An FPTAS for the Volume of Some $\mathcal{V}$-polytopes—It is Hard to Compute the Volume of the Intersection of Two Cross-Polytopes

Ei Ando[1(✉)] and Shuji Kijima[2,3]

[1] Sojo University, 4-22-1, Ikeda, Nishi-Ku, Kumamoto 860-0082, Japan
ando-ei@cis.sojo-u.ac.jp

[2] Kyushu University, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan
kijima@inf.kyushu-u.ac.jp

[3] JST PRESTO, 744 Motooka, Nishi-ku, Fukuoka 819-0395, Japan

Abstract. Given an n-dimensional convex body by a membership oracle in general, it is known that any polynomial-time *deterministic* algorithm cannot approximate its volume within ratio $(n/\log n)^n$. There is a substantial progress on *randomized* approximation such as Markov chain Monte Carlo for a high-dimensional volume, and for many #P-hard problems, while only a few #P-hard problems are known to yield deterministic approximation. Motivated by the problem of deterministically approximating the volume of a $\mathcal{V}$-polytope, that is a polytope with a small number of vertices and (possibly) exponentially many facets, this paper investigates the problem of computing the volume of a "knapsack dual polytope," which is known to be #P-hard due to Khachiyan (1989). We reduce an approximate volume of a knapsack dual polytope to that of the *intersection of two cross-polytopes*, and give FPTASs for those volume computations. Interestingly, computing the volume of the intersection of two cross-polytopes (i.e., L_1-balls) is #P-hard, unlike the cases of L_∞-balls or L_2-balls.

Keywords: #P-hard · Deterministic approximation · FPTAS · $\mathcal{V}$-polytope · Intersection of L_1-balls

1 Introduction

1.1 Approximation of a High Dimensional Volume: Randomized vs. Deterministic

A high dimensional volume is hard to compute, even for approximation. When an n-dimensional convex body is given by a *membership oracle*, no polynomial-time *deterministic* algorithm can approximate its volume within ratio $(n/\log n)^n$ [3,6,10,21]. The impossibility comes from the fact that the volume of an n-dimensional L_∞-ball (i.e., hypercube) is exponentially large to

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the volume of its inscribed L_2-ball or L_1-ball, despite that the L_2-ball (L_1-ball as well) is convex and touches each facet of the L_∞-ball (see e.g., [23]). Lovász said in [21] for a convex body K that "*If K is a polytope, then there may be much better ways to compute* Vol(K)." Unfortunately, computing an exact volume is often #P-hard, even for a relatively simple polytope. For instance, computing the *volume* of a knapsack polytope $K(b) = \{\boldsymbol{x} \in [0,1]^n \mid \sum_{i=1}^n a_i x_i \leq b\}$, where $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ is the "item sizes" and $b \in \mathbb{Z}_{\geq 0}$ is the "knapsack capacity", is a well-known #P-hard problem [8].

The difficulty caused by the exponential gap between L_∞-ball and L_1-ball also does harm a simple Monte Carlo algorithm. Then, the Markov chain Monte Carlo (MCMC) method achieves a great success for approximating the high dimensional volume. Dyer, Frieze and Kannan [9] gave the first fully polynomial-time randomized approximation scheme (FPRAS) for the volume computation of a general convex body[1]. They employed a *grid-walk*, which is efficiently implemented with a membership oracle, and showed it is rapidly mixing, then they gave an FPRAS runs in $\mathrm{O}^*(n^{23})$ time where O^* ignores $\mathrm{poly}(\log n)$ and $1/\epsilon$ factors. After several improvements, Lovász and Vempala [22] improved the time complexity to $\mathrm{O}^*(n^4)$ in which they employ hit-and-run walk, and recently Cousins and Vempala [5] gave an $\mathrm{O}^*(n^3)$-time algorithm. Many randomized techniques, including MCMC, also have been developed for designing FPRAS for #P-hard problems.

In contrast, the development of *deterministic* approximations for #P-hard problems is a current challenge, and not many results seem to be known. A remarkable progress is the *correlation decay* argument due to Weitz [25]; he designed a *fully polynomial time approximation scheme* (*FPTAS*) for counting independent sets in graphs whose maximum degree is at most 5. A similar technique is independently presented by Bandyopadhyay and Gamarnik [2], and there are several recent developments on the technique, e.g., [4,11,17,18,20]. For counting knapsack solutions[2], Gopalan, Klivans and Meka [12], and Štefankovič, Vempala and Vigoda [24] gave deterministic approximation algorithms based on dynamic programming (see also [13]), in a similar way to the simple random sampling algorithm by Dyer [7]. (He showed a deterministic dynamic programming and a random sampling algorithm in [7].) Modifying dynamic programming in [24], Li and Shi [19] gave an FPTAS that can approximate the volume of a knapsack polytope. Their algorithm runs in $\mathrm{O}((n^3/\epsilon^2)\mathrm{poly}\log b)$ time where b is the knapsack capacity. Motivated by a different approach, Ando and Kijima [1] gave another FPTAS for the volume of a knapsack polytope.

[1] Precisely, they are concerned with a "well-rounded" convex body, after an affine transformation of a general finite convex body.

[2] Given $\boldsymbol{a} \in \mathbb{Z}_{>0}^n$ and $b \in \mathbb{Z}_{>0}$, the problem is to compute $|\{\boldsymbol{x} \in \{0,1\}^n \mid \sum_{i=1}^n a_i x_i \leq b\}|$. Remark that it is computed in polynomial time when all the inputs a_i ($i = 1, \ldots, n$) and b are bounded by $\mathrm{poly}(n)$, using a version of the standard dynamic programming for knapsack problem (see e.g., [7,13]). It should be worth noting that [12,24] needed special techniques, different from ones for optimization problems, to design FPTASs for the counting problem.

Their scheme is based on a classical approximate convolution and runs in $\mathrm{O}(n^3/\epsilon)$ time. The running time is independent of the size of items and the knapsack capacity if we assume that the basic arithmetic operations can be performed in constant time.

1.2 $\mathcal{H}$-polytope and $\mathcal{V}$-polytope

An $\mathcal{H}$-polyhedron is an intersection of finitely many closed half-spaces in $\mathbb{R}^n$. An $\mathcal{H}$-polytope is a bounded $\mathcal{H}$-polyhedron. A $\mathcal{V}$-polytope is a convex hull of a finite point set in $\mathbb{R}^n$ [23]. From the view point of computational complexity, a major difference between an $\mathcal{H}$-polytope and a $\mathcal{V}$-polytope is the measure of their 'input size.' An $\mathcal{H}$-polytope given by linear inequalities defining half-spaces may have vertices exponentially many to the number of the inequalities, e.g., an n-dimensional hypercube is given by $2n$ linear inequalities as an $\mathcal{H}$-polytope, and has 2^n vertices. In contrast, a $\mathcal{V}$-polytope given by a point set may have facets exponentially many to the number of vertices, e.g., an n-dimensional cross-polytope (that is an L_1-ball, in fact) is given by a set of $2n$ points as a $\mathcal{V}$-polytope, and it has 2^n facets.

There are many interesting properties between $\mathcal{H}$-polytope and $\mathcal{V}$-polytope [23]. A membership query is polynomial time for both $\mathcal{H}$-polytope and $\mathcal{V}$-polytope. It is still unknown about the complexity of a query if a given pair of $\mathcal{V}$-polytope and $\mathcal{H}$-polytope are identical. Linear programming (LP) on a $\mathcal{V}$-polytope is trivially polynomial time since it is sufficient to check the objective value of all vertices and hence LP is usually concerned with an $\mathcal{H}$-polytope.

1.3 Volume of $\mathcal{V}$-polytope

Motivated by a hardness of the volume computation of a $\mathcal{V}$-polytope, Khachiyan [15] is concerned with the following $\mathcal{V}$-polytope: Suppose a vector $\boldsymbol{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is given. Then let

$$P_{\boldsymbol{a}} \stackrel{\text{def}}{=} \mathrm{conv}\,\{\pm \boldsymbol{e}_1, \ldots, \pm \boldsymbol{e}_n, \boldsymbol{a}\} \tag{1}$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are the standard basis vectors in $\mathbb{R}^n$. This paper calls $P_{\boldsymbol{a}}$ *knapsack dual polytope*[3]. Khachiyan [15] showed that computing $\mathrm{Vol}(P_{\boldsymbol{a}})$ is $\#P$-hard[4]. The hardness is given by a Cook reduction from counting set partitions, of which the decision version is a celebrated *weakly* NP-hard problem. It is not known if

[3] See [23] for the duality of polytopes. In fact, $P_{\boldsymbol{a}}$ itself is not the dual of a knapsack polytope in a canonical form, but it is obtained by an affine transformation from a dual of knapsack polytope under some assumptions. Khachiyan [16] says that computing $\mathrm{Vol}(P_{\boldsymbol{a}})$ 'is "polar" to determining the volume of the intersection of a cube and a halfspace.' .

[4] If all a_i $(i = 1, \ldots, n)$ are bounded by $\mathrm{poly}(n)$, it is computed in polynomial time, so did the counting knapsack solutions. See also footnote 1 for counting knapsack solutions.

we can have an efficient approximation algorithm for computing $\mathrm{Vol}(P_{\boldsymbol{a}})$ immediately from the approximation algorithm in e.g., [1] by exploiting that $P_{\boldsymbol{a}}$ is a dual of a knapsack polytope.

1.4 Contribution

Motivated by a development of techniques for *deterministic* approximation of the volumes of $\mathcal{V}$-polytopes, this paper investigates the knapsack dual polytope $P_{\boldsymbol{a}}$ given by (1). The main goal of the paper is to establish the following theorem.

Theorem 1. *For any ϵ ($0 < \epsilon < 1$), there exists a* deterministic *algorithm that outputs a value $\widehat{V}$ satisfying* $(1-\epsilon)\mathrm{Vol}(P_{\boldsymbol{a}}) \leq \widehat{V} \leq (1+\epsilon)\mathrm{Vol}(P_{\boldsymbol{a}})$ *in* $\mathrm{O}(n^{10}\epsilon^{-6})$ *time.*

As far as we know, this is the first result on designing an FPTAS for computing the volume of a $\mathcal{V}$-polytope which is known to be #P-hard. We also discuss some topics related to the volume of $\mathcal{V}$-polytopes appearing in the proof process. Let us briefly explain the outline of the paper.

Technique/Organization. The first step for Theorem 1 is a transformation of the *approximation problem* to another one: An approximate volume of $P_{\boldsymbol{a}}$ is reduced to the volume of a union of geometric sequence of cross-polytopes (Sect. 3.1), and then it is reduced to the volume of the intersection of two cross-polytopes (Sect. 3.2). We remark that the former reduction is just for approximation, and is useless for proving #P-hardness. A technical point of this step is that the latter reduction is based on a subtraction—if you are familiar with an approximation, you may worry that a subtraction may destroy an approximation ratio[5]. It requires careful tuning of a parameter (β in Sect. 3) which plays conflicting functions in Sects. 3.1 and 3.2: the larger β, the better approximation in Sect. 3.1, while the smaller β, the better in Sect. 3.2. Then, Sect. 3.3 claims by giving an appropriate β that if we have an FPTAS for the volume of an *intersection of two cross-polytopes* then we have an FPTAS of $\mathrm{Vol}(P_{\boldsymbol{a}})$.

Section 4 shows an FPTAS for the volume of the intersection of two cross-polytopes (i.e., L_1-balls). The scheme is based on a modified version of the technique developed in [1], which is based on a classical approximate convolution. At a glance, the volume of the intersection of two-balls may seem easy. It is true for two L_∞-balls (i.e., axis-aligned hypercubes), or L_2-balls (i.e., Euclidean balls). However, we show in Sect. 5 that computing the volume of the intersection of cross-polytopes is #P-hard. Intuitively, this interesting fact may come from the fact that the $\mathcal{V}$-polytope, meaning that an n-dimensional cross-polytope, has 2^n facets. In Sect. 6, we extend the technique in Sect. 4 to the intersection of any constant number of cross-polytopes.

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[5] Suppose you know that x is approximately 49 within 1% error. Then, you know that $x+50$ is approximately 99 within 1% error. However, it is difficult to say $50-x$ is approximately 1. Even when additionally you know that x does not exceed 50, $50-x$ may be 2, 1, 0.1 or smaller than 0.001, meaning that the approximation ratio is unbounded.

2 Preliminaries

This section presents some notation. Let $\mathrm{conv}(S)$ denote the convex hull of $S \subseteq \mathbb{R}^n$, where S is not restricted to a finite point set. A *cross-polytope* $C(\boldsymbol{c}, r)$ of radius $r \in \mathbb{R}_{>0}$ centered at $\boldsymbol{c} \in \mathbb{R}^n$ is given by

$$C(\boldsymbol{c}, r) \stackrel{\mathrm{def}}{=} \mathrm{conv}\{\boldsymbol{c} \pm r\boldsymbol{e}_i \ i = 1, \ldots, n\}$$

where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ are the standard basis vectors in $\mathbb{R}^n$. Clearly, $C(\boldsymbol{c}, r)$ has $2n$ vertices. In fact, $C(\boldsymbol{c}, r)$ is an L_1-ball in $\mathbb{R}^n$ described by

$$C(\boldsymbol{c}, r) = \{\boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{c}\|_1 \leq r\} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \langle \boldsymbol{x} - \boldsymbol{c}, \boldsymbol{\sigma} \rangle \leq r \ (\forall \boldsymbol{\sigma} \in \{-1, 1\}^n)\}$$

where $\|\boldsymbol{u}\|_1 = \sum_{i=1}^n |u_i|$ for $\boldsymbol{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and $\langle \boldsymbol{u}, \boldsymbol{v} \rangle = \sum_{i=1}^n u_i v_i$ for $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^n$. Note that $C(\boldsymbol{c}, r)$ has 2^n facets. It is not difficult to see that the volume of a cross-polytope in n-dimension is $\mathrm{Vol}(C(\boldsymbol{c}, r)) = \frac{2^n}{n!} r^n$ for any $r \geq 0$ and $\boldsymbol{c} \in \mathbb{R}^n$, where $\mathrm{Vol}(S)$ for $S \subseteq \mathbb{R}^n$ denotes the (n-dimensional) volume of S.

3 FPTAS for Knapsack Dual Polytope

This section reduces an approximation of $\mathrm{Vol}(P_{\boldsymbol{a}})$ to that of the intersection of two cross-polytopes. In Sect. 4, we will give an FPTAS for the volume of the intersection of two cross-polytopes, accordingly we obtain Theorem 1.

3.1 Reduction to a Geometric Series of Cross-Polytopes

Let β be a parameter[6] satisfying $0 < \beta < 1$, and let $Q_0, Q_1, Q_2, \ldots$ be a sequence of cross-polytopes defined by

$$Q_k \stackrel{\mathrm{def}}{=} C((1 - \beta^k)\boldsymbol{a}, \beta^k) \tag{2}$$

for $k = 0, 1, 2, \ldots$. Remark that $Q_0 = C(\boldsymbol{0}, 1), Q_1 = C((1 - \beta)\boldsymbol{a}, \beta), Q_\infty = C(\boldsymbol{a}, 0) = \{\boldsymbol{a}\}$. The goal of Sect. 3.1 is to establish the following. Here $1 \pm \epsilon$ is the final relative approximation ratio that we aim to achieve.

Lemma 1. *Let ϵ satisfy $0 < \epsilon < 1$. If $1 - \beta \leq \dfrac{c_1 \epsilon}{n\|\boldsymbol{a}\|_1}$ where $0 < c_1 < 1$, then $(1 - c_1\epsilon)\mathrm{Vol}(P_{\boldsymbol{a}}) \leq \mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) \leq \mathrm{Vol}(P_{\boldsymbol{a}})$.*

Figure 1 illustrates the approximation of $P_{\boldsymbol{a}}$ by this infinite sequence of cross-polytopes. The second inequality in Lemma 1 is relatively easy by the following lemma. [7]

Lemma 2. $\bigcup_{k=0}^{\infty} Q_k \subseteq P_{\boldsymbol{a}}$

To prove the first inequality in Lemma 1, we need the following lemmas.

Lemma 3. $\bigcup_{k=0}^{\infty} \mathrm{conv}(Q_k \cup Q_{k+1}) \cup \{\boldsymbol{a}\} \supseteq P_{\boldsymbol{a}}$

Lemma 4. *If $1 - \beta \leq \dfrac{c_1 \epsilon}{n\|\boldsymbol{a}\|_1}$, then $\mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) \geq (1 - c_1\epsilon)\mathrm{Vol}(P_{\boldsymbol{a}})$.*

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[6] We will set $\beta = 1 - \dfrac{\epsilon}{2n\|\boldsymbol{a}\|_1}$, later.

[7] Most of the proofs cannot be included due to the space limit.

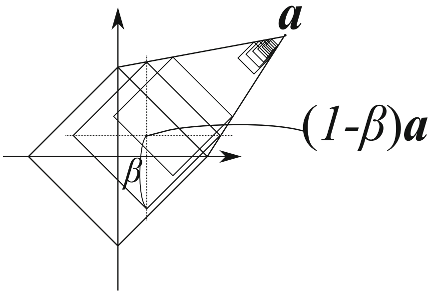

Fig. 1. Approximating $P_{\boldsymbol{a}}$ by an infinite sequence of cross-polytopes.

3.2 Reduction to the Intersection of Two Cross-Polytopes

We here claim the following.

Lemma 5. $\mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) = \frac{1}{1-\beta^n}\left(\frac{2^n}{n!} - \mathrm{Vol}(Q_1 \cap Q_0)\right)$

The first step of the proof is the following recursive formula.

Lemma 6. $\bigcup_{k=0}^{m} Q_k = \left(\dot{\bigcup}_{k=0}^{m-1} Q_k \setminus Q_{k+1}\right) \dot{\cup}\, Q_m$ *where* $A \dot{\cup} B$ *denotes the disjoint union of* A *and* B*, meaning that* $A \dot{\cup} B = A \cup B$ *and* $A \cap B = \emptyset$*.*

The second step is the following lemma.

Lemma 7. $\mathrm{Vol}(Q_k \setminus Q_{k+1}) = \beta^{nk}\mathrm{Vol}(Q_0 \setminus Q_1)$*.*

By using Lemmas 6 and 7, we can prove Lemma 5 as follows.

Proof (Proof of Lemma 5).

$$\begin{aligned}
\mathrm{Vol}\left(\bigcup_{k=0}^{\infty} Q_k\right) &= \mathrm{Vol}\left(\left(\dot{\bigcup}_{k=0}^{\infty} Q_k \setminus Q_{k+1}\right) \dot{\cup}\, Q_\infty\right) \\
&= \sum_{k=0}^{\infty} \mathrm{Vol}(Q_k \setminus Q_{k+1}) + \mathrm{Vol}(Q_\infty) = \sum_{k=0}^{\infty} \beta^{nk}\mathrm{Vol}(Q_0 \setminus Q_1) \\
&= \frac{1}{1-\beta^n}\mathrm{Vol}(Q_0 \setminus Q_1) = \frac{1}{1-\beta^n}\left(\frac{2^n}{n!} - \mathrm{Vol}(Q_1 \cap Q_0)\right)
\end{aligned}$$

□

A reader who are familiar with approximation may worry about the subtraction $\frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)$ in Lemma 5. We claim the following.

Lemma 8. *If* $1 - \beta \geq \dfrac{c_2\epsilon}{n\|\boldsymbol{a}\|_1}$ *and* $0 < c_2\epsilon < 1$*, then* $\mathrm{Vol}(Q_0 \cap Q_1) \leq \dfrac{1}{1 + \frac{c_2\epsilon}{2n}} \dfrac{2^n}{n!}$.

Intuitively, Lemma 8 implies that $\frac{2^n}{n!} - \mathrm{Vol}(Q_0 \cap Q_1)$ is large enough, and an approximation of $\mathrm{Vol}(Q_0 \cap Q_1)$ provides a good approximation of $\mathrm{Vol}(\bigcup_{k=0}^{\infty} Q_k)$, and hence $\mathrm{Vol}(P_{\boldsymbol{a}})$.

3.3 Approximation Algorithm and Analysis

Based on Lemma 1 in Sect. 3.1 and Lemma 5 in Sect. 3.2, we give an FPTAS for $\mathrm{Vol}(P_{\boldsymbol{a}})$ where we assume an algorithm to approximate $\mathrm{Vol}(Q_0 \cap Q_1)$.

Algorithm 1 ($(1 \pm \epsilon)$-approximation $(0 < \epsilon \leq 1/2)$)
Input: $\boldsymbol{a} \in \mathbb{Z}_+^n$;
1. Set parameter $\beta := 1 - \dfrac{\epsilon}{2n\|\boldsymbol{a}\|_1}$;
2. Approximate $I \stackrel{\text{def}}{=} \mathrm{Vol}(C(\mathbf{0},1) \cap C((1-\beta)\boldsymbol{a}, \beta))$ by Z such that $I \leq Z \leq \left(1 + \frac{\epsilon^2}{4n}\right) I$;
3. Output $\widehat{V} = \frac{1+\epsilon}{1-\beta^n}\left(\frac{2^n}{n!} - Z\right)$.

Lemma 9. *The output $\widehat{V}$ of Algorithm 1 satisfies*

$$(1-\epsilon)\,\mathrm{Vol}(P_{\boldsymbol{a}}) \leq \widehat{V} \leq (1+\epsilon)\mathrm{Vol}(P_{\boldsymbol{a}}).$$

4 The Volume of the Intersection of Two Cross-Polytopes

This section gives an FPTAS for the volume of the intersection of two cross-polytopes in the n-dimensional space. Without loss of generality[8], we are concerned with $\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c}, r))$ for $\boldsymbol{c} \geq \mathbf{0}$ and r $(0 < r \leq 1)$. This section establishes the following.

Theorem 2. *For any δ $(0 < \delta < 1)$, there exists a* deterministic *algorithm which outputs a value Z satisfying $\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c}, r)) \leq Z \leq (1+\delta)\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c}, r))$ for any input $\boldsymbol{c} \geq \mathbf{0}$ and r $(0 < r \leq 1)$ satisfying $\|\boldsymbol{c}\|_1 \leq r$, and runs in $O(n^7\delta^{-3})$ time.*

The assumption that $\|\boldsymbol{c}\|_1 \leq r$ implies both centers $\mathbf{0}$ and $\boldsymbol{c}$ are contained in the intersection $C(\mathbf{0},1) \cap C(\boldsymbol{c}, r)$. Note that the assumption does not harm to our main goal Theorem 1 (recall Algorithm 1 in Sect. 3.3). We show in Sect. 5 that Computing $\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c}, r))$ remains $\#P$-hard even on the assumption.

4.1 Preliminaries: Convolution for the Volume

As a preliminary step, Sect. 4.1 gives a convolution which provides $\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c}, r))$. Let $\Psi_0 \colon \mathbb{R}^2 \to \mathbb{R}$ be given by $\Psi_0(u,v) = 1$ if $u \geq 0$ and $v \geq 0$, otherwise $\Psi_0(u,v) = 0$. Inductively, we define $\Psi_i \colon \mathbb{R}^2 \to \mathbb{R}$ for $i = 1, 2, \ldots, n$ by

$$\Psi_i(u,v) \stackrel{\text{def}}{=} \int_{-1}^{1} \Psi_{i-1}(u - |s|, v - |s - c_i|)\mathrm{d}s \tag{3}$$

for $u, v \in \mathbb{R}$. We remark that $\Psi_i(u,v) = 0$ holds if $u \leq 0$ or $v \leq 0$, for any $i = 1, 2, \ldots, n$ by the definition.

[8] Remark that $\mathrm{Vol}(C(\boldsymbol{c}, r) \cap C(\boldsymbol{c}', r')) = r^n \mathrm{Vol}\left(C(\mathbf{0},1) \cap C\left(\frac{(\boldsymbol{c}-\boldsymbol{c}')^+}{r}, \frac{r'}{r}\right)\right)$ holds for any $\boldsymbol{c}, \boldsymbol{c}' \in \mathbb{R}^n$ and $r, r' \in \mathbb{R}_{>0}$, where $(\boldsymbol{c}-\boldsymbol{c}')^+ = (|c_1 - c_1'|, |c_2 - c_2'|, \ldots, |c_n - c_n'|)$.

Lemma 10. $\Psi_n(1,r) = \mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c},r))$

To prove Lemma 10, it might be helpful to introduce a probability space. Let $\boldsymbol{X} = (X_1,\ldots,X_n)$ be a uniform random variable over $[-1,1]^n$, i.e., X_i $(i=1,\ldots,n)$ are mutually independent. Then, $\Pr\left[X \in C(\mathbf{0},1) \cap C(\boldsymbol{c},r)\right] = \frac{\mathrm{Vol}(C(\mathbf{0},1)\cap C(\boldsymbol{c},r))}{\mathrm{Vol}([-1,1]^n)} = \frac{1}{2^n}\mathrm{Vol}(C(\mathbf{0},1) \cap C(\boldsymbol{c},r))$ holds.

Lemma 11. *For any $u,v \in \mathbb{R}$ and any $i = 1,2,\ldots,n$,*

$$\frac{1}{2^i}\Psi_i(u,v) = \Pr\left[\left(\sum_{j=1}^{i} |X_j| \leq u\right) \wedge \left(\sum_{j=1}^{i} |X_j - c_j| \leq v\right)\right].$$

Now, Lemma 10 is easy from Lemma 11.

4.2 Idea for Approximation

Our FPTAS is based on an approximation of $\Psi_i(u,v)$. Let $G_0(u,v) = \Psi_0(u,v)$ for any $u,v \in \mathbb{R}$, i.e., $G_0(u,v) = 1$ if $u \geq 0$ and $v \geq 0$, otherwise $G_0(u,v) = 0$. Inductively assuming $G_{i-1}(u,v)$, we define

$$\overline{G}_i(u,v) \stackrel{\text{def}}{=} \int_{-1}^{1} G_{i-1}(u-|s|, v-|s-c_i|)\mathrm{d}s \tag{4}$$

for $u,v \in \mathbb{R}$, for convenience. Then, let $G_i(u,v)$ be a staircase approximation of $\overline{G}_i(u,v)$, given by

$$G_i(u,v) \stackrel{\text{def}}{=} \begin{cases} \overline{G}_i\left(\frac{1}{M}k, \frac{r}{M}\ell\right) & \left(\begin{array}{c}\text{if } \frac{1}{M}(k-1) < u \leq \frac{1}{M}k \ (k=1,2,\ldots), \text{ and} \\ \frac{r}{M}(\ell-1) < v \leq \frac{r}{M}\ell \ \ (\ell=1,2,\ldots).\end{array}\right) \\ 0 & \text{(otherwise)} \end{cases} \tag{5}$$

for any $u,v \in \mathbb{R}$. Thus, we remark that

$$G_i(u,v) = G_i\left(\tfrac{1}{M}\lceil Mu\rceil, \tfrac{r}{M}\left\lceil \tfrac{M}{r}v\right\rceil\right) \tag{6}$$

holds for any $u,v \in \mathbb{R}$, by the definition. Section 4.3 will show that $G_i(u,v)$ approximates $\Psi_i(u,v)$ well.

In the rest of Sect. 4.2, we briefly comment on the computation of G_i. First, remark that (4) implies that $\overline{G}_i(u,v)$ is computed only from $G_{i-1}(u',v')$ for $u' \leq u$ and $v' \leq v$, i.e., we do not need to know $G_{i-1}(u',v')$ for $u' > u$ or $v' > v$. Second, remark (6) implies that $G_i(u,v)$ for $u \leq 1$ and $v \leq r$ takes (at most) $(M+1)^2$ different values. Precisely, let

$$\Gamma \stackrel{\text{def}}{=} \left\{\frac{1}{M}(k, r\ell) \mid k = 0,1,2,\ldots,M,\ \ell = 0,1,2,\ldots,M\right\}$$

then $G_i(u,v)$ for $(u,v) \in \Gamma$ provides all possible values of $G_i(u,v)$ for $u \leq 1$ and $v \leq r$, since (6).

Then, we explain how to compute $G_i(u,v)$ for $(u,v) \in \Gamma$ from G_{i-1}. For an arbitrary $(u,v) \in \Gamma$, let

$$\begin{aligned} S(u) &\stackrel{\text{def}}{=} \left\{ s \in [-1,1] \mid u - |s| = \tfrac{1}{M} k \ (k = 0,1,2,\ldots,M) \right\} \\ &= \left\{ s \in [-1,1] \mid s = \pm(u - \tfrac{1}{M}k) \ (k = 0,1,2,\ldots,M) \right\}, \end{aligned}$$

let

$$\begin{aligned} S_i(v) &\stackrel{\text{def}}{=} \left\{ s \in [-1,1] \mid v - |s - c_i| = \tfrac{r}{M} \ell \ (\ell = 0,1,2,\ldots,M) \right\} \\ &= \left\{ s \in [-1,1] \mid s = c_i \pm (v - \tfrac{r}{M}\ell) \ (\ell = 0,1,2,\ldots,M) \right\}, \end{aligned}$$

and let $T_i(u,v) \stackrel{\text{def}}{=} S(u) \cup S_i(v) \cup \{-1, 0, c_i, 1\}$. Suppose $t_0, t_1, \ldots, t_m$ be an ordering of all elements of $T_i(u,v)$ such that $t_i \leq t_{i+1}$ for any $i = 0,1,\ldots,m$, where $m = |T_i(u,v)|$. Then, we can compute $G_i(u,v)$ for any $(u,v) \in \Gamma$ by $G_i(u,v) = \overline{G}_i(u,v)$, which can be transformed into

$$\begin{aligned} \overline{G}_i(u,v) &= \int_{-1}^{1} G_{i-1}(u - |s|, v - |s - c_i|) \mathrm{d}s \\ &= \sum_{j=0}^{m-1} (t_{j+1} - t_j) G_{i-1} \left(\tfrac{1}{M} \lceil M(u - |t_{j+1}|) \rceil, \tfrac{r}{M} \left\lceil \tfrac{M}{r}(v - |t_{j+1} - c_i|) \right\rceil \right) \end{aligned} \tag{7}$$

where we remark again that the terms of (7) consist of $G_{i-1}(u,v)$ for $(u,v) \in \Gamma$.

4.3 Algorithm and Analysis

Based on the arguments in Sect. 4.2, our algorithm is described as follows.

Algorithm 2 (for $(1+\delta)$-approximation $(0 < \delta \leq 1)$)
Input: $\boldsymbol{c} \in \mathbb{Q}^n_{\geq 0}$, $r \in \mathbb{Q}$ $(0 \leq r \leq 1)$;
1. Set $M := \lceil 4n^2\delta^{-1} \rceil$;
2. Set $G_0(u,v) := 1$ *for* $(u,v) \in \Gamma$, *otherwise* $G_0(u,v) := 0$;
3. For $i := 1, \ldots, n$,
4. For $(u,v) \in \Gamma$,
5. Compute $G_i(u,v)$ *from* G_{i-1} *by (7);*
6. Output $G_n(1,r)$.

Lemma 12. *The running time of Algorithm 2 is* $\mathrm{O}(n^7\delta^{-3})$.

Theorem 2 is immediate from Lemma 12 and the following Lemma 13.

Lemma 13. $\Psi_n(1,r) \leq G_n(1,r) \leq (1+\delta)\Psi_n(1,r)$.

The proof sketch of Lemma 13 is the following.

Proof (Proof Sketch of Lemma 13). The first inequality is immediate. Then, we show the latter inequality. We can prove that

$$\frac{\Psi_n(1,r)}{\Psi_n(1+\frac{n}{M}, r(1+\frac{n}{M}))} \geq \left(\frac{M}{M+n}\right)^{2n} = \left(\frac{1}{1+\frac{n}{M}}\right)^{2n} \geq \left(1-\frac{n}{M}\right)^{2n}$$
$$\geq \left(1-\frac{\delta}{4n}\right)^{2n} \geq 1-2n\frac{\delta}{4n} = 1-\frac{\delta}{2}.$$

Then, $\frac{\Psi_n(1+\frac{n}{M}, r(1+\frac{n}{M}))}{\Psi_n(1,r)} \leq \frac{1}{1-\frac{\delta}{2}} \leq 1+\delta$ for any $\delta \leq 1$, and we obtain the claim. □

5 Hardness of the Volume of the Intersection of Two Cross-Polytopes

This section establishes the following.

Theorem 3. *Given a vector $\boldsymbol{c} \in \mathbb{Z}^n_{>0}$ and integers $r_1, r_2 \in \mathbb{Z}_{>0}$, computing the volume of $C(\mathbf{0}, r_1) \cap C(\boldsymbol{c}, r_2)$ is #P-hard, even when each cross-polytopes contains the center of the other one, i.e., $\mathbf{0} \in C(\boldsymbol{c}, r_2)$ and $\boldsymbol{c} \in C(\mathbf{0}, r_1)$.*

The proof of Theorem 3 is a reduction of counting set partitions, which is a well-known #P-hard problem. To be precise, we reduce the following problem, which is a version of counting set partition (for the #P-hardness of counting set partition, see e.g., [14]).

Problem 1 (#LARGE SET). Given an integer vector $\boldsymbol{a} \in \mathbb{Z}^n_{>0}$ such that $\|\boldsymbol{a}\|_1$ is even, meaning that $\|\boldsymbol{a}\|_1/2$ is an integer, the problem is to compute

$$|\{\boldsymbol{\sigma} \in \{-1,1\}^n \mid \langle \boldsymbol{\sigma}, \boldsymbol{a} \rangle > 0\}|. \tag{8}$$

Note that $|\{\boldsymbol{\sigma} \in \{-1,1\}^n \mid \langle \boldsymbol{\sigma}, \boldsymbol{a} \rangle = 0\}| = \left|\left\{S \subseteq \{1,\ldots,n\} \mid \sum_{i\in S} a_i = \frac{\|\boldsymbol{a}\|_1}{2}\right\}\right|$ holds: if $\boldsymbol{\sigma} \in \{-1,1\}^n$ satisfies $\langle \boldsymbol{\sigma}, \boldsymbol{a} \rangle = 0$, then let $S \subseteq \{1,\ldots,n\}$ be the set of indices of $\sigma_i = 1$ then $\sum_{i\in S} a_i = \|\boldsymbol{a}\|_1/2$ holds. Using the following simple observation, we see that Problem 1 is equivalent to counting set partitions.

Observation 1. *For any $\boldsymbol{\sigma} \in \{-1,1\}^n$, $\langle \boldsymbol{\sigma}, \boldsymbol{a} \rangle > 0$ if and only if $\langle -\boldsymbol{\sigma}, \boldsymbol{a} \rangle < 0$.*

By Observation 1, we see that $|\{\boldsymbol{\sigma} \in \{-1,1\}^n \mid \langle \boldsymbol{\sigma}, \boldsymbol{a} \rangle = 0\}|$ is equal to $2^n - 2\,|\{\boldsymbol{\sigma} \in \{-1,1\}^n \mid \langle \boldsymbol{\sigma}, \boldsymbol{a} \rangle > 0\}|$.

In the following, let $\boldsymbol{a} \in \mathbb{Z}^n_{>0}$ be an instance of Problem 1. Roughly speaking, our proof of Theorem 3 claims that the volume of $(C(\delta\boldsymbol{a}, 1) \cap C(\mathbf{0}, 1+\epsilon)) \setminus C(\mathbf{0}, 1)$ is proportional to the answer of #LARGE SET when $0 < \epsilon < \delta \ll 1/\|\boldsymbol{a}\|_1$. If we could compute the volume of the intersection of two cross-polytopes exactly, then we obtain $\mathrm{Vol}((C(\delta\boldsymbol{a}, 1) \cap C(\mathbf{0}, 1+\epsilon)) \setminus C(\mathbf{0}, 1)) = \mathrm{Vol}(C(\delta\boldsymbol{a}, 1) \cap C(\mathbf{0}, 1+\epsilon)) - \mathrm{Vol}(C(\delta\boldsymbol{a}, 1) \cap C(\mathbf{0}, 1))$, which would solve #LARGE SET.

6 Intersection of a Constant Number of Cross-Polytopes

Let $\boldsymbol{p}_i \in \mathbb{R}^n$, $r_i \in \mathbb{R}_{\geq 0}$ and $C(\boldsymbol{p}_i, r_i)$ for $i = 1, \ldots, k$, where $C(\boldsymbol{p}, r)$ is a cross-polytope (L_1-ball) with center $\boldsymbol{p} \in \mathbb{R}^n$ and radius $r \in \mathbb{R}_{\geq 0}$. Then, we are to compute the following polytope given by $S(\Pi, \boldsymbol{r}) = \bigcap_{i=1}^{k} C(\boldsymbol{p}_i, r_i)$, where Π is an $n \times k$ matrix $\Pi = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k)$ and $\boldsymbol{r} = (r_1, \ldots, r_k)$. For the analysis, we assume that $\boldsymbol{p}_1, \ldots, \boldsymbol{p}_k$ are internal points of $S(\Pi, \boldsymbol{r})$.

Theorem 4. *There is an algorithm that outputs an approximation Z of $\mathrm{Vol}(S(\Pi, \boldsymbol{r}))$ in $O(k^{k+2} n^{2k+3} / \delta^{k+1})$ time satisfying $\mathrm{Vol}(S(\Pi, \boldsymbol{r})) \leq Z \leq (1 + \delta)\mathrm{Vol}(S(\Pi, \boldsymbol{r}))$.*

7 Conclusion

Motivated by the problem of deterministically approximating the volume of a $\mathcal{V}$-polytope, this paper gave an FPTAS for the volume of the knapsack dual polytope $\mathrm{Vol}(P_{\boldsymbol{a}})$. In the process, we showed that computing the volume of the intersection of L_1-balls is #P-hard, and gave an FPTAS. As we remarked, the volume of the intersection of two L_q-balls are easy for $q = 2, \infty$. The complexity of the volume of the intersection of two L_q-balls for other $q > 0$ is interesting. The problem seems difficult even for approximation in the case of $q \in (0, 1)$, since L_q-ball is no longer convex. Our FPTAS for the intersection of two cross-polytopes assumes that each cross-polytope contains the center of the other one. It is open if an FPTAS exists without the assumption.

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