Chapter 2
Impact of Differing Grammatical Structures in Mathematics Teaching and Learning

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2.1 Introduction

Mathematics is taught in many different languages around the world. In some countries mathematics is taught in only one language, for example in the medium of English in England, but in other countries in two or more languages. In countries such as Tunisia and Papua New Guinea, the language of instruction is dependent on the year level. Multilingualism is increasingly becoming the norm for many communities, and for others has been a way of life for thousands of years. However the fact that a community is multilingual does not imply that every member of the community can speak all the represented languages (Trinick, 2015). Therefore a constant issue in multilingual communities is the choice of what language or languages to use in schools, a choice which is 'closely bound up with issues of access, power and dominance' (Barwell, 2003, p. 37; see also Setati, 2008). For example, a widely held view by early policy makers, generally representing the colonising power, was that linguistic diversity, that is, multilingualism, presented obstacles for national development, while linguistic homogeneity was associated with modernisation and Westernisation (Ricento, 2000). As a result there has been a worldwide trend for

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schooling to be in English or in another of the world's dominant languages such as French, Russian, Spanish or Chinese, both as a direct colonial legacy, and in order to access the dominant discourse.

In more recent times, where populations have attempted to throw off the colonial yoke, in several countries there have been attempts to modernise the indigenous language or elaborate the national language (other than the colonising language) to enable Western school mathematics to be taught (Trinick, 2015). Therefore, mathematics is being taught in a linguistic spectrum from those with a recently created mathematics register such as Māori, to those who have had a formal tradition of teaching mathematics for several centuries, such as French or English. Not only are there differences in linguistic traditions, but there are also contexts in which children from these various linguistic traditions are all learning in the same mathematics classroom in a language which is not their mother tongue.

This chapter concerns itself with issues regarding the different grammatical structures of the languages used in mathematics learning: both languages of instruction and other languages that are used in classrooms. The linguistic term that refers to the particular kind of language used in mathematics is the mathematics register (see Halliday, 1978). This chapter examines features of the mathematics register and the evolution of the mathematics register for a few selected languages. Mathematics is understood to operate in and through language, and different languages offer different resources with which to do this. Building on the earlier work of Barton (2009), this chapter also considers the nature of the mathematics that is made possible by the linguistic expressions of different languages. It discusses linguistic differences that occur in the key mathematical areas of logic, reasoning, space and number, in a general manner and with specific examples from languages from different parts of the world. In most of the cases discussed, the impact of grammatical structures on mathematical teaching and learning has not been empirically investigated. In this chapter, we point out potential impacts and make suggestions for educators and researchers. There remains much scope for future research both on the cognitive impacts of differing grammatical structures and their implications for mathematics education.

The analysis in this chapter is influenced by Whorf's (1956) *linguistic relativity* hypothesis, that is, the idea that the structure of a language can affect the thought processes of speakers of that language. While there are many similarities in how diverse languages have developed their grammars, there are also many remarkable differences. If the forms and constructions of one language do not always have exact counterparts in other languages, this may suggest that the thinking processes of the speakers of one language will differ from those of a speaker of any other language. While there are few modern proponents of 'linguistic determinism' in its strongest form, many linguists have accepted a more moderate linguistic relativity, namely that the ways in which we see the world may be influenced by the kind of language we use (Chandler, 2004). For example, there are certain areas, such as perception of space, where some Whorfian effects have been demonstrated by empirical investigation. Research shows various indigenous Australian language perceptions of space are incongruent with spatial descriptions in European languages (see Levinson, 2003; Levinson & Wilkins, 2006; Edmonds-Wathen, 2011).

Culture, language and cognition are intertwined in a complex manner. More general indications of the effect of culture on mathematical learning can be found in Gay and Cole's (1967) a much quoted study of the Kpelle of Liberia. According to Gay and Cole, the Kpelle are proficient in types of mathematical reasoning for which they have cultural uses, such as estimation, but less proficient in other areas for which they do not have cultural needs (see Austin & Howson, 1979). However, in this chapter we are more concerned with the impact of language features on mathematical thinking than with the cultural practices that have led to the development of these language features.

Lucy (1992) points out that linguistic relativity effects relate to habitual thought, rather than potential thought. There is evidence that people can perceive and reason in ways that their languages do not facilitate, such as enumerative capacity amongst people who do not have number words (Butterworth, Reeve, Reynolds, & Lloyd, 2008). Not having number words does not mean that they do not have the potential to enumerate. However, it does mean that they are likely to apply different strategies to solve certain problems than those who do have number words (Butterworth, Reeve, & Reynolds, 2011).

The issue for mathematics with which we are concerned is not just whether a certain concept can be expressed in a certain language, but the ease of expression of the concept: that is, how the grammatical structure facilitates or impedes this expression. Barton (2012) suggests that 'we bring mathematics into existence by talking about it, and the way we talk about it changes the questions we can ask' (p. 227). Becoming aware of differences between languages can help teachers and learners avoid confusion as well as enrich the learning environment. Although the chapter describes both limitations and facilitations of individual language features, there are more limitations discussed because it is when difficulties occur that language differences are investigated as a possible factor. However, there are also pedagogical opportunities to be exploited in the relations between language and mathematics, particularly when there are language-derived alternative ways of approaching aspects of conventional mathematics (Barton, 2009, 2012). These linguistic issues are also important for curriculum development particularly when it is occurring in a language that has not previously had a formal mathematics register.

All of the authors of this chapter have worked in multilingual environments, encountering, and addressing these issues at first hand. Our own language backgrounds and our histories influence how we approach this topic, including preconceptions which we may not be aware of ourselves. Cris is an English-speaking Australian who has taught and researched in remote schools in northern Australia where Australian Indigenous-language-speaking students are taught in English. Tony is a bilingual Māori/English speaker, a lecturer in Māori-medium mathematics initial teacher education and provides professional learning support to Māori-medium schools. His main area of research is the complex relationship between language and learning mathematics including the mathematics register. Viviane is a French-speaking teacher and researcher who has worked for many years with PhD students from the francophone area of Africa, such as Tunisia and Cameroon.

2.2 The Mathematics Register

A significant body of research examining language issues in the learning and teaching of mathematics in schools has recognised that language use in school differs in some important general ways from language use outside of school and, moreover, those subjects such as mathematics are characterised by specific registers (see Halliday, 1978; Halliday & Hasan, 1985). Discussion on the features and definitions of the *mathematics register* can be traced back to studies on register theory and the much broader field of Systematic Functional Linguistics (SFL), sometimes known as Hallidayan linguistics (Schleppegrell, 2004). Essentially, the four theoretical claims of SFL are (1) that language is functional, (2) two of the functions are to make meaning, and to develop and maintain relationships, (3) these functions are influenced by the social and cultural contexts in which the interactions occur and (4) the process of using language is a semiotic process, the process of making meanings by choice (Eggins, 2004). According to Halliday (1996), the term *register* refers to specific lexical and grammatical choices made by speakers, with varying degrees of consciousness, depending on the situational context, the participants in the conversation and the function of the language in the discourse.

Although researchers have long recognised the vital role that language plays in learning and teaching (Aiken, 1972), it was not until at least the 1970s that they began to highlight its importance in the process of acquiring mathematical knowledge and skills (e.g. Cocking & Mestre, 1988; Mousley & Marks, 1991; Pimm, 1987). Similarly, interest in the problems of mathematics learners whose first language differs from the language of instruction was also brought to the fore in the early 1970s, particularly by the work of Halliday (1975). He addressed language difference and distance as instructional obstacles and described a *register of mathematics*, which to this day is considered definitive in discussions about language and mathematics (Schleppegrell, 2007). In a subsequent publication, Halliday (1978) extended his description of the mathematics register, highlighting that the kind of mathematics that students need to develop through schooling uses language in new ways to serve new functions. This is not just a question of learning new words, but also new 'styles of meaning and modes of argument […] and of combining existing elements into new combinations' (p. 196). Halliday defined the mathematics register as:

> a set of meanings that is appropriate to a particular function of language, together with the words and structures that express these meanings. We can refer to 'mathematics register', in the sense of the meanings that belong to the language of mathematics (the mathematical use of natural language, that is: not mathematics itself), and that a language must express if it is being used for mathematical purposes. (p. 195)

The primary motivation to consider the features of the mathematical register has its roots in research considering issues to do with the language of the learner, aspects of the register that are challenging for learners, and the relationship between thought and language (Trinick, 2015). Interest in the relationship between language and the development of thinking is not new and has been studied by many psychologists, for example Bruner (1966) and Vygotsky (1978). Although not mathematically oriented, the work of theorists such as the linguist Whorf (1956), who suggested that language affects habitual thought, also influenced Halliday and his theories on the register.

Collectively this work influenced researchers examining mathematics learning in a second language, for example, the research work of Cuevas (1984) and Spanos, Rhodes, Dale, and Crandall (1988). Concerned with the considerable underachievement of Hispanic second language learners in the United States, a group of language educators including Spanos et al. (1988) categorised the linguistic features of mathematical problem-solving. To support the development of a framework to examine the language of mathematical problems, they resurrected and mathematised a model first proposed by Morris (1955) in his seminal work in semiotics and adopted by Carnap (1955) to categorise the linguistic features of particular scientific domains (Spanos et al., 1988). The Morris (1938, 1955) model distinguished between the following three linguistic categories:

1. Syntactics, the study of how linguistic signs, or symbols, behave in relation to each other.
2. Semantics, the study of how linguistic signs behave in relation to the objects or concepts they refer to or their senses or how 'meaning' is conveyed through signs and language (Halliday, 1978).
3. Pragmatics, the study of how linguistic signs are used and interpreted by speakers (Spanos et al., 1988) and the study of how context affects meaning (Leech, 1983).

It is important to note that the terms such as semantics and pragmatics have contested definitions and meanings and are a study in their own right (see Levinson, 1993). According to Da Costa (1997), it is necessary to take account of these three aspects for a proper understanding of mathematical logic.

2.2.1 Register Development: Modern European Languages

The language of modern mathematics is part of the continuum of technical language development that had began in Europe by at least the seventeenth century. Modern mathematics has also drawn heavily on the ancient language stocks in Europe, Asia Minor and North Africa, and may be said to represent the cumulative technical language development of diverse peoples over thousands of years (Closs, 1977). The evolution of mathematics was/is also the evolution of the grammatical resources of the natural languages by which Western mathematics came to be constructed (Halliday & Martin, 1993).

Inevitably, this involved the introduction of ways of referring to new objects or new properties, processes, functions and relations. Halliday (1978) suggested that the most typical procedure in contemporary European languages for the creation of new technical terms was to create new words out of non-native stock, and that these terms are not normally used in everyday situations. For example, some mathematical terms such as *quadrilateral* and *parallelogram* are made up out of Latin and Greek elements, even if the actual word did not exist in the original languages. Using these terms also requires using specific grammatical patterns (Schleppegrell, 2007). The mathematics register also imbued existing everyday words with specific mathematical meanings, such as *constant*. Sometimes this reinterpretation of existing words changed their grammatical category and function; for example numbers in ordinary English function somewhat as adjectives, but in mathematics discourse can serve as nouns (Pimm, 1987).

A particularly notable feature of the mathematics register is *nominalisation*, in which processes, originally verbs, become packed into noun phrases (Halliday, 2004). These dense noun phrases are then used in complex relational clauses (Schleppegrell, 2007). One of the consequences of this is that it reconstructs processes as objects. Nominalisation both concretises, though turning processes into things, and is part of what is considered in mathematics education to be abstraction. A *concept* in mathematics is an abstract noun. This is worth noting because different languages place different emphases on the roles of nouns and verbs.

2.2.2 Register Development: Multilingual Contexts

More recently, the discussion on the mathematics register has moved from a focus on issues confronting monolingual and bilingual students to multilingual contexts as migrants from different countries move around the world or within countries that have different regional languages seeking employment and or educational opportunities (see Barwell, Barton, & Setati, 2007). Additionally, there has been the rise of minority and indigenous peoples' movements, movements that usually incorporate a strong educational focus as the means to political and economic emancipation (Smith, 1999). These political and educational movements frequently involve the teaching and learning of mathematics in the indigenous language (Barton, Fairhall, & Trinick, 1998; Meaney, 2002).

Indigenous groups attempting to modernise their indigenous language have been confronted by a range of challenges, including linguistic ones. Literature has highlighted the limitations of the lexicons of indigenous languages to express modern Western mathematics. For example, some languages such as Igbo and Yoruba do not have simple word equivalents to 'zero', a concept which plays a central role in most mathematics (Austin & Howson, 1979; Verran, 2001). Yet 'zero' was a late arrival upon the European mathematical scene (Austin & Howson, 1979). It was an invention in the Hindu–Arabic number system to mark empty places in graphical representations of numerals, before evolving into a number. Verran (2001) claims it is not surprising that exclusively oral number systems would not have developed ‘zero’, a number that developed in response to a need in written mathematics.

While many modernising indigenous languages did not initially have the range of terminology necessary to teach Western mathematics, the deficiencies in mathematics vocabulary can be applied at some point in time to all languages, including the English language:

> History repeats itself. Where people are now wondering whether mathematics can ever be adequately taught through the medium of Australian Aboriginal languages, and many are stating the opinion that these languages are “too primitive” and mathematics can only be effectively taught through the medium of English, it is worthwhile to remember that back in the 1500s in England people had to fight hard to be allowed to teach mathematics and other subjects in English. The language of instruction was Latin, and English, which is a creole, was considered inadequate to convey the higher forms of learning. However, it was argued that if the common people could learn in their mother tongue (English) they would learn better and more of them would be able to take advantage of the education offered (Harris, 1980, p. 2).

Over time, English has borrowed words such as ‘cosine’, ‘sine’ and the symbol for zero from other languages (Pimm, 1995). However, while the mathematics register in English has developed incrementally over hundreds of years, from time-to-time borrowing words from other languages, this developmental process has not been similarly accorded to indigenous languages, particularly those that are endangered. From a language planning perspective, Kaplan and Baldauf (1997) contend that all languages have some mechanism for elaboration. Finlayson and Madiba (2002) noted that there is a substantial body of research work that shows that languages will develop through use. According to Cooper (1989), form will always follow function. The post-colonial experience in various African countries has shown that successful language planning and development, while eminently possible, needs to be supported at all levels from government to grassroots (Bamgbose, 1999; Bokamba, 1995). Cumulatively this research suggests languages have the ability to develop a mathematics register to meet the demand of modern mathematics given the functional need.

2.3 Grammatical Systems

While attention to register development in mathematics often focusses on creating or borrowing mathematical terminology, the ease with which this is done depends in part on the grammatical system of the developing language. Languages tend to have both closed and open classes of words. Open classes can be easily added to as needs arise, and in many (although not all) languages include nouns and verbs. Grammatical functions tend to be performed by words in closed classes, which in English and French include pronouns and prepositions. It is much more difficult for languages to add to these classes. Where a language has a mathematical process or function grammatically encoded such additions can be performed with relative ease by a speaker of the language. Where a mathematical function is not encoded in the grammar, the articulation of the function can be more difficult. It can also be difficult for speakers to accept the addition of the function to a language if it involves adding to a closed class.

As well differing in the mathematical functionality in these closed grammatical systems, languages classify mathematical ideas and functions in different grammatical categories. For example, in this section, we look at whether a language classifies numbers as nouns, adjectives, verbs or something of all three.

2.4 Number

This section discusses two aspects of number that differ in the grammatical systems of language. The first is the matter of into which syntactic category a language puts its numbers. The second is the transparency and regularity of the number system, including how well it articulates with the written system. This section does not propose to cover the many different counting systems that exist in the world (e.g. see Zaslavsky, 1979).

First however, is the controversial question of what constitutes a number system. The Amazonian language Pirahã has been described as not even having a word for 'exactly one', but only for 'approximately one' (Gordon, 2004; Frank, Everett, Fedorenko, & Gibson, 2008). There are whole groups of languages which have been characterised as having very few numbers. Writers such as Von Brandenstein (1970) and Blake (1981) stated that no Australian language has a word for a number higher than four, and Dixon (1980) described a typical Australian number system as having the numbers 'one', 'two', possibly 'three' and 'many' (Dixon, 1980). However, there are Australian languages with extensive number systems, such as Anindilyakwa (Stokes, 1982) and Tiwi (McRoberts, 1990). Harris (1982, 1987) warned that characterising Australian languages as non-counting may seriously misrepresent the mathematical systems and abilities of indigenous Australians. Additionally Meaney, Trinick, and Fairhall (2012) maintained that in some cases, indigenous mathematics may have been done in ways that were different to Western cultural norms and thus remain unrecognised by the researchers, who have been predominately European and very few in number, making this research contentious.

Researchers with a positive view of the capacity of indigenous languages to be modernised, such as Bender and Heller (2006), critiqued the earlier work and showed that traditional indigenous mathematics systems have been more than adequate to cope with their traditional cultural demands (Trinick, 2015). In their review of literature of the mathematics concepts of the Native Americans, Schindler and Davison (1985) noted that amongst the different groups there was little functional use for large numbers. Some cultures have not developed extensive number systems because they have not had the need for them, not because they could not do so (Harris, 1987; Lancy, 1983). In those Australian languages which do have small number systems, such as Warlpiri and Iwaidja, the numbers are additive and larger numbers can be created if necessary (Hale, 1975). For example 'four' in Iwaidja is *ngarrkarrk lda ngarrkarrk* 'two and two'. However, this is clearly unwieldy and if a business or educational need for larger numbers arises, it is a matter of either importing the numbers from another language, or of creating new numbers. Mendes (2011) describes the creation of numbers in Kaibi, a language of Brazil, due to the desire of the Kaibi people for their own numbers, rather than the continued use of Portuguese numbers.

We are thus less concerned with whether an individual language has an extensive number system, but are interested in the grammatical features associated with the number system.

2.4.1 Syntactic Category

When we talk about syntactic category with number, we are talking about whether numbers operate as nouns, verbs or adjectives. The term *natural number* implies a relation between numbers and nature (Verran, 2001). This is likely to make one think that there is something natural about how one's own languages express and use numbers. In fact, numbers can operate syntactically in very different ways in different languages. In Yoruba, numbers are nominalised verb phrases that function modally. Verran (2001) says that the Yoruba *Ó rí ajá méta* 'He saw three dogs' would be better translated 'He saw dogmatter in the mode of a group in the mode of three' (p. 69).

Barton (2009) describes some of the variety of roles that number can take in different languages. In English, numbers can not only act as nouns (in much mathematical discourse) or as adjectives (in much everyday discourse) but also form their own grammatical class. In Kankana-ey, a language spoken in the northern Philippines, numbers can act as adjectives. Numbers in Polynesian languages such as Māori are in their own grammatical class, but have more of the nature of verbs. Barton (2009) gives the example of how the Māori request *Homai kia rima nga pene* 'Give me five pens' would be more literally translated as 'Give me, let them be fiv-ing, the pens' (p. 43). Verbal numbers have also been described for North American languages such as Mi'kmaq, where they must be conjugated according to what is being counted, as well as distinguished for animacy or inanimacy (Lunney Borden, 2010).

The educational implication of the varied syntactic roles of numbers in different languages is that some languages find certain mathematical expressions, or uses of numbers, far more easy to deal with than others (Barton, 2009). Because numbers in English can function in varied ways, but also because they are strongly noun-like, they concord with the way numbers are used in mathematics. In contrast, in the Māori language, numbers are verb-like, and thus the grammar of numerical quantification (as opposed to geometry) are treated as verbal sentences, rather than as nouns, like in English.

The way numbers operate can also vary in *numeral classifier languages* (Allan, 1977), where the way of counting depends on the type of thing to be counted, which might include shape, as in Chinese, or animacy, as in Yucatec Mayan. Things classified differently may not be able to be added together easily, as Lancy (1983) notes of some languages of Papua New Guinea, such as Loboda. This may need to be taken into account in teaching mathematics in these languages.

2.4.2 Transparency and Regularity

Number systems can also vary in the concordance of their names with the base system used, and whether this in turn concords with the symbolic written notation. The base-ten written notation is dominant in the world today, but even some languages that predominantly use a base-ten system have irregularities. For example, the numbers between 11 and 19 are irregular in English, with 'eleven' and 'twelve' hiding their 'one' (and ten) and 'two' (and ten) origins. While French has a regular *dix-neuf* 'ten nine' for 19, 80 is *quatre-vingts* 'four twenties'.

Research that links the transparency and regularity of the number system to better performance in arithmetic calculation can be taken as working within a framework of linguistic relativity. Chinese and Vietnamese both have base-ten number systems that are regular and transparent, such that the spoken number in these languages explicitly corresponds to the base-ten composition of the number, so for example, 14 is said *ten-four*, and 44 as *four-ten*(*s*)-*four* (Miura, Kim, Chang, & Okamoto, 1988; Nguyen & Grégoire, 2011). This transparency has been linked to the ease of acquisition of counting and place value understanding (Geary, Bow-Thomas, Fan, & Siegler, 1993; Nguyen & Grégoire, 2011).

Some languages have complex multibase systems. Yoruba, for instance, uses a primary base of 20, with secondary bases of 10 and five. It also uses subtraction more than addition, so that 47 can be decomposed as $(-3-10+(20\times 3))$ (Verran, 2001). There are multiple ways of deriving large numbers; Verran lists seven ways of deriving 19,669. While this system would be very complicated to write, and particularly to take account of multiple representations of large numbers, it facilitates mental computation. Verran claims that

> a Yoruba numerator, with a well-honed memory of factorial relations, would scorn the cumbersome graphic processes that must be adopted to remember where you are when calculating with a base-ten system. For a reckoner skilled in the Yoruba system, writing things down would constitute a significant interference in working the system. (p. 64)

As seen from the examples above, how well a number system fits with the requirements of mathematics education can depend on syntactic category and the regularity of the system. While these things can be and are at times modified for educational or other goals, attention must also be paid to what could be lost, such as the ease of mental calculation in Yoruba, or a dynamic world view that prioritises process (numbers as verbs) over objectivity (numbers are nouns).

2.5 Logic and Reasoning

According to Hunter (1990), the issue of whether or not logic, an underpinning of mathematical behaviour, is governed by language, was first raised by Whorf (1956). Logical connectives are one of the resources that a language uses to link and sequence ideas. As well as ‘if’ and ‘then,’ additional connectors include ‘because, for example, but, either, or’. When students read problems they must be able to recognise logical connectors and what situation they signal (Dale & Cuevas, 1987). These situations include similarity, contradiction, cause and effect, and logical sequence. Dawe (1983) found that the knowledge of logical connectives in the language of instruction was the most important variable on a test of deductive reasoning for bilinguals from four different countries. Logical connectives tend to be a closed grammatical class in a language. Hence the presence or absence of particular logical connectives in a language can facilitate or impede reasoning.

Gay and Cole’s (1967) famous study of mathematical reasoning among the Kpelle people of Liberia found that the Kpelle performed better on tests of logical disjunction than English-speaking US college students, but performed less proficiently on tests of implication. Gale and Cole attributed these differences between groups to differences in the class of logical operators in the Kpelle and English languages: ‘the precision of the Kpelle language with respect to disjunction aids them with this task’ (p. 82). Kpelle has words for both ‘inclusive or’ and ‘exclusive or’, whereas everyday English has one word that includes both concepts. On the other hand, there is no easy way to express a condition such as ‘if and only if’ in Kpelle.

On the other hand, Iwaidja speakers of North West Arnhem Land in northern Australia seem to be adopting common conjunctions such as ‘but’ and ‘or’ from English into Iwaidja. Traditionally, Iwaidja had only a single conjunction *lda*, which fulfilled roles such as ‘and’, ‘but’ and ‘or’. A construction such as ‘A or B’ was rendered ‘maybe A and/or/but maybe B’. It now appears that *bad* ‘but’ and *u* ‘or’ tend to be quite common in Iwaidja speech.

The syntax of mathematics is often seen as the language that describes relationships (Carrasquillo & Rodriguez, 1996). Traditional *te reo Māori* (Māori language) already had a great quantity of logical connectives that could be used in mathematical discussion. For example, numbers are related to other numbers by such relations as ‘greater than’ (*nui ake*), ‘less than’ (*iti iho*) and ‘equal to’ (*ōrite ki*). While the Māori language has an abundance of logical connectives, how they are used in mathematics classrooms has implications not only for the learning of mathematics but also for the cultural teaching in the classroom. For example, the word ‘relation’ can be translated as either *whanaunga* or *pānga*. However, both these words are context specific. *Whanaunga* is a generic term applied to kin of both sexes related by marriage, adoption and/or descent. This word implies some human kinship relation. Thus *whanau* terms are inappropriate to use when describing ‘relationships between mathematical objects’ and it is more appropriate to use terms like *pānga* (a connection) or *tūhono* (join), for non-kinship/human relations (Trinick, 1999).

2.5.1 Negation

How a language expresses negation can also affect mathematical reasoning. Kazima's (2007) study of Chichewa students' understandings of the language of probability in English found that their attribution of meanings to words such as 'likely' and 'unlikely' was influenced by the meanings of these words in Chichewa. In Chichewa, 'unlikely' is *zokayikitsa*. 'Likely' is *zosakayikitsa*, the negative of 'unlikely' and so means 'not unlikely'. Modifying 'likely' thus can create double or complex negatives where 'not very likely' translates as 'not very not unlikely'. Kazima suggests that teachers need to be aware of preconceptions about mathematical meanings that students bring to the mathematics classrooms from their home languages. She also points out that students need opportunities to construct for themselves the meanings they need in mathematics classrooms, as opposed to being just presented with definitions, and that this needs to be done through multiple examples of how the words are used in their mathematics lessons.

In the Māori language, negation is a complex phenomenon. For example, mathematical practices, such as quantification and location in time and space, are treated like verbal sentences, hence the term *kāore* 'not' to negate is used. Non-verbal sentences are negated by terms such as *ehara* 'not'. Therefore, a sentence such as 'there are <u>not</u> three in the group' is translated as *<u>kāore</u> e toru kei roto i te rōpū*. A sentence such as 'the group is <u>not</u> big' is translated *<u>ehara</u> te rōpū i te nui*. The teacher needs to be aware that this makes the presentation and discussion of examples of negation more complex than in a language where there is a single construction to negate a statement.

In French, negation presents unexpected mathematical challenges, both for students whose French is not the preferred language and for native speakers, due to the relationship between syntax and semantics. In singular sentences, negation is applied to the verb using 'ne… pas', such that *7 divise 27* '7 divides 27' is negated *7 ne divise pas 27* '7 does not divide 27'. When sentences involve an existential quantifier, applying the negation on the verb does not provide a logical negation. Both the statements *certains nombres entiers sont pairs* 'some integers are even' and *certains nombres entiers ne sont pas pairs* 'some integers are not even' are true, so that while the second sentence is negative (syntax) the truth values are not exchanged (semantics). This is not specific to French; it occurred in ancient Greek, as Aristotle noted in *On Interpretation* (Organon, Book 2). Another problem, more specific to French, appears with sentences involving a universal quantifier. Accordingly, using the French linguistic norm, applying the negation on the verb provides the negation of the sentence: *tous les entiers sont pairs* 'all integers are even' and *tous les entiers ne sont pas pairs* 'all integers are not even' exchange their truth values. However, the substitution of *sont impairs* 'are odd' for *ne sont pas pairs* 'are not even' modifies the meaning of the sentence. *Tous les entiers sont impairs* 'all integers are odd' is a false statement, while using the norm *tous les entiers ne sont pas pairs* 'all integers are not even' is true.

Even for French native speakers, such sentences are ambiguous (Durand-Guerrier & Njomgang-Ngansop, 2009).

In Tunisia, mathematics is taught in Arabic until the end of the *Ecole de base* (Grade 8) and then in French at secondary school. A study by Ben Kilani (2005) showed clearly that these ambiguities were reinforced by the specific linguistic context (Durand-Guerrier & Ben Kilani, 2004). In the Arabic language, when the negation is inside the sentence, its scope is not the sentence but the verb or the predicate, so that in a word-to-word translation, the meaning is changed; for example the statement *tous les entiers ne sont pas pairs* 'all the integers are not even', that according to the norm means 'not all integers are even', will be interpreted as *tous les entiers sont non-pairs* 'all integers are odd'. Ben Kilani's (2005) study showed that for most students, the universal sentences with an internal negation were not interpreted as the negation of the corresponding universal affirmative sentence, but as its *contrary* in Aristotle's sense.

The ongoing research of Njomgang-Ngansop in Cameroon shows that the grammatical structure of negation in Ewondo also differs from French, leading to ambiguities or inadequate interpretation of negative sentences (See Njomgang-Ngansop & Durand-Guerrier, 2011; this volume, Chap. 5).

Hence for formal language in a variety of languages, the logical formalisation of such statements can point out the grammatical difference, and help teachers and advanced students to become aware of such phenomena, and be less susceptible to misinterpretations of the intended meaning.

2.5.2 Formal Semantics

A relationship between syntax and semantics is an ancient discussion, posited by Aristotle, in terms of opposition in *On interpretation* (Organon, Book 2), and concerning the relationship between truth and validity in *Prior Analysis* (Organon, Book 3; see Durand-Guerrier, 2008). The modern introduction of semantics into logic was undertaken by Frege (1984), further developed by Wittgenstein (1921) and Tarski (1944), and then influenced Morris (1938, 1955).

While Tarski (1944) had considered that his semantic definition of truth did not apply to languages allowing self-reference, including natural languages, Montague (1974) argued that there were no theoretical differences between natural and formal languages. He applied Tarski's theoretical model to natural language, and introduced a unifying mathematical theory known as *Montague Grammar* encompassing the syntax and semantics of both kinds of languages (Montague, 1974). This originated formal methods in linguistics such as the Discourse Representative Theory (Kamp, 1981; Kamp & Reyle, 1993). These formal methods can be useful in mathematics education, where natural language and formal language are used concomitantly. Logical analysis can offer conceptual clarification (Quine, 1997), and in multilingual contexts offers a common reference for comparison.

As an example, we discuss briefly here the well-known *Donkey sentences* problem concerning quantification and anaphora. The Donkey sentences owe their name to a famous example in Kamp (1981) of how to represent in predicate calculus, the sentence:

$$\text{Every farmer who owns a donkey beats it.} \tag{1}$$

To formalise this sentence, we must determine what type of quantifier should be used to formalise ‘a donkey’, existential or universal. A common choice would be to represent the ‘a’ in ‘a donkey’ existentially, and that the bounded quantifier ‘every farmer’ introduces an implication, so that (1) is paraphrased by:

For all x, if x is a farmer and there exists y such that y is a donkey and x owns y, then x beats y,

and is formalised as:

$$\forall x\,[[\text{Farmer}\,(x)\ \&\ \exists y\,[\text{Donkey}\,(y)\ \&\ \text{Owns}\ (x,y)]] \Rightarrow \text{Beats}\,(x,y)] \tag{2}$$

However, from a formal point of view, there is a problem, since in (2) ‘y’ is a free variable, while it should be within the scope of the existential quantifier.

An alternative is to represent ‘a’ with a universal quantifier and to formalise (1) as

$$\forall x\,\forall y\,[[\text{Farmer}\,(x)\ \&\ \text{Donkey}\,(y)\ \&\ \text{Owns}\,(x,y)] \Rightarrow \text{Beats}\,(x,y)] \tag{3}$$

The linguistic question is why ‘a’ should be sometimes represented by an existential quantifier, and sometimes by a universal quantifier. Kamp (1981) assumes that it is necessary to modify the language use for representing such sentences. The Discourse Representative Theory has been designed for this purpose, and other theories have since been elaborated (i.e. Abbott, 1999). Discourse Representative Theory is a non-quantificational approach aiming to extend the narrow conception of meaning as truth conditions to a more dynamic notion of meaning relative to context. In particular, it assumes that indefinites (syntax) introduce discourse referents (semantics) remaining in mental representation (pragmatics), which are accessible to anaphoric elements, such as pronouns.

Mathematical discourse at all levels includes natural language, where phenomena such as anaphora are common. In anaphora words take meaning from relationships with other parts of a statement. Because the meaning is not directly contained in the word, anaphora offers the potential for confusion to students. Durand-Guerrier (1996) reports an experiment with students beginning with the university in France given a questionnaire on implication. One of the questions was:

(u_n) is the name of a sequence of real numbers determined by a recursive law of type '$u_{n+1}=f(u_n)$', where f is a continuous function on the set of real numbers.
One has then the following theorem:

$$\text{If the sequence } (u_n) \text{ converges to the real number } L, \\ \text{then } L \text{ is a solution for equation } (\mathrm{E}): \text{'}f(x)=x\text{'} \tag{4}$$

Questions:
What can be said about the convergence of the sequence (u_n) if:

(a) The equation (E) has no solution?
(b) The equation (E) has at least one solution?

What can be said about eventual solutions for the equation (E) if:

(c) The sequence (u_n) converges?
(d) The sequence (u_n) does not converge?

The structure of sentence (4) is more complex than the Donkey sentence previously mentioned. Three variables are needed (sequence, function, limit) and there are various relationships between these objects. Unlike sentence (1) above, the implication is already introduced and the universal quantifier is implicit. The answer to both (b) and (d) is 'one cannot say anything': it is possible that the sequence converges/that the equation has at least a solution, and it is also possible that the sequence does not converge/that the equation has no solution. For (a), we can deduce that the sequence does not converge; and for (c) that the equation has at least a solution.

The first appearance of L in the theorem introduces an anaphoric discourse referent that remains in the mental representation. With (b) and (d), although there is no more referent for L, many students seem to consider that L is a given element as in the following answers to those questions:

(b) 'If equation (E) has at least a solution, this solution might be either L, or not be L. We can't conclude about convergence of this sequence.'
(d) 'If sequence u doesn't converge, then it is possible that there exists one or several solutions to (E), but none of them is L.'

For others, the uniqueness of the limit is moved to uniqueness for the equation:

(b) 'u may converge to only one limit or not at all. Hence there exists a unique limit for u if (E) has a unique solution.'
(b) 'If the equation (E) has a solution, then u converges; if the equation (E) has more than one solution, then u diverges.'

Anaphora contributes to the misunderstanding of the sentence and to difficulties in using it in inferences. Teachers should become aware of such didactical phenomena. Moreover, formalising the sentence (4) with students could open discussion on features which provide clues to both the concept and to the logical connectives and quantifiers involved.

2.6 Space and Geometry

Space is a fundamental part of many areas of mathematics including graphing, geometry, calculus and mechanics, among others (Lean & Clements, 1981). Spatial skills are also used to manage information on the page or in the mind when performing complex computations (Booth & Thomas, 1999; Wheatley, 1998). Spatial visualisation can be used to solve number and fraction problems (Lean & Clements, 1981). Spatial language has also been historically assumed to reflect a natural order of perception of the world (Miller & Johnson-Laird, 1976) and thus to vary less than some other parts of language. Children's development of spatial thought has also been taken to follow a regular trajectory (Piaget and Inhelder 1948/1956). However, linguistic research into spatial frames of reference has revealed more variation in spatial language than was previously thought (Levinson, 2003; Levinson & Wilkins, 2006; Pederson et al., 1998). Piaget and Inhelder (1956) described three main stages in conceptual spatial development: topological, projective and Euclidean. These stages parallel the acquisition of spatial frames of reference in European languages, but the same order of acquisition does not apply to the Mayan languages Tzeltal (Brown & Levinson, 2000) or Tzotzil (De León, 1994). The stages are thus in part language-dependent rather than universal as has been often assumed. Variations in spatial language have consequences for mathematics beyond the clearly spatial fields. A great deal of the operational language of mathematics uses metaphorical extensions of spatial language, so variations in spatial language will affect how mathematical processes can be described. The topics of topological language and spatial frames of references are discussed below.

2.6.1 Spatial Frame of Reference

European languages among others favour the use in small-scale space of an egocentric or *relative* spatial frame of reference, which uses left and right, and front and back that are projected from the speaker's viewpoint (Levinson, 2003). In written mathematics, we constantly use left and right to place and order things on the page or screen. Many Australian languages such as Warlpiri (Laughren, 1978) and Guugu Yimithirr (Levinson, 1997) favour *absolute* frames of reference, using terms for north, south, east and west constantly, including most importantly for mathematics learning, in small-scale space. Speakers of these languages tend not to use the relative left and right. They might approach the written organisation of mathematics differently to speakers of strongly relative languages. An absolute-preferring speech community might consider developing a convention of assigning absolute axes to the page/screen/workspace. If you were to do that in English, north would be assigned to the top of a page, but there are many speech communities with a sun orientation (see O'Grady, 1998) who might prefer to use east. Aymara speakers who construe the past as in front of them, use a word meaning 'front' for the east (Núñez & Cornejo, 2012). This connection of the past with the east is also seen in the Kuuk Thaayorre speakers of Pormpuraaw in northern Australia (Boroditsky & Gaby, 2010), suggesting that a preference for absolute conceptions of space sometimes leads to absolute-oriented conceptions of time.

The Māori spatial concepts are multilayered and are derived from a range of traditions all now merged into one. For example, some direction terms are derived from the concept of the North Island of Aotearoa (New Zealand) being a fish, therefore the head of the fish is 'up' (south) and the tail of the fish 'down' (north). The sky is also referred to as 'up' (north) and the land as 'down' (south). Traditionally many spatial terms were very localised, but one of the consequences of standardising the Māori-medium mathematics language has been to decontextualise spatial terms.

People who speak languages that favour an *intrinsic* frame of reference, like the Australian language Iwaidja, and talk about things in terms of their relation to each other, but not to the speaker or external referent (Edmonds-Wathen, 2011), might organise objects mentally or on the written page differently again. Discussing the implications of the favouring of the intrinsic frame of reference in Mopan, a Mayan language, Danziger (1996) discussed mental rotation activities that are often given as spatial mathematics problems in schools, such as deciding whether pictured dice are the same or not. She suggested that such problems could be solved using the intrinsic frame of reference without mentally rotating.

There is certainly scope for further research on spatial frames of reference in mathematics education, particularly in terms of children's development of spatial language from a cross-linguistic perspective.

2.6.2 Topological Language

Topological information in language includes concepts such as closure, proximity, separation and continuity (Piaget and Inhelder 1948/1956). In English and some other European languages, this is provided predominantly with prepositions such as 'in', 'on', 'at', 'by', 'under', 'behind' and 'in front of'. Doing mathematics, the primary spatial meaning of many of these terms is extended metaphorically. The prepositions are used to indicate the roles of numbers or other operands (Barton, 2009). Some languages do not have this range of prepositions. Iwaidja has one general locative preposition *wuka*, which means 'in', 'on', 'at' or 'by', or 'in the vicinity of'. Iwaidja speakers using English may have trouble differentiating the prepositions, and use the English word ‘where’ in the general way that *wuka* is used in Iwaidja, that is, to mean ‘in’, ‘on’ or ‘at’.

Other languages encode topological information using case-marking rather than separate words. In these languages, the roles are marked on the dependent nouns. Turkish is an example of a case-marking language where topological relationships are marked with a suffix. For example ‘A in B’ is expressed ‘A B-*in*’, where *–in* is not a separate word. Johnston and Slobin (1979) studied order of acquisition of certain spatial terms in English, Italian, Turkish and Serbo-Croatian, finding a similar order for each language but different ages of acquisition which they attributed to linguistic factors such as morphological complexity and, in the case of Turkish, to the fact that these terms are ‘postpositions’ in Turkish, rather than prepositions. There is definitely scope for mathematics education research comparing prepositional and ‘postpositional’ or case-marking languages.

2.7 Suggested Directions for Teachers and Researchers

So how can teachers and researchers in multilingual contexts become aware of grammatical differences among the languages in their context, and what can they do with this awareness? Throughout this chapter we have pointed out potential avenues for future research, particularly in logical and spatial areas. We have pointed out that mathematical terminology may fall into different syntactic categories in different languages. Teachers might consider the general linguistic features of the languages of their students: are relationships between words shown by a fixed word order, or by affixes on keywords? Are the languages verb-rich and productive, or are verbs a closed class? How are logical relations expressed? Are features such as evidentiality (such as whether the speaker has personal evidence for what is being said or not) grammaticalised or optional? These aspects of language can all potentially affect how mathematical ideas are processed and expressed and manipulated in the language. As educators, we want to give our students access to the richness of mathematical discourse. In many of the world’s dominant languages, this discourse is especially marked by the process of nominalisation. Halliday (2004) suggests that the language of science, including, we might imagine, the language of mathematics will ‘back off from its present extremes of nominalisation and grammatical metaphor and go back to being more preoccupied with processes and more tolerant of indeterminacy and flux’ (p. 224). While he states that this would be unlikely to be able to be achieved by design, teachers are in a position to influence the relative status of noun phrases and verb groups in their mathematics classroom. If they continue to accord status to complex nominalisations as evidence of abstraction and higher-order thinking, then these types of language use will continue to be privileged in mathematical discourse.

Teaching mathematics in the medium of Māori has supported the maintenance of cultural knowledge as a functional system that has applicability to everyday functional use. For example, Māori has multiple quantitative pronouns in comparison to

English or French. These terms take into account the hierarchical relationship between the speaker and the listener, for example *tōku hoa* (my friend-higher status) versus *taku hoa* (my friend-equal status), but also the relationship between the listener and other people. In addition, Māori has a special set of plural pronouns that refer to two persons only. *Tāua* (us two) includes the speaker and listener, while *māua* (us two) includes the speaker, another person but excludes the listener. Similarly, *tātau* includes the listener, the speaker, and others, but *mātau* includes the speaker and others, but not the listener. In English, both of these would be translated like *us*. Also, in Māori there is no distinction of case or gender.

Verran (2001) described the disconcertment that she experienced in Nigerian classrooms while observing the practices with number of Yoruba teachers of mathematics and science, who were her students in a teacher education programme. Rather than trying to explain away this disconcertment, she set out to compare and contrast the different 'generalizing logics' of English and Yoruba. Eventually, she says:

> I learned to trust my students' classes and to trust them as teachers and their pupils as learners. Encouraging my students to do science and mathematics lessons in practical ways, bringing to the fore the actual doing of the little rituals of the quantifying with hands, eyes, water, string, and rulers as well as with utterances turned out to be a useful and generative way to deal with the generative tensions between English and Yoruba logics of numbering. (pp. 235–236)

Similarly, in this chapter we have not sought to explain away grammatical differences in languages that influence mathematical learning, but to see how this diversity of expression might be comprehended and utilised by teachers and researchers. Finally, mathematicians might like to investigate more deeply this diversity, which can point to ways to create new mathematics (Barton, 2009).

2.8 Conclusion

The impacts of features of grammatical structures on mathematical thinking are still underresearched. We have shown that languages express mathematical ideas in diverse ways. These different ways of exploring mathematical ideas provide an opportunity to enrich the mathematical experiences of learners in multilingual contexts. They can also introduce ambiguities or misunderstanding between teachers and students and impede the process of mathematical learning. While at times, multilingualism and/or teaching mathematics in the medium of indigenous languages has been considered from deficit perspectives, this chapter considers these challenges as more enabling and enriching. Teaching mathematics in indigenous languages supports the revitalisation and maintenance of the languages, particularly those that are endangered. This revitalisation may or may not involve grammatical changes.

We have also shown that languages such as English and French, with long traditions of developing a mathematics register, nevertheless contain some grammatical features which are not always ideal mathematically. If opportunities arise for linguistic–mathematical innovations in these world languages, language planners might like to consider innovating for features which research shows facilitate mathematics learning.

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