῶν χωρίων καὶ τὰ διαστήματα τῶν γραμμῶν ὁρίζειν δύναται. διόπερ ἴσων μὲν τῶν καθέτων οὐσῶν ἴσα τὰ διαστήματα τῶν εὐθειῶν, μειζόνων δὲ καὶ ἐλαττόνων γινομένων καὶ ἡ ἀπόστασις ἐλασσοῦται καὶ συνεύουσιν ἀλλήλαις, ἐφ' ἃ μέρη εἰσὶν αἱ κάθετοι ἐλάσσονες.

The type of definition here is a locus definition such as Euclid’s definition of ‘circle’ (see § 1). What does Proclus hope to gain by this definition? First, it does not, in fact, avoid the parallel postulate.[95] One can construct a line with two points equidistant from a given line—draw two perpendiculars equal to each other and connect them, but it will not necessarily be the case that every other perpendicular will be equal nor that the angles at the points from which the perpendiculars are drawn will be right angles. I have no idea if this is significant, since all attempts to prove the parallel postulate must fail. Nonetheless, neither al-Nayrizi nor Proclus suggest that the motivation was the parallel postulate, nor do they mention it again. Furthermore, al-Nayrizi[96] mentions another definition by a contemporary of

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[93] Cf. al-Nairīzī (2009, pp. 16–17) followed by (2003, p. 88).

[94] Either both are based on Heron’s commentary on Euclid, or, as seems more likely, Proclus and Heron (whom al-Nairīzī uses) are each based on Geminus.

[95] Euclid, *Elements* I post. 5: and that if a straight line falls on two lines makes the interior angles on the same parts smaller than two right angles, then if the two lines are extended infinitely on the parts where the angles are smaller than two right angles, they meet. Euclid tacitly assumes the converse (cf. *Elements* I 17), if the lines meet the angles are less than two right angles. Note, per my example, that there is nothing in Posidonius' definition about whether parallel or non-parallel lines do or do not meet when extended.

[96] al-Nairīzī (2003, pp. 88–9). Even if the source is Geminus, modifying Posidonius, the point remains that this definition is the one used by Agapius (see next two notes).

Simplicius, Agapius,[97] that parallel lines are equidistant even if extended ad infinitum on both sides. This definition then forms the basis, after I 27,[98] of attempts to prove the parallel postulate. As to Proclus, when he turns to the parallel postulate after I 29,[99] Posidonius is absent. So Posidonius' divergence from Euclid might not have been due his concern to avoid the parallel postulate.

More to our point, the construction avoids the need for an infinite universe. The Euclidean definition, that the lines when extended infinitely will not converge, presupposes that the lines could be extended infinitely and be seen not to converge. The parallel postulate guarantees that the exercise need not be undertaken, but the definition requires that it could be.

Let's assume that Posidonius has taken care of the parallel postulate. Then one can show that two lines are parallel simply by constructing arbitrary perpendiculars between them or their extensions (finite) and showing that any two are equal, i.e., if the lines can be extended in some direction to where there are perpendiculars. One can similarly show that they 'converge' by showing that they are unequal, where this is all 'converge' will mean.[100] Of course, what else our Aristotelianizing Stoic hopes to gain, we do not know. He may want it possible to extend all converging lines to where they actually meet (and so leave the parallel postulate where it is), or he may have required a strictly finite geometrical world. Nor do we know if a finitism, corresponding his views on void, even was his intent. But, either way, it would seem more reasonable to put Posidonius in the class of abstractionists. Well, another nice story.

So far as I can tell, no other text that is remotely mathematical suggests Alexander's puzzle. Proclus mentions three instances where someone does raise a difficulty for a construction in a problem because there isn't place (τόπος) to do it:

I 2 To put a line equal to a given line at a given point. (*In Eucl*. 225.16, 225.8–227.8)

I 9 To bisect a given finite line. (*In Eucl*. 275.7, 275.7–277.4)

I 12 To draw a straight perpendicular line to a given infinite line from a given point. (*In Eucl*. 289.18–20, 289.16–290.13), where there isn't enough place on the other side of the given line)

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[97] On the identity of Agapius, cf. Lo Bello in al-Nairīzī (2003, n. L5: 224–229).

[98] al-Nairīzī (2003, pp. 157–6).

[99] Proclus, *In Eucl*. 365–73, cf. 362–3.

[100] Of course, in a finite spherical universe, all parallel lines could be shown to be parallel in this way, with some condition that will allow that there might be points on one line from which perpendiculars cannot be drawn to the other. But some non-parallel lines cannot be extended so that perpendiculars can be drawn between them. One suspects that the geometry would end up with more postulates to deal with what could easily be proved by treating the spherical world as a subspace of an infinite one.

We should also put in this list:

I 11 To draw a straight line at right angles to a given straight-line from point given on it. (*In Eucl*. 281.6–7)

Here one is asked to construct the line from an end point without extending the given line. One could surely do this if there were place (τόπος). A modern reader, I suspect, will tend to read these discussions as posing an objection to Euclid's solution to the respective problem. Proclus (*In Eucl.* 389.7–15, ad I 12), however, is careful to distinguish between an objection (ἔνστασις) and a case (πτῶσις—different configurations in proving a theorem or doing a problem) and rightly, I think, treats the difficulty as a demand for a case. Proclus puts up a small barrier in the way of our using the four examples, namely that the commentators did not distinguish between cases and objections. So perhaps he and I are wrong, and someone did think that the lack of place (τόπος) was an objection to the constructions.

It turns out that our commentators might well be Heron of Alexandria. In his commentary on Euclid, al-Nayrizi (Anaritius) preserves a construction identical to that of Proclus' commentary on I 11 (2003, 128–9) and mentions Hero as the author. So, Heath[101] suggests that we add several proofs in al-Nayrizi to the list, besides I 11,

I 16 In every triangle, when one side is extended, the outside angle is larger than each of the opposite and interior angles. (*In Eucl*. 305.21–6, with Hero mentioned)

I 20 In every triangle, two sides, taken in every way, are larger than the remaining side. (2003, 140–4) (*In Eucl*. 323.5–326.5, with Hero and Porphyry mentioned)

I 48 converse of the Pythagorean theorem (2003, 202–3, with Heron mentioned) (cf. *In Eucl*. 430.9–431.14)

Since al-Nayrizi mentions the 'objection' to I 16, we should be cautious. I am not sure why Heath includes it in his list, since Proclus merely says that Heron reports an objection of Philip of Mende that triangles qua triangles do not have exterior angles. Whatever Philip's objection may have been, it apparently has nothing to do with lack of place (τόπος) and everything to do with the nature of theorems about triangles.

Whereas Euclid's construction for I 20 involves two lines outside the triangle, Proclus gives three proofs, the second incomplete, of I 20 which he attributes to Heron and Porphyry, where each, but the first especially, is genuinely simpler in its construction than Euclid's proof, as it uses one line inside the triangle. So the lack of a construction exterior to the triangle may have nothing to do with lack of place (τόπος). A similar remark may be made for I 48. Here, it is enough to show that any two triangles with sides a, b such that $T(a) + T(b) = T(\text{the third side})$ are congruent.[102] So one takes a triangle a, b, c where $T(a) + T(b) = T(c)$ and constructs a

[101] Heath (1926, pp. 22–3).

right triangle with legs equal to *a*, *b*. Heron's version builds the right triangle with *a* as one of the legs, and then shows that the other leg is the same line as *b*. I don't know if the proof is simpler, but it fits with an aesthetic that minimizes auxiliary constructions and seeks to show that two lines with different basic properties are the same line. In sum, none of these proofs need have anything to do with issues of adequate places.

If my argument so far is unsound, then one could argue Heron shows a concern similar to Alexander's. Can one do Euclid's *Elements* in such a way that the size of the universe does not matter? For each of these is a theorem, and there is no reason to worry about adequate room in proving a theorem unless you believe:

> The proof of the theorem about some figure either employs a construction that fits every possible location and size of the figure or breaks up into cases that it covers every possible location and size.

If so, we would have a motivation to see in Heron the influence of some naturalist school, perhaps Strato, who, I noted, seems to have influenced Heron's pneumatics and who held that the universe is finite but not surrounded by infinite void in the way that Stoics believed. Presumably, this is something like what Hintikka supposed in suggesting that Heron was troubled to take into account a finite universe.[103]

As delightful as this speculation might be, I fear that there is a much simpler explanation of Heron's concerns about adequate place (τόπος). Suppose that the only propositions where Heron expressed a concern for adequate space were problems. In our texts, these are I 2, 9, 11, and 12. I need to draw a line at right angles to the end of a given line. But the line goes into a wall. I cannot draw it with the construction in Euclid. So I use Euclid's constructions to construct a square, the fourth line of which is my desired line. I need to construct a perpendicular to a line, but there is no room on the other side. Well, one might suspect a bit of a joke here. I have an infinite line, and there isn't room on the other side of the line. For he doesn't drop the condition that the line be infinite. Nonetheless, the situation is real enough. I have a very long wall and need to draw a perpendicular to the wall and cannot get to the other side. If so, Proclus is right to treat these as cases and not as objections. But what sort of cases?

Now there is something quaint in all this, something that might even be familiar to readers of Heron. I have no idea whether Euclid's constructions might be practical in the real world. I'll leave that to historians of architecture, while noting that sometimes they actually are, and at other times aren't. It is not just that we expect this from Heron. In a context of pure geometry where these issues do not

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[102] 'T(x)' is 'the square on x'.

[103] Hintikka (1973, pp. 121), "Aristotle's compunctions about geometrical constructions were apparently shared by at least one well-known mathematician of antiquity. Heron mechanicus tried to dispense with the production of particular straight lines as much as possible, motivated by the idea that there might not always be enough space available to carry out such a production. (It does not matter for my purposes whether Heron was himself worried about this or whether he was trying to reassure others)…"

arise, he brings in constructions that look to applications in tight spaces. Or is it rather that in his commentary, Heron is bringing geometry closer to applied geometry? Of course, I don't know the answer to this question. Yet, it is clear that there is nothing wrong with saying, "Here is a way of constructing a perpendicular," and saying, "By the way, when you apply geometry, you might find this construction useful."

The Arabic translation of Pappus, *Collectio* VIII on mechanics has a remarkably similar concern, where the problems are also close to the selection in Proclus and al-Nayrizi.[104] Suppose you wish to do a construction that requires a compass, but your compass just isn't big enough. The first problems (two methods) presumes that the compass can be smaller, where you want to draw a perpendicular from a given point on it. For the rest you use the maximum width of the compass: to bisect a given line, to trisect it, etc.; to add to a line a segment equal to it; to extend another line with an equal to a given line (3 cases); to construct a triangle from given lines, where any pair is larger than the third—this gets checked (2 cases: where two sides are equal, where none is equal). In fact, the second case of the first problem is a construction from the end-point, where there is no place. Here, again, it is plausible that the issue is about ordinary conditions and not cosmic ones. It is very reasonable to guess that the ultimate source for the Arabic translation was either Heron or a treatise in the Heronian tradition.[105]

The most geometrical of Heron's extant treatises is the *Metrica*, in the sense that the ingredients should be shapes that are to be measured (see above § 1). Except as an appendix to Book I (measurement of areas) and II (measurement of volume) on the measurement of disorderly figures, the figures are taken from the rich mathematical tradition. Here, Heron does distinguish between the metrical and the geometrical (see note 20). So the practical concerns expressed in the commentary on Euclid and the Arabic translation of Pappus are metamathematical. They explain why one might be interested in investigating certain constructions—there might be a wall; the compass might not be big enough.

Furthermore, we can look elsewhere for implicit concerns about physical limitations on geometrical constructions that are quite ordinary. As Sidoli and Saito (2009) observe, in *De sphaera*, Theodosius will employ many sorts of constructions involving the interiors of spheres, even in the demonstration of problems, while the distinction between problems and theorems is not rigorous, except in one respect. With the exception of finding the center of the sphere (I 2), necessary for the rest of the treatise, the other six problems themselves can be accomplished by working on

[104] Jackson (1980). These are sometimes described as rusty compass problems, but, as I indicate, this would not seem to be the concern. Cf. Section F (p. 527), "Now that is not easy to do using the construction mentioned by Euclid in his *Elements*, since we have only one small pair of compasses with which to work."

[105] Jackson (1980) makes an excellent argument that the excerpt belongs in Pappus, *Collectio* VII, but it is much less clear how it belongs, except as a completely distinct topic. He reports (p. 524) that it occurs between § 44, which concerns inscribing seven hexagons in a circle, six about the seventh (constructions that make the demonstration obvious) and § 45, about the juxtaposition of gears.

the surface of the sphere or outside it. As far as the abstract stereometry of spheres is concerned, this is irrelevant, but if one is working with practical, bronze or marble spheres, where lines need to be drawn for that high market star globe, one needs to avoid penetration. So, Heron is well within a tradition that treats problems as practical. Indeed, if we look, we will find other examples, such as Eratosthenes criticism of earlier duplications of the cube due to their impracticality and his praise of his own.[106]

Recall that I did not want to make generalizations about philosophies of Greek geometers. Nonetheless, we have seen that Euclid has no difficulty stipulating an infinite line. Of course, what Euclid intended by that stipulation we cannot know. It is enough to note that we have no reason to suppose that Euclid abjured the actual infinite. Secondly, Apollonius happily uses actual infinite lines. Thirdly, we saw that Heron and Theodosius seem to bring applied geometrical considerations into their treatment of problems in texts usually thought to be pure geometry. That dictates restrictions on a solution to a problem, but says little about how they think of geometry or the world, except that geometrical constructions should be practical. Of course, this too could be wrong. Heron might even have concerns about a finite universe, although, if he did, it would yet more curious why he thinks that there might be no place (τόπος) on the other side of an *infinite* line.

7 Conclusion

I began my discussion with concerns about four basic spatial notions in Greek philosophy and mathematics, well two significant, place (τόπος) and position (θέσις), and two relatively insignificant in mathematics, void (κενόν) and room (χώρα). Here I argued that ordinary notions bereft of philosophical analysis were adequate to describe how mathematicians treated objects. Constructions operate in ordinary worlds without rich notions of space. I then turned to Aristotle's views on place and how his views on mathematical objects can hang together with his views about place. I did not come up with a single answer to the question, but noted that for Aristotle there is a sense in which geometrical objects have place, but that it had to be different at least from the way in which physical, locomotive objects have place. This seems more a reflection of mathematical usage than a profound new way of thinking about mathematical objects. Quite the contrary, Aristotle's conception of mathematical objects as perceptible magnitudes qua magnitudes leaves open the question what gets removed by the 'qua' operator, in our case, 'place' or 'absolute direction'. This is what made Alexander's puzzle about the finite world poignant. I then turned to more plausible candidates for philosophers interacting with Greek mathematicians. Not surprisingly, we did not find much, although many philosophers, Epicurean and Stoic, seem to have tried. However, I also tried to illustrate my point that we expect different Greek mathematicians to have their own

[106] Euctocius, *In Arch. de sphaer. et cyl.* 90.4–14.

quirks. In the case of Euclid, he was happy to request an infinite line; Apollonius, an actual infinite line. In the case of Heron (or ps.-Heron), limitations of place (τόπος) came in, but this was not the cosmic limitation that might have bothered Posidonius. It was the limitations that come into consideration from an interest in practical problems, that also occurs in Theodosius and Eratosthenes. But Heron may also have been a physicalist, in strong contrast to the abstractionism of post-Stoic Aristotelians and Platonists. His attitudes certainly influences the Heronian corpus and tradition. I am sorry if I have disappointed my reader in two ways. First, I did not find any grand theory of place in Greek mathematics. I also did not find a view about infinite lines in Greek mathematics. Instead, I found individual mathematicians introducing the infinite in different ways and without comment. I found variety, and I like that.[107]

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