Chapter 2
General Anisotropic Elasticity

Abstract This Chapter is an introduction to general anisotropic elasticity, i.e. to the elasticity of 3D anisotropic bodies. The main classical topics of the matter are treated in detail: starting from the Hooke's law for anisotropic bodies, the two principal notations of Voigt and Kelvin are introduced and the reasons for the use of the last one are argued. Then, after an explanation of the mechanical meaning of the anisotropic elastic constants, the key topic of elastic symmetries is treated in detail. The technical elastic constants are then introduced as well as the elastic bounds for the components of the elastic tensor and for the technical constants. After an observation on the decomposition of the strain energy for anisotropic bodies, the Chapter ends with the determination of the symmetry planes, the curvilinear anisotropy and some examples of typical anisotropic materials.

2.1 The Hooke's Law for Anisotropic Bodies

Be Ω a body acted upon by body forces $\mathbf{f}$ and by surface tractions $\mathbf{t}$ on its frontier $\partial\Omega$ whose outward unit normal is $\mathbf{n}$. We consider a small arbitrary variation $\delta\mathbf{u}$ of the displacement field on Ω, compatible with the given boundary conditions and satisfying the kinematical conditions (1.25). The total mechanical work dW done by the applied forces can be transformed as follows

$$\begin{aligned} dW = &\int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\,d\omega + \int_{\partial\Omega}\mathbf{t}\cdot\delta\mathbf{u}\,ds = \int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\,d\omega + \int_{\partial\Omega}\boldsymbol{\sigma}\mathbf{n}\cdot\delta\mathbf{u}\,ds = \\ &\int_\Omega \mathbf{f}\cdot\delta\mathbf{u}\,d\omega + \int_{\partial\Omega}\boldsymbol{\sigma}\delta\mathbf{u}\cdot\mathbf{n}\,ds = \int_\Omega[\mathbf{f}\cdot\delta\mathbf{u} + \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u})]\,d\omega = \\ &\int_\Omega[(\mathbf{f}+\mathrm{div}\boldsymbol{\sigma})\cdot\delta\mathbf{u} + \boldsymbol{\sigma}\cdot\nabla\delta\mathbf{u}]\,d\omega = \int_\Omega \boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}\,d\omega \end{aligned} \tag{2.1}$$

P. Vannucci, *Anisotropic Elasticity*, Lecture Notes in Applied and Computational Mechanics 85, DOI 10.1007/978-981-10-5439-6_2

In establishing Eq. (2.1) we have used successively the Cauchy's stress theorem (1.26), the symmetry of $\boldsymbol{\sigma}$, the Gauss theorem, two standard results for tensor and vector fields[1] and the motion equation (1.27).

The quantity

$$\delta V=\boldsymbol{\sigma}\cdot\delta\boldsymbol{\varepsilon}=\sigma_{ij}\delta\varepsilon_{ij}, \tag{2.2}$$

represents the variation of the internal energy of the body per unit of volume produced by a small variation of the strain state.

Following the energetic approach of Green (1839), we define as *elastic* a body for which the total variation ΔV of the internal energy due to a finite transformation from a state A to a state B is independent from the integration path. In particular, ΔV must then be null for any transformation where A=B:

$$\Delta V=\int_A^B\delta V=V_B-V_A. \tag{2.3}$$

Hence, for an elastic body δV must be the exact differential dV of a scalar function $V(\boldsymbol{\varepsilon})$, the *strain energy density* or *elastic potential*[2]:

$$V=V(\boldsymbol{\varepsilon}):\ \ dV=\frac{\partial V}{\partial\varepsilon_{ij}}d\varepsilon_{ij}. \tag{2.4}$$

In such a case, Eq. (2.2) must be rewritten as

$$dV=\boldsymbol{\sigma}\cdot d\boldsymbol{\varepsilon}=\sigma_{ij}d\varepsilon_{ij}, \tag{2.5}$$

so that we get the *Green's formula*

$$\sigma_{ij}=\frac{\partial V}{\partial\varepsilon_{ij}}. \tag{2.6}$$

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[1]Namely, we have used the identity

$$\mathrm{div}(\mathbf{L}^\top\mathbf{v})=\mathrm{div}\mathbf{L}\cdot\mathbf{v}+\mathbf{L}\cdot\nabla\mathbf{v},$$

with $\mathbf{L}$ a second-rank tensor field and $\mathbf{v}$ a vector field, see (Gurtin 1981, p. 30), and the fact that, $\forall\mathbf{L}:\mathbf{L}=\mathbf{L}^\top$,

$$\mathbf{L}\cdot\frac{\nabla\mathbf{v}+\nabla^\top\mathbf{v}}{2}=\frac{1}{2}(\mathbf{L}\cdot\nabla\mathbf{v}+\mathbf{L}\cdot\nabla\mathbf{v}^\top)=\frac{1}{2}(\mathbf{L}\cdot\nabla\mathbf{v}+\mathbf{L}^\top\cdot\nabla\mathbf{v}^\top)=\mathbf{L}\cdot\nabla\mathbf{v},$$

because of the property of tensor scalar product:

$$\mathbf{A}\cdot\mathbf{B}=A_{ij}B_{ij}=\mathbf{A}^\top\cdot\mathbf{B}^\top.$$

[2]The existence of such a function can be established upon physical arguments, using the first law of thermodynamics for adiabatic transformations or the second law for the isothermal ones, (Love 1944, p. 94).

We now postulate that in the initial state, i.e. when the body is not acted upon by forces, $\boldsymbol{\varepsilon} = \mathbf{O}$ and $\boldsymbol{\sigma} = \mathbf{O}$, i.e. the body is unstrained and unstressed in its initial state. Then, developing $V(\boldsymbol{\varepsilon})$ in a Taylor series about $\boldsymbol{\varepsilon} = \mathbf{O}$ we get

$$V(\boldsymbol{\varepsilon}) = V(\boldsymbol{\varepsilon} = \mathbf{O}) + \left.\frac{\partial V}{\partial \varepsilon_{ij}}\right|_{\boldsymbol{\varepsilon}=\mathbf{O}} \varepsilon_{ij} + \frac{1}{2} \left.\frac{\partial^2 V}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}\right|_{\boldsymbol{\varepsilon}=\mathbf{O}} \varepsilon_{ij}\varepsilon_{kl} + \ldots \tag{2.7}$$

Choosing arbitrarily $V(\boldsymbol{\varepsilon} = \mathbf{O}) = 0$, always possible for a potential, and limiting the development to the first non null term, which is correct for small strains, gives

$$V = \frac{1}{2} \left.\frac{\partial^2 V}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}\right|_{\boldsymbol{\varepsilon}=\mathbf{O}} \varepsilon_{ij}\varepsilon_{kl}; \tag{2.8}$$

the second derivatives in the above equation are linear operators depending upon four indexes; they are the components of a fourth-rank tensor $\mathbb{E}$,

$$E_{ijkl} := \left.\frac{\partial^2 V}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}\right|_{\boldsymbol{\varepsilon}=\mathbf{O}}, \tag{2.9}$$

the *(stiffness) elasticity tensor*, so that

$$V = \frac{1}{2} E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbb{E}\boldsymbol{\varepsilon}. \tag{2.10}$$

Collecting all the parameters describing the elastic response of the material, $\mathbb{E}$ is the operator that describes the elastic response of the continuum. It has 81 Cartesian components, the *elastic moduli* E_{ijkl}. Nevertheless, the number of *independent* elastic moduli is far less than 81. In fact, first of all, by the Schwarz theorem we get

$$E_{ijkl} = \frac{\partial^2 V}{\partial \varepsilon_{kl} \partial \varepsilon_{ij}} = \frac{\partial^2 V}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}} = E_{klij}; \tag{2.11}$$

the above 15 relations are known as *major symmetries* and reduce the number of independent Cartesian components of $\mathbb{E}$ from 81 to 66.

Now, if we apply the Green's formula (2.6) to Eq. (2.10) we get

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl} \ \rightarrow \ \boldsymbol{\sigma} = \mathbb{E}\boldsymbol{\varepsilon}. \tag{2.12}$$

This is the *Hooke's law* (1660), establishing a linear relation between stress and strain.[3] This linearity is a direct consequence of the quadratic structure of V and of the Green's formula; though initially formulated for isotropic bodies, it is the basic

[3] An alternative, classical, approach to elasticity is to postulate the Hooke's law and the existence of V; once obtained the Green's formula, using the Schwarz theorem gives again the major symmetries, while the minor ones are still given by the symmetry of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$. Then, the expression of V is obtained integrating dV:

law of elasticity also for the more general case of anisotropic continua. Using now the Hooke's law, we obtain another reduction of the number of independent elastic moduli due to the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$:

$$\sigma_{ij}=\sigma_{ji} \text{ and } \varepsilon_{ij}=\varepsilon_{ji} \quad\Rightarrow\quad E_{ijkl}=E_{jikl}=E_{ijlk}=E_{jilk}\ \forall i,j,k,l\in\{1,2,3\}. \tag{2.13}$$

The above 45 relations among the components of $\mathbb{E}$ are called the *minor symmetries* and reduce the number of independent elastic moduli to only 21. This is the highest number of independent moduli that an elastic material can have. In such a case, the material is completely anisotropic or *triclinic*. Further reductions of the number of independent moduli can be obtained only if special conditions, not universal but depending upon the material type, are introduced. Such conditions are called *elastic symmetries*, and indicate the invariance of some elastic moduli under some geometric transformations. Injecting the Hooke's law into the expression (2.10) of V we get also

$$V=\frac{1}{2}\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}. \tag{2.14}$$

Let us now consider the inverse of the Hooke's law:

$$\varepsilon=\mathbb{Z}\sigma,\quad \mathbb{Z}=\mathbb{E}^{-1}, \tag{2.15}$$

with $\mathbb{Z}$ the *compliance elasticity tensor*; introducing this last equation for $\boldsymbol{\varepsilon}$ into (2.10) gives

$$V=\frac{1}{2}\boldsymbol{\sigma}\cdot\mathbb{Z}\boldsymbol{\sigma}, \tag{2.16}$$

an expression called *stress energy density* in the literature.

A last remark: in this section, the word *symmetry* has been used for denoting the equivalence of the positions of an index for two or more components of the elasticity tensor; to make the distinction with the concept of *elastic symmetry*, the expression *tensor* or *index symmetry* could be used. Anyway, the reader should be aware of the fact that the same word, *symmetry*, can have two rather different meanings in our context.

2.1.1 The Voigt's Notation

The general, tensorial, expression of the Hooke's law needs the use of quantities with four indexes,

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(Footnote 3 continued)

$$dV=\boldsymbol{\sigma}\cdot d\boldsymbol{\varepsilon}=\mathbb{E}\boldsymbol{\varepsilon}\cdot d\boldsymbol{\varepsilon}=E_{ijkl}\varepsilon_{kl}d\varepsilon_{ij}\ \rightarrow\ V=\frac{1}{2}E_{ijkl}\varepsilon_{ij}\varepsilon_{kl}=\frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbb{E}\boldsymbol{\varepsilon}.$$

$$\sigma_{ij} = E_{ijkl}\varepsilon_{kl}, \tag{2.17}$$

which can be somewhat cumbersome and heavy. That is why some simplified notations have been proposed. In particular, they allow for a matrix representation of (2.17); these formalisms switch the algebra from that of a fourth-rank tensor to that of a 6×6 square symmetric matrix.

The most well known of the matrix formalisms for anisotropic elasticity is that of Voigt (1910): the stress and strain tensors are written as follows:

$$\{\sigma\} = \left\{\begin{array}{c} \sigma_1 = \sigma_{11} \\ \sigma_2 = \sigma_{22} \\ \sigma_3 = \sigma_{33} \\ \sigma_4 = \sigma_{23} \\ \sigma_5 = \sigma_{31} \\ \sigma_6 = \sigma_{12} \end{array}\right\}, \quad \{\varepsilon\} = \left\{\begin{array}{c} \varepsilon_1 = \varepsilon_{11} \\ \varepsilon_2 = \varepsilon_{22} \\ \varepsilon_3 = \varepsilon_{33} \\ \varepsilon_4 = 2\varepsilon_{23} \\ \varepsilon_5 = 2\varepsilon_{31} \\ \varepsilon_6 = 2\varepsilon_{12} \end{array}\right\}. \tag{2.18}$$

Equation (2.18) shows the relations and order established for the indexes by the Voigt's notation:

$$11 \to 1, \quad 22 \to 2, \quad 33 \to 3, \quad 23 \to 4, \quad 31 \to 5, \quad 12 \to 6. \tag{2.19}$$

The introduction of the coefficient 2 for the terms ε_4, ε_5 and ε_6 is needed for taking into account for the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in the Hooke's law. This fact imposes some accuracy in the use of the Voigt's notation, because the algebras for tensors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are not completely the same, namely for their transformation upon axes rotation and tensor inversion.

The Voigt's notation transforms hence second rank symmetric tensors into column vectors; correspondingly, the fourth-rank elasticity tensor is transformed into a 6×6 symmetric square matrix, the symmetry of such a matrix corresponding to the major symmetries of $\mathbb{E}$. According to the index transformation rule (2.19), the matrix form of the Hooke's law with the Voigt's notation is

$$\{\sigma\} = [C]\{\varepsilon\} \quad \to \quad \left\{\begin{array}{c} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{array}\right\} = \left[\begin{array}{cccccc} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{array}\right] \left\{\begin{array}{c} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{array}\right\}. \tag{2.20}$$

The name $[C]$ is usually preferred to $\mathbb{E}$ to make a clear distinction between the tensor and matrix representation (for the same reason, we will name differently the compliance tensor and matrix). Thanks to the introduction of coefficients 2 in $(2.18)_2$, there is a perfect coincidence between the E_{ijkl} and the C_{pq}; it is sufficient to remind rule (2.19) to make correspond to each pair of indexes ij and kl in E_{ijkl} the correct p and q in C_{pq}; for instance, $E_{2312} = C_{46}$, $E_{1322} = C_{52}$ and so on.

Let us now consider the inverse of the Hooke's law (2.15), that we will write in the Voigt's notation as

$$\{\varepsilon\} = [S]\{\sigma\} \quad \rightarrow \quad \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix}. \tag{2.21}$$

Unlike the case of the stiffness matrix $[C]$, not all the components of $[S]$ are equal to the corresponding ones of $\mathbb{Z}$. This is a consequence of the introduction of the factors 2 in $(2.18)_2$ and the correct transformation is

$$[S_{ij}] = \left[\begin{array}{c|c} Z_{ppqq} & 2Z_{pprs} \\ \hline 2Z_{pprs} & 4Z_{pqrs} \end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccc|ccc} S_{11}=Z_{1111} & S_{12}=Z_{1122} & S_{13}=Z_{1133} & S_{14}=2Z_{1123} & S_{15}=2Z_{1131} & S_{16}=2Z_{1112} \\ & S_{22}=Z_{2222} & S_{23}=Z_{2233} & S_{24}=2Z_{2223} & S_{25}=2Z_{2231} & S_{26}=2Z_{2212} \\ & & S_{33}=Z_{3333} & S_{34}=2Z_{3323} & S_{35}=2Z_{3331} & S_{36}=2Z_{3312} \\ \hline & & & S_{44}=4Z_{2323} & S_{45}=4Z_{2331} & S_{46}=4Z_{2312} \\ & sym & & & S_{55}=4Z_{3131} & S_{56}=4Z_{3112} \\ & & & & & S_{66}=4Z_{1212} \end{array}\right]. \tag{2.22}$$

The above equations show that passing to the Voigt's notation implies a different algebra for stiffness and compliance, as anticipated above. Mathematically, matrices $[C]$ and $[S]$ do not represent some second order tensors in $\mathbb{R}^6$. Practically, the use of the Voigt's notation needs some carefulness, not only in the differences between $[C]$ and $[S]$, but also in the transformation of these matrices produced by axes rotation.

2.1.2 The Kelvin's Notation

The Kelvin's notation (by somebody named *Mandel's notation*) was proposed by W. Thomson, baron of Kelvin, as early as 1856 (Kelvin 1856, 1878), but, probably because making use of irrational quantities, it has not known an as large success as the Voigt's notation. Nevertheless, rather recently a new attention has been brought by scientists on it, mainly for its algebraic properties: the Kelvin's notation has not the drawbacks of the Voigt's one, as it will be shown below.

The Kelvin's notation is different from the Voigt's one in that the coefficients 2 affecting $\boldsymbol{\varepsilon}$, Eq. (2.18), are equally distributed over $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, in such a way their product still amounts to 2:

$$\{\sigma\}=\left\{\begin{array}{c}\sigma_1=\sigma_{11}\\ \sigma_2=\sigma_{22}\\ \sigma_3=\sigma_{33}\\ \sigma_4=\sqrt{2}\sigma_{23}\\ \sigma_5=\sqrt{2}\sigma_{31}\\ \sigma_6=\sqrt{2}\sigma_{12}\end{array}\right\},\quad \{\varepsilon\}=\left\{\begin{array}{c}\varepsilon_1=\varepsilon_{11}\\ \varepsilon_2=\varepsilon_{22}\\ \varepsilon_3=\varepsilon_{33}\\ \varepsilon_4=\sqrt{2}\varepsilon_{23}\\ \varepsilon_5=\sqrt{2}\varepsilon_{31}\\ \varepsilon_6=\sqrt{2}\varepsilon_{12}\end{array}\right\}. \tag{2.23}$$

In this way, Eqs. (2.20) and (2.21) still hold but there is no difference between $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in the transition from the tensor to the matrix representation; in particular, the way the components of $[C]$ and $[S]$ are deduced from the corresponding ones of $\mathbb{E}$ and $\mathbb{Z}$, are exactly the same (no summation over dummy indexes):

$$\left[C_{ij}\right]=\left[\begin{array}{c|c}E_{ppqq} & \sqrt{2}E_{pprs}\\ \hline \sqrt{2}E_{pprs} & 2E_{pqrs}\end{array}\right],\quad \left[S_{ij}\right]=\left[\begin{array}{c|c}Z_{ppqq} & \sqrt{2}Z_{pprs}\\ \hline \sqrt{2}Z_{pprs} & 2Z_{pqrs}\end{array}\right]. \tag{2.24}$$

The above symbolic relations can be easily translated in the detailed expressions of C_{ij} and S_{ij}, applying a scheme quite similar to that detailed in Eq. (2.22).

Merhabadi and Cowin (1990), have shown that the Kelvin's notation gives a representation of elasticity by matrices, $[C]$ and $[S]$, representing second-rank symmetric tensors in $\mathbb{R}^6$, which is not the case with the Voigt's notation. Hence, the Kelvin's notation transfers the algebra of elasticity from fourth-rank tensors in $\mathbb{R}^3$, to second-rank tensors in $\mathbb{R}^6$. This fact has some advantages, for instance the rotation of matrices $[C]$ and $[S]$ is made in the same way, unlike with the Voigt's notation. For these reasons, the Kelvin's notation is preferred in this text.

2.1.3 The Mechanical Meaning of the Anisotropic Elastic Constants

In the most general case of a triclinic material, the number of independent elastic moduli is as great as 21; it is important to understand the mechanical meaning of these parameters, because, unlike in the case of an isotropic material, *some unusual, strange mechanical effects can arise in anisotropy*. To discover these effects and connect them to particular elastic parameters, it is worth to use the compliances, i.e. the components of $[S]$ or $\mathbb{Z}$.

Let us consider a cube of a triclinic material, submitted to the only traction σ_1, see Fig. 2.1. In such a case, Eq. (2.21) gives $\varepsilon_k=S_{k1}\sigma_1\ \ \forall k=1,\dots,6$ (or, equivalently, $\varepsilon_{ij}=Z_{ij11}\sigma_{11}\ \ \forall i,j=1,2,3$).

So, while in an isotropic solid only the terms ε_1, corresponding to the direct stretching effect, and $\varepsilon_2=\varepsilon_3$, corresponding to the *Poisson's effect*, i.e. the deformation in a plane orthogonal to the direction of the normal stress, are not null for a uniaxial traction, in a completely anisotropic body all the components of $\boldsymbol{\varepsilon}$ are not null: a normal stress produces also shearing strains. The coupling effects are hence

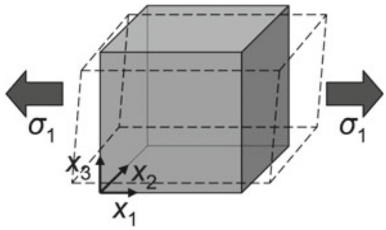


Fig. 2.1 Anisotropic stretched cube

not restricted to the only Poisson's effect, due to the terms $S_{ij},\ i, j = 1, 2, 3,\ i \neq j$: in the anisotropic case, there is also a coupling between normal stress and shear strain, due to terms $S_{kl},\ k = 4, 5, 6,\ l = 1, 2, 3$. In addition, generally speaking $S_{12} \neq S_{23} \neq S_{31}$, so that the Poisson's effect is different in the orthogonal directions, i.e. $\varepsilon_2 \neq \varepsilon_3$. In the same way, usually $S_{4k} \neq S_{5k} \neq S_{6k},\ k = 1, 2, 3$, so that also for the shearing stresses it is $\varepsilon_4 \neq \varepsilon_5 \neq \varepsilon_6$. Finally, the anisotropic cube does not only change its volume under the unique action of a traction, like in isotropic bodies, but it changes also its form: it becomes a prism with no orthogonal faces.

Let us now consider the same cube submitted to a unique shear stress, say σ_5; Eq. (2.21) gives then $\varepsilon_k = S_{k5}\sigma_5 \quad \forall k = 1, \ldots, 6$ (or, equivalently, $\varepsilon_{ij} = Z_{ij31}\sigma_{31} \quad \forall i, j = 1, 2, 3$). This time, we can observe a coupling between shear stresses and extension strains, due to the terms $S_{lk},\ k = 4, 5, 6,\ l = 1, 2, 3$ and also a coupling between a shear stress and the shearing strains in orthogonal planes, due to the terms $S_{ij},\ i, j = 4, 5, 6,\ i \neq j$. This last effect is called the *Chentsov's effect*: it is completely analogous to the Poisson's effect, but it concerns shear stresses and strains in the place of tractions and extensions. Also in this case, the couplings shear stress-extensions and the Chentsov's effect are not necessarily the same in all the planes, because generally speaking $S_{l4} \neq S_{l5} \neq S_{l6},\ l = 1, 2, 3$ and $S_{45} \neq S_{56} \neq S_{64}$. It is then apparent that, submitted to simple shear stress, the cube changes not only its shape, but also its volume, unlike in the case of isotropic bodies.

Finally, the compliance matrix can be subdivided into parts in charge of a particular effect, like in Fig. 2.2. It is immediately recognized that a similar partition is possible also for the stiffness matrix $[C]$.

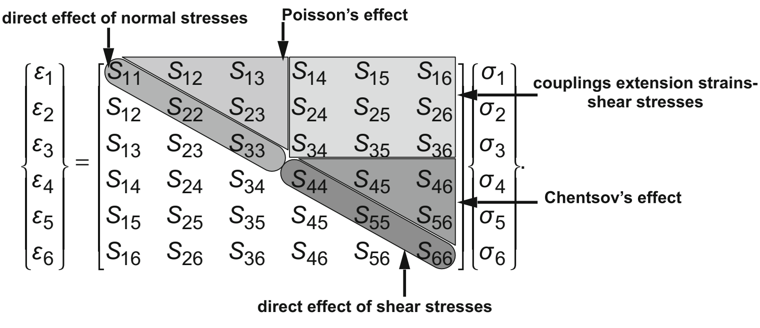


Fig. 2.2 Partition of the compliance matrix by mechanical effects

2.2 Elastic Symmetries

2.2.1 Taking into Account for Elastic Symmetries

Recalling what said in Sect. 1.1, when some kind of symmetry in the behavior is present, then some *equivalent* directions exist, whereon the behavior is the same. The figure formed with these directions is a symmetrical figure allowing all the covering operations of a certain group.

Just because the behavior is the same along equivalent directions, the forms of the elasticity matrix $[C]$[4] and of the strain energy are the same in two frames related by a covering operations. This gives some relations among the components of $[C]$, e.g. some of them are null.[5] Let us sketch the procedure for obtaining such relations:

- the expressions of the strain energy in two frames $\mathscr{R}$ and $\mathscr{R}'$ related by a covering operation are[6]
$$V=\frac{1}{2}\{\varepsilon\}^{\top}[C]\{\varepsilon\},\quad V'=\frac{1}{2}\{\varepsilon'\}^{\top}[C]\{\varepsilon'\};\tag{2.25}$$
- the strain tensor $\{\varepsilon'\}$ can be written in the frame $\mathscr{R}$:
$$\{\varepsilon'\}=[R]\{\varepsilon\},\tag{2.26}$$
with $[R]$ the orthogonal matrix corresponding to the covering operation, i.e. to the symmetry of the material;
- injecting Eq. (2.26) into V', Eq. (2.25), and putting $V=V'$, gives the equation
$$\{\varepsilon\}^{\top}[C]\{\varepsilon\}=([R]\{\varepsilon\})^{\top}[C][R]\{\varepsilon\}\quad\forall\{\varepsilon\};\tag{2.27}$$
- this unique scalar equation gives all the relations that must be true for the components of $[C]$ *exactly because it is independent from the strain state*, i.e. because it is true $\forall\{\varepsilon\}$.

Now, the question is: which is the orthogonal matrix $[R]$ corresponding to a given covering operation? This will be the matter of the following Sections.

Before, just a last remark: the procedure sketched above is not the only one; in fact, in place of working with the strain energy, one could directly state that $[C]$,

[4]The use of $[C]$ is here preferred to that of $\mathbb{E}$ because it facilitates calculations; of course, the results found for components C_{ij} are immediately valid also for the E_{pqrs}, see Eqs. (2.22) and (2.24).

[5]The same is true for the stress energy; in such a case one can obtain relations among the components of $[S]$ that are exactly the same ones found for $[C]$.

[6]We denote here by a prime a component in $\mathscr{B}'$ or also, for the sake of shortness though with a slight abuse of notation, a vector or tensor whose components are intended to be given in $\mathscr{B}'$. $\mathscr{B}=\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\mathscr{B}'=\{\mathbf{e}_1',\mathbf{e}_2',\mathbf{e}_3'\}$ are two orthonormal bases of the vector space of translations $\mathscr{V}$ associated with the ordinary euclidean space $\mathscr{E}$, and they are associated with the frames $\mathscr{R}=\{o;\mathscr{B}\}$ and $\mathscr{R}'=\{o';\mathscr{B}'\}$ respectively, $o,o'\in\mathscr{E}$.

or $[S]$, does not change when passing from $\mathscr{R}$ to $\mathscr{R}'$. This approach is practically equivalent to the previous one, but it gives six scalar equations. For instance, for $[C]$ we have:

$$\begin{aligned}&\{\sigma\} = [C]\{\varepsilon\},\ and\ \{\sigma'\} = [C]\{\varepsilon'\} \ \rightarrow\ [R]\{\sigma\} = [C][R]\{\varepsilon\} \ \rightarrow\\ &\{\sigma\} = [R]^\top[C][R]\{\varepsilon\} \ \Rightarrow\ [C] = [R]^\top[C][R].\end{aligned} \tag{2.28}$$

2.2.2 Rotation of Axes

Let us consider two orthonormal bases $\mathscr{B} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\mathscr{B}' = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ and let us suppose that these two bases are related by the orthogonal tensor $\mathbf{U}$.[7]

We define $\mathbf{U}$ as the tensor such that

$$\mathbf{e}_i = \mathbf{U}\mathbf{e}'_i \ \Rightarrow\ \mathbf{e}'_i = \mathbf{U}^\top\mathbf{e}_i; \tag{2.29}$$

with this definition, it is easy to show that

$$\mathbf{U} = \begin{bmatrix} \mathbf{e}'_1 \\ \hline \mathbf{e}'_2 \\ \hline \mathbf{e}'_3 \end{bmatrix}, \tag{2.30}$$

i.e. the matrix representing $\mathbf{U}$ in the basis $\mathscr{B}'$ has for rows the Cartesian components of the vectors of $\mathscr{B}'$, components expressed in the base $\mathscr{B}$. Algebraically, these components are the director cosines of the angles between two corresponding axes in $\mathscr{B}$ and $\mathscr{B}'$.

Using the above equations, the components in $\mathscr{B}'$ of a tensor of any rank r can be expressed as a linear combination of its components in $\mathscr{B}$, the coefficients of the combination being products of r components of $\mathbf{U}$. In fact, considering that

$$\mathbf{e}_i = \mathbf{U}\mathbf{e}'_i = U_{pq}(\mathbf{e}'_p \otimes \mathbf{e}'_q)\mathbf{e}'_i = U_{pq}\delta_{qi}\mathbf{e}'_p = U_{pi}\mathbf{e}'_p, \tag{2.31}$$

then, for a vector ($r = 1$) it is

$$\mathbf{w} = w_i\mathbf{e}_i = w_i U_{ki}\mathbf{e}'_k \ \Rightarrow\ w'_k = U_{ki}w_i, \tag{2.32}$$

for a second-rank tensor ($r = 2$) it is

$$\begin{aligned}&\mathbf{L} = L_{ij}\mathbf{e}_i \otimes \mathbf{e}_j = L_{ij}U_{mi}\mathbf{e}'_m \otimes U_{nj}\mathbf{e}'_n = U_{mi}U_{nj}L_{ij}\mathbf{e}'_m \otimes \mathbf{e}'_n \ \Rightarrow\\ &L'_{mn} = U_{mi}U_{nj}L_{ij}\end{aligned} \tag{2.33}$$

[7] $\mathbf{U}$ is not necessarily a proper rotation, because reflections are possible too.

and finally for a fourth-rank tensor ($r = 4$) it is[8]

$$\mathbb{E} = E_{ijkl}\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l = E_{ijkl}U_{mi}\mathbf{e}'_m \otimes U_{nj}\mathbf{e}'_n \otimes U_{pk}\mathbf{e}'_p \otimes U_{ql}\mathbf{e}'_q =$$
$$U_{mi}U_{nj}U_{pk}U_{ql}E_{ijkl}\mathbf{e}'_m \otimes \mathbf{e}'_n \otimes \mathbf{e}'_p \otimes \mathbf{e}'_q \quad \Rightarrow \quad E'_{mnpq} = U_{mi}U_{nj}U_{pk}U_{ql}E_{ijkl}. \tag{2.34}$$

Given $\mathbf{A}, \mathbf{B} \in \text{Lin}$, the *conjugation product* $\mathbf{A} \boxtimes \mathbf{B}$ is the fourth-rank tensor defined by the operation

$$(\mathbf{A} \boxtimes \mathbf{B})\mathbf{C} := \mathbf{A}\mathbf{C}\mathbf{B}^\top \ \ \forall \mathbf{C} \in \text{Lin}. \tag{2.35}$$

It is worth to remark that Eq. (2.35) implies that for the vectors of a basis it is

$$(\mathbf{e}_i \otimes \mathbf{e}_j) \boxtimes (\mathbf{e}_k \otimes \mathbf{e}_l) = \mathbf{e}_i \otimes \mathbf{e}_k \otimes \mathbf{e}_j \otimes \mathbf{e}_l, \tag{2.36}$$

which gives

$$(\mathbf{A} \boxtimes \mathbf{B})_{ijkl} = A_{ik}B_{jl}. \tag{2.37}$$

Once defined the transpose $\mathbb{A}^\top$ of a fourth-rank tensor $\mathbb{A}$ as the unique tensor such that

$$\mathbf{L} \cdot (\mathbb{A}\mathbf{M}) = \mathbf{M} \cdot (\mathbb{A}^\top\mathbf{L}) \ \ \forall \mathbf{L}, \mathbf{M} \in \text{Lin}, \tag{2.38}$$

it is immediate to show that

$$\begin{aligned}(\mathbf{A} \otimes \mathbf{B})^\top &= \mathbf{B} \otimes \mathbf{A},\\ (\mathbf{A} \boxtimes \mathbf{B})^\top &= \mathbf{A}^\top \boxtimes \mathbf{B}^\top.\end{aligned} \tag{2.39}$$

Like for tensors in Lin, also a tensor $\mathbb{A} \in \mathbb{L}\text{in}$ is said to be *symmetric* $\iff \mathbb{A} = \mathbb{A}^\top$. It is simple to check that

$$\mathbb{A} = \mathbb{A}^\top \ \ \Rightarrow \ \ A^\top_{ijkl} = A_{klij}, \tag{2.40}$$

i.e., the major symmetries of the elastic tensors $\mathbb{E}$ and $\mathbb{Z}$ actually coincide with the definition of symmetric tensor in $\mathbb{L}$in.

For an orthogonal second-rank tensor $\mathbf{U}$, we define its *orthogonal conjugator* $\mathbb{U}$ as

$$\mathbb{U} := \mathbf{U} \boxtimes \mathbf{U}; \tag{2.41}$$

it is not difficult to show that just as $\mathbf{U}$ preserves scalar products of elements in $\mathscr{V}$, its associated orthogonal conjugator $\mathbb{U}$ preserves scalar products in Lin:

$$\mathbb{U}\mathbf{A} \cdot \mathbb{U}\mathbf{B} = \mathbf{A} \cdot \mathbf{B} \ \ \forall \mathbf{A}, \mathbf{B} \in \text{Lin}. \tag{2.42}$$

[8] $\forall$ $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{L}$ $\in$Lin, $\mathbf{A} \otimes \mathbf{B}$ is the fourth-rank tensor defined by the operation $(\mathbf{A} \otimes \mathbf{B})\mathbf{L} := (\mathbf{B} \cdot \mathbf{L})\mathbf{A}$. Applying this rule to the dyads of a basis, we get a fundamental result: $(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l)(\mathbf{e}_p \otimes \mathbf{e}_q) = (\mathbf{e}_k \otimes \mathbf{e}_l) \cdot (\mathbf{e}_p \otimes \mathbf{e}_q)(\mathbf{e}_i \otimes \mathbf{e}_j) = \delta_{kp}\delta_{lq}(\mathbf{e}_i \otimes \mathbf{e}_j)$.

In other words, $\mathbb{U}$ is an *orthogonal tensor* in $\mathbb{L}$in (Podio-Guidugli 2000, p. 55). Introducing the *identity* of $\mathbb{L}$in,

$$\mathbb{I} = \mathbf{I} \boxtimes \mathbf{I} \ \ \Rightarrow \ \ \mathbb{I} = I_{ijkl}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l) = \delta_{ik}\delta_{jl}(\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l), \tag{2.43}$$

it is easy to recognize that also for rotations in $\mathbb{L}$in

$$\mathbb{U}\mathbb{U}^\top = \mathbb{U}^\top\mathbb{U} = \mathbb{I}. \tag{2.44}$$

Be $\mathbf{n} \in \mathscr{V}$ a unit vector and let us suppose that $\mathbf{n}$ is orthogonal to a symmetry plane. Then

$$\mathbf{U} := \mathbf{I} - 2\mathbf{n} \otimes \mathbf{n}, \ \ \mathbf{U} = \mathbf{U}^\top, \ \ \mathbb{U} = \mathbf{U} \boxtimes \mathbf{U} = \mathbb{U}^\top, \tag{2.45}$$

is the orthogonal tensor describing the symmetry in the plane whose normal is $\mathbf{n}$. In fact,

$$\begin{aligned} &\mathbf{U}\mathbf{n} = -\mathbf{n} \\ &\mathbf{U}\mathbf{m} = \mathbf{m} \ \ \forall \mathbf{m} \in \mathscr{V} : \ \mathbf{m} \cdot \mathbf{n} = 0, \ |\mathbf{m}| = 1. \end{aligned} \tag{2.46}$$

Thanks to these last definitions, it is possible to give a compact form to results (2.32), (2.33) and (2.34):

$$\begin{aligned} &\mathbf{w}' = \mathbf{U}\mathbf{w}, \\ &\mathbf{L}' = \mathbf{U}\mathbf{L}\mathbf{U}^\top = (\mathbf{U} \boxtimes \mathbf{U})\mathbf{L} = \mathbb{U}\mathbf{L}, \\ &\mathbb{E}' = (\mathbf{U} \boxtimes \mathbf{U})\mathbb{E}(\mathbf{U} \boxtimes \mathbf{U})^\top = \mathbb{U}\mathbb{E}\mathbb{U}^\top. \end{aligned} \tag{2.47}$$

Using Eq. (2.23) and the result of Eq. (2.37), we can now obtain the matrix $[R]$ that corresponds, in the Kelvin's notation, to tensor $\mathbb{U}$; the calculations are rather tedious and a little bit long, but the final result is (Mehrabadi and Cowin 1990),

$$[R] = \begin{bmatrix} U_{11}^2 & U_{12}^2 & U_{13}^2 & \sqrt{2}U_{12}U_{13} & \sqrt{2}U_{13}U_{11} & \sqrt{2}U_{11}U_{12} \\ U_{21}^2 & U_{22}^2 & U_{23}^2 & \sqrt{2}U_{22}U_{23} & \sqrt{2}U_{23}U_{21} & \sqrt{2}U_{21}U_{22} \\ U_{31}^2 & U_{32}^2 & U_{33}^2 & \sqrt{2}U_{32}U_{33} & \sqrt{2}U_{33}U_{31} & \sqrt{2}U_{31}U_{32} \\ \sqrt{2}U_{21}U_{31} & \sqrt{2}U_{22}U_{32} & \sqrt{2}U_{23}U_{33} & U_{23}U_{32}+U_{22}U_{33} & U_{33}U_{21}+U_{31}U_{23} & U_{31}U_{22}+U_{32}U_{21} \\ \sqrt{2}U_{31}U_{11} & \sqrt{2}U_{32}U_{12} & \sqrt{2}U_{33}U_{13} & U_{32}U_{13}+U_{33}U_{12} & U_{31}U_{13}+U_{33}U_{11} & U_{31}U_{12}+U_{32}U_{11} \\ \sqrt{2}U_{11}U_{21} & \sqrt{2}U_{12}U_{22} & \sqrt{2}U_{13}U_{23} & U_{12}U_{23}+U_{13}U_{22} & U_{11}U_{23}+U_{13}U_{21} & U_{11}U_{22}+U_{12}U_{21} \end{bmatrix}. \tag{2.48}$$

The above matrix $[R]$ allows for the change of basis of any second-rank tensor in the Kelvin's notation. In particular for $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$:

$$\{\sigma'\} = [R]\{\sigma\}, \quad \{\varepsilon'\} = [R]\{\varepsilon\}. \tag{2.49}$$

It can be checked that, when $\mathbf{U}$ is an orthogonal tensor, then

$$[R][R]^\top = [R]^\top[R] = [I], \tag{2.50}$$

i.e., $[R]$ is an orthogonal matrix in $\mathbb{R}^6$; this is not the case with the Voigt's notation. Hence, $[R]$ represents, in the given basis, an orthogonal tensor of Lin over a manifold of dimension six.

It is impossible to put the result of Eq. (2.34) in matrix form, because also in the Kelvin's notation it depends upon four indexes; nevertheless, it is of course possible to express all the components of such an operator, but actually in the most general case these components have so extremely complicate and long expressions that it is practically impossible to write down all of them, so they are omitted here.

2.2.3 A Tensorial Characterization of Elastic Symmetries

The results of the previous section give us the possibility of characterizing in an elegant tensorial form the existence of elastic symmetries in a solid (Podio-Guidugli 2000, p. 56).

Let us suppose that a material has a given elastic symmetry and that the two bases $\mathscr{B}$ and $\mathscr{B}'$ correspond to equivalent directions with respect to the symmetry of concern. Physically, this means that it is not possible to detect the change from $\mathscr{B}$ to $\mathscr{B}'$ by experiments measuring stresses, because the behavior is exactly the same in the two cases: $\mathbb{E} = \mathbb{E}'$. Then, applying Eq. $(2.47)_2$ to $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$,

$$\boldsymbol{\sigma}' = \mathbb{U}\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}' = \mathbb{U}\boldsymbol{\varepsilon}, \tag{2.51}$$

and the Hooke's law, Eq. (2.12), we get, because $\mathbb{E} = \mathbb{E}'$,

$$\boldsymbol{\sigma}' = \mathbb{E}\boldsymbol{\varepsilon}' \;\rightarrow\; \mathbb{U}\boldsymbol{\sigma} = \mathbb{E}\mathbb{U}\boldsymbol{\varepsilon} \;\rightarrow \mathbb{U}\mathbb{E}\boldsymbol{\varepsilon} = \mathbb{E}\mathbb{U}\boldsymbol{\varepsilon} \;\Rightarrow\; \mathbb{U}\mathbb{E} = \mathbb{E}\mathbb{U}. \tag{2.52}$$

Hence, an orthogonal transformation $\mathbf{U}$ is in the elastic symmetry group of the material *if and only if* $\mathbb{E}$ *and* $\mathbb{U}$ *commute*, $\mathbb{U}$ being the orthogonal conjugator of $\mathbf{U}$.

The result of Eq. (2.52) constitutes also a way for determining the number and type of independent elastic moduli, i.e. the distinct components of $\mathbb{E}$; this is the way sketched, with reference to matrix $[C]$, in the last paragraph of Sect. 2.2.1, Eq. (2.28); however, as said thereon, the energetic approach is preferred in this text.

2.2.4 Triclinic Bodies

A triclinic body has no material symmetries, so Eq. (2.27) cannot be written. As a consequence, it is not possible to reduce the number of independent elastic components, that remains fixed to 21. Matrix $[C]$ appears hence as

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ & & C_{33} & C_{34} & C_{35} & C_{36}\\ & & & C_{44} & C_{45} & C_{46}\\ & sym & & & C_{55} & C_{56}\\ & & & & & C_{66}\end{bmatrix}. \tag{2.53}$$

2.2.5 Monoclinic Bodies

The only symmetry of a monoclinic body is a reflection in a plane. Without loss in generality, we can suppose $x_3=0$ to be the symmetry plane. In such a case it is, see Eqs. (2.45) and (2.48),

$$\mathbf{U}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&-1\end{bmatrix}\ \Rightarrow\ [R]=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&-1&0&0\\0&0&0&0&-1&0\\0&0&0&0&0&1\end{bmatrix}, \tag{2.54}$$

that applied to Eq. (2.27) gives the condition

$$\begin{aligned}&C_{14}\varepsilon_1\varepsilon_4+C_{24}\varepsilon_2\varepsilon_4+C_{34}\varepsilon_3\varepsilon_4+C_{15}\varepsilon_1\varepsilon_5+\\&C_{25}\varepsilon_2\varepsilon_5+C_{35}\varepsilon_3\varepsilon_5+C_{46}\varepsilon_4\varepsilon_6+C_{56}\varepsilon_5\varepsilon_6=0,\end{aligned} \tag{2.55}$$

which is satisfied $\forall\boldsymbol{\varepsilon}\iff$

$$C_{14}=C_{24}=C_{34}=C_{15}=C_{25}=C_{35}=C_{46}=C_{56}=0. \tag{2.56}$$

Hence, a monoclinic body depends upon only 13 distinct elastic moduli:

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ & C_{22} & C_{23} & 0 & 0 & C_{26}\\ & & C_{33} & 0 & 0 & C_{36}\\ & & & C_{44} & C_{45} & 0\\ & sym & & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix}. \tag{2.57}$$

2.2.6 Orthotropic Bodies

Let us now add another plane of symmetry orthogonal to the previous one, say the plane $x_2 = 0$. Following the same procedure, we get successively:

$$\mathbf{U} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{2.58}$$

$$\begin{aligned}&(C_{14}\varepsilon_1 + C_{24}\varepsilon_2 + C_{34}\varepsilon_3 + C_{45}\varepsilon_5)\varepsilon_4 + \\ &(C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + C_{36}\varepsilon_3 + C_{56}\varepsilon_5)\varepsilon_6 = 0 \ \ \forall \boldsymbol{\varepsilon} \iff \\ &C_{14} = C_{24} = C_{34} = C_{45} = C_{16} = C_{26} = C_{36} = C_{56} = 0.\end{aligned} \tag{2.59}$$

So, the existence of the second plane of symmetry has added the four supplementary conditions

$$C_{16} = C_{26} = C_{36} = C_{45} = 0 \tag{2.60}$$

to the previous eight ones, reducing hence to only 9 the number of distinct elastic moduli. Let us now suppose the existence of a third plane of symmetry, orthogonal to the previous ones, the plane $x_1 = 0$. With the same procedure, we get:

$$\mathbf{U} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \tag{2.61}$$

$$\begin{aligned}&(C_{15}\varepsilon_1 + C_{25}\varepsilon_2 + C_{35}\varepsilon_3 + C_{45}\varepsilon_4)\varepsilon_5 + \\ &(C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + C_{36}\varepsilon_3 + C_{46}\varepsilon_4)\varepsilon_6 = 0 \ \ \forall \boldsymbol{\varepsilon} \iff \\ &C_{15} = C_{25} = C_{35} = C_{45} = C_{16} = C_{26} = C_{36} = C_{46} = 0.\end{aligned} \tag{2.62}$$

Rather surprisingly, this new symmetry condition does not give any supplementary condition to those in (2.56) and (2.60). Because the procedure does not depend upon the order of the symmetries, as it is immediately recognized, the only consequence is that the existence of two orthogonal planes of elastic symmetry is physically impossible: only the presence of one or three mutually orthogonal planes of symmetry is admissible. A material having three planes of symmetry is called *orthotropic*. The class of orthotropic materials is very important, because a lot of materials or structures belong to it. An orthotropic material depends hence upon 9 distinct elastic moduli and its matrix [C] looks like

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}. \tag{2.63}$$

2.2.7 Axially Symmetric Bodies

We have seen in Sect. 1.4.2 that there are only four possible cases of axial symmetries for crystals: the 2-, 3-, 4- and 6-fold axis of symmetry. In elasticity, there is another possibility, that will be examined in the next Section. Let us then consider the above four cases, in the order, taking as symmetry axis, without loss in generality, the axis x_3.

Let us begin with a 2-fold axis of symmetry; the covering operation corresponds hence to a rotation of π about x_3. In such a case, we have

$$\mathbf{U}=\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R]=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.64}$$

and we can observe that the matrix $[R]$ is the same of the monoclinic case, Eq. (2.54). Hence, a 2-fold axis of symmetry coincides with a plane of symmetry.

For a 3-fold axis of symmetry, the covering operation corresponds to a rotation of $2\pi/3$ about x_3, which gives

$$\mathbf{U}=\begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R]=\begin{bmatrix} \frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & -\sqrt{\frac{3}{8}} \\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & \sqrt{\frac{3}{8}} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \sqrt{\frac{3}{8}} & -\sqrt{\frac{3}{8}} & 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix}; \tag{2.65}$$

in this case, condition (2.27) is very long and omitted here, but finally it gives 14 conditions on the components of $[C]$:

$$\begin{aligned}
&C_{16} = C_{26} = C_{34} = C_{35} = C_{36} = C_{45} = 0,\\
&C_{22} = C_{11},\quad C_{55} = C_{44},\quad C_{23} = C_{13},\quad C_{24} = -C_{14},\\
&C_{25} = -C_{15},\quad C_{56} = \sqrt{2}C_{14},\quad C_{46} = \sqrt{2}C_{15},\quad C_{66} = C_{11} - C_{12}.
\end{aligned} \tag{2.66}$$

So, there are only 7 distinct elastic moduli:

$$[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & 0\\
 & C_{11} & C_{13} & -C_{14} & -C_{15} & 0\\
 & & C_{33} & 0 & 0 & 0\\
 & & & C_{44} & 0 & -\sqrt{2}C_{15}\\
 & sym & & & C_{44} & \sqrt{2}C_{14}\\
 & & & & & C_{11} - C_{12}
\end{bmatrix}. \tag{2.67}$$

For a 4-fold axis of symmetry, the covering operation corresponds to a rotation of $\pi/2$ about x_3, which gives

$$\mathbf{U} = \begin{bmatrix} 0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix} \Rightarrow [R] = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -1 & 0\\
0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -1
\end{bmatrix}; \tag{2.68}$$

we omit also in this case the long expression of Eq. (2.27), but the final result are 14 conditions different from the (2.66):

$$\begin{aligned}
&C_{14} = C_{24} = C_{34} = C_{15} = C_{25} = C_{35} = C_{45} = C_{36} = C_{46} = C_{56} = 0,\\
&C_{22} = C_{11},\quad C_{55} = C_{44},\quad C_{23} = C_{13},\quad C_{26} = -C_{16},
\end{aligned} \tag{2.69}$$

leaving an elastic matrix $[C]$ still depending upon only 7 distinct moduli, but different from the previous case, Eq. (2.67):

$$[C] = \begin{bmatrix}
C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\
 & C_{11} & C_{13} & 0 & 0 & -C_{16}\\
 & & C_{33} & 0 & 0 & 0\\
 & & & C_{44} & 0 & 0\\
 & sym & & & C_{44} & 0\\
 & & & & & C_{66}
\end{bmatrix}. \tag{2.70}$$

The last case is that of a 6-fold axis of symmetry, with as covering operation a rotation of $\pi/3$ about x_3, which gives

$$\mathbf{U}=\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ 0 & 0 & 1\end{bmatrix}\ \Rightarrow\ [R]=\begin{bmatrix}\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 & \sqrt{\frac{3}{8}}\\ \frac{3}{4} & \frac{1}{4} & 0 & 0 & 0 & -\sqrt{\frac{3}{8}}\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0\\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ -\sqrt{\frac{3}{8}} & \sqrt{\frac{3}{8}} & 0 & 0 & 0 & -\frac{1}{2}\end{bmatrix}; \tag{2.71}$$

condition (2.27), omitted because too long, gives in this case 16 conditions:

$$\begin{aligned}&C_{14}=C_{24}=C_{34}=C_{15}=C_{25}=C_{35}=\\&C_{45}=C_{16}=C_{26}=C_{36}=C_{46}=C_{56}=0,\\&C_{22}=C_{11},\ \ C_{55}=C_{44},\ \ C_{23}=C_{13},\ \ C_{66}=C_{11}-C_{12},\end{aligned} \tag{2.72}$$

for a final elastic matrix $[C]$ depending upon only 5 moduli:

$$[C]=\begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & 0\\ & C_{11} & C_{13} & 0 & 0 & 0\\ & & C_{33} & 0 & 0 & 0\\ & & & C_{44} & 0 & 0\\ & sym & & & C_{44} & 0\\ & & & & & C_{11}-C_{12}\end{bmatrix}. \tag{2.73}$$

2.2.8 Transversely Isotropic Bodies

Let us now consider the case of a material with an axis of cylindrical symmetry, i.e. an axis of symmetry where the covering operation is a rotation by any angle θ; such a material is called *transversely isotropic*, and many materials belong to this class, like for instance wood, fiber reinforced composites, laminated steel and so on. Also in this case we can proceed in the usual way. Denoting, for shortening the expressions, $c=\cos\theta,\ \ s=\sin\theta$, we get:

$$\mathbf{U}=\begin{bmatrix}c & s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{bmatrix}\ \Rightarrow\ [R]=\begin{bmatrix}c^2 & s^2 & 0 & 0 & 0 & \sqrt{2}cs\\ s^2 & c^2 & 0 & 0 & 0 & -\sqrt{2}cs\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & c & -s & 0\\ 0 & 0 & 0 & s & c & 0\\ -\sqrt{2}cs & \sqrt{2}cs & 0 & 0 & 0 & c^2-s^2\end{bmatrix}. \tag{2.74}$$

In this case we obtain exactly the same 16 conditions (2.72); this means that, elastically, the 6-fold axis of symmetry is strictly identical to an axis of cylindrical symmetry. Hence, two such materials cannot be distinguished using only the results of tests on stress or strain. This must not surprise, because this fact is in perfect accordance with the Neumann's principle, as the 6-fold axis of symmetry is contained in the more general case of cylindrical symmetry.

Finally, Eq. (2.73) represents also the elastic matrix of a transversely isotropic material, who has 5 distinct elastic moduli.

2.2.9 Isotropic Bodies

Iostropy is the complete symmetry: all the directions are equivalent. The conditions of isotropy could be find following the usual procedure, imposing that Eq. (2.27) is valid for any orthogonal transformation $[R]$. However, this general approach, that can be followed using for instance the Euler angles for expressing a generic $[R]$, results to be very cumbersome and computationally heavy.

A more direct approach is the following one: for a transversely isotropic body, all the directions orthogonal to the axis of symmetry, say x_3, are equivalent. In other words, fixing the axes of x_1 and x_2 is completely arbitrary. Let us then suppose that, besides the equivalence of all the directions in the plane perpendicular to x_3, also x_1 and x_3 are equivalent. We then impose to a material described by a transversely isotropic elastic matrix, Eq. (2.73), this further equivalence, which is described by

$$\mathbf{U}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\ \Rightarrow\ [R]=\begin{bmatrix}0&0&1&0&0&0\\0&1&0&0&0&0\\1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&0&0&1&0\\0&0&0&1&0&0\end{bmatrix}. \tag{2.75}$$

This gives three new conditions:

$$C_{13}=C_{12},\quad C_{33}=C_{11},\quad C_{44}=C_{66}, \tag{2.76}$$

which reduce the number of distinct elastic constants from 5 to only 2:

$$[C]=\begin{bmatrix}C_{11}&C_{12}&C_{12}&0&0&0\\&C_{11}&C_{12}&0&0&0\\&&C_{11}&0&0&0\\&&&C_{11}-C_{12}&0&0\\&sym&&&C_{11}-C_{12}&0\\&&&&&C_{11}-C_{12}\end{bmatrix}. \tag{2.77}$$

Because x_1 is any direction, all the directions of the space are equivalent; this can be proved showing that the elastic matrix (2.77) is insensitive to any change of basis leaving x_2 unchanged, i.e.

$$\mathbf{U}=\begin{bmatrix} c & 0 & s\\ 0 & 1 & 0\\ -s & 0 & c\end{bmatrix} \Rightarrow [R]=\begin{bmatrix} c^2 & 0 & s^2 & 0 & \sqrt{2}cs & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ s^2 & 0 & c^2 & 0 & -\sqrt{2}cs & 0\\ 0 & 0 & 0 & c & 0 & -s\\ -\sqrt{2}cs & 0 & \sqrt{2}cs & 0 & c^2-s^2 & 0\\ 0 & 0 & 0 & s & 0 & c\end{bmatrix}, \quad (2.78)$$

which gives as only condition $C_{44}=C_{11}-C_{12}$, a condition already contained in the previous ones, Eq. (2.72) and (2.76). This proves that nothing is added to the previous conditions and hence that all the directions in any meridian plane are equivalent, i.e. that the body is isotropic.

There is another, more elegant and direct way to prove that an isotropic body depends upon only two distinct moduli: because of isotropy, the elastic response is insensitive to a change of frame, so the elastic moduli of an isotropic material cannot be frame-dependent. This means that for an isotropic material, V cannot depend upon the ε_{ij}, that are frame-dependent quantities, but rather on the *invariants* of $\boldsymbol{\varepsilon}$.[9] As a consequence, being V a quadratic function of the ε_{ij}, its general expression is of the type

$$V=\frac{1}{2}c_1 I_1^2+c_2 I_2, \quad (2.79)$$

with[10]

$$I_1=\mathrm{tr}\boldsymbol{\varepsilon}=\varepsilon_{ii}, \quad I_2=\frac{\mathrm{tr}^2\boldsymbol{\varepsilon}-\mathrm{tr}\boldsymbol{\varepsilon}^2}{2}=\frac{\varepsilon_{ii}\ \varepsilon_{ii}-\varepsilon_{ij}\ \varepsilon_{ji}}{2}. \quad (2.80)$$

The third order invariant of $\boldsymbol{\varepsilon}$, i.e. $\det\boldsymbol{\varepsilon}$, cannot enter in the expression of V, because it is a cubic function of the ε_{ij}, while V must be a quadratic function of the ε_{ij}. Then,

$$V=\frac{1}{2}[(c_1+c_2)\varepsilon_{ii}\ \varepsilon_{ii}-c_2\ \varepsilon_{ij}\ \varepsilon_{ji}], \quad (2.81)$$

so that[11]

$$\begin{aligned}\sigma_{\underline{ii}}&=\frac{\partial V}{\partial\varepsilon_{\underline{ii}}}=(c_1+c_2)\varepsilon_{ii}-c_2\ \varepsilon_{\underline{ii}},\\ \sigma_{ij}&=\frac{\partial V}{\partial\varepsilon_{ij}}=-c_2\ \varepsilon_{ji}=-c_2\ \varepsilon_{ij}.\end{aligned} \quad (2.82)$$

[9] The elastic potential V is, as any other quantity derived by a scalar product, an invariant, i.e. it is not frame-dependent. Hence, because $[C]$ for an isotropic material is frame independent, the expression of V cannot depend upon frame-dependent quantities, the ε_{ij}, but only upon frame-independent functions of the ε_{ij}: the invariants of $\boldsymbol{\varepsilon}$.

[10] $\boldsymbol{\varepsilon}^2=\boldsymbol{\varepsilon}\boldsymbol{\varepsilon}=\varepsilon_{ij}\mathbf{e}_i\otimes\mathbf{e}_j\ \varepsilon_{hk}\mathbf{e}_h\otimes\mathbf{e}_k=\varepsilon_{ij}\ \varepsilon_{hk}\ \mathbf{e}_j\cdot\mathbf{e}_h(\mathbf{e}_i\otimes\mathbf{e}_k)=\varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}(\mathbf{e}_i\otimes\mathbf{e}_k)\rightarrow\mathrm{tr}\boldsymbol{\varepsilon}^2=\varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}\mathrm{tr}(\mathbf{e}_i\otimes\mathbf{e}_k)=\varepsilon_{ij}\ \varepsilon_{hk}\ \delta_{jh}\delta_{ik}=\varepsilon_{ij}\ \varepsilon_{ji}$.

[11] Following a common practice, when an index is underlined, it is not a dummy index: no summation over it.

For instance:

$$\begin{aligned}\sigma_{11} &= \frac{\partial V}{\partial \varepsilon_{11}} = (c_1 + c_2)(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}) - c_2\, \varepsilon_{11},\\ \sigma_{12} &= \frac{\partial V}{\partial \varepsilon_{12}} = -c_2\, \varepsilon_{12} \quad \text{etc.}\end{aligned} \tag{2.83}$$

We see hence that in the case of isotropic materials, only two constants are sufficient to characterize the elastic behavior.

2.2.10 Some Remarks About Elastic Symmetries

Some remarks can be done about the results found in the previous Sections. First of all, the results given, in all the cases, for $[C]$ are completely valid also for $[S]$; this is not the case with the Voigt's notation, where for some symmetries, not each S_{ij} has the same expression of the corresponding C_{ij}.

A mechanically important remark is the fact that typically some coupling components disappear in a symmetry basis. The case of orthotropic bodies is emblematic: in the orthotropic frame, the skyline of $[C]$ is exactly the same of an isotropic body and the only coupling is the Poisson's effect. Nevertheless, this is no longer true in any other basis: *in a generic basis, all the anisotropic materials, regardless of their symmetries, behave like a triclinic body, i.e. they have all the coupling terms (generally speaking, their elastic matrix is complete, none of its terms vanishes).*

The only exception to this fact is isotropy; in fact, for an isotropic body the matrices $[C]$ and $[S]$ are completely invariant, i.e. *their only two distinct moduli are tensor invariants* and the only possible coupling is the Poisson's effect. This is the obvious consequence of the fact that all the directions of the space are equivalent. Physically, the fact that the least number of independent elastic constants is two means that in a stressed elastic body there are, in general, at least two distinct and independent deformation effects.

2.2.11 Elasticity of Crystals and Elastic Syngonies

Crystals have an elastic behavior that belongs to one of the cases above or is a combination of these cases. Examining their cases, allows us for entirely defining the ten elastic syngonies introduced in Sect. 1.4.2. In particular, referring to the Voigt's classification, Table 1.1, it is[12]:

[12]We recall that the following classification is based upon the definition of elastic syngony as a class of materials sharing the same number and type of independent elastic moduli, see Sect. 1.4.2.

1. classes 1 and 2 belong to the triclinic case, with 21 constants; their matrix $[C]$ is like in Eq. (2.53) and this crystal syngony corresponds with the *triclinic elastic syngony*;
2. classes 3, 4 and 5 belong to the monoclinic case, with 13 constants; their matrix $[C]$ is like in Eq. (2.57) and this crystal syngony corresponds with the *monoclinic elastic syngony*;
3. classes 6, 7 and 8 of the orthorhombic syngony belong to the orthotropic case, with 9 constants; their matrix $[C]$ is like in Eq. (2.63) and the orthorhombic syngony corresponds hence entirely with the *orthotropic elastic syngony*;
4. classes 12 and 13 of the trigonal syngony belong to the 3-fold rotational symmetry case, with 7 constants; they have a matrix $[C]$ as in Eq. (2.67) and they constitute the *trigonal elastic syngony with 7 constants*;
5. classes 17, 18 and 20 of the tetragonal syngony belong to the 4-fold rotational symmetry case, with 7 constants; their matrix is like in Eq. (2.70) and they constitute the *tetragonal elastic syngony with 7 constants*;
6. classes 9, 10 and 11 of the trigonal syngony are a combination of the 3-fold rotational symmetry and the monoclinic symmetry cases: if the plane of symmetry is the plane $x_1 = 0$, then the usual procedure applied to the matrix (2.67) gives $C_{15} = 0$, and matrix (2.67) becomes

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{11} & C_{13} & -C_{14} & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & \sqrt{2}C_{14} \\ & & & & & C_{11} - C_{12} \end{bmatrix}; \tag{2.84}$$

if it is $x_2 = 0$ the plane of symmetry, then it is $C_{14} = 0$ and matrix (2.67) becomes

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{11} & C_{13} & 0 & -C_{15} & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & -\sqrt{2}C_{15} \\ & sym & & & C_{44} & 0 \\ & & & & & C_{11} - C_{12} \end{bmatrix}; \tag{2.85}$$

these cases constitute the *trigonal elastic syngony with 6 constants*;
7. classes 14, 15, 16 and 19 of the tetragonal syngony are a particular case of the orthotropic symmetry: they have identical elastic properties along the axis x_1 and x_2, which gives the three supplementary conditions $C_{22} = C_{11},\ C_{23} = C_{13},\ C_{55} = C_{44}$, so reducing matrix (2.63) to

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{66} \end{bmatrix}; \tag{2.86}$$

these cases constitute the *tetragonal elastic syngony with 6 constants*;

8. classes of the hexagonal syngony, from the 21 to the 27, belong to the 6-fold rotational symmetry, with 5 constants; together with transversely isotropic materials, that do not exist as crystals, they form the *axe-symmetric elastic syngony*, with $[C]$ as in Eq. (2.73);
9. classes of the cubic syngony, from the 28 to the 32, are a particular case of the orthotropic symmetry: they have identical properties along the three axes, which gives the six supplementary conditions $C_{33} = C_{22} = C_{11},\ C_{23} = C_{13} = C_{12},\ C_{66} = C_{55} = C_{44}$, so reducing matrix (2.63) to

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}; \tag{2.87}$$

   the cubic crystal syngony corresponds entirely with the *cubic elastic syngony*;
10. the last elastic syngony is the *isotropic elastic syngony*; of course, no crystal syngonies belong to this case; nevertheless, a huge number of materials have an isotropic behavior. Though the texts on crystals and anisotropy usually forget to consider the isotropic case, this one actually exists and for the sake of completeness we prefer here to consider it as an elastic syngony; the isotropic matrix (2.77) can be obtained as a particular case of the cubic one, (2.87), when $C_{44} = C_{11} - C_{12}$.

2.3 The Technical Constants of Elasticity

In practical applications, engineers usually prefer to replace the use of the elastic stiffness matrix components by the so-called *technical elasticity constants* or *engineer moduli*.

Technical constants quantify an effect, a direct or a coupling one, whose mechanical meaning is immediate and that can be easily identified and measured in simple laboratory tests, like for instance unidirectional tensile tests.

Of course, the set of technical constants must be equivalent to the set of independent elastic moduli, which means, on one side, that the number of technical constants and distinct elastic moduli must be the same, i.e. 21, and that the technical constants must represent all the mechanical effects in a stressed body.

Though replacing the components of the stiffness elastic matrix $[C]$, the technical constants are defined as functions of the S_{ij}. Unlike the elastic moduli, only 6 technical constants are *moduli*: they measure a *direct effect* i.e. they correspond to terms on the diagonal of $[C]$, and are homogenous to a stress. The remaining 15 technical constants are *coefficients*: they are dimensionless quantities because constructed as the ratio between two strain components and they measure a *coupling effect*, i.e. they correspond to terms out of the diagonal of $[C]$. Let us introduce all of them.

2.3.1 The Young's Moduli

The three *Young's moduli* generalize to anisotropy the analogous isotropic modulus and are defined in a similar way:

$$E_i := \frac{\sigma_i}{\varepsilon_i}, \quad i = 1, 2, 3, \quad \sigma_i \neq 0, \ \sigma_j = 0 \text{ for } j \neq i, \ j = 1, \ldots, 6. \tag{2.88}$$

As a consequence, from Eq. (2.21) we get the relations (no summation over dummy indexes)

$$S_{ii} = Z_{iiii} = \frac{1}{E_i}, \quad i = 1, 2, 3. \tag{2.89}$$

The mechanical meaning of the Young's moduli is self-evident: each one of them measures the extension stiffness along the direction of one of the frame axes, i.e. the stress to be applied in the direction x_i to stretch the same direction with a strain equal to unity. Generally speaking, the three Young's moduli are different, i.e. in anisotropy the directions of the space have different stiffnesses.

2.3.2 Shear Moduli

Also in this case, the three *shear moduli* generalize to anisotropy the isotropic concept of shear modulus[13]:

$$G_{ij} := \frac{\sigma_k}{2\varepsilon_k}, \quad i, j = 1, 2, 3, \ i \neq j, \ k = 4, 5, 6, \ \sigma_k \neq 0, \ \sigma_h = 0 \text{ for } h \neq k, \ h = 1, \ldots, 6. \tag{2.90}$$

[13]The reader should consider that the definition of the shear moduli normally found in the literature is

$$G_{ij} := \frac{\sigma_{ij}}{\gamma_{ij}},$$

where γ_{ij} is the so-called *technical shear strain*, $\gamma_{ij} := 2\varepsilon_{ij}$. The above equation in the Kelvin's notation just corresponds to Eq. (2.90).

To be remarked that in the literature there is a strange discrepancy in the nomenclature of the $G_{ij}s$: in fact, the Kelvin notation is used for σ_k and ε_k but in G_{ij} the indexes are those indicating the directions. The correspondences between k and ij are of course those indicated by Eq. (2.19). As a consequence, from Eqs. (2.21) and (2.24) we get the relations (no summation over dummy indexes)

$$2S_{kk} = 4Z_{ijij} = \frac{1}{G_{ij}}, \quad i, j = 1, 2, 3, \ i \neq j, \ k = 4, 5, 6. \tag{2.91}$$

The mechanical meaning of the shear moduli is completely analogous to that of the Young's moduli, but it concerns shear stress and strain, and the same remarks can be done.

2.3.3 Poisson's Coefficients

The definition of the *Poisson's coefficients* or *ratios* in anisotropy is quite similar to that given for isotropic bodies:

$$\nu_{ij} := -\frac{\varepsilon_j}{\varepsilon_i}, \quad i, j = 1, 2, 3, \ \sigma_i \neq 0, \ \sigma_h = 0 \text{ for } h \neq i, \ h = 1, \ldots, 6. \tag{2.92}$$

Like for shear moduli, also in this case the nomenclature makes use, in the same formula, of the Kelvin's notation along with the classical tensorial one.

From the Young's moduli definition, Eq. (2.88), we get

$$\varepsilon_j = -\nu_{ij}\varepsilon_i = -\nu_{ij}\frac{\sigma_i}{E_i}, \quad i, j = 1, 2, 3. \tag{2.93}$$

Through Eq. (2.21) this gives (no summation over dummy indexes)

$$S_{ji} = Z_{jjii} = -\frac{\nu_{ij}}{E_i} \quad \Rightarrow \quad \nu_{ij} = -\frac{S_{ji}}{S_{ii}}, \quad i, j = 1, 2, 3. \tag{2.94}$$

Finally, the symmetry of matrix $[S]$, consequence of the major symmetries of $\mathbb{Z}$, gives the *reciprocity relations*

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad i, j = 1, 2, 3, \tag{2.95}$$

which reduce the number of distinct Poisson's coefficients from 6 to only 3.

Some remarks about the Poisson's coefficients: they measure the Poisson's effect, i.e. the deformation in a direction transversal to that of the normal stress. Because, generally speaking, the three Poisson's coefficients are different, the Poisson's effect is different in the different directions. Also, due to the dependence upon the frame orientation, Eq. (2.34), it is possible that in some directions $\nu_{ij} \leq 0$, i.e. directions with null or negative Poisson's coefficients are quite possible in anisotropic elasticity. To end, it is worth to remark that some authors exchange the place of suffixes i and j in the definition of ν_{ij}.

2.3.4 Chentsov's Coefficients

The *Chentsov's coefficients* $\mu_{ij,kl}$ play the same role of the Poisson's coefficients with respect to shear stress and strain. They are defined as follows:

$$\begin{aligned}&\mu_{ij,kl} := \frac{\varepsilon_{ij}}{\varepsilon_{kl}}, \quad i, j, k, l = 1, 2, 3, \ i \neq j, \ k \neq l,\\ &\sigma_{kl} \neq 0, \ \sigma_{pq} = 0 \text{ for } pq \neq kl, \ \ p, q = 1, 2, 3.\end{aligned} \tag{2.96}$$

Hence, coefficient $\mu_{ij,kl}$ measures the Chentsov's effect in the plane ij due to the shear stress σ_{kl}, i.e. the ratio between the shear strain components ε_{ij} and ε_{kl}. By the definition of the G_{ij}s, Eq. (2.90), it follows that (no summation over dummy indexes)

$$\varepsilon_{ij} = \mu_{ij,kl}\varepsilon_{kl} = \mu_{ij,kl}\frac{\sigma_{kl}}{2G_{kl}} \ \ i, j, k, l = 1, 2, 3, \tag{2.97}$$

and through Eqs. (2.21) and (2.24) we get

$$2S_{pq} = 4Z_{ijkl} = \frac{\mu_{ij,kl}}{G_{kl}} \ \ \Rightarrow \ \ \mu_{ij,kl} = \frac{S_{pq}}{S_{qq}}, \ i, j, k, l = 1, 2, 3, \ \ p, q = 4, 5, 6, \tag{2.98}$$

with p that corresponds to the couple ij and q to kl according to the scheme (2.19). The symmetry of $[S]$ gives the reciprocity relations

$$\frac{\mu_{ij,kl}}{G_{kl}} = \frac{\mu_{kl,ij}}{G_{ij}}, \tag{2.99}$$

that, along with the minor symmetries of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ reduce to only three the number of distinct Chentsov's coefficients. Finally, the remarks done for the ν_{ij}s can be rephrased *verbatim* for the $\mu_{ij,kl}$s.

2.3.5 Coefficients of Mutual Influence of the First Type

These coefficients characterize the normal strain ε_{ii} due to the shear σ_{jk} (no summation over dummy indexes):

$$\eta_{i,jk} := \frac{\varepsilon_{ii}}{2\varepsilon_{jk}} \ \ i,j,k = 1,2,3, \ \ j \neq k, \ \ \sigma_{jk} \neq 0, \ \ \sigma_{pq} = 0 \text{ for } pq \neq jk, \ \ p,q = 1,2,3. \tag{2.100}$$

By the definition of the G_{ij}s, Eq. (2.90), it follows that

$$\varepsilon_{ii} = 2\eta_{i,jk}\varepsilon_{jk} = \eta_{i,jk}\frac{\sigma_{jk}}{G_{jk}}, \tag{2.101}$$

and through Eqs. (2.21) and (2.24) we get

$$\sqrt{2}S_{ip} = 2Z_{iijk} = \frac{\eta_{i,jk}}{G_{jk}} \ \ \Rightarrow \ \ \eta_{i,jk} = \frac{S_{ip}}{\sqrt{2}S_{pp}}, \ \ i,j,k = 1,2,3, \ \ p = 4,5,6, \tag{2.102}$$

with p that corresponds to the couple jk according to the scheme (2.19). For the symmetry of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$, the exchange of suffixes j and k has no effects, so the number of distinct coefficients is only 9.

2.3.6 Coefficients of Mutual Influence of the Second Type

These coefficients characterize the shear strain ε_{ij} due to the normal stress σ_{kk} (no summation over dummy indexes):

$$\eta_{ij,k} := \frac{2\varepsilon_{ij}}{\varepsilon_{kk}} \ \ i,j,k = 1,2,3, \ \ i \neq j, \ \ \sigma_{kk} \neq 0, \ \ \sigma_{pq} = 0 \text{ for } pq \neq kk, \ \ p,q = 1,2,3. \tag{2.103}$$

By the definition of the E_is, Eq. (2.88), it follows that

$$2\varepsilon_{ij} = \eta_{ij,k}\varepsilon_{kk} = \eta_{ij,k}\frac{\sigma_{kk}}{E_k}, \tag{2.104}$$

and through Eqs. (2.21) and (2.24) we get

$$\sqrt{2}S_{pk} = 2Z_{ijkk} = \frac{\eta_{ij,k}}{E_k} \ \ \Rightarrow \ \ \eta_{ij,k} = \sqrt{2}\frac{S_{pk}}{S_{kk}}, \ \ i,j,k = 1,2,3, \ \ p = 4,5,6, \tag{2.105}$$

with p that corresponds to the couple ij according to the scheme (2.19). Like for the coefficients of the first type, the symmetries of $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ reduce the number of distinct coefficients of the second type to only 9.

The coefficients of the second type are not independent from those of the first type; in fact, the symmetry of $[S]$ gives immediately the reciprocity relations

$$\frac{\eta_{ij,k}}{E_k} = \frac{\eta_{k,ij}}{G_{ij}}, \ \ i,j,k = 1,2,3. \tag{2.106}$$

So the use of the coefficients of the first or of the second type is arbitrary and equally valid. Also for the coefficients of the first and second type can be repeated almost *verbatim* the remarks done about the other coefficients.

2.3.7 Some Remarks About the Technical Constants

The relations between a technical constant and the corresponding component of $\mathbb{Z}$, given in the previous Sections, are valid regardless of the notation used, i.e. they are the same also with the Voigt's notation. On the contrary, the relations with the components S_{ij} depends upon the notation, and those found above are not completely identical with the Voigt's notation, see for instance (Jones 1999, p. 79).

It is possible, of course, to express also the components of $[C]$ as functions of the technical constants; this necessitates the inversion of $[S]$ and in the most general case it gives so complicate and long expressions that it is impossible to write them.

Nevertheless, in the important case of orthotropic materials the transformation is rather simple. In fact, in the orthotropic frame, the inverse of matrix $[S]$, which is perfectly analogous to matrix (2.63), is given by (no summation on the dummy indexes)

$$\begin{aligned}
&C_{ii} = \frac{S_{jj}S_{kk} - S_{jk}^2}{S} = \frac{1 - \nu_{jk}\nu_{kj}}{\Delta}E_i, \quad i, j, k = 1, 2, 3, \ i \neq j \neq k,\\
&C_{ij} = \frac{S_{ik}S_{kj} - S_{ij}S_{kk}}{S} = \frac{\nu_{ij} + \nu_{ik}\nu_{kj}}{\Delta}E_j, \quad i, j, k = 1, 2, 3, \ i \neq j \neq k,\\
&C_{44} = \frac{1}{S_{44}} = 2G_{23}, \quad C_{55} = \frac{1}{S_{55}} = 2G_{31}, \quad C_{66} = \frac{1}{S_{66}} = 2G_{12}, \quad \text{with}\\
&S = S_{11}S_{22}S_{33} - S_{11}S_{23}^2 - S_{22}S_{13}^2 - S_{33}S_{12}^2 + 2S_{12}S_{23}S_{13},\\
&\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{32}\nu_{21}\nu_{13}.
\end{aligned} \tag{2.107}$$

In matrix form we have

$$[C] = \begin{bmatrix} \frac{S_{22}S_{33}-S_{23}^2}{S} & \frac{S_{13}S_{32}-S_{12}S_{33}}{S} & \frac{S_{12}S_{23}-S_{13}S_{22}}{S} & 0 & 0 & 0\\ & \frac{S_{11}S_{33}-S_{13}^2}{S} & \frac{S_{21}S_{13}-S_{23}S_{11}}{S} & 0 & 0 & 0\\ & & \frac{S_{11}S_{22}-S_{12}^2}{S} & 0 & 0 & 0\\ & & & \frac{1}{S_{44}} & 0 & 0\\ & sym & & & \frac{1}{S_{55}} & 0\\ & & & & & \frac{1}{S_{66}} \end{bmatrix}, \tag{2.108}$$

and with the technical constants

$$[C]=\begin{bmatrix}\frac{1-\nu_{23}\nu_{32}}{\Delta}E_1 & \frac{\nu_{12}+\nu_{13}\nu_{32}}{\Delta}E_2 & \frac{\nu_{13}+\nu_{12}\nu_{23}}{\Delta}E_3 & 0 & 0 & 0\\ & \frac{1-\nu_{13}\nu_{31}}{\Delta}E_2 & \frac{\nu_{23}+\nu_{21}\nu_{13}}{\Delta}E_3 & 0 & 0 & 0\\ & & \frac{1-\nu_{12}\nu_{21}}{\Delta}E_3 & 0 & 0 & 0\\ & & & 2G_{23} & 0 & 0\\ & sym & & & 2G_{13} & 0\\ & & & & & 2G_{12}\end{bmatrix} \tag{2.109}$$

and conversely

$$[S]=\begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0\\ & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0\\ & & \frac{1}{E_3} & 0 & 0 & 0\\ & & & \frac{1}{2G_{23}} & 0 & 0\\ & sym & & & \frac{1}{2G_{13}} & 0\\ & & & & & \frac{1}{2G_{12}}\end{bmatrix}. \tag{2.110}$$

It is also worth to specify these results also for the isotropic case

$$[C]=\begin{bmatrix}\frac{(1-\nu)E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ & \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} & \frac{\nu E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ & & \frac{(1-\nu)E}{(1-2\nu)(1+\nu)} & 0 & 0 & 0\\ & & & \frac{E}{1+\nu} & 0 & 0\\ & sym & & & \frac{E}{1+\nu} & 0\\ & & & & & \frac{E}{1+\nu}\end{bmatrix}, \tag{2.111}$$

and conversely

$$[S]=\begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ & & \frac{1}{E} & 0 & 0 & 0\\ & & & \frac{1+\nu}{E} & 0 & 0\\ & sym & & & \frac{1+\nu}{E} & 0\\ & & & & & \frac{1+\nu}{E}\end{bmatrix}. \tag{2.112}$$

To remark that the Voigt's notation can be obtained simply dividing by a factor 2 the components C_{44}, C_{55} and C_{66}, and multiplying by 2 the components S_{44}, S_{55} and S_{66}.

2.4 Bounds on the Elastic Constants

2.4.1 General Conditions and Results

Elastic constants cannot take any value: they are bounded because of the physical fact that the deformation of an elastic body Ω cannot produce energy, i.e. the overall work $\mathscr{L}_{ext}$ done by the applied forces *must be positive*. From the Clapeyron's Theorem

$$\mathscr{L}_{ext} = 2V_{tot} = 2\left(\frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}\, d\Omega\right), \tag{2.113}$$

we get hence the condition that the total strain energy V_{tot} must be positive. Assuming the strain as independent field over Ω, then the overall condition is

$$V_{tot} = \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon}\, d\Omega > 0 \;\; \forall \boldsymbol{\varepsilon} \neq \mathbf{O}. \tag{2.114}$$

The above constraint on the deformation of an elastic body is a strong condition. By a procedure of limit towards small volumes, it is easy to see that it must be true also locally, i.e. $\forall p \in \Omega$; it is just the local form of (2.114) that gives the bounds on the elastic constants of a material. In fact, getting the local form of (2.114) and injecting the Hooke's law (2.12) gives

$$V = \frac{1}{2}\boldsymbol{\sigma} \cdot \boldsymbol{\varepsilon} = \frac{1}{2}\boldsymbol{\varepsilon} \cdot \mathbb{E}\boldsymbol{\varepsilon} > 0 \;\; \forall \boldsymbol{\varepsilon} \neq \mathbf{O}. \tag{2.115}$$

Equation (2.115) is the mathematical condition corresponding to the thermodynamical fact that no energy can be produced deforming an elastic body: the elasticity stiffness tensor $\mathbb{E}$ must be *positive definite*.

If the $\boldsymbol{\sigma}$ is taken as independent field over Ω in place of $\boldsymbol{\varepsilon}$, we get a similar restriction on the stress energy and finally the condition that the elasticity compliance tensor $\mathbb{Z}$ must be positive definite. Of course, the two approaches give in the end the same results for the elastic constants.

2.4.2 Mathematical Conditions for the Elastic Matrices

It is easier to obtain practical results for the components of matrices $[C]$ and $[S]$ than for tensors $\mathbb{E}$ and $\mathbb{Z}$, so let us rewrite condition (2.115) in its equivalent matrix form:

$$\frac{1}{2}\{\varepsilon\}^{\top}[C]\{\varepsilon\} > 0 \;\; \forall\{\varepsilon\} \neq \{O\}. \tag{2.116}$$

Of course, the above condition states the positive definiteness of matrix $[C]$, and an analogous condition can be written for matrix $[S]$.

Mathematically, the problem is hence clear: being $[C]$ symmetric, so, by the Spectral Theorem, with real eigenvalues λ_i, condition (2.116) corresponds to impose that all the eigenvalues are positive:

$$\frac{1}{2}\{\varepsilon\}^\top[C]\{\varepsilon\} > 0 \ \ \forall\{\varepsilon\} \neq \{O\} \quad \Longleftrightarrow \quad \lambda_i > 0 \ \forall i = 1, \ldots, 6. \tag{2.117}$$

The above result is almost useless, because the Laplace's equation of $[C]$ is an algebraic equation of degree 6. Hence, generally speaking, it is not possible to get an analytic expression of the roots of this equation, the eigenvalues λ_i, for obtaining the searched bounds on the C_{ij}.

Nevertheless, a first qualitative result is that the number of conditions on the C_{ij}s is 6. As distinct components are, in the most general case, 21, the conditions on the C_{ij}s are not necessarily simple bounds but at least some of them are necessarily relations among some of the components. Also, for the hexagonal, cubic and isotropic syngonies the number of conditions is redundant with respect to the distinct elastic constants, so some of them have lower and upper bounds and/or some of the bounds are redundant (this, anyway, can be true also for other syngonies).

Though the approach by eigenvalues is practically impossible, there is another mathematical approach which is completely general and feasible. To this purpose, let us introduce the following definitions and theorems of matrix algebra.

A *principal minor* of a matrix $[A]$ is the determinant of the sub-matrix extracted from $[A]$ removing an equal number of rows and columns having the same indexes, i.e. preserving the leading diagonal.

A *leading principal minor of order* r is the determinant of a principal $r \times r$ sub-matrix whose rows and columns are the first r rows and columns of $[A]$. Hence, a $n \times n$ matrix has n leading principal minors.

Theorem 1 (Necessary condition for a symmetric matrix to be positive definite) *All the principal minors of a positive definite* $n \times n$ *symmetric matrix* $[A]$ *are positive.*

Proof By the hypothesis,

$$\{x\}^\top[A]\{x\} > 0 \ \ \forall\{x\} \neq \{0\}. \tag{2.118}$$

Then, for a principal $r \times r$ sub-matrix $[A^r]$ extracted from $[A]$ deleting $n - r$ rows and columns with the same indexes, we have

$$\{x^r\}^\top[A^r]\{x^r\} = \{x\}^\top[A]\{x\} > 0 \ \ \forall\{x\}, \{x^r\} \neq \{0\} \tag{2.119}$$

where $\{x\}$ is any vector whose components corresponding to the removed rows of $[A]$ are null and with at least one of the other components different from zero, while $\{x^r\}$ is the $r-$dimensional vector obtained removing from $\{x\}$ the components corresponding to the removed rows of $[A]$.

Hence, $[A^r]$ is positive definite and because it is symmetric, its eigenvalues are positive, see Eq. (2.117). Then, its determinant, which is the product of its eigenvalues, is positive too. This proves that $\det[A^r] > 0$ and hence the theorem.

Theorem 2 (Necessary and sufficient condition for a symmetric matrix to be positive definite) *For a $n \times n$ symmetric matrix $[A]$ to be positive definite it is necessary and sufficient that its n leading principal minors are all positive.*

The proof of this theorem is non trivial and the reader is referred to (Hohn 1958, p. 340).

The six principal minors of $[C]$ are

$$
\begin{aligned}
&M_1 = C_{11}, \quad M_2 = \begin{vmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{vmatrix}, \quad M_3 = \begin{vmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}, \\
&M_4 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{12} & C_{22} & C_{23} & C_{24} \\ C_{13} & C_{23} & C_{33} & C_{34} \\ C_{14} & C_{24} & C_{34} & C_{44} \end{vmatrix}, \quad M_5 = \begin{vmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} \end{vmatrix}, \\
&M_6 = \det[C].
\end{aligned}
\tag{2.120}
$$

Contrarily to the eigenvalues, it is always possible to explicit the above expressions and hence the 6 conditions

$$
M_i > 0, \ i = 1, \ldots, 6. \tag{2.121}
$$

That is why the use of Theorem 2 is more interesting than condition (2.117), though to write down the 6 conditions in the most general case of a triclinic material gives so long expressions that they are omitted here.

We can, however, consider the different elastic syngonies and because they have a simpler form of $[C]$ than in the triclinic case, also conditions (2.121) will be simpler. In particular, let us consider here some cases whose results are particularly simple (the bounds are written for matrix $[C]$, but similar results can be written for $[S]$ too; redundant bounds have been omitted):

- orthotropic elastic syngony, Eq. (2.63):

$$
\begin{aligned}
&C_{ii} > 0, \quad i = 1, 4, 5, 6, \\
&C_{11}C_{22} - C_{12}^2 > 0, \\
&C_{11}C_{22}C_{33} - C_{33}C_{12}^2 - C_{11}C_{23}^2 - C_{22}C_{13}^2 + 2C_{12}C_{13}C_{23} > 0;
\end{aligned}
\tag{2.122}
$$

- tetragonal elastic syngony with 6 constants, Eq. (2.86):

$$
\begin{aligned}
&C_{44} > 0,\\
&C_{66} > 0,\\
&C_{11}^2 - C_{12}^2 > 0,\\
&(C_{11} - C_{12})\left[C_{33}(C_{11} + C_{12}) - 2C_{13}^2\right] > 0;
\end{aligned}
\tag{2.123}
$$

- axially symmetric elastic syngony, Eq. (2.73):

$$
\begin{aligned}
&C_{44} > 0,\\
&C_{11}^2 - C_{12}^2 > 0,\\
&(C_{11} - C_{12})\left[C_{33}(C_{11} + C_{12}) - 2C_{13}^2\right] > 0;
\end{aligned}
\tag{2.124}
$$

- cubic elastic syngony, Eq. (2.87)

$$
\begin{aligned}
&C_{44} > 0,\\
&C_{11} - C_{12} > 0,\\
&C_{11} + 2C_{12} > 0;
\end{aligned}
\tag{2.125}
$$

- isotropic elastic syngony, Eq. (2.77):

$$
\begin{aligned}
&C_{11} - C_{12} > 0,\\
&C_{11} + 2C_{12} > 0.
\end{aligned}
\tag{2.126}
$$

2.4.3 A Mechanical Approach

The bounds on the elastic constants can be found also by a direct mechanical approach, based upon the fact that the strain energy must be positive *for each possible choice of the strain field* $\boldsymbol{\varepsilon}$. This allows for choosing particularly simple strain fields, giving some direct, simple results. Let us see how (no summation over dummy indexes): choose a field $\{\varepsilon\}$ with only one component $\varepsilon_i \neq 0$. Then,

$$
V > 0 \iff C_{ii} > 0,\ i = 1, \ldots, 6;
\tag{2.127}
$$

we get hence six conditions. Unfortunately, each one of them is only a *necessary* condition for the strain energy be positive, so the (2.127) *do not constitute a set of necessary and sufficient* conditions for the positiveness of V. Nevertheless, they give us a precious information: *all the moduli responsible for the direct effects are strictly positive*. Using the stress energy instead of the strain energy, it is immediately recognized that it is also:

$$
S_{ii} > 0\ \forall i = 1, \ldots, 6.
\tag{2.128}
$$

2.4.4 Bounds on the Technical Constants

The results of Eqs. (2.89), (2.91) and (2.128) give immediately

$$E_i > 0, \quad G_{ij} > 0 \;\; \forall i, j = 1, 2, 3 : \tag{2.129}$$

all the Young's and shear moduli are strictly positive quantities, result that is valid for any kind of elastic syngony.

To these necessary conditions some other relations for the technical constants can be added. First of all, let us consider a spherical state of stress; it is then easy to see that

$$\begin{aligned} &\{\sigma\} = \sigma\{I\} \;\;\Rightarrow\;\; \{\sigma\}^\top[S]\{\sigma\} > 0 \;\;\iff \\ &S_{11} + S_{22} + S_{33} + 2(S_{13} + S_{32} + S_{21}) > 0. \end{aligned} \tag{2.130}$$

Replacing in the above result the expressions of the S_{ij}s from Eqs. (2.89) and (2.94) gives the condition

$$\frac{1-2\nu_{12}}{E_1} + \frac{1-2\nu_{23}}{E_2} + \frac{1-2\nu_{31}}{E_3} > 0. \tag{2.131}$$

This result is valid regardless of the elastic syngony; for the cubic and isotropic syngonies it becomes the well known bound $\nu < 1/2$ on the Poisson's coefficient. A simpler but rougher estimation can be obtained from bound (2.131), (Lekhnitskii 1950, p. 85):

$$\begin{aligned} &\frac{3-2(\nu_{12}+\nu_{23}+\nu_{31})}{\min\{E_1, E_2, E_3\}} > \frac{1-2\nu_{12}}{E_1} + \frac{1-2\nu_{23}}{E_2} + \frac{1-2\nu_{31}}{E_3} > 0 \;\Rightarrow \\ &\nu_{12} + \nu_{23} + \nu_{31} < \frac{3}{2}. \end{aligned} \tag{2.132}$$

Some other necessary conditions can be given expressing the C_{ii} in terms of the technical parameters. This is impossible in the most general case of the triclinic syngony, for the calculations are too complicate. However, this can be done for the orthotropic syngony; the supplementary bounds can be found expressing the (2.127) as functions of the technical constants through Eq. (2.107) and taking into account the positivity of the Young's moduli, Eq. (2.129):

$$\begin{aligned} &1 - \nu_{ij}\nu_{ji} > 0 \;\; \forall i, j = 1, 2, 3; \\ &\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{32}\nu_{21}\nu_{13} > 0. \end{aligned} \tag{2.133}$$

Condition $(2.133)_2$ can be transformed to

$$\nu_{32}\nu_{21}\nu_{13} < \frac{1}{2}\left(1 - \nu_{32}^2\frac{E_2}{E_3} - \nu_{21}^2\frac{E_1}{E_2} - \nu_{13}^2\frac{E_3}{E_1}\right) < \frac{1}{2}. \tag{2.134}$$

Through the reciprocity conditions on the Poisson's coefficients, Eq. (2.95), conditions $(2.133)_1$ can be written also as

$$|\nu_{ij}| < \sqrt{\frac{E_i}{E_j}}\ \ \forall i, j = 1, 2, 3, \tag{2.135}$$

or equivalently

$$|S_{ij}| < \sqrt{S_{ii}S_{jj}}\ \ \forall i, j = 1, 2, 3. \tag{2.136}$$

Some remarks to end this part; first of all, the bounds concern frame dependent quantities, and of course they are more easily written in a frame composed by symmetry directions. Then, the only, general, necessary and sufficient conditions are the (2.121), that can always be written and used in numerical applications, e.g. for checking the validity of the results of experimental tests.

In the case of orthotropic materials, a set of conditions on the technical constants can be easily written, but it is still questionable whether or not it constitutes a set of necessary *and* sufficient conditions for the positivity of the strain energy, a point never treated in the literature. Finally, bounds on the Chentsov's and mutual influence coefficients are apparently unknown in the literature.

In the case of isotropic materials, the conditions of positivity of the strain energy reduce to the well known three bounds on E and ν

$$E > 0, \quad -1 < \nu < \frac{1}{2}, \tag{2.137}$$

but when the bounds are written for the two distinct components of $[C]$, C_{11} and C_{12}, then rather surprisingly the bounds are only two, see Eq. (2.126):

$$C_{11} - C_{12} > 0, \quad C_{11} + 2C_{12} > 0. \tag{2.138}$$

Also when the isotropic constitutive law is written under the form of the Lamé's equations

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon} + \lambda \mathrm{tr}\boldsymbol{\varepsilon}\,\mathbf{I}, \tag{2.139}$$

it is easy to show that the only two bounds on the Lamé's constants λ and μ are

$$\mu > 0, \quad 2\mu + 3\lambda > 0, \tag{2.140}$$

that corresponds exactly to bounds (2.138). This fact shows that the number of necessary and sufficient conditions for the strain energy to be positive depends upon the choice of the elastic constants and that, anyway, it is quite hard to establish a priori its value, whose maximum remains however 6.

A last remark: all the bounds and conditions written in this Section are written on *frame dependent quantities*, apart those written for the isotropic case, Eq. (2.126), of course. In particular, conditions (2.122) to (2.125) are valid exclusively in the symmetry frame where the respective matrices $[C]$ have been written. In the plane case, we will see that it is possible, with the polar formalism, to give *completely invariant bounds*, i.e. bounds established on tensor invariants, which are not yet known for the general 3D case.

2.5 An Observation About the Decomposition of the Strain Energy

Let us consider a point which is true at least for isotropic materials but often thought as generally true also for other elastic syngonies: is it possible to decompose the strain, or stress, energy into *spherical* and *deviatoric* parts? In other words, we ponder whether or not it is *always* possible to write

$$V = V_{sph} + V_{dev}, \tag{2.141}$$

where V_{sph}, the *spherical* part of V is produced *exclusively* by the spherical part of $\boldsymbol{\varepsilon}$ and by its corresponding part of $\boldsymbol{\sigma}$, i.e.

$$V_{sph} = \frac{1}{2}\boldsymbol{\varepsilon}_{sph} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{sph}, \tag{2.142}$$

and V_{dev}, the *deviatoric* part of V is produced *exclusively* by the deviatoric part of $\boldsymbol{\varepsilon}$ and by its corresponding part of $\boldsymbol{\sigma}$, i.e.

$$V_{dev} = \frac{1}{2}\boldsymbol{\varepsilon}_{dev} \cdot \mathbb{E}\boldsymbol{\varepsilon}_{dev}. \tag{2.143}$$

Mechanically, such a decomposition means that V can be considered as the sum of two parts: one, V_{sph}, due to volume changes not accompanied by shape changes, the other one, V_{dev}, produced by isochoric shape changes. This decomposition is, for instance, at the basis of the Hüber–Hencky–von Mises criterion, where the only V_{dev} is considered to be responsible of yielding.

We recall that it is always possible to decompose $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ into a spherical and a deviatoric part

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}_{sph} + \boldsymbol{\sigma}_{dev}, \quad \boldsymbol{\sigma}_{sph} = \frac{1}{3}\mathrm{tr}\boldsymbol{\sigma}\,\mathbf{I}, \quad \boldsymbol{\sigma}_{dev} = \boldsymbol{\sigma} - \boldsymbol{\sigma}_{sph}, \\ \boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}_{sph} + \boldsymbol{\varepsilon}_{dev}, \quad \boldsymbol{\varepsilon}_{sph} = \frac{1}{3}\mathrm{tr}\boldsymbol{\varepsilon}\,\mathbf{I}, \quad \boldsymbol{\varepsilon}_{dev} = \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_{sph}, \end{aligned} \tag{2.144}$$

and that any spherical part is orthogonal to any deviatoric part:

$$\begin{aligned}\boldsymbol{\sigma}_{sph}\cdot\boldsymbol{\varepsilon}_{dev}&=\frac{1}{3}\text{tr}\boldsymbol{\sigma}\ \mathbf{I}\cdot\left(\boldsymbol{\varepsilon}-\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \mathbf{I}\right)=\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \text{tr}\boldsymbol{\sigma}-\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \text{tr}\boldsymbol{\sigma}=0,\\ \boldsymbol{\sigma}_{dev}\cdot\boldsymbol{\varepsilon}_{sph}&=\left(\boldsymbol{\sigma}-\frac{1}{3}\text{tr}\boldsymbol{\sigma}\ \mathbf{I}\right)\cdot\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \mathbf{I}=\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \text{tr}\boldsymbol{\sigma}-\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \text{tr}\boldsymbol{\sigma}=0.\end{aligned}\tag{2.145}$$

Using decomposition (2.144) we have

$$\begin{aligned}V=&\frac{1}{2}\boldsymbol{\varepsilon}\cdot\mathbb{E}\boldsymbol{\varepsilon}=\frac{1}{2}(\boldsymbol{\varepsilon}_{sph}+\boldsymbol{\varepsilon}_{dev})\cdot\mathbb{E}(\boldsymbol{\varepsilon}_{sph}+\boldsymbol{\varepsilon}_{dev})=\\ &\frac{1}{2}\boldsymbol{\varepsilon}_{sph}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{sph}+\frac{1}{2}\boldsymbol{\varepsilon}_{dev}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{dev}+\frac{1}{2}\boldsymbol{\varepsilon}_{sph}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{dev}+\frac{1}{2}\boldsymbol{\varepsilon}_{dev}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{sph}.\end{aligned}\tag{2.146}$$

For the decomposition (2.141) to be true, it is necessary and sufficient that

$$\boldsymbol{\varepsilon}_{sph}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{dev}=0\ \Rightarrow\ \text{tr}\left[\boldsymbol{\varepsilon}_{sph}^{\top}(\mathbb{E}\boldsymbol{\varepsilon}_{dev})\right]=0\ \ \forall\boldsymbol{\varepsilon}.\tag{2.147}$$

In fact, whenever Eq. (2.147) is satisfied, for definition (2.38) it is

$$\boldsymbol{\varepsilon}_{dev}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{sph}=\mathbb{E}^{\top}\boldsymbol{\varepsilon}_{dev}\cdot\boldsymbol{\varepsilon}_{sph}=\boldsymbol{\varepsilon}_{sph}\cdot\mathbb{E}\boldsymbol{\varepsilon}_{dev},\tag{2.148}$$

because of the symmetry of $\mathbb{E}$, i.e. for its major symmetries. This result shows that the two mixed terms in (2.146) are identical.

Through (2.144), condition (2.147) can be written as

$$\text{tr}\left[\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \mathbf{I}(\mathbb{E}\boldsymbol{\varepsilon}_{dev})\right]=0\ \ \forall\boldsymbol{\varepsilon}\quad\Longleftrightarrow\quad\text{tr}(\mathbb{E}\boldsymbol{\varepsilon}_{dev})=0.\tag{2.149}$$

The components of $\mathbb{E}$ must satisfy Eq. (2.149) for the decomposition (2.141) to be possible. It can be rewritten as

$$\text{tr}\left[\mathbb{E}\left(\boldsymbol{\varepsilon}-\frac{1}{3}\text{tr}\boldsymbol{\varepsilon}\ \mathbf{I}\right)\right]=0\ \Rightarrow\ 3\text{tr}(\mathbb{E}\boldsymbol{\varepsilon})-\text{tr}\boldsymbol{\varepsilon}\ \text{tr}(\mathbb{E}\mathbf{I})=0\ \ \forall\boldsymbol{\varepsilon}.\tag{2.150}$$

Actually, it is easy to check that condition (2.147) corresponds to impose that

$$\boldsymbol{\sigma}_{dev}=\mathbb{E}\boldsymbol{\varepsilon}_{dev},\quad\boldsymbol{\sigma}_{sph}=\mathbb{E}\boldsymbol{\varepsilon}_{sph}.\tag{2.151}$$

Condition (2.150) can be written by components:

$$E_{hhkk}\varepsilon_{ii}-3E_{jjpq}\varepsilon_{pq}=0\ \ \forall\varepsilon_{mn},\quad i,j,h,k,p,q,m,n=1,2,3.\tag{2.152}$$

Generally speaking, this quantity does not vanish for any possible choice of $\boldsymbol{\varepsilon}$. As a consequence, *for a generic anisotropic material decomposition of the strain energy into a spherical and deviatoric part is not possible*. Nevertheless, it can be checked that for the cubic syngony Eq. (2.152) is always satisfied. In fact, for an orthotropic material condition (2.152) becomes

$$\begin{aligned}&\frac{1}{3}[E_{1111}\,(2\varepsilon_{11}-\varepsilon_{22}-\varepsilon_{33})+E_{2222}\,(2\varepsilon_{22}-\varepsilon_{11}-\varepsilon_{33})+\\&E_{3333}\,(2\varepsilon_{33}-\varepsilon_{22}-\varepsilon_{11})]+\\&\frac{2}{3}[E_{1122}\,(\varepsilon_{11}+\varepsilon_{22}-2\varepsilon_{33})+E_{1133}\,(\varepsilon_{11}+\varepsilon_{33}-2\varepsilon_{22})+\\&E_{2233}\,(\varepsilon_{22}+\varepsilon_{33}-2\varepsilon_{11})]=0,\end{aligned}\tag{2.153}$$

condition which is not yet satisfied, generally speaking, but which is always satisfied when

$$E_{1111}=E_{2222}=E_{3333},\quad E_{1122}=E_{2233}=E_{1133},\tag{2.154}$$

i.e. by cubic materials. Hence, for materials of the cubic syngony, decomposition (2.141) is always possible and, a fortiori, by isotropic materials too, as well known, because they can be considered as a special case of the cubic syngony, see Sect. 2.2.11, point 10.

2.6 Determination of Symmetry Planes

The classification in elastic syngonies presupposes that, for a given material, the existing equivalent directions are known, so as to write $\mathbb{E}$, or equivalently $[C]$, in a symmetry frame, which makes some of the E_{ijkl}s, and the corresponding C_{ij}s, vanish.

But when a material is completely unknown, the independent measures to be done in experimental tests to characterize the material are as much as 21; practically, it is very complicate to do all of these tests. Nevertheless, the existence of possible symmetry planes remains unknown also once all the C_{ij}s are known, if $[C]$ is a full matrix.

The problem is hence the following one: given a general matrix $[C]$, is it possible to determine if some planes of symmetry exist and which they are? We will see in Sect. 4.1 that in the planar case it is very simple to determine whether or not the material has some kind of elastic symmetry and the symmetry directions using the polar formalism. In the three dimensional case, the problem is much more complicate; it has been solved by Cowin and Mehrabadi in two works, (Cowin and Mehrabadi 1987; Cowin 1989), successively completed by Ting (1996). We give here a brief account of these results.

Be $\mathbf{n}$ and $\mathbf{m}$ two vectors such that $||\mathbf{n}||=||\mathbf{m}||=1,\ \mathbf{m}\cdot\mathbf{n}=0$, with $\mathbf{n}$ orthogonal to a symmetry plane for a material whose elastic tensor is $\mathbb{E}$. Consider the following second-rank symmetric tensors: $\mathbf{V}=\mathbb{E}\mathbf{I}$, $\mathbf{W}$ the acoustic[14] tensor relative to the basis

[14]The *acoustic* or *Green-Christoffel tensor* $\mathbf{A_u}$ relative to the direction $\mathbf{u}$ is the unique tensor such that $\mathbf{A_u w}=\mathbb{E}(\mathbf{w}\otimes\mathbf{u})\mathbf{u}\ \forall\mathbf{w}\in\mathscr{V}$.

direction $\mathbf{e}_p$, $\mathbf{X}$ and $\mathbf{Y}$ the acoustic tensors relative to $\mathbf{n}$ and $\mathbf{m}$, respectively.[15] We can now state the following:

Theorem 3 *The following statements are equivalent ($\lambda_i \in \mathbb{R},\ i = 1, \ldots, 6$):*

1. *the material has a plane of symmetry whose normal is* $\mathbf{n}$*;*
2. $\mathbf{Vn} = \lambda_1 \mathbf{Yn} = \lambda_2 \mathbf{n}$*;*
3. $\mathbf{Wn} = \lambda_3 \mathbf{Yn} = \lambda_4 \mathbf{n}$*;*
4. $\mathbf{Xn} = \lambda_5 \mathbf{Yn} = \lambda_6 \mathbf{n}$.

Proof Without loss of generality, let us suppose that $\mathbf{n} = \mathbf{e}_1$ and $\mathbf{m} = \cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_3$. When $\mathbf{n}$ is an eigenvector of $\mathbf{V}$, $\mathbf{W}$, $\mathbf{X}$ or $\mathbf{Y}$ then

$$\begin{aligned}
&\mathbf{Vn} = \lambda_v \mathbf{n} \ \rightarrow\ E_{i1qq}\mathbf{e}_i = \lambda_v \mathbf{e}_1,\\
&\mathbf{Wn} = \lambda_w \mathbf{n} \ \rightarrow\ E_{ip1p}\mathbf{e}_i = \lambda_w \mathbf{e}_1,\\
&\mathbf{Xn} = \lambda_x \mathbf{n} \ \rightarrow\ E_{i111}\mathbf{e}_i = \lambda_x \mathbf{e}_1,\\
&\mathbf{Yn} = \lambda_y \mathbf{n} \ \rightarrow\\
&\left[E_{i212}\cos^2\theta + E_{i313}\sin^2\theta + (E_{i213} + E_{i312})\sin\theta\cos\theta\right]\mathbf{e}_i = \lambda_y \mathbf{e}_1\ \forall\theta.
\end{aligned} \tag{2.155}$$

For $i = 1$, the above results give the values of the respective eigenvalues, but for $i = 2, 3$ we get, respectively,

$$\begin{aligned}
&E_{21qq} = E_{31qq} = 0,\\
&E_{2p1p} = E_{3p1p} = 0,\\
&E_{2111} = E_{3111} = 0,\\
&E_{2212}\cos^2\theta + E_{2313}\sin^2\theta + (E_{2213} + E_{2312})\sin\theta\cos\theta =\\
&E_{3212}\cos^2\theta + E_{3313}\sin^2\theta + (E_{3213} + E_{3312})\sin\theta\cos\theta = 0\ \forall\theta.
\end{aligned} \tag{2.156}$$

Passing to the C_{ij}s for the sake of convenience, and writing down in the order all the above relations, we get

$$\begin{aligned}
&C_{15} + C_{25} + C_{35} = C_{16} + C_{26} + C_{36} = 0,\\
&C_{15} + C_{35} + \frac{C_{46}}{\sqrt{2}} = C_{16} + C_{26} + \frac{C_{45}}{\sqrt{2}} = 0,\\
&C_{15} = C_{16} = 0,\\
&C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0.
\end{aligned} \tag{2.157}$$

[15]It is simple to verify that

$$\begin{aligned}
&\mathbf{V} = \mathbb{E}\mathbf{I} = E_{ikqq}\mathbf{e}_i \otimes \mathbf{e}_k,\\
&\mathbf{W} = E_{ipkp}\mathbf{e}_i \otimes \mathbf{e}_k,\\
&\mathbf{X} = E_{ilkm}n_l n_m \mathbf{e}_i \otimes \mathbf{e}_k,\\
&\mathbf{Y} = E_{ijkh}m_j m_h \mathbf{e}_i \otimes \mathbf{e}_k.
\end{aligned}$$

.

If the material has $x_1 = 0$ as unique plane of symmetry, it belongs to the monoclinic syngony and its matrix $[C]$ is given by, see Sect. 2.2.5,

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ & C_{22} & C_{23} & C_{24} & 0 & 0 \\ & & C_{33} & C_{34} & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}, \tag{2.158}$$

that is:

$$C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{46} = 0. \tag{2.159}$$

It is then clear that conditions $(2.157)_{1,4}$, $(2.157)_{2,4}$ or $(2.157)_{3,4}$ imply (2.159) and vice-versa, which proves the theorem.

This theorem states that the material has a plane of symmetry whose normal is $\mathbf{n}$ if and only if $\mathbf{n}$ is the eigenvector of $\mathbf{Y}$ and of at least another tensor among $\mathbf{V}$, $\mathbf{W}$ or $\mathbf{X}$.

2.6.1 Physical Interpretations

A physical interpretation of Theorem 3 is possible in the frame of the acoustics theory, see (Ting 1996, p. 61): tensor $\mathbf{X}$ is the acoustic tensor for the elastic waves that propagate in the direction of $\mathbf{n}$. An elastic wave is a *longitudinal wave* whenever $\mathbf{n}$ is an eigenvector of $\mathbf{X}$; in such a case, $\mathbf{n}$ is called a *specific direction* of $\mathbf{X}$. It has been proved by Kolodner that in an anisotropic material there exist always at least three different specific directions (Kolodner 1966). When $\mathbf{n}$ is an eigenvector of $\mathbf{Y}$, then the wave is *transversal*, $\mathbf{m}$ is the direction of the wave propagation and $\mathbf{n}$ is called the *specific axis*.

Then conditions $(2.157)_{3,4}$, i.e. when $\mathbf{n}$ is an eigenvector of $\mathbf{X}$ and $\mathbf{Y}$, are equivalent to say that $\mathbf{n}$ is at the same time a specific direction and a specific axis, i.e. it is simultaneously the direction of propagation of longitudinal waves and the transversal direction of transversal waves propagating along the direction of $\mathbf{m}$ orthogonal to $\mathbf{n}$.

A statical interpretation has also been given by Hayes and Norris. It traduces the above acoustics conditions into equivalent statical conditions. They have been resumed in the following

Theorem 4 *A material has a plane of symmetry if and only if at least two orthogonal planes of pure shear exist, sharing a common shear direction which is the normal to the plane of symmetry.*

For the proof of this Theorem, rather articulated, we address the reader to the original paper (Hayes and Norris 1991).

2.7 Curvilinear Anisotropy

When in a body there are directions that are not parallel but mechanically equivalent, then the body possesses a *curvilinear anisotropy*. It is still possible to write the Hooke's law in a rectangular coordinate system, as done until now. However, in doing so, the components of $[C]$ or $[S]$ are no more constants, but vary with the position according to the variation of the coordinate directions with respect to the equivalent directions.

Be $\{\xi, \eta, \zeta\}$ the coordinate directions of the curvilinear coordinates that coincide with the mechanically equivalent directions. With self-evident meaning of the symbols, the Hooke's law can be written in the curvilinear coordinate system as (Lekhnitskii 1950, p. 64),

$$\begin{Bmatrix} \sigma_{\xi\xi} \\ \sigma_{\eta\eta} \\ \sigma_{\zeta\zeta} \\ \sqrt{2}\sigma_{\eta\zeta} \\ \sqrt{2}\sigma_{\zeta\xi} \\ \sqrt{2}\sigma_{\xi\eta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\xi\xi} \\ \varepsilon_{\eta\eta} \\ \varepsilon_{\zeta\zeta} \\ \sqrt{2}\varepsilon_{\eta\zeta} \\ \sqrt{2}\varepsilon_{\zeta\xi} \\ \sqrt{2}\varepsilon_{\xi\eta} \end{Bmatrix}, \tag{2.160}$$

where the C_{ij}s are constants. In some cases of non homogenous bodies, the C_{ij}s can depend upon the coordinates $\{\xi, \eta, \zeta\}$. Of course, if some type of elastic symmetry is present in the body, then some of the C_{ij}s can be null, as in the ordinary cases of the elastic syngonies.

A special case of curvilinear anisotropy is that of *cylindrical anisotropy*: the body has an axis of symmetry, not necessarily inside the body itself, all the directions crossing this axis at right angles are equivalent, as well as all the directions parallel to the axis and the directions orthogonal to the first two directions.

Using a customary set of cylindrical coordinates $\{r, \theta, z\}$, with z the axis of symmetry, then the Hooke's law can be written as

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{\theta z} \\ \sqrt{2}\sigma_{zr} \\ \sqrt{2}\sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{\theta z} \\ \sqrt{2}\varepsilon_{zr} \\ \sqrt{2}\varepsilon_{r\theta} \end{Bmatrix}. \tag{2.161}$$

A special case of cylindrical anisotropy is that of *cylindrical orthotropy*: each plane which is radial, tangential or orthogonal to the symmetry axis is a plane of symmetry. In such a case matrix $[C]$ in Eq. (2.161) is simplified:

$$\begin{Bmatrix} \sigma_{rr} \\ \sigma_{\theta\theta} \\ \sigma_{zz} \\ \sqrt{2}\sigma_{\theta z} \\ \sqrt{2}\sigma_{zr} \\ \sqrt{2}\sigma_{r\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{rr} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{zz} \\ \sqrt{2}\varepsilon_{\theta z} \\ \sqrt{2}\varepsilon_{zr} \\ \sqrt{2}\varepsilon_{r\theta} \end{Bmatrix}. \tag{2.162}$$

It is worth noting that cylindrical orthotropy is *not* equivalent to transverse isotropy (that in fact depends upon only 5 constants, not upon 9). Actually, transverse isotropy is a special case of cylindrical orthotropy, because not only the radial and tangential directions are equivalent, but all the directions lying in a plane orthogonal to the symmetry axis are equivalent directions.

Some examples of cylindrical anisotropy are a block of wood with regular yearly cylindrical layers, or metallic pipes, for their manufacturing process, or a circular reinforced concrete slab with steel bars disposed radially and circumferentially, a bicycle wheel, when homogenized, a circular stone arch and so on.

Another, less common, type of curvilinear anisotropy is that of *spherical anisotropy*: there is a center of symmetry, not necessarily belonging to the body, and all the rays emanating from it are equivalent directions. Also, the tangents to the meridians and to the parallels are equivalent directions too. Using a standard spherical coordinate systems $\{\rho, \theta, \varphi\}$, where the directions of the coordinate axes coincide with the equivalent directions, Eq. (2.160) becomes

$$\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sqrt{2}\sigma_{\theta\varphi} \\ \sqrt{2}\sigma_{\varphi\rho} \\ \sqrt{2}\sigma_{\rho\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & sym & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\rho\rho} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{\varphi\varphi} \\ \sqrt{2}\varepsilon_{\theta\varphi} \\ \sqrt{2}\varepsilon_{\varphi\rho} \\ \sqrt{2}\varepsilon_{\rho\theta} \end{Bmatrix}. \tag{2.163}$$

The case of *spherical orthotropy* is get when each meridian and tangential plane is a plane of symmetry as well as each plane orthogonal to these two planes. Then, Eq. (2.163) becomes

$$\begin{Bmatrix} \sigma_{\rho\rho} \\ \sigma_{\theta\theta} \\ \sigma_{\varphi\varphi} \\ \sqrt{2}\sigma_{\theta\varphi} \\ \sqrt{2}\sigma_{\varphi\rho} \\ \sqrt{2}\sigma_{\rho\theta} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & sym & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{\rho\rho} \\ \varepsilon_{\theta\theta} \\ \varepsilon_{\varphi\varphi} \\ \sqrt{2}\varepsilon_{\theta\varphi} \\ \sqrt{2}\varepsilon_{\varphi\rho} \\ \sqrt{2}\varepsilon_{\rho\theta} \end{Bmatrix}. \tag{2.164}$$

To remark the difference between isotropy and spherical orthotropy: isotropy is a special case of spherical orthotropy, because all the directions are equivalent, not only those emanating from the centre of symmetry. This reduces the number of independent elastic constants from 9 to only 2.

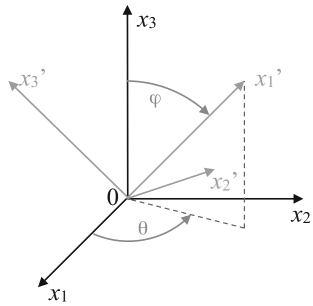


Fig. 2.3 Scheme of the frame rotation for tracing the elastic constants 3D-graphics

2.8 Some Examples of Anisotropic Materials

To end this Chapter, we give in this Section some examples of anisotropic materials, showing the matrix $[C]$ (in GPa) and the 3D-directional diagrams of some of the technical constants. These last have been obtained as the value get by the constant on the axis of x_1' of a frame $\{x_1', x_2', x_3'\}$ rotated with respect to the frame $\{x_1, x_2, x_3\}$ where the matrix $[C]$ is known, see Fig. 2.3. The rotation matrix $[R]$ is obtained according to Eq. (2.48), with a rotation tensor $\mathbf{U}$ that is

$$\mathbf{U} = \begin{bmatrix} \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\varphi\cos\theta & -\cos\varphi\sin\theta & \sin\varphi \end{bmatrix}. \tag{2.165}$$

So, with this choice axis x_2 lies always in the horizontal plane.

The compliance matrix $[S']$ in the rotated frame can be obtained using the inverse of relations (2.49):

$$\begin{aligned} &\{\varepsilon\} = [S]\{\sigma\} \ \rightarrow \ [R]^\top\{\varepsilon'\} = [S][R]^\top\{\sigma'\} \ \rightarrow \\ &\{\varepsilon'\} = [R][S][R]^\top\{\sigma'\} \ \Rightarrow \ [S'] = [R][S][R]^\top. \end{aligned} \tag{2.166}$$

This result can be applied to $[C]$ too, and it is the matrix corresponding of Eq. (2.34). Once the S_{ij}s known, the technical constants can be easily calculated using the results of Sect. 2.3.

Through Eqs. (2.165) and (2.166) it can be shown that for the materials of the hexagonal elastic syngony it is always

$$S_{14} = S_{16} = S_{24} = S_{26} = S_{34} = S_{36} = S_{45} = S_{56} = 0. \tag{2.167}$$

For these materials, the only Chentsov's and mutual influence coefficients that are not identically null are $\mu_{23,12}$, $\eta_{1,31}$, $\eta_{2,31}$, $\eta_{3,31}$, $\eta_{31,1}$, $\eta_{31,2}$ and $\eta_{31,3}$.

Different cases are considered below; for each one of them, the directional 3D-graphics of E_1, G_{12}, ν_{12}, $\mu_{23,12}$, $\eta_{1,31}$ and $\eta_{31,1}$ are traced. For the four last constants, when needed a blue translucent sphere is also traced: it represents the surface whereon the property traced in the graphics vanishes. So, the part of the 3D-graphics inside the sphere corresponds to negative values of the property itself. The graphics show clearly that the Poisson's, Chentsov's and mutual influence coefficients can get negative values. The values of the C_{ij}s, E_1 and G_{12} are in GPa. To remember that $[C]$ is given in the Kelvin's notation, Eq. (2.24).

From the graphics below, one can appreciate the extreme variety of forms of the technical constants. It can be remarked how anisotropy properties change very quickly for small changes of direction (Fig. 2.9, 2.10, 2.11, 2.12, 2.13 and 2.14).

The case of the hexagonal syngony is very articulated, and it can be shown that there are as much as 8 possible different profiles of $E_1(\varphi)$, (Vannucci 2015) (of course E_1, like all the other properties, does not depend upon θ, because the hexagonal elastic syngony is equivalent to transverse isotropy).

- Anorthite ($CaAl_2Si_2O_8$)
  Crystal syngony: Monoclinic, $N = 13$, plane of symmetry: $x_2 = 0$.
  Source: Evans and Grove (2004)

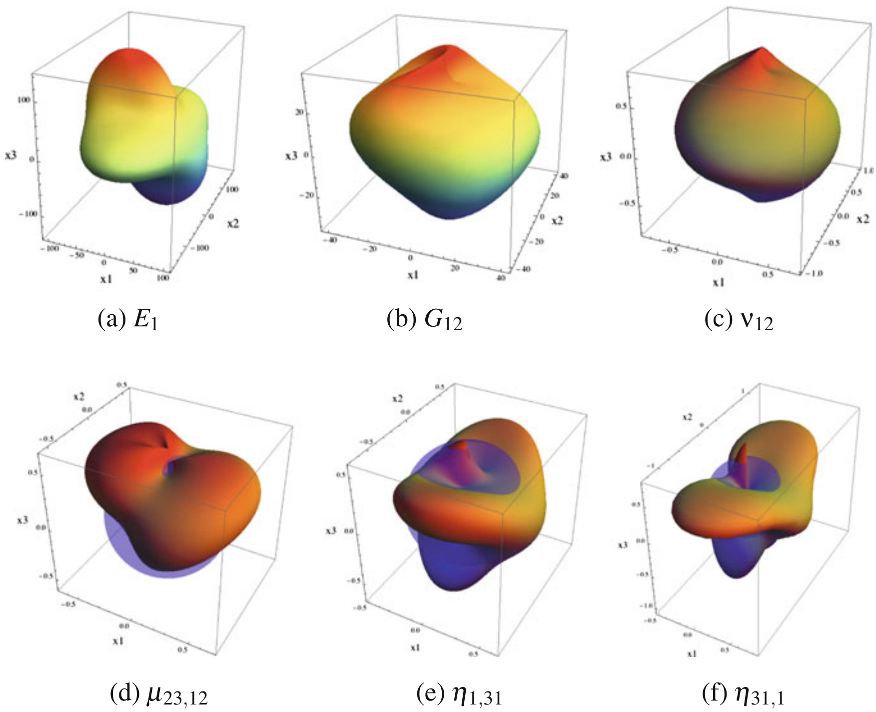
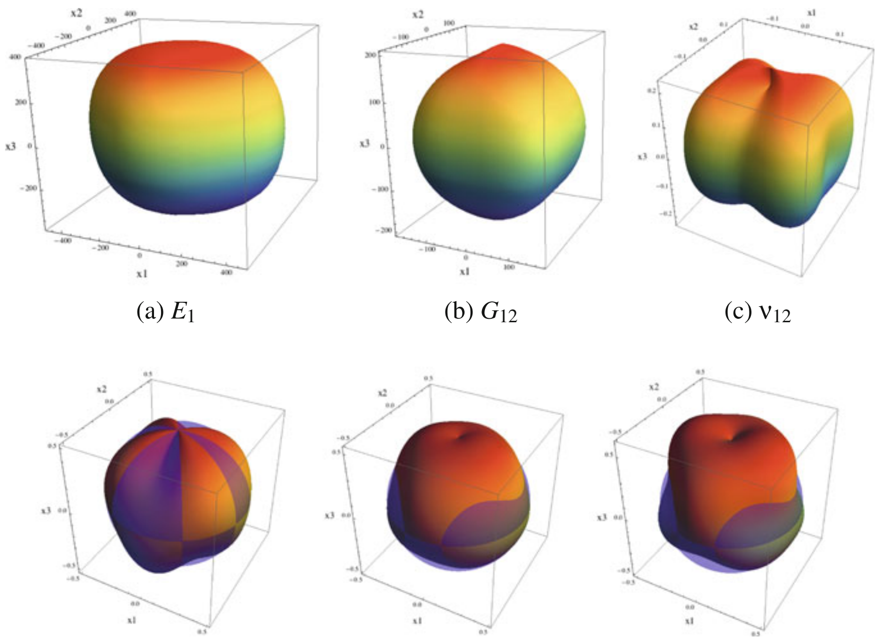


Fig. 2.4 Directional 3D-graphics of Anorthite

$$[C] = \begin{bmatrix} 124 & 66 & 50 & 0 & -26.9 & 0 \\ & 205 & 42 & 0 & -9.9 & 0 \\ & & 156 & 0 & -25.4 & 0 \\ & & & 48 & 0 & -2 \\ & sym & & & 80 & 0 \\ & & & & & 84 \end{bmatrix}$$

- Perovskite ($CaTiO_3$)
  Crystal syngony: Orthorhombic, $N = 9$.
  Source: Evans and Grove (2004)

$$[C] = \begin{bmatrix} 515 & 117 & 117 & 0 & 0 & 0 \\ & 525 & 139 & 0 & 0 & 0 \\ & & 435 & 0 & 0 & 0 \\ & & & 48 & 0 & 0 \\ & sym & & & 404 & 0 \\ & & & & & 350 \end{bmatrix}$$



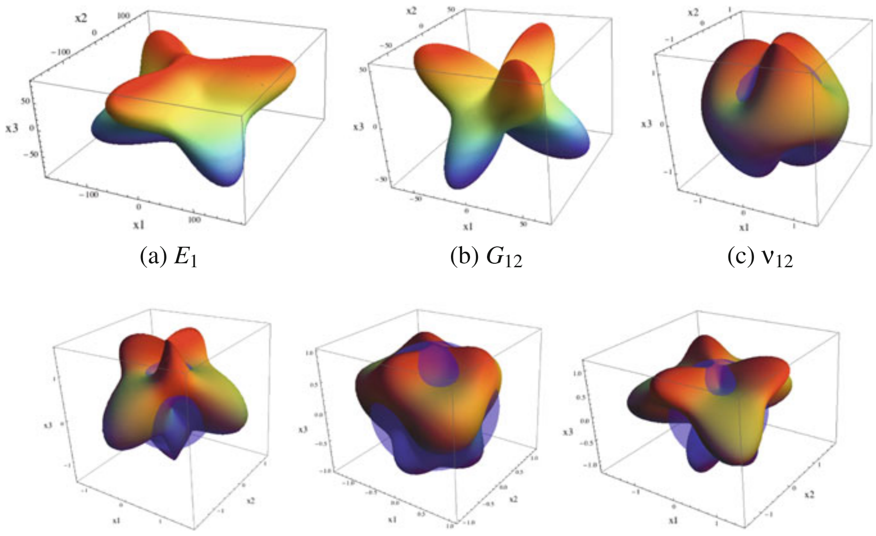
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.5 Directional 3D-graphics of Perovskite

- Dolomite ($CaMg(CO_3)_2$) (*estimated)
  Crystal syngony: Trigonal, $N = 7$.
  Source: Bakri and Zaoui (2011)

$$[C] = \begin{bmatrix} 196.6 & 64.4 & 54.7 & 31.7 & 25.3^* & 0 \\ & 196.6 & 54.7 & -31.7 & -25.3^* & 0 \\ & & 110 & 0 & 0 & 0 \\ & & & 83.2 & 0 & -35.84 \\ & sym & & & 83.2 & 44.8 \\ & & & & & 132.2 \end{bmatrix}$$



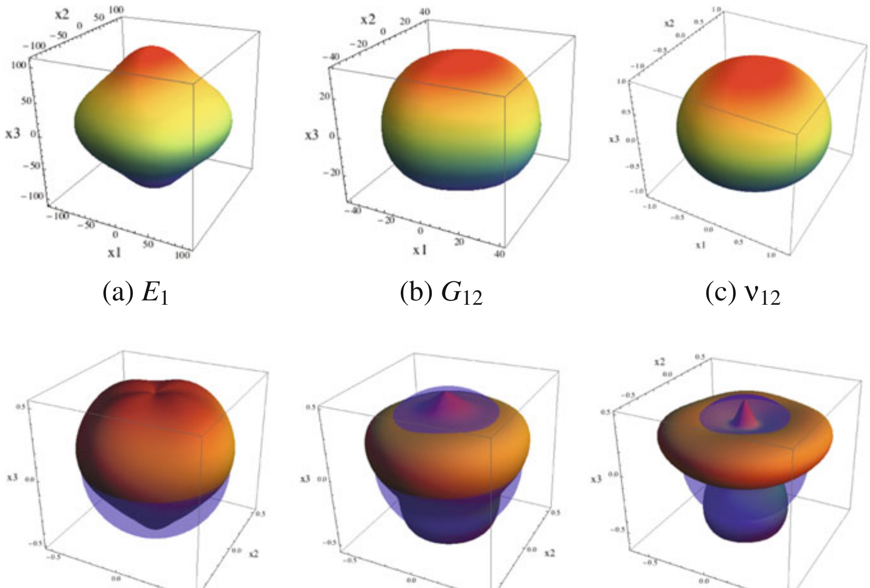
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.6 Directional 3D-graphics of Dolomite

- Calcium Tungstate ($CaWO_4$)
  Crystal syngony: Tetragonal, $N = 7$.
  Source: Landolt and Börnstein (1992)

$$
[C] = \begin{bmatrix}
141 & 61 & 41 & 0 & 0 & 1.9 \\
 & 141 & 41 & 0 & 0 & -1.9 \\
 & & 125 & 0 & 0 & 0 \\
 & & & 67.4 & 0 & 0 \\
 & sym & & & 67.4 & 0 \\
 & & & & & 81.4
\end{bmatrix}
$$



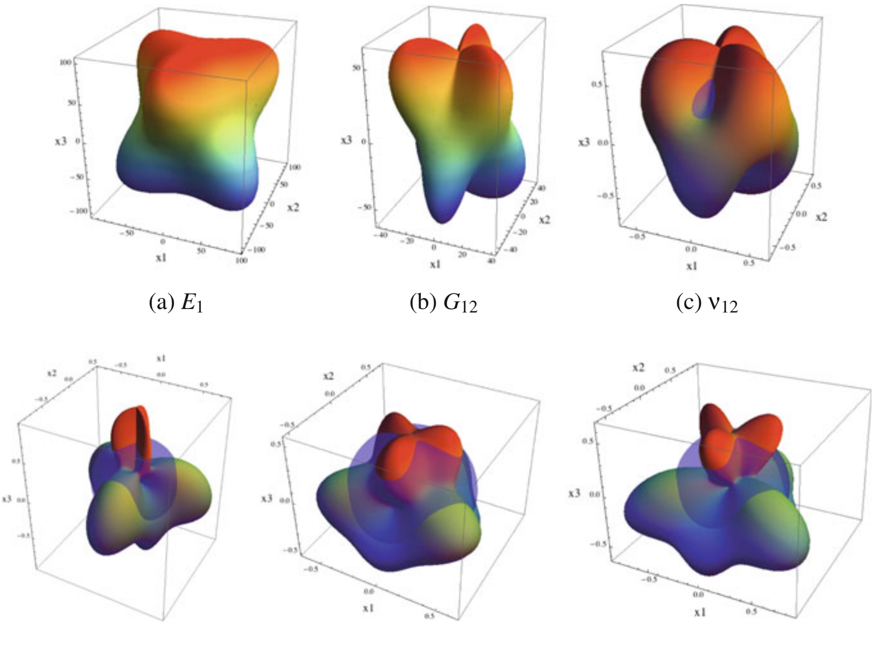
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.7 Directional 3D-graphics of Calcium Tungstate

- Quartz (SiO_2)
  Crystal syngony: Trigonal, $N = 6$.
  Source: Landolt and Börnstein (1992)

$$[C] = \begin{bmatrix} 86.8 & 7.1 & 14.4 & 24.3 & 0 & 0 \\ & 86.8 & 14.4 & -24.3 & 0 & 0 \\ & & 107.5 & 0 & 0 & 0 \\ & & & 116.4 & 0 & 0 \\ & sym & & & 116.4 & 34.4 \\ & & & & & 79.7 \end{bmatrix}$$



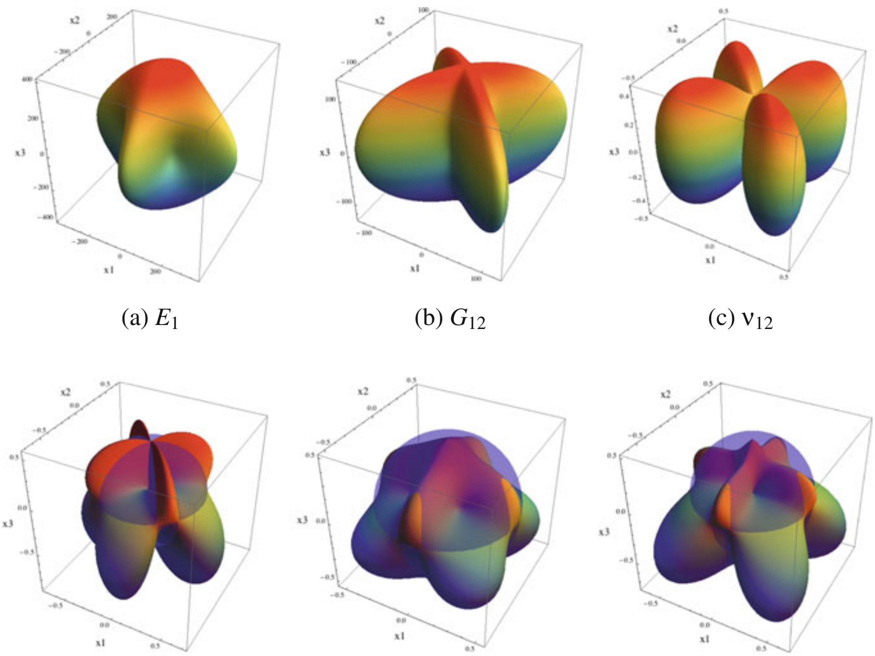
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.8 Directional 3D-graphics of Quartz

- Zircon ($ZrSiO_4$)
  Crystal syngony: Tetragonal, $N = 6$.
  Source: Evans and Grove (2004)

$$[C] = \begin{bmatrix} 424 & 70 & 149 & 0 & 0 & 0 \\ & 424 & 149 & 0 & 0 & 0 \\ & & 489 & 0 & 0 & 0 \\ & & & 262 & 0 & 0 \\ & sym & & & 262 & 0 \\ & & & & & 96 \end{bmatrix}$$



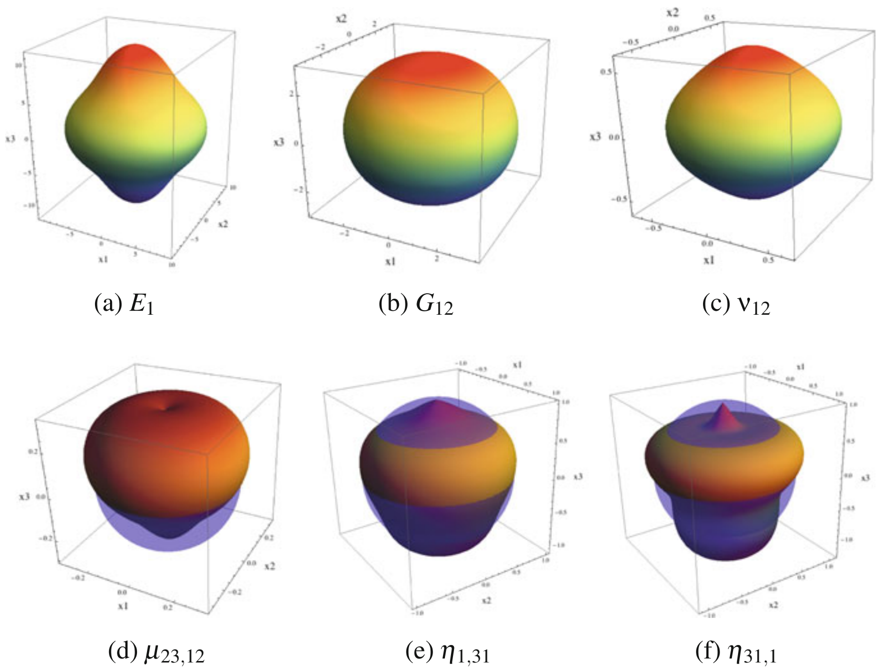
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.9 Directional 3D-graphics of Zircon

- Ice (H_2O)
  Crystal syngony: Hexagonal, $N = 5$.
  Source: Evans and Grove (2004)

$$[C] = \begin{bmatrix} 13.5 & 6.5 & 6 & 0 & 0 & 0 \\ & 13.5 & 6 & 0 & 0 & 0 \\ & & 15 & 0 & 0 & 0 \\ & & & 6 & 0 & 0 \\ & sym & & & 6 & 0 \\ & & & & & 7 \end{bmatrix}$$



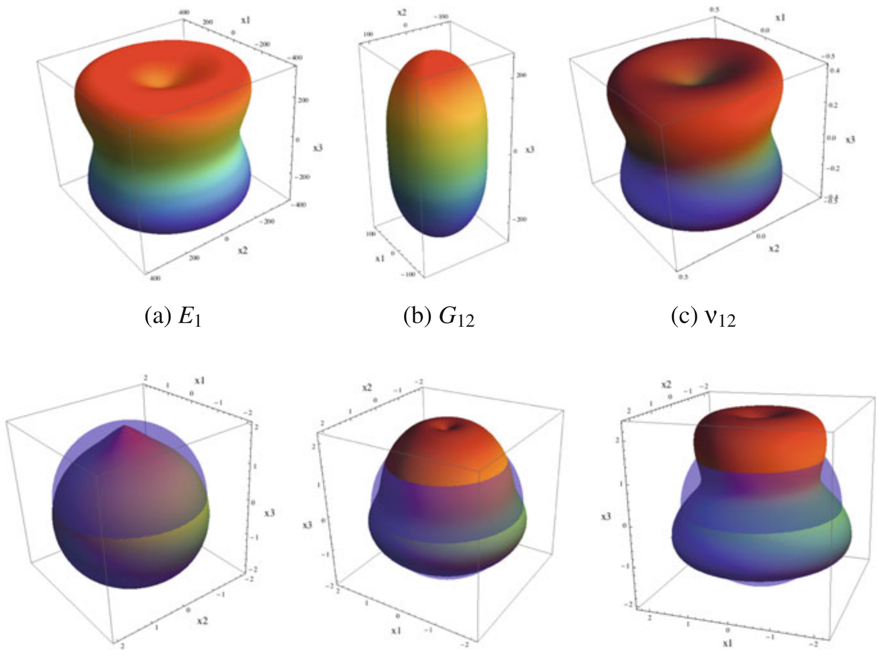
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.10 Directional 3D-graphics of Ice

- Titanium Boride (TiB_2)
  Crystal syngony: Hexagonal, $N = 5$.
  Source: Landolt and Börnstein (1992)

$$[C] = \begin{bmatrix} 648.3 & 404.2 & 317.7 & 0 & 0 & 0 \\ & 648.3 & 317.7 & 0 & 0 & 0 \\ & & 439.3 & 0 & 0 & 0 \\ & & & 500 & 0 & 0 \\ & sym & & & 500 & 0 \\ & & & & & 244.1 \end{bmatrix}$$



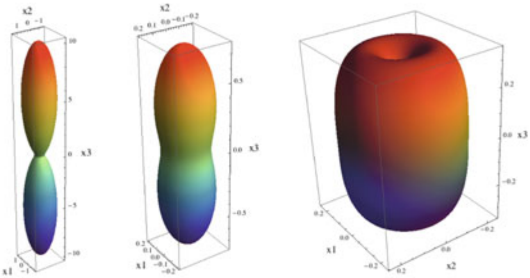
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

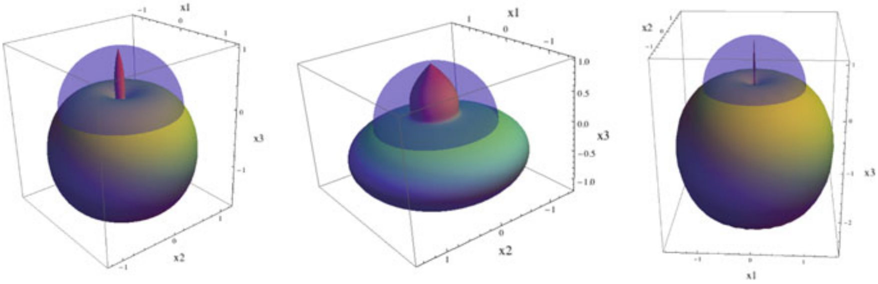
Fig. 2.11 Directional 3D-graphics of Titanium Boride

- Pine Wood
  Transversely isotropic, $N = 5$.
  Source: Lekhnitskii (1950)

$$[C] = \begin{bmatrix} 0.45 & 0.11 & 0.13 & 0 & 0 & 0 \\ & 0.45 & 0.13 & 0 & 0 & 0 \\ & & 10.1 & 0 & 0 & 0 \\ & & & 1.5 & 0 & 0 \\ & sym & & & 1.5 & 0 \\ & & & & & 0.34 \end{bmatrix}$$



(a) E_1 (b) G_{12} (c) ν_{12}

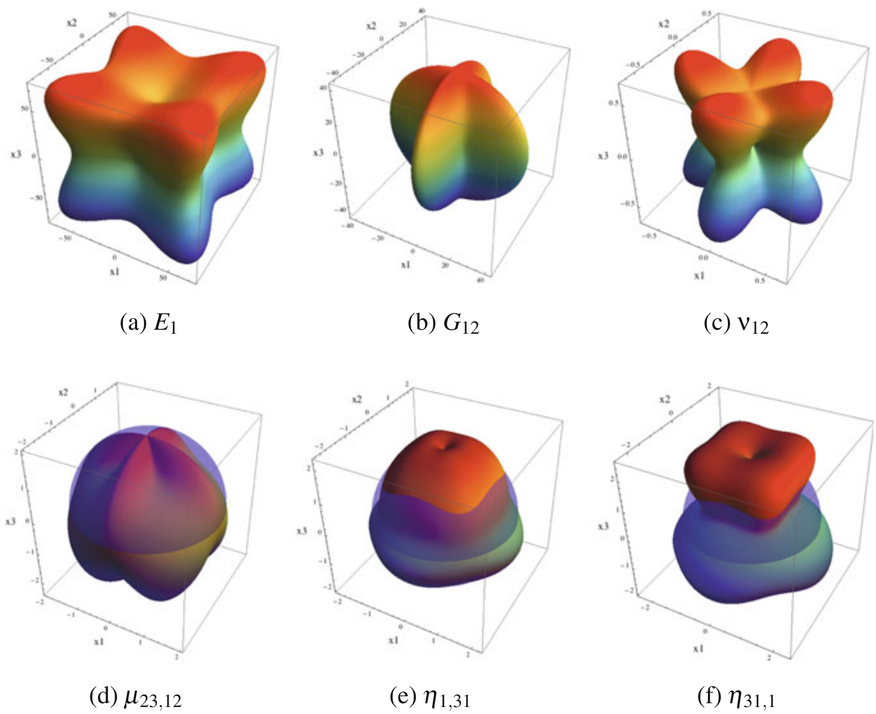


(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.12 Directional 3D-graphics of Pine Wood

- Gold (Au)
  Crystal syngony: Cubic, $N = 3$.
  Source: Evans and Grove (2004)

$$[C] = \begin{bmatrix} 191 & 162 & 162 & 0 & 0 & 0 \\ & 191 & 162 & 0 & 0 & 0 \\ & & 191 & 0 & 0 & 0 \\ & & & 84 & 0 & 0 \\ & sym & & & 84 & 0 \\ & & & & & 84 \end{bmatrix}$$



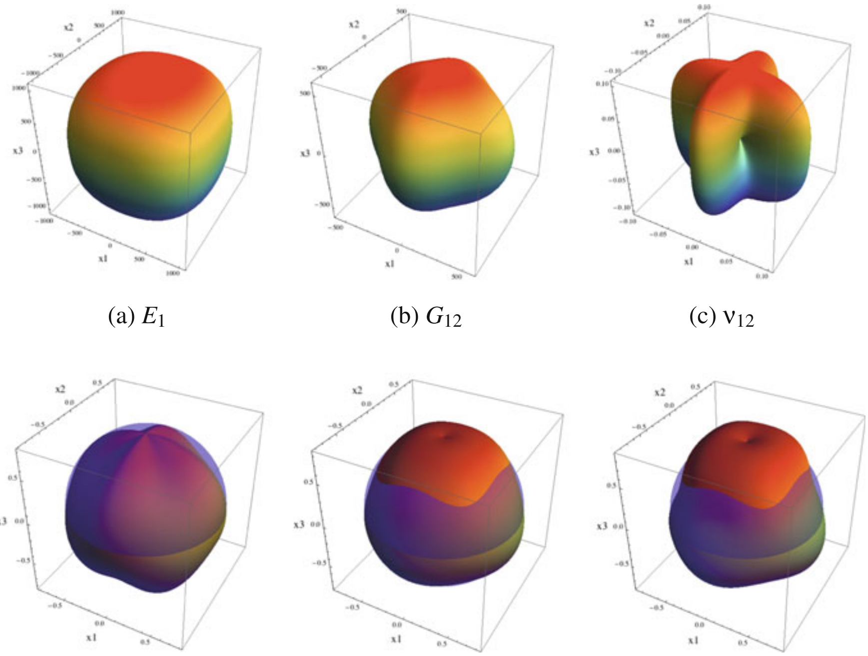
(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.13 Directional 3D-graphics of Gold

- Diamond (C)
  Crystal syngony: Cubic, $N = 3$.
  Source: Evans and Grove (2004)

$$
[C] = \begin{bmatrix}
1079 & 124 & 124 & 0 & 0 & 0 \\
 & 1079 & 124 & 0 & 0 & 0 \\
 & & 1079 & 0 & 0 & 0 \\
 & & & 1156 & 0 & 0 \\
 & sym & & & 1156 & 0 \\
 & & & & & 1156
\end{bmatrix}
$$



(a) E_1 (b) G_{12} (c) ν_{12}

(d) $\mu_{23,12}$ (e) $\eta_{1,31}$ (f) $\eta_{31,1}$

Fig. 2.14 Directional 3D-graphics of Diamond

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http://www.springer.com/978-981-10-5438-9

Anisotropic Elasticity
Vannucci, P.
2018, XVII, 426 p. 126 illus., 38 illus. in color.,
Hardcover
ISBN: 978-981-10-5438-9