

# Linear Tikhonov Inversions



# Outline

- Statement of the inverse problem
- App for a linear problem
- Data misfit
- Model norms and their importance
- Tikhonov approach to inversion
- A flow chart to keep you on track

#### Our statement of the inverse problem

- Given observations:  $d_j^{obs}$ , j = 1, ..., N
  - Uncertainties:  $\epsilon_j$
  - Ability for forward modelling:  $\mathcal{F}[m] = d$









#### Linear problem

$$d_j = \int_v g_j(x)m(x)dx \qquad g_j(x) = e^{jpx}\cos(2\pi jqx)$$

- $d_j$ : j-th datum
- $g_j$ : kernel function for j-th datum
- m: model



Each datum is an inner product of the model with the kernel function.

For this case

$$d_j = \mathbf{g} \cdot \mathbf{m} = 4.89$$

# Analogy with 1D frequency domain EM

- FDEM system (Resolve)
- Signals: sinusoids at 5 frequencies
- Penetration depth depends upon frequency
- Measurements are fields from buried conductors









HCP frequencies: 382, 1822, 7970, 35920 and 130100 Hz

$$g_j(x) = e^{jpx} cos(2\pi jqx)$$
 5

#### Forward Problem

- Datum is defined as:  $d_i = \int_0^1 g_i(x)m(x)dx$
- Discretize model:  $\mathbf{m} = (m_1, m_2, ..., m_M)$

$$d_{i} = \int_{0}^{X_{1}} g_{i}(x)m_{1}dx + \int_{X_{1}}^{X_{2}} g_{i}(x)m_{2}dx + ..$$
$$= \sum_{j=1}^{M} \left(\int g_{i}(x)dx\right)m_{j}$$

• In matrix form:  $\mathbf{d} = \mathbf{G}\mathbf{m}$ 



**G**:  $(N \times M)$  matrix **d**:  $(N \times 1)$  vector **m**:  $(M \times 1)$  vector

# Linear inversion app (demo)

- It will help us
  - Develop a model
  - Consider kernels
  - Generate data
- Model: m(x)
- Kernels (physics):

$$g_j(x) = e^{jpx} \cos(2\pi jqx)$$

• Data:

$$d_j = \int_v g_j(x)m(x)dx$$



# Models with the app (demo)

#### Build a 1D model

- Background
- Box car
- Gaussian



# Kernels with the app (demo)

• Kernels for FEM are decaying sinusoids

$$g_j(x) = e^{jpx} \cos(2\pi jqx)$$

- N: number of data
- M: number of cells
- p: controls decay
- q: frequency
- j1: starting j value
- jN: ending j value
- ymin, ymax (plot limits)



#### Real observation includes noise

• Data: d = Gm



10

#### Real observation includes noise

• Data: d = Gm



11

## Inversion



#### Inverse problem

- Observed data:  $d_j^{obs}$ , j = 1, ..., N
- Uncertainty:  $\epsilon_j$
- Ability to simulate data:  $\mathcal{F}[m] = d$
- Find the model which fits the observation



• For linear problem:  $\mathbf{Gm} = \mathbf{d}$ 

 $\mathbf{m} \in \mathbb{R}^M$  $\mathbf{d} \in \mathbb{R}^N$  $\mathbf{G} \in \mathbb{R}^{M imes N}$  M > N

more unknowns than data (underdetermined system)

# Inversion using Misfit criterion

Forward modelling	$\mathbf{Gm} = \mathbf{d}$
Data	$\mathbf{d}^{obs} = \mathbf{d} + \boldsymbol{\epsilon}$
Noise	$\epsilon$

• Gaussian errors with standard deviation,  $\epsilon_j$ 

• Misfit measure: 
$$\phi_d = \sum_{j=1}^N \left(\frac{d_j - d_j^{obs}}{\epsilon_j}\right)^2$$

- Expected value of  $\phi_d$  is  $E[\phi_d] = N$
- Data are fit when  $\phi_d \simeq \phi_d^*$

 $\phi_d^*$ : target misfit $\phi_d^* = N$  14

#### Inversion with misfit only



$$\phi_d = \sum_{j=1}^N \left(\frac{d_j - d_j^{obs}}{\epsilon_j}\right)^2$$

15

#### Inversion app (demo)

• With accurate data

• With noisy data.

🕨 🔍 🧧 LinearInversion	×			<b>.</b>
$\dot{\mathbf{r}}  ightarrow \mathbf{C}$ $\mathbf{\hat{C}}$ $\mathbf{\hat{O}}$ localhost:8888			r	¥ 🜔 🔝 🖘 🗄
📁 jupyter LinearInve	rsion (unsaved changes)			Logout
File Edit View Insert	Cell Kernel Wi	dgets Help	Trusted	Python 3 O
E + ≫ 4 F. ↑ ↓	N Run ■ C Mari	kdown 🗘 📼		
In [37]: Q2 = app.int	ceract_plot_model()			
m_backgro	a	.00		
m1 -	1	.00		
m2 -	2	.00		
m1_center	C	.20		
dm1		.20		
m2_center		.75		
sigma_2	model	,U/	kornol	
option	add noise	uata	Kerner	
percentage	0.1			
floor	0.1			
1.0 0.0 0.0 0.0 0.0 0.0 0.2	s of matrix G 2 1 5 0 -1 -2 0.0	Model	Data 40 20 0 5 10 3	15

# The basic problem of non-uniqueness

- Each datum is a "volumetric" response
- Data are

$$d_i = \sum_{j=1}^M G_{ij} m_j$$

 $\mathbf{d} = \mathbf{Gm} \qquad \qquad \mathbf{G} : (N \times M)$ 

- In the app
  - M=100
  - N=20

- So, M>N (underdetermined problem)  $\rightarrow$  infinitely many solutions

## Questions to consider

 Consider the simple problem that involves two unknowns: m<sub>1</sub> and m<sub>2</sub>
 Any point here

- We have one datum:  $m_1 + 2m_2 = 2$ 

- What is the value of  $m_1$  and  $m_2$ ?
  - (1, 0.5)
  - (2,0)

- . . . .



#### Possible norms

• Smallness

$$\phi_m = \|\mathbf{m}\|^2 = \sum m_j^2$$

• Smallness with reference

$$\phi_m = \|\mathbf{m} - \mathbf{m}_{\text{ref}}\|^2 = \sum_j (m_j - m_{\text{ref}\ j})^2$$

• Smoothness

$$\phi_m = \left\|\frac{d\mathbf{m}}{d\mathbf{x}}\right\|^2 = \sum_j (m_{j+1} - m_j)^2$$

$$\phi_{m} = m_{1}^{2} + m_{2}^{2}$$

$$\Rightarrow m = (0.392, 0.784)$$

$$\phi_{m} = (m_{1} - m_{ref})^{2} + (m_{2} - m_{ref})^{2}$$

$$\Rightarrow m = (0.8, 0.61) \quad m_{ref} = 1$$

$$\phi_{m} = (m_{1} - m_{2})^{2}$$

$$\Rightarrow m = (2/3, 2/3)$$
19

# How to pick "one" from "many"

- Define: ruler to measure size (norm) of models
- Choose the "smallest"
- e.g. height



#### Model norms: Generally finding functions

Smallest model: 
$$\phi_m = \int m^2 dx$$

Smallest with reference: 
$$\phi_m = \int (m - m_{ref})^2 dx$$

$$\phi_m = \int \left(\frac{dm}{dx}\right)^2 dx$$

Combination: 
$$\phi_m = \alpha_s \int (m - m_{ref})^2 dx + \alpha_x \int \left(\frac{dm}{dx}\right)^2 dx$$

Discretize:  $\phi_m = \alpha_s \|\mathbf{W}_s(\mathbf{m} - \mathbf{m}_{ref})\|_2^2 + \alpha_x \|\mathbf{W}_x(\mathbf{m})\|_2^2$ 

# Summary

- Norms provide ways to select a specific solution
- Also need to address data errors and a misfit criterion



# Inversion using Misfit criterion

Forward modelling	$\mathbf{Gm} = \mathbf{d}$
Data	$\mathbf{d}^{obs} = \mathbf{d} + \boldsymbol{\epsilon}$
Noise	$\epsilon$

• Gaussian errors with standard deviation,  $\epsilon_j$ 

• Misfit measure: 
$$\phi_d = \sum_{j=1}^N \left(\frac{d_j - d_j^{obs}}{\epsilon_j}\right)^2$$

- Expected value of  $\phi_d$  is  $E[\phi_d] = N$
- Data are fit when  $\phi_d \simeq \phi_d^*$

 $\phi_d^*$ : target misfit $\phi_d^* = N$  23

# Combining misfit and model norm

- Goal for our inverse problem: find the model *m* that
  - produces an acceptable misfit ( $\phi_d < \phi_d^*$ )
  - minimizes the model norm,  $\phi_m$
- Re-cast as an optimization:

minimize  $\phi_d + \beta \phi_m$  where  $0 < \beta < \infty$ 

•  $\beta$  : trade-off (Tikhonov) parameter

# The role of $\beta$

Analogy: an optimization problem with two requirements

- Travelling from A to B
  - minimize time taken
  - minimize fuel consumption

 $\phi = \operatorname{time} + \beta \cdot \operatorname{fuel}$ 



- both time and fuel consumption are functions of speed

- $\beta = 0$ : minimize time (regardless of fuel)
- large  $\beta$ : minimize fuel (but still get there)



# The role of $\beta$

- A typical problem might be:
  - Minimize fuel consumption
  - Subject to getting there in 2 hours





# The role of $\beta$ : managing misfit

- Our inverse problem
  - Find the model (m): minimize  $\phi_d + \beta \phi_m$
  - Which beta to use?
  - If standard deviations of data are known,

$$E[\phi_d] = N$$

- Desired misfit is  $\phi_d^* \simeq N$
- Choose  $\beta$  so that  $\phi_d(m) = \phi_d^*$



#### Inversion app (demo)



# $\beta$ is the trade-off parameter

• Solve

minimize  $\phi_d + \beta \phi_m$ 

- β too large → underfitting
   Structural information lost
- β too small → overfitting the data
   Noise becomes imaged as structure
- $\beta$  just right  $\rightarrow$  optimal fit  $\phi_d(m) \simeq N$ 
  - Best estimate of a model which adequately re-creates the observations



# $\beta$ is the trade-off parameter

• Solve

minimize  $\phi_d + \beta \phi_m$ 

- *β* too large → underfitting

   Structural information lost
- $\beta$  too small  $\rightarrow$  overfitting the data
  - Noise becomes imaged as structure
- $\beta$  just right  $\rightarrow$  optimal fit  $\phi_d(m) \simeq N$ 
  - Best estimate of a model which adequately re-creates the observations



# $\beta$ is the trade-off parameter

• Solve

minimize  $\phi_d + \beta \phi_m$ 

- β too large → underfitting
   Structural information lost
- β too small → overfitting the data
   Noise becomes imaged as structure
- $\beta$  just right  $\rightarrow$  optimal fit  $\phi_d(m) \simeq N$ 
  - Best estimate of a model which adequately re-creates the observations



# Flow chart for inverse problem



# Summary

• Solving linear inverse problem

minimize  $\phi_d + \beta \phi_m$ 



Trade-off parameter,  $\beta$ 



Data misfit

$$\phi_d = \sum_{j=1}^N \left(\frac{d_j - d_j^{obs}}{\epsilon_j}\right)^2 \quad \begin{cases} \phi_d \simeq \phi_d^* \\ \phi_d^* = N \end{cases}$$

 $\phi_m = \alpha_s \int (m - m_{ref})^2 dx + \alpha_x \int \left(\frac{dm}{dx}\right)^2 dx$ 

Model objective function (or regularization)

# Summary

• Solving linear inverse problem

minimize  $\phi_d + \beta \phi_m$ 



Trade-off parameter,  $\beta$ 

Data misfit

function

Model objective

Data misfit  

$$\phi_{d} = \sum_{j=1}^{N} \left(\frac{d_{j} - d_{j}^{obs}}{\epsilon_{j}}\right)^{2} \quad \left\{\begin{array}{l} \phi_{d} \simeq \phi_{d}^{*} \\ \phi_{d}^{*} = N \end{array}\right.$$
Model objective function (or regularization)  

$$\phi_{m} = \alpha_{s} \int (m - m_{ref})^{2} dx + \alpha_{x} \int \left(\frac{dm}{dx}\right)^{2} dx$$
How do we solve this optimization problem?

# Outline

- Statement of the inverse problem
- App for a linear problem
- Data misfit
- Model norms and their importance.
- Tikhonov approach to inversion
- A flow chart to keep you on track

# Homework problem

#### Accessing app

- Link: https://mybinder.org/v2/gh/geoscixyz/geosci-labs/master?filepath=notebooks%2Findex.ipynb
- Wait until you see index page

<image/> <ul> <li>If the the rest of t</li></ul>	Cjupyter index (autosaved)	🔶
<image/> <section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><section-header><text></text></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header></section-header>	File Edit View Insert Cell Kernel Widgets Help	Trusted Python 3 C
<section-header><section-header><section-header><section-header><section-header><text><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><list-item><text><text><text><text><text><text></text></text></text></text></text></text></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></list-item></text></section-header></section-header></section-header></section-header></section-header>	1 + 3< 2 E ↑ ↓ H Run ■ C > Markdown ‡	
<section-header><section-header><text><text><list-item><list-item><text><text><text><list-item><list-item><text><list-item><list-item><list-item><text></text></list-item></list-item></list-item></text></list-item></list-item></text></text></text></list-item></list-item></text></text></section-header></section-header>		
<section-header><section-header><text><text><list-item><list-item><text><text><text><list-item><text><list-item><text></text></list-item></text></list-item></text></text></text></list-item></list-item></text></text></section-header></section-header>		
<ul> <li>Independent of the end relation of the original of th</li></ul>	GeoSci-labs	
geophysics. They support the open source textbooks : 9 geo.geosel.xyz, a resource for algeophysics. These notebooks are powered by <u>SimPEG</u> , an open source framework for Simulation and Parameter Estimation in Cophysics. Up to have feedback, we would like to hear from you! 9 Georefissues 9 Join the development CIP [ EM ] GPR   Inversion   Mag   Seismic   Gravity Contents	The purpose of these notebooks is to provide tools for you to investigate fundamental concepts in ammplied	
<ul> <li>9:09:09:09:00:01, a resource for applied geophysics.</li> <li>9:00:00:01, a resource for applied geophysics.</li> <li>Provide the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:00:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:00:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:00:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:00:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:00:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:01:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:01:00:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:01:00:01, and the observed by <u>SIMPEG</u>, an open source framework for Simulation and Parameter Estimation in Cophysics.</li> <li>9:00:01:01:01:01:01:01:01:01:01:01:01:01:</li></ul>	geophysics. They support the open source textbooks :	
<ul> <li>emgesselays, a resource for electromagnetic geophysics.</li> <li>These notebooks are powered by <u>SimPEG</u>, an open source framework for Simulation and Parameter Estimation in Geophysics.</li> <li>Tyou have feedback, we would like to hear from you!</li> <li>Contact us</li> <li>Boond issues</li> <li>Join the development</li> </ul> CIP [EM] [QPR] [Inversion   Mag   Seismic   Gravity Contents	<ul> <li>gpg,geosci.xyz, a resource for applied geophysics</li> </ul>	
These notebooks are powered by <u>SimPEG</u> , an open source framework for Simulation and Parameter Estimation in Gophysics. If you have feedback, we would like to hear from you! 9. Boont issues 9. Join the development DCIP! [EM] [QPR] [Invariant] Mag [Seismic] Gravity: Contents	<ul> <li>em.geosci.xyz, a resource for electromagnetic geophysics.</li> </ul>	
If you have feedback, we would like to hear from you!  Contact us  Begord usual  Contact us  Contents  Contents	These notebooks are powered by <u>SimPEG</u> , an open source framework for Simulation and Parameter Estimation in Geophysics	
<ul> <li>in you have reacted and to have it only you</li> <li>Branon lissues</li> <li>Join the development</li> </ul> DCIP   EM   GPR   Inversion   Mag   Selamic   Gravity Contents	If you have feadback, we would like to have from you!	
e Joint Luis     e Join the development  DCIP   EM   GPR   Inversion   Mag   Seismic   Gravity  Contents		
Join the development  DCIP   EM   GPR   Inversion   Mag   Seismic   Gravity  Contents	Contact us     Report issues	
DCIP   EM   GPR   Inversion   Mag   Seismic   Gravity Contents	Join the development	
Contents	DCID LEM LODP Linuxreion LMag L Salemin L Gravity	
Contents		
<b>?</b>	Contents	
<b>?</b>		
<b>?</b>		
	<b></b>	





# Homework problem

- Question:
  - Use default parameters in LinearInversion.ipynb
  - Except data error (%) = 0 (keep floor = 0.01)
  - What is the value of  $\beta$  that corresponds to the inflection point of the Tikhonov curve (corner of the L-curve)

