Inverse Theory

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& the UBC-GIF team

https://courses.geosci.xyz/lapis2019

Thanks!

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Collaborators

Seogi

Lindsey

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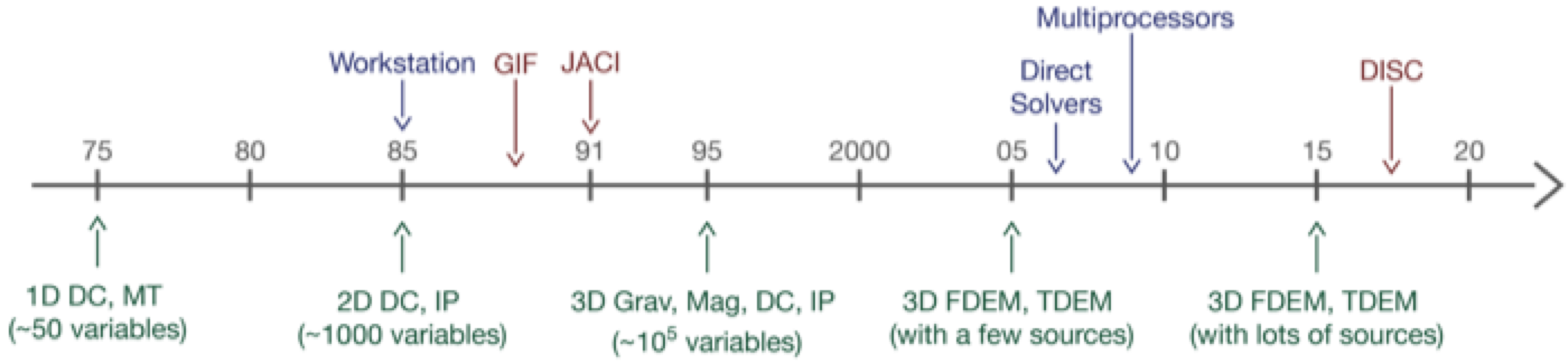
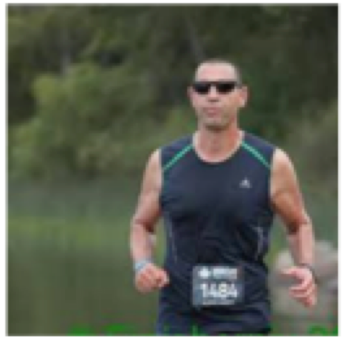
Some backgrounds

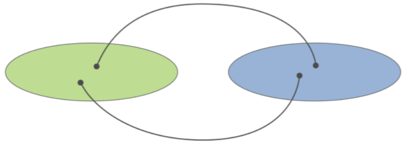
- Doug inspired by Bob Parker, Freeman Gilbert and George Backus: The Geophysical Inverse Problem





Result: Computing power + advances in inversion methodology
→ we can now solve most EM geophysics problems

Inverse Theory: Overview



Outline

- Problems where inversion is needed
- Define some terms
- Linear problem
- Understand origin of non-uniqueness
- Ill-conditioning
- Types of constraint information
- Topics for course

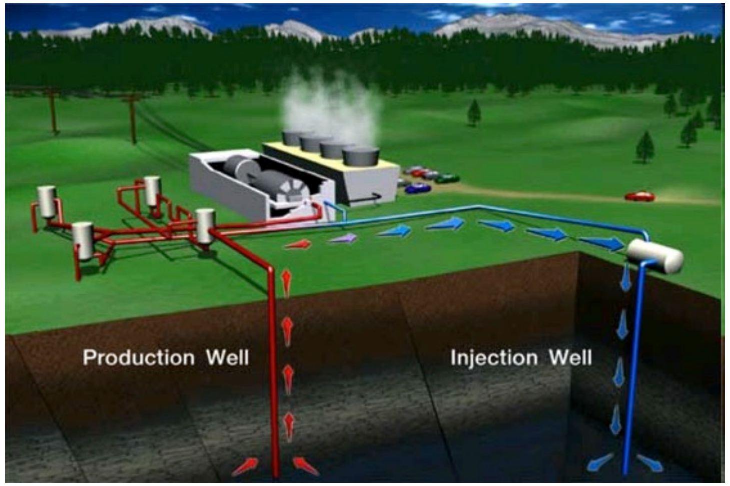
some problems of relevance …

Finding resources

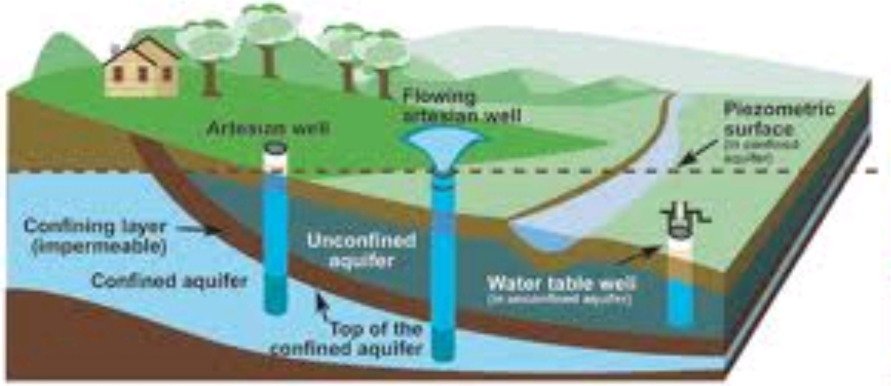
Hydrocarbons

Minerals

Geothermal Energy



Ground Water



Natural hazards

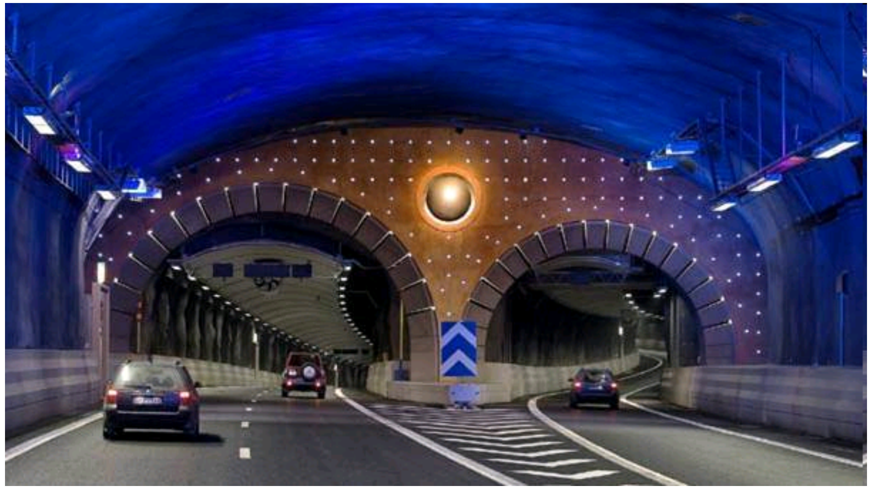
Volcanoes

Tsunami

Earthquakes

Geotechnical engineering

Tunnels



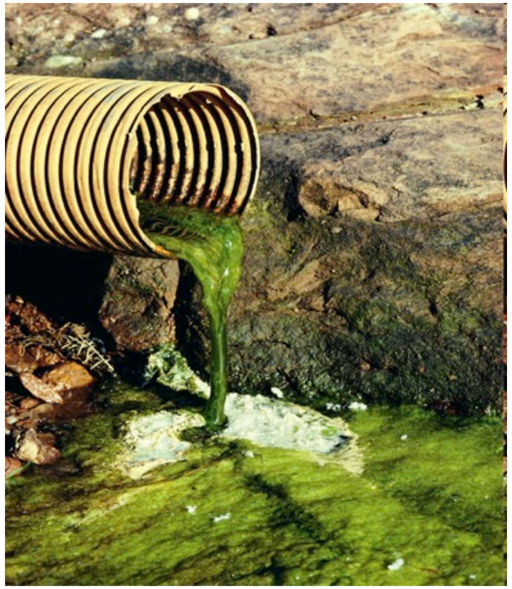
Slope stability

In-mine safety

Environmental

Salt water intrusion

Water contamination

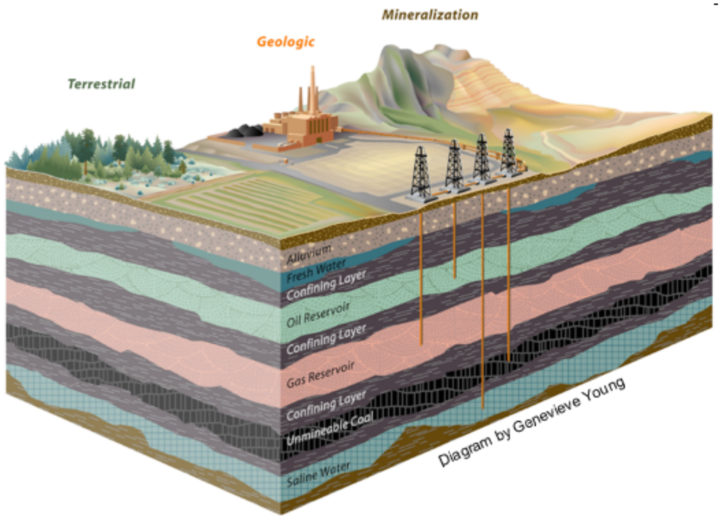


Unexploded Ordnance (UXO)

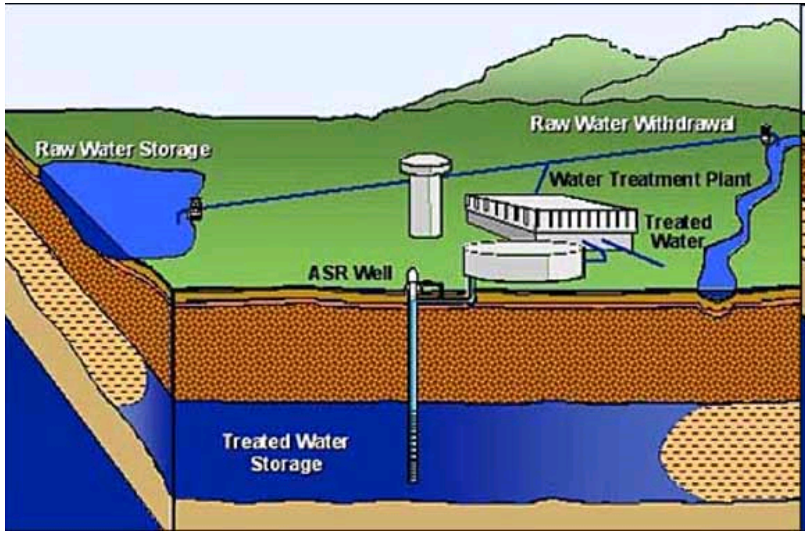
http://www.centennialofflight.gov

Surface or underground storage

CO2 sequestration



Aquifer Storage and Recover

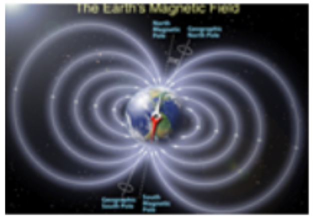


Industrial Waste Disposal

What do these problems have in common?

All require ways to see into the earth without direct sampling.

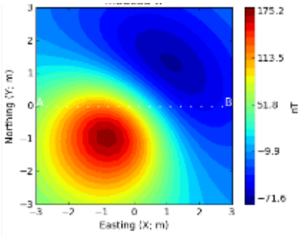
Source



Input energy

measure

Data

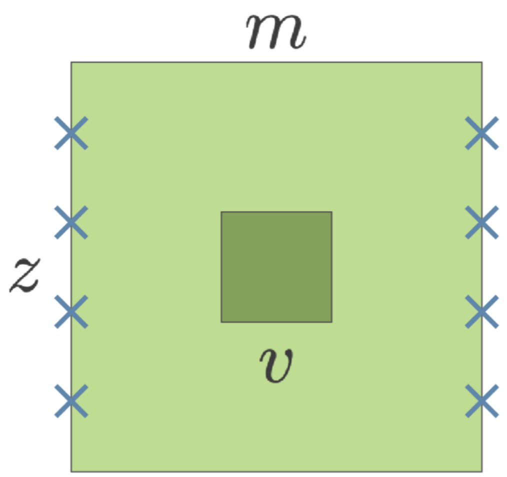


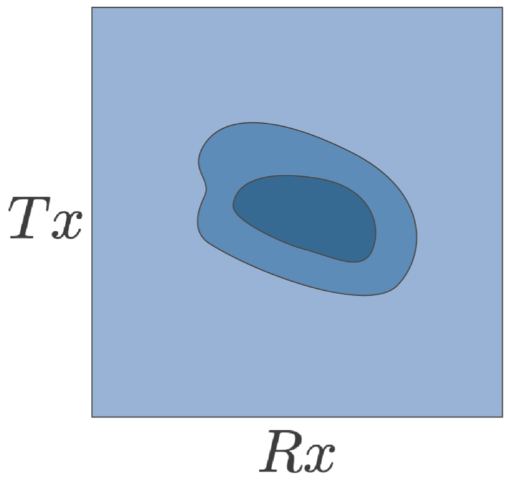
Physical Properties

σ, μ, ε

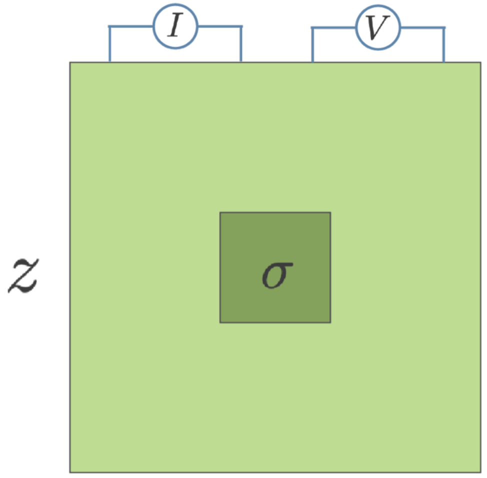
Some experiments and data

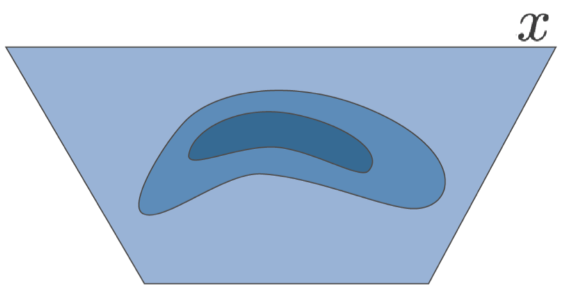
Cross-well tomography





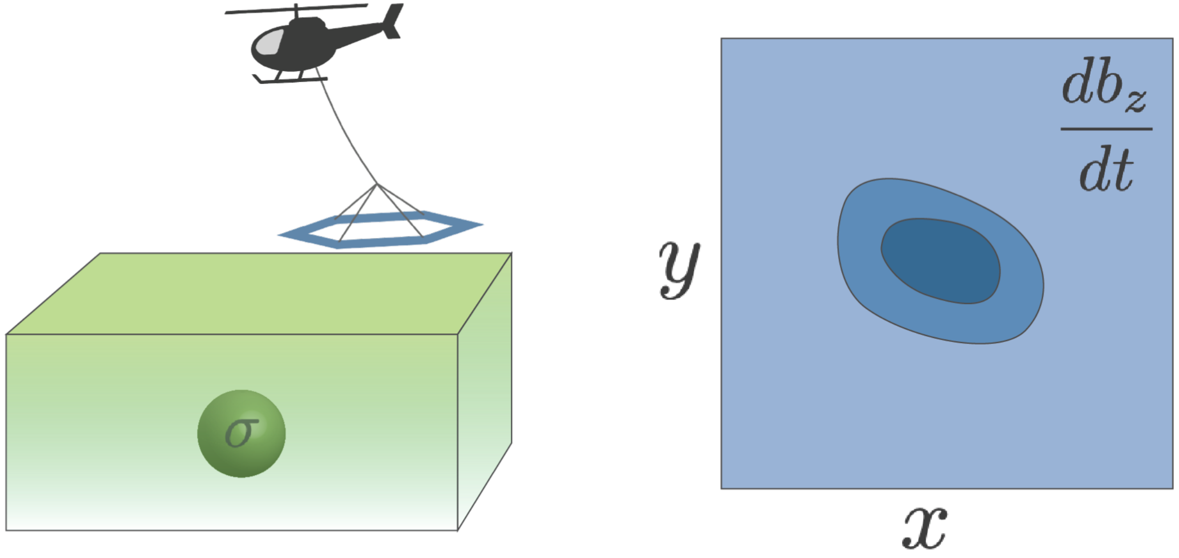
Direct Current (DC) Resistivity



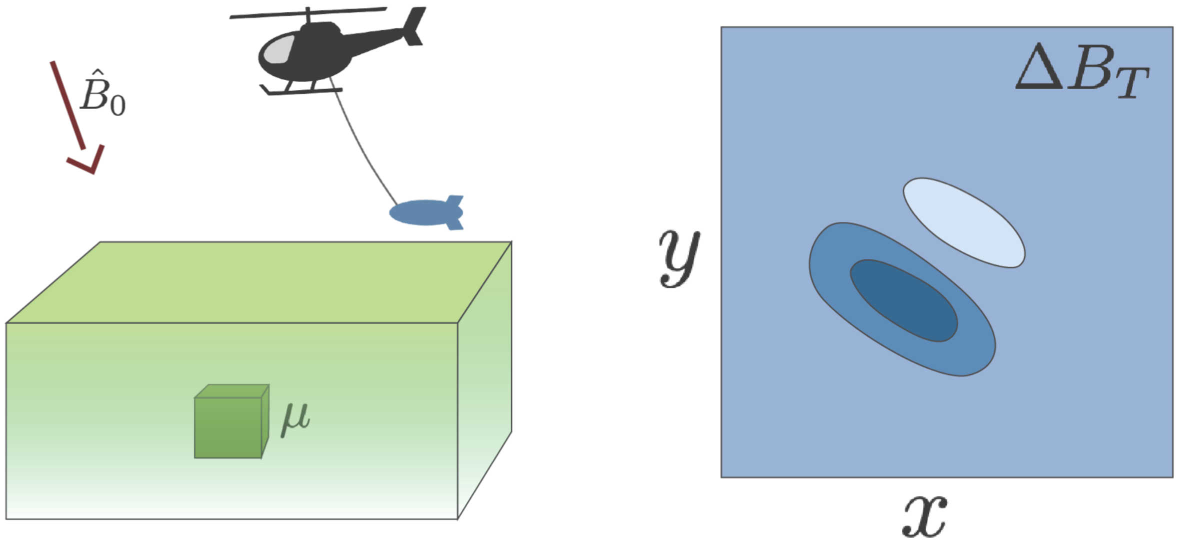


Some experiments and data

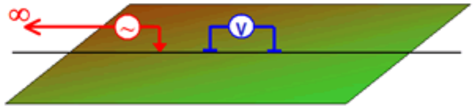
Airborne Time Domain EM

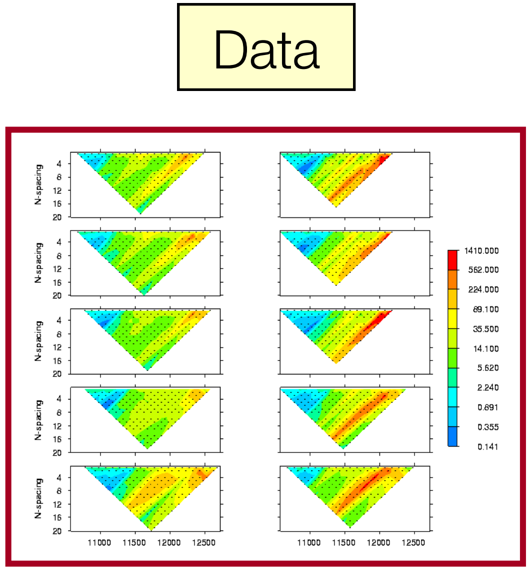


Magnetics

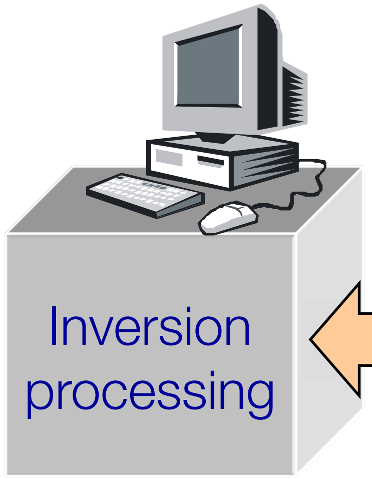


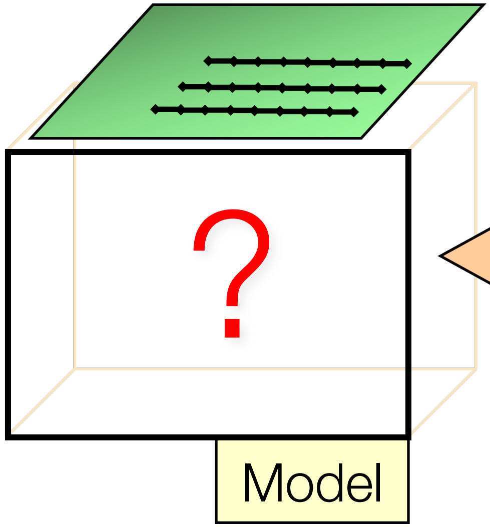
Inversion





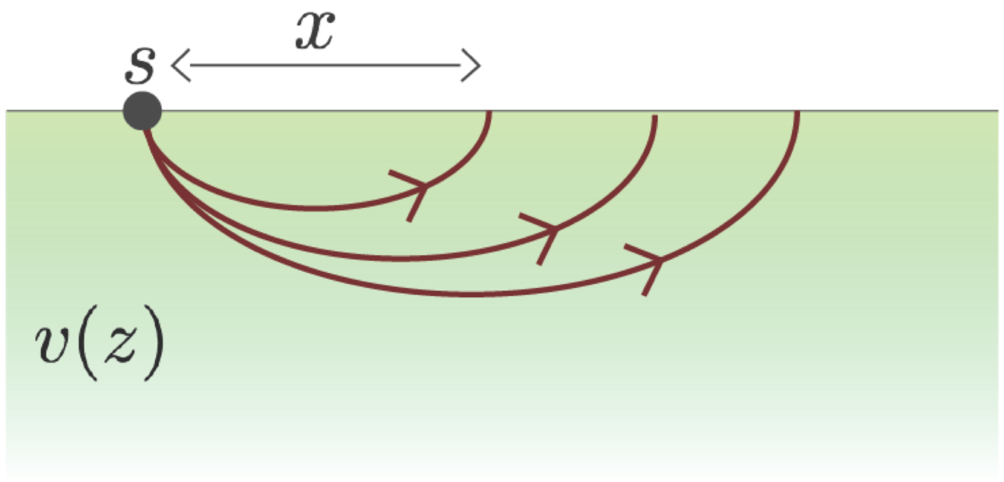
Measurements over the Earth are data.

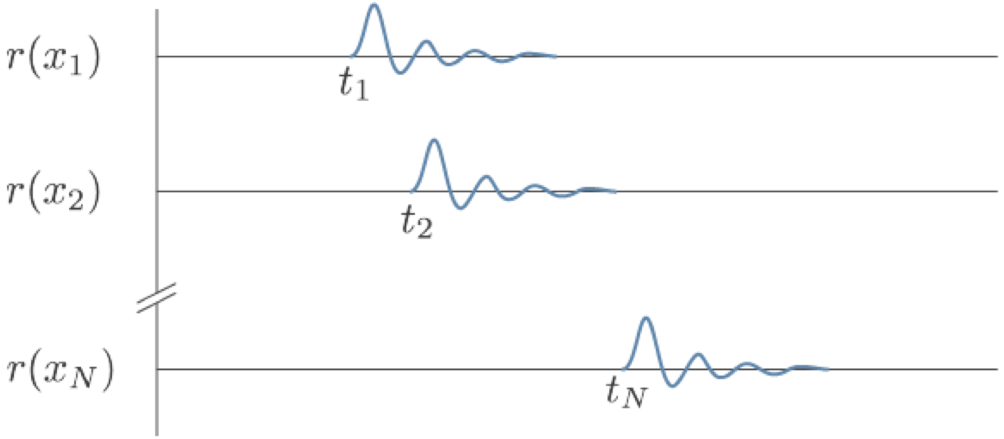


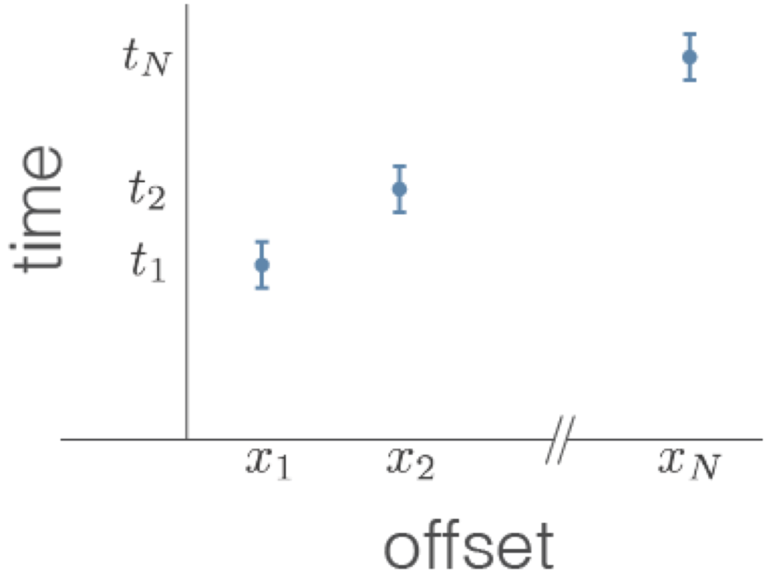


Inversion estimates Earth models based upon data and prior knowledge.

Seismic refraction





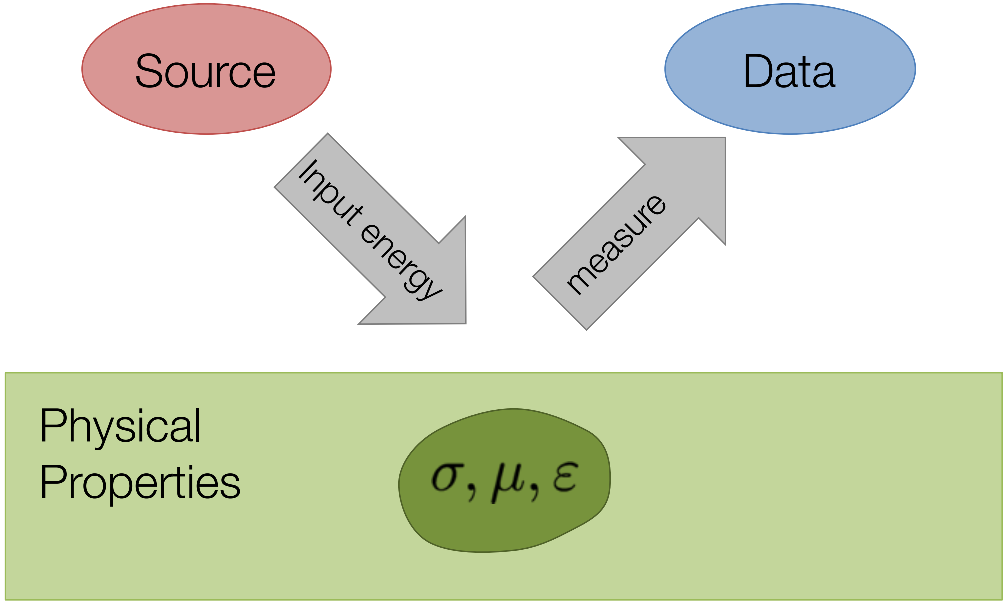


Inverse problem

- Given data $t_j (j = 1, N)$
- Estimates of uncertainties (data errors)
- Ability to forward model

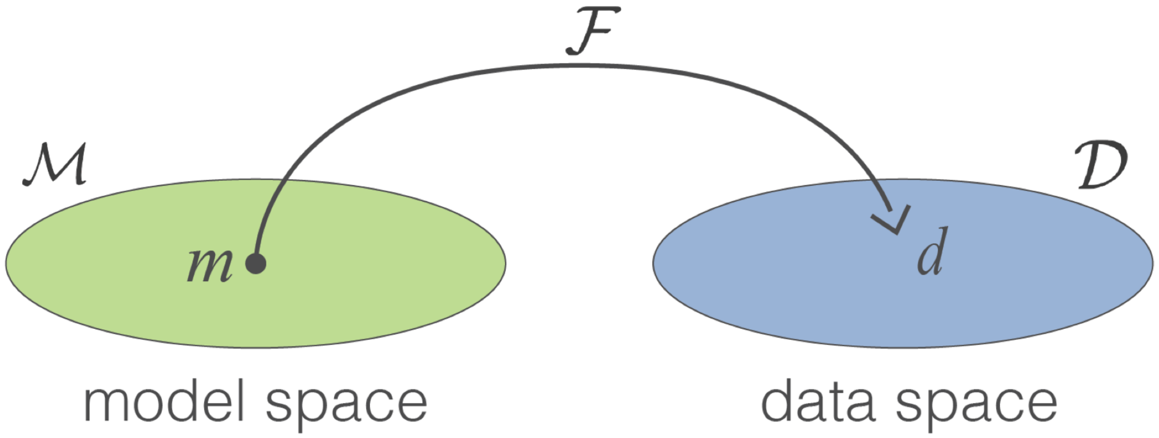
What is the model / velocity structure, $v(z)$ that produce d the data?

Forward problem: simulate data



- Earth model: physical properties
- Experiment: transmitters, receivers, geometry
- Physics: how energy propagates

Convenient notation



$\mathcal{M}$: model space

m : any element in $\mathcal{M}$

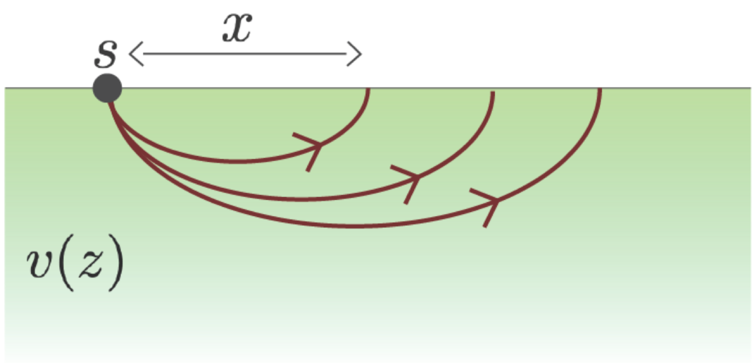
$\mathcal{D}$: model space

d : any element in $\mathcal{D}$

$\mathcal{F}$: forward mapping, $\mathcal{F}[m] = d$

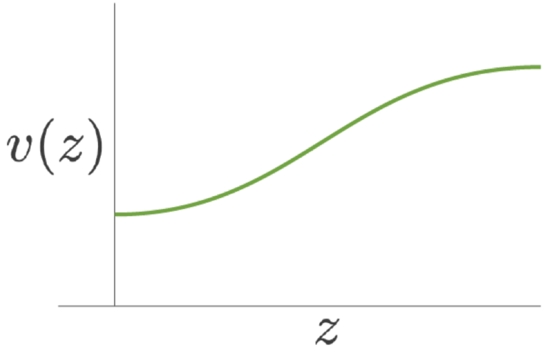
Model space

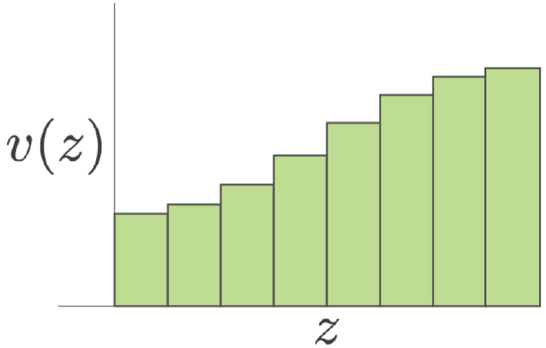
Seismic



model: function of $v(z)$

or discretize $\mathbf{v} = (v_1, ..., v_M)$

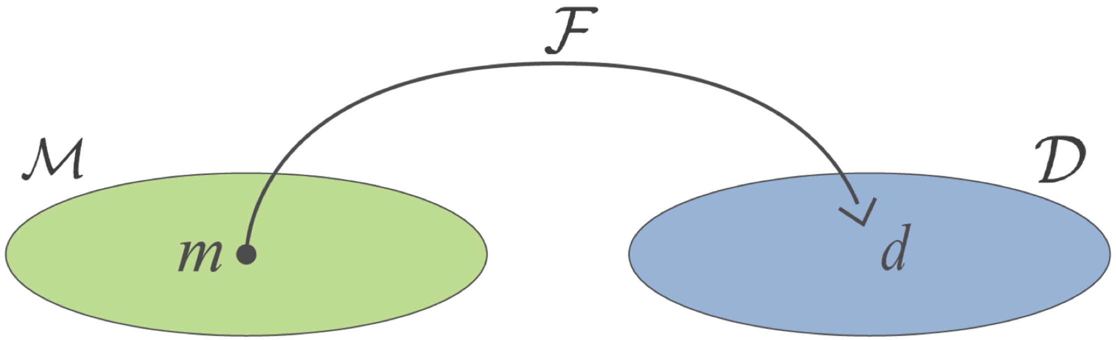




Model space is a vector space with

- rules of inner products (angles between vector elements)
- norms to measure length

Data space



Elements of data space:

$$\mathbf{d} = (d_1, \ldots, d_N) \qquad \in \mathbb{R}^N$$

Each datum:

$$d_j = \mathcal{F}_j[m]$$

$\mathcal{F}_j[\cdot]$: forward mapping

Depends upon:

- Physics of experiment
- Geometry
- Details about source and receiver

Two types of mapping: linear and non-linear

Linear mapping: If f and g are two elements of model space, α and β are constants

$$\mathcal{F}[\alpha f + \beta g] = \alpha \mathcal{F}[f] + \beta \mathcal{F}[g]$$

If it does not hold then $\mathcal{F}$ is **non-linear** mapping

Examples of linear functionals:

$$d_j = \int_a^b g_j(x) m(x) dx$$

$$\begin{cases} d_j: & j\text{-th data} \\ g_j: & \text{kernel} \\ m: & \text{model} \end{cases}$$

Biot-Savart: $\vec{B}(\vec{r_j}) = \dfrac{\mu_0}{4\pi} \displaystyle\int_v \dfrac{\vec{J}(\vec{r}\,') \times \hat{r}}{|\vec{r_j} - \vec{r}\,'|^2} dv$ $\quad m = \vec{J}$

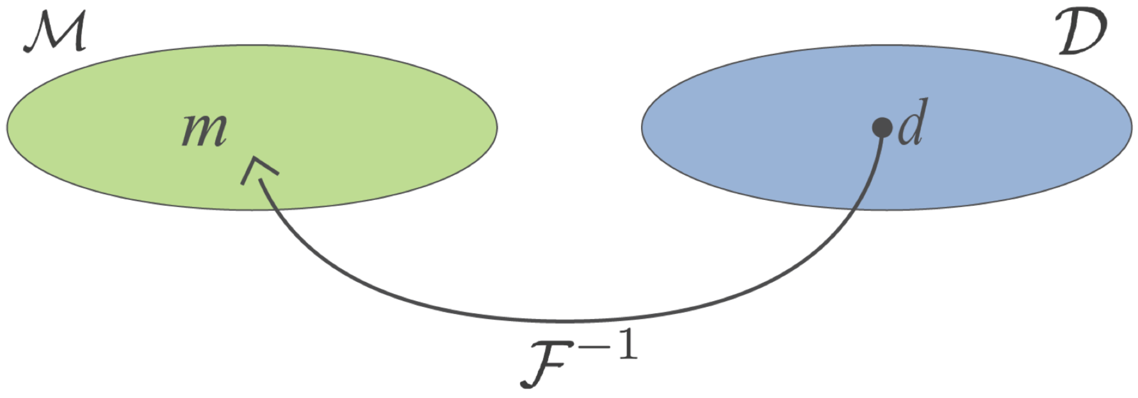
Convolution: $y(t_j) = \displaystyle\int_{-\infty}^{\infty} h(\tau) w(t_j - \tau) d\tau$ $\quad m = h$

RMS-to-interval velocity: $V_{rms}^2(t_j) = \dfrac{1}{t_j} \displaystyle\int_0^{t_{max}} v_{int}^2(t) H(t - t_j) dt$ $\quad m = v_{int}^2$

So, two models: $m_1(x), \quad m_2(x)$

Linear	$d_j^{(2)} = \int_a^b g_j(x) m_2(x) dx$ $d_j = \int_a^b g_j(x) \Big[\alpha m_1(x) + \beta m_1(x)\Big] dx$ $d_j = \alpha d_j^{(2)} + \beta d_j^{(2)}$	$d_j^{(1)} = \int_a^b g_j(x) m_1(x) dx$
Non-linear	$d_j = \int_a^b g_j(x) m^3(x) dx$ $m(x) \quad \rightarrow \quad 2m(x)$ $d_j \quad \rightarrow \quad \delta d_j$	Most PDE's are non-linear (e.g. DC resistivity) $\nabla \cdot \sigma \nabla V = -\delta(\vec{r} - \vec{r}\,')$ $\mathcal{F}[\sigma] = V$ $\mathcal{F}[2\sigma] \neq 2V$

Inverse Problem



$$\mathcal{F}^{-1} : \mathcal{D} \rightarrow \mathcal{M}$$

- Find the model that give rise to the data
- For the RMS problem

$$V_{rms}^2(t_j) = \frac{1}{t_j} \int_0^{t_{max}} v_{int}^2(t) H(t - t_j) dt$$

- Forward problem: Given v_{int} compute V_{rms}
- Inverse problem: Given V_{rms} find v_{int}

Questions:

- Does a solution exist?
- How to construct a solution?
- Is it unique?
- How to handle non-uniqueness

Answers depend upon available data:

- infinite amount of accurate data
- finite amount of accurate
- finite amount of inaccurate data

Inversion with perfect data

Forward problem:

$$V_{rms}^2(t) = \frac{1}{t}\int_0^t v_{int}^2(u)du$$

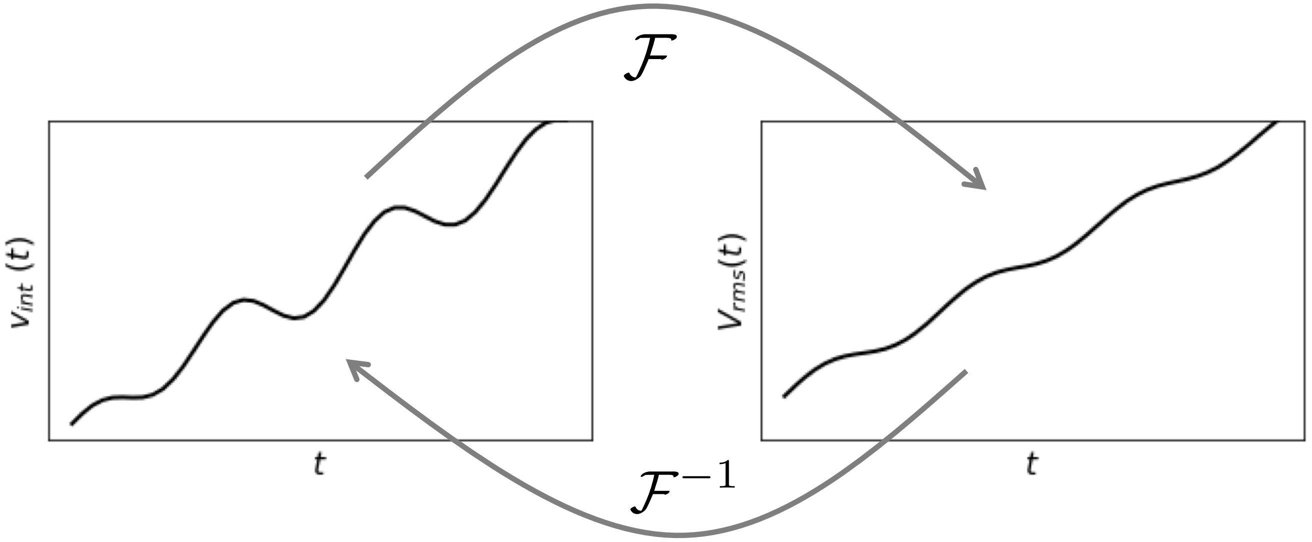
V_{rms}: rms velocity
v_{int}: interval velocity

Analytic inverse:

$$v_{int} = V_{rms}\left(1 + \frac{2tV'_{rms}(t)}{V_{rms}}\right)^{1/2}$$

Term $tV_{rms}{}'$ magnifies small changes in V_{rms} at late times

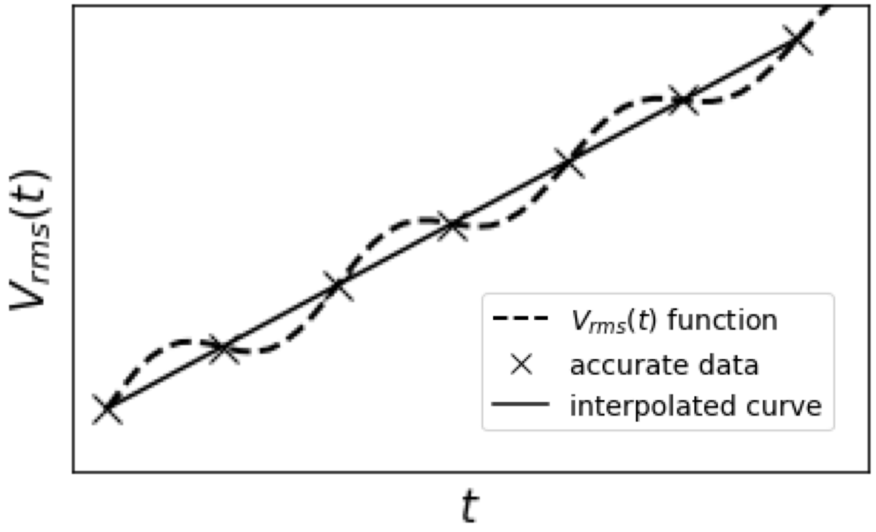
→ large changes in v_{int}

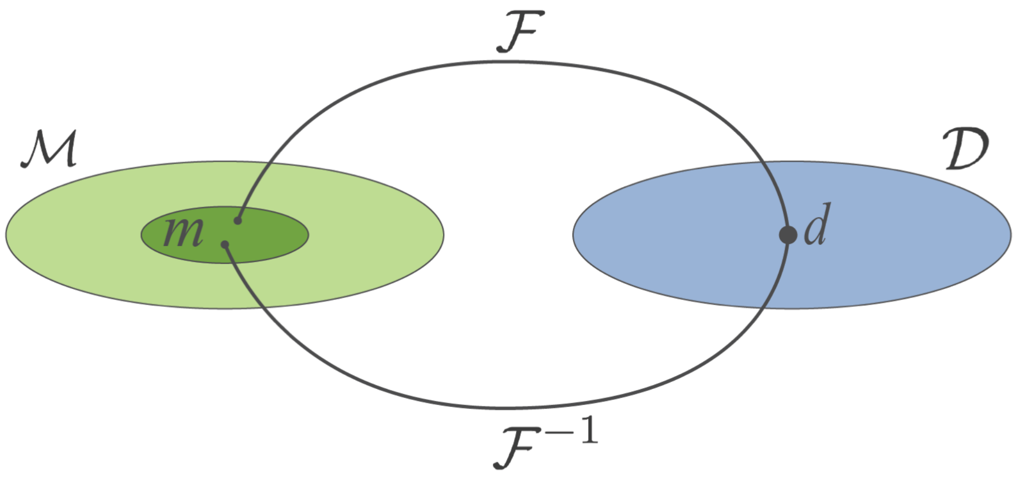


What is learned

- V_{rms} known exactly → v_{int} recovered uniquely
- Forward problem "smooths"
- Inverse problem must "roughen"

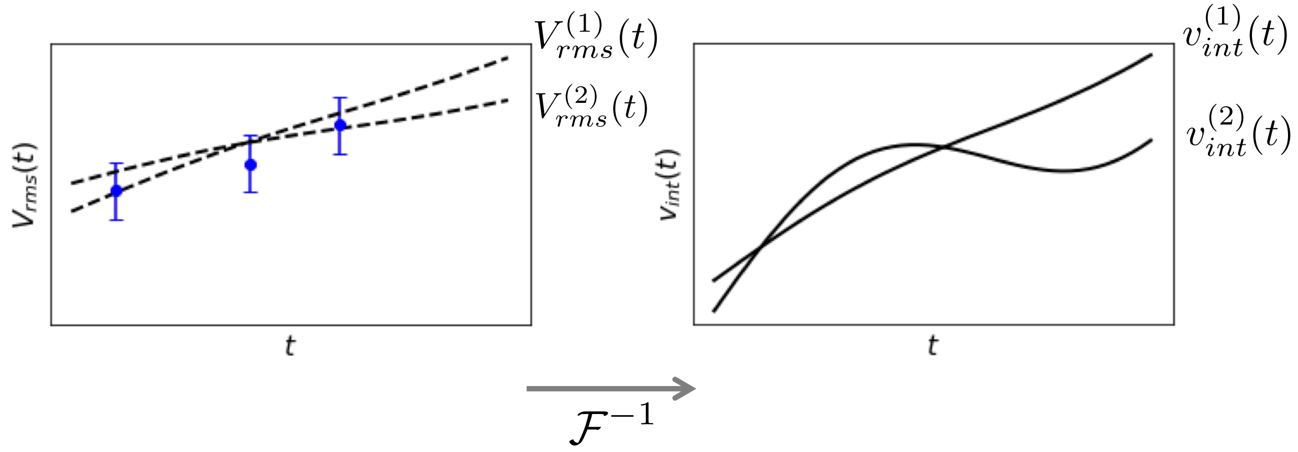
Inversion with finite number of **accurate** data

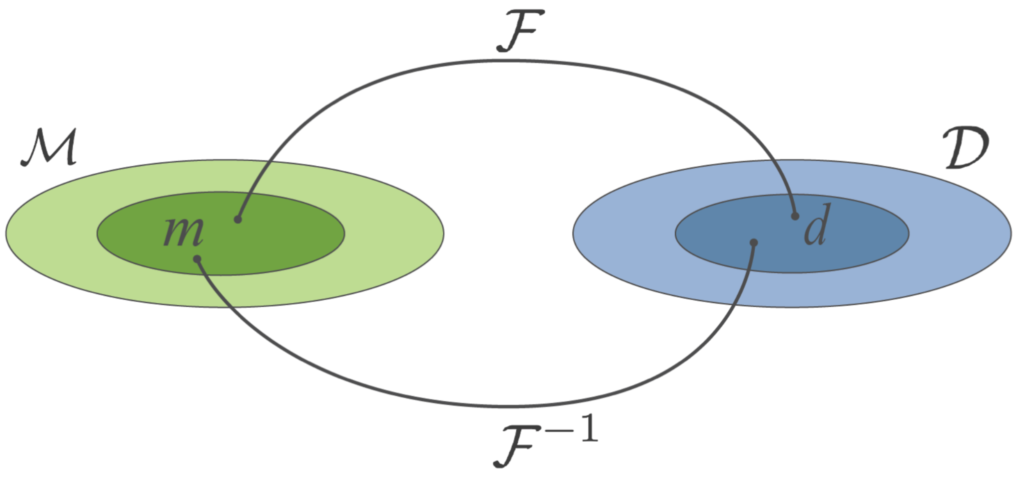




- Interpolate to obtain $V_{rms}(t)$, then use analytic solution:
- Each interpolation → different $v_{int}(t)$
- **Non-unique**, even with **accurate data**

Inversion with finite number of **inaccurate** data

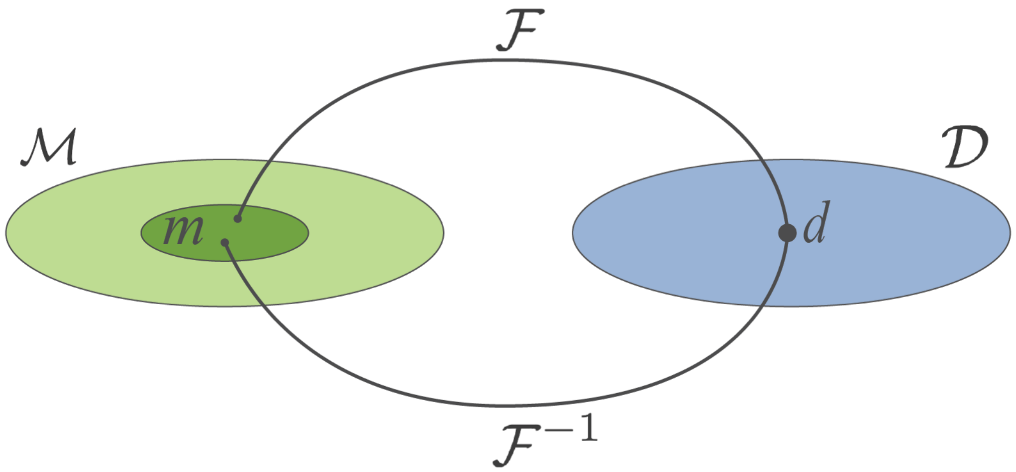




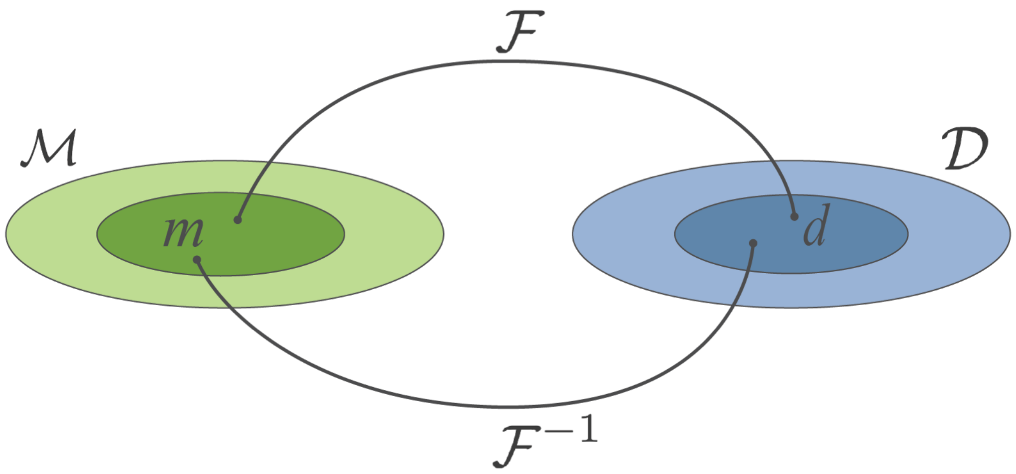
- Increases non-uniqueness
- Infinite number of ways to interpolate
- …

Visualizing non-uniqueness

accurate data



inaccurate data



Ill-conditioning

Let $$d^{obs} = d^{true} + \delta d$$

$$\mathcal{F}^{-1}[d^{true}] = m_c \quad \text{(constructed model)}$$

$$\mathcal{F}^{-1}[d^{obs}] = \mathcal{F}^{-1}[d^{true} + \delta d] = m_c + \delta m$$

Suppose $\|\delta d\|$ is small

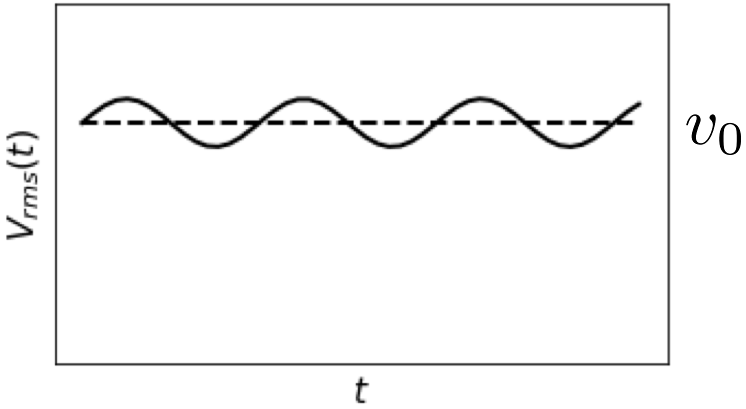
Does that mean $\|\delta m\|$ is small?

Return to RMS problem

Suppose

$$v_{int}(t) = v_0 \quad \text{(constant)}$$

$$V_{rms}(t) = v_0$$



Perturbation

$$V_{rms}(t) = v_0 + asin(\omega_0 t)$$

$$V_{rms}\,'(t) = a\omega_0 cos(\omega_0 t)$$

$$\begin{cases} a\text{: constant} \\ \omega_0\text{: arbitrary value} \end{cases}$$

Analytic solution

$$v_{int} = V_{rms}\left(1 + \frac{2tV'_{rms}(t)}{V_{rms}}\right)^{1/2}$$

$$\delta v_{int} \propto \frac{ta\omega_0}{v_0}$$

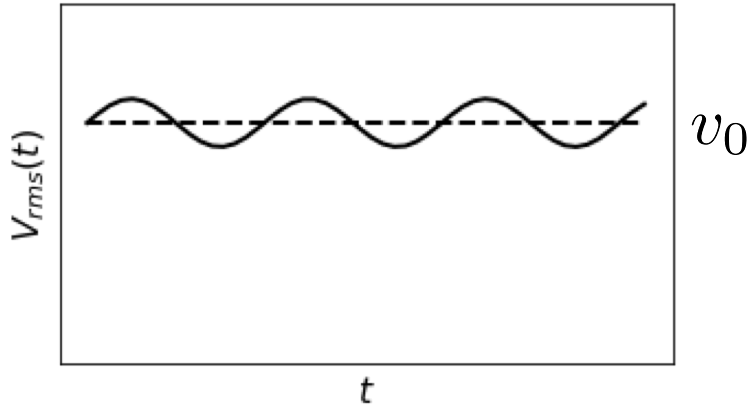
Can be arbitrarily large even if ‘a’ is small

Return to RMS problem

Suppose

$$v_{int}(t) = v_0 \quad \text{(constant)}$$

$$V_{rms}(t) = v_0$$



Perturbation

$$V_{rms}(t) = v_0 + asin(\omega_0 t)$$

$$V_{rms}\,'(t) = a\omega_0 cos(\omega_0 t)$$

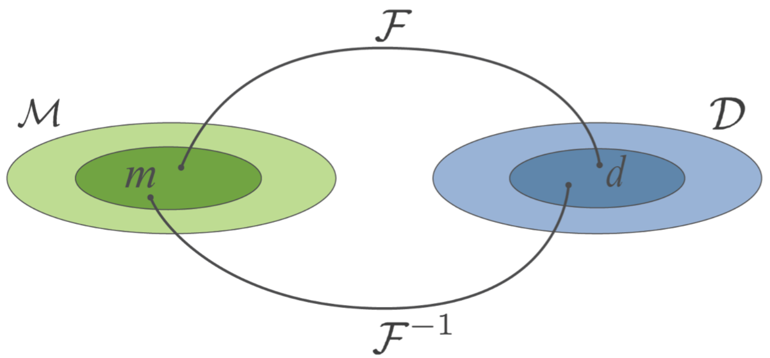
$$\begin{cases} a\text{: constant} \\ \omega_0\text{: arbitrary value} \end{cases}$$

Small changes in the data can result in large changes in the recovered model
Inverse problem is ill-conditioned

$$\delta v_{int} \propto \frac{ta\omega_0}{v_0}$$

Can be arbitrarily large even if ‘a’ is small

Summary for inverse problem



- Non-unique
- Ill-conditioned

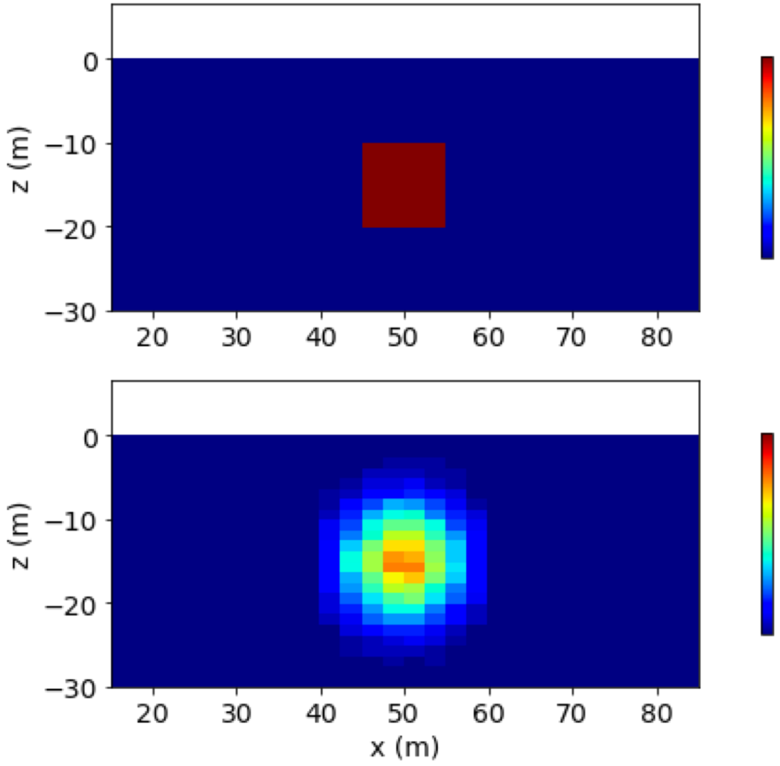
The Inverse Problem is ill-posed

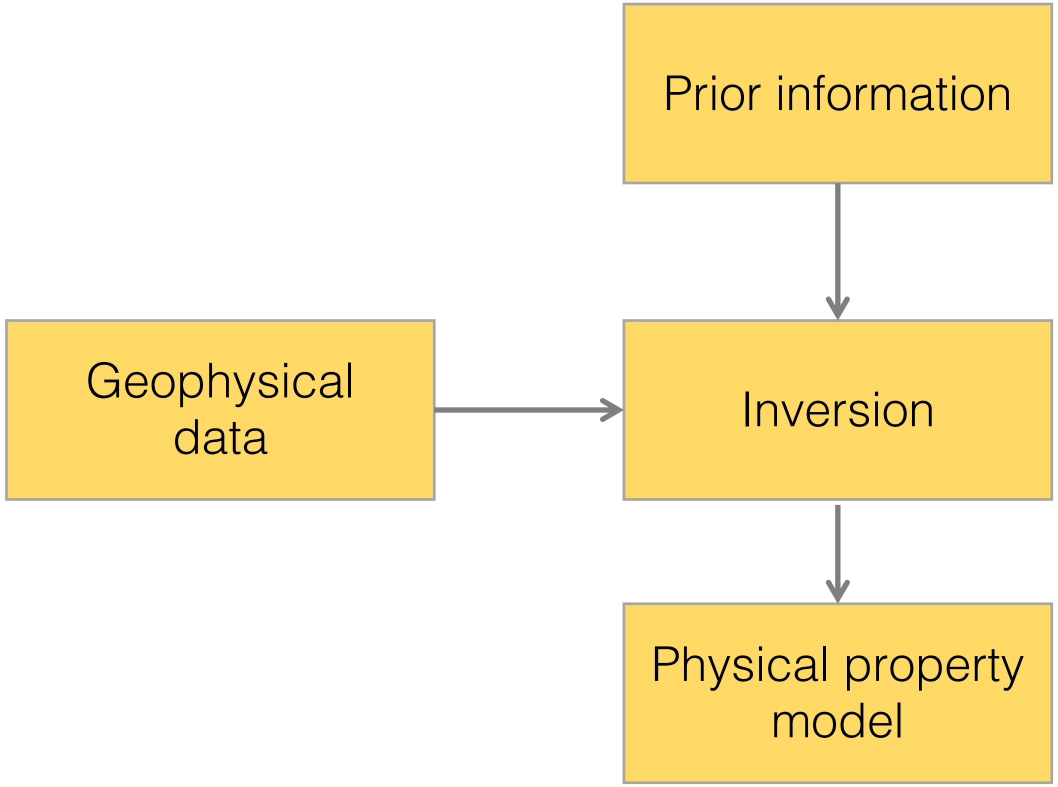
Any inversion approach must address these issues

Constraining the inversion

What information is available?

Structural characteristics

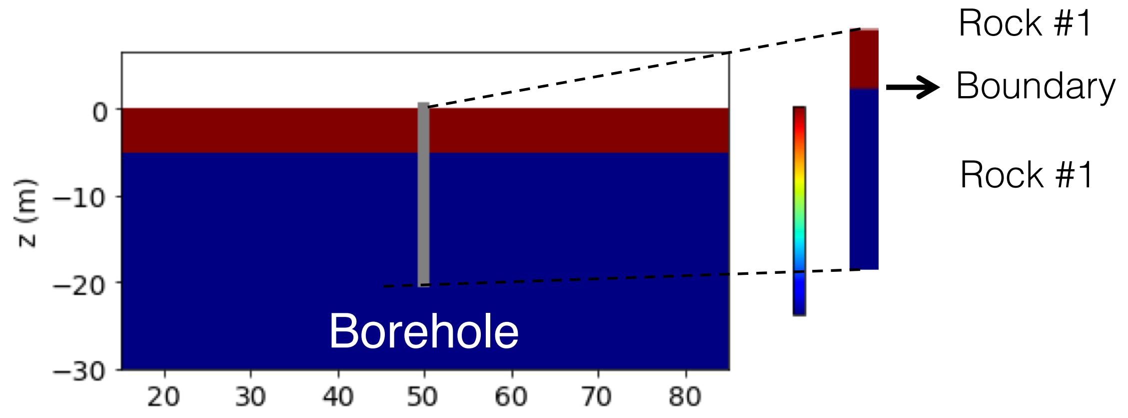


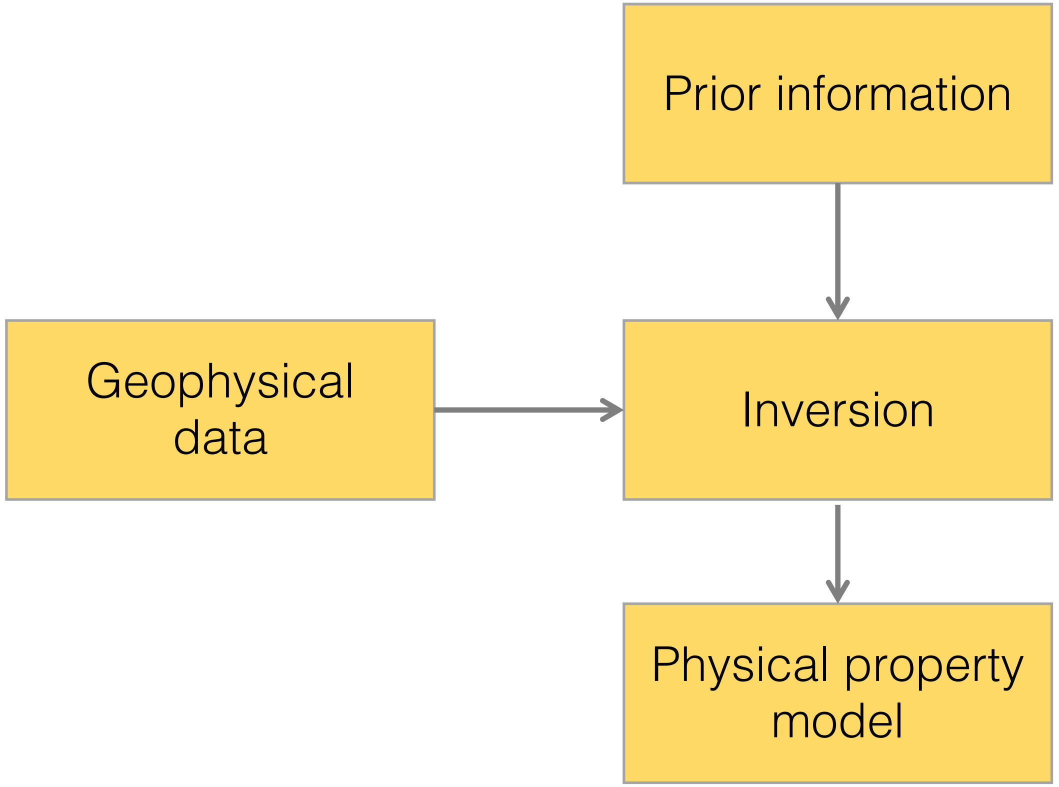


Constraining the inversion

What information is available?

Geologic constraints

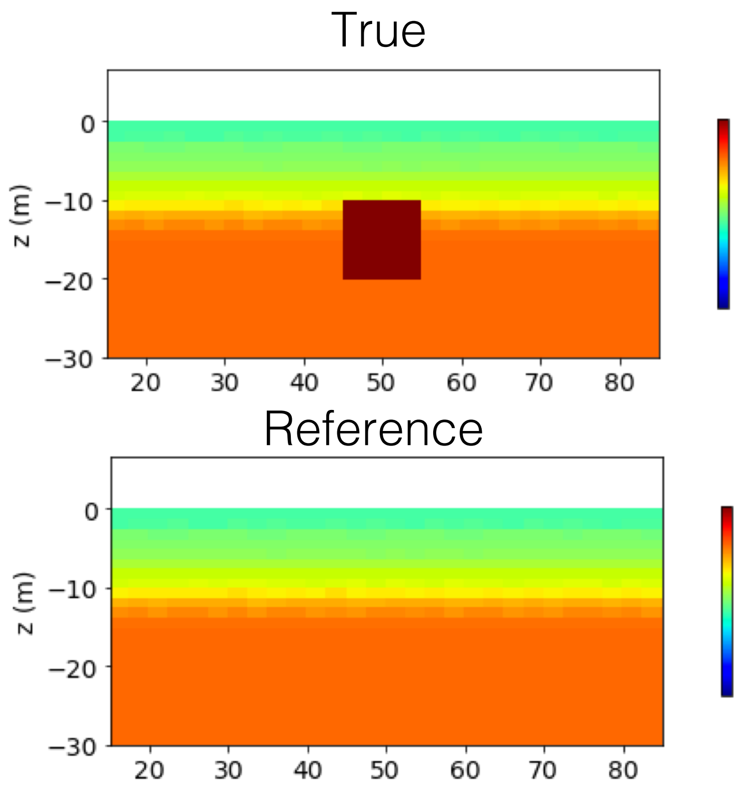


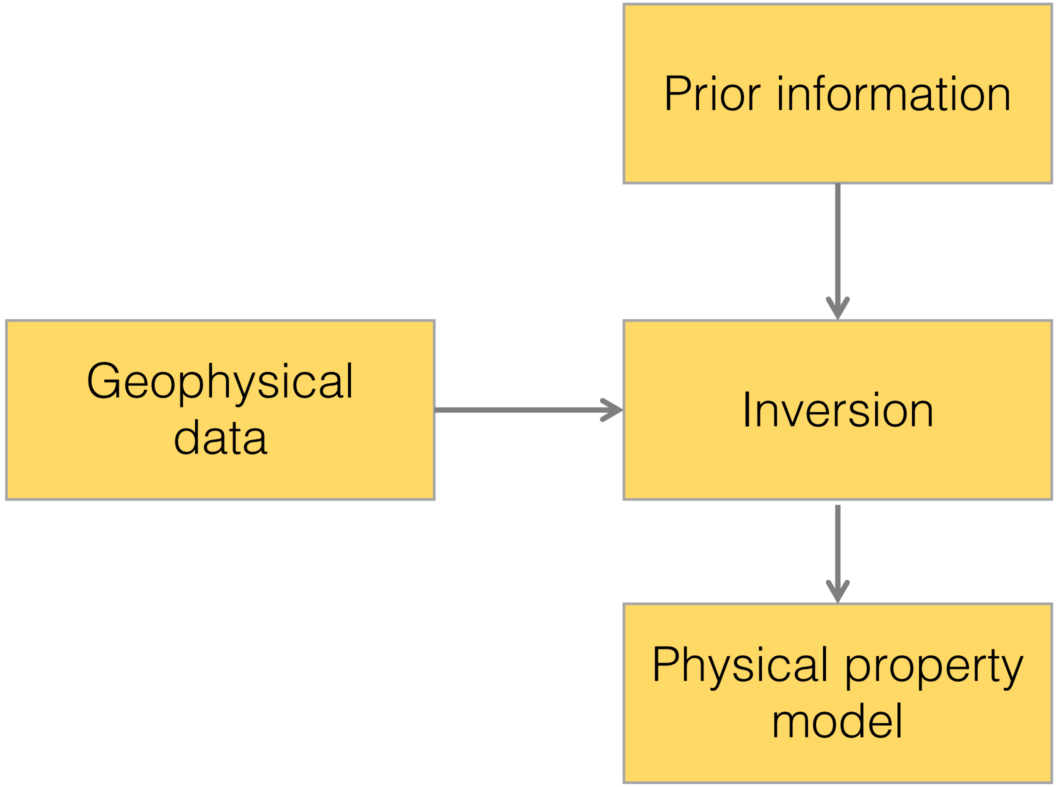


Constraining the inversion

What information is available?

Reference model





Constraining the inversion

What information is available?

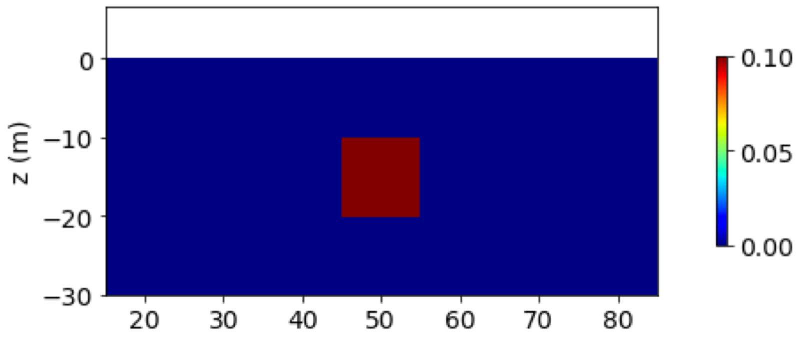
Physics

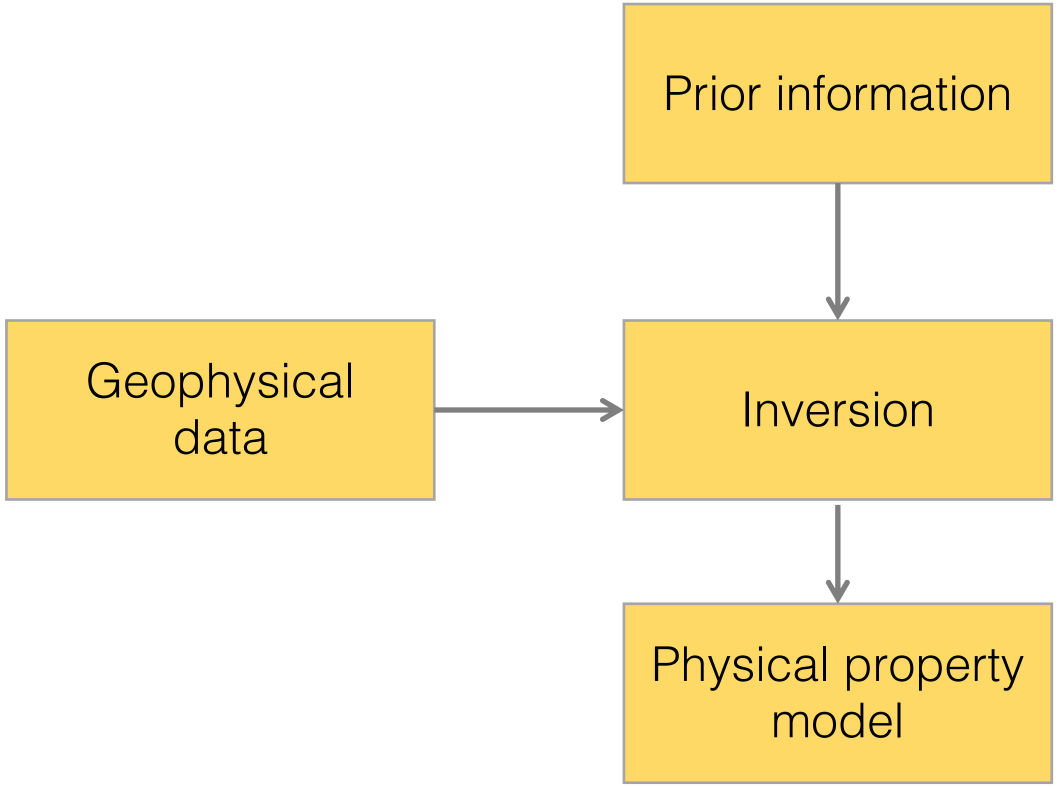
- Magnetic susceptibility is positive

$$\kappa \geq 0$$

- Or values must lie between certain bounds

$$m_l \leq m \leq m_u$$

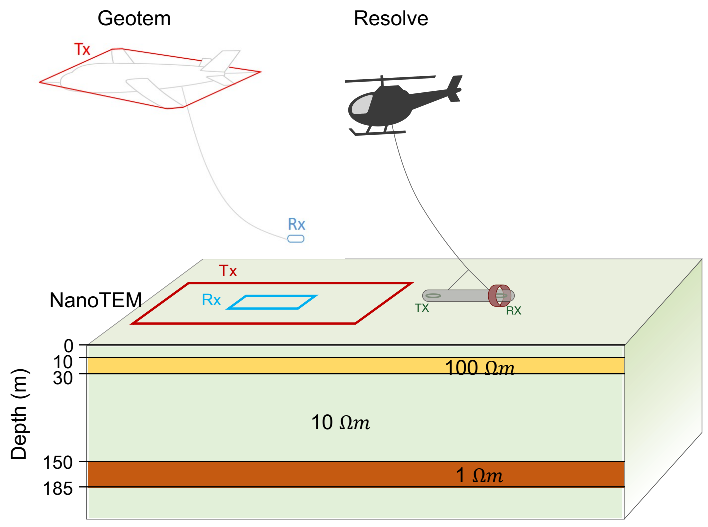


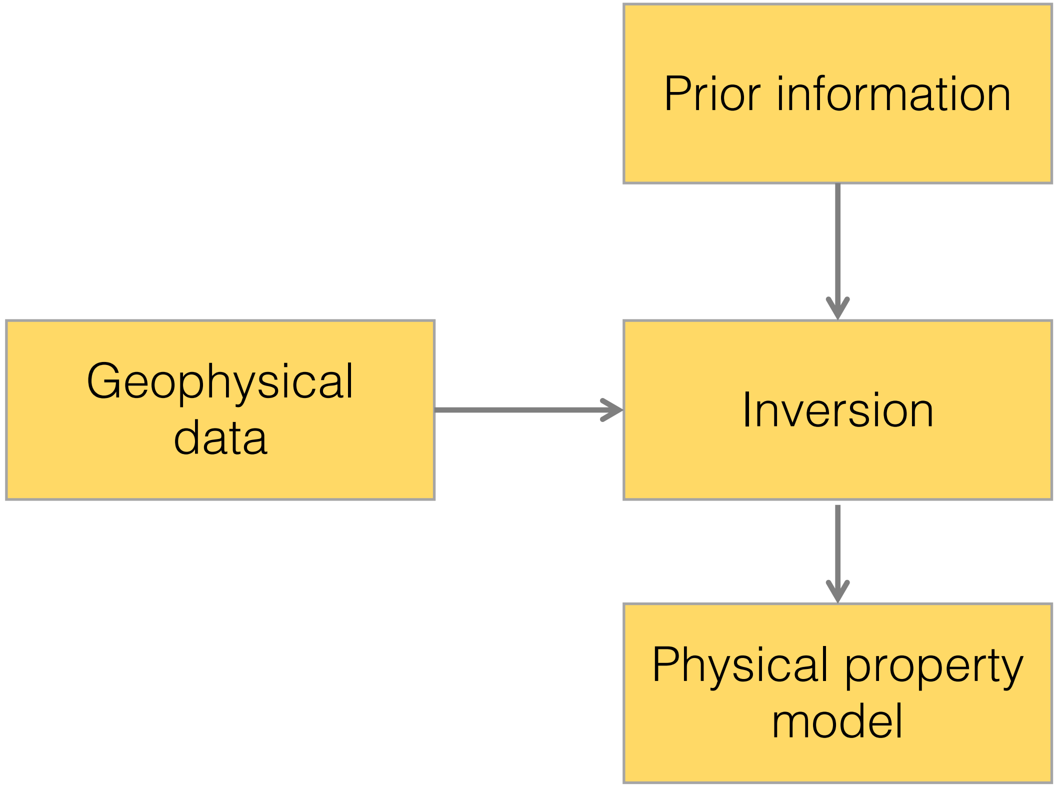


Constraining the inversion

What information is available?

- Joint data sets connected with same property
  (e.g. electric conductivity)



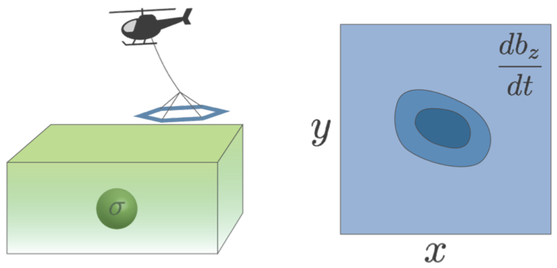


Constraining the inversion

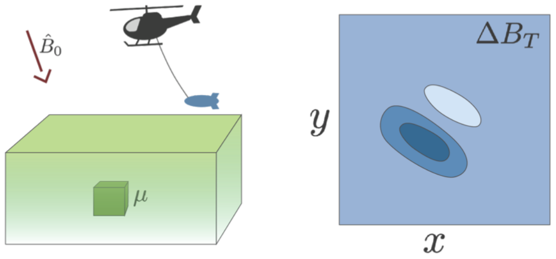
What information is available?

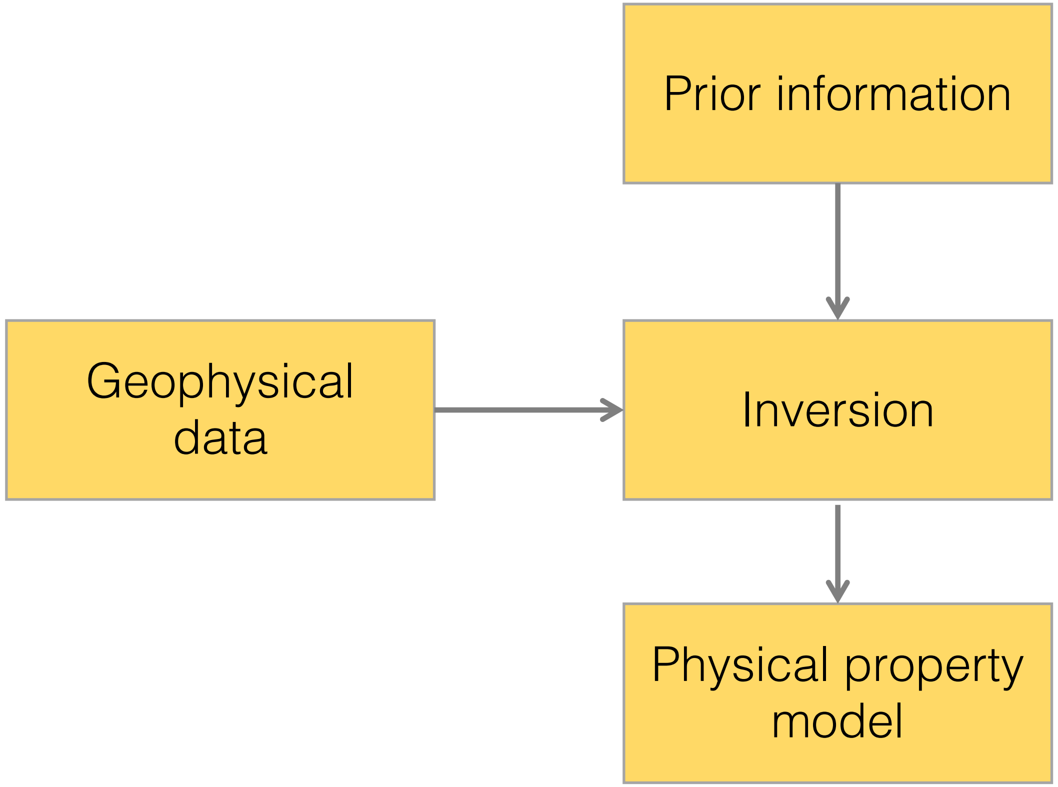
- Joint data sets from different properties

EM conductivity



Magnetics - susceptibility



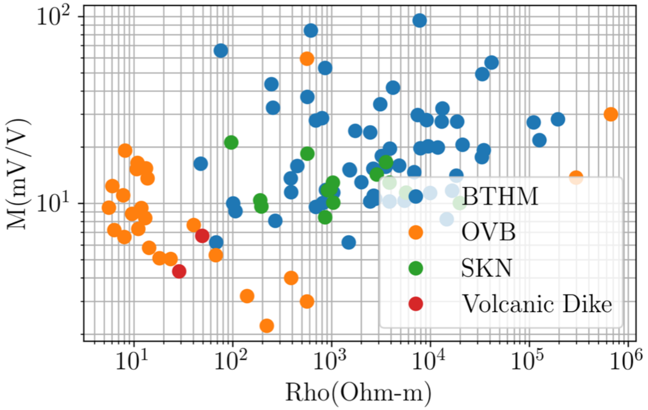


Constraining the inversion

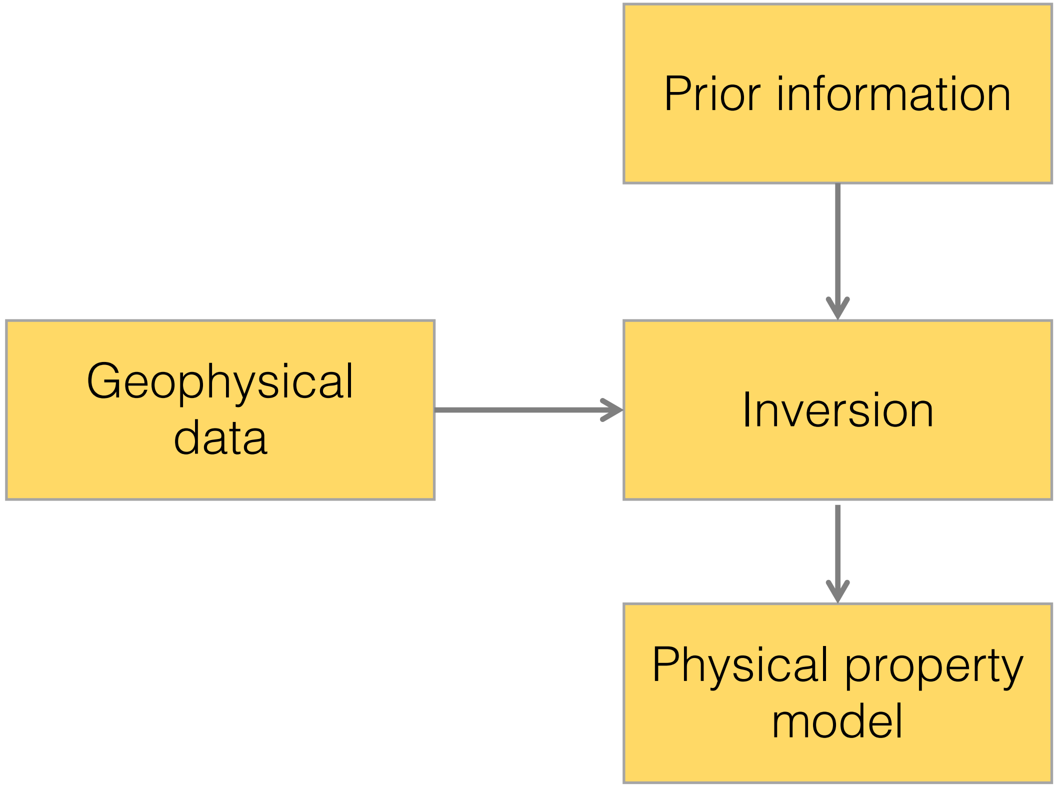
What information is available?

- Petrophysics
- Well-logs

Multiple properties



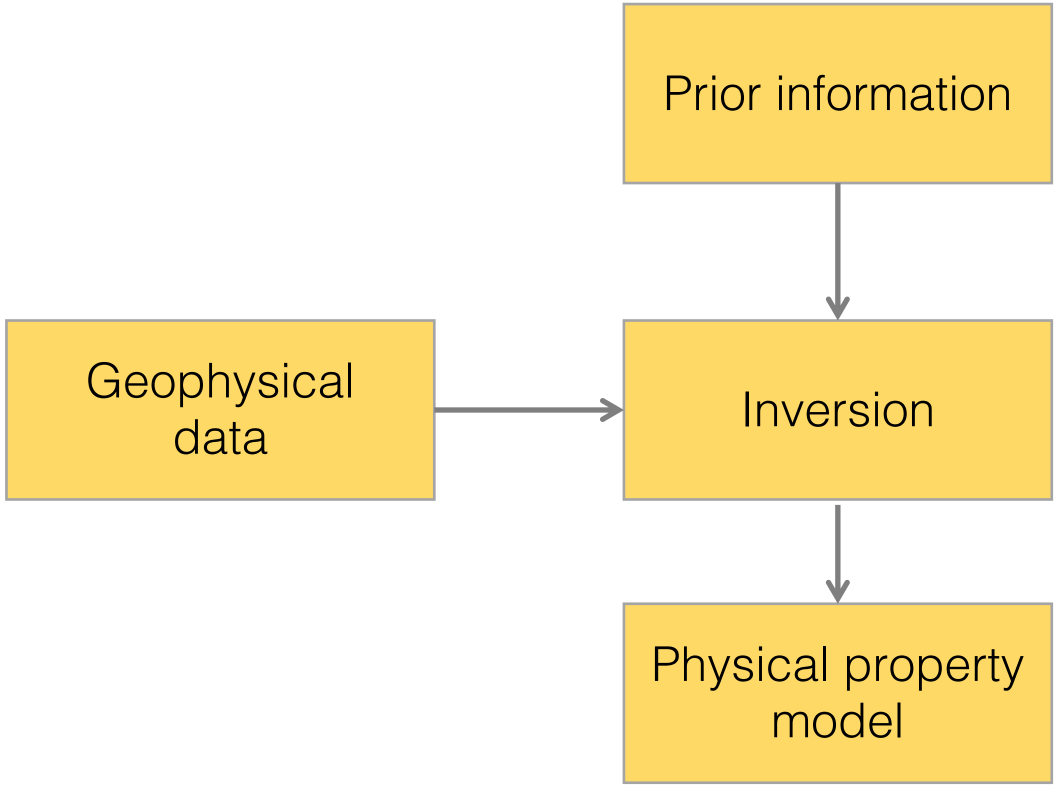
Drill cores



Constraining the inversion

What information is available?

- Geologic structure
- Geologic constraints
- Reference model
- Bounds
- Multiple data sets
- Physical property measurements



Framework for Inverse Problem

Tikhonov (deterministic)

Find a single "best" solution by solving optimization

$$\text{minimize} \quad \phi = \phi_d + \beta\phi_m$$

$$\text{subject to} \quad m_L < m < m_H$$

$$\begin{cases} \phi_d\text{: data misfit} \\ \phi_m\text{: regularization} \\ \beta\text{: trade-off parameter} \\ m_L, m_H\text{: lower and upper bounds} \end{cases}$$

Bayesian (probabilistic)

Use Bayes' theorem

$$P(m|d^{obs}) \propto P(d^{obs}|m)P(m)$$

$$\begin{cases} P(m)\text{: prior information about } m \\ P(d^{obs}|m)\text{: probability about the data errors (likelihood)} \\ P(m|d^{obs})\text{: posterior probability for the model} \end{cases}$$

Two approaches:

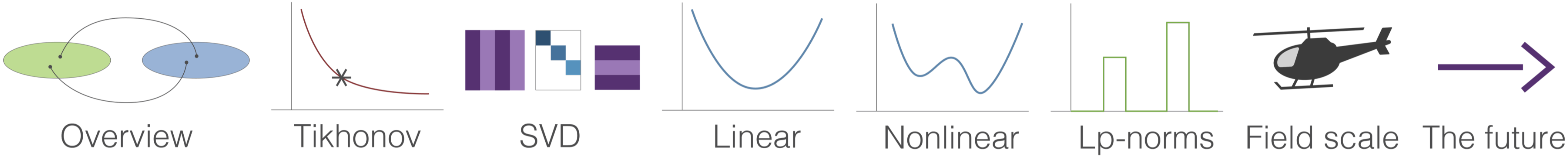
(a) Characterize $P(m|d^{obs})$

(b) Find a particular solution that maximizes $P(m|d^{obs})$ (MAP: (maximum a posteriori) estimate

This course

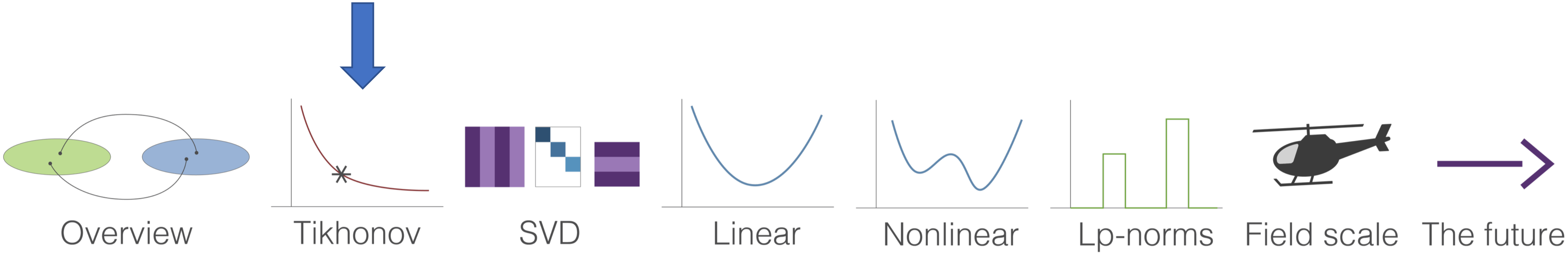
- Tikhonov style inversion
- Linear and non-linear problems
- Progressively incorporate different types of a priori information
- Thematic examples (1D, 2D, 3D)
- Case histories
- Open source resources (Jupyter notebooks)

Major components for the week



https://courses.geosci.xyz/lapis2019

Next up


Overview
Tikhonov
SVD
Linear
Nonlinear
Lp-norms
Field scale
The future