Chapter 1

Uncertain Measure

Uncertainty theory was founded by Liu [122] in 2007 and subsequently studied by many researchers. Nowadays uncertainty theory has become a branch of axiomatic mathematics for modeling belief degrees. This chapter will present normality, duality, subadditivity and product axioms of uncertainty theory. From those four axioms, this chapter will also introduce an uncertain measure that is a fundamental concept in uncertainty theory. In addition, product uncertain measure and conditional uncertain measure will be explored at the end of this chapter.

1.1 Measurable Space

From the mathematical viewpoint, uncertainty theory is essentially an alternative theory of measure. Thus uncertainty theory should begin with a measurable space. In order to learn uncertainty theory, let us introduce algebra, σ-algebra, measurable set, Borel algebra, Borel set, and measurable function. The main results in this section are well-known. For this reason the credit references are not provided. You may skip this section if you are familiar with them.

Definition 1.1 *Let Γ be a nonempty set (sometimes called universal set). A collection $\mathcal{L}$ consisting of subsets of Γ is called an algebra over Γ if the following three conditions hold: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \in \mathcal{L}$, then*

$$\bigcup_{i=1}^{n} \Lambda_i \in \mathcal{L}. \tag{1.1}$$

The collection $\mathcal{L}$ is called a σ-algebra over Γ if the condition (c) is replaced

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with closure under countable union, i.e., when $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, *we have*

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}. \tag{1.2}$$

Example 1.1: The collection $\{\emptyset, \Gamma\}$ is the smallest σ-algebra over Γ, and the power set (i.e., all subsets of Γ) is the largest σ-algebra.

Example 1.2: Let Λ be a proper nonempty subset of Γ. Then $\{\emptyset, \Lambda, \Lambda^c, \Gamma\}$ is a σ-algebra over Γ.

Example 1.3: Let $\mathcal{L}$ be the collection of all finite disjoint unions of all intervals of the form

$$(-\infty, a], \quad (a, b], \quad (b, \infty), \quad \emptyset. \tag{1.3}$$

Then $\mathcal{L}$ is an algebra over $\Re$ (the set of real numbers), but not a σ-algebra because $\Lambda_i = (0, (i-1)/i] \in \mathcal{L}$ for all i but

$$\bigcup_{i=1}^{\infty} \Lambda_i = (0, 1) \notin \mathcal{L}. \tag{1.4}$$

Example 1.4: A σ-algebra $\mathcal{L}$ is closed under countable union, countable intersection, difference, and limit. That is, if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then

$$\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \bigcap_{i=1}^{\infty} \Lambda_i \in \mathcal{L}; \quad \Lambda_1 \setminus \Lambda_2 \in \mathcal{L}; \quad \lim_{i\to\infty} \Lambda_i \in \mathcal{L}. \tag{1.5}$$

Definition 1.2 *Let* Γ *be a nonempty set, and let* $\mathcal{L}$ *be a* σ*-algebra over* Γ*. Then* $(\Gamma, \mathcal{L})$ *is called a measurable space, and any element in* $\mathcal{L}$ *is called a measurable set.*

Example 1.5: Let $\Re$ be the set of real numbers. Then $\mathcal{L} = \{\emptyset, \Re\}$ is a σ-algebra over $\Re$. Thus $(\Re, \mathcal{L})$ is a measurable space. Note that there exist only two measurable sets in this space, one is $\emptyset$ and another is $\Re$. Keep in mind that the intervals like $[0, 1]$ and $(0, +\infty)$ are not measurable!

Example 1.6: Let $\Gamma = \{a, b, c\}$. Then $\mathcal{L} = \{\emptyset, \{a\}, \{b, c\}, \Gamma\}$ is a σ-algebra over Γ. Thus $(\Gamma, \mathcal{L})$ is a measurable space. Furthermore, $\{a\}$ and $\{b, c\}$ are measurable sets in this space, but $\{b\}, \{c\}, \{a, b\}, \{a, c\}$ are not.

Definition 1.3 *The smallest* σ*-algebra* $\mathcal{B}$ *containing all open intervals is called the Borel algebra over the set of real numbers, and any element in* $\mathcal{B}$ *is called a Borel set.*

Example 1.7: It has been proved that intervals, open sets, closed sets, rational numbers, and irrational numbers are all Borel sets.

Example 1.8: There exists a non-Borel set over $\Re$. Let $[a]$ represent the set of all rational numbers plus a. Note that if $a_1 - a_2$ is not a rational number, then $[a_1]$ and $[a_2]$ are disjoint sets. Thus $\Re$ is divided into an infinite number of those disjoint sets. Let A be a new set containing precisely one element from them. Then A is not a Borel set.

Definition 1.4 *A function f from a measurable space $(\Gamma, \mathcal{L})$ to the set of real numbers is said to be measurable if*

$$f^{-1}(B) = \{\gamma \in \Gamma \mid f(\gamma) \in B\} \in \mathcal{L} \tag{1.6}$$

for any Borel set B of real numbers.

Continuous function and monotone function are instances of measurable function. Let $f_1, f_2, \cdots$ be a sequence of measurable functions. Then the following functions are also measurable:

$$\sup_{1 \le i < \infty} f_i(\gamma); \quad \inf_{1 \le i < \infty} f_i(\gamma); \quad \limsup_{i \to \infty} f_i(\gamma); \quad \liminf_{i \to \infty} f_i(\gamma). \tag{1.7}$$

Especially, if $\lim_{i \to \infty} f_i(\gamma)$ exists for each γ, then the limit is also a measurable function.

1.2 Event

Let $(\Gamma, \mathcal{L})$ be a measurable space. Recall that each element Λ in $\mathcal{L}$ is called a measurable set. The first action we take is to rename measurable set as *event* in uncertainty theory.

How do we understand those terminologies? Let us illustrate them by an indeterminate quantity (e.g. bridge strength). At first, the universal set Γ consists of all possible outcomes of the indeterminate quantity. If we believe that the possible bridge strengths range from 80 to 120 in tons, then the universal set is

$$\Gamma = [80, 120]. \tag{1.8}$$

Note that you may replace the universal set with an enlarged interval, and it would have no impact.

The σ-algebra $\mathcal{L}$ should contain all events we are concerned about. Note that event and proposition are synonymous although the former is a set and the latter is a statement. Assume the first event we are concerned about corresponds to the proposition "the bridge strength is less than or equal to 100 tons". Then it may be represented by

$$\Lambda_1 = [80, 100]. \tag{1.9}$$

Also assume the second event we are concerned about corresponds to the proposition "the bridge strength is more than 100 tons". Then it may be represented by

$$\Lambda_2 = (100, 120]. \tag{1.10}$$

If we are only concerned about the above two events, then we may construct a σ-algebra $\mathcal{L}$ containing the two events Λ_1 and Λ_2, for example,

$$\mathcal{L} = \{\emptyset,\ \Lambda_1,\ \Lambda_2,\ \Gamma\}. \tag{1.11}$$

In this case, we totally have four events: $\emptyset$, Λ_1, Λ_2 and Γ. However, please note that the subsets like $[80, 90]$ and $[110, 120]$ are not events because they do not belong to $\mathcal{L}$.

Keep in mind that different σ-algebras are used for different purposes. The minimum requirement of a σ-algebra is that it contains all events we are concerned about. It is suggested to take the minimum σ-algebra that contains those events.

1.3 Uncertain Measure

Let us define an uncertain measure $\mathcal{M}$ on the σ-algebra $\mathcal{L}$. That is, a number $\mathcal{M}\{\Lambda\}$ will be assigned to each event Λ to indicate the belief degree with which we believe Λ will happen. There is no doubt that the assignment is not arbitrary, and the uncertain measure $\mathcal{M}$ must have certain mathematical properties. In order to rationally deal with belief degrees, Liu [122] suggested the following three axioms:

Axiom 1. *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2. *(Duality Axiom)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *(Subadditivity Axiom) For every countable sequence of events* Λ_1, $\Lambda_2, \cdots$, *we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}. \tag{1.12}$$

Remark 1.1: Uncertain measure is interpreted as the personal belief degree (not frequency) of an uncertain event that may happen. It depends on the personal knowledge concerning the event. The uncertain measure will change if the state of knowledge changes.

Remark 1.2: Duality axiom is in fact an application of the law of truth conservation in uncertainty theory. The property ensures that the uncertainty theory is consistent with the law of excluded middle and the law of contradiction. In addition, the human thinking is always dominated by the duality. For example, if someone says a proposition is true with belief degree

0.6, then all of us will think that the proposition is false with belief degree 0.4.

Remark 1.3: Given two events with known belief degrees, it is frequently asked that how the belief degree for their union is generated from the individuals. Personally, I do not think there exists any rule to make it. A lot of surveys showed that, generally speaking, the belief degree of a union of events is neither the sum of belief degrees of the individual events (e.g. probability measure) nor the maximum (e.g. possibility measure). Perhaps there is no explicit relation between the union and individuals except for the subadditivity axiom.

Remark 1.4: Pathology occurs if subadditivity axiom is not assumed. For example, suppose that a universal set contains 3 elements. We define a set function that takes value 0 for each singleton, and 1 for each event with at least 2 elements. Then such a set function satisfies all axioms but subadditivity. Do you think it is strange if such a set function serves as a measure?

Remark 1.5: Although probability measure satisfies the above three axioms, probability theory is not a special case of uncertainty theory because the product probability measure does not satisfy the fourth axiom, namely the product axiom on Page 17.

Definition 1.5 *(Liu [122]) The set function* $\mathcal{M}$ *is called an uncertain measure if it satisfies the normality, duality, and subadditivity axioms.*

Exercise 1.1: Let $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. It is clear that there exist 8 events in the σ-algebra

$$\mathcal{L} = \{\emptyset, \{\gamma_1\}, \{\gamma_2\}, \{\gamma_3\}, \{\gamma_1, \gamma_2\}, \{\gamma_1, \gamma_3\}, \{\gamma_2, \gamma_3\}, \Gamma\}. \tag{1.13}$$

Assume c_1, c_2, c_3 are nonnegative numbers satisfying the consistency condition

$$c_i + c_j \le 1 \le c_1 + c_2 + c_3, \quad \forall i \ne j. \tag{1.14}$$

Define

$$\mathcal{M}\{\gamma_1\} = c_1, \quad \mathcal{M}\{\gamma_2\} = c_2, \quad \mathcal{M}\{\gamma_3\} = c_3,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 1 - c_3, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 1 - c_2, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 1 - c_1,$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

Show that $\mathcal{M}$ is an uncertain measure.

Exercise 1.2: Suppose that $\lambda(x)$ is a nonnegative function on $\Re$ (the set of real numbers) such that

$$\sup_{x \in \Re} \lambda(x) = 0.5. \tag{1.15}$$

Define a set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \sup\limits_{x\in\Lambda} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) < 0.5 \\ 1 - \sup\limits_{x\in\Lambda^c} \lambda(x), & \text{if } \sup\limits_{x\in\Lambda} \lambda(x) = 0.5 \end{cases} \tag{1.16}$$

for each Borel set Λ. Show that $\mathcal{M}$ is an uncertain measure on $\Re$.

Exercise 1.3: Suppose $\rho(x)$ is a nonnegative and integrable function on $\Re$ (the set of real numbers) such that

$$\int_{\Re} \rho(x)\mathrm{d}x \ge 1. \tag{1.17}$$

Define a set function

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda} \rho(x)\mathrm{d}x < 0.5 \\ 1 - \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x, & \text{if } \displaystyle\int_{\Lambda^c} \rho(x)\mathrm{d}x < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.18}$$

for each Borel set Λ. Show that $\mathcal{M}$ is an uncertain measure on $\Re$.

Theorem 1.1 *(Monotonicity Theorem) Uncertain measure $\mathcal{M}$ is a monotone increasing set function. That is, for any events $\Lambda_1 \subset \Lambda_2$, we have*

$$\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}. \tag{1.19}$$

Proof: The normality axiom says $\mathcal{M}\{\Gamma\} = 1$, and the duality axiom says $\mathcal{M}\{\Lambda_1^c\} = 1 - \mathcal{M}\{\Lambda_1\}$. Since $\Lambda_1 \subset \Lambda_2$, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$. By using the subadditivity axiom, we obtain

$$1 = \mathcal{M}\{\Gamma\} \le \mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2\} = 1 - \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}.$$

Thus $\mathcal{M}\{\Lambda_1\} \le \mathcal{M}\{\Lambda_2\}$.

Theorem 1.2 *Suppose that $\mathcal{M}$ is an uncertain measure. Then the empty set $\emptyset$ has an uncertain measure zero, i.e.,*

$$\mathcal{M}\{\emptyset\} = 0. \tag{1.20}$$

Proof: Since $\emptyset = \Gamma^c$ and $\mathcal{M}\{\Gamma\} = 1$, it follows from the duality axiom that

$$\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 1 - 1 = 0.$$

Theorem 1.3 *Suppose that $\mathcal{M}$ is an uncertain measure. Then for any event Λ, we have*

$$0 \le \mathcal{M}\{\Lambda\} \le 1. \tag{1.21}$$

Proof: It follows from the monotonicity theorem that $0 \le \mathcal{M}\{\Lambda\} \le 1$ because $\emptyset \subset \Lambda \subset \Gamma$ and $\mathcal{M}\{\emptyset\} = 0$, $\mathcal{M}\{\Gamma\} = 1$.

Theorem 1.4 *Let $\Lambda_1, \Lambda_2, \cdots$ be a sequence of events with $\mathcal{M}\{\Lambda_i\} \to 0$ as $i \to \infty$. Then for any event Λ, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda \cup \Lambda_i\} = \lim_{i\to\infty} \mathcal{M}\{\Lambda\backslash\Lambda_i\} = \mathcal{M}\{\Lambda\}. \tag{1.22}$$

Especially, an uncertain measure remains unchanged if the event is enlarged or reduced by an event with uncertain measure zero.

Proof: It follows from the monotonicity theorem and subadditivity axiom that

$$\mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda \cup \Lambda_i\} \le \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda_i\}$$

for each i. Thus we get $\mathcal{M}\{\Lambda \cup \Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$. Since $(\Lambda\backslash\Lambda_i) \subset \Lambda \subset ((\Lambda\backslash\Lambda_i) \cup \Lambda_i)$, we have

$$\mathcal{M}\{\Lambda\backslash\Lambda_i\} \le \mathcal{M}\{\Lambda\} \le \mathcal{M}\{\Lambda\backslash\Lambda_i\} + \mathcal{M}\{\Lambda_i\}.$$

Hence $\mathcal{M}\{\Lambda\backslash\Lambda_i\} \to \mathcal{M}\{\Lambda\}$ by using $\mathcal{M}\{\Lambda_i\} \to 0$.

Theorem 1.5 *(Asymptotic Theorem) For any events $\Lambda_1, \Lambda_2, \cdots$, we have*

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0, \quad \text{if } \Lambda_i \uparrow \Gamma, \tag{1.23}$$

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} < 1, \quad \text{if } \Lambda_i \downarrow \emptyset. \tag{1.24}$$

Proof: Assume $\Lambda_i \uparrow \Gamma$. Since $\Gamma = \cup_i \Lambda_i$, it follows from the subadditivity axiom that

$$1 = \mathcal{M}\{\Gamma\} \le \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

Since $\mathcal{M}\{\Lambda_i\}$ is increasing with respect to i, we have $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} > 0$. If $\Lambda_i \downarrow \emptyset$, then $\Lambda_i^c \uparrow \Gamma$. It follows from the first inequality and the duality axiom that

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1 - \lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} < 1.$$

The theorem is proved.

Example 1.9: Assume Γ is the set of real numbers. Let α be a number with $0 < \alpha \le 0.5$. Define a set function as follows,

$$\mathcal{M}\{\Lambda\} = \begin{cases} 0, & \text{if } \Lambda = \emptyset \\ \alpha, & \text{if } \Lambda \text{ is upper bounded} \\ 0.5, & \text{if both } \Lambda \text{ and } \Lambda^c \text{ are upper unbounded} \\ 1-\alpha, & \text{if } \Lambda^c \text{ is upper bounded} \\ 1, & \text{if } \Lambda = \Gamma. \end{cases} \tag{1.25}$$

It is easy to verify that $\mathcal{M}$ is an uncertain measure. Write $\Lambda_i = (-\infty, i]$ for $i = 1, 2, \cdots$ Then $\Lambda_i \uparrow \Gamma$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = \alpha$. Furthermore, we have $\Lambda_i^c \downarrow \emptyset$ and $\lim_{i\to\infty} \mathcal{M}\{\Lambda_i^c\} = 1 - \alpha$.

1.4 Uncertainty Space

Definition 1.6 *(Liu [122]) Let* Γ *be a nonempty set, let* $\mathcal{L}$ *be a* σ*-algebra over* Γ*, and let* $\mathcal{M}$ *be an uncertain measure. Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space.*

For practical purposes, the study of uncertainty spaces is sometimes restricted to complete uncertainty spaces.

Definition 1.7 *An uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called complete if for any* $\Lambda_1, \Lambda_2 \in \mathcal{L}$ *with* $\mathcal{M}\{\Lambda_1\} = \mathcal{M}\{\Lambda_2\}$ *and any subset* A *with* $\Lambda_1 \subset A \subset \Lambda_2$*, one has* $A \in \mathcal{L}$*. In this case, we also have*

$$\mathcal{M}\{A\} = \mathcal{M}\{\Lambda_1\} = \mathcal{M}\{\Lambda_2\}. \tag{1.26}$$

Exercise 1.4: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be a complete uncertainty space, and let Λ be an event with $\mathcal{M}\{\Lambda\} = 0$. Show that A is an event and $\mathcal{M}\{A\} = 0$ whenever $A \subset \Lambda$.

Exercise 1.5: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be a complete uncertainty space, and let Λ be an event with $\mathcal{M}\{\Lambda\} = 1$. Show that A is an event and $\mathcal{M}\{A\} = 1$ whenever $A \supset \Lambda$.

Definition 1.8 *(Gao [48]) An uncertainty space* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called continuous if for any events* $\Lambda_1, \Lambda_2, \cdots$*, we have*

$$\mathcal{M}\left\{\lim_{i\to\infty} \Lambda_i\right\} = \lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} \tag{1.27}$$

provided that $\lim_{i\to\infty} \Lambda_i$ exists.

Exercise 1.6: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be a continuous uncertainty space. For any events $\Lambda_1, \Lambda_2, \cdots$, show that

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 1, \quad \text{if } \Lambda_i \uparrow \Gamma, \tag{1.28}$$

$$\lim_{i\to\infty} \mathcal{M}\{\Lambda_i\} = 0, \quad \text{if } \Lambda_i \downarrow \emptyset. \tag{1.29}$$

1.5 Product Uncertain Measure

Product uncertain measure was defined by Liu [125] in 2009, thus producing the fourth axiom of uncertainty theory. Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ Write

$$\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots \tag{1.30}$$

that is the set of all ordered tuples of the form $(\gamma_1, \gamma_2, \cdots)$, where $\gamma_k \in \Gamma_k$ for $k = 1, 2, \cdots$ A measurable rectangle in Γ is a set

$$\Lambda = \Lambda_1 \times \Lambda_2 \times \cdots \tag{1.31}$$

where $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots$ The smallest σ-algebra containing all measurable rectangles of Γ is called the product σ-algebra, denoted by

$$\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots \tag{1.32}$$

Then the product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}$ is defined by the following product axiom (Liu [125]).

Axiom 4. *(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \cdots$ The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{1.33}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \cdots$, respectively.

Remark 1.6: Note that (1.33) defines a product uncertain measure only for rectangles. How do we extend the uncertain measure $\mathcal{M}$ from the class of rectangles to the product σ-algebra $\mathcal{L}$? For each event $\Lambda \in \mathcal{L}$, we have

$$\mathcal{M}\{\Lambda\} = \begin{cases} \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 1 - \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}, \\ \qquad \text{if} \displaystyle\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} > 0.5 \\ 0.5, \qquad \text{otherwise.} \end{cases} \tag{1.34}$$

Remark 1.7: Note that the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is always less than or equal to 1, i.e.,

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \le 1.$$

This means that at most one of

$$\sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\} \quad \text{and} \quad \sup_{\Lambda_1 \times \Lambda_2 \times \cdots \subset \Lambda^c} \min_{1 \le k < \infty} \mathcal{M}_k\{\Lambda_k\}$$

is greater than 0.5. Thus the expression (1.34) is reasonable.

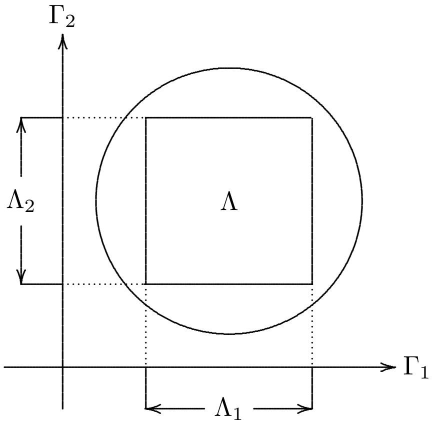

Figure 1.1: Extension from Rectangles to Product σ-Algebra. The uncertain measure of Λ (the disk) is essentially the acreage of its inscribed rectangle $\Lambda_1 \times \Lambda_2$ if it is greater than 0.5. Otherwise, we have to examine its complement Λ^c. If the inscribed rectangle of Λ^c is greater than 0.5, then $\mathcal{M}\{\Lambda^c\}$ is just its inscribed rectangle and $\mathcal{M}\{\Lambda\} = 1 - \mathcal{M}\{\Lambda^c\}$. If there does not exist an inscribed rectangle of Λ or Λ^c greater than 0.5, then we set $\mathcal{M}\{\Lambda\} = 0.5$. Reprinted from Liu [129].

Remark 1.8: If the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is just 1, i.e.,

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\} + \sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\} = 1,$$

then the product uncertain measure (1.34) is simplified as

$$\mathcal{M}\{\Lambda\} = \sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}. \tag{1.35}$$

Theorem 1.6 *(Peng and Iwamura [185]) The product uncertain measure defined by (1.34) is an uncertain measure.*

Proof: In order to prove that the product uncertain measure (1.34) is indeed an uncertain measure, we should verify that the product uncertain measure satisfies the normality, duality and subadditivity axioms.

Step 1: The product uncertain measure is clearly normal, i.e., $\mathcal{M}\{\Gamma\} = 1$.

Step 2: We prove the duality, i.e., $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\} > 0.5.$$

Then we immediately have

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}<0.5.$$

It follows from (1.34) that

$$\mathcal{M}\{\Lambda\}=\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\},$$

$$\mathcal{M}\{\Lambda^c\}=1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset(\Lambda^c)^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}=1-\mathcal{M}\{\Lambda\}.$$

The duality is proved. Case 2: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

This case may be proved by a similar process. Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5$$

and

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5.$$

It follows from (1.34) that $\mathcal{M}\{\Lambda\}=\mathcal{M}\{\Lambda^c\}=0.5$ which proves the duality.

Step 3: Let us prove that $\mathcal{M}$ is an increasing set function. Suppose Λ and Δ are two events in $\mathcal{L}$ with $\Lambda\subset\Delta$. The argument breaks down into three cases. Case 1: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

Then

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\ge\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}>0.5.$$

It follows from (1.34) that $\mathcal{M}\{\Lambda\}\le\mathcal{M}\{\Delta\}$. Case 2: Assume

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}>0.5.$$

Then

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\ge\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}>0.5.$$

Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\}&=1-\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\\&\le 1-\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}=\mathcal{M}\{\Delta\}.\end{aligned}$$

Case 3: Assume

$$\sup_{\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda}\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le 0.5$$

and

$$\sup_{\Delta_1\times\Delta_2\times\cdots\subset\Delta^c}\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\le 0.5.$$

Then

$$\mathcal{M}\{\Lambda\}\le 0.5\le 1-\mathcal{M}\{\Delta^c\}=\mathcal{M}\{\Delta\}.$$

Step 4: Finally, we prove the subadditivity of $\mathcal{M}$. For simplicity, we only prove the case of two events Λ and Δ. The argument breaks down into three cases. Case 1: Assume $\mathcal{M}\{\Lambda\}<0.5$ and $\mathcal{M}\{\Delta\}<0.5$. For any given $\varepsilon>0$, there are two rectangles

$$\Lambda_1\times\Lambda_2\times\cdots\subset\Lambda^c,\quad \Delta_1\times\Delta_2\times\cdots\subset\Delta^c$$

such that

$$1-\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}\le\mathcal{M}\{\Lambda\}+\varepsilon/2,$$

$$1-\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\le\mathcal{M}\{\Delta\}+\varepsilon/2.$$

Note that

$$(\Lambda_1\cap\Delta_1)\times(\Lambda_2\cap\Delta_2)\times\cdots\subset(\Lambda\cup\Delta)^c.$$

It follows from the duality and subadditivity axioms that

$$\begin{aligned}\mathcal{M}_k\{\Lambda_k\cap\Delta_k\}&=1-\mathcal{M}_k\{(\Lambda_k\cap\Delta_k)^c\}=1-\mathcal{M}_k\{\Lambda_k^c\cup\Delta_k^c\}\\&\ge 1-(\mathcal{M}_k\{\Lambda_k^c\}+\mathcal{M}_k\{\Delta_k^c\})\\&=1-(1-\mathcal{M}_k\{\Lambda_k\})-(1-\mathcal{M}_k\{\Delta_k\})\\&=\mathcal{M}_k\{\Lambda_k\}+\mathcal{M}_k\{\Delta_k\}-1\end{aligned}$$

for any k. Thus

$$\begin{aligned}\mathcal{M}\{\Lambda\cup\Delta\}&\le 1-\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\cap\Delta_k\}\\&\le 1-\min_{1\le k<\infty}\mathcal{M}_k\{\Lambda_k\}+1-\min_{1\le k<\infty}\mathcal{M}_k\{\Delta_k\}\\&\le\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}+\varepsilon.\end{aligned}$$

Letting $\varepsilon\to 0$, we obtain

$$\mathcal{M}\{\Lambda\cup\Delta\}\le\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Delta\}.$$

Case 2: Assume $\mathcal{M}\{\Lambda\}\ge 0.5$ and $\mathcal{M}\{\Delta\}<0.5$. When $\mathcal{M}\{\Lambda\cup\Delta\}=0.5$, the subadditivity is obvious. Now we consider the case $\mathcal{M}\{\Lambda\cup\Delta\}>0.5$, i.e., $\mathcal{M}\{\Lambda^c\cap\Delta^c\}<0.5$. By using $\Lambda^c\cup\Delta=(\Lambda^c\cap\Delta^c)\cup\Delta$ and Case 1, we get

$$\mathcal{M}\{\Lambda^c\cup\Delta\}\le\mathcal{M}\{\Lambda^c\cap\Delta^c\}+\mathcal{M}\{\Delta\}.$$

Thus

$$\mathcal{M}\{\Lambda \cup \Delta\} = 1 - \mathcal{M}\{\Lambda^c \cap \Delta^c\} \le 1 - \mathcal{M}\{\Lambda^c \cup \Delta\} + \mathcal{M}\{\Delta\}$$
$$\le 1 - \mathcal{M}\{\Lambda^c\} + \mathcal{M}\{\Delta\} = \mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\}.$$

Case 3: If both $\mathcal{M}\{\Lambda\} \ge 0.5$ and $\mathcal{M}\{\Delta\} \ge 0.5$, then the subadditivity is obvious because $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Delta\} \ge 1$. The theorem is proved.

Definition 1.9 *Assume* $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ *are uncertainty spaces for* $k = 1, 2, \cdots$ *Let* $\Gamma = \Gamma_1 \times \Gamma_2 \times \cdots$, $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ *and* $\mathcal{M} = \mathcal{M}_1 \wedge \mathcal{M}_2 \wedge \cdots$ *Then the triplet* $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called a product uncertainty space.*

1.6 Independence

Definition 1.10 *(Liu [129]) The events* $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ *are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\} \tag{1.36}$$

where Λ_i^* *are arbitrarily chosen from* $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \cdots, n$, *respectively, and* Γ *is the sure event.*

Remark 1.9: Especially, two events Λ_1 and Λ_2 are independent if and only if

$$\mathcal{M}\{\Lambda_1^* \cap \Lambda_2^*\} = \mathcal{M}\{\Lambda_1^*\} \wedge \mathcal{M}\{\Lambda_2^*\} \tag{1.37}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c\}$, $i = 1, 2$, respectively. That is, the following four equations hold:

$$\mathcal{M}\{\Lambda_1 \cap \Lambda_2\} = \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2\},$$
$$\mathcal{M}\{\Lambda_1^c \cap \Lambda_2\} = \mathcal{M}\{\Lambda_1^c\} \wedge \mathcal{M}\{\Lambda_2\},$$
$$\mathcal{M}\{\Lambda_1 \cap \Lambda_2^c\} = \mathcal{M}\{\Lambda_1\} \wedge \mathcal{M}\{\Lambda_2^c\},$$
$$\mathcal{M}\{\Lambda_1^c \cap \Lambda_2^c\} = \mathcal{M}\{\Lambda_1^c\} \wedge \mathcal{M}\{\Lambda_2^c\}.$$

Example 1.10: The impossible event $\emptyset$ is independent of any event Λ because $\emptyset^c = \Gamma$ and

$$\mathcal{M}\{\emptyset \cap \Lambda\} = \mathcal{M}\{\emptyset\} = \mathcal{M}\{\emptyset\} \wedge \mathcal{M}\{\Lambda\},$$
$$\mathcal{M}\{\emptyset^c \cap \Lambda\} = \mathcal{M}\{\Lambda\} = \mathcal{M}\{\emptyset^c\} \wedge \mathcal{M}\{\Lambda\},$$
$$\mathcal{M}\{\emptyset \cap \Lambda^c\} = \mathcal{M}\{\emptyset\} = \mathcal{M}\{\emptyset\} \wedge \mathcal{M}\{\Lambda^c\},$$
$$\mathcal{M}\{\emptyset^c \cap \Lambda^c\} = \mathcal{M}\{\Lambda^c\} = \mathcal{M}\{\emptyset^c\} \wedge \mathcal{M}\{\Lambda^c\}.$$

Example 1.11: The sure event Γ is independent of any event Λ because $\Gamma^c = \emptyset$ and

$$\begin{aligned}\mathcal{M}\{\Gamma \cap \Lambda\} &= \mathcal{M}\{\Lambda\} = \mathcal{M}\{\Gamma\} \wedge \mathcal{M}\{\Lambda\},\\ \mathcal{M}\{\Gamma^c \cap \Lambda\} &= \mathcal{M}\{\Gamma^c\} = \mathcal{M}\{\Gamma^c\} \wedge \mathcal{M}\{\Lambda\},\\ \mathcal{M}\{\Gamma \cap \Lambda^c\} &= \mathcal{M}\{\Lambda^c\} = \mathcal{M}\{\Gamma\} \wedge \mathcal{M}\{\Lambda^c\},\\ \mathcal{M}\{\Gamma^c \cap \Lambda^c\} &= \mathcal{M}\{\Gamma^c\} = \mathcal{M}\{\Gamma^c\} \wedge \mathcal{M}\{\Lambda^c\}.\end{aligned}$$

Example 1.12: Generally speaking, an event Λ is not independent of itself because

$$\mathcal{M}\{\Lambda \cap \Lambda^c\} \neq \mathcal{M}\{\Lambda\} \wedge \mathcal{M}\{\Lambda^c\}$$

whenever $\mathcal{M}\{\Lambda\}$ is neither 1 nor 0.

Theorem 1.7 *(Liu [129]) The events $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^*\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\} \tag{1.38}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \cdots, n$, respectively, and $\emptyset$ is the impossible event.

Proof: Assume $\Lambda_1, \Lambda_2, \cdots, \Lambda_n$ are independent events. It follows from the duality of uncertain measure that

$$\mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^{*c}\right\} = 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^{*c}\} = \bigvee_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\}$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \emptyset\}$, $i = 1, 2, \cdots, n$, respectively. The equation (1.38) is proved. Conversely, if the equation (1.38) holds, then

$$\mathcal{M}\left\{\bigcap_{i=1}^{n} \Lambda_i^*\right\} = 1 - \mathcal{M}\left\{\bigcup_{i=1}^{n} \Lambda_i^{*c}\right\} = 1 - \bigvee_{i=1}^{n} \mathcal{M}\{\Lambda_i^{*c}\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\Lambda_i^*\}.$$

where Λ_i^* are arbitrarily chosen from $\{\Lambda_i, \Lambda_i^c, \Gamma\}$, $i = 1, 2, \cdots, n$, respectively. The equation (1.36) is true. The theorem is proved.

Theorem 1.8 *(Liu [137]) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces and $\Lambda_k \in \mathcal{L}_k$ for $k = 1, 2, \cdots, n$. Then the events*

$$\Gamma_1 \times \cdots \times \Gamma_{k-1} \times \Lambda_k \times \Gamma_{k+1} \times \cdots \times \Gamma_n, \quad k = 1, 2, \cdots, n \tag{1.39}$$

are always independent in the product uncertainty space. That is, the events

$$\Lambda_1,\ \Lambda_2,\ \cdots,\Lambda_n \tag{1.40}$$

are always independent if they are from different uncertainty spaces.

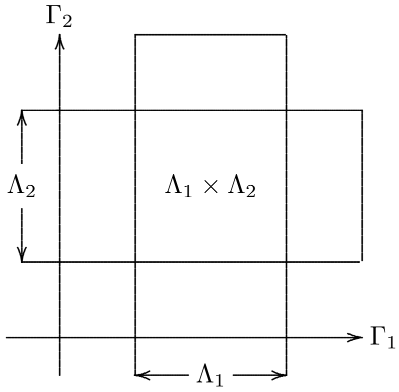

Figure 1.2: $(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2) = \Lambda_1 \times \Lambda_2$

Proof: For simplicity, we only prove the case of $n = 2$. It follows from the product axiom that the product uncertain measure of the intersection is

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{\Lambda_1 \times \Lambda_2\} = \mathcal{M}_1\{\Lambda_1\} \wedge \mathcal{M}_2\{\Lambda_2\}.$$

By using $\mathcal{M}\{\Lambda_1 \times \Gamma_2\} = \mathcal{M}_1\{\Lambda_1\}$ and $\mathcal{M}\{\Gamma_1 \times \Lambda_2\} = \mathcal{M}_2\{\Lambda_2\}$, we obtain

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \wedge \mathcal{M}\{\Gamma_1 \times \Lambda_2\}.$$

Similarly, we may prove that

$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)\} = \mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c\} \wedge \mathcal{M}\{\Gamma_1 \times \Lambda_2\},$$
$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2) \cap (\Gamma_1 \times \Lambda_2)^c\} = \mathcal{M}\{\Lambda_1 \times \Gamma_2\} \wedge \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\},$$
$$\mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c \cap (\Gamma_1 \times \Lambda_2)^c\} = \mathcal{M}\{(\Lambda_1 \times \Gamma_2)^c\} \wedge \mathcal{M}\{(\Gamma_1 \times \Lambda_2)^c\}.$$

Thus $\Lambda_1 \times \Gamma_2$ and $\Gamma_1 \times \Lambda_2$ are independent events. Furthermore, since Λ_1 and Λ_2 are understood as $\Lambda_1 \times \Gamma_2$ and $\Gamma_1 \times \Lambda_2$ in the product uncertainty space, respectively, the two events Λ_1 and Λ_2 are also independent.

1.7 Polyrectangular Theorem

Let $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ and $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ be two uncertainty spaces, $\Lambda_1 \in \mathcal{L}_1$ and $\Lambda_2 \in \mathcal{L}_2$. It follows from the product axiom that the rectangle $\Lambda_1 \times \Lambda_2$ has an uncertain measure

$$\mathcal{M}\{\Lambda_1 \times \Lambda_2\} = \mathcal{M}_1\{\Lambda_1\} \wedge \mathcal{M}_2\{\Lambda_2\}. \tag{1.41}$$

This section will extend this result to a more general case.

Definition 1.11 *(Liu [137]) Let* $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ *and* $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ *be two uncertainty spaces. A set on* $\Gamma_1 \times \Gamma_2$ *is called a polyrectangle if it has the form*

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i}) \tag{1.42}$$

where $\Lambda_{1i} \in \mathcal{L}_1$ *and* $\Lambda_{2i} \in \mathcal{L}_2$ *for* $i = 1, 2, \cdots, m$, *and*

$$\Lambda_{11} \subset \Lambda_{12} \subset \cdots \subset \Lambda_{1m}, \tag{1.43}$$

$$\Lambda_{21} \supset \Lambda_{22} \supset \cdots \supset \Lambda_{2m}. \tag{1.44}$$

A rectangle $\Lambda_1 \times \Lambda_2$ is clearly a polyrectangle. In addition, a "cross"-like set is also a polyrectangle. See Figure 1.3.

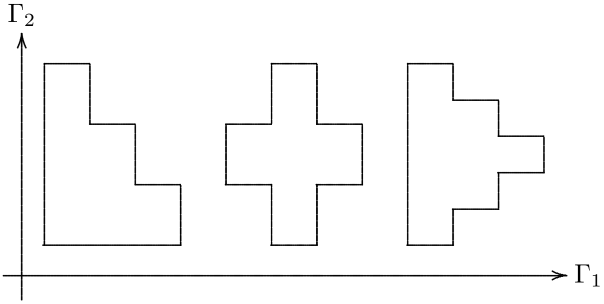

Figure 1.3: Three Polyrectangles

Theorem 1.9 *(Liu [137], Polyrectangular Theorem) Let* $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1)$ *and* $(\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ *be two uncertainty spaces. Then the polyrectangle*

$$\Lambda = \bigcup_{i=1}^{m} (\Lambda_{1i} \times \Lambda_{2i}) \tag{1.45}$$

on the product uncertainty space $(\Gamma_1, \mathcal{L}_1, \mathcal{M}_1) \times (\Gamma_2, \mathcal{L}_2, \mathcal{M}_2)$ *has an uncertain measure*

$$\mathcal{M}\{\Lambda\} = \bigvee_{i=1}^{m} \mathcal{M}_1\{\Lambda_{1i}\} \wedge \mathcal{M}_2\{\Lambda_{2i}\}. \tag{1.46}$$

Proof: It is clear that the maximum rectangle in the polyrectangle Λ is one of $\Lambda_{1i} \times \Lambda_{2i}$, $i = 1, 2, \cdots, n$. Denote the maximum rectangle by $\Lambda_{1k} \times \Lambda_{2k}$. Case I: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_1\{\Lambda_{1k}\},$$

then the maximum rectangle in Λ^c is $\Lambda_{1k}^c \times \Lambda_{2,k+1}^c$, and

$$\mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = \mathcal{M}_1\{\Lambda_{1k}^c\} = 1 - \mathcal{M}_1\{\Lambda_{1k}\}.$$

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1k}^c \times \Lambda_{2,k+1}^c\} = 1.$$

Case II: If

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} = \mathcal{M}_2\{\Lambda_{2k}\},$$

then the maximum rectangle in Λ^c is $\Lambda_{1,k-1}^c \times \Lambda_{2k}^c$, and

$$\mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = \mathcal{M}_2\{\Lambda_{2k}^c\} = 1 - \mathcal{M}_2\{\Lambda_{2k}\}.$$

Thus

$$\mathcal{M}\{\Lambda_{1k} \times \Lambda_{2k}\} + \mathcal{M}\{\Lambda_{1,k-1}^c \times \Lambda_{2k}^c\} = 1.$$

No matter what case happens, the sum of the uncertain measures of the maximum rectangles in Λ and Λ^c is always 1. It follows from the product axiom that (1.46) holds.

Remark 1.10: Note that the polyrectangular theorem is also applicable to the polyrectangles that are unions of infinitely many rectangles. In this case, the polyrectangles may become the shapes in Figure 1.4.

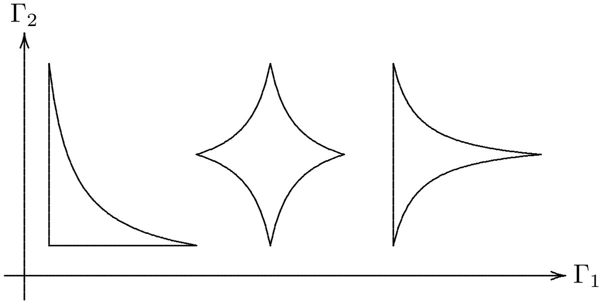

Figure 1.4: Three Deformed Polyrectangles

1.8 Conditional Uncertain Measure

We consider the uncertain measure of an event A after it has been learned that some other event B has occurred. This new uncertain measure of A is called the *conditional uncertain measure* of A given B.

In order to define a conditional uncertain measure $\mathcal{M}\{A|B\}$, at first we have to enlarge $\mathcal{M}\{A \cap B\}$ because $\mathcal{M}\{A \cap B\} < 1$ for all events whenever $\mathcal{M}\{B\} < 1$. It seems that we have no alternative but to divide $\mathcal{M}\{A \cap B\}$ by $\mathcal{M}\{B\}$. Unfortunately, $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ is not always an uncertain measure. However, the value $\mathcal{M}\{A|B\}$ should not be greater than $\mathcal{M}\{A \cap B\}/\mathcal{M}\{B\}$ (otherwise the normality will be lost), i.e.,

$$\mathcal{M}\{A|B\} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \tag{1.47}$$

On the other hand, in order to preserve the duality, we should have

$$\mathcal{M}\{A|B\} = 1 - \mathcal{M}\{A^c|B\} \ge 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}. \tag{1.48}$$

Furthermore, since $(A \cap B) \cup (A^c \cap B) = B$, we have $\mathcal{M}\{B\} \le \mathcal{M}\{A \cap B\} + \mathcal{M}\{A^c \cap B\}$ by using the subadditivity axiom. Thus

$$0 \le 1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \le 1. \tag{1.49}$$

Hence any numbers between $1-\mathcal{M}\{A^c\cap B\}/\mathcal{M}\{B\}$ and $\mathcal{M}\{A\cap B\}/\mathcal{M}\{B\}$ are reasonable values that the conditional uncertain measure may take. Based on the maximum uncertainty principle (Liu [122]), we have the following conditional uncertain measure.

Definition 1.12 *(Liu [122]) Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and* $A, B \in \mathcal{L}$. *Then the conditional uncertain measure of* A *given* B *is defined by*

$$\mathcal{M}\{A|B\} = \begin{cases} \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 1 - \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}}, & \text{if } \dfrac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} < 0.5 \\ 0.5, & \text{otherwise} \end{cases} \tag{1.50}$$

provided that $\mathcal{M}\{B\} > 0$.

Remark 1.11: It follows immediately from the definition of conditional uncertain measure that

$$1 - \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \le \mathcal{M}\{A|B\} \le \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}. \tag{1.51}$$

Furthermore, the conditional uncertain measure obeys the maximum uncertainty principle, and takes values as close to 0.5 as possible.

Remark 1.12: The conditional uncertain measure $\mathcal{M}\{A|B\}$ yields the posterior uncertain measure of A after the occurrence of event B.

Theorem 1.10 *Let* $(\Gamma, \mathcal{L}, \mathcal{M})$ *be an uncertainty space, and let* B *be an event with* $\mathcal{M}\{B\} > 0$. *Then* $\mathcal{M}\{\cdot|B\}$ *defined by (1.50) is an uncertain measure, and* $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ *is an uncertainty space.*

Proof: It is sufficient to prove that $\mathcal{M}\{\cdot|B\}$ satisfies the normality, duality and subadditivity axioms. At first, it satisfies the normality axiom, i.e.,

$$\mathcal{M}\{\Gamma|B\} = 1 - \frac{\mathcal{M}\{\Gamma^c \cap B\}}{\mathcal{M}\{B\}} = 1 - \frac{\mathcal{M}\{\emptyset\}}{\mathcal{M}\{B\}} = 1.$$

For any event A, if

$$\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} \geq 0.5, \quad \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}} \geq 0.5,$$

then we have $\mathcal{M}\{A|B\} + \mathcal{M}\{A^c|B\} = 0.5 + 0.5 = 1$ immediately. Otherwise, without loss of generality, suppose

$$\frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} < 0.5 < \frac{\mathcal{M}\{A^c \cap B\}}{\mathcal{M}\{B\}},$$

then we have

$$\mathcal{M}\{A|B\} + \mathcal{M}\{A^c|B\} = \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}} + \left(1 - \frac{\mathcal{M}\{A \cap B\}}{\mathcal{M}\{B\}}\right) = 1.$$

That is, $\mathcal{M}\{\cdot|B\}$ satisfies the duality axiom. Finally, for any countable sequence $\{A_i\}$ of events, if $\mathcal{M}\{A_i|B\} < 0.5$ for all i, it follows from (1.51) and the subadditivity axiom that

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \mid B\right\} \leq \frac{\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \cap B\right\}}{\mathcal{M}\{B\}} \leq \frac{\sum_{i=1}^{\infty} \mathcal{M}\{A_i \cap B\}}{\mathcal{M}\{B\}} = \sum_{i=1}^{\infty} \mathcal{M}\{A_i|B\}.$$

Suppose there is one term greater than 0.5, say

$$\mathcal{M}\{A_1|B\} \geq 0.5, \quad \mathcal{M}\{A_i|B\} < 0.5, \quad i = 2, 3, \cdots$$

If $\mathcal{M}\{\cup_i A_i|B\} = 0.5$, then we immediately have

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \mid B\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i|B\}.$$

If $\mathcal{M}\{\cup_i A_i|B\} > 0.5$, we may prove the above inequality by the following facts:

$$A_1^c \cap B \subset \bigcup_{i=2}^{\infty} (A_i \cap B) \cup \left(\bigcap_{i=1}^{\infty} A_i^c \cap B\right),$$

$$\mathcal{M}\{A_1^c \cap B\} \leq \sum_{i=2}^{\infty} \mathcal{M}\{A_i \cap B\} + \mathcal{M}\left\{\bigcap_{i=1}^{\infty} A_i^c \cap B\right\},$$

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} A_i \mid B\right\} = 1 - \frac{\mathcal{M}\left\{\bigcap_{i=1}^{\infty} A_i^c \cap B\right\}}{\mathcal{M}\{B\}},$$

$$\sum_{i=1}^{\infty}\mathcal{M}\{A_i|B\} \ge 1 - \frac{\mathcal{M}\{A_1^c \cap B\}}{\mathcal{M}\{B\}} + \frac{\displaystyle\sum_{i=2}^{\infty}\mathcal{M}\{A_i \cap B\}}{\mathcal{M}\{B\}}.$$

If there are at least two terms greater than 0.5, then the subadditivity is clearly true. Thus $\mathcal{M}\{\cdot|B\}$ satisfies the subadditivity axiom. Hence $\mathcal{M}\{\cdot|B\}$ is an uncertain measure. Furthermore, $(\Gamma, \mathcal{L}, \mathcal{M}\{\cdot|B\})$ is an uncertainty space.

1.9 Bibliographic Notes

When no samples are available to estimate a probability distribution, we have to invite some domain experts to evaluate the belief degree that each event will happen. Perhaps some people think that the belief degree is subjective probability or fuzzy concept. However, Liu [131] declared that it is usually inappropriate because both probability theory and fuzzy set theory may lead to counterintuitive results in this case.

In order to rationally deal with belief degrees, uncertainty theory was founded by Liu [122] in 2007 and perfected by Liu [125] in 2009 with the normality axiom, duality axiom, subadditivity axiom, and product axiom of uncertain measure.

Furthermore, uncertain measure was also actively investigated by Gao [48], Liu [129], Zhang [268], Peng and Iwamura [185], and Liu [137], among others. Since then, the tool of uncertain measure was well developed and became a rigorous footstone of uncertainty theory.

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