Chapter 2
On the Quantization Problem

2.1 Introduction

In 1925 Max Born and Pascual Jordan set out to give a rigorous mathematical basis to Werner Heisenberg's newly born "matrix mechanics". This led them led to state a quantization rule for monomials; that rule associates to the product $x^r p^s$ the operator

$$\mathrm{Op}_{\mathrm{BJ}}(x^r p^s) = \frac{1}{s+1} \sum_{k=0}^{s} \widehat{p}^{s-k} \widehat{x}^r \widehat{p}^k \tag{2.1}$$

where $\widehat{x}$ and $\widehat{p}$ are operators satisfying the canonical commutation relation $[\widehat{x}, \widehat{p}] = i\hbar$. For historical and technical reasons we do not discuss here, Born and Jordan's rule was quickly superseded by a more general rule proposed by Hermann Weyl. Elaborating on the Fourier inversion formula

$$a(x, p) = \frac{1}{2\pi\hbar} \int e^{\frac{i}{\hbar}(x_0 x + p_0 p)} Fa(x_0, p_0) dp_0 dx_0$$

Weyl defined the operator $\mathrm{Op}_{\mathrm{W}}(a)$ associated to an observable (or "symbol") a by formally replacing x and p by $\widehat{x}$ and $\widehat{p}$ in the formula above:

$$\mathrm{Op}_{\mathrm{W}}(a) = \frac{1}{2\pi\hbar} \int e^{\frac{i}{\hbar}(x_0 \widehat{x} + p_0 \widehat{p})} Fa(x_0, p_0) dp_0 dx_0. \tag{2.2}$$

McCoy showed in [15] Weyl's rule leads to the formula

$$\mathrm{Op}_{\mathrm{W}}(x^r p^s) = \frac{1}{2^s} \sum_{k=0}^{s} \binom{s}{k} \widehat{p}^{s-k} \widehat{x}^r \widehat{p}^k \tag{2.3}$$

which is different from Born and Jordan's rule as soon as $r, s \geq 2$ (they however coincide when $r = s = 1$, leading in both cases to the operator $\frac{1}{2}(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})$).

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The Weyl rule was rediscovered and developed in the 1970s by mathematicians working on the theory of pseudo-differential operators and partial differential equations. It turns out that the Weyl quantization rule is mathematically speaking: is very attractive because of its simplicity; in addition it enjoys a very interesting symmetry property (symplectic covariance; i.e. covariance under linear canonical transformations). It is also intimately related to the Wigner transform, which allows a phase space representation of quantum mechanics. The resulting "Weyl–Wigner" formalism is a well-studied topic in both mathematics and quantum mechanics. So far so good. However, an inconsistency arises when Weyl quantization is used. It comes from the following fact: it is conventional wisdom in physics that the Schrödinger and Heisenberg pictures of quantum mechanics are equivalent (the Schrödinger picture is based on Schrödinger's equation which predicts the time-evolution of the quantum state, and the Heisenberg picture views states as constant in time, and considers the observable to evolve). But, and this has been unnoticed, for this equivalence to hold we *must* use the Born–Jordan scheme, and this because the Heisenberg picture breaks down if we use any other quantization rule. That is, the Schrödinger and Heisenberg pictures are inequivalent if one uses Weyl quantization (or any other ordering rule for that). We are thus left with only one possible conclusion, which might be unwelcome for many physicists: *the right quantization rule for observables is that proposed in 1925 by Born and Jordan.*

From a mathematical point of view, the Born–Jordan pseudo-differential operators are obtained as follows. There are infinitely many ways to associate an operator to a given symbol (or "classical observable") a. For instance, one can use the Kohn–Nirenberg prescription

$$A_{\mathrm{KN}}\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}p(x-y)}a(x,p)\psi(y)d^n p d^n y$$

which is very popular among mathematicians working in the theory of partial differential equations, or in time-frequency analysis. Or one can use the Weyl prescription, which is given in pseudo-differential form by the formula

$$A_{\mathrm{W}}\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}(x-y)p}a(\tfrac{1}{2}(x+y),p)\psi(y)d^n p d^n y;$$

the latter is very popular among physicists for the reasons discussed above. There is also the anti-normal ordering, which we just mention in passing (it is not widely used). And then, there is the so-called Shubin prescription: for every real number τ one associates a pseudo-differential operator A_τ to the symbol a by the formula

$$A_\tau\psi(x) = \left(\tfrac{1}{2\pi\hbar}\right)^n \iint e^{\frac{i}{\hbar}(x-y)p}a(((1-\tau)x+\tau y),p)\psi(y)d^n p d^n y.$$

Obviously, choosing $\tau = 1$ one recovers the Kohn–Nirenberg operator A_{KN}, and choosing $\tau = \frac{1}{2}$ one recovers the Weyl operator A_{W}, so the Shubin operators are just a generalization of known schemes. Now, we *define* the Born–Jordan operator A_{BJ} with symbol a as the average

$$A_{\mathrm{BJ}} = \int_0^1 A_\tau d\tau$$

of all Shubin operators A_τ on the interval [0, 1]; this formula should be interpreted as

$$A_{\mathrm{BJ}}\psi(x) = \int_0^1 A_\tau\psi(x)d\tau$$

for $\psi \in \mathcal{S}(\mathbb{R}^n)$. This definition leads to a completely new pseudo-differential calculus, whose properties are different from those of the operators A_τ (and hence, in particular, from those of the Weyl operator $\widehat{A}_{\mathrm{W}}$). For instance, as opposed to what happens with Weyl or Shubin calculus, it is not obvious that every continuous operator $\mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$ can be represented as a Born–Jordan operator A_{BJ}; the usual argument using Schwartz kernel theorem does not work here (put differently "there might be quantum observables which have no classical analogue"). It also turns out that in Born–Jordan quantization the zero operator can correspond to a non-zero symbol; this particularity raises concerns about the uniqueness of "dequantization"; these matters will be studied in detail.

2.2 The Ordering Problem

Already in the early days of quantum mechanics physicists were confronted with the ordering problem for products of observables (i.e. of symbols, in mathematical language). While it was agreed that the correspondence rule

$$x_j \longrightarrow x_j, \quad p_j \longrightarrow -i\hbar\partial/\partial x_j$$

could be successfully be applied to the position and momentum variables, thus turning the Hamiltonian function

$$H = \sum_{j=1}^n \frac{1}{2m_j} p_j^2 + V(x_1, .., x_n) \tag{2.4}$$

into the partial differential operator

$$\widehat{H} = \sum_{j=1}^n -\frac{\hbar^2}{2m_j}\frac{\partial^2}{\partial x_j^2} + V(x_1, .., x_n) \tag{2.5}$$

it quickly became apparent that these rules lead to ambiguities when applied to more general observables involving products of the variables x_j and p_j. For instance, what should the operator corresponding to the magnetic Hamiltonian

$$H = \sum_{j=1}^{n} \frac{1}{2m_j} \left(p_j - A_j(x_1, .., x_n)\right)^2 + V(x_1, .., x_n) \tag{2.6}$$

be? Even in the simple case of the product $x_j p_j = p_j x_j$ the correspondence rule led to the a priori equally good answers $-i\hbar x_j \partial/\partial x_j$ and $-i\hbar(\partial/\partial x_j)x_j$ which differ by the quantity $i\hbar$; things became even more complicated when one came (empirically) to the conclusion that the right answer should in fact be the "average rule"

$$x_j p_j \longrightarrow -\tfrac{1}{2} i\hbar \left(x_j \tfrac{\partial}{\partial x_j} + \tfrac{\partial}{\partial x_j} x_j\right) \tag{2.7}$$

corresponding to the splitting $x_j p_j = \frac{1}{2}(x_j p_j + p_j x_j)$. To better understand the issue, we have to go back a few years in time, to 1925. That year Heisenberg wrote a seminal paper [13] which defined what we today call "matrix mechanics"; in an attempt to understand Heisenberg's ideas, and to put them on a firm mathematical basis, Born and Jordan [1] wrote a comprehensive paper where they addressed the ordering problem: assume that some quantization process associated to the canonical variables x (position) and p (momentum) two operators $\widehat{x}$ and $\widehat{p}$ satisfying Max Born's canonical commutation rule $\widehat{x}\widehat{p} - \widehat{p}\widehat{x} = i\hbar$. A natural and simple choice (but of course not the only possible one) is to choose the unbounded operators on $\mathbb{R}^n$

$$\widehat{x} = x,\ \widehat{p} = -i\hbar\partial/\partial x_j.$$

What should then the operator $a_{rs}(\widehat{x}, \widehat{p})$ associated to the monomial $a_{rs}(x, p) = x^r p^s$ be? Born and Jordan's answer was

$$a_{rs}(\widehat{x}, \widehat{p}) = \frac{1}{s+1} \sum_{k=0}^{s} \widehat{p}^{s-k} \widehat{x}^r \widehat{p}^k. \tag{2.8}$$

They subsequently addressed the multi-dimensional case in a joint work [2] with Heisenberg himself. In [8] we have analyzed in detail Born and Jordan's argument, and shown that their approach to Heisenberg's matrix mechanics becomes effective if and only if one uses the quantization rule (2.8) for monomials. Born and Jordan's derivation has actually been discussed by many authors (see for instance Fedak and Prentis [10], Castellani [3], Crehan [5]), but to the best of our knowledge none has taken up the logical need for the rule (2.8). Approximately at the same time Hermann Weyl had started to develop his ideas about how to quantize the observables of a physical system, and communicated them to Max Born and Pascual Jordan (see Scholz [17] for a historical account). His basic ideas of a group theoretical approach were published two years later [19, 20]. A very interesting novelty in Weyl's approach was that he proposed to associate to an observable of a physical system what we would call today a pseudo-differential operator in Weyl form. In fact, writing the observable as an inverse Fourier transform

$$a(x, p) = \int e^{i(ps+xt)} Fa(s, t) ds dt \tag{2.9}$$

he defined its operator analogue by the formal substitution $x \longrightarrow \widehat{x}$, $p \longrightarrow \widehat{p}$, which yields

$$a(\widehat{x}, \widehat{p}) = \int e^{i(\widehat{p}s+\widehat{x}t)} Fa(s, t) ds dt; \tag{2.10}$$

this is essentially the modern definition of a pseudo-differential operator in terms of Heisenberg operators. Now, Weyl's theory immediately yields the symmetrized quantization rule

$$a(\widehat{x}, \widehat{p}) = \frac{1}{2}(\widehat{x}\widehat{p} + \widehat{p}\widehat{x})$$

(as does Born Jordan's algebraic constructions) and one finds that more generally (McCoy [15], 1932)

$$a_{rs}(\widehat{x}, \widehat{p}) = \frac{1}{2^s} \sum_{k=0}^{s} \binom{s}{k} \widehat{p}^{s-k} \widehat{x}^r \widehat{p}^k \tag{2.11}$$

for a monomial $a_{rs}(x, p) = x^r p^s$.

We now make an essential observation. It turns out that Weyl's quantization rule (2.11) for monomials is a particular case of the so-called "τ-ordering" introduced by Shubin [18]: for any real number τ one defines the operator

$$a_{rs}^{\tau}(\widehat{x}, \widehat{p}) = \sum_{k=0}^{s} \binom{s}{k} (1-\tau)^k \tau^{s-k} \widehat{p}^{s-k} \widehat{x}^r \widehat{p}^k \tag{2.12}$$

which is identical to Weyl's prescription when one chooses $\tau = \frac{1}{2}$. When $\tau = 0$ one gets the "normal ordering" $\widehat{x}^r \widehat{p}^s$ familiar from the elementary theory of partial differential equations while $\tau = 1$ yields the "anti-normal ordering" $\widehat{p}^s \widehat{x}^r$. We now make the following fundamental observation: the Born–Jordan prescription (2.8) is obtained by averaging (2.12) on the interval [0, 1]. In fact, noting that

$$\int_0^1 (1-\tau)^k \tau^{s-k} d\tau = \frac{k!(s-k)!}{(s+1)!}$$

we get

$$\int_0^1 a_{rs}^{\tau}(\widehat{x}, \widehat{p})d\tau = \frac{1}{s+1} \sum_{k=0}^{s} \widehat{p}^{s-k} \widehat{x}^r \widehat{p}^k \tag{2.13}$$

which is precisely (2.8).

In physics as well as in mathematics, the question of a "good" choice of quantization is more than just academic. For instance, different choices may lead to different spectral properties. The following example is due to Crehan [5]. Consider the Hamiltonian function

$$H(z) = \tfrac{1}{2}(p^2 + x^2) + \lambda(p^2 + x^2)^3.$$

The term that gives an ordering problem is evidently $(p^2 + x^2)^3$; Crehan then shows that the most general quantization invariant under the substitution $(x, p) \longmapsto (p, -x)$ is

$$\widehat{H} = \frac{1}{2}(\widehat{p}^2 + \widehat{x}^2) + \lambda(\widehat{p}^2 + \widehat{x}^2)^3 + \lambda(3\alpha\hbar^2 - 4)(\widehat{p}^2 + \widehat{x}^2).$$

It is easy to see that the eigenfunctions of $\widehat{H}$ are those of the harmonic oscillator $\widehat{H}_0 = \frac{1}{2}(\widehat{p}^2 + \widehat{x}^2)$ (they are thus the Hermite functions) and do not depend on the choices of the parameters λ and α. However the corresponding eigenvalues do: they are the numbers

$$E_N = (N + \tfrac{1}{2})\hbar + \lambda\hbar(2N+1)^3 + \lambda\hbar(2N+1)(3\alpha\hbar^2 - 4)$$

for $N = 0, 1, 2, ...$, which clearly shows the dependence of the spectrum on the parameters α and λ, and hence of the chosen quantization. This example clearly shows that the choice of a quantization is not just an academic problem, but has deep consequences when one looks for the correct spectra in physics. There are more subtle issues associated with the choice of quantization, and these will be discussed later on in this book.

We note that the ordering problem for monomials is still not closed, as witnessed by recent research (see for instance Domingo and Galapon [9]).

2.3 What Is Quantization?

In physics "quantization" refers to a mathematical procedure designed to describe a quantum system using its formulation as a classical system. We have been loosely talking about "quantization" as a process which allows one to associate an operator acting on some function space to a function; the latter is supposed to represent a dynamical variable, for instance energy, or position, or momentum; for a detailed and interesting discussion of the historical development of quantization, see Mehra and Rechenberg's treatise [16]. In the case of monomials the approach seems to be more abstract, because we associate to expressions like $x^r p^s$ a formal product of operators $\widehat{x}$ and $\widehat{p}$. It would therefore be useful to have a solid working mathematical definition of the notion of quantization. Let us immediately note that there is no consensus in the literature about what a "good" definition should be. We are going to give below a definition of quantization which is rather minimalistic, but sufficient for our purposes (and probably also the most reasonable from a physical point of view). But let us first explain what properties a quantization cannot satisfy; this will give us the opportunity of debunking what we called "urban legends" in [8]. The first of these properties is the so-called Dirac rule: any quantization $a \leftrightarrow \mathrm{Op}(a)$ should satisfy the relation

$$[\mathrm{Op}(a), \mathrm{Op}(b)] = i\hbar \mathrm{Op}(\{a, b\}) \tag{2.14}$$

where $\{a, b\}$ is the Poisson bracket of the two observables a, b. It is however well-known (the Groenewold–van Hove theorem, see [11, 12]) that (2.14) cannot hold for polynomials with degree > 2. Kauffmann gives in [14] an excellent analysis of Dirac's correspondence, and in [3] Castellani analyzes the (non-)existence of quantization rules satisfying (2.14). The second quantization rule that cannot be satisfied is von Neumann's condition

$$\mathrm{Op}(a^N) = (\mathrm{Op}(a))^N. \tag{2.15}$$

In fact, Cohen [4] has proven that this condition would prohibit the existence of a quasi-probability distribution $\rho(x, p)$ satisfying the marginal conditions

$$\int \rho(x, p) d^n p = |\psi(x)|^2, \quad \int \rho(x, p) d^n x = |F\psi(p)|^2 \tag{2.16}$$

and the average value formula

$$\langle g(\mathrm{Op}(a))\psi|\psi\rangle = \int g(a)(x, p)\rho(x, p) dp dx. \tag{2.17}$$

This would, among other unwanted consequences, prohibit the existence of the Wigner distribution and of a Weyl type phase space quantum mechanics.

So, now that we know what a quantization cannot be, let us list a few properties we would like a quantization to have.

Let us denote by $\mathrm{Class}(n)$ the vector space of all (real or complex) functions defined on phase space $\mathbb{R}^{2n}$; we do not assume any particular smoothness property for the elements of $\mathrm{Class}(n)$ (which we call "observables", or "symbols"). We will denote by $\mathrm{Quant}(n)$ the complex vector space of all continuous linear operators $\widehat{A} : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}'(\mathbb{R}^n)$. We call *quantization* any linear mapping

$$\mathrm{Op} : \mathrm{Class}(n) \longrightarrow \mathrm{Quant}(n)$$

having the following properties:

- **Triviality axiom**:

$$\operatorname{Op}(1) = I_{\mathrm{d}}, \quad \operatorname{Op}(x_j) = \widehat{x}_j, \quad \operatorname{Op}(p_j) = \widehat{p}_j$$

  (I_{d} the identity operator);
- **Self-adjointness**: if $a = a(x, p)$ is real, then $\operatorname{Op}(a)$ is self-adjoint; more generally:

$$\operatorname{Op}(a^*) = \operatorname{Op}(a)^\dagger$$

  where a^* is the complex conjugate of a.

These two first properties are well-known, and very “reasonable”; the third axiom seems a little bit artificial, but helps maintain a relatively small class of possible quantizations:

- **Reduced Dirac correspondence**:

$$[\widehat{x}_j, \operatorname{Op}(a)] = i\hbar \operatorname{Op}(\{x_j, a\})$$
$$[\widehat{p}_j, \operatorname{Op}(a)] = i\hbar \operatorname{Op}(\{p_j, a\})$$

  for every $a \in \operatorname{Class}(n)$ and $j = 1, ..., n$.

It turns out that, at least as far as monomials or polynomials are concerned, the property above allows one to give very explicit expressions for $\operatorname{Op}(a)$; in particular one can prove the existence of a function f such that $f(0)$ and

$$\operatorname{Op}(x^r p^s) = \sum_{j=0}^{\min(r,s)} f^{(j)}(0) \binom{s}{j} \binom{r}{j} j! \hbar^j \widehat{p}^{s-j} \widehat{x}^{r-j} \tag{2.18}$$

(see Domingo and Galapon [9]). This property makes it easy to connect quantization—in the general case—with the theory of the Cohen classes, which plays an essential role in phase space quantum mechanics (and in its cousin, time-frequency analysis). We will come back to this property in Chap. 3.

A quantization scheme satisfying these three properties is called by some authors a “generalized Weyl correspondence”; we will not use this terminology because it gives the impression that the Weyl correspondence plays a privileged and central role in quantization. While it is true that the Weyl correspondence is in a sense the simplest quantization scheme, and that other quantization schemes can be studied in terms of it, it is not necessarily the best one in physics, as our discussion below will show.

2.4 Motivation for Born–Jordan Quantization

As shortly argued above there are many reasons to believe that the Born–Jordan ordering, which leads to the Born–Jordan pseudo-differential calculus is the correct physical quantization scheme. We have shown in [7, 8] that the equivalence of the Schrödinger and Heisenberg pictures of quantum mechanics (which is taken for granted in quantum physics) requires that the " observables" be quantized using the Born–Jordan rule. In fact, close scrutiny of Born and Jordan's argument shows that their quantization rule (2.13) is not only sufficient, but also *necessary* for their definitions to be mathematically consistent.

In the Schrödinger picture of quantum mechanics (wave mechanics), the operators are constant (unless they are explicitly time-dependent), and the states evolve in time: $\psi(t) = U(t, t_0)\psi(t_0)$ where

$$U(t, t_0) = e^{iH_{\mathcal{S}}(t-t_0)/\hbar} \tag{2.19}$$

is a family of unitary operators (the propagator); the time evolution of ψ is thus governed by Schrödinger's equation

$$i\hbar\frac{\partial\psi}{\partial t} = H_{\mathcal{S}}\psi; \tag{2.20}$$

$H_{\mathcal{S}}$ is an operator associated with the classical Hamiltonian function H by some "quantization rule". In the Heisenberg picture (matrix mechanics), the state vectors are time-independent operators that incorporate a dependency on time, while an observable $A_{\mathcal{S}}$ in the Schrödinger picture becomes a time-dependent operator $A_{\mathcal{H}}(t)$ in the Heisenberg picture; this time dependence satisfies the Heisenberg equation

$$i\hbar\frac{dA_{\mathcal{H}}}{dt} = i\hbar\frac{\partial A_{\mathcal{H}}}{\partial t} + [A_{\mathcal{H}}, H_{\mathcal{H}}]. \tag{2.21}$$

Schrödinger [6] (and, independently, Eckart [5]) attempted to prove shortly after the publication of Heisenberg's result that wave mechanics and matrix mechanics were mathematically equivalent. Both proofs contained flaws, and one had to wait until von Neumann's [7] seminal work for a rigorous proof of the equivalence of both theories. We will not bother with the technical shortcomings of Schrödinger's and Eckart's approaches here, but rather focus on one, perhaps more fundamental, aspect which seems to have been overlooked in the literature. We observe that it is possible to go from the Heisenberg picture to the Schrödinger picture (and back) using the following simple argument: a ket

$$|\psi_{\mathcal{S}}(t)\rangle = U(t, t_0)|\psi_{\mathcal{S}}(t_0)\rangle \tag{2.22}$$

in the Schrödinger picture becomes, in the Heisenberg picture, the constant ket

$$|\psi_{\mathcal{H}}\rangle = U(t, t_0)^{\dagger}|\psi_{\mathcal{S}}(t)\rangle = |\psi_{\mathcal{S}}(t_0)\rangle \tag{2.23}$$

whereas an observable $A_{\mathcal{S}}$ becomes

$$A_{\mathcal{H}}(t) = U(t, t_0)^{\dagger} A_{\mathcal{S}} U(t, t_0); \tag{2.24}$$

in particular the Hamiltonian is

$$H_{\mathcal{H}}(t) = U(t, t_0)^{\dagger} H_{\mathcal{S}} U(t, t_0). \tag{2.25}$$

Taking $t = t_0$ this relation implies that $H_{\mathcal{H}}(t_0) = H_{\mathcal{S}}$; now in the Heisenberg picture energy is constant, so the Hamiltonian operator $H_{\mathcal{H}}(t)$ must be a constant of the motion. It follows that $H_{\mathcal{H}}(t) = H_{\mathcal{S}}$ for all times t and hence both operators $H_{\mathcal{H}}$ and $H_{\mathcal{S}}$ must be quantized using the *same* rules. A consequence of this property is that if we believe that Heisenberg's "matrix mechanics" is correct and is equivalent to Schrödinger's theory, then the Hamiltonian operator appearing in the Schrödinger equation (2.20) *must* be quantized using the Born–Jordan rule, and not, as is usual in quantum mechanics, the Weyl quantization rule.

Now, why should we then choose the Born–Jordan quantization scheme, and not, for instance, the Weyl correspondence? It turns out that Born and Jordan's argument only works if one uses the quantization scheme that they proposed. We have explained this in detail in [8]; for completeness we reproduce here the argument (with some simplifications). A close scrutiny of the arguments in Born and Jordan [1] and its follow-up [2] by Born et al. shows that the key to their approach lies in the differentiation rule for products of non-commuting variables. They actually give two definitions, and prove thereafter that both coincide if and only if one makes an essential assumption on the ordering of the quantization of monomials. The first definition is algebraic: if

$$y = \prod_{m=1}^{s} y_{\ell_m} = y_{\ell_1} y_{\ell_2} \cdots y_{\ell_s} \tag{2.26}$$

is a product of non-commuting variables y_ℓ then, if $k \in \{\ell_1, \ell_2, ..., \ell_s\}$, the derivative of y with respect to y_k is given by what they call a "differential quotient of first type":

$$\left(\frac{\partial y}{\partial y_k}\right)_1 = \sum_{r=1}^{s} \delta_{\ell_r k} \prod_{m=r+1}^{s} y_{\ell_m} \prod_{m=1}^{r-1} y_{\ell_m} \tag{2.27}$$

($\delta_{\ell_r k}$ the Kronecker delta). In words: pick a factor x_k in (2.26) and form the product of all the following factors, and thereafter the product of the preceding factors (in that order). When y is a monomial $\widehat{p}^s \widehat{x}^r$ this rule yields

$$\left(\frac{\partial}{\partial\widehat{p}}(\widehat{p}^s\widehat{x}^r)\right)_1 = \sum_{\ell=0}^{s-1}\widehat{p}^{s-1-\ell}\widehat{x}^r\widehat{p}^\ell \tag{2.28}$$

$$\left(\frac{\partial}{\partial\widehat{x}}(\widehat{p}^s\widehat{x}^r)\right)_1 = \sum_{j=0}^{r-1}\widehat{x}^{r-1-j}\widehat{p}^s\widehat{x}^j. \tag{2.29}$$

The second definition (explicitly given in formula (3) of [2]) is similar to that of an ordinary partial derivative:

$$\left(\frac{\partial y}{\partial y_k}\right)_2 = \lim_{\alpha\to 0}\frac{f(\cdots, y_k+\alpha, \cdots)}{\alpha}. \tag{2.30}$$

With this definition formulas (2.28) and (2.29) become

$$\left(\frac{\partial}{\partial\widehat{p}}(\widehat{p}^s\widehat{x}^r)\right)_2 = s\widehat{p}^{s-1}\widehat{x}^r$$
$$\left(\frac{\partial}{\partial\widehat{x}}(\widehat{p}^s\widehat{x}^r)\right)_2 = r\widehat{p}^s\widehat{x}^{r-1}.$$

Their next step consists in identifying both notions of partial derivative; more specifically they want that the quantization $\widehat{H}$ (still to be defined) of a Hamiltonian function satisfies the equalities

$$\left(\frac{\partial\widehat{H}}{\partial\widehat{p}}\right)_1 = \left(\frac{\partial\widehat{H}}{\partial\widehat{p}}\right)_2, \left(\frac{\partial\widehat{H}}{\partial\widehat{x}}\right)_1 = \left(\frac{\partial\widehat{H}}{\partial\widehat{x}}\right)_2. \tag{2.31}$$

They thereafter show quite explicitly (in the footnote (1) of [2]) that these equations hold if the quantization $\widehat{H}$ of $H = p^s x^r$ is the self-adjoint operator given by

$$\widehat{H} = \frac{1}{r+1}\sum_{j=0}^{r}\widehat{x}^{r-j}\widehat{p}^s\widehat{x}^j = \frac{1}{s+1}\sum_{\ell=0}^{s}\widehat{p}^{s-\ell}\widehat{x}^r\widehat{p}^\ell. \tag{2.32}$$

They do not, however, show that it is the only possibility leading to a self-adjoint operator $\widehat{H}$. This is however the case, as we have shown in [8].

To be complete, let us explain why Born and Jordan needed these constructions. They assumed that the equations of motion for $\widehat{p}$ and $\widehat{x}$ are formally the same as in Hamiltonian mechanics, namely

$$\frac{d\widehat{x}}{dt} = \frac{\partial\widehat{H}}{\partial\widehat{p}}, \quad \frac{d\widehat{p}}{dt} = -\frac{\partial\widehat{H}}{\partial\widehat{x}}. \tag{2.33}$$

Pursuing this classical analogy, they require in addition that the Hamilton equations, written in terms of Poisson brackets

$$\frac{dx}{dt} = \{x, H\}, \quad \frac{dp}{dt} = \{p, H\}$$

should be replaced with the operator relations

$$\frac{d\widehat{x}}{dt} = [\widehat{x}, \widehat{H}], \quad \frac{d\widehat{p}}{dt} = [\widehat{p}, \widehat{H}];$$

to be consistent with the Hamilton equations (2.33) one must thus have

$$[\widehat{x}, \widehat{H}] = i\hbar\frac{\partial \widehat{H}}{\partial \widehat{p}}, \quad [\widehat{p}, \widehat{H}] = i\hbar\frac{\partial \widehat{H}}{\partial \widehat{x}}. \tag{2.34}$$

This last step in Born and Jordan's construction also requires that the operator $\widehat{H}$ must be given by the rule (2.32) above.

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