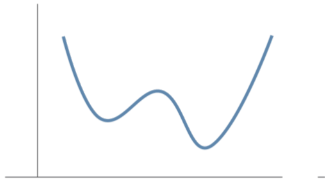
Non-linear inversion



Nonlinear

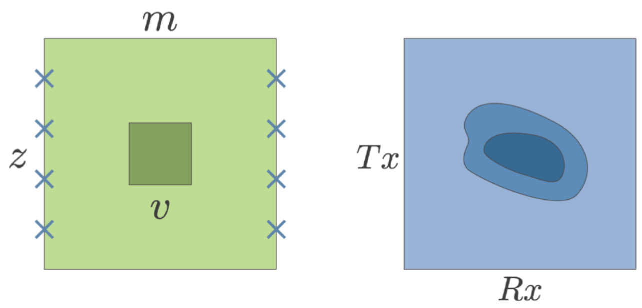
Outline

- Non-linear forward problem
- Non-linear optimization (Newton's)
  - Quadratic convergence
- Example problem: DC resistivity
  - Physics
  - Discretization
  - Optimization
  - 2D synthetic inversion
  - 3D field example: Mt. Isa

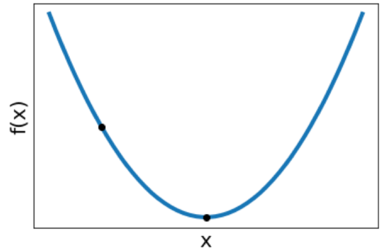
Non-linear inversion

- Set inverse problem minimize $\phi = \phi_d + \beta\phi_m$

$$\phi_d = \sum_{j=1}^{N} \left(\frac{\mathcal{F}_j(m) - d_j^{obs}}{\epsilon_j} \right)^2$$



- For linear problem: $d = Gm$
  - And quadratic regularization: $\int_v (m - m_{\text{ref}})^2 dv$
  - This is quadratic so we can solve in one step



- Problem becomes non-linear if:

(i) $\mathcal{F}[m]$ is non-linear
(ii) ϕ_d is not l_2 (e.g. $\sum \left| \frac{\mathcal{F}_i[m]-d_i}{\epsilon_i} \right|$)
(iii) ϕ_m is not quadratic

Non-linear inversion

- Data: $d = \mathcal{F}[m]$

  If $\mathcal{F}[\cdot]$ is Linear operator $\Rightarrow$ $d = Gm$

  Non-linear: $\mathcal{F}[am_1 + bm_1] \neq a\mathcal{F}[m_1] + b\mathcal{F}[m_2]$

- Examples:
  - Seismic
  - Maxwell's (1st order wave equation)
  - DC resistivity

- Solve optimization problem: $\text{minimize } \phi(m) = \phi_d + \beta\phi_m(m)$

Non-linear inversion

- Data: $d = \mathcal{F}[m]$

$\text{If } \mathcal{F}[\cdot] \text{ is Linear operator} \Rightarrow d = Gm$

$\text{Non-linear: } \mathcal{F}[am_1 + bm_1] \neq a\mathcal{F}[m_1] + b\mathcal{F}[m_2]$

- Examples:
  - Seismic
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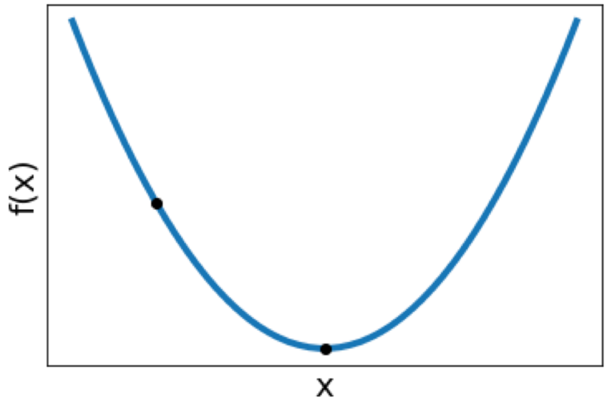
- Solve optimization problem: $\text{minimize } \phi(m) = \phi_d + \beta\phi_m(m)$

(1) Discretize PDE and solve forward problem
(2) Iteration

Non-linear optimization

- Single variable x: $\quad \text{minimize } f(x)$ $\quad f(\cdot)$: function



- Case I: f is quadratic

$$f(x) = \frac{1}{4}x^2 - 3x + 9 = (\frac{1}{2}x - 3)^2$$

$$f'(x) = (\frac{1}{2}x - 3) = 0 \quad \Rightarrow \quad x = 6$$

$$\text{Minimum: } f'(x) = 0$$
$$f''(x) > 0$$

- Suppose

$$f(x) = (\frac{1}{2}x - 3)^2 + ax^3 + bx^4$$

Non-linear optimization

- Single variable x: $\quad \text{minimize } f(x) \quad f(\cdot)\text{: function}$

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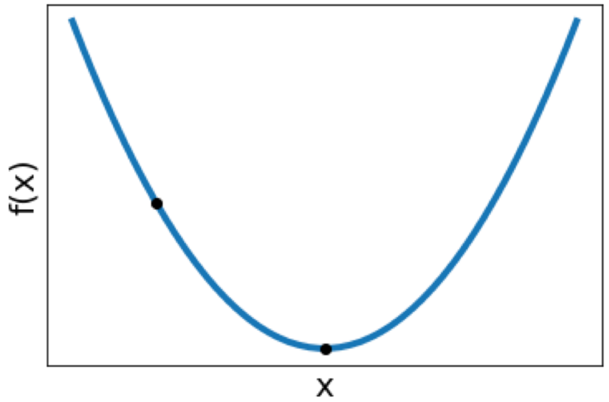

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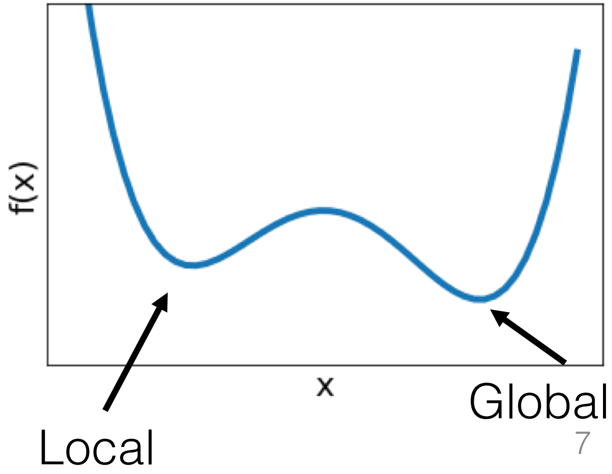
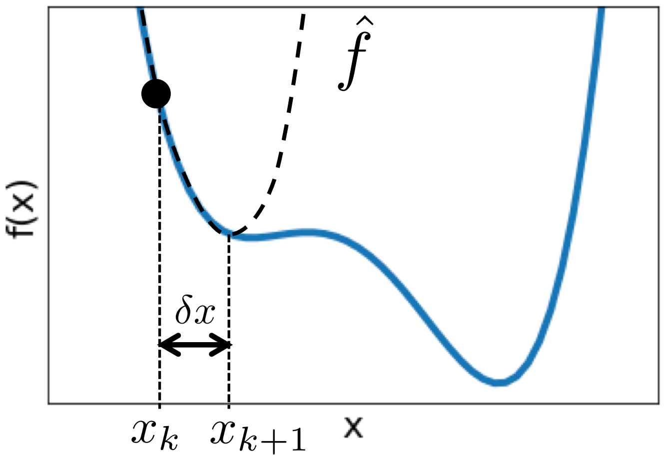


Non-linear optimization

- Newton's Method
  i. Begin with x_k
  ii. Solve a local quadratic for δx
  iii. $x_{k+1} = x_k + \delta x$



Local Quadratic: $\hat{f}(x_k + \delta x) = f(x_k) + f'(x_k)\delta x + \frac{1}{2} f''(x_k)\delta x^2 + \cancel{\mathcal{O}(\delta x^3)}$

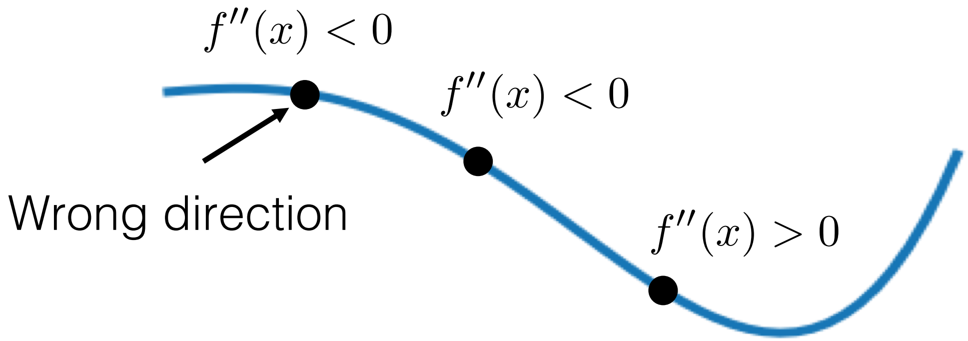
δx that minimizes $\hat{f}(x_k + \delta x)$ $\Rightarrow$

$$\boxed{\begin{aligned} f''(x_k)\delta x &= -f'(x_k) \\ \text{or } \delta x &= -\frac{f'(x_k)}{f''(x_k)} \end{aligned}}$$

Quadratic convergence: $|x_{k+1} - x^*| < c|x_k - x^*|^2$

- Things can go wrong

$$\delta x = -\frac{f'(x_k)}{f''(x_k)}$$



- If wrong direction, negative curvature

→ use gradient: $\delta x = -cf'(x)$

c: constant

- If right direction but $f''(x)$ is wrong

→ the step length is too large

Choose c

$|f(x + \delta x) - f(x)| > \varepsilon$

Scale

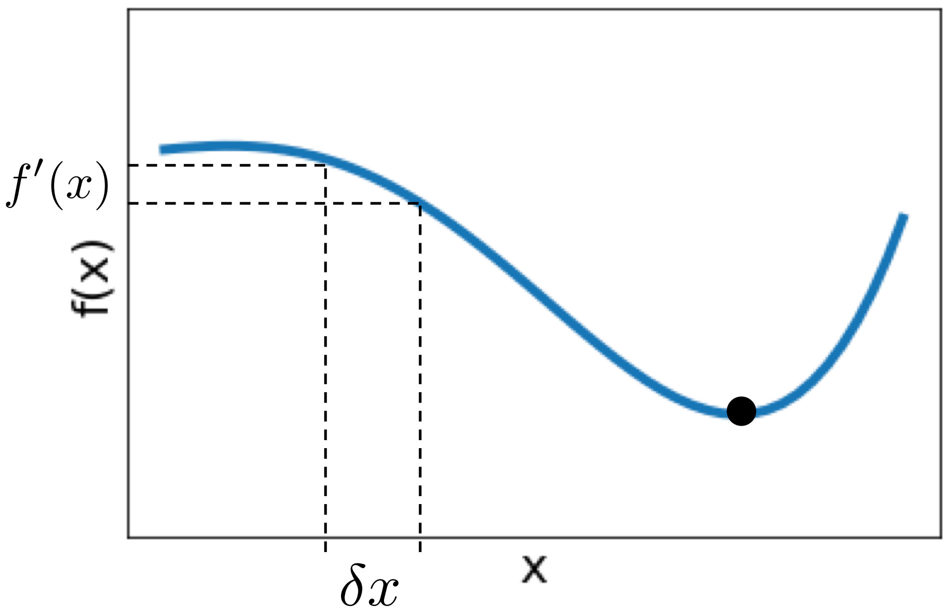
$x_{k+1} = x_k + \alpha\delta x \quad \alpha < 1$

Convergence conditions

$$f'(x) < \text{tolerance}$$

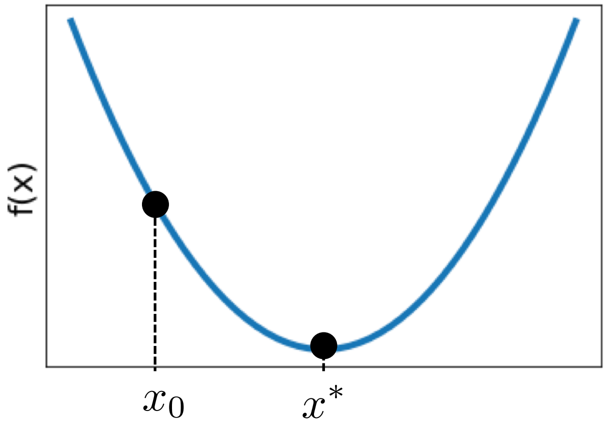
$$\|\delta x\| < \text{tolerance}$$

$$f''(x) > 0$$



Summary: Newton’s method

Linear

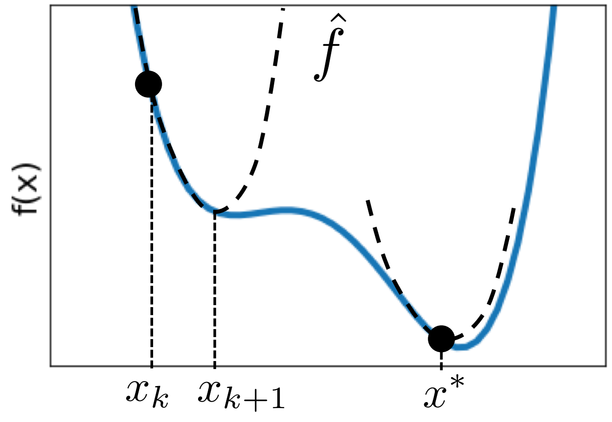


$$f''(x)\delta x = -f'(x)$$

$$x^* = -\frac{f'(x)}{f''(x)}$$

Solution in one step

Non-linear



$$f''(x_k)\delta x = -f'(x_k)$$

$$\delta x = -\frac{f'(x_k)}{f''(x_k)}$$

$$x_{k+1} = x_k + \alpha\delta x \qquad \alpha < 1$$

Iterate to convergence

Multivariate functions

Minimize $\phi(m)$ $m \in \{m_1, m_2, ..., m_M\}$

Taylor expansion

$$\phi(m + \delta m) = \phi(m) + (\nabla_m \phi(m))^T \delta m + \frac{1}{2} \nabla_m \nabla_m \phi(m) \delta m + \mathcal{O}(\delta m^3)$$

Note similarity to single variable

$$f(x + \delta x) = f(x) + f'(x)\delta x + \frac{1}{2} f''(x) \delta x^2 + \mathcal{O}(\delta x^3)$$

Define

Gradient: $g(m) = \nabla_m \phi = \begin{pmatrix} \frac{\partial \phi}{\partial m_1} \\ \vdots \\ \frac{\partial \phi}{\partial m_M} \end{pmatrix}$ $\qquad g \in \mathbb{R}^M$

Hessian:

$$H(m) = \nabla_m \nabla_m \phi = \begin{pmatrix} \frac{\partial^2 \phi}{\partial m_1^2} & \frac{\partial^2 \phi}{\partial m_1 \partial m_2} & \cdots & \frac{\partial^2 \phi}{\partial m_1 \partial m_M} \\ \frac{\partial^2 \phi}{\partial m_2 \partial m_1} & \frac{\partial^2 \phi}{\partial m_2^2} & \cdots & \frac{\partial^2 \phi}{\partial m_2 \partial m_M} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 \phi}{\partial m_M \partial m_1} & \frac{\partial^2 \phi}{\partial m_M \partial m_2} & \cdots & \frac{\partial^2 \phi}{\partial m_M^2} \end{pmatrix} \qquad H \in \mathbb{R}^{M \times M}$$

Symmetric

$$H_{i,j} = \frac{\partial^2 \phi}{\partial m_i \partial m_j}$$

Minimum defined: $\boxed{\begin{array}{l} g(m^*) = 0 \\ H(m^*) \text{ is positive definite} \end{array}}$

Finding a solution

(i) Begin with $m^{(k)}$

(ii) Solve $H\big(m^{(k)}\big)\delta m = -g\big(m^{(k)}\big)$ c.f. $\{f''(x)\delta x = -f'(x)\}$

(iii) $m^{(k+1)} = m^{(k)} + \alpha\delta m$

Our inversion

Minimize: $$\phi(m) = \frac{1}{2}\|\mathcal{F}[m] - d^{obs}\|^2 + \frac{\beta}{2}\|m\|^2$$

Sensitivity:

$$\nabla_m \mathcal{F}(m) = J$$

$$J_{ij} = \frac{\partial \mathcal{F}_i[m]}{\partial m_j}$$

Gradient: $$g(m) = \nabla_m \phi = J^T\left(\mathcal{F}[m] - d^{obs}\right) + \beta m$$

Hessian: $$H(m) = \nabla_m g(m) = J^T J + \cancel{(\nabla_m J)^T\left(\mathcal{F}[m] - d^{obs}\right)} + \beta$$

neglect

Final $$H\delta m = -g$$

$$(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$$

$$\delta d = \mathcal{F}[m] - d^{obs}$$

Comparison to linear problem

Nonlinear Problem: $(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$

$(\mathcal{F}[m] = d)$

$\delta d = \mathcal{F}[m] - d^{obs}$

Linear Problem: $(G^T G + \beta)\delta m = -(G^T d + \beta m)$

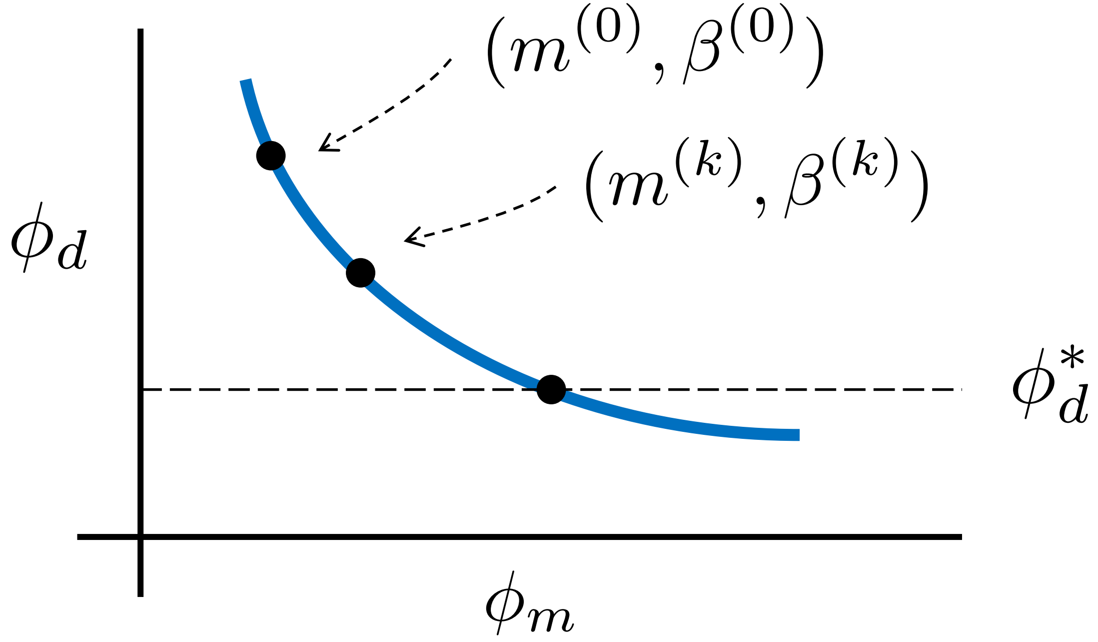
$(Gm = d)$

Sensitivity J acts as a local linear for non-linear operator

$$J\delta m = \delta d$$

General algorithm:

$$\text{minimize } \phi = \phi_d + \beta\phi_m$$



Initialize $m^{(0)}, \beta^{(0)}$

until convergence

$$H\delta m = -g$$

$$m^{(k+1)} = m^{(k)} + \alpha\delta m \ \text{(line search)}$$

$$\beta^{(k+1)} = \frac{\beta^{(k)}}{\gamma} \quad \text{(cooling)}$$

Many variants:
- Solving system
- Cooling rate
- ....

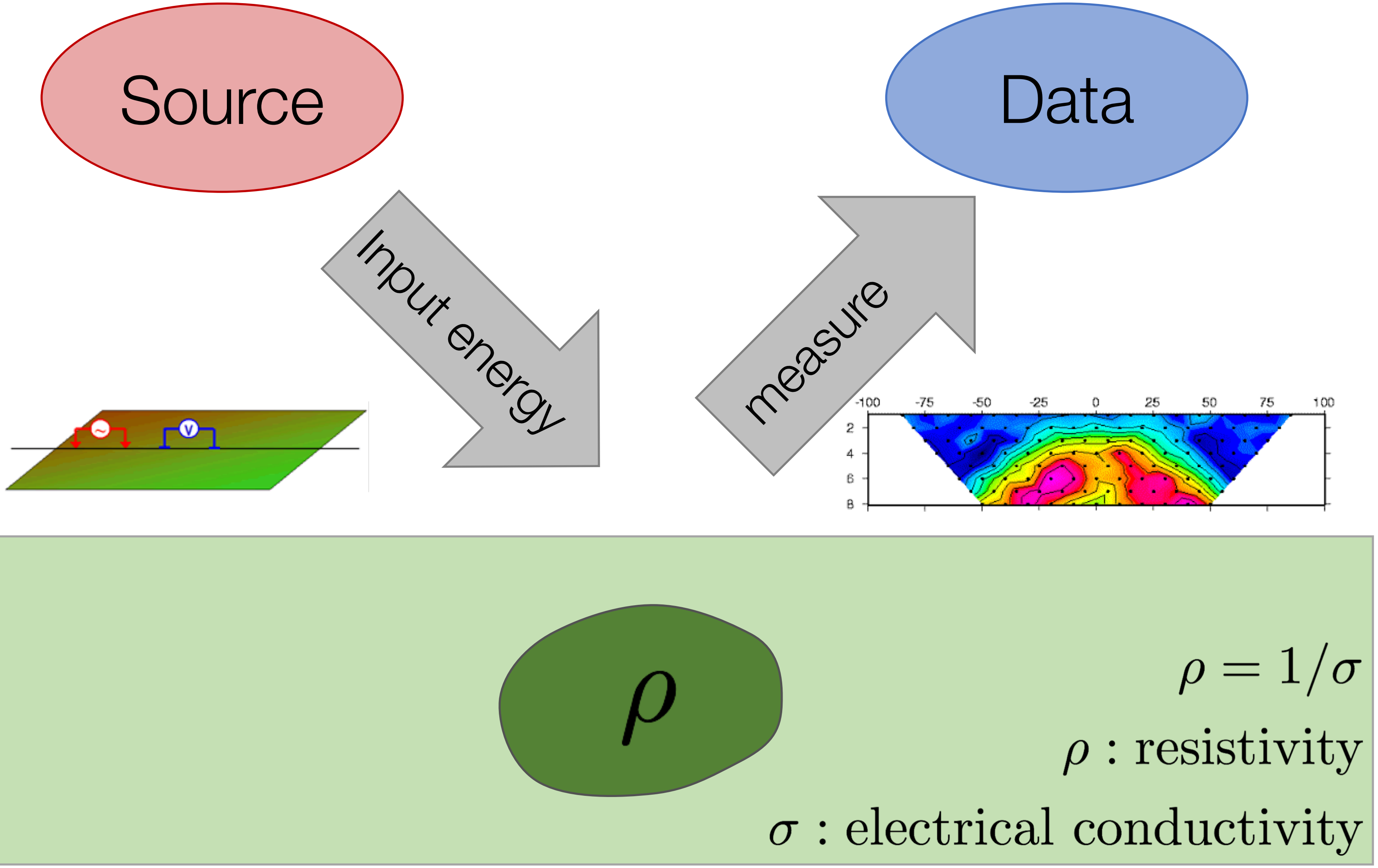
Summary

$$\phi(m) = \frac{1}{2}\|\mathcal{F}[m] - d^{obs}\|^2 + \frac{\beta}{2}\|m\|^2$$

Linear	Non-linear
$d = Gm$	$d = \mathcal{F}[m]$
$(G^T G + \beta)\delta m = -(G^T d + \beta m)$	$(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$
	$\delta d = \mathcal{F}[m] - d^{obs}$
	$m^{(k+1)} = m^{(k)} + \alpha \delta m$

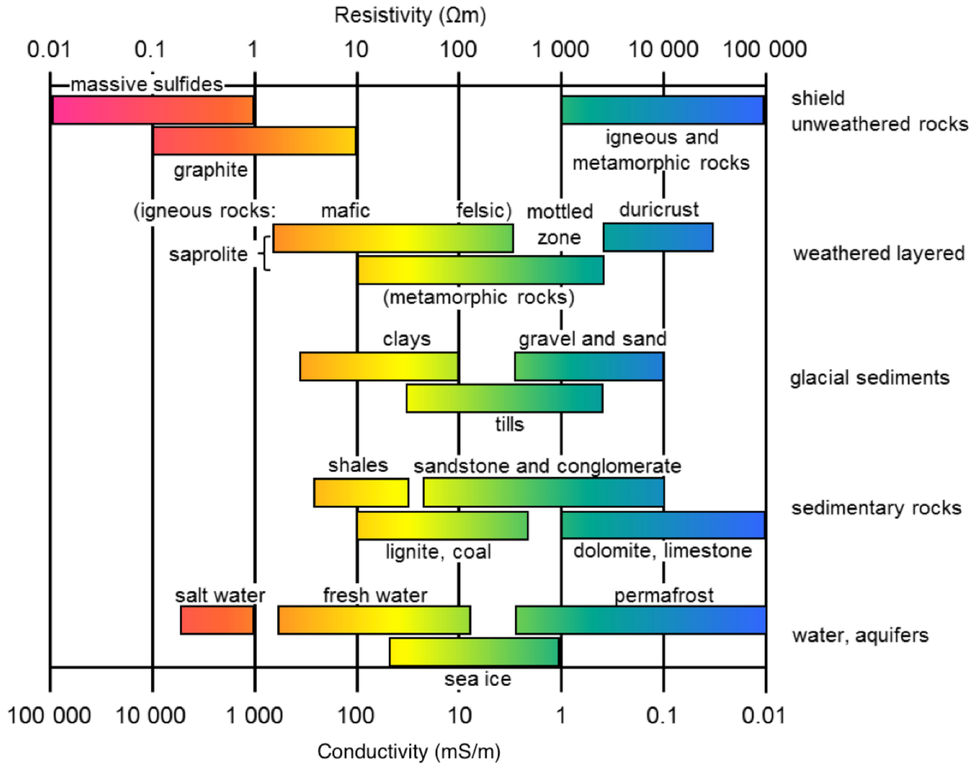
All understanding from linear problems is valid for nonlinear problems

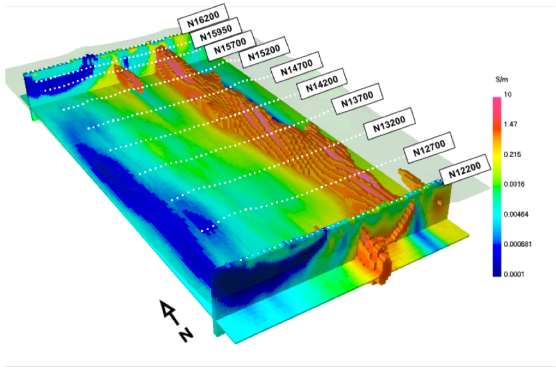
DC resistivity



Electrical conductivity

- DC resistivity is sensitive to:
  - σ: Conductivity [S/m]
  - ρ: Resistivity [Ωm]
  - $\sigma = 1/\rho$
- Varies over many orders of magnitude
- Depends on many factors:
  - Rock type. porosity, fluid…
- Has many applications: Mineral exploration, ground water, geotechnical, …:



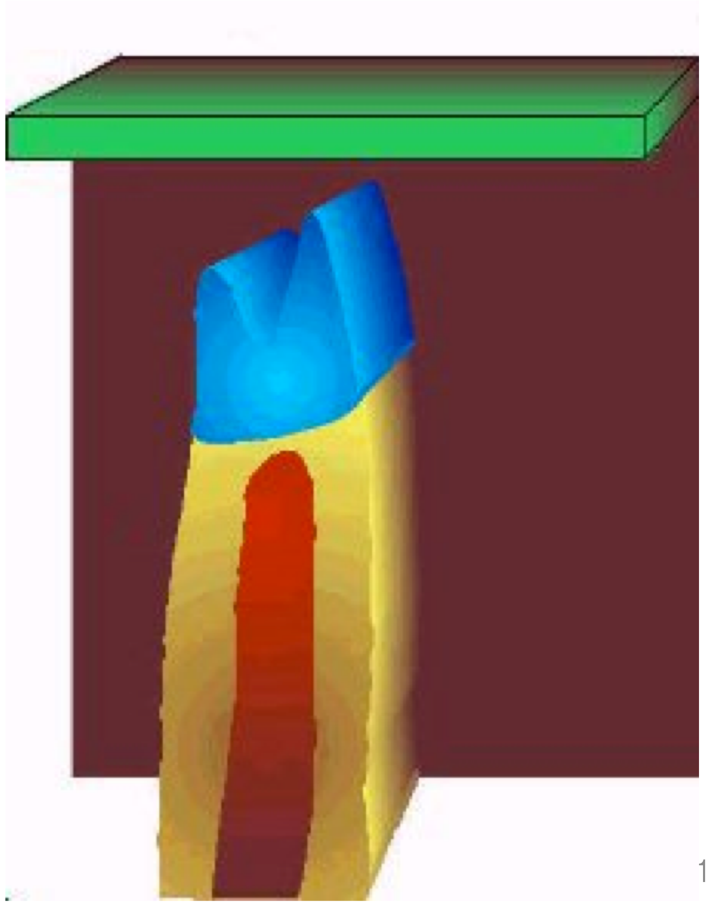


Basic Experiment

- Target:
  - Ore body. Mineralized regions less resistive than host

Elura Orebody Electrical resistivities

Rock Type	*Ohm-m*
Overburden	12
Host rocks	200
Gossan	420
Mineralization (pyritic)	0.6
Mineralization (pyrrhotite)	0.6



Basic Experiment

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- **Setup:**
    - Tx: Current electrodes
    - Rx: Potential electrodes

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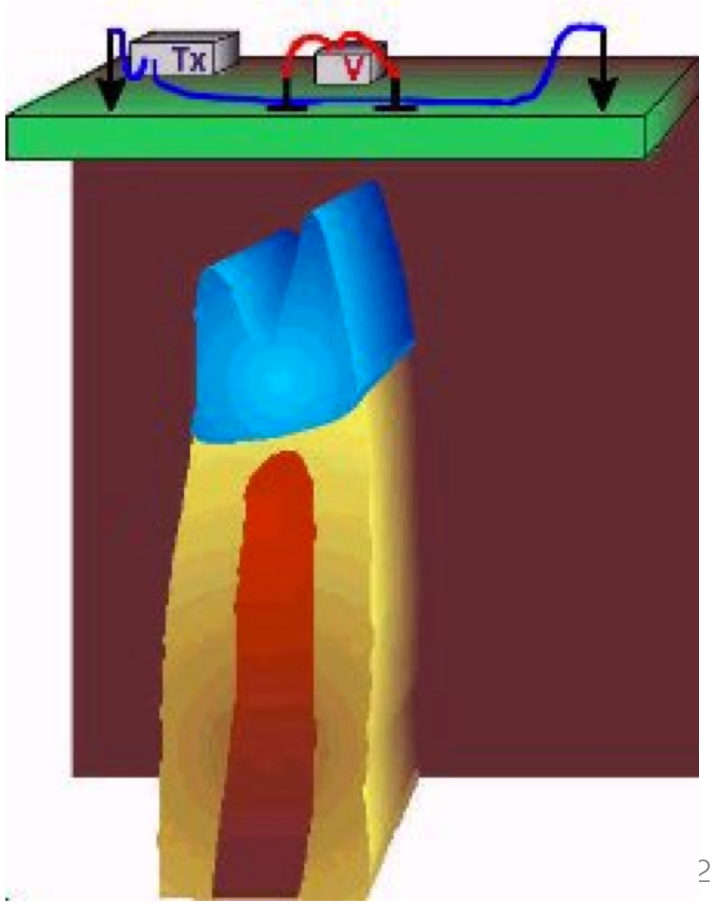

Basic Experiment

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  - Preferentially flow through conductors

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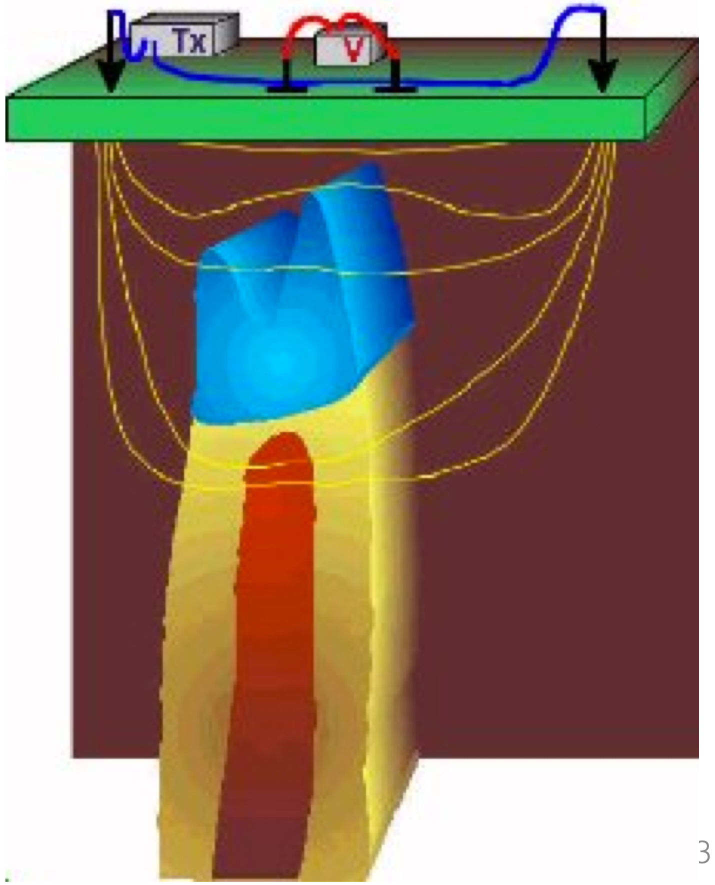

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- Target:
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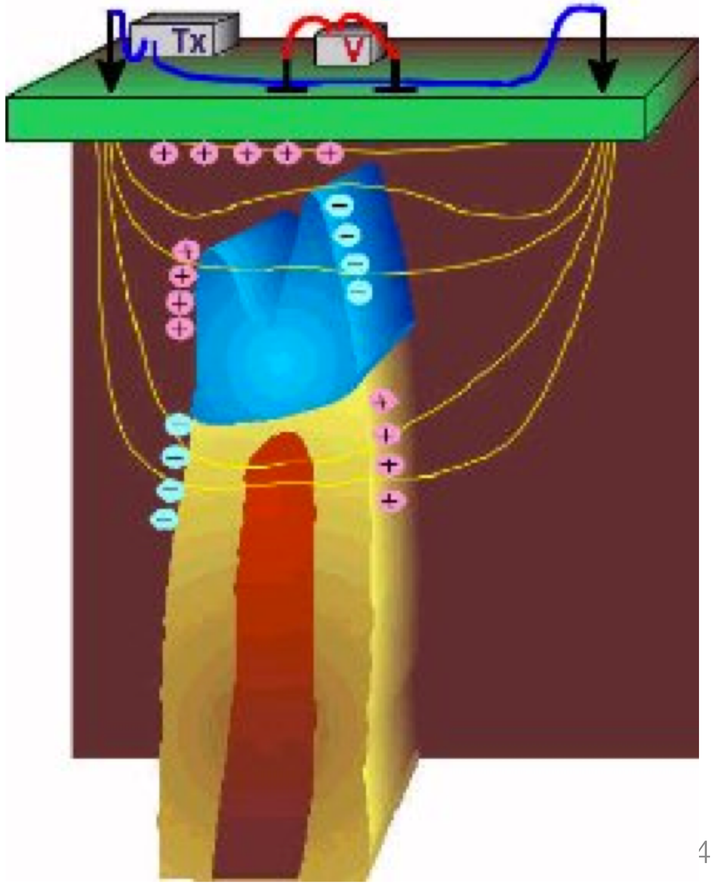

Basic Experiment

- Target:
  - Ore body. Mineralized regions less resistive than host
- Setup:
  - Tx: Current electrodes
  - Rx: Potential electrodes
- Currents:
  - Preferentially flow through conductors
- Charges:
  - Build up at interfaces
- Potentials:
  - Associated with the charges are measured at the surface

Elura Orebody Electrical resistivities

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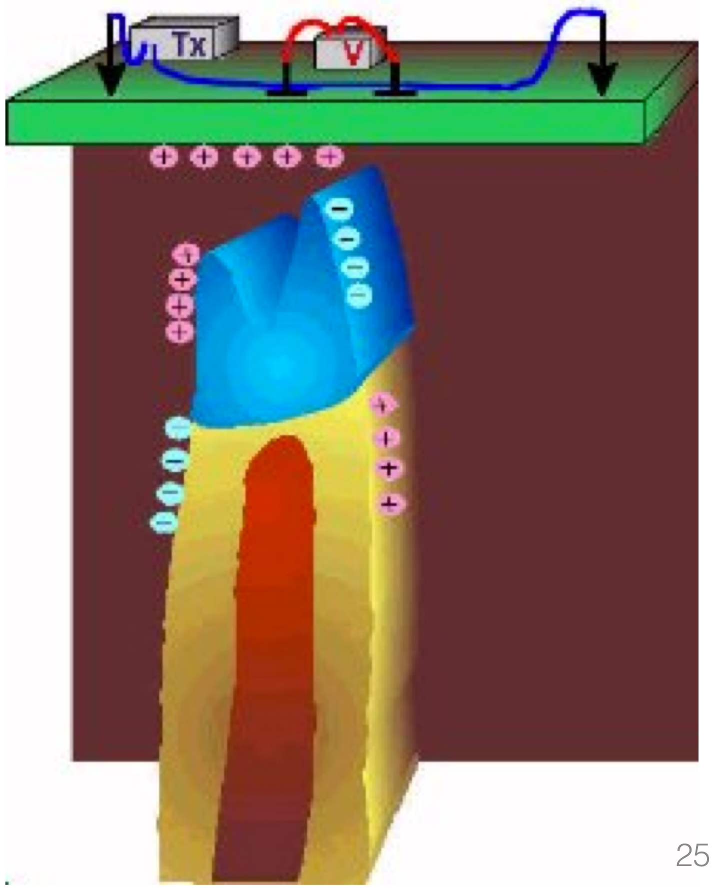
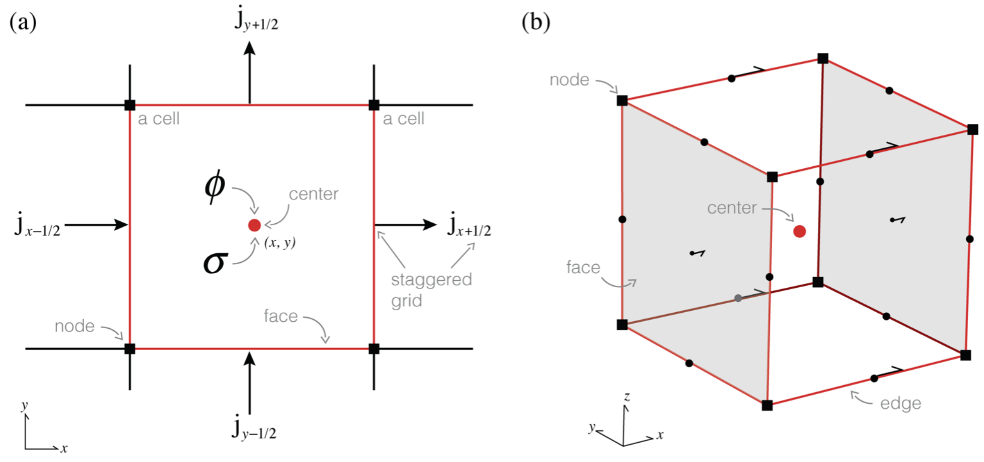


Forward problem



Continuous

$$\nabla \cdot \sigma \nabla V = I\delta(r) = q$$

$$\vec{j} = \sigma \vec{e}$$

$$\vec{e} = -\nabla V$$

$$\vec{j} \cdot \hat{n} = 0 \text{ at boundary}$$

Discrete (FV)

$$\mathbf{G}^T \mathbf{M}_\sigma \mathbf{G} \mathbf{u} = \mathbf{q}$$

generically,

$$\mathbf{A}(\mathbf{m})\mathbf{u} = \mathbf{q}$$

$\mathbf{G}$: gradient matrix
$\mathbf{M}_\sigma$: conductivity inner product matrix
$\mathbf{q}$: source term

Forward problem

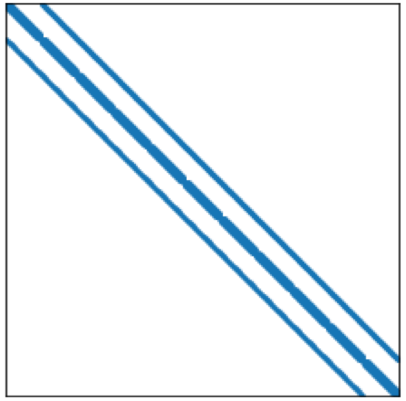
$$\mathbf{A}(\mathbf{m})\mathbf{u} = \mathbf{q}$$

$\mathbf{A}$: system matrix $(nN \times nN)$
$\mathbf{m}$: model $(M \times 1)$
$\mathbf{u}$: potentials to solve $(nN \times 1)$
$\mathbf{q}$: righthand side $(nN \times 1)$

- Solve forward problem:

$$\mathbf{u} = \mathbf{A}(\mathbf{m})^{-1}\mathbf{q}$$

- Matrix $\mathbf{A}$:
  - Sparse
  - Symmetric
  - Positive definite
  - Real-valued (can be complex sometimes)



Use iterative or direct solver

Inverse problem

- Solve $$(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$$

Predicted data: $d = \mathcal{F}[m]$

Residual: $\delta d = \mathcal{F}[m] - d^{obs}$

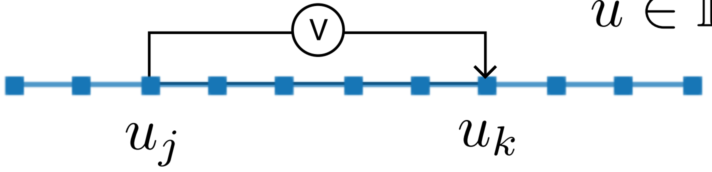
- How to compute sensitivity matrix J

$$J_{ij} = \frac{\partial d_i}{\partial m_j}$$

- Forward modelling: $A(m)u = q$

u: potential on the nodes

$u \in \mathbb{R}^{nN}$



- data are potential difference between two nodes

$$P = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ \vdots & & \ddots & & & & \vdots \end{pmatrix}$$

$$d = Pu$$

P: projection matrix

Inverse problem: computing sensitivity

- Find $\nabla_m u(m)$

$$A(m)u = q$$

$$d = Pu$$

$$J_{ij} = \frac{\partial d_i}{\partial m_j}$$

$$i = 1, 2, \ldots, N$$
$$j = 1, 2, \ldots, M$$

- Chain rule:

$$\nabla_m \big[A(m)u(m)\big] = \underbrace{\nabla_m \big[A(m)u(m)_{\text{fixed}}\big]}_{\mathcal{G}(m,u)} + A(m)\nabla_m u(m) = 0$$

$$\mathcal{G}(m,u) + A(m)\nabla_m u(m) = 0$$

$$\nabla_m u(m) = -A(m)^{-1}\mathcal{G}(m,u)$$

$$\boxed{\begin{gathered} J = -PA(m)^{-1}\mathcal{G}(m,u) \\ J \in \mathbb{R}^{M \times N} \end{gathered}}$$

Examine sensitivity

$$J = -PA(m)^{-1}\mathcal{G}(m,u)$$

P: projection matrix
$A(m)^{-1}$: forward modelling

- For DC problem

$$\mathbf{A} = \mathbf{G}^T\mathbf{M}_\sigma\mathbf{G}$$

$$\mathbf{M}_\sigma = \mathbf{diag}\big(\mathbf{Av}^T\mathbf{diag}(\boldsymbol{\sigma}\odot\mathbf{vol})\big)$$

$$\mathcal{G}(m,u) = \nabla_m\big[A(m)u(m)_{\text{fixed}}\big] = \mathbf{G}^T\mathbf{diag}\big(\mathbf{Gu}\big)\mathbf{Av}^T\mathbf{diag}(\mathbf{vol})$$

$\mathbf{Av}$: averaging matrix
$\mathbf{vol}$: volume of cells

Putting everything together

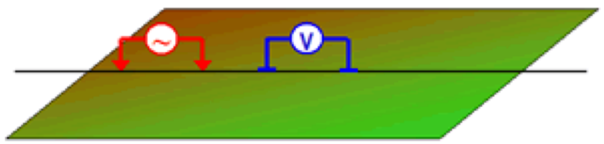
- Solve $$(J^T J + \beta)\delta m = -(J^T \delta d + \beta m)$$

- Using CG we need: $$J^T y \qquad y \in \mathbb{R}^N$$ $$J v \qquad v \in \mathbb{R}^M$$

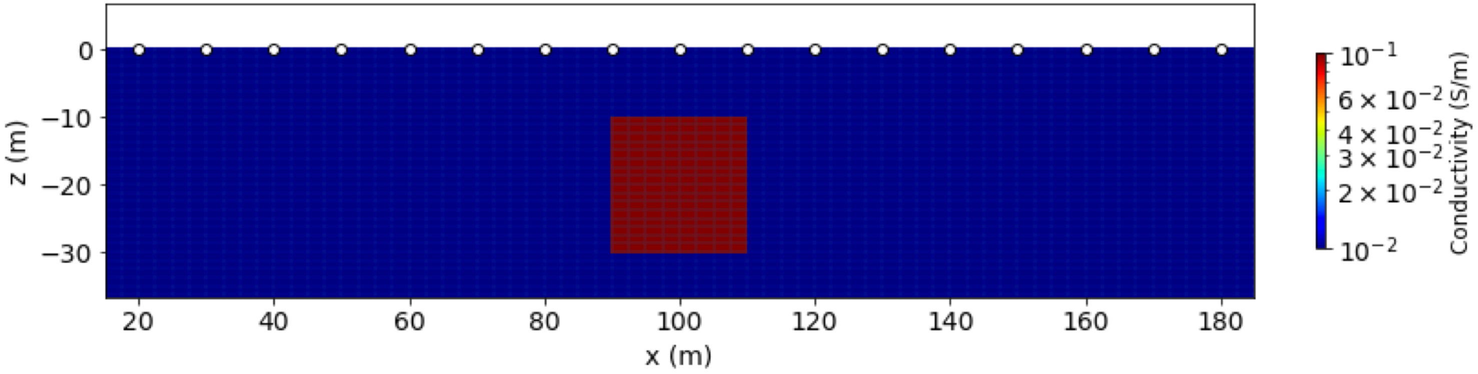
- but, $J = -PA(m)^{-1}\mathcal{G}(m,u)$ all sparse matrices

Example 2D DC resistivity

Dipole-Dipole

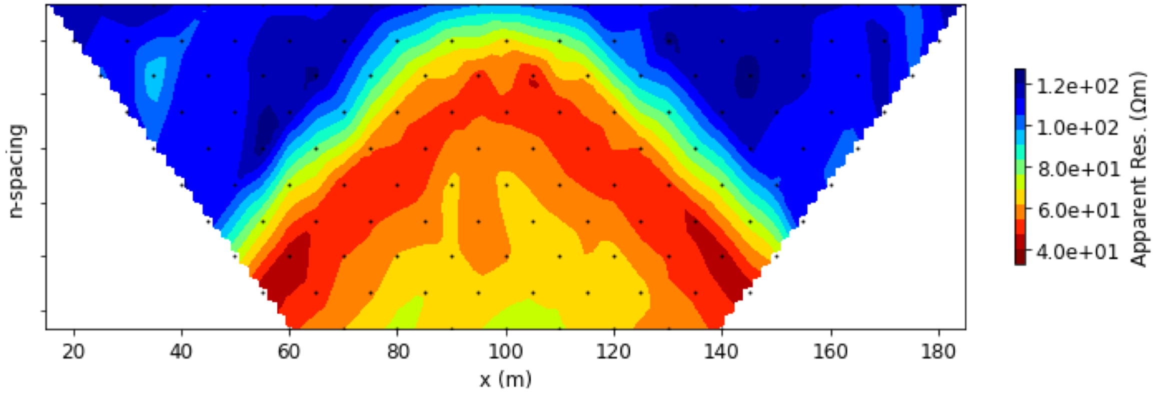


Conductivity model

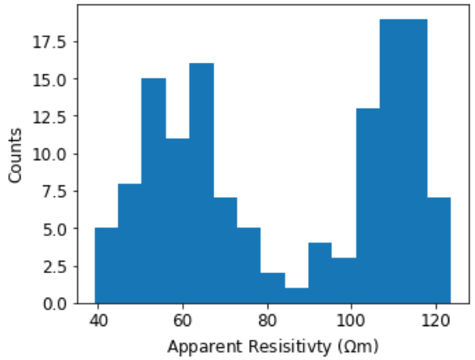


- Dipole-dipole array
  - n-spacing = 10
  - Electrode-spacing = 10m
  - # of data = 135
- 5% Gaussian noise added

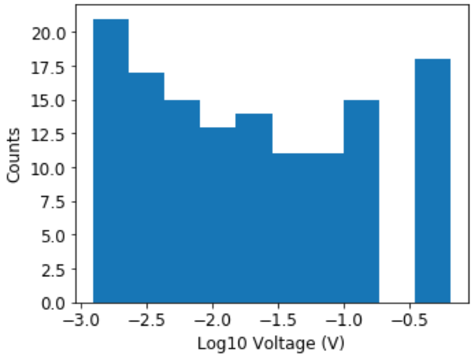
Apparent resistivity pseudo-section



Data histogram

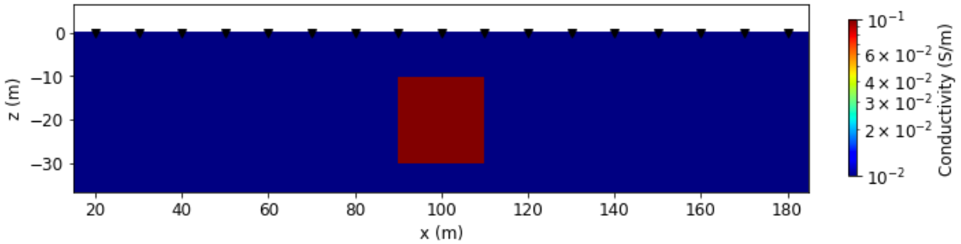


Voltage

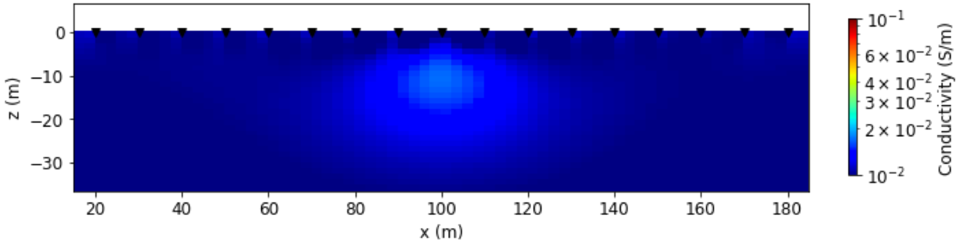


DC inversion: iteration 1

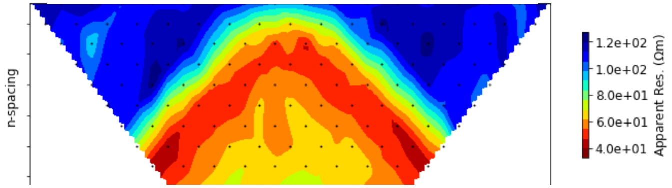
True model



Predicted model



Observed data



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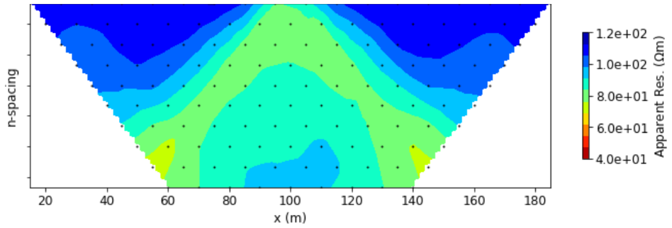
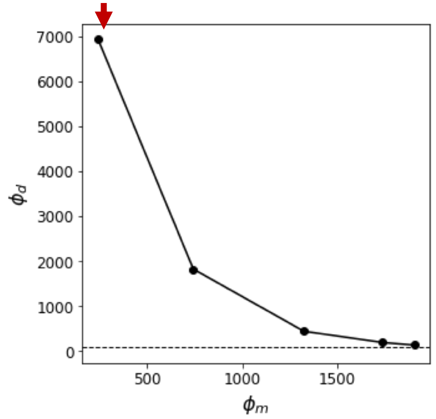
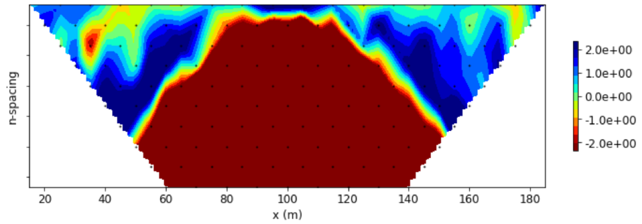


Tikhonov curve

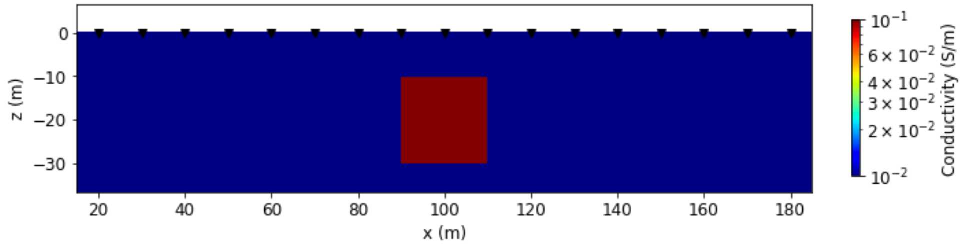


Normalized misfit

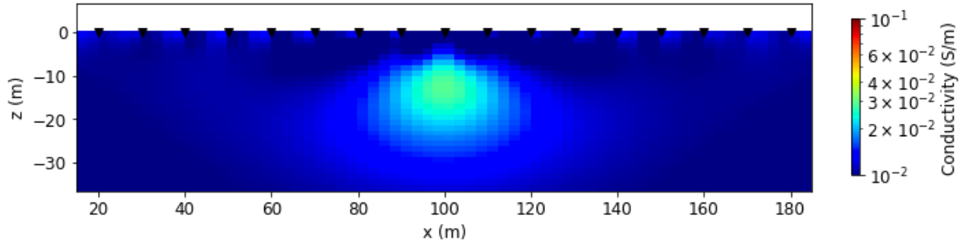


DC inversion: iteration 2

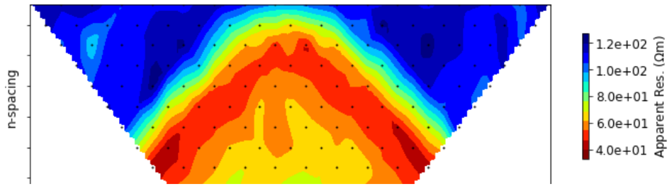
True model



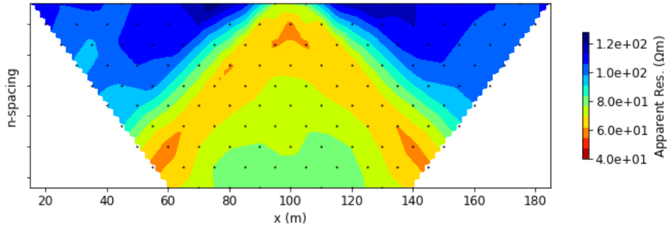
Predicted model



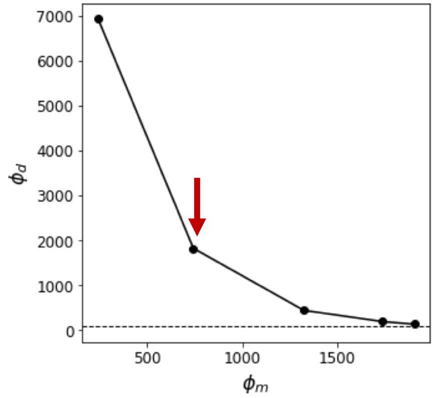
Observed data



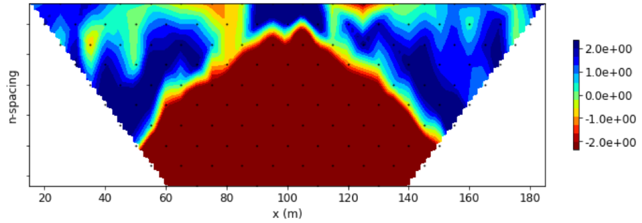
Predicted data



Tikhonov curve

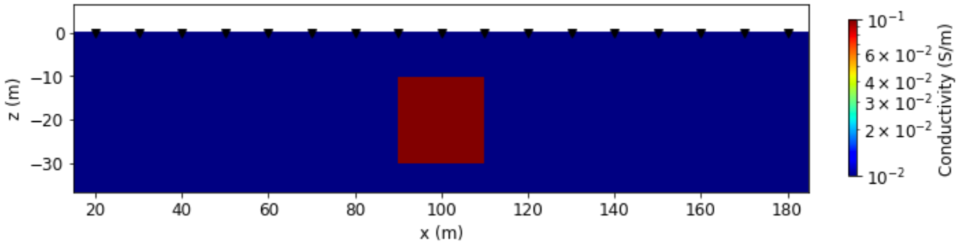


Normalized misfit

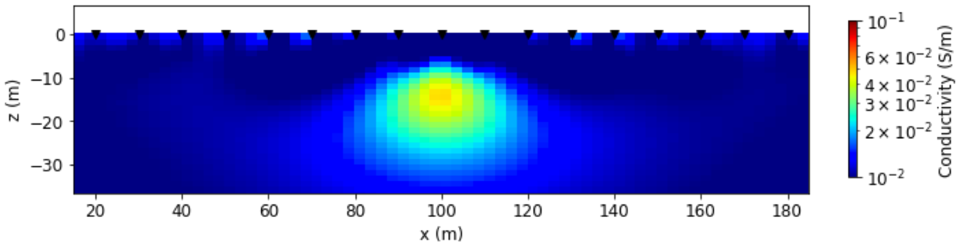


DC inversion: iteration 3

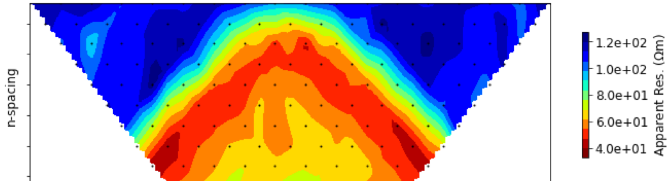
True model



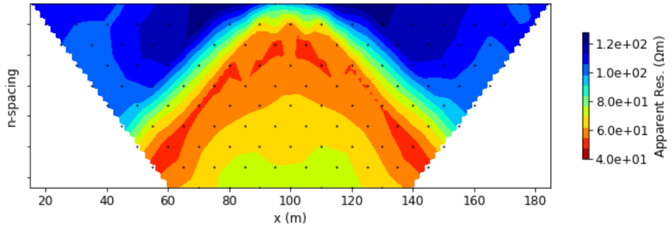
Predicted model



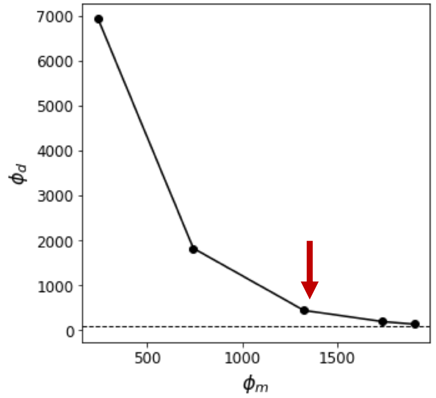
Observed data



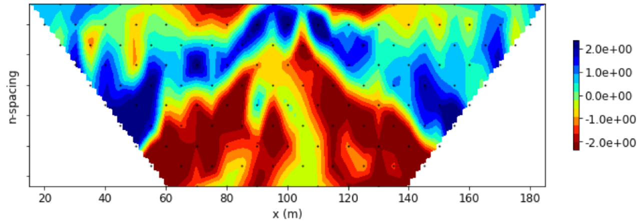
Predicted data



Tikhonov curve

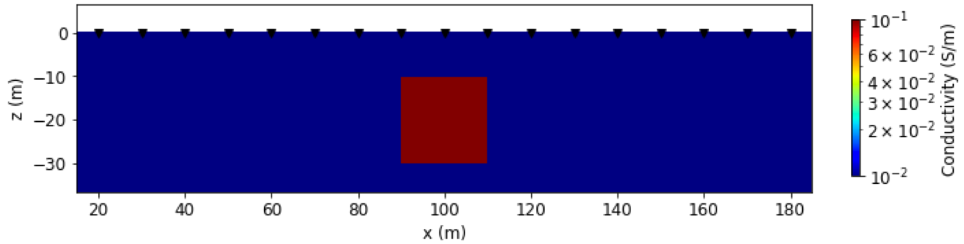


Normalized misfit

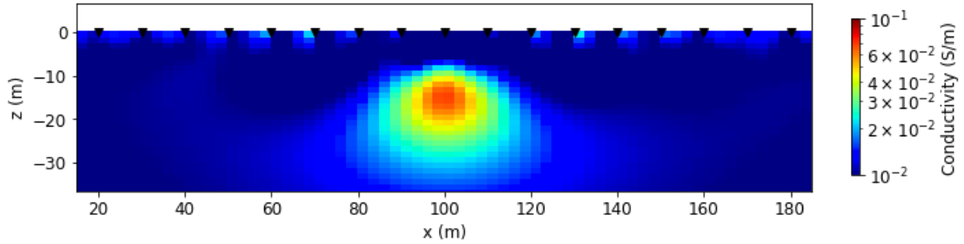


DC inversion: iteration 4

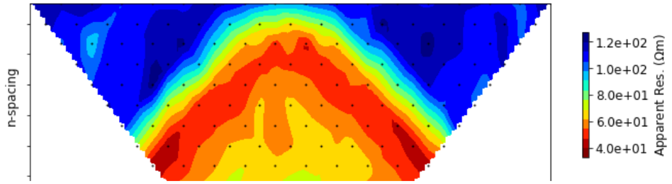
True model



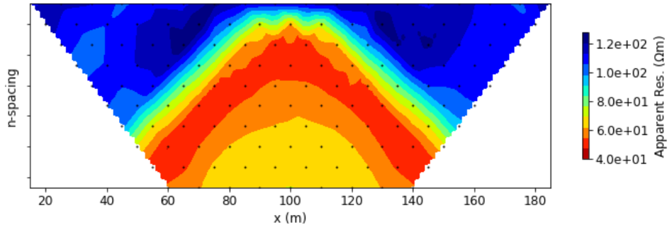
Predicted model



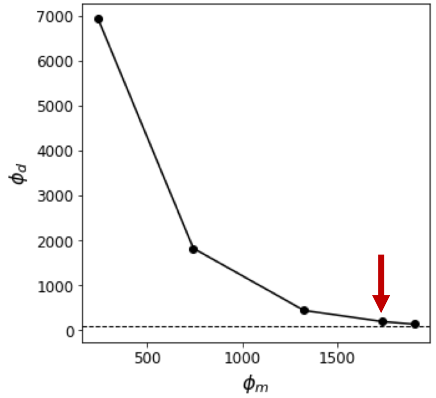
Observed data



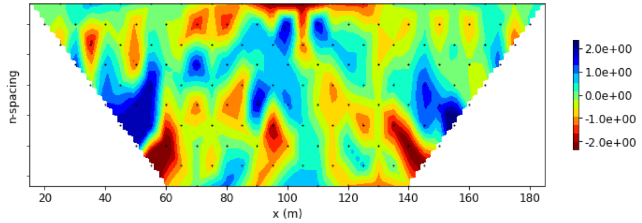
Predicted data



Tikhonov curve

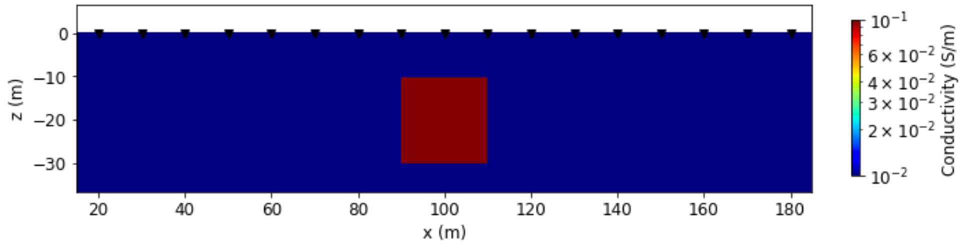


Normalized misfit

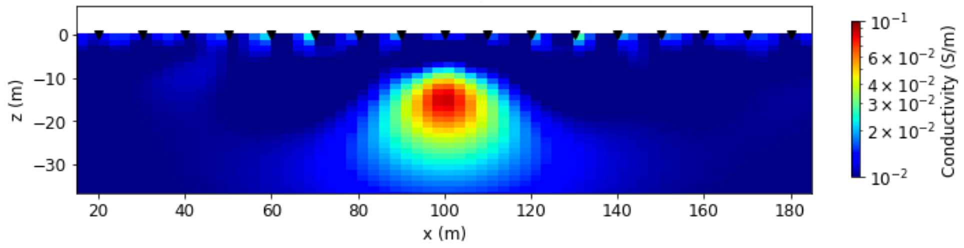


DC inversion: iteration 5

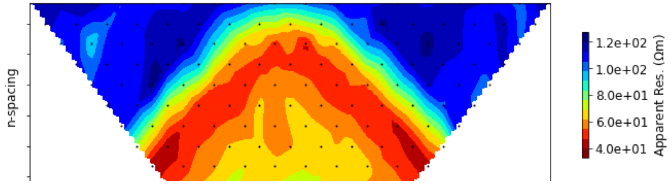
True model



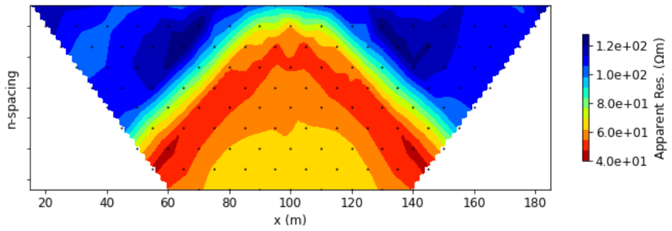
Predicted model



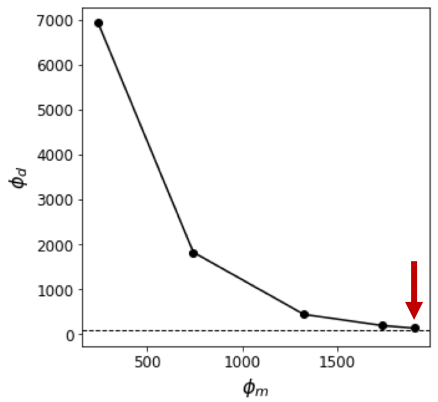
Observed data



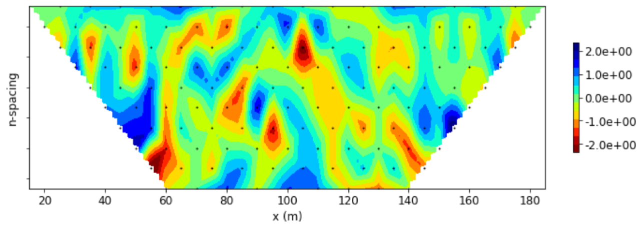
Predicted data



Tikhonov curve

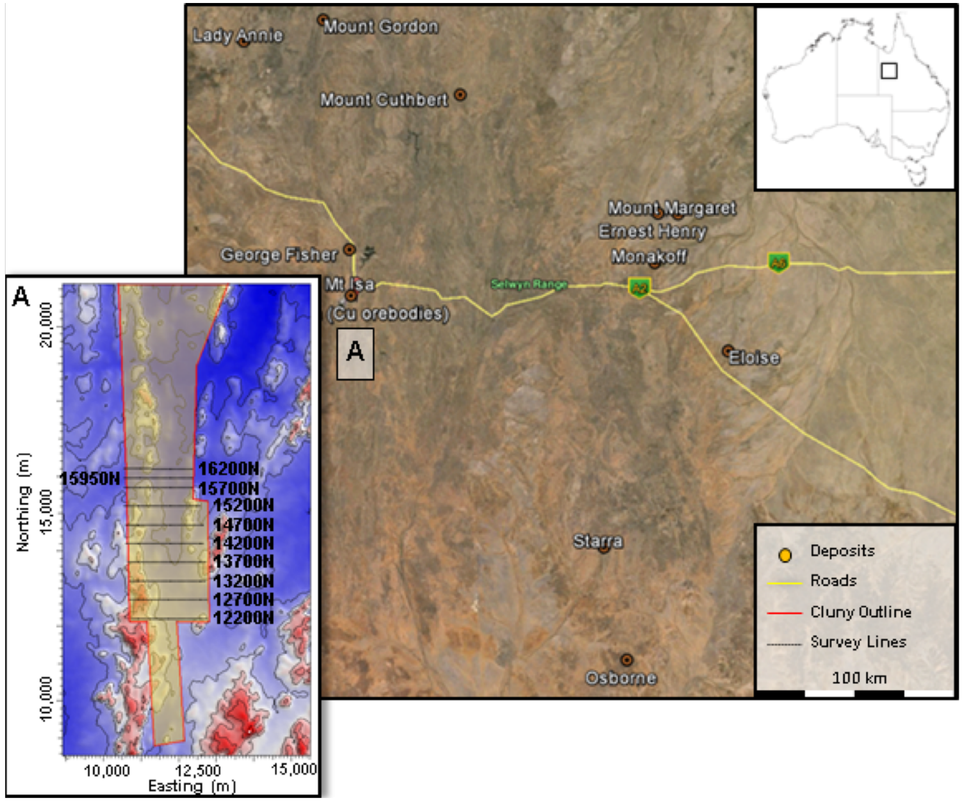


Normalized misfit

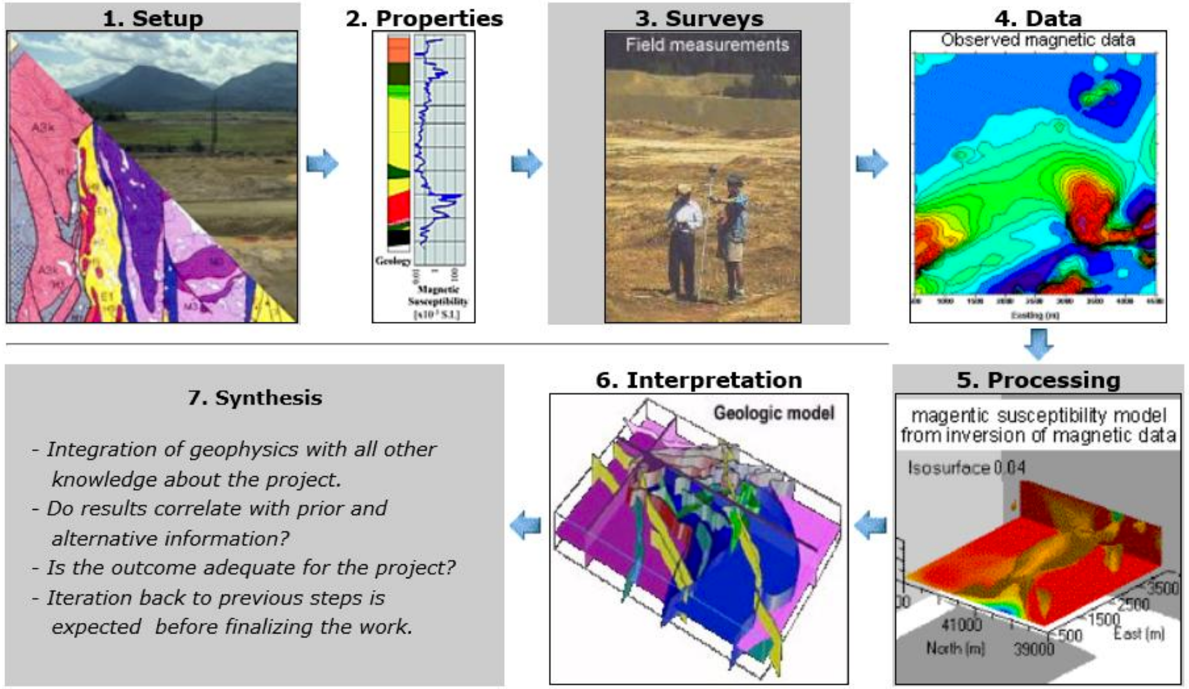


DCR Case History: Mt. Isa

Mt. Isa (Cluny prospect)

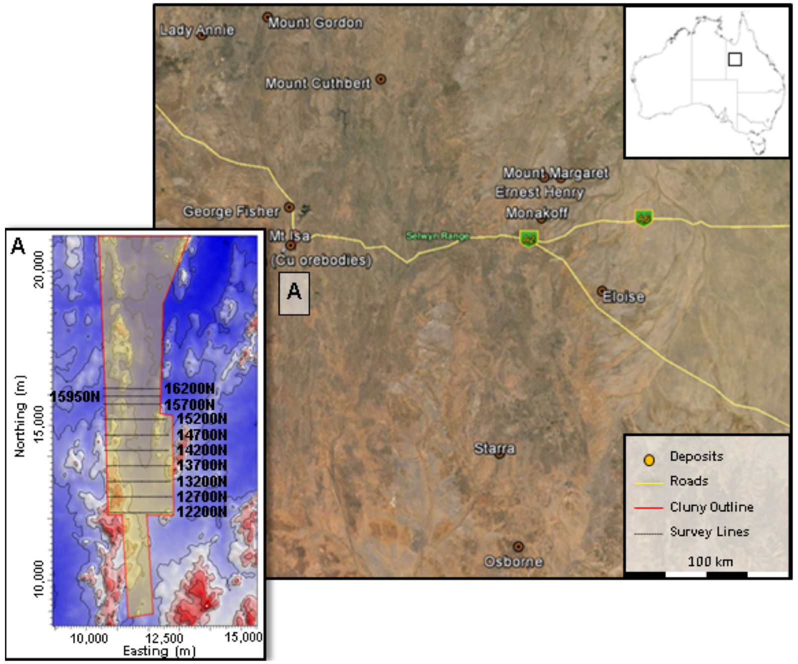


Seven Steps

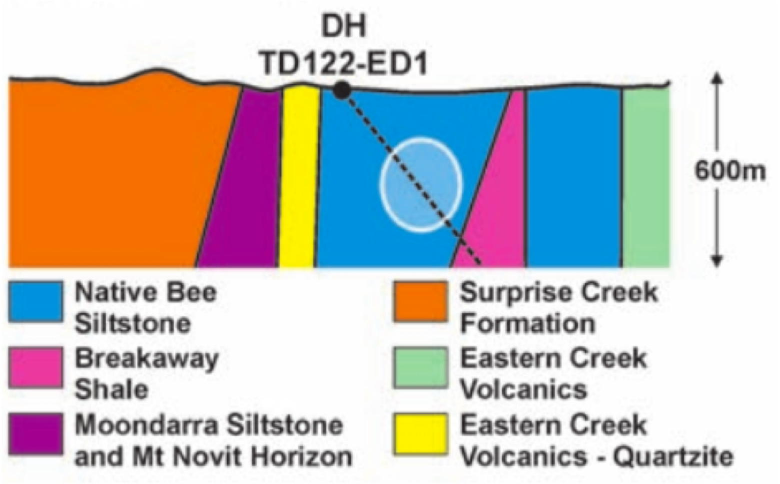


Setup

Mt. Isa (Cluny prospect)



Geologic model

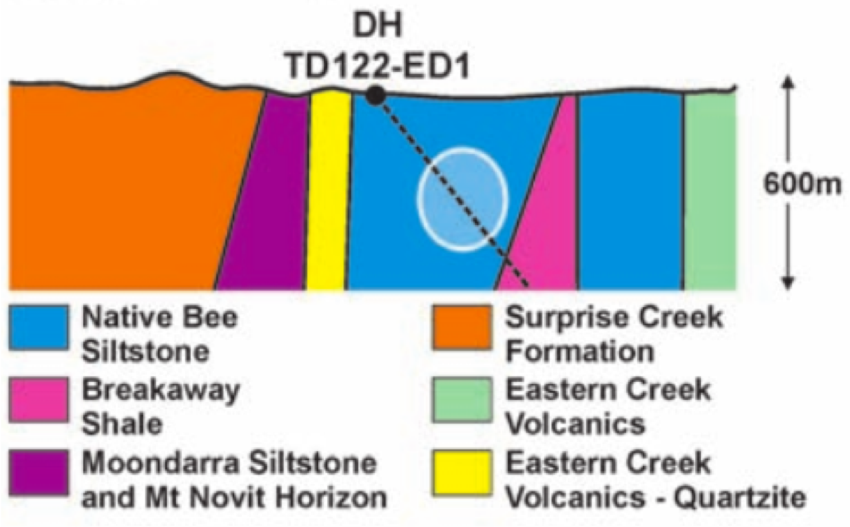


Question

- Can conductive units, which would be potential targets within the siltstones, be identified with DC data?

Properties

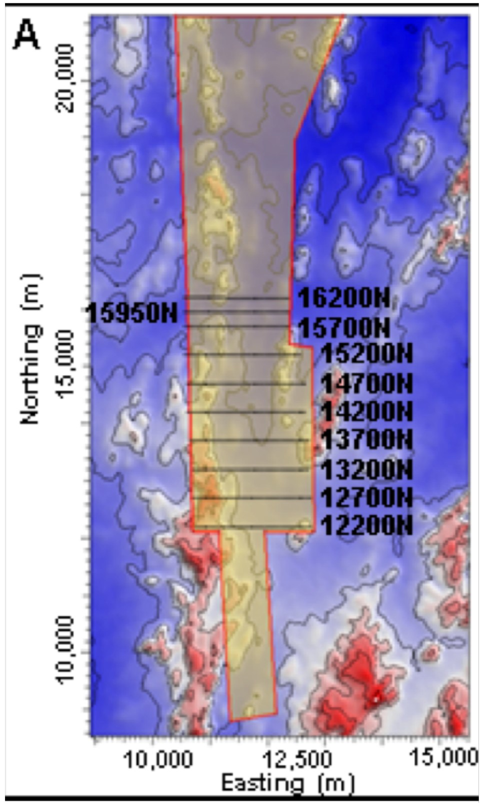
Geologic model



Conductivity table

Rock Unit	Conductivity
Native Bee Siltstone	Moderate
Moondarra Siltstone	Moderate
Breakaway Shale	Very High
Mt Novit Horizon	High
Surprise Creek Formation	Low
Eastern Creek Volcanics	Low

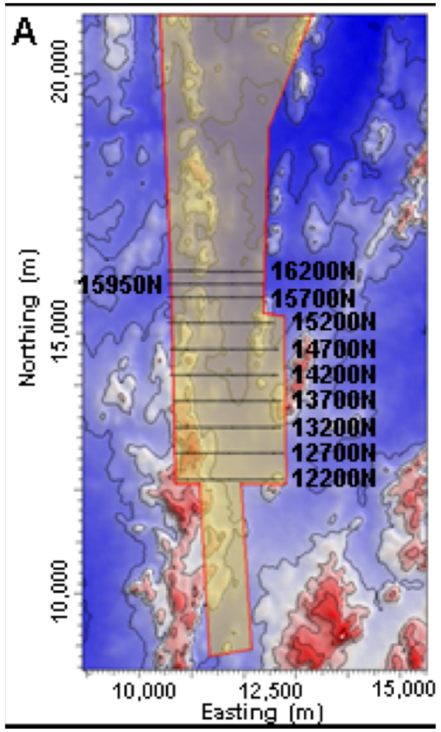
Surface topography



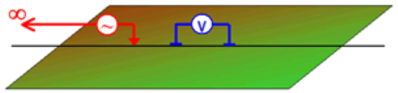
Survey and Data

- Eight survey lines
- Two survey configurations.

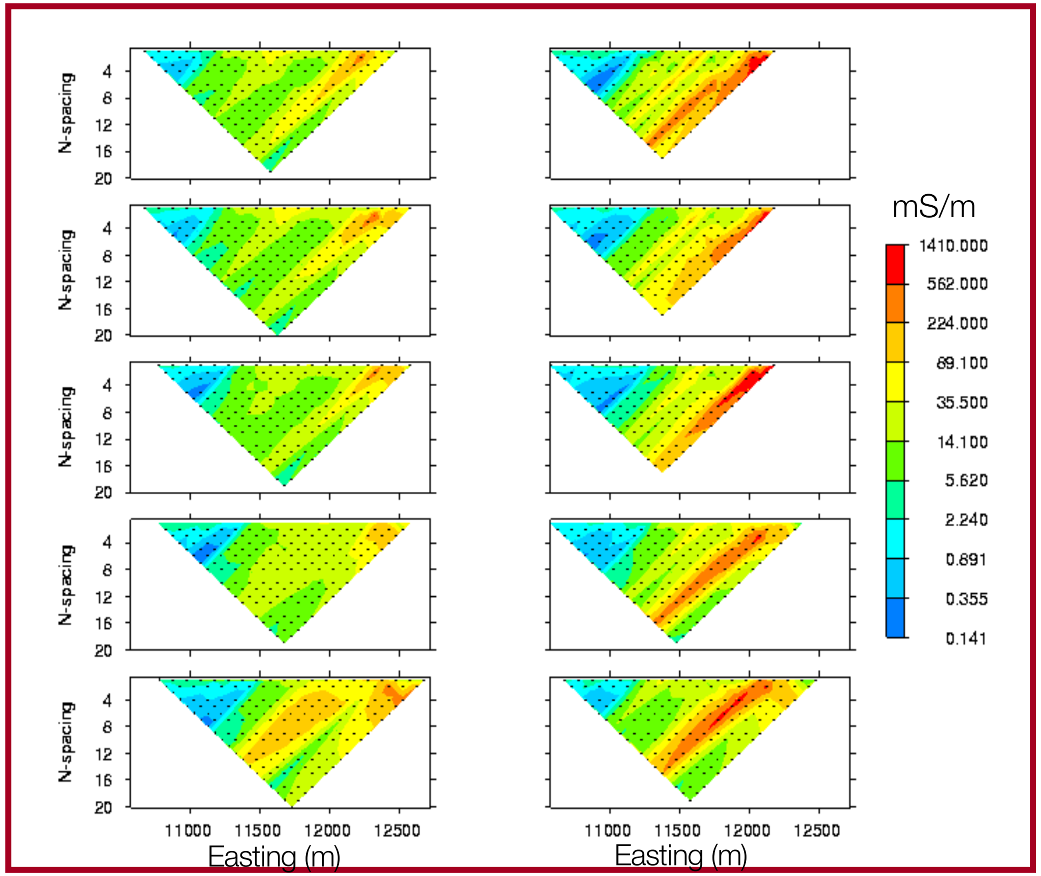
Surface topography



Data set #1:



Apparent resistivity, pole - dipole.



Survey and Data

- Eight survey lines
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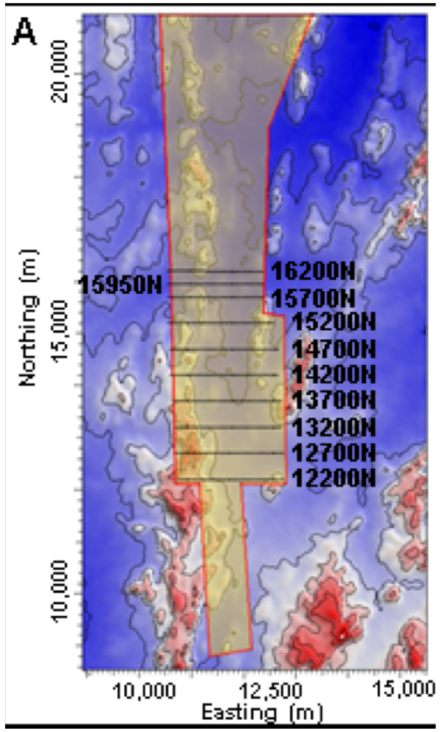
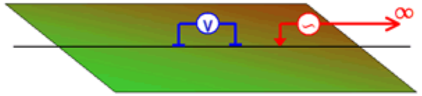
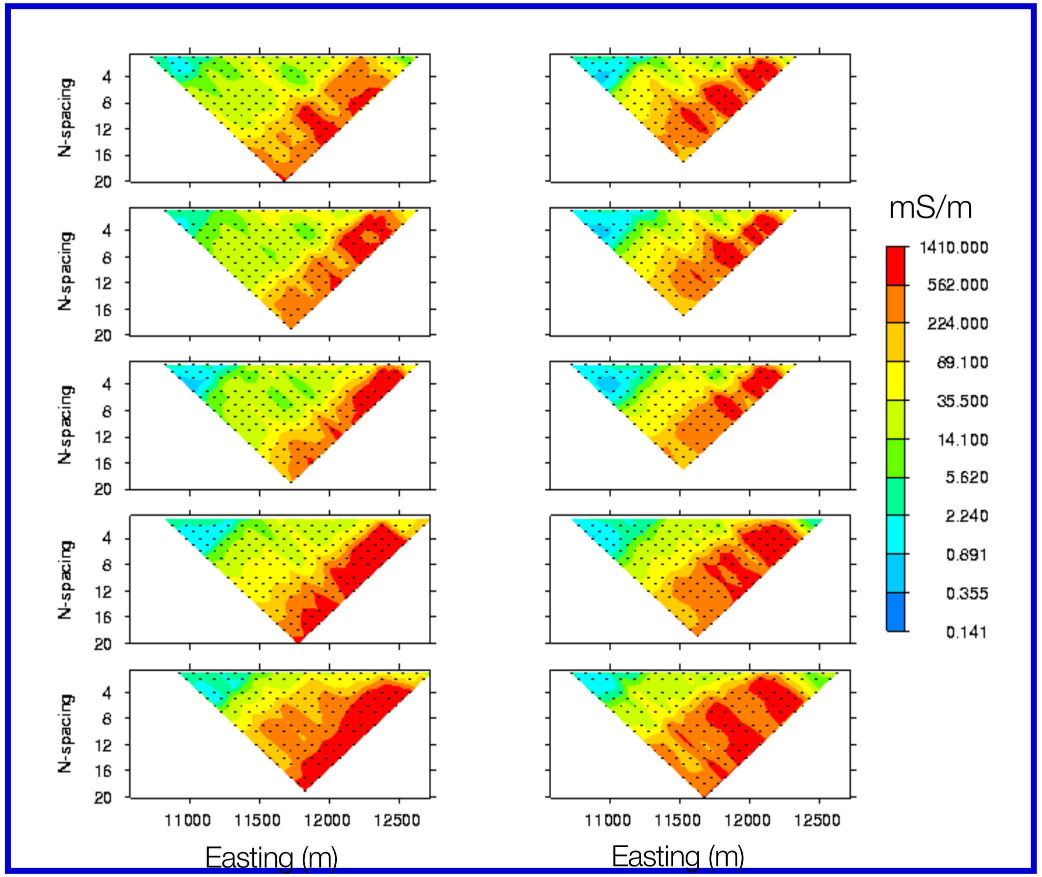
Surface topography



Data set #2:

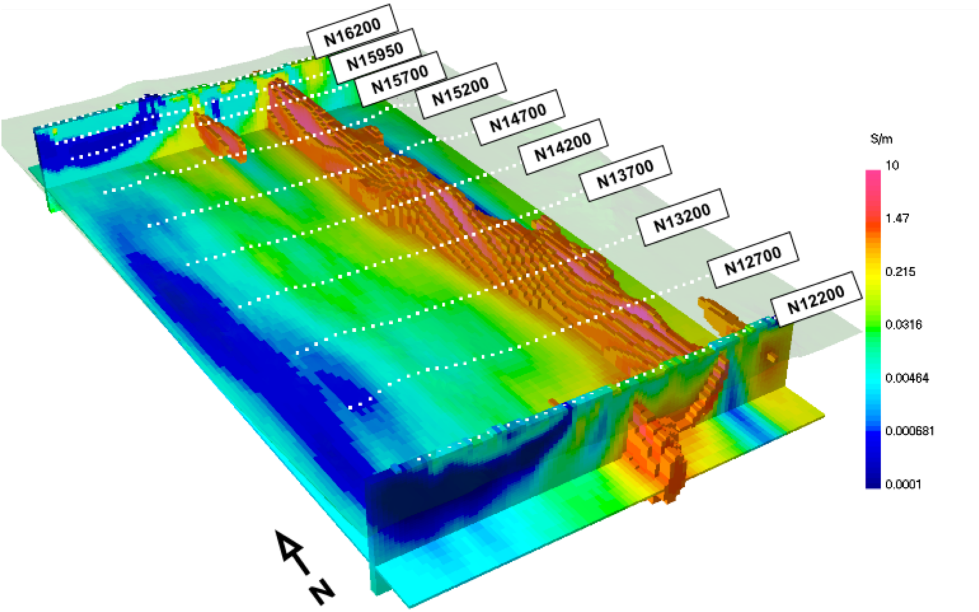


Apparent resistivity, dipole - pole

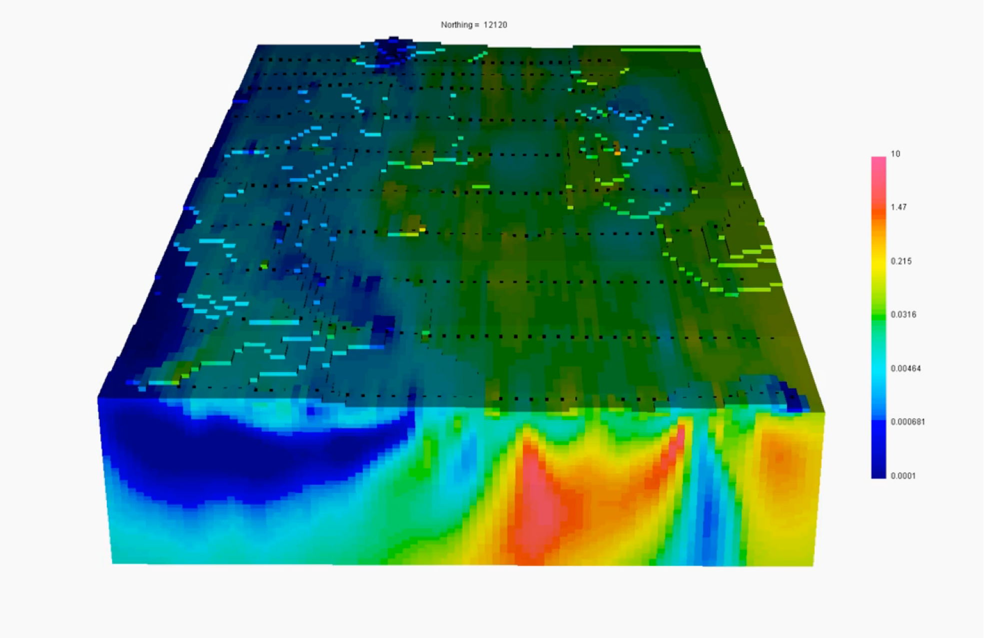


Processing and interpretation

3D resistivity model



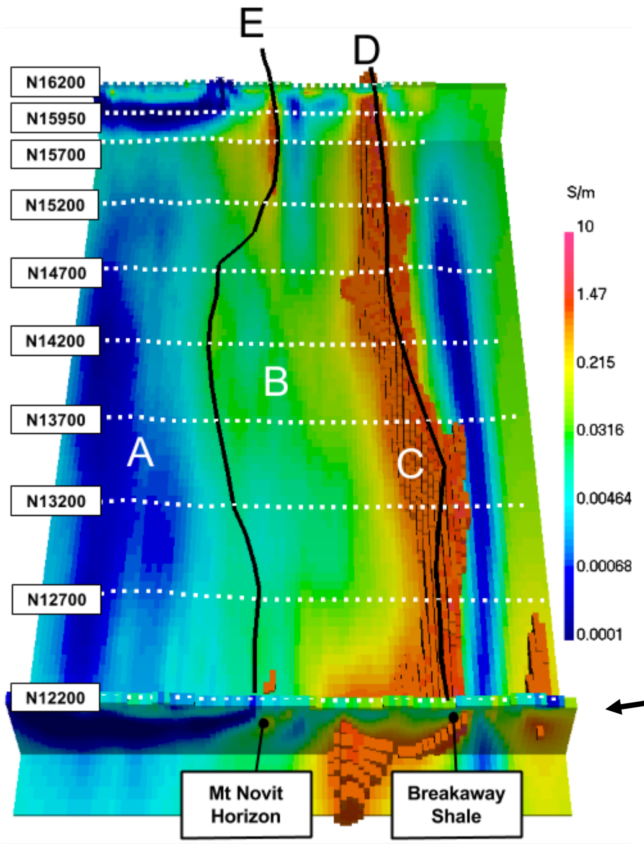
Animation



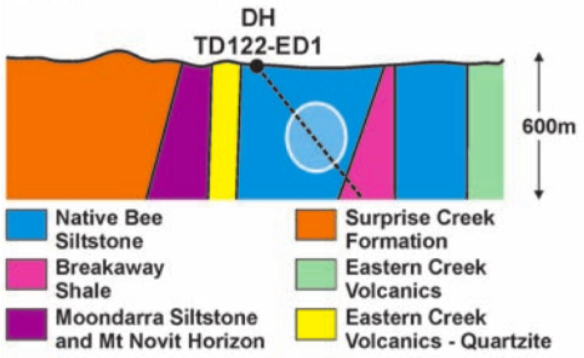
Synthesis

- Identified a major conductor → black shale unit
- Some indication of a moderate conductor

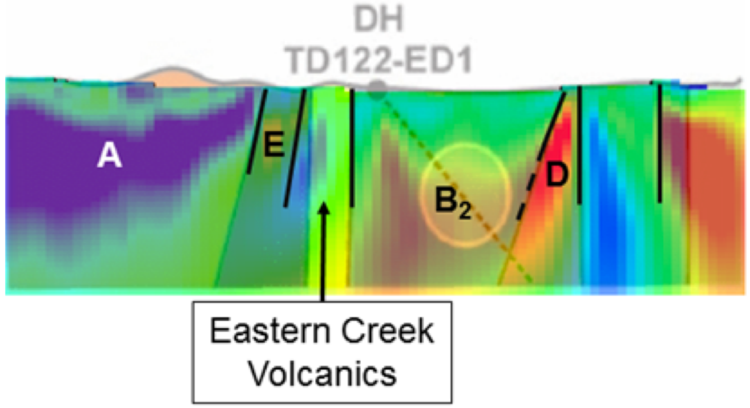
3D resistivity model



Geologic section



Resistivity section



Summary

- Non-linear forward problem
- Non-linear optimization (Newton's)
    - Quadratic convergence
- Example problem: DC resistivity
    - Physics
    - Discretization
    - Optimization
    - 2D synthetic inversion
    - 3D field example: Mt. Isa

Next up …


Overview
Tikhonov
SVD
Linear
Nonlinear
Lp-norms
Field scale
The future

The end