Chapter 2
Copson-Type Inequalities

Archimedes will be remembered when Aeschylus is forgotten, because languages die and mathematical ideas do not. "Immortality" may be a silly word, but probably a mathematician has the best chance of whatever it may mean.

Godfrey Harold Hardy (1877–1947).

This chapter considers time scale versions of Copson type inequalities and their converses. We prove extensions of Copson type inequalities proved by Walsh on discrete time scales and we also consider converses of these inequalities. This chapter is divided into four sections and is organized as follows. In Sect. 2.1, we prove a time scale version of a Copson inequality and in Sect. 2.2, we consider generalized Copson type inequalities. In Sect. 2.3, we present converses of Copson-type inequalities. Section 2.4 considers extensions of Copson inequalities by adapting the Walsh method.

2.1 Copson-Type Inequalities I

In 1927, Copson [48] proved that if $g(x) > 0, p > 1$ and that $g^p(x)$ is integrable over $(0, \infty)$, then $\int_x^\infty (g(t)/t)dt$ converges if $x > 0$ and

$$\int_0^\infty \left(\int_x^\infty \frac{g(t)}{t}dt\right)^p dx \leq p^p \int_0^\infty g^p(x)dx. \tag{2.1.1}$$

In 1928, Copson [49] (see also [77, Theorem 344]) proved the discrete version of (2.1.1) which is given by

$$\sum_{n=1}^{\infty} \left(\sum_{k=n}^{n} \frac{a(k)}{k}\right)^p \leq p^p \sum_{n=1}^{\infty} a^p(n), \tag{2.1.2}$$

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where $p > 1$ and $a(n) > 0$ for $n \geq 1$. The inequalities switch order when $0 < p < 1$ and the constant p^p is the best possible. Hardy [65] had earlier stated a weak version of (2.1.2) in the case $p = 2$, and as a result (2.1.2) is sometimes called the Copson-Hardy inequality.

In 1927, Copson in [48] extended the inequality (2.1.2) by adapting Elliott's proof [57] and bringing into play the dual of Hardy's inequality, a result now known as the Copson inequality: if $p > 1$, $a(n) > 0$ and $\lambda(n) > 0$, $\Lambda(n) = \sum_{i=1}^{n} \lambda(i)$ and $\sum_{n=1}^{\infty} \lambda(n)a^p(n)$ is convergent then

$$\begin{aligned}\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{i=n}^{\infty} \frac{a(i)\lambda(i)}{\Lambda(i)} \right)^p &\leq p^p \sum_{n=1}^{\infty} \lambda(n)a(n) \left(\sum_{i=n}^{\infty} \frac{a(i)\lambda(i)}{\Lambda(i)} \right)^{p-1} \\ &\leq p^p \sum_{n=1}^{\infty} \lambda(n)a^p(n). \end{aligned} \tag{2.1.3}$$

The constant p^p again is best possible.

In this section, we state and prove two different forms of (2.1.3) on time scales. The results in this section are adapted from [163].

Theorem 2.1.1. *Let $\mathbb{T}$ be a time scale with $a \in [0, \infty)_{\mathbb{T}}$ and $p > 1$. Let $\Lambda(t) := \int_a^t \lambda(s)\Delta s$ and define*

$$\Phi(t) := \int_t^{\infty} \frac{\lambda(s)g(s)}{\Lambda^{\sigma}(s)} \Delta s, \quad \textit{for } t \in [a, \infty)_{\mathbb{T}}. \tag{2.1.4}$$

Then

$$\int_a^{\infty} \lambda(t)(\Phi(t))^p \Delta t \leq p \int_a^{\infty} \Phi^{p-1}(t)\lambda(t)g(t)\Delta t, \tag{2.1.5}$$

and

$$\int_a^{\infty} \lambda(t)(\Phi(t))^p \Delta t \leq p^p \int_a^{\infty} \lambda(t)g^p(t)\Delta t. \tag{2.1.6}$$

Proof. First we prove the inequality (2.1.5). Consider the integral $\int_a^{\infty} \lambda(t)\Phi^p(t)\Delta t$, and integrating by the parts formula (1.1.4) with $v(t) = \Lambda(t)$, and $u(t) = \Phi^p(t)$, we obtain

$$\int_a^{\infty} \lambda(t)(\Phi(t))^p \Delta t = \Lambda(t)\Phi^p(t)|_a^{\infty} + \int_a^{\infty} (\Lambda^{\sigma}(t)) \left(-\Phi^p(t)\right)^{\Delta} \Delta t. \tag{2.1.7}$$

Using $\Phi(\infty) = 0$ and $\Lambda(a) = 0$, we get that

$$\int_a^{\infty} \lambda(t)\Phi^p(t)\Delta t = \int_a^{\infty} (\Lambda^{\sigma}(t)) \left(-\Phi^p(t)\right)^{\Delta} \Delta t. \tag{2.1.8}$$

Applying the chain rule (1.1.5), we see that there exists $d \in [t, \sigma(t)]$ such that

$$-\left(\Phi^{p}(t)\right)^{\Delta} = -p\Phi^{p-1}(d)(\Phi^{\Delta}(t)). \tag{2.1.9}$$

Since $\Phi^{\Delta}(t) = -\lambda(t)g(t)/\Lambda^{\sigma}(t) \leq 0$, and $d \geq t$, we have

$$-\left(\Phi^{p}(t)\right)^{\Delta}\left(\Lambda^{\sigma}(t)\right) \leq p\lambda(t)g(t)(\Phi(t))^{p-1}. \tag{2.1.10}$$

Substituting (2.1.10) into (2.1.8), we have

$$\int_{a}^{\infty} \lambda(t)\Phi^{p}(t)\Delta t \leq p\int_{a}^{\infty} (\Phi(t))^{p-1}\lambda(t)g(t)\Delta t, \tag{2.1.11}$$

which is the desired inequality (2.1.5). The inequality (2.1.11) can be written in the form

$$\int_{a}^{\infty} \lambda(t)\left(\Phi(t)\right)^{p}\Delta t \leq p\int_{a}^{\infty} \frac{\lambda(t)g(t)}{(\lambda(t))^{\frac{p-1}{p}}}\left[(\lambda(t))^{\frac{p-1}{p}}(\Phi(t))^{p-1}\right]\Delta t. \tag{2.1.12}$$

Applying the Hölder inequality (1.1.8) on the term

$$\int_{a}^{\infty}\left[(\lambda^{-1}(t))^{\frac{p-1}{p}}\lambda(t)g(t)\right]\left[(\lambda(t))^{\frac{p-1}{p}}(\Phi(t))^{p-1}\right]\Delta t,$$

with indices p and $p/(p-1)$, we see that

$$\begin{aligned}
&\int_{a}^{\infty} \frac{\lambda(t)g(t)}{(\lambda(t))^{\frac{p-1}{p}}}\left[(\lambda(t))^{\frac{p-1}{p}}(\Phi(t))^{p-1}\right]\Delta t \\
&\leq \left[\int_{a}^{\infty}\left[\frac{\lambda(t)g(t)}{(\lambda(t))^{\frac{p-1}{p}}}\right]^{p}\Delta t\right]^{1/p}\left[\int_{a}^{\infty}\lambda(t)(\Phi(t))^{p}\Delta t\right]^{1-1/p}.
\end{aligned} \tag{2.1.13}$$

Substituting (2.1.13) into (2.1.12), we have

$$\begin{aligned}
&\int_{a}^{\infty}\lambda(t)(\Phi(t))^{p}\Delta t \\
&\leq p\left[\int_{a}^{\infty}\lambda(t)\left(g(t)\right)^{p}\Delta t\right]^{1/p}\left[\int_{a}^{\infty}\lambda(t)(\Phi(t))^{p}\Delta t\right]^{1-\frac{1}{p}}.
\end{aligned} \tag{2.1.14}$$

This implies that

$$\int_{a}^{\infty}\lambda(t)(\Phi(t))^{p}\Delta t \leq p^{p}\int_{a}^{\infty}\lambda(t)\left(g(t)\right)^{p}\Delta t. \tag{2.1.15}$$

which is the desired inequality (2.1.6). The proof is complete. ■

Remark 2.1.1. As a special case of Theorem 2.1.1 when $\mathbb{T} = \mathbb{R}$ and $p > 1$, we have the following integral inequality of Copson type (note that when $\mathbb{T} = \mathbb{R}$, we have $\Phi^{\sigma}(t) = \Phi(t),\ \Lambda^{\sigma}(t) = \Lambda(t)$ and $\mu(t) = 0$)

$$\int_a^{\infty} \lambda(t) \left(\int_t^{\infty} \frac{\lambda(s)g(s)}{\Lambda(s)} ds \right)^p dt \leq p^p \int_a^{\infty} \lambda(t) g^p(t) dt\,,\ p > 1.$$

As a special case when $\lambda(t) = 1$, we have inequality (2.1.1) due to Copson.

Remark 2.1.2. Assume that $\mathbb{T} = \mathbb{N}$ in Theorem 2.1.1 and $a = 1$. Then using $\int_a^{\sigma(t)} f(t)\Delta t = \sum_{s=a}^{\sigma(n)-1} f(s)$, we see that

$$\Lambda^{\sigma}(n) = \Lambda(n+1) = \sum_{s=1}^{n+1-1} \lambda(s) = \sum_{s=1}^{n} \lambda(s).$$

Assume that $\sum_{s=1}^{\infty} \lambda(n) g^p(n)$ is convergent and $p > 1$. Then the inequality (2.1.5) becomes the discrete Copson inequality (2.1.3), namely

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{i=n}^{\infty} \frac{\lambda(i)g(i)}{\Lambda(i)} \right)^p \leq p^p \sum_{n=1}^{\infty} \lambda(n) g^p(n), \quad p > 1, \tag{2.1.16}$$

where $\Lambda(n) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$.

In the following theorem, we prove a time scale version of Copson's type inequality (2.1.3) on time scales. In this theorem we will replace $\Lambda^{\sigma}(s)$ by $\Lambda(s)$.

Theorem 2.1.2. *Let $\mathbb{T}$ be a time scale with $a \in [0, \infty)_{\mathbb{T}}$ and $\Lambda(t) := \int_a^t \lambda(s)\Delta s$ and define*

$$\Phi(t) := \int_t^{\infty} \frac{\lambda(s)g(s)}{\Lambda(s)} \Delta s, \quad \text{for } t \in [a, \infty)_{\mathbb{T}}. \tag{2.1.17}$$

If $p > 1$, then

$$\int_a^{\infty} \lambda(t)(\Phi(t))^p \Delta t \leq p \int_a^{\infty} (\Phi(t))^{p-1} \frac{\Lambda^{\sigma}(t)}{\Lambda(t)} \lambda(t) g(t) \Delta t, \tag{2.1.18}$$

and

$$\int_a^{\infty} \lambda(t)(\Phi(t))^p \Delta t \leq p^p \int_a^{\infty} \frac{\Lambda^{\sigma}(t)}{\Lambda(t)} \lambda(t) g^p(t) \Delta t. \tag{2.1.19}$$

Proof. We proceed as in the proof of Theorem 2.1.1 and integrating by the parts formula (1.1.4) with $v^{\Delta}(t) = \lambda(t),\ u(t) = (\Phi(t))^p$, and using $\Phi(\infty) = 0$ and $\Lambda(a) = 0$, we obtain

$$\int_a^\infty \lambda(t)(\Phi(t))^p \Delta t \le p \int_a^\infty (\Phi(t))^{p-1} \frac{\Lambda^\sigma(t)}{\Lambda(t)} \lambda(t) g(t) \Delta t,$$

which is the desired inequality (2.1.18). This inequality implies, by the Hölder inequality (1.1.8), that

$$\int_a^\infty \lambda(t)(\Phi(t))^p \Delta t \le p \left[\int_a^\infty \frac{\Lambda^\sigma(t)}{\Lambda(t)} \lambda(t) g^p(t) \Delta t\right]^{1/p} \times \left[\int_a^\infty \lambda(t) \Phi^p(t) \Delta t\right]^{1-\frac{1}{p}}.$$

This implies

$$\left[\int_a^\infty \lambda(t)(\Phi(t))^p \Delta t\right]^{1-(1-\frac{1}{p})} \le p \left[\int_a^\infty \frac{\Lambda^\sigma(t)}{\Lambda(t)} \lambda(t) g^p(t) \Delta t\right]^{1/p},$$

and hence

$$\int_a^\infty \lambda(t)(\Phi(t))^p \Delta t \le p^p \int_a^\infty \frac{\Lambda^\sigma(t)}{\Lambda(t)} \lambda(t) g^p(t) \Delta t,$$

which is the desired inequality (2.1.19). The proof is complete. ■

If we assume that

$$\inf_t \frac{\Lambda(t)}{\Lambda^\sigma(t)} = L > 0, \tag{2.1.20}$$

and use it in Theorem 2.1.2 then we obtain the following result.

Corollary 2.1.1. *Let $\mathbb{T}$ be a time scale with $a \in [0, \infty)_{\mathbb{T}}$ and $p > 1$. Let $\Lambda(t)$ and $\Phi(t)$ be defined as in Theorem 2.1.2 such that (2.1.20) holds. Then*

$$\int_a^\infty \lambda(t)(\Phi(t))^p \Delta t \le \frac{p^p}{L} \int_a^\infty \lambda(t) g^p(t) \Delta t.$$

Remark 2.1.3. As a special case of (2.1.18), when $\mathbb{T} = \mathbb{R}$ and $p > 1$, we have the following integral inequality of Copson type

$$\int_a^\infty \lambda(t) \left(\int_t^\infty \frac{\lambda(s) g(s)}{\Lambda(s)} ds\right)^p dt \le p^p \int_a^\infty \lambda(t) g^p(t) dt, \ p > 1.$$

Remark 2.1.4. Assume that $\mathbb{T} = \mathbb{N}$ in Theorem 2.1.2, $p > 1$ and $a = 1$. In this case inequality (2.1.18) becomes the following discrete Copson type inequality

$$\sum_{n=1}^{\infty} \lambda(n) \left(\sum_{s=n}^{\infty} \frac{\lambda(s)g(s)}{\Lambda(s)} \right)^p \leq p^p \sum_{n=1}^{\infty} \frac{\Lambda(n+1)}{\Lambda(n)} \lambda(n) g^p(n), \ p > 1,$$

where $\Lambda(n) = \sum_{n=1}^{n-1} \lambda(s)$.

2.2 Copson-Type Inequalities II

In 1928, Copson [49, Theorems 1.1, 1.2] extended the inequality (2.1.3) and proved that if $a(n) > 0$, $\lambda(n) > 0$ for $n \geq 1$ and $p \geq c > 1$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=1}^{n} a(i)\lambda(i) \right)^p \leq \left(\frac{p}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n), \tag{2.2.1}$$

where $\Lambda(n) = \sum_{i=1}^{n} \lambda(i)$. He also proved that if $0 \leq c < 1 < p$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)} \left(\sum_{i=n}^{\infty} \lambda(i)a(i) \right)^p \leq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \Lambda^{p-c}(n) a^p(n). \tag{2.2.2}$$

The original motivation for Copson type inequalities (2.2.1) and (2.2.2), was a desire to generalize the inequalities (1.3.2) and (1.3.1) due to Hardy and Littlewood. When $\lambda(k) = 1$ for all k and $c = p$, inequality (2.2.1) becomes the Hardy inequality (1.2.1). The constants are best possible.

In Eq. (2.2.1) Copson assumed that $p > c$ and Bennett [22] observed that this inequality continues to hold for $c > p$ with constant $(p/(p-1))^p$ instead of $(p/(c-1))^p$. In [60] Gao proved that the inequality (2.2.1) holds with the best constant $(p/(c-1))^p$.

In 1976, Copson [50, Theorems 1 and 3] proved the continuous counterparts of these inequalities. In particular he proved that if $p \geq 1$ and $c > 1$, then

$$\int_0^{\infty} \frac{\lambda(t)}{\Lambda^c(t)} \Phi^p(t) dt \leq \left(\frac{p}{c-1} \right)^p \int_0^{\infty} \frac{\lambda(t)}{\Lambda^{c-p}(t)} g^p(t) dt, \tag{2.2.3}$$

where $\Lambda(t) = \int_0^t \lambda(s) ds$ and $\Phi(t) = \int_0^t \lambda(s) g(s) ds$, and if $p > 1$ and $0 \leq c < 1$, then

$$\int_a^{\infty} \frac{\lambda(t)}{\Lambda^c(t)} \Psi^p(t) dt \leq \left(\frac{p}{1-c} \right)^p \int_a^{\infty} \frac{\lambda(t)}{\Lambda^{c-p}(t)} g^p(t) dt, \tag{2.2.4}$$

where $\Psi(t) = \int_t^{\infty} \lambda(s) g(s) ds$.

In this section, we prove inequalities on time scales which as special cases contain Copson inequalities (2.2.1)–(2.2.4). As special cases when $\mathbb{T} = \mathbb{R}$, we obtain Hardy-Littlewood type inequalities (1.3.3) and (1.3.4). The results in this section are adapted from [162].

Theorem 2.2.1. *Let $\mathbb{T}$ be a time scale with $a \in [0, \infty)_{\mathbb{T}}$ and $p \geq c > 1$. Let*

$$\Lambda(t) := \int_a^t \lambda(s)\Delta s, \; \Phi(t) := \int_a^t \lambda(s)g(s)\Delta s, \; \textit{for} \;\; t \in [a, \infty)_{\mathbb{T}}. \tag{2.2.5}$$

Then

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \leq \frac{p}{c-1}\int_a^\infty \Lambda^{1-c}(t)\lambda(t)g(t)(\Phi(t))^{p-1}\Delta t, \tag{2.2.6}$$

and

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \leq \left(\frac{p}{c-1}\right)^p \int_a^\infty \frac{(\Lambda^\sigma(t))^{(p-1)c}}{(\Lambda(t))^{p(c-1)}}\lambda(t)g^p(t)\Delta t. \tag{2.2.7}$$

Proof. Integrating the left hand side of (2.2.6) using the integration by parts formula (1.1.4) with $v^\sigma(t) = (\Phi^\sigma(t))^p$, and $u^\Delta(t) = \lambda(t)/\left(\Lambda^\sigma(t)\right)^c$, we obtain

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t = u(t)\Phi^p(t)|_a^\infty + \int_a^\infty (-u(t))\left(\Phi^p(t)\right)^\Delta \Delta t, \tag{2.2.8}$$

where

$$u(t) := -\int_t^\infty \frac{\lambda(s)}{(\Lambda^\sigma(s))^c}\Delta s = -\int_t^\infty \left(\Lambda^\sigma(s)\right)^{-c}\left(\Lambda^\Delta(s)\right)\Delta s. \tag{2.2.9}$$

By the chain rule (1.1.6) and the fact that $\Lambda^\Delta(t) = \lambda(t) \geq 0$, we see that

$$\begin{aligned} -\left(\Lambda^{1-c}(t)\right)^\Delta &= -(1-c)\int_0^1 \frac{\Lambda^\Delta(t)}{[h\Lambda^\sigma(t) + (1-h)\Lambda(t)]^c}dh \\ &\geq (c-1)\frac{\Lambda^\Delta(t)}{(\Lambda^\sigma(t))^c}. \end{aligned}$$

This implies that

$$\left(\Lambda^\sigma(s)\right)^{-c}\Lambda^\Delta(t) \leq \frac{-1}{c-1}\left(\Lambda^{1-c}(t)\right)^\Delta,$$

and then, we have

$$
\begin{aligned}
-u(t) &= \int_t^\infty \frac{\lambda(s)}{(\Lambda^\sigma(s))^c}\Delta s \le \frac{-1}{c-1}\int_t^\infty \frac{\Delta s}{(\Lambda^{c-1}(s))^\Delta} \\
&\le \frac{1}{(c-1)\,\Lambda^{c-1}(t)}.
\end{aligned}
\tag{2.2.10}
$$

Applying the chain rule (1.1.5), we see that there exists $d \in [t, \sigma(t)]$ such that

$$
(\Phi^p(t))^\Delta = p\Phi^{p-1}(d)(\Phi^\Delta(t)).
\tag{2.2.11}
$$

Since $\Phi^\Delta(t) = \lambda(t)g(t) \ge 0$ and $\sigma(t) \ge d$, we have

$$
(\Phi^p(t))^\Delta \le p\lambda(t)g(t)(\Phi^\sigma(t))^{p-1}.
\tag{2.2.12}
$$

Using $\Phi(a) = 0$, $u(\infty) = 0$ and combining (2.2.8), (2.2.10) and (2.2.12), we get that

$$
\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \le \frac{p}{c-1}\int_a^\infty \frac{\lambda(t)g(t)(\Phi^\sigma(t))^{p-1}}{\Lambda^{c-1}(t)}\Delta t,
\tag{2.2.13}
$$

which is the desired inequality (2.2.6). Now, we prove (2.2.7). From (2.2.13), we see that

$$
\begin{aligned}
&\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Phi^\sigma(t))^p\Delta t \\
&\le \frac{p}{c-1}\int_a^\infty \frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\left(\frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\right)^{\frac{p-1}{p}}\Delta t.
\end{aligned}
\tag{2.2.14}
$$

Applying the Hölder inequality (1.1.8) on the integral

$$
\int_a^\infty \left[\frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\right]\left[\left(\frac{\lambda(t)}{(\Lambda^\sigma(t))^c}\right)^{\frac{p-1}{p}}(\Phi^\sigma(t))^{p-1}\right]\Delta t,
$$

with indices p and $p/(p-1)$, we see that

$$
\begin{aligned}
&\int_a^\infty \left[\frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\right]\left[\left(\frac{\lambda(t)}{(\Lambda^\sigma(t))^c}\right)^{\frac{p-1}{p}}(\Phi^\sigma(t))^{p-1}\right]\Delta t \\
&\le \left[\int_a^\infty \left[\frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\right]^p\Delta t\right]^{\frac{1}{p}} \\
&\quad\times\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Phi^\sigma(t))^p\Delta t\right]^{\frac{p-1}{p}}.
\end{aligned}
\tag{2.2.15}
$$

Substituting (2.2.15) into (2.2.14), we have

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \leq \frac{p}{c-1}\left[\int_a^\infty \left[\frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t\right]^{1/p}$$
$$\times\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Phi^\sigma(t))^p\Delta t\right]^{\frac{p-1}{p}}.$$

This implies that

$$\left[\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t\right]^{1-\frac{p-1}{p}}$$
$$\leq \frac{p}{c-1}\left[\int_a^\infty \left[\frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t\right]^{1/p},$$

and then, we get

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \leq \left(\frac{p}{c-1}\right)^p \int_a^\infty \left[\frac{\Lambda^{1-c}(t)\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t.$$

This leads to

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \leq \left(\frac{p}{c-1}\right)^p \int_a^\infty \frac{(\Lambda^\sigma(t))^{(p-1)c}}{(\Lambda(t))^{p(c-1)}}\lambda(t)g^p(t)\Delta t,$$

which is the desired inequality (2.2.7). The proof is complete. ■

From Theorem 2.2.1, using condition (2.1.20), we have the following result.

Corollary 2.2.1. *Let $\mathbb{T}$ be a time scale and $p \geq c > 1$. Let $\Lambda(t)$ and $\Phi(t)$ be defined as in Theorem 2.2.1 and assume (2.1.20) holds. Then*

$$\int_a^\infty \frac{\lambda(t)(\Phi^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \leq \left(\frac{p}{c-1}\right)^p L^{(p-1)c}\int_a^\infty (\Lambda(t))^{(p-c)}\lambda(t)g^p(t)\Delta t. \tag{2.2.16}$$

Remark 2.2.1. As a special case of Theorem 2.2.1 when $\mathbb{T}=\mathbb{R}$ and $p > 1$ and $c > 1$, we have the following Copson integral inequality (note that when $\mathbb{T}=\mathbb{R}$, we have $\sigma(t)=t$ and then $\Lambda^\sigma(t)=\Lambda(t)=\int_a^t \lambda(s)ds$))

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}\left(\int_a^t \lambda(s)g(s)ds\right)^p dt \leq \left(\frac{p}{c-1}\right)^p \int_a^\infty \Lambda^{p-c}(t)\lambda(t)g^p(t)dt.$$

As a special case when $\lambda(t) = 1$ and $a = 0$, we have the Hardy-Littlewood type inequality

$$\int_0^\infty \frac{1}{t^c}\left(\int_0^t g(s)ds\right)^p dt \le \left(\frac{p}{c-1}\right)^p \int_0^\infty \frac{1}{t^{c-p}} g^p(t)dt,$$

and when $c = p$, we have the classical Hardy inequality

$$\int_0^\infty \frac{1}{t^p}\left(\int_0^t g(s)ds\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^\infty g^p(t)dt. \tag{2.2.17}$$

Let $G(t) = \int_a^t g(s)ds$ in (2.2.17). Thus, we have (note that $G(a) = 0$) that

$$\int_a^\infty \frac{1}{t^p}(G(t))^p\, dt \le \left(\frac{p}{1-p}\right)^\lambda \int_a^\infty (G'(t))^p dt,$$

which can be considered as a generalization of Wirtinger's inequality. Note also that when $p = 2,\ a = 0$ and replacing ∞ by 1, we get the well-known inequality

$$\int_0^1 \left(G'(t)\right)^2 dt \ge \frac{1}{4}\int_0^1 \frac{1}{t^2} G^2(t), \text{ with } G(0) = 0,$$

due to Hardy with the best constant $1/4$.

Remark 2.2.2. Assume that $\mathbb{T} = \mathbb{N}$ in Corollary 2.2.1, and $p > 1,\ a = 1$ and define $\Lambda_1(n) = \sum_{k=1}^{n} \lambda(k)$. In this case the inequality (2.2.16) becomes the following discrete Copson type inequality (where $p > 1,\ c > 1$)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda_1^c(n)}\left(\sum_{s=1}^{n} \lambda(s)g(s)\right)^p \le \left(\frac{p}{c-1}\right)^p L^{(p-1)c} \sum_{n=1}^{\infty} \Lambda_1^{p-c}(n)\lambda(n)g^p(n).$$

In the following, we prove a time scale version of the Copson type inequality (2.2.2) which as a special case contains the inequality (1.3.3) due to Hardy and Littlewood.

Theorem 2.2.2. *Let $\mathbb{T}$ be a time scale with $a \in [0,\infty)_{\mathbb{T}}$, $p > 1$ and $0 \le c < 1$. Let $\Lambda(t)$ be defined as in (2.2.5) and define*

$$\overline{\Phi}(t) := \int_t^\infty \lambda(s)g(s)\Delta s, \quad \textit{for } t \in [a,\infty)_{\mathbb{T}}. \tag{2.2.18}$$

Then

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Phi}(t))^p \Delta t \le \frac{p}{1-c}\int_a^\infty (\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)(\overline{\Phi}(t))^{p-1}\Delta t, \tag{2.2.19}$$

and

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Phi}(t))^p \Delta t \le \left(\frac{p}{1-c}\right)^p \int_a^\infty (\Lambda^\sigma(t))^{p-c}\lambda(t)g^p(t)\Delta t. \tag{2.2.20}$$

Proof. Integrating the left hand side of (2.2.19) using integration by parts formula (1.1.4) with $u(t) = (\overline{\Phi}(t))^p$, and $v^\Delta(t) = \lambda(t)/\left(\Lambda^\sigma(t)\right)^c$, we obtain

$$\int_a^\infty \frac{\lambda(t)(\overline{\Phi}(t))^p}{(\Lambda^\sigma(t))^c}\Delta t = v(t)\overline{\Phi}^p(t)\Big|_a^\infty + \int_a^\infty (v^\sigma(t))\left(-\overline{\Phi}^p(t)\right)^\Delta \Delta t, \tag{2.2.21}$$

where

$$v(t) := \int_a^t \frac{\lambda(s)}{(\Lambda^\sigma(t))^c}\Delta s = \int_a^t (\Lambda^\sigma(t))^{-c}\left(\Lambda^\Delta(s)\right)\Delta s. \tag{2.2.22}$$

By the chain rule (1.1.6) and the fact that $\Lambda^\Delta(t) = \lambda(t) \ge 0$, we see that

$$\begin{aligned}\left(\Lambda^{1-c}(t)\right)^\Delta &= (1-c)\int_0^1 \frac{\Lambda^\Delta(t)}{[h\Lambda^\sigma(t) + (1-h)\Lambda(t)]^c}dh \\ &\ge (1-c)\frac{\Lambda^\Delta(t)}{[\Lambda^\sigma(t)]^c}.\end{aligned}$$

This implies that

$$\begin{aligned}v^\sigma(t) &= \int_a^{\sigma(t)} \frac{\lambda(s)}{(\Lambda^\sigma(t))^c}\Delta s \le \left(\frac{1}{1-c}\right)\int_a^{\sigma(t)} \left(\Lambda^{1-c}(s)\right)^\Delta \Delta s \\ &\le \left(\frac{1}{1-c}\right)(\Lambda^\sigma(t))^{1-c}.\end{aligned} \tag{2.2.23}$$

Applying the chain rule (1.1.5), we see that there exists $d \in [t, \sigma(t)]$ such that

$$-\left(\overline{\Phi}^p(t)\right)^\Delta = -p\overline{\Phi}^{p-1}(d)(\overline{\Phi}^\Delta(t)). \tag{2.2.24}$$

Since $\overline{\Phi}^\Delta(t) = -\lambda(t)g(t) \le 0$ and $d \ge t$, we have

$$-\left(\overline{\Phi}^p(t)\right)^\Delta \le p\lambda(t)g(t)(\overline{\Phi}(t))^{p-1}. \tag{2.2.25}$$

Using $\overline{\Phi}(\infty) = 0$, $\Lambda(a) = 0$ and substituting (2.2.23) and (2.2.25) into (2.2.21), we get that

$$\int_a^\infty \frac{\lambda(t)(\overline{\Phi}(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \le \frac{p}{1-c}\int_a^\infty (\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)(\overline{\Phi}(t))^{p-1}\Delta t, \tag{2.2.26}$$

which is the desired inequality (2.2.19). Now, we use (2.2.26) to prove (2.2.20). The inequality (2.2.26) can be written in the form

$$\begin{aligned}
&\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Phi}(t))^p \Delta t \\
&\leq \frac{p}{1-c}\int_a^\infty \frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\left(\frac{\lambda(t)(\overline{\Phi}(t))^p}{(\Lambda^\sigma(t))^c}\right)^{\frac{p-1}{p}} \Delta t. \qquad (2.2.27)
\end{aligned}$$

Applying the Hölder inequality (1.1.8) on the integral

$$\int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\right]\left[\left(\frac{\lambda(t)}{(\Lambda^\sigma(t))^c}\right)^{\frac{p-1}{p}}(\overline{\Phi}(t))^{p-1}\right]\Delta t,$$

with indices p and $p/(p-1)$, we see that

$$\begin{aligned}
&\int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\right]\left[\left(\frac{\lambda(t)}{(\Lambda^\sigma(t))^c}\right)^{\frac{p-1}{p}}(\overline{\Phi}(t))^{p-1}\right]\Delta t \\
&\leq \left[\int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t\right]^{1/p} \\
&\quad\times\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Phi}(t))^p \Delta t\right]^{\frac{p-1}{p}}. \qquad (2.2.28)
\end{aligned}$$

Substituting (2.2.28) into (2.2.27), we have

$$\begin{aligned}
&\int_a^\infty \frac{\lambda(t)(\overline{\Phi}(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \\
&\leq \frac{p}{1-c}\left[\int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t\right]^{1/p} \\
&\quad\times\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Phi}(t))^p \Delta t\right]^{\frac{p-1}{p}}.
\end{aligned}$$

Hence

$$\left[\int_a^\infty \frac{\lambda(t)(\overline{\Phi}^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t\right]^{1-\frac{p-1}{p}} \leq \frac{p}{1-c}\left[\int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t\right]^{1/p},$$

and then

$$\int_a^\infty \frac{\lambda(t)(\overline{\Phi}(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \le \left(\frac{p}{1-c}\right)^p \int_a^\infty \left[\frac{(\Lambda^\sigma(t))^{1-c}\lambda(t)g(t)}{((\Lambda^\sigma(t))^{-c}\,\lambda(t))^{\frac{p-1}{p}}}\right]^p \Delta t.$$

This implies

$$\int_a^\infty \frac{\lambda(t)(\overline{\Phi}^\sigma(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \le \left(\frac{p}{1-c}\right)^p \int_a^\infty (\Lambda^\sigma(t))^{p-c}\lambda(t)g^p(t)\Delta t,$$

which is the desired inequality (2.2.20). The proof is complete. ■

Remark 2.2.3. As a special case of Theorem 2.2.2 when $\mathbb{T} = \mathbb{R}$, $p > 1$ and $0 \le c < 1$, we have the following integral inequality which can be considered as a generalization of Hardy's inequality (note that when $\mathbb{T} = \mathbb{R}$, we have $\sigma(t) = t$)

$$\int_a^\infty \frac{\lambda(t)}{\Lambda^c(t)}\left(\int_t^\infty \lambda(s)g(s)ds\right)^p dt \le \left(\frac{p}{1-c}\right)^p \int_a^\infty \Lambda^{p-c}(t)\lambda(t)g^p(t)dt.$$

As a special case when $\lambda(t) = 1$ and $a = 0$, we have a Hardy-Littlewood type inequality

$$\int_0^\infty \frac{1}{t^c}\left(\int_t^\infty g(s)ds\right)^p dt \le \left(\frac{p}{1-c}\right)^p \int_0^\infty \frac{1}{t^{c-p}} g^p(t)dt.$$

Remark 2.2.4. Assume that $\mathbb{T} = \mathbb{N}$ in Theorem 2.2.2, $p > 1$, $a = 1$ and define $\Lambda(n) = \sum_{k=1}^{n} \lambda(k)$. In this case inequality (2.2.20) becomes the following discrete Copson-type inequality (where $p > 1$ and $0 \le c < 1$)

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{\Lambda^c(n)}\left(\sum_{s=n}^{\infty} \lambda(s)g(s)\right)^p \le \left(\frac{p}{(1-c)}\right)^p \sum_{n=1}^{\infty} \Lambda^{p-c}(n)\lambda(n)g^p(n).$$

Remark 2.2.5. In [22] Bennett used the discrete inequality (2.2.2) due to Copson to prove a generalization of the Littlewood inequality of the form

$$\sum_{n=1}^{\infty} a_n^p A_n^q \left(\sum_{k=n}^{\infty} a_k^{1+p/q}\right)^r \le \left(\frac{p(q+r)-q}{p}\right)^q \sum_{n=1}^{\infty} \left[a_n^p A_n^q\right]^{1+r/q},\ p,\ q,\ r \ge 1. \tag{2.2.29}$$

We mentioned here that the inequality (2.2.20) can be used to prove the time scale version of (2.2.29) which takes the form

$$\int_{t_0}^\infty a^p(t)\,(A^\sigma(t))^q\,\Lambda^r(t)\Delta t \le \left(\frac{p(q+r)-q}{p}\right)^r \int_{t_0}^\infty [a^p(t)(A^\sigma(t))^q]^{1+r/q}\Delta t, \tag{2.2.30}$$

where

$$A^{\sigma}(t) = \int_{t_0}^{\sigma(t)} a(s)\Delta s, \;\; \Lambda(t) := \int_{t}^{\infty} a^{1+p/q}(s)\Delta s, \; \textit{for } t \in [t_0, \infty)_{\mathbb{T}}. \tag{2.2.31}$$

This will be discussed later in Chap. 4.

2.3 Converses of Copson-Type Inequalities

In 1928, Copson [49, Theorem 2.3] proved converses of the inequality (2.2.2) which is given by

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda(n))^c} \left(\sum_{i=n}^{\infty} \lambda(i) g(i) \right)^p \geq \left(\frac{p}{1-c} \right)^p \sum_{n=1}^{\infty} \lambda(n) \left(\sum_{i=1}^{n} \lambda(i) \right)^{p-c} g^p(n), \tag{2.3.1}$$

where $g(n) > 0$, $\lambda(n) > 0$ for $n \geq 1$, $\Lambda(n) = \sum_{i=1}^{n} \lambda(i)$ and $0 < p < 1$ and $c < 0$. The original motivation for Copson's inequality (2.3.1), was a desire to generalize the following inequality

$$\sum_{n=1}^{\infty} n^{-c} \left(\sum_{k=n}^{\infty} g(k) \right)^p \geq K \sum_{n=1}^{\infty} n^{-c} \left(n g(n) \right)^p, p > 0, c < 1,$$

which is the converse of the inequality (1.3.2) due to Hardy and Littlewood [74]; here K is a positive constant depending on p and c and $g(n) > 0$ for $n \geq 1$. In 1976, Copson [50] proved that if $0 < p \leq 1$ and $c < 1$, then

$$\int_0^{\infty} \frac{\lambda(t)}{(\Lambda(t))^c} \left(\int_t^{\infty} \lambda(s) g(s) ds \right)^p dt \geq \left(\frac{p}{1-c} \right)^p \int_0^{\infty} \lambda(t) \left(\Lambda(t) \right)^{p-c} g^p(t) dt. \tag{2.3.2}$$

He also proved that if $0 < p \leq 1$ and $c > 1$, and $\Lambda(t) \to \infty$ as $t \to \infty$, then

$$\int_a^{\infty} \frac{\lambda(t)}{(\Lambda(t))^c} \left(\int_a^{t} \lambda(s) g(s) ds \right)^p dt \geq \left(\frac{p}{1-c} \right)^p \int_0^{\infty} \lambda(t) \left(\Lambda(t) \right)^{p-c} g^p(t) dt. \tag{2.3.3}$$

In 1987, Bennett [22] (see also Leindler [109]) proved the discrete version of (2.3.3) which is given by

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{(\Lambda(n))^c} \left(\sum_{i=1}^{n} \lambda(i) g(i) \right)^p \geq \left(\frac{pL}{c-1} \right)^p \sum_{n=1}^{\infty} \lambda(n) \left(\Lambda(n) \right)^{p-c} g^p(n), \tag{2.3.4}$$

where $c > 1 > p > 0$, $\Lambda(n) \to \infty$ as $n \to \infty$, and $L = \inf \frac{\lambda(n)}{\lambda(n+1)}$. In this section, we prove some dynamic inequalities which as special cases contain the inequalities (2.3.1)–(2.3.4). The results in this section are adapted from [166].

Theorem 2.3.1. *Let $\mathbb{T}$ be a time scale with $a \in (0, \infty)_{\mathbb{T}}$ and $c \leq 0 < p < 1$. Let $\Lambda(t) = \int_a^t \lambda(s)\Delta s$, and*

$$\overline{\Psi}(t) = \int_t^\infty \lambda(s)g(s)\Delta s. \tag{2.3.5}$$

Then

$$\int_a^\infty \frac{\lambda(t)(\overline{\Psi}(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \geq \left(\frac{p}{1-c}\right)^p \int_a^\infty \lambda(t)(\Lambda^\sigma(t))^{p-c} g^p(t)\Delta t. \tag{2.3.6}$$

Proof. Integrating the left hand side of (2.3.6) by the parts formula (1.1.4) with $v^\Delta(t) = \lambda(t)/\left(\Lambda^\sigma(t)\right)^c$, and $u(t) = \left(\overline{\Psi}(t)\right)^p$, we obtain

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Psi}(t))^p\Delta t = v(t)\overline{\Psi}^p(t)\Big|_a^\infty + \int_a^\infty (v^\sigma(t))(-\overline{\Psi}^p(t))^\Delta\Delta t, \tag{2.3.7}$$

where $v(t) = \int_a^t \frac{\lambda(s)}{(\Lambda^\sigma(s))^c}\Delta s$. From the inequality (2.3.7) and $\overline{\Psi}(\infty) = \Lambda(a) = 0$, we have

$$\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\overline{\Psi}(t))^p\Delta t = \int_a^\infty v^\sigma(t)(-\overline{\Psi}^p(t))^\Delta\Delta t. \tag{2.3.8}$$

Applying the chain rule (1.1.5), we see that there exists $d \in [t, \sigma(t)]$ such that

$$-(\overline{\Psi}^p(t))^\Delta = \frac{-p}{\overline{\Psi}^{1-p}(d)}(\overline{\Psi}^\Delta(t)) = \frac{p\lambda(t)g(t)}{\overline{\Psi}^{1-p}(d)}. \tag{2.3.9}$$

Since $\overline{\Psi}^\Delta(t) = -\lambda(t)g(t) \leq 0$, and $d \geq t$, we see that $\overline{\Psi}(t) \geq \overline{\Psi}(d)$, and then

$$\frac{p\lambda(t)g(t)}{\overline{\Psi}^{1-p}(d)} \geq \frac{p\lambda(t)g(t)}{(\overline{\Psi}(t))^{1-p}}, \quad \text{(note } 0 < p < 1).$$

This and (2.3.9) imply that

$$\left(-\overline{\Psi}^p(t)\right)^\Delta) \geq \frac{pg(t)\lambda(t)}{(\overline{\Psi}(t))^{1-p}}. \tag{2.3.10}$$

From the chain rule (1.1.6) and the fact that $(\Lambda(t))^\Delta = \lambda(t) \geq 0$ and $c \leq 0$, we see that

$$\begin{aligned}\left((\Lambda(t))^{1-c}\right)^\Delta &= (1-c)\int_0^1 \frac{\lambda(t)}{[h\Lambda^\sigma(t) + (1-h)\Lambda(t)]^c}dh \\ &\leq (1-c)\frac{\lambda(t)}{[\Lambda^\sigma(t)]^c}.\end{aligned}$$

This implies that

$$v^{\sigma}(t) = \int_a^{\sigma(t)} \frac{\lambda(s)}{(\Lambda^{\sigma}(s))^{c}} \Delta s \geq \left(\frac{1}{1-c}\right) \int_a^{\sigma(t)} \left(\Lambda^{1-c}(s)\right)^{\Delta} \Delta s$$
$$= \left(\frac{1}{1-c}\right) (\Lambda^{\sigma}(t))^{1-c}. \tag{2.3.11}$$

Substituting (2.3.10) and (2.3.11) into (2.3.8) we have that

$$\left(\int_a^{\infty} \frac{\lambda(t)}{(\Lambda^{\sigma}(t))^{c}} \left(\overline{\Psi}(t)\right)^{p} \Delta t\right)^{p}$$
$$\geq \left(\frac{p}{1-c}\right)^{p} \left[\int_a^{\infty} \left(\frac{g^{p}(t)\lambda^{p}(t)}{(\overline{\Psi}(t))^{p(1-p)}(\Lambda^{\sigma}(t))^{p(c-1)}}\right)^{1/p} \Delta t\right]^{p}. \tag{2.3.12}$$

Applying the Hölder inequality

$$\int_a^b F(t)G(t)\Delta t \leq \left[\int_a^b F^{q}(t)\Delta t\right]^{\frac{1}{q}} \left[\int_a^b G^{h}(t)\Delta t\right]^{\frac{1}{h}},$$

on the term

$$\left[\int_a^{\infty} \left(\frac{g^{p}(t)\lambda^{p}(t)}{(\Lambda^{\sigma}(t))^{p(c-1)}\,(\Psi(t))^{p(1-p)}}\right)^{1/p} \Delta t\right]^{p},$$

with indices $q = 1/p > 1$, $h = 1/(1-p)$ (note that $\frac{1}{q} + \frac{1}{h} = 1$, where $q > 1$) and

$$F(t) = \frac{g^{p}(t)\lambda^{p}(t)}{(\Lambda^{\sigma}(t))^{p(c-1)}\,(\Psi(t))^{p(1-p)}}, \text{ and } G(t) = \left(\frac{\lambda(t)}{(\Lambda^{\sigma})^{c}(t)}\right)^{1-p} (\Psi(t))^{p(1-p)},$$

we see that

$$\left(\int_a^{\infty} F^{1/p}(t)\Delta t\right)^{p}$$
$$= \left[\int_a^{\infty} \left(\frac{g^{p}(t)\lambda^{p}(t)}{(\Lambda^{\sigma}(t))^{p(c-1)}\,(\Psi(t))^{p(1-p)}}\right)^{1/p} \Delta t\right]^{p} \geq \frac{\int_a^{\infty} F(t)G(t)\Delta t}{\left[\int_a^{\infty}(G(t))^{\frac{1}{1-p}}\Delta t\right]^{1-p}}$$
$$= \left[\int_a^{\infty} \frac{g^{p}(t)\left(\lambda(t)(\Lambda^{\sigma}(t))^{-c}\right)^{1-p}\lambda^{p}(t)(\Psi(t))^{p(1-p)}\Delta t}{(\Lambda^{\sigma}(t))^{p(c-1)}\,(\Psi(t))^{p(1-p)}}\right]$$

$$\times \left[\int_a^\infty ((\frac{\lambda(t)}{(\Lambda^\sigma(t))^c})^{1-p}(\Psi(t))^{p(1-p)}))^{\frac{1}{1-p}}\Delta t\right]^{p-1}$$

$$= \left[\int_a^\infty \frac{\lambda(t)g^p(t)}{(\Lambda^\sigma(t))^{p(c-1)}\,((\Lambda^\sigma(t))^c)^{1-p}}\Delta t\right]\left[\int_a^\infty \frac{\lambda(t)}{((\Lambda^\sigma(t))^c}(\Psi(t))^p\Delta t\right]^{p-1}$$

$$= \left[\int_a^\infty \frac{g^p(t)\lambda(t)}{(\Lambda^\sigma(t))^{c-p}}\Delta t\right]\frac{1}{\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Psi(t))^p\Delta t\right]^{1-p}}.$$

This implies that

$$\left[\int_a^\infty \left(\frac{g^p(t)\lambda^p(t)(\Psi(t))^{-p(1-p)}}{(\Lambda^\sigma(t))^{p(c-1)}}\right)^{1/p}\Delta t\right]^p \geq \frac{\int_a^\infty g^p(t)(\Lambda^\sigma(t))^{p-c}(t)\lambda(t)\Delta t}{\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Psi(t))^p\Delta t\right]^{1-p}}. \tag{2.3.13}$$

Substituting (2.3.13) into (2.3.12), we get

$$\left(\int_a^\infty \frac{\lambda(t)(\Psi(t))^p}{(\Lambda^\sigma(t))^c}\Delta t\right)^p \geq \left(\frac{p}{1-c}\right)^p \frac{\int_a^\infty \lambda(t)\,(\Lambda^\sigma(t))^{p-c}\,g^p(t)\Delta t}{\left[\int_a^\infty \frac{\lambda(t)}{(\Lambda^\sigma(t))^c}(\Psi(t))^p\Delta t\right]^{1-p}}.$$

This gives that

$$\int_a^\infty \frac{\lambda(t)(\Psi(t))^p}{(\Lambda^\sigma(t))^c}\Delta t \geq \left(\frac{p}{1-c}\right)^p\left[\int_a^\infty \lambda(t)\,(\Lambda^\sigma(t))^{p-c}\,g^p(t)\Delta t\right],$$

which is the desired inequality (2.3.6). The proof is complete. ■

Remark 2.3.1. Assume that $\mathbb{T} = \mathbb{R}$ in Theorem 2.3.1, $c \leq 0 < p < 1$ and $a = 1$. In this case, we have the following integral inequality of Bennett-Leindler type (note that when $\mathbb{T} = \mathbb{R}$, we have $\Lambda^\sigma(t) = \Lambda(t) = \int_a^t \lambda(s)ds$)

$$\int_1^\infty \frac{\lambda(t)}{(\Lambda(t))^c}\left(\int_t^\infty \lambda(s)g(s)ds\right)^p dt \geq \left(\frac{p}{1-c}\right)^p \int_1^\infty \lambda(t)\,(\Lambda(t))^{p-c}\,g^p(t)dt.$$

Remark 2.3.2. Assume that $\mathbb{T} = \mathbb{N}$ in Theorem 2.3.1, $c \leq 0 < p < 1$ and $a = 1$. In this case inequality (2.3.6) becomes the following discrete Bennett-Leindler inequality

$$\sum_{n=1}^\infty \frac{\lambda(n)}{(\Lambda(n))^c}\left(\sum_{k=n}^\infty \lambda(k)g(k)\right)^p \geq \left(\frac{p}{1-c}\right)^p \sum_{n=1}^\infty \lambda(n)\left(\sum_{k=1}^n \lambda(k)\right)^{p-c} g^p(n),$$

where $\Lambda(n) = \sum_{k=1}^n \lambda(n)$.

In the following theorem, we prove a time scale version of the Bennett-Leindler type inequality (2.3.4) on time scales.

Theorem 2.3.2. *Let $\mathbb{T}$ be a time scale with $a \in (0,\infty)_{\mathbb{T}}$ and $0 < p \leq 1 < c$. Let $\Lambda(t) = \int_a^t \lambda(s)\Delta s$, such that*

$$L := \inf_{t\in\mathbb{T}} \frac{\Lambda(t)}{\Lambda^{\sigma}(t)} > 0, \tag{2.3.14}$$

and define $\overline{\Phi}(t) := \int_a^t \lambda(t)g(s)\Delta s$. Then

$$\int_a^{\infty} \frac{\lambda(t)(\overline{\Phi}^{\sigma}(t))^p}{(\Lambda^{\sigma}(t))^c}\Delta t \geq \left(\frac{pL^{1-c}}{c-1}\right)^p \int_a^{\infty} \lambda(t)(\Lambda^{\sigma}(t))^{p-c} g^p(t)\Delta t. \tag{2.3.15}$$

Proof. Integrating the left hand side of (2.3.6) by the parts formula (1.1.4) with $u^{\Delta}(t) = \frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}$, and $v^{\sigma}(t) = \left(\overline{\Phi}^{\sigma}(t)\right)^p$, we obtain

$$\int_a^{\infty} \frac{\lambda(t)(\overline{\Phi}^{\sigma}(t))^p}{(\Lambda^{\sigma}(t))^c}\Delta t = u(t)\overline{\Phi}^p(t)\Big|_a^{\infty} + \int_a^{\infty} (-u(t))(\overline{\Phi}^p(t))^{\Delta}\Delta t,$$

where $u(t) = \int_a^t \frac{\lambda(s)}{(\Lambda^{\sigma}(s))^c}\Delta s$. From the inequality (2.3.7) and $\overline{\Phi}(\infty) = u(a) = 0$, we have

$$\int_a^{\infty} \frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}(\overline{\Phi}^{\sigma}(t))^p\Delta t = \int_a^{\infty} (-u(t))(\overline{\Phi}^p(t))^{\Delta}\Delta t. \tag{2.3.16}$$

Applying the chain rule (1.1.5), we see that there exists $d \in [t, \sigma(t)]$ such that

$$\left(\overline{\Phi}^p(t)\right)^{\Delta} = \frac{p}{\overline{\Phi}^{1-p}(d)}(\overline{\Phi}^{\Delta}(t)) = \frac{p\lambda(t)g(t)}{\overline{\Phi}^{1-p}(d)}. \tag{2.3.17}$$

Since $\overline{\Phi}^{\Delta}(t) = \lambda(t)g(t) \geq 0$, and $\sigma(t) \geq d$, we see that $\overline{\Phi}^{\sigma}(t) \geq \overline{\Phi}(d)$, and then

$$\frac{p\,\lambda(t)\,g(t)}{\overline{\Phi}^{1-p}(d)} \geq \frac{p\lambda(t)g(t)}{(\overline{\Phi}^{\sigma}(t))^{1-p}} \quad \text{(note } 0 < p < 1\text{)}.$$

This and (2.3.17) implies that

$$\left(\overline{\Phi}^p(t)\right)^{\Delta} \geq \frac{p\lambda(t)g(t)}{(\overline{\Phi}^{\sigma}(t))^{1-p}}. \tag{2.3.18}$$

From the chain rule (1.1.6) and the fact that $(\Lambda(t))^{\Delta} = \lambda(t) \geq 0$ and $c > 1$, we see that

$$
\begin{aligned}
\left((\Lambda(t))^{1-c}\right)^{\Delta} &= (1-c)\int_0^1 \frac{\lambda(t)}{[h\Lambda^{\sigma}(t)+(1-h)\Lambda(t)]^c}dh \\
&\leq -(c-1)\frac{\lambda(t)}{[\Lambda^{\sigma}(t)]^c}.
\end{aligned}
$$

This and (2.3.14) imply that

$$
\begin{aligned}
-u(t) = -\int_a^t \frac{\lambda(s)}{(\Lambda^{\sigma}(s))^c}\Delta s &\geq \left(\frac{1}{c-1}\right)\int_a^t \left(\Lambda^{1-c}(s)\right)^{\Delta}\Delta s \\
&= \left(\frac{1}{c-1}\right)\left(\frac{\Lambda(t)}{\Lambda^{\sigma}(t)}\right)^{1-c}(\Lambda^{\sigma}(t))^{1-c} \\
&\geq \left(\frac{L^{1-c}}{c-1}\right)(\Lambda^{\sigma}(t))^{1-c}. \qquad (2.3.19)
\end{aligned}
$$

Substituting (2.3.19) and (2.3.18) into (2.3.16) yields

$$
\begin{aligned}
&\left(\int_a^{\infty}\frac{\lambda(t)}{(\Lambda^{\sigma}(t))^c}\left(\overline{\Phi}^{\sigma}(t)\right)^p\Delta t\right)^p \\
&\geq \left(\frac{pL^{1-c}}{c-1}\right)^p\left[\int_a^{\infty}\left(\frac{g^p(t)\lambda^p(t)}{(\overline{\Phi}^{\sigma}(t))^{p(1-p)}(\Lambda^{\sigma}(t))^{p(c-1)}}\right)^{1/p}\Delta t\right]^p.
\end{aligned}
$$

The rest of the proof is similar to the proof of Theorem 2.3.1 and hence is omitted. The proof is complete. ■

Remark 2.3.3. Assume that $\mathbb{T}=\mathbb{R}$ in Theorem 2.3.2, $0<p\leq 1<c$ and $a=1$. In this case, we have the following integral inequality of Leindler type (note that when $\mathbb{T}=\mathbb{R}$, we have $\Lambda^{\sigma}(t)=\Lambda(t)=\int_a^t\lambda(s)ds$)

$$
\int_1^{\infty}\frac{\lambda(t)}{(\Lambda(t))^c}\left(\int_a^t\lambda(t)g(s)ds\right)^p dt \geq \left(\frac{p}{c-1}\right)^p\int_1^{\infty}\lambda(t)(\Lambda(t))^{p-c}g^p(t)dt. \qquad (2.3.20)
$$

Remark 2.3.4. Assume that $\mathbb{T}=\mathbb{N}$ in Theorem 2.3.2, $0<p\leq 1<c$ and $a=1$. In this case inequality (2.3.15) becomes the following discrete Bennett-Leindler type inequality,

$$
\sum_{n=1}^{\infty}\frac{\lambda(n)}{(\Lambda(n+1))^c}\left(\sum_{k=1}^{n}\lambda(k)g(k)\right)^p\Delta t \geq \left(\frac{pL^{1-c}}{c-1}\right)^p\sum_{n=1}^{\infty}\lambda(n)(\Lambda(n+1))^{p-c}g^p(n), \qquad (2.3.21)
$$

where $\Lambda(n+1)=\sum_{k=1}^{n}\lambda(n)$.

2.4 Extensions of Copson-Type Inequalities

In 1940, Walsh [188] extended the Copson type inequality (2.2.1) and proved that if $b \geq 1$, $c \geq 1$, and for all $n \geq 0$, $a(n) > 0,\ \lambda(n) > 0,\ p(n) > 0$, and

$$p(n)\Lambda(n)/P(n) \geq p(n+1)\Lambda(n+1)/P(n+1) \tag{2.4.1}$$

where

$$\Lambda(n) = \sum_{s=a}^{n} \lambda(s),\ A(n) = \sum_{s=a}^{n} \lambda(s)g(s),\ P(n) = \sum_{s=a}^{n} p(s)\lambda(s),$$

then

$$\begin{aligned}&\sum_{n=a}^{\infty} \frac{\lambda(n)\,(A(n))^{c}\,p^{b}(n)\,(\Lambda(n))^{b-1}}{(P(n))^{b+c-1}} \\ &\leq \left(\frac{c}{c-1}\right)^{b} \sum_{n=a}^{\infty} \lambda(n)\,(P(n))^{1-c}\,(\Lambda(n))^{b-1}\,g^{b}(n)\,(A(n))^{c-b}.\end{aligned} \tag{2.4.2}$$

The inequality (2.4.2) was proved by employing a general algebraic inequality designed and proved by Walsh of the form

$$c(1-x)^{b} \geq b\alpha^{b-1}(1-\beta^{1-c}x^{c}) + b(c-1)\alpha^{b-1}(1-\beta) + c(1-b)\alpha^{b}, \tag{2.4.3}$$

where

$$b \geq 1,\ c \geq 1,\ 0 \leq \alpha \leq 1,\ 0 \leq \beta \leq 1$$

and x is assumed to lie in the range $(0, 1)$.

In this section, we follow Walsh's method to prove some dynamic inequalities of the form (2.4.2) on discrete time scales, i.e., where the domain of the unknown function is a so-called discrete time scale $\mathbb{T}$. The inequalities, as special cases, contain the discrete Walsh inequality (2.4.2) when $\mathbb{T} = \mathbb{N}$, and can be used to derive discrete inequalities when $\mathbb{T} = h\mathbb{N}$ and other types of discrete time scales when $\mu(t) \neq 0$. The results in this section are adapted from [165].

Theorem 2.4.1. *Let $\mathbb{T}$ be a discrete time scale and $b > 1,\ c > 1$ and define*

$$\begin{cases} \Lambda(t) := \int_a^t \lambda(s)\Delta s,\ A(t) := \int_a^t g(s)\lambda(s)\Delta s, \\ P(t) := \int_a^t p(s)\lambda(s)\Delta s, \end{cases} \quad \textit{for } t \in [a, \infty)_{\mathbb{T}}. \tag{2.4.4}$$

If

$$\left(P^{\sigma}(t)\right)^{1-c}\left(A^{\sigma}(t)\right)^{c} \geq (P(t))^{1-c}(A(t))^{c}, \tag{2.4.5}$$

then

$$\int_{a}^{\infty} \frac{\lambda(t)p^{b}(t)\left(A^{\sigma}(t)\right)^{c}}{\left(P^{\sigma}(t)\right)^{b+c-1}} \Delta t \leq \left(\frac{c}{c-1}\right)^{b} \int_{a}^{\infty} \frac{\lambda(t)g^{b}(t)\left(A^{\sigma}(t)\right)^{c-b}}{\left(P^{\sigma}(t)\right)^{c-1}} \Delta t. \tag{2.4.6}$$

Proof. Let

$$x := \frac{A(t)}{A^{\sigma}(t)} < 1, \quad \beta := \frac{P(t)}{P^{\sigma}(t)} < 1, \text{ and } \alpha := \eta \frac{\lambda(t)}{\Lambda^{\sigma}(t)}, \tag{2.4.7}$$

where η is positive and will be determined later such that $\alpha < 1$. As a result of these substitutions in (2.4.3), we get that

$$\begin{aligned}
& c\left(1-\frac{A(t)}{A^{\sigma}(t)}\right)^{b} \\
&= c\left(\frac{\mu(t)A^{\Delta}(t)}{A^{\sigma}(t)}\right)^{b} = c\left(\frac{\mu(t)g(t)\lambda(t)}{A^{\sigma}(t)}\right)^{b} \\
&\geq b\left(\eta\frac{\lambda(t)}{\Lambda^{\sigma}(t)}\right)^{b-1}\left(1-\left(\frac{P(t)}{P^{\sigma}(t)}\right)^{1-c}\left(\frac{A(t)}{A^{\sigma}(t)}\right)^{c}\right) \\
&\quad + b(c-1)\frac{\lambda^{b-1}(t)\mu(t)p(t)\lambda(t)}{\left(\Lambda^{\sigma}(t)\right)^{b-1}P^{\sigma}(t)}\eta^{b-1} + \frac{c(1-b)\lambda^{b}(t)}{\left(\Lambda^{\sigma}(t)\right)^{b}}\eta^{b}.
\end{aligned} \tag{2.4.8}$$

Multiplying both sides by

$$\lambda^{1-b}(t)\left(A^{\sigma}(t)\right)^{c}\left(\Lambda^{\sigma}(t)\right)^{b-1}\left(P^{\sigma}(t)\right)^{1-c}, \tag{2.4.9}$$

gives

$$\begin{aligned}
& \frac{c\lambda(t)(g(t)\mu(t))^{b}\left(A^{\sigma}(t)\right)^{c-b}}{\left(P^{\sigma}(t)\right)^{c-1}\left(\Lambda^{\sigma}(t)\right)^{1-b}} \\
&\geq b\eta^{b-1}\left[\left(P^{\sigma}(t)\right)^{1-c}\left(A^{\sigma}(t)\right)^{c}-(P(t))^{1-c}(A(t))^{c}\right] + \lambda(t)\left(A^{\sigma}(t)\right)^{c} \\
&\quad \times\left[\frac{b(c-1)\mu(t)p(t)}{\left(P^{\sigma}(t)\right)^{c}}\eta^{b-1} + \frac{c(1-b)\eta^{b}}{\left(P^{\sigma}(t)\right)^{c-1}\Lambda^{\sigma}(t)}\right].
\end{aligned} \tag{2.4.10}$$

Now, we consider the last term on the right hand side, namely

$$f(\eta) := M\eta^{b} + K\eta^{b-1}, \tag{2.4.11}$$

as a function of η, where

$$K := \frac{b(c-1)\mu(t)p(t)}{(P^{\sigma}(t))^{c}}, \quad \text{and } M := \frac{c(1-b)\,(P^{\sigma}(t))^{1-c}}{\Lambda^{\sigma}(t)}.$$

By differentiation, we see that the function $f(\eta)$ has a maximum value at

$$\eta = \frac{c-1}{c}\frac{p(t)\mu(t)\Lambda^{\sigma}(t)}{P^{\sigma}(t)} > 0, \tag{2.4.12}$$

and the maximum value of $f(\eta)$ is given by

$$\max_{\eta>0} f(\eta) = \frac{(c-1)^{b}}{c^{b-1}}\frac{p^{b}(t)\mu^{b}(t)\,(\Lambda^{\sigma}(t))^{b-1}}{(P^{\sigma}(t))^{b+c-1}}. \tag{2.4.13}$$

From (2.4.7) and (2.4.12), we see that

$$\begin{aligned}\alpha &= \eta\frac{\lambda(t)}{\Lambda^{\sigma}(t)} = \frac{c-1}{c}\frac{p(t)\mu(t)\lambda(t)}{P^{\sigma}(t)} = \frac{c-1}{c}\frac{\mu(t)P^{\Delta}(t)}{P^{\sigma}(t)} \\ &= \frac{c-1}{c}\frac{P^{\sigma}(t)-P(t)}{P^{\sigma}(t)},\end{aligned}$$

satisfies $0 < \alpha < 1$. Then (2.4.10) and (2.4.13) imply that

$$\begin{aligned}&\frac{c\lambda(t)(g(t)\mu(t))^{b}\,(A^{\sigma}(t))^{c-b}}{(P^{\sigma}(t))^{c-1}\,(\Lambda^{\sigma}(t))^{1-b}} \\ &\geq b\eta^{b-1}\left[(P^{\sigma}(t))^{1-c}\,(A^{\sigma}(t))^{c} - (P(t))^{1-c}\,(A(t))^{c}\right] \\ &\quad + \frac{(c-1)^{b}}{c^{b-1}}\frac{\lambda(t)\mu^{b}(t)p^{b}(t)\,(A^{\sigma}(t))^{c}\,(\Lambda^{\sigma}(t))^{b-1}}{(P^{\sigma}(t))^{b+c-1}}.\end{aligned}$$

Using the condition (2.4.5), we have that

$$\frac{c\lambda(t)g^{b}(t)\,(A^{\sigma}(t))^{c-b}}{(P^{\sigma}(t))^{c-1}} \geq \frac{(c-1)^{b}}{c^{b-1}}\frac{\lambda(t)\,(A^{\sigma}(t))^{c}}{p^{-b}(t)(P^{\sigma}(t))^{b+c-1}}. \tag{2.4.14}$$

This implies after integration from a to T that

$$c\int_{a}^{T}\frac{\lambda(t)g^{b}(t)\,(A^{\sigma}(t))^{c-b}}{(P^{\sigma}(t))^{c-1}}\Delta t \geq \frac{(c-1)^{b}}{c^{b-1}}\int_{a}^{T}\frac{\lambda(t)\,(A^{\sigma}(t))^{c}}{p^{-b}(t)(P^{\sigma}(t))^{b+c-1}}\Delta t. \tag{2.4.15}$$

Thus

$$\int_a^T \frac{\lambda(t)\,(A^\sigma(t))^c}{(P^\sigma(t))^{b+c-1}p^{-b}(t)}\Delta t \le \left(\frac{c}{c-1}\right)^b \int_a^T \frac{\lambda(t)g^b(t)\,(A^\sigma(t))^{c-b}}{(P^\sigma(t))^{c-1}}\Delta t.$$

Finally, letting $T \to \infty$, we have that

$$\int_a^\infty \frac{\lambda(t)\,(A^\sigma(t))^c\, p^b(t)}{(P^\sigma(t))^{b+c-1}}\Delta t \le \left(\frac{c}{c-1}\right)^b \int_a^\infty \frac{\lambda(t)g^b(t)\,(A^\sigma(t))^{c-b}}{(P^\sigma(t))^{c-1}}\Delta t,$$

which is the desired inequality (2.4.6). The proof is complete. ■

As a special case of Theorem 2.4.1, when $c > b$, we have the following result.

Corollary 2.4.1. *Let $\mathbb{T}$ be a discrete time scale and $c > b > 1$ and $\Lambda(t)$, $A(t)$ and $P(t)$ are defined as in Theorem 2.4.1. If (2.4.5) holds, then*

$$\int_a^\infty \frac{\lambda(t)\,(A^\sigma(t))^c\, p^b(t)}{(P^\sigma(t))^{b+c-1}}\Delta t \le \left(\frac{c}{b-1}\right)^c \int_a^\infty \frac{\lambda(t)g^b(t)\,(A^\sigma(t))^{c-b}}{(P^\sigma(t))^{c-1}}\Delta t. \tag{2.4.16}$$

As a special case when $p(t) = 1$ for all t, we see that $P^\sigma(t) = \Lambda^\sigma(t)$ and then (2.4.16) reduces to

$$\int_a^\infty \frac{\lambda(t)\,(A^\sigma(t))^c}{(\Lambda^\sigma(t))^b}\Delta t \le \left(\frac{c}{b-1}\right)^c \int_a^\infty \lambda(t)(A^\sigma(t))^{c-b}g^b(t)\Delta t. \tag{2.4.17}$$

As a special case when $b = c$ and $p(t) = 1$ for all t, we see that $P(t) = \Lambda(t)$ and then (2.4.17) reduces to

$$\int_a^\infty \frac{\lambda(t)\,(A^\sigma(t))^c}{(\Lambda^\sigma(t))^c}\Delta t \le \left(\frac{c}{c-1}\right)^c \int_a^\infty \lambda(t)g^c(t)\Delta t. \tag{2.4.18}$$

As a special case when $\lambda(t) = 1$, we have the inequality

$$\int_a^\infty \left(\frac{1}{\sigma(t)-a}\int_a^{\sigma(t)} g(s)\Delta s\right)^c \Delta t \le \left(\frac{c}{c-1}\right)^c \int_a^\infty g^c(t)\Delta t. \tag{2.4.19}$$

As a special case when $\mathbb{T} = \mathbb{N}$, we see that inequality (2.4.17) becomes a Copson type inequality

$$\sum_{n=a}^\infty \frac{\lambda(n)A^c(n)}{\Lambda^b(n)} \le \left(\frac{c}{b-1}\right)^c \sum_{n=a}^\infty \lambda(n)(A(n))^{c-b}g^c(n), \tag{2.4.20}$$

where $c > 1$ and $b > 1$, and for all n, $a(n) > 0,\ \lambda(n) > 0$ and

$$A(n) = \sum_{s=a}^{n} \lambda(s)g(s), \text{ and } \Lambda(n) = \sum_{s=a}^{n} \lambda(s). \tag{2.4.21}$$

As a special case when $\mathbb{T} = \mathbb{N}$, we see that inequality (2.4.18) becomes the following Copson discrete inequality

$$\sum_{n=a}^{\infty} \lambda(n) \left(\frac{A(n)}{\Lambda(n)}\right)^c \leq \left(\frac{c}{c-1}\right)^c \sum_{n=a}^{\infty} \lambda(n) g^c(n), \tag{2.4.22}$$

where $c > 1$ and $A(n)$ and $\Lambda(n)$ are defined as in (2.4.21). As a special case when $\mathbb{T} = \mathbb{N}$ and $a = 1$, we see that inequality (2.4.18) becomes the following Hardy discrete inequality

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{s=1}^{n} g(s)\right)^c \leq \left(\frac{c}{c-1}\right)^c \sum_{n=1}^{\infty} g^c(n).$$

Theorem 2.4.2. *Let $\mathbb{T}$ be a discrete time scale and $c > 1,\ b > 1$ and $\Lambda(t)$, $A(t)$ and $P(t)$ are defined as in Theorem 2.4.1. If*

$$\frac{p^{\sigma}(t)\Lambda^{\sigma^2}(t)}{P^{\sigma^2}(t)} \leq \frac{p(t)\Lambda^{\sigma}(t)}{P^{\sigma}(t)}, \tag{2.4.23}$$

then

$$\begin{aligned} &\int_a^{\infty} \frac{\lambda(t)\mu^{b-1}(t) \left(A^{\sigma}(t)\right)^c p^b(t)}{(P^{\sigma}(t))^{b+c-1} \left(\Lambda^{\sigma}(t)\right)^{1-b}} \Delta t \\ \leq\ & \left(\frac{c}{b-1}\right)^c \int_a^{\infty} \frac{\lambda(t)\mu^{b-1}(t) g^b(t) \left(A^{\sigma}(t)\right)^{c-b}}{\left(P^{\sigma}(t)\right)^{c-1} \left(\Lambda^{\sigma}(t)\right)^{1-b}} \Delta t. \end{aligned} \tag{2.4.24}$$

Proof. We proceed as in the proof of Theorem 2.4.1, to get

$$\begin{aligned} &c\frac{(\mu(t)g(t))^b \lambda^b(t)}{(A^{\sigma}(t))^b} \\ &\geq b\,(\eta(t))^{b-1} \frac{\lambda^{b-1}(t) \left(\left(P^{\sigma}(t)\right)^{1-c} \left(A^{\sigma}(t)\right)^c - \left(P(t)\right)^{1-c} \left(A(t)\right)^c\right)}{\left(P^{\sigma}(t)\right)^{1-c} \left(A^{\sigma}(t)\right)^c \left(\Lambda^{\sigma}(t)\right)^{b-1}} \\ &\quad + b(c-1)\frac{\lambda^{b-1}(t)\mu(t)p(t)\lambda(t)}{\left(\Lambda^{\sigma}(t)\right)^{b-1} P^{\sigma}(t)} \eta^{b-1} + \frac{c(1-b)\lambda^b(t)}{\left(\Lambda^{\sigma}(t)\right)^b} \eta^b. \end{aligned} \tag{2.4.25}$$

Multiplying both sides of (2.4.25) by

$$\frac{\lambda^{1-b}(t)\,(A^\sigma(t))^c\,(\Lambda^\sigma(t))^{b-1}\,(P^\sigma(t))^{1-c}}{\mu(t)},$$

we get that

$$\begin{aligned}
&\frac{c\lambda(t)(g(t)\mu(t))^b\,(A^\sigma(t))^{c-b}}{\mu(t)\,(P^\sigma(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}} \\
&\geq b\,(\eta(t))^{b-1}\left[(P(t))^{1-c}\,(A(t))^c\right]^\Delta + \frac{\lambda(t)\,(A^\sigma(t))^c}{\mu(t)} \\
&\quad\times\left[\frac{b(c-1)\mu(t)p(t)}{(P^\sigma(t))^c}\eta^{b-1} + \frac{c(1-b)\eta^b}{(P^\sigma(t))^{c-1}\,(\Lambda^\sigma(t))}\right]. \qquad (2.4.26)
\end{aligned}$$

Now, we consider the last term on the right hand side, namely

$$f_1(\eta) := M_1\eta^b + K_1\eta^{b-1},$$

as a function of η, where

$$K_1 := \frac{b(c-1)\mu(t)p(t)}{(P^\sigma(t))^c}, \quad \text{and } M_1 := \frac{c(1-b)\,(P^\sigma(t))^{1-c}}{\Lambda^\sigma(t)}.$$

By differentiation, we see that the function $f_1(\eta)$ has a maximum value at

$$\eta = \frac{c-1}{c}\frac{p(t)\mu(t)\Lambda^\sigma(t)}{P^\sigma(t)} > 0,$$

and the maximum value of $f_1(\eta)$ is given by

$$\max_{\eta>0} f_1(\eta) = \frac{(c-1)^b}{c^{b-1}}\frac{p^b(t)\mu^b(t)\,(\Lambda^\sigma(t))^{b-1}}{(P^\sigma(t))^{b+c-1}}.$$

Integrating from a to T, we obtain

$$\begin{aligned}
&c\int_a^T \frac{\lambda(t)(g(t)\mu(t))^b\,(A^\sigma(t))^{c-b}}{\mu(t)\,(P^\sigma(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}}\Delta t \\
&\geq b\int_a^T (\eta(t))^{b-1}\left[(P(t))^{1-c}\,(A(t))^c\right]^\Delta \Delta t \\
&\quad+\frac{(c-1)^b}{c^{b-1}}\int_a^T \frac{\lambda(t)\mu^{b-1}(t)\,(A^\sigma(t))^c\,(\Lambda^\sigma(t))^{b-1}}{p^{-b}(t)(P^\sigma(t))^{b+c-1}}\Delta t. \qquad (2.4.27)
\end{aligned}$$

Using the integration by parts formula, we see that

$$\begin{aligned}
&\int_a^T (\eta(t))^{b-1}\left[(P(t))^{1-c}(A(t))^c\right]^\Delta \Delta t \\
&= \int_a^T (\eta(t))^{b-1}\left[(P(t))^{1-c}(A(t))^c\right]^\Delta \Delta t = (\eta(T))^{b-1}\left[(P(T))^{1-c}(A(T))^c\right] \\
&\quad - \int_a^T \left((\eta(t))^{b-1}\right)^\Delta (P^\sigma(t))^{1-c}(A^\sigma(t))^c \Delta t,
\end{aligned}$$

where $A(a)P(a) = 0$. From the chain rule and condition (2.4.23), we see that

$$\begin{aligned}
\left((\eta(t))^{b-1}\right)^\Delta &= (b-1)(\eta(t))^\Delta \int_0^1 [h\eta^\sigma + (1-h)\eta]^{b-2} dh \\
&= \frac{(b-1)}{\mu(t)}\left(\frac{p^\sigma(t)\Lambda^{\sigma^2}(t)}{P^{\sigma^2}(t)} - \frac{p(t)\Lambda^\sigma(t)}{P^\sigma(t)}\right) \\
&\quad \times \int_0^1 [h(\eta)^\sigma + (1-h)\eta]^{b-2} dh \leq 0.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\int_a^T (\eta(t))^{b-1} \frac{\left[(P^\sigma(t))^{1-c}(A^\sigma(t))^c - (P(t))^{1-c}(A(t))^c\right]}{\mu(t)} \Delta t \\
&= \int_a^T (\eta(t))^{b-1}\left[(P(t))^{1-c}(A(t))^c\right]^\Delta \Delta t \geq 0.
\end{aligned}$$

Using this in (2.4.27), we obtain that

$$\begin{aligned}
&\int_a^T \frac{c\lambda(t)g^b(t)\mu^{b-1}(t)(A^\sigma(t))^{c-b}}{(P^\sigma(t))^{c-1}(\Lambda^\sigma(t))^{1-b}} \Delta t \\
&\geq \frac{(c-1)^b}{c^{b-1}} \int_a^T \frac{\lambda(t)\mu^{b-1}(t)(A^\sigma(t))^c(\Lambda^\sigma(t))^{b-1}}{p^{-b}(t)(P^\sigma(t))^{b+c-1}} \Delta t,
\end{aligned}$$

which is the desired inequality (2.4.24). The proof is complete. ■

Remark 2.4.1. As a special case when $\mathbb{T} = \mathbb{N}$, we see that $\sigma(t) = t + 1$ and then condition (2.4.23) becomes

$$\frac{p(n)\Lambda(n+1)}{P(n+1)} \geq \frac{p(n+1)\Lambda(n+2)}{P(n+2)}, \tag{2.4.28}$$

where $\Lambda(n) = \sum_{s=a}^{n-1} \lambda(n)$ and $P(n) = \sum_{s=a}^{n-1} \lambda(n)p(n)$. One can easily see that condition (2.4.28) is the same condition (2.4.1) imposed by Walsh.

In the following, we prove a companion of inequality (2.4.6) on discrete time scales which contains a discrete inequality of Walsh type.

Theorem 2.4.3. *Let $\mathbb{T}$ be a discrete time scale and $b > 1,\ c > 1$ and define*

$$\begin{cases} \Lambda(t) = \int_a^t \lambda(s)\Delta s,\ \Gamma(t) := \int_t^\infty g(s)\lambda(s)\Delta s, \\ F(t) := \int_t^\infty p(s)\lambda(s)\Delta s, \end{cases} \quad \textit{for } t \in [a,\infty)_{\mathbb{T}}. \tag{2.4.29}$$

If

$$(F(t))^{1-c}(\Gamma(t))^c \geq (F^\sigma(t))^{1-c}(\Gamma^\sigma(t))^c, \tag{2.4.30}$$

then

$$\begin{aligned} &\int_a^\infty \frac{\lambda(t)(\Gamma(t))^c (p(t))^b (\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t \\ &\leq \left(\frac{c}{c-1}\right)^b \int_a^\infty \frac{\lambda(t)(g(t))^b(\Gamma(t))^{c-b}}{(F(t))^{c-1}(\Lambda^\sigma(t))^{1-b}}\Delta t. \end{aligned} \tag{2.4.31}$$

Proof. Let

$$x := \frac{\Gamma^\sigma(t)}{\Gamma(t)} < 1, \quad \beta := \frac{F^\sigma(t)}{F(t)} < 1, \text{ and } \alpha := \eta\frac{\lambda(t)}{\Lambda^\sigma(t)}, \tag{2.4.32}$$

where η is positive will be determined later such that $\alpha < 1$. As a result of these substitutions in (2.4.3), we get that

$$\begin{aligned} &c\left(1-\frac{\Gamma^\sigma(t)}{\Gamma(t)}\right)^b \\ &= c\left(\frac{-\mu(t)\Gamma^\Delta(t)}{\Gamma(t)}\right)^b = c\left(\frac{\mu(t)g(t)\lambda(t)}{\Gamma(t)}\right)^b \\ &\geq b\left(\eta\frac{\lambda(t)}{\Lambda^\sigma(t)}\right)^{b-1}\left(1-\left(\frac{F^\sigma(t)}{F(t)}\right)^{1-c}\left(\frac{\Gamma^\sigma(t)}{\Gamma(t)}\right)^c\right) \\ &\quad + b(c-1)\eta^{b-1}\frac{\lambda^{b-1}(t)\mu(t)p(t)\lambda(t)}{(\Lambda^\sigma(t))^{b-1}F(t)} + \frac{c(1-b)\lambda^b(t)}{(\Lambda^\sigma(t))^b}\eta^b. \end{aligned} \tag{2.4.33}$$

Multiplying both sides by

$$\lambda^{1-b}(t)\,(\Gamma(t))^{c}\,(\Lambda^{\sigma}(t))^{b-1}\,(F(t))^{1-c}\,, \tag{2.4.34}$$

gives

$$\begin{aligned}
&\frac{c\lambda(t)(g(t)\mu(t))^{b}\,(\Gamma(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^{\sigma}(t))^{1-b}} \\
&\geq b\,(\eta(t))^{b-1}\left[(F(t))^{1-c}\,(\Gamma(t))^{c}-(F^{\sigma}(t))^{1-c}\,(\Gamma^{\sigma}(t))^{c}\right]+\lambda(t)\,(\Gamma(t))^{c} \\
&\times\left[\frac{b(c-1)p(t)\mu(t)}{(F(t))^{c}}\eta^{b-1}+\frac{c(1-b)}{(F(t))^{c-1}\,(\Lambda^{\sigma}(t))}\eta^{b}\right].
\end{aligned} \tag{2.4.35}$$

Now, we consider the last term on the right hand side, namely $g(\eta):=M_1\eta^{b}+K_1\eta^{b-1}$, as a function of η, where

$$K_1:=\frac{b(c-1)p(t)\mu(t)}{F^{c}(t)}\text{ and }M_1:=\frac{c(1-b)}{(F(t))^{c-1}\,\Lambda^{\sigma}(t)}.$$

The function $g(\eta)$ has a maximum value at

$$\eta:=\frac{c-1}{c}\frac{p(t)\mu(t)\Lambda^{\sigma}(t)}{F(t)}>0, \tag{2.4.36}$$

and the maximum value is given by

$$\max_{\eta\geq 0}g(\eta)=\frac{(c-1)^{b}}{c^{b-1}}\frac{(p(t)\mu(t))^{b}\,(\Lambda^{\sigma}(t))^{b-1}}{(F(t))^{b+c-1}}. \tag{2.4.37}$$

From (2.4.32) and (2.4.36), we see that

$$\begin{aligned}
\alpha&=\eta\frac{\lambda(t)}{\Lambda^{\sigma}(t)}=\frac{c-1}{c}\frac{p(t)\mu(t)\lambda(t)}{F(t)}=\frac{c-1}{c}\frac{-\mu(t)F^{\Delta}(t)}{F(t)} \\
&=\frac{c-1}{c}\frac{-(F^{\sigma}(t)-F(t))}{F(t)}=\frac{c-1}{c}\frac{F(t)-F^{\sigma}(t)}{F(t)},
\end{aligned}$$

satisfies $0<\alpha<1$. Then (2.4.30), (2.4.35) and (2.4.37) imply that

$$\begin{aligned}
&\frac{c\lambda(t)(g(t)\mu(t))^{b}\,(\Gamma(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^{\sigma}(t))^{1-b}} \\
&\geq b\,(\eta(t))^{b-1}\left[(F(t))^{1-c}\,(\Gamma(t))^{c}-(F^{\sigma}(t))^{1-c}\,(\Gamma^{\sigma}(t))^{c}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(c-1)^b}{c^{b-1}}\frac{\lambda(t)\,(p(t)\mu(t))^b\,(\Gamma(t))^c\,(\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}} \\
&\geq \frac{(c-1)^b}{c^{b-1}}\frac{\lambda(t)\,(p(t)\mu(t))^b\,(\Gamma(t))^c\,(\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}}.
\end{aligned}
$$

This implies after integration from a to T that

$$
\begin{aligned}
&c\int_a^T \frac{\lambda(t)(g(t))^b\,(\Gamma(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}}\Delta t \\
&\quad\geq \frac{(c-1)^b}{c^{b-1}}\int_a^T \frac{\lambda(t)\,(\Gamma(t))^c\,p^b(t)\,(\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_a^T \frac{\lambda(t)\,(\Gamma(t))^c\,p^b(t)\,(\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t \\
&\quad\leq \left(\frac{c}{c-1}\right)^b \int_a^T \frac{\lambda(t)g^b(t)\,(\Gamma(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}}\Delta t.
\end{aligned}
$$

Finally, letting $T\to\infty$, we have that

$$
\begin{aligned}
&\int_a^\infty \frac{\lambda(t)\,(\Gamma(t))^c\,p^b(t)\,(\Lambda^\sigma(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t \\
&\quad\leq \left(\frac{c}{c-1}\right)^b \int_a^\infty \frac{\lambda(t)g^b(t)\,(\Gamma(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^\sigma(t))^{1-b}}\Delta t,
\end{aligned}
$$

which is the desired inequality (2.4.29). The proof is complete. ■

Remark 2.4.2. As a special case of Theorem 2.4.3 when $\mathbb{T}=\mathbb{N}$, we see that $\mu(t)=1$ and

$$
\begin{aligned}
&\Lambda(n)\ :=\sum_{s=a}^{n}\lambda(s)\Delta s,\quad \Gamma(n)=\sum_{s=n}^{\infty}g(s)\lambda(s), \\
&F(n)=\sum_{s=n}^{\infty}p(s)\lambda(s),
\end{aligned}
$$

and condition (2.4.30) becomes

$$
(F(n))^{1-c}\,(\Gamma(n))^c \geq (F(n+1))^{1-c}\,(\Gamma(n+1))^c\,.
$$

In this time scale *the inequality (2.4.31) reduces to*

$$\sum_{n=a}^{\infty} \frac{\lambda(n)\,(\Gamma(n))^{c}\,p^{b}(n)\,(\Lambda_1(n))^{b-1}}{(F(n))^{b+c-1}} \leq \left(\frac{c}{c-1}\right)^{b} \sum_{n=a}^{\infty} \frac{\lambda(n)g^{b}(n)\,(\Gamma(n))^{c-b}}{(F(n))^{c-1}\,(\Lambda(n))^{1-b}}.$$

Theorem 2.4.4. *Let* $\mathbb{T}$ *be a discrete time scale and* $b > 1,\ c > 1$ *and define*

$$\begin{cases} \Lambda(t) = \int_a^t \lambda(s)\Delta s, \ \ A(t) = \int_a^t a(s)\lambda(s)\Delta s, \\ \\ F(t) = \int_t^{\infty} p(s)\lambda(s)\Delta s, \end{cases} \quad \text{for } t \in [a,\infty)_{\mathbb{T}}. \tag{2.4.38}$$

If

$$(F(t))^{1-c}\,(A^{\sigma}(t))^{c} \geq (F^{\sigma}(t))^{1-c}\,(A(t))^{c}, \tag{2.4.39}$$

then

$$\int_a^{\infty} \frac{\lambda(t)\,(A^{\sigma}(t))^{c}\,p^{b}(t)\,(\Lambda^{\sigma}(t))^{b-1}}{(F(t))^{b+c-1}}\Delta t \leq \left(\frac{c}{c-1}\right)^{b} \int_a^{\infty} \frac{\lambda(t)g^{b}(t)\,(A^{\sigma}(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^{\sigma}(t))^{1-b}}\Delta t. \tag{2.4.40}$$

Proof. Let

$$x := \frac{A(t)}{A^{\sigma}(t)} < 1, \quad \beta := \frac{F^{\sigma}(t)}{F(t)} < 1, \ \text{ and } \ \alpha := \eta\frac{\lambda(t)}{\Lambda^{\sigma}(t)}, \tag{2.4.41}$$

where η is positive will be determined later such that $\alpha < 1$. As a result of these substitutions in (2.4.3), we get that

$$\begin{aligned} &c\left(1-\frac{A(t)}{A^{\sigma}(t)}\right)^{b} \\ &= c\left(\frac{\mu(t)A^{\Delta}(t)}{A^{\sigma}(t)}\right)^{b} = c\left(\frac{\mu(t)g(t)\lambda(t)}{A^{\sigma}(t)}\right)^{b} \\ &\geq b\left(\eta\frac{\lambda(t)}{\Lambda^{\sigma}(t)}\right)^{b-1}\left(1-\left(\frac{F^{\sigma}(t)}{F(t)}\right)^{1-c}\left(\frac{A(t)}{A^{\sigma}(t)}\right)^{c}\right) \\ &\quad + b(c-1)\eta^{b-1}\frac{\lambda^{b-1}(t)p(t)\mu(t)\lambda(t)}{(\Lambda^{\sigma}(t))^{b-1}F(t)} + \alpha\frac{c(1-b)\eta^{b}\lambda^{b}(t)}{(\Lambda^{\sigma}(t))^{b}}. \end{aligned}$$

Multiplying both sides by $\lambda^{1-b}(t)\,(A^{\sigma}(t))^{c}\,(\Lambda^{\sigma}(t))^{b-1}\,(F(t))^{1-c}$, we have that

$$\begin{aligned}
&\frac{c\lambda(t)(g(t)\mu(t))^{b}\,(A^{\sigma}(t))^{c-b}}{(F(t))^{c-1}\,(\Lambda^{\sigma}(t))^{1-b}} \\
&\geq b\,(\eta(t))^{b-1}\left[(F(t))^{1-c}\,(A^{\sigma}(t))^{c}-(F^{\sigma}(t))^{1-c}\,(A(t))^{c}\right] \\
&\quad+\lambda(t)\,(A^{\sigma}(t))^{c}\left[\frac{b(c-1)p(t)\mu(t)}{(F(t))^{c}}\eta^{b-1}+\frac{c(1-b)\eta^{b}}{(F(t))^{c-1}\,(\Lambda^{\sigma}(t))}\right].
\end{aligned}$$

The rest of the proof is similar to the proof of Theorem 2.4.1. The proof is complete. ■

Remark 2.4.3. As a special case in Theorem 2.4.4, when $\mathbb{T}=\mathbb{N}$ *and* $b>1,\ c>1$, we see that

$$\Lambda(n)\ :\ =\sum_{s=1}^{n-1}\lambda(s),\ A(n)=\sum_{s=1}^{n-1}g(s)\lambda(s),$$

$$F(n)=\sum_{s=n}^{\infty}p(s)\lambda(s),\quad \textit{for } n\geq 1,$$

and condition (2.4.39) becomes

$$(F(n))^{1-c}\,(A(n+1))^{c}\geq (F(n+1))^{1-c}\,(A(n))^{c}.$$

Then, we have the following discrete inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\lambda(n)\,(\Lambda(n+1))^{b-1}\,p^{b}(n)}{(F(n))^{b+c-1}}\left(\sum_{i=1}^{n}\lambda(i)g(i)\right)^{c} \\
&\leq\left(\frac{c}{c-1}\right)^{b}\sum_{n=1}^{\infty}\left(\sum_{i=1}^{n}\lambda(i)g(i)\right)^{c-b}\frac{\lambda(n)(g(n))^{b}}{(F(n))^{c-1}\,(\Lambda(n+1))^{1-b}}.
\end{aligned}$$

If $c=b$ and $p(n)=1$ for all n, then we have the following inequality

$$\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\lambda(n)\,(\Lambda(n+1))^{c-1}}{(F(n))^{2c-1}}\left(\sum_{i=1}^{n}\lambda(i)g(i)\right)^{c} \\
&\leq\left(\frac{c}{c-1}\right)^{c}\sum_{n=1}^{\infty}\frac{\lambda(n)(g(n))^{c}}{(F(n))^{c-1}\,(\Lambda(n+1))^{1-c}}.
\end{aligned}$$

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