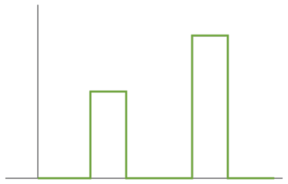
L_p norm inversion



Outline

- Why beyond L_2?
- L_p norms
- 2D cross-well tomography
- 3D magnetics
- Case history: 3D gravity at Kevitsa

L_p-norms and Constraints

$$\phi_d = \|\mathbf{W}_d(F[\mathbf{m}] - \mathbf{d}^{obs})\|_2^2$$

minimize $\phi = \phi_d + \beta\phi_m$

$F[\mathbf{m}]$: linear or non-linear function

$$\phi_m = \alpha_s \int_v (\mathbf{m} - \mathbf{m}_{\text{ref}})^2 dv + \sum_{j=x,y,z} \alpha_j \int_v \left(\frac{d}{dj}(\mathbf{m} - \mathbf{m}_{\text{ref}})^2\right) dj$$

Discretize $\phi_m = \sum_{j=s,x,y,z} \alpha_j \|\mathbf{W}_j(\mathbf{m} - \mathbf{m}_{\text{ref}})\|_2^2$

Change them and explore model space.
Is that enough?

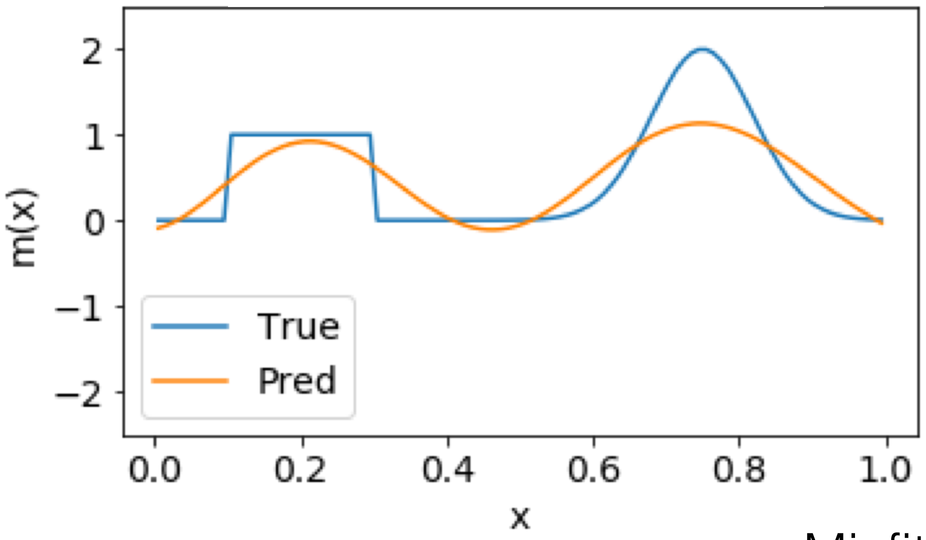
Flexibility

$\mathbf{m}_{\text{ref}}$ Prior knowledge

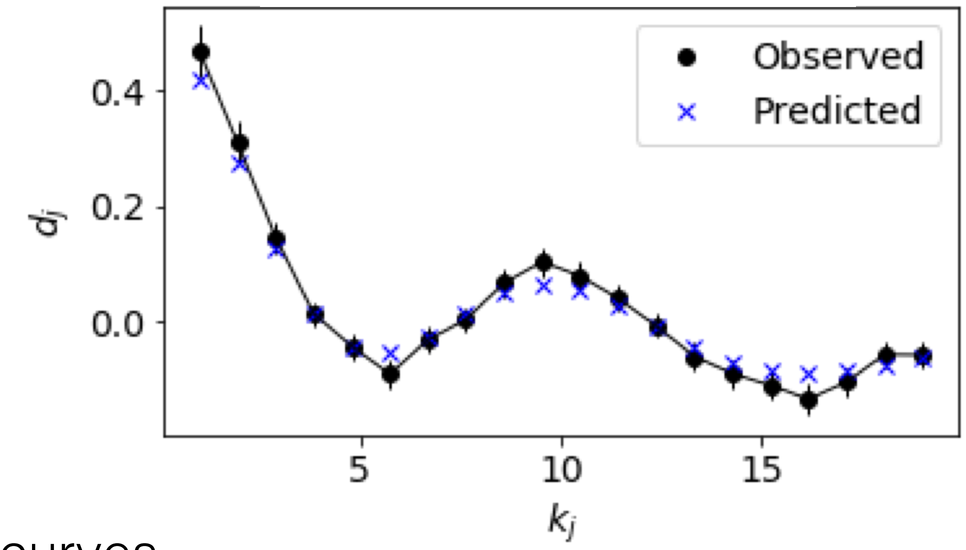
$\alpha_s, \alpha_x, \alpha_y, \alpha_z$ Relative importance of the norms

1D inversion

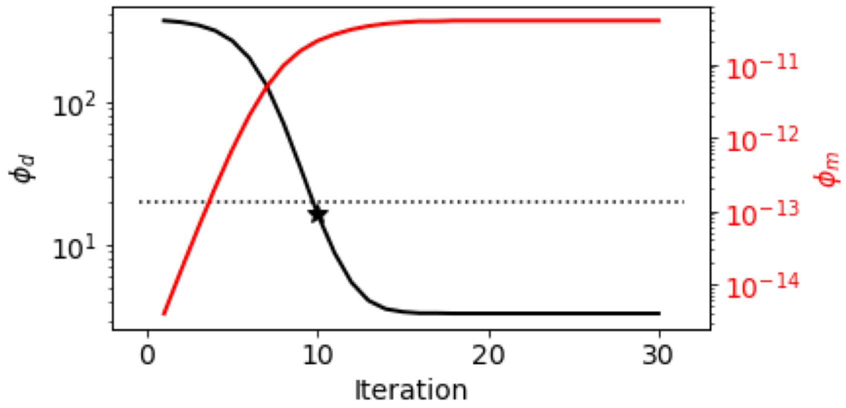
Model



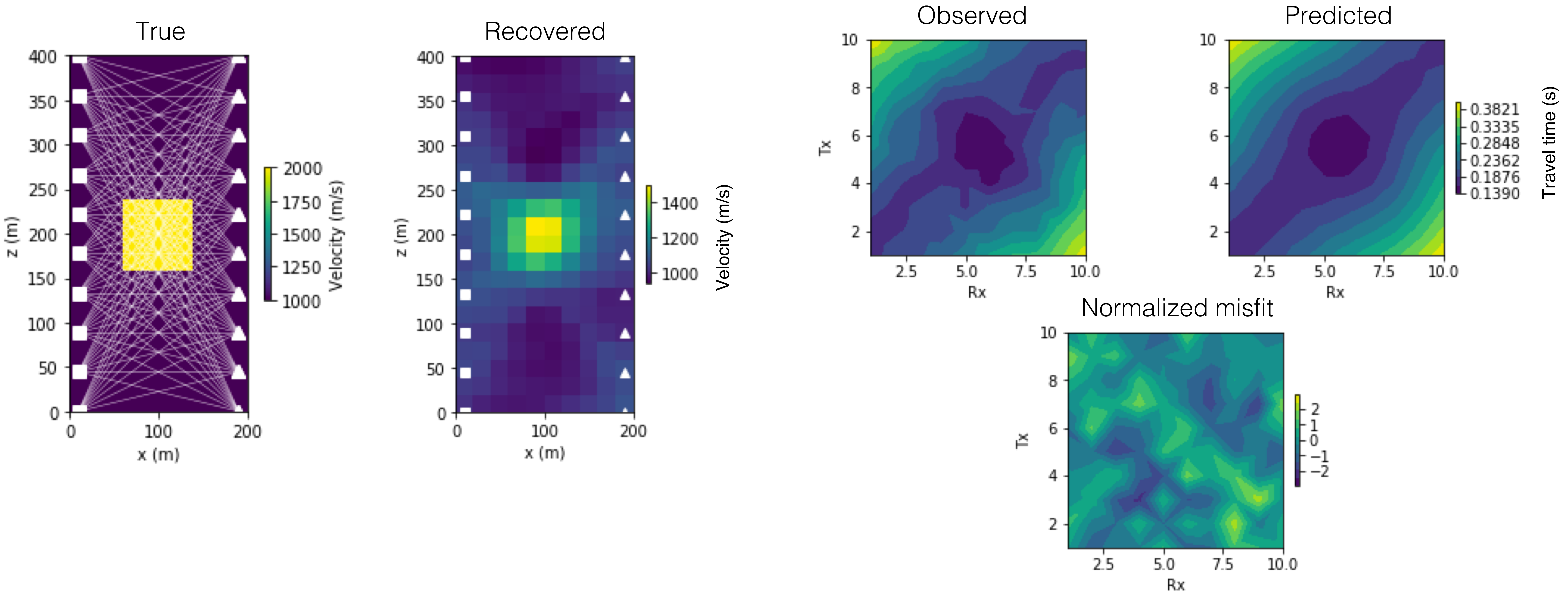
Data



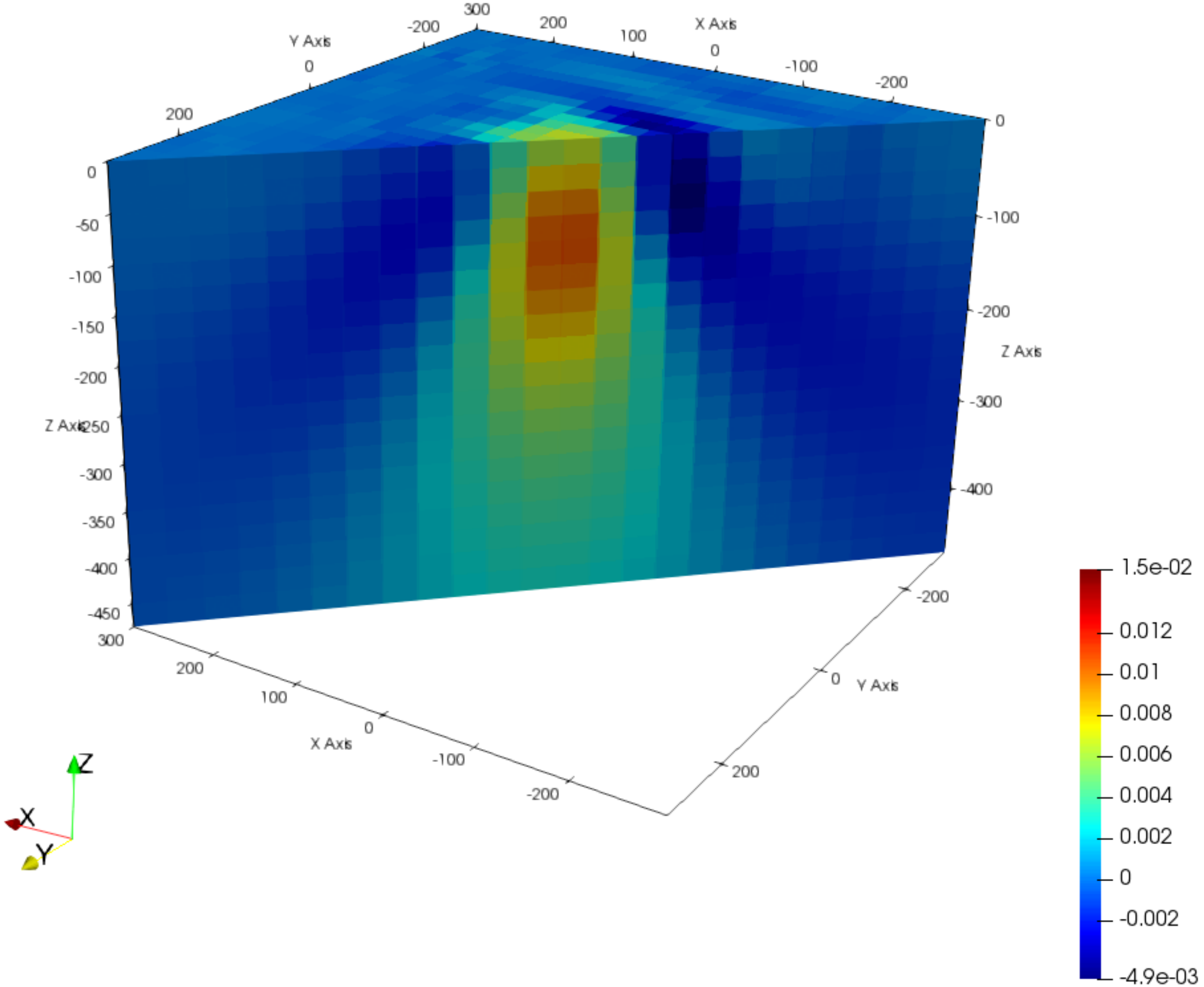
Misfit curves



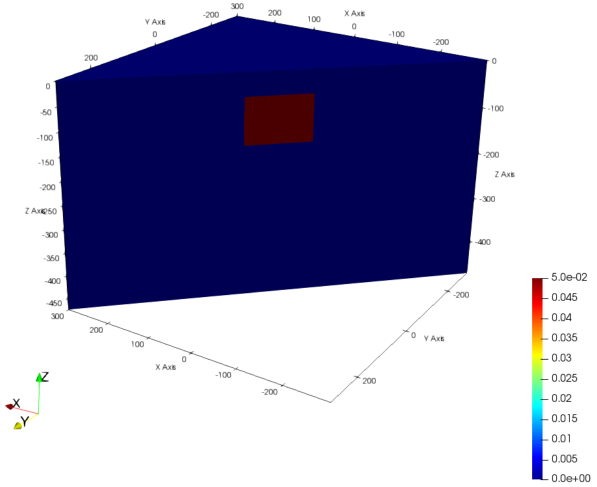
2D crosswell inversion



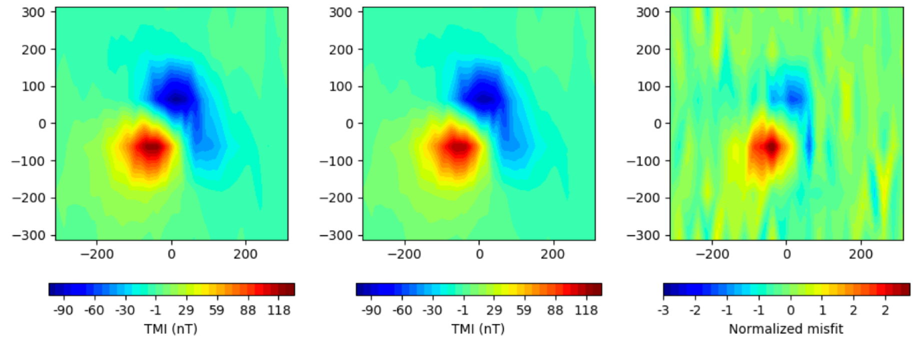
3D magnetic inversion



True model

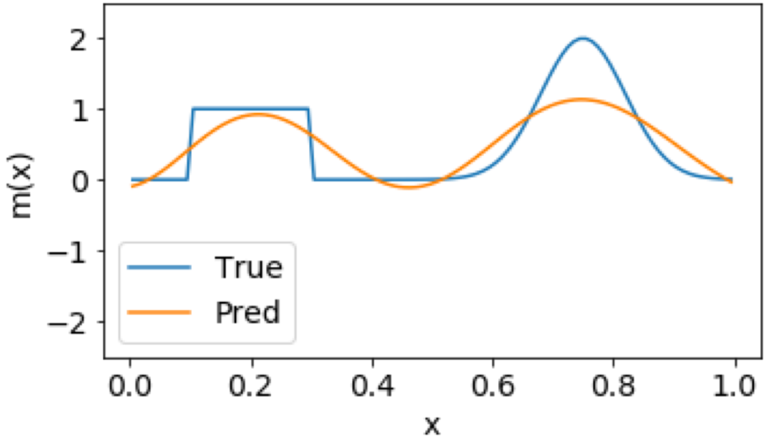


Obs. Pred. Misfit

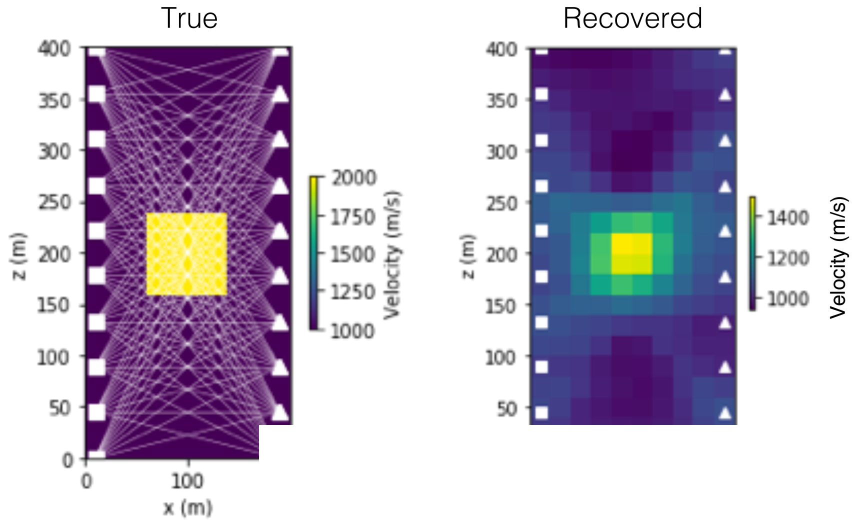


What is missing?

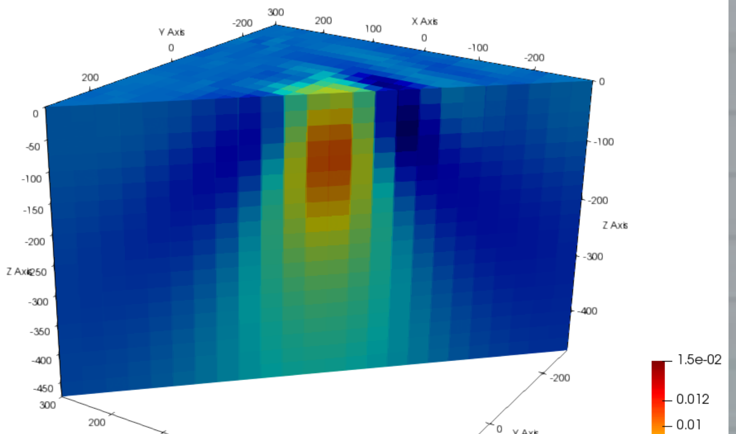
1D



2D



- L_2 norm: “smallness” and “smoothness”
  - Smooths sharp boundaries
  - May have artefacts
  - Violate physics (negative susceptibility)



What is missing?

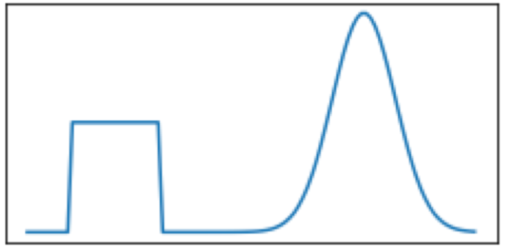
- Work so far we have used L_2 norms

Discretize

$$\phi_m = \int m^2(x)dx \quad \Longrightarrow \quad \phi_m = \sum_{i=1}^{M} m_i^2 v_i$$

- General L_p-norm

$$\phi_m = \sum_{i=1}^{M} |m_i|^p v_i \qquad 0 \le p \le 2$$



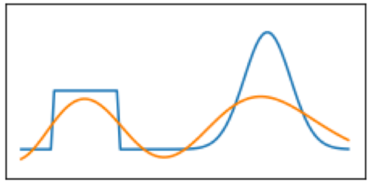
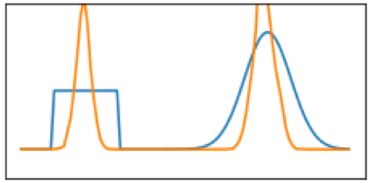
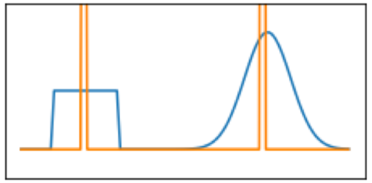
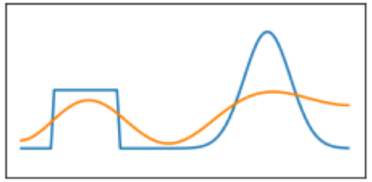
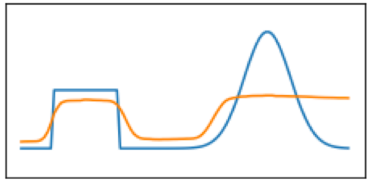
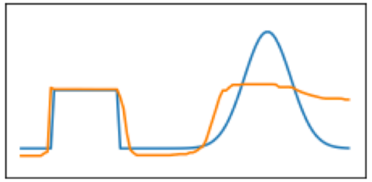
	p=2	p=1	p=0.5	p=0
ϕ_m	69	55	54	100

General character

$$\phi_m = \sum_{i=1}^{M} |m_i|^p v_i$$

- Geometric character
  - p=2: all elements close to zero
  - p=1: sparse solution, # of non-zero elements are ≤ # of data
  - p=0: minimum support, model with the fewest number of elements

- 1D problem

	p=2	p=1	p=0
only ϕ_s			
only ϕ_x			

General Lp objective function

- Each component of a 3D objective function can have its own Lp-norm

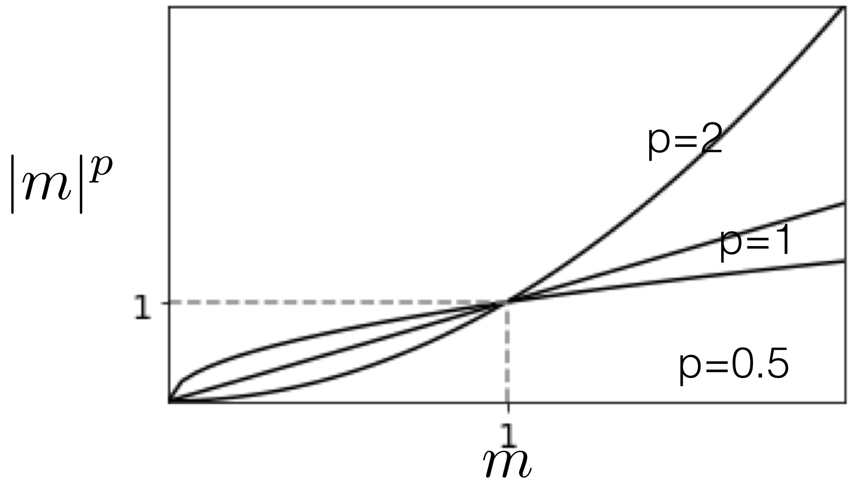
$$\phi_m = \sum_{j=s,x,y,z} \alpha_j \int_v |f_j(m)|^{p_j}\, dv \qquad 0 \le p_j \le 2$$

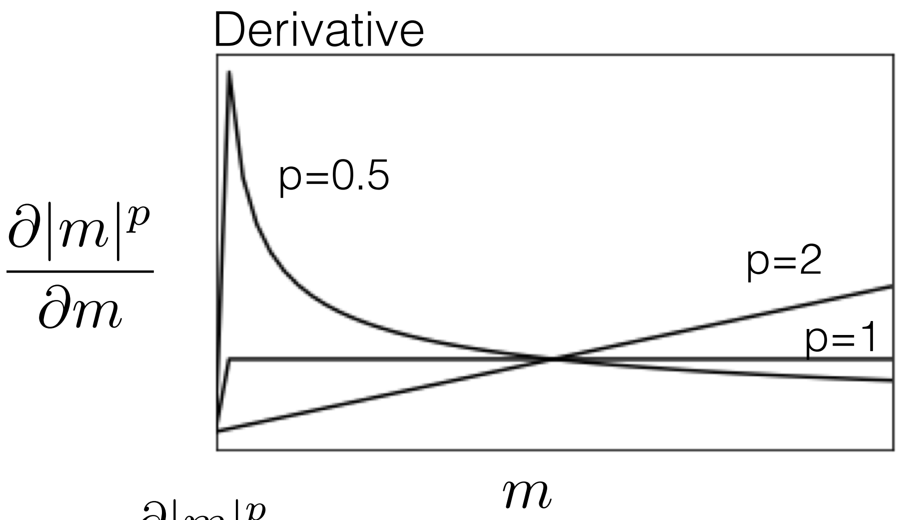
$$= \phi_s^{p_s} + \phi_x^{p_x} + \phi_y^{p_y} + \phi_z^{p_z}$$

$$f_s(m) = m \quad f_x(m) = \frac{dm}{dx} \quad f_y(m) = \frac{dm}{dy} \quad f_z(m) = \frac{dm}{dz}$$

Implementation

Single variable m





$$\frac{\partial |m|^p}{\partial m} = p|m|^{p-1} sgn(m)$$

$\text{singluar } p < 1$

Derivatives for m → 0 are ill-defined

Various approaches to handle this: Huber, Ekblom, Lawson

E.g. Lawson:

$$|m|_L^p \approx \frac{m^2}{(m^2 + \epsilon_m^2)^{1-p/2}}$$

$\epsilon_m\text{: small perterbation}$

Solve: IRLS (Iterative Reweighted Least Squares

$$\sum_i |m_i|^p \simeq \sum_i \frac{m_i^2}{(m^2+\epsilon_m^2)^{1-p/2}} = \|\mathbf{R}^{(k)}\mathbf{m}\|^2 \qquad \mathbf{R}^{(k)} = \mathbf{diag}\Big(\frac{1}{(\mathbf{m}^2+\boldsymbol{\epsilon}_m^2)^{1-p/2}}\Big)$$

$\mathbf{R}^{(k)}$ is an weighting matrix evaluated at current model, $\mathbf{m}^{(k)}$

For the linear inverse problem ($\mathbf{Gm}=\mathbf{d}$)

$$\phi = \|\mathbf{Gm}^{(k+1)} - \mathbf{d}\|^2 + \beta\|\mathbf{R}^{(k+1)}\mathbf{m}^{(k+1)}\|^2$$

$\nabla_m \phi = 0$ - - - - → $\big(\mathbf{G}^T\mathbf{G} + \beta\mathbf{R}^{(k)^T}\mathbf{R}^{(k)}\big)\mathbf{m}^{(k+1)} = \mathbf{G}^T\mathbf{d}^{obs}$

Iterate until $\|\mathbf{m}^{(k+1)} - \mathbf{m}^{(k)}\| < \eta$

General norm

$$\phi_m = \sum_{j=s,x,y,z} \alpha_j \int_v |f_j(m)|^{p_j} \, dv$$

$$f_s(m) = (m - m_{\text{ref}}), \; f_x(m) = \frac{dm}{dx}, \; f_y(m) = \frac{dm}{dy}, \; f_z(m) = \frac{dm}{dz}$$

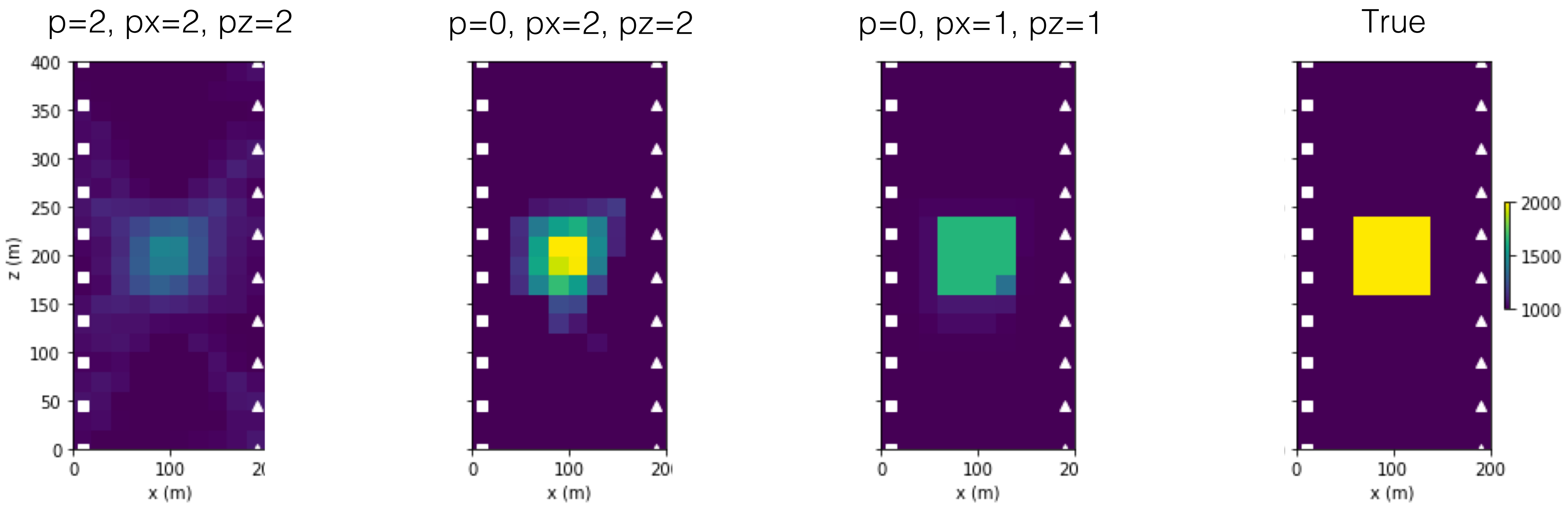
Lawson $\quad |f_j(m)|^{p_j} \longrightarrow \dfrac{\Big(f_j(m)\Big)^2}{\Big(f_j(m)^2 + \epsilon_m^2\Big)^{1-p/2}}$

Solve using Iterative Re-weighted Least Square (IRLS)

2D crosswell example

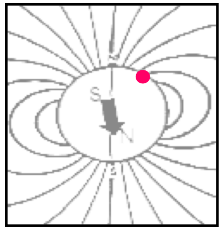
$$\phi_m = \alpha_s \int_v |\mathbf{m} - \mathbf{m}_{\text{ref}}|^{p_s} dv + \alpha_x \int_v \left| \frac{d}{dx}(\mathbf{m} - \mathbf{m}_{\text{ref}}) \right|^{p_x} dx + \alpha_z \int_v \left| \frac{d}{dz}(\mathbf{m} - \mathbf{m}_{\text{ref}}) \right|^{p_z} dz$$

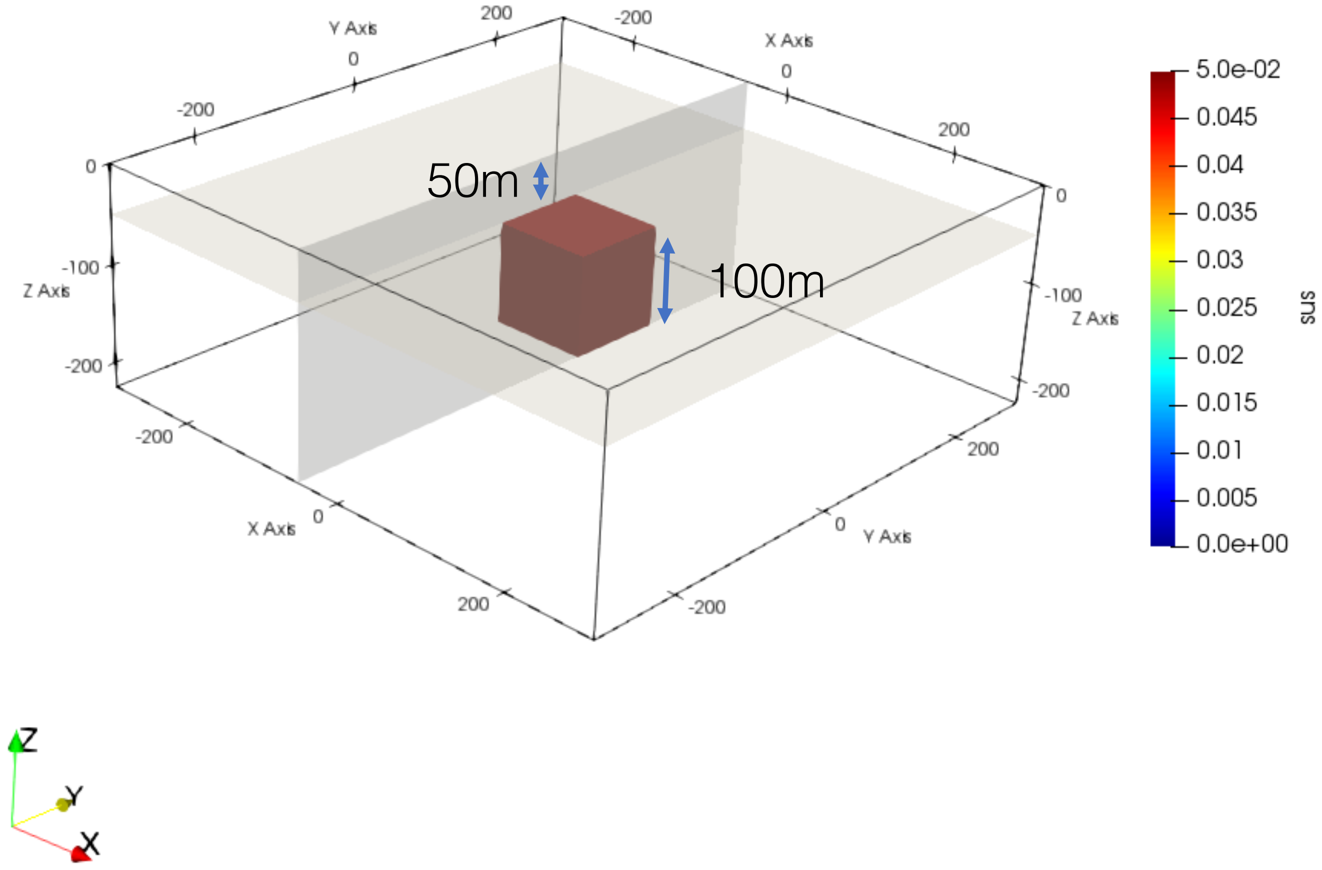
Each term has three adjustable parameters: (α, p, ϵ_m)

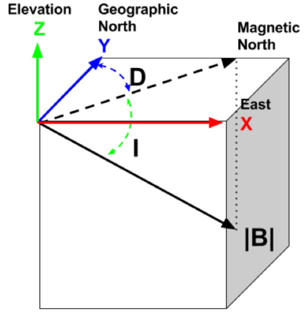


3D magnetic example

Synthetic susceptibility model





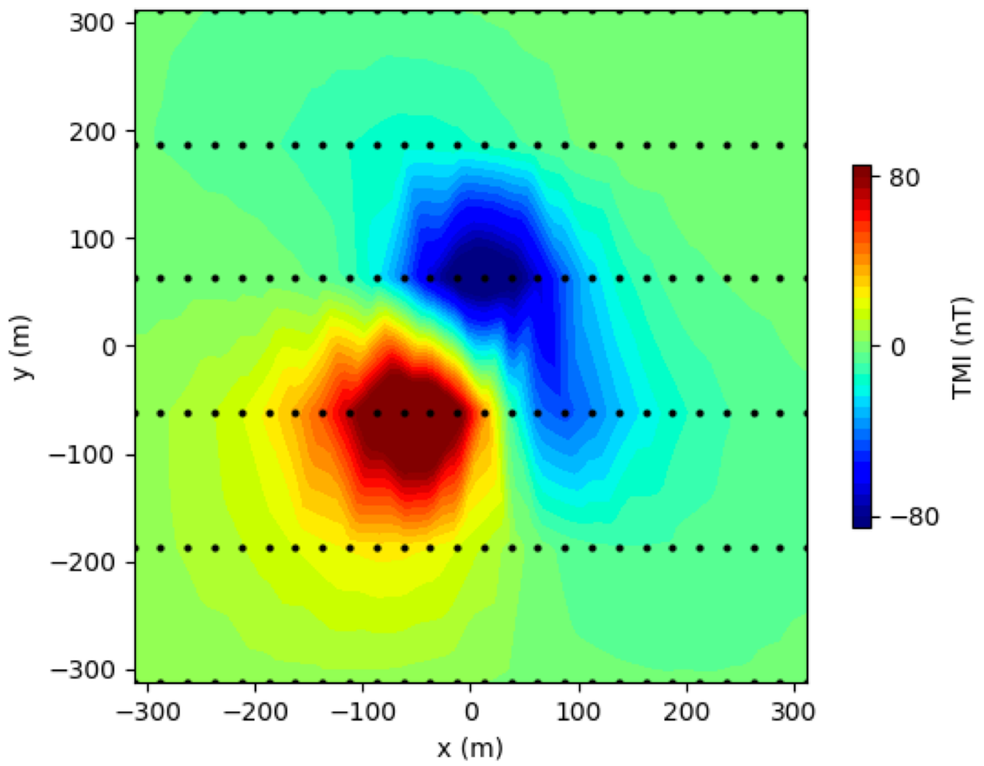


- Earth field
  - Inclination: 30°
  - Declination: 45 °
  - $|B_0|$ = 50,000 nT
- Susceptible block
  - 100m x 100m x 100m block
  - Block susceptibility = 0.5
  - Block top = 50m

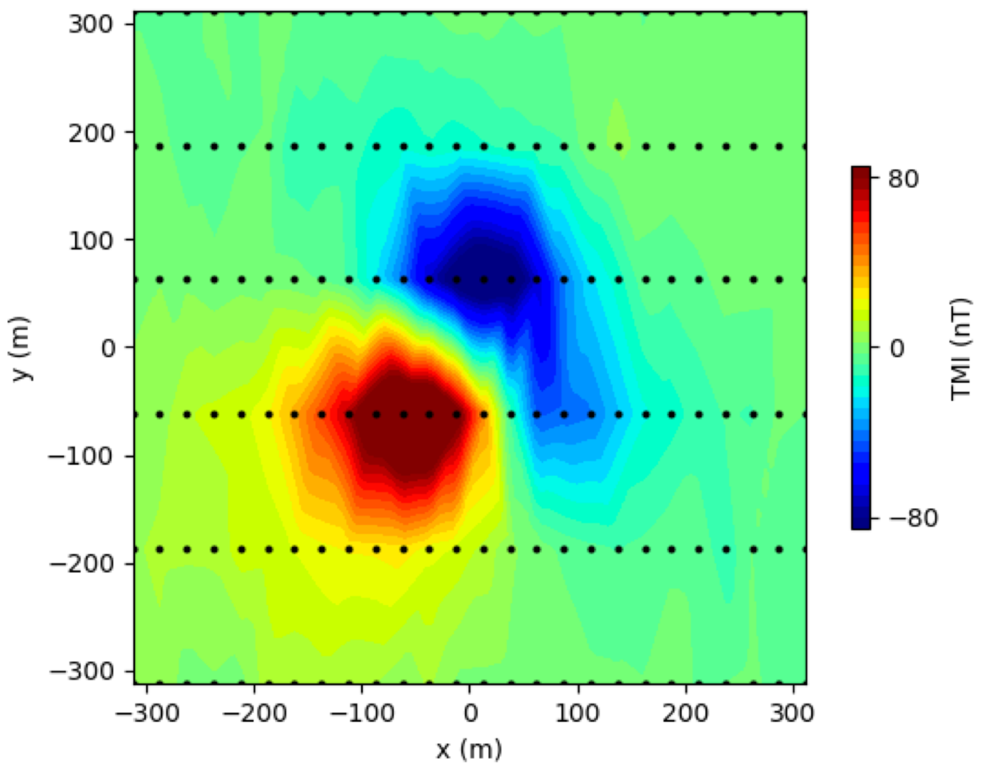
Synthetic survey

Survey parameters:
- 100 m line spacing.
- 25 station spacing.
- N=156 (elevation= 2m)

Contoured "perfect" data



Contoured data with noise = 2 nT

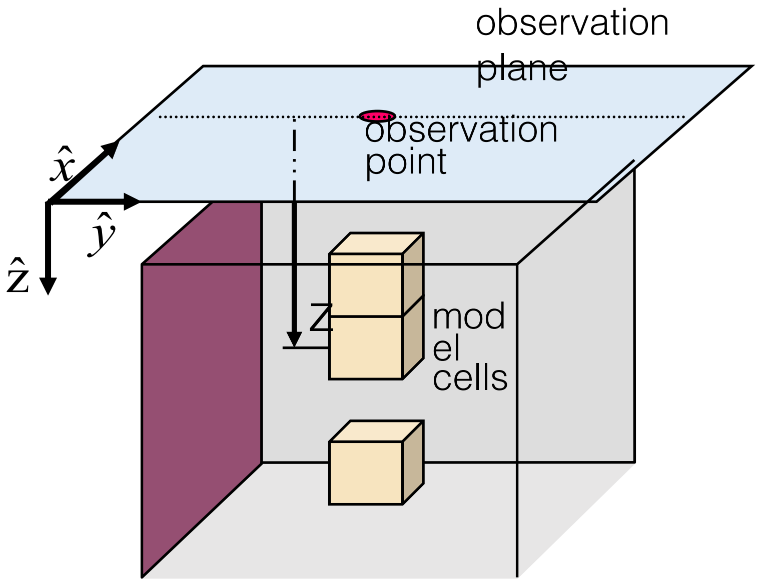


Solving inverse problem **with weighting**

Model objective function

$$\phi_m = \alpha_s \int_v w_s \left(\kappa - \kappa_{\text{ref}}\right)^2 dv + \alpha_x \int_v w_x \left(\frac{d\kappa}{dx}\right)^2 dx + \alpha_y \int_v w_y \left(\frac{d\kappa}{dy}\right)^2 dy + \alpha_z \int_v w_z \left(\frac{d\kappa}{dz}\right)^2 dz$$

Data misfit $\quad \phi_d = \sum_{j=1}^{N} \left(\frac{G_{ij}\kappa_j - d_j^{obs}}{\epsilon_j}\right)$



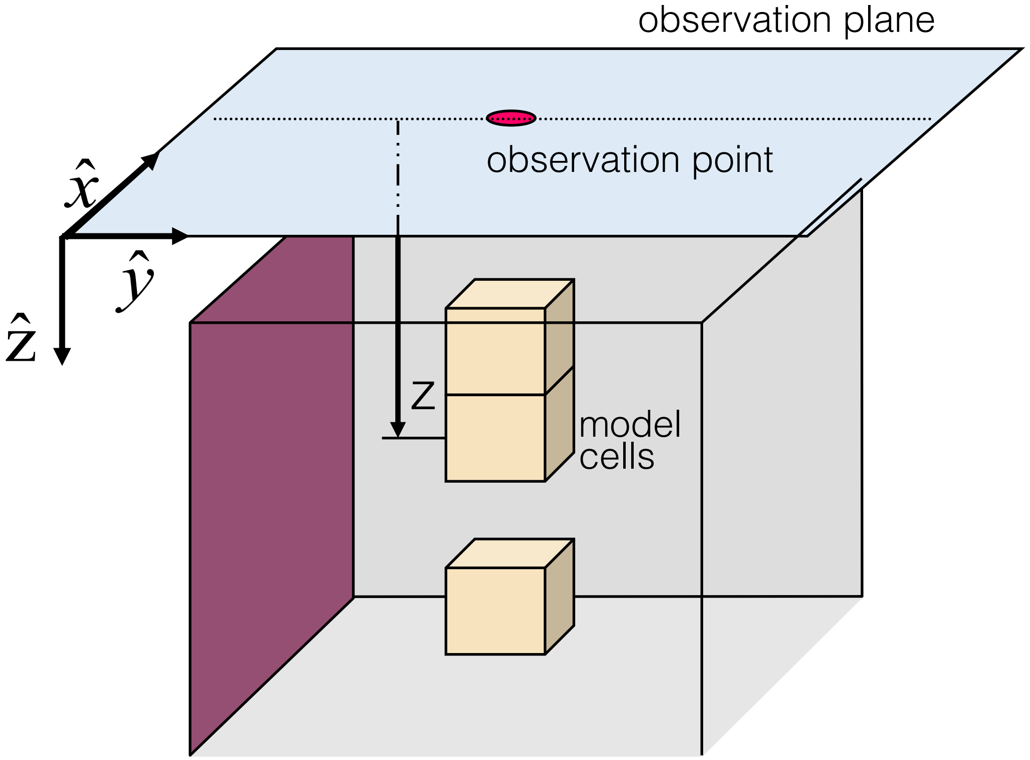
The Inverse problem is:

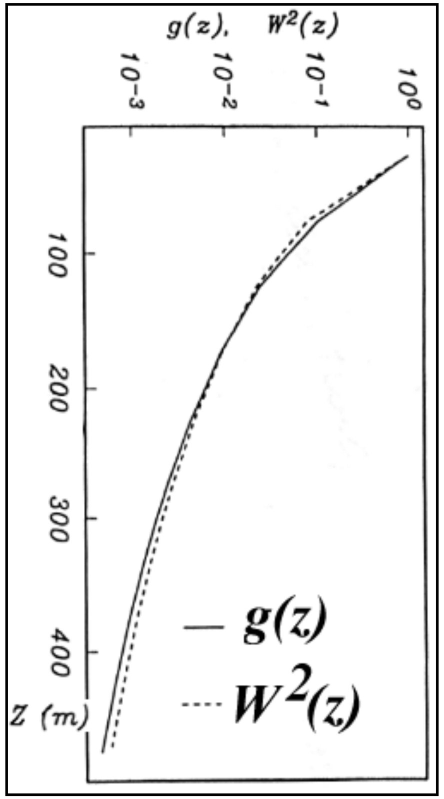
$$\text{minimize} \quad \phi = \phi_d + \beta\phi_m$$

$$\text{find } \beta \text{ such that } \phi_d = \phi_d^* \text{ where } \phi_d = N$$

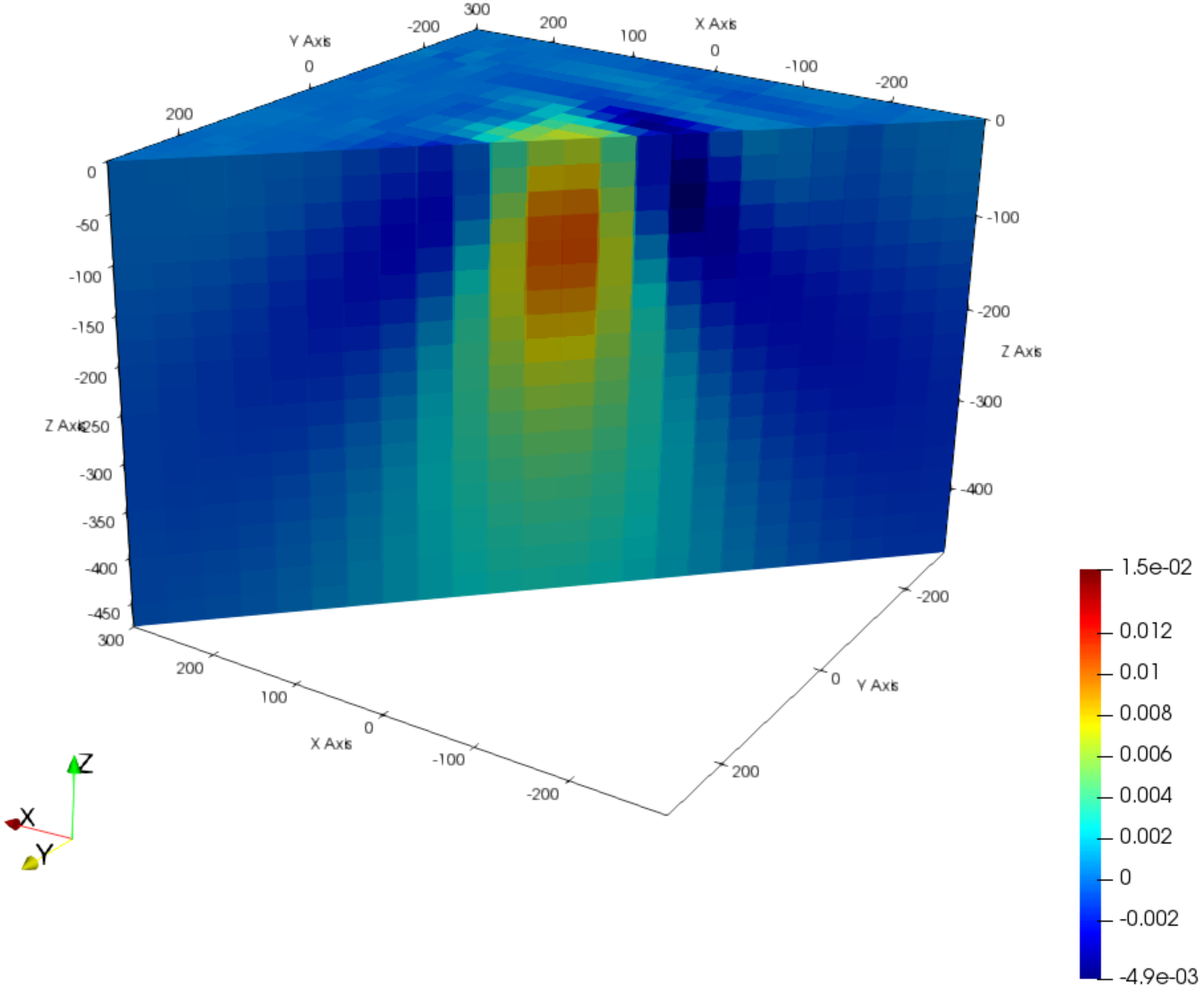
Depth weighting

- Decays with depth: $w(z) = \dfrac{1}{(z + z_o)^{3/2}}$
- z_0: by least-squares fit between g(z) and $w^2(z)$



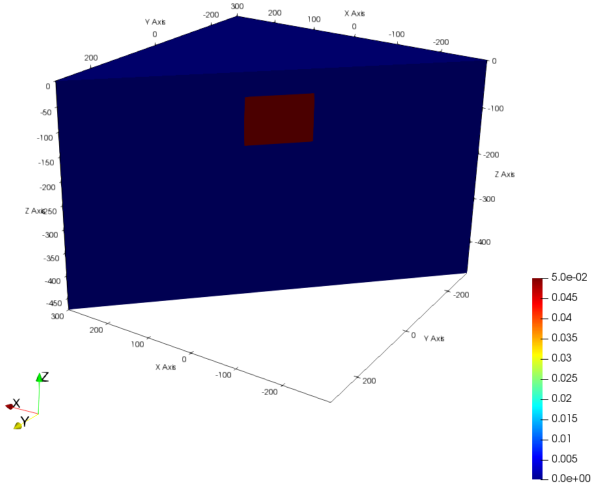


Inversion with sensitivity weighting

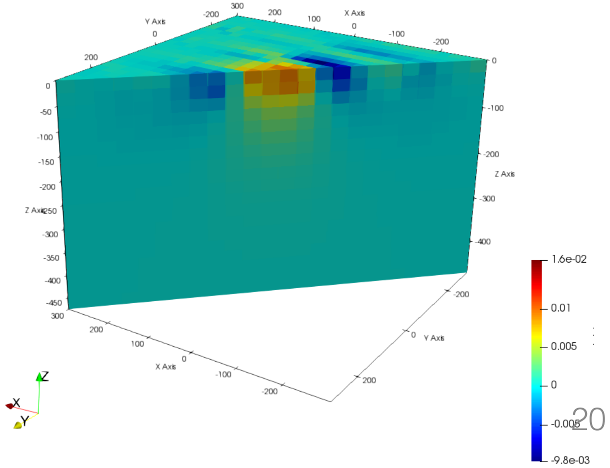


Negative

True model



Without sensitivity weight



Positivity

- Susceptibility is positive
- The inverse problem is stated as

$$\begin{aligned}&\text{minimize} \quad \phi = \phi_d + \beta\phi_m \\ &\text{subject to } m \geq 0\end{aligned}$$

- The problem becomes non-linear. Need to address this.

Bound Constraints

- Physical property bounds in each cell

$$\mathbf{m}_L \leq \mathbf{m} \leq \mathbf{m}_H$$

Eldad's book 2015

- Projected Gradient Gauss-Newton (Kelly, 1999; Haber,)
  - At each GN iteration

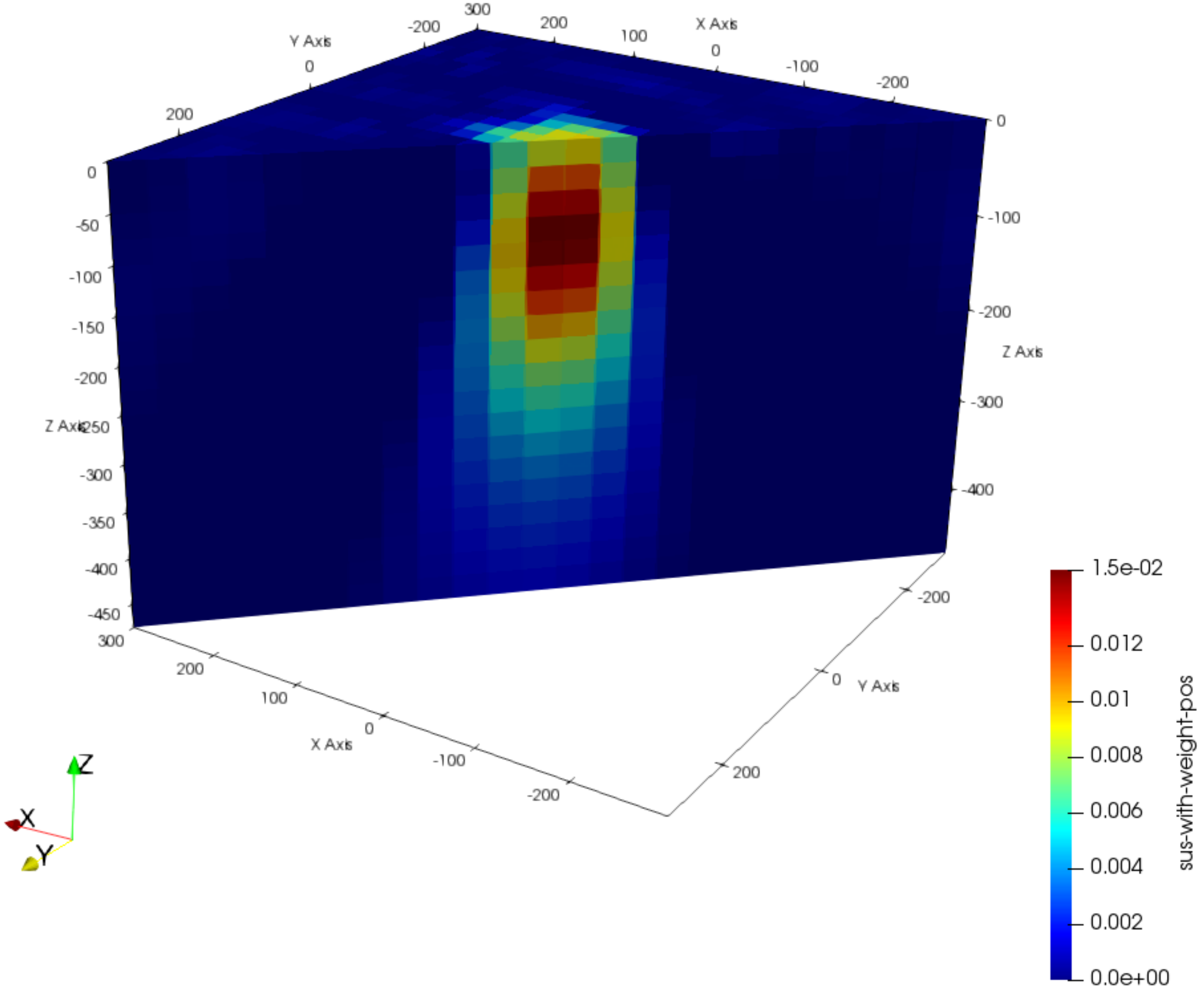
$$\delta\mathbf{m} = \mathbf{H}^{-1}\delta\mathbf{d} + \alpha\mathbf{g}$$

$\mathbf{H}$: Hessian for cells not at the bounds
$\mathbf{g}$: gradient for cells at the bounds
α: scalar

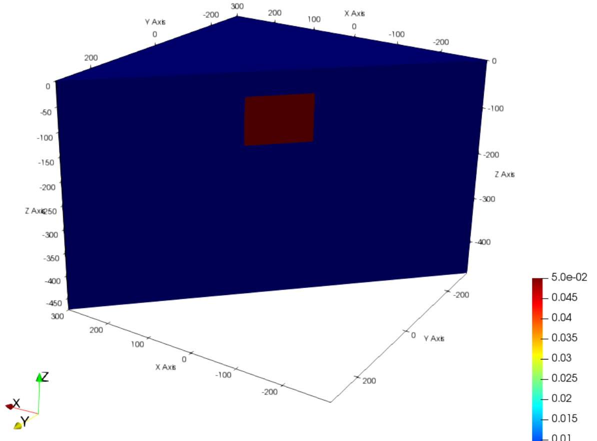
Positivity of susceptibility $\mathbf{m} \geq 0$

3D magnetic inversion:

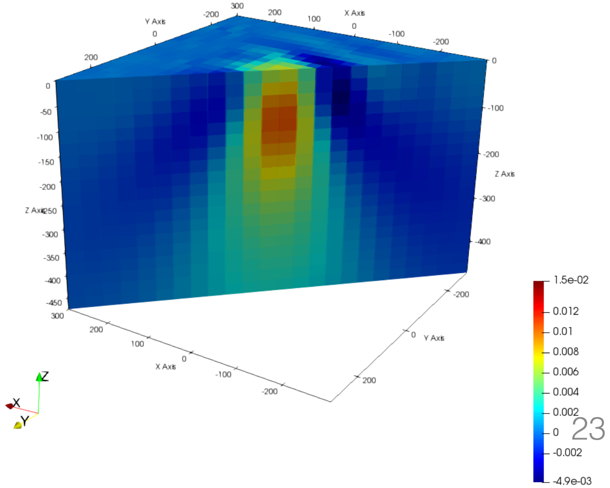
- depth (or sensitivity) weighting
- positivity (bounds): $\mathbf{m} \geq 0$



True model



Without positivity



Magnetic inversion with Lp norms

Model objective function

$$\phi_m = \alpha_s \int_v w_s \left|\kappa - \kappa_{\text{ref}}\right|^{p_s} dv + \alpha_x \int_v w_x \left|\frac{d\kappa}{dx}\right|^{p_x} dx + \alpha_y \int_v w_y \left|\frac{d\kappa}{dy}\right|^{p_y} dy + \alpha_z \int_v w_z \left|\frac{d\kappa}{dz}\right|^{p_z} dz$$

Data misfit $\quad \phi_d = \sum_{j=1}^{N} \left(\frac{G_{ij}\kappa_j - d_j^{obs}}{\epsilon_j}\right)$

The Inverse problem is:

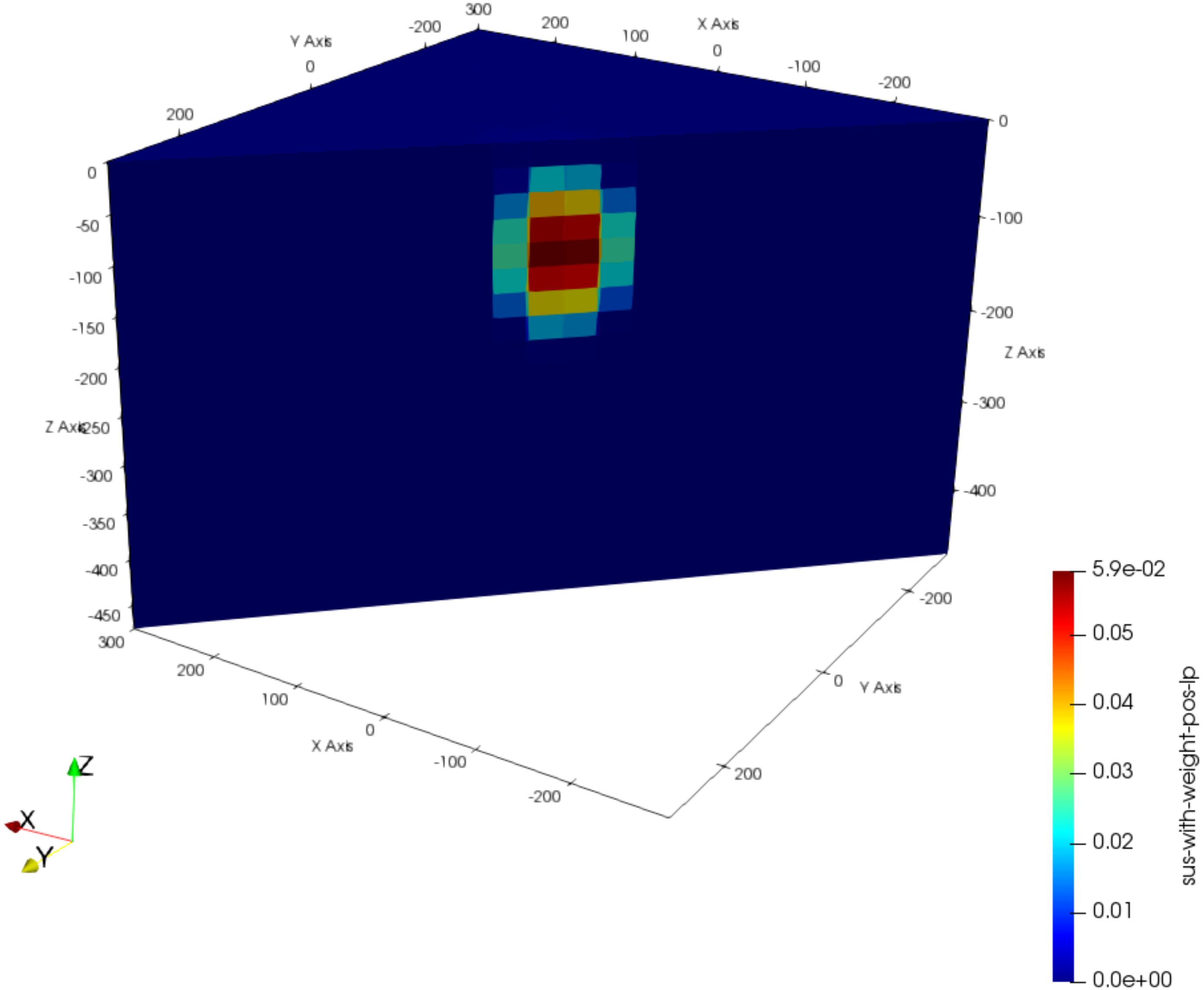
$$\text{minimize} \quad \phi = \phi_d + \beta\phi_m$$

find β such that $\phi_d = \phi_d^*$ where $\phi_d = N$

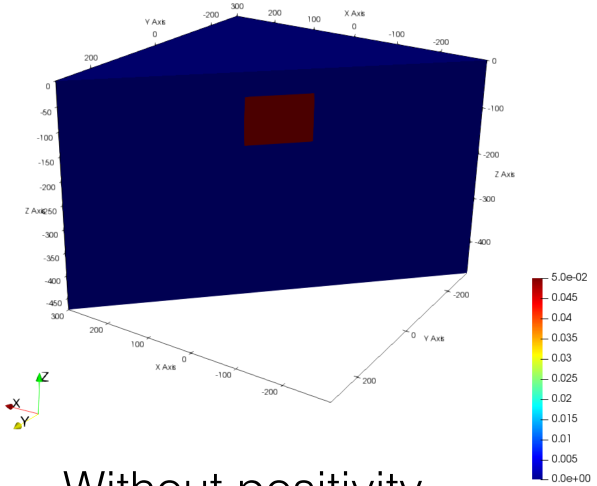
subject to $\kappa \geq 0$

3D magnetic inversion:

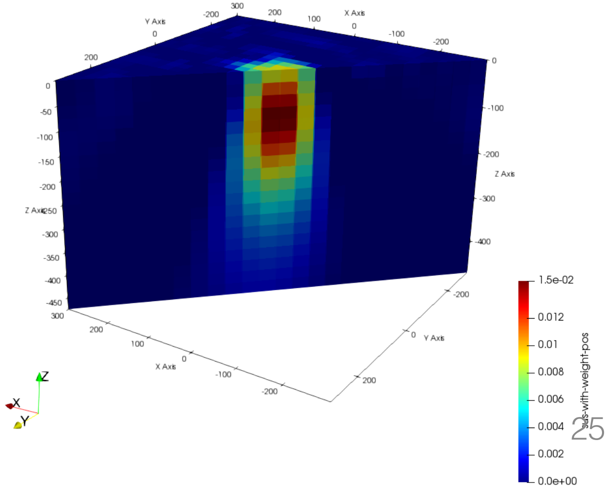
- depth (or sensitivity) weighting
- positivity (bounds): $\mathbf{m} \geq 0$
- L_p norm ($\mathbf{p_s=0}$, $p_x=p_y=p_z=2$)



True model

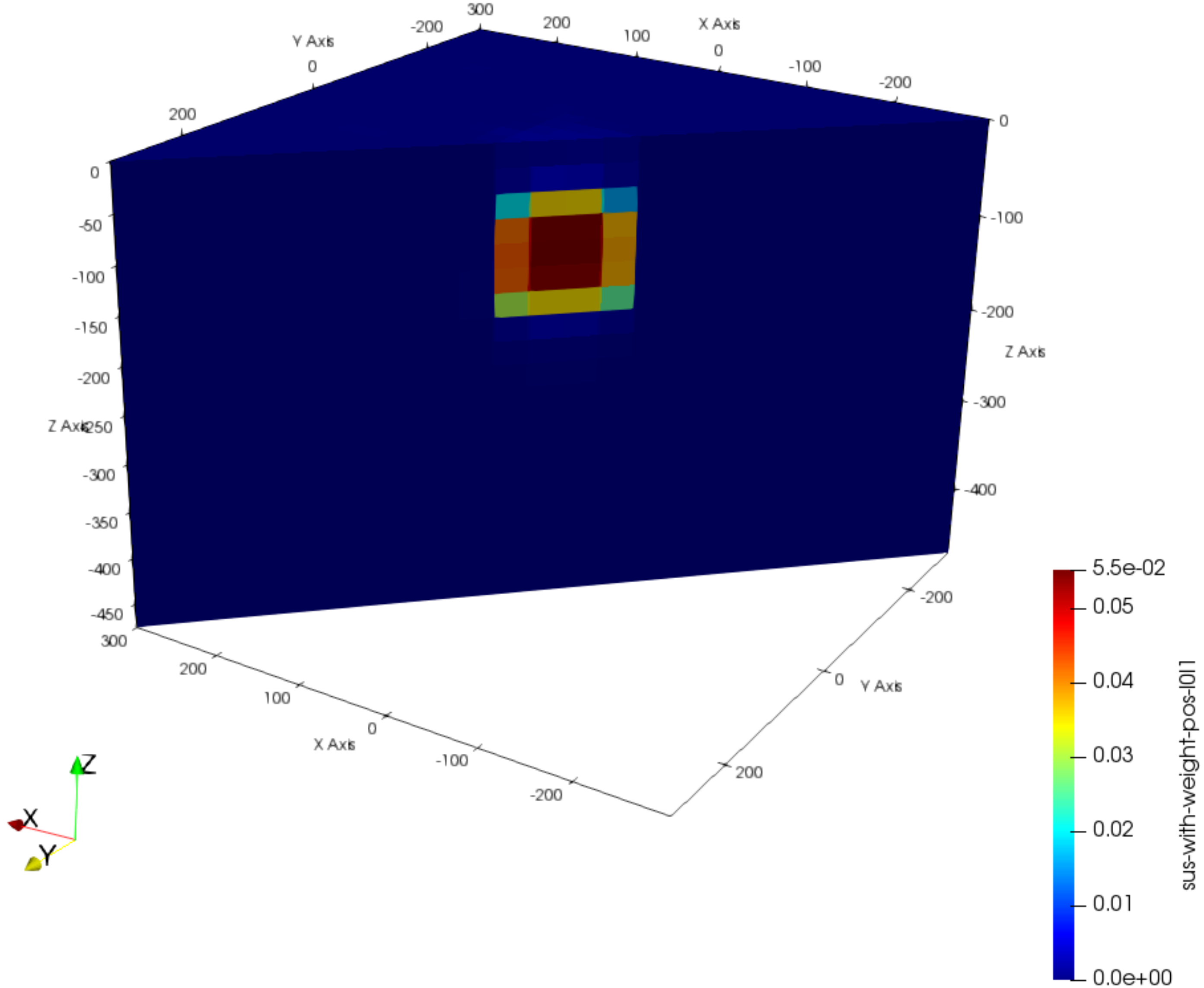


Without positivity

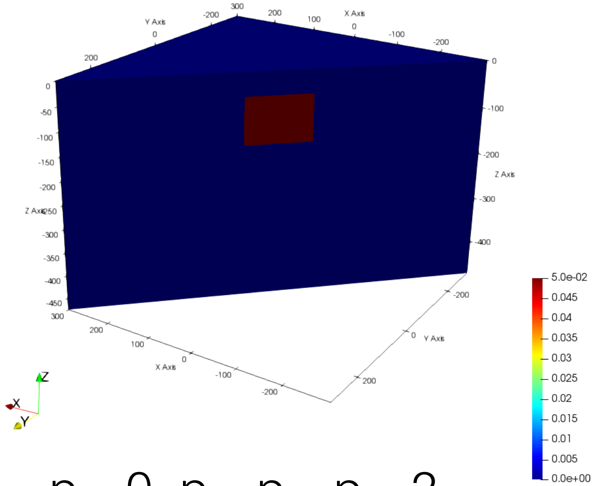


3D magnetic inversion:

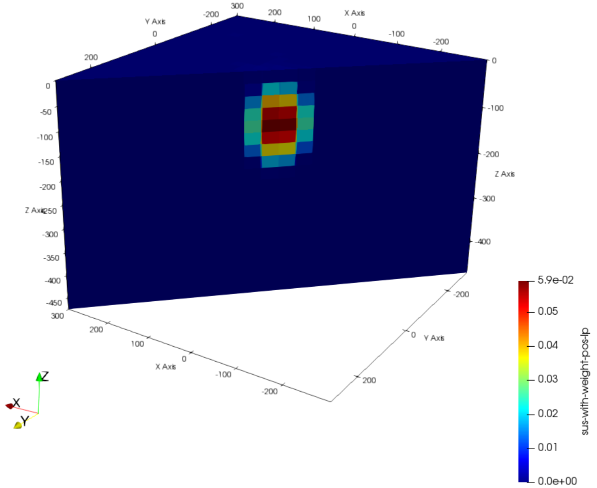
- depth (or sensitivity) weighting
- positivity (bounds): $\mathbf{m} \geq 0$
- L_p norm (p_s=0, p_x=p_y=p_z=1)



True model

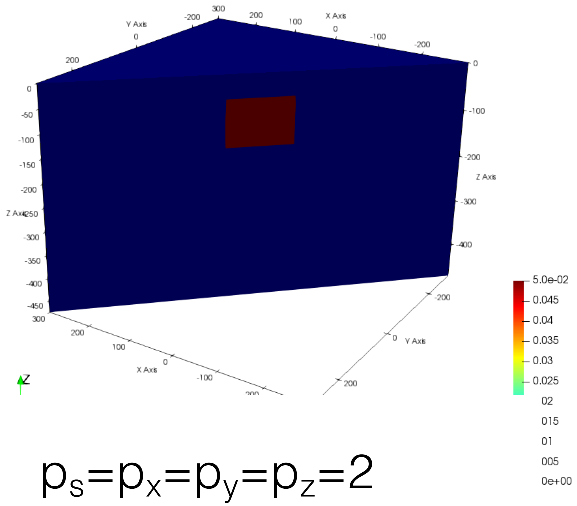


p_s=0, p_x=p_y=p_z=2

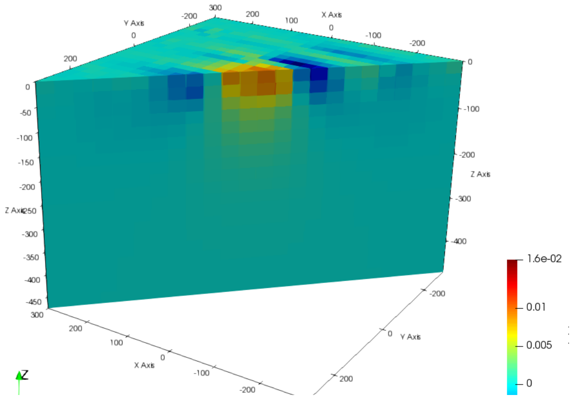


Summary: magnetic inversion

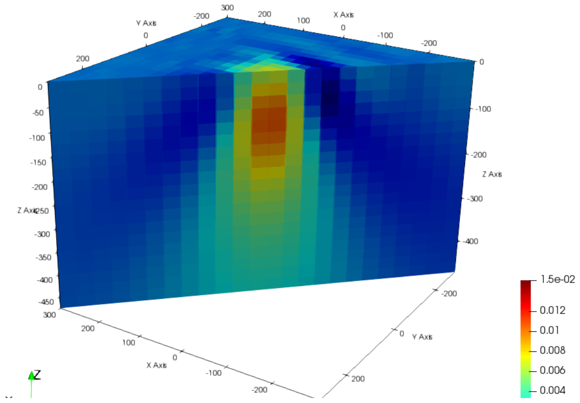
True model



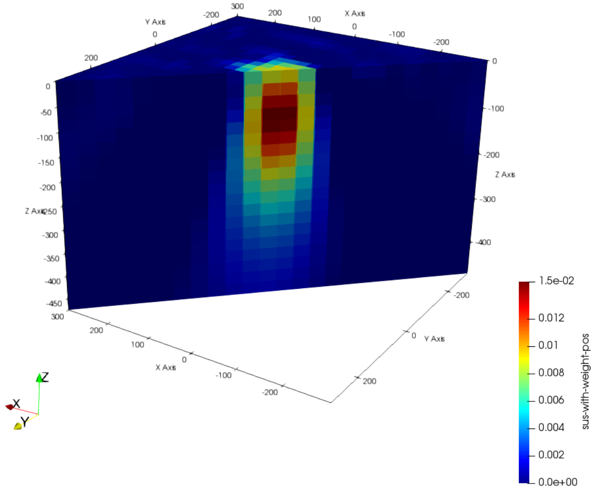
No weight



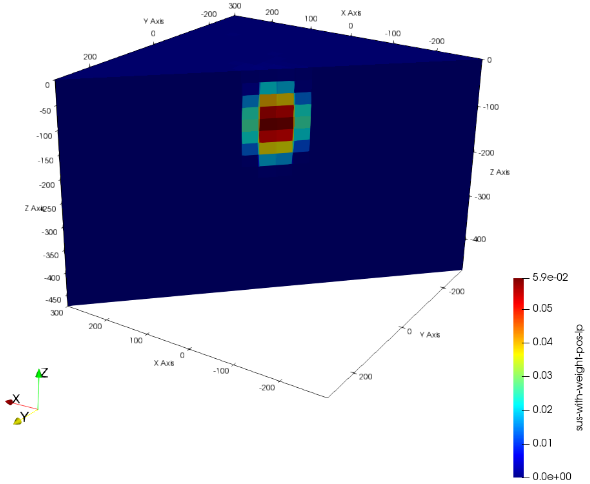
Sensitivity weight



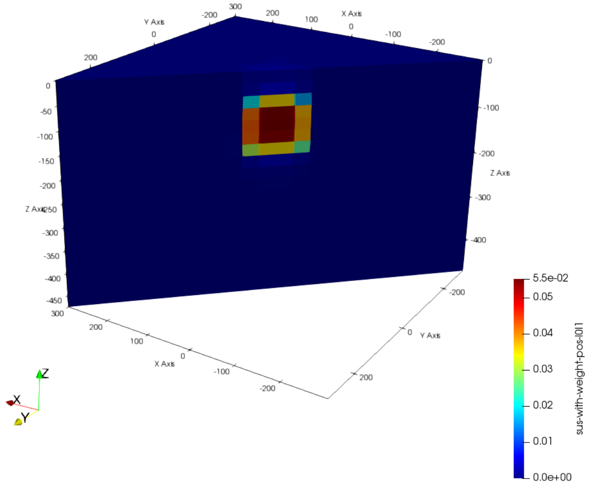
$p_s=p_x=p_y=p_z=2$



$p_s=0,\ p_x=p_y=p_z=2$

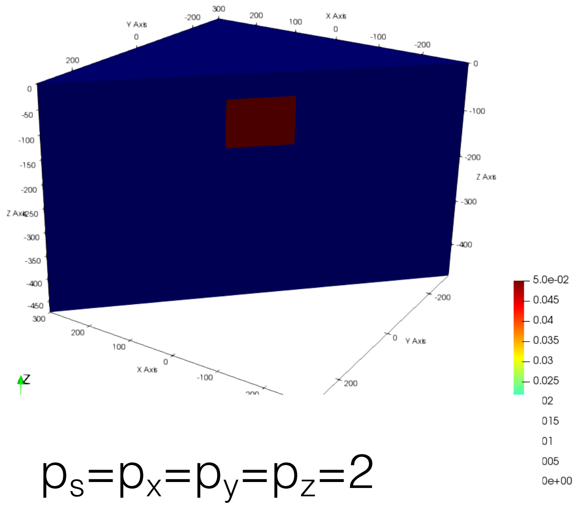


$p_s=0,\ p_x=p_y=p_z=1$

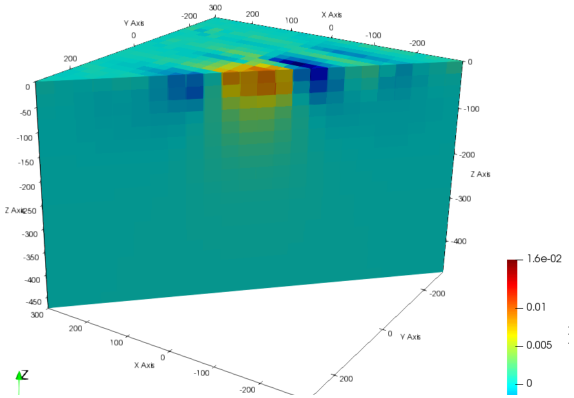


Summary: magnetic inversion

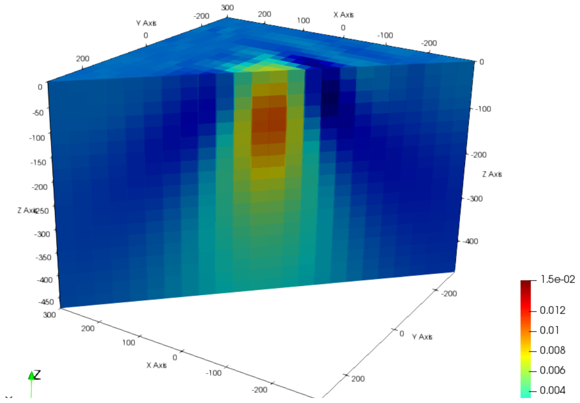
True model



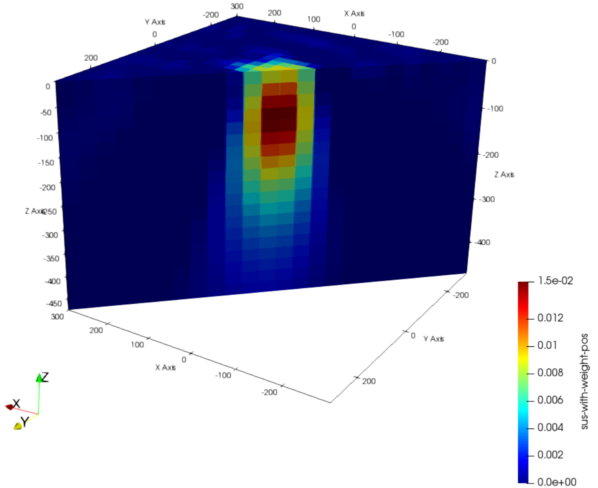
No weight



Sensitivity weight



$p_s=p_x=p_y=p_z=2$



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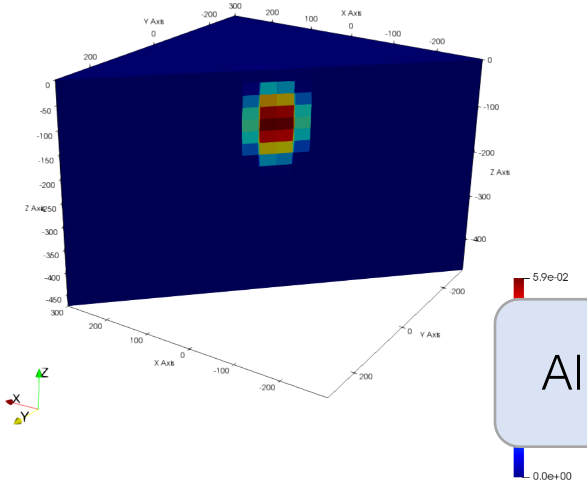
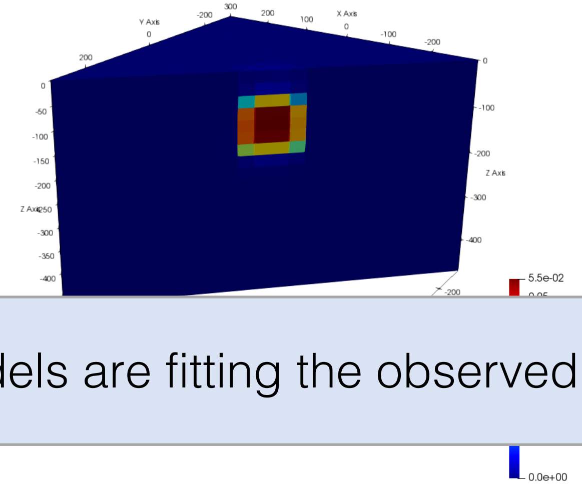


$p_s=0,\ p_x=p_y=p_z=1$



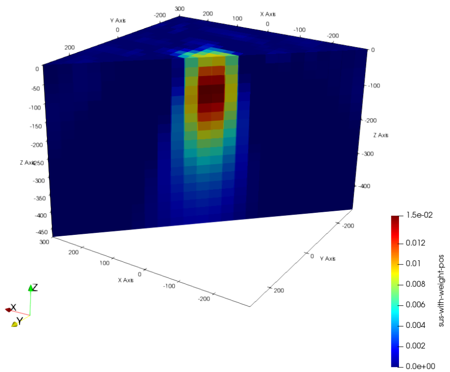
All models are fitting the observed data

Summary: L_p-norms

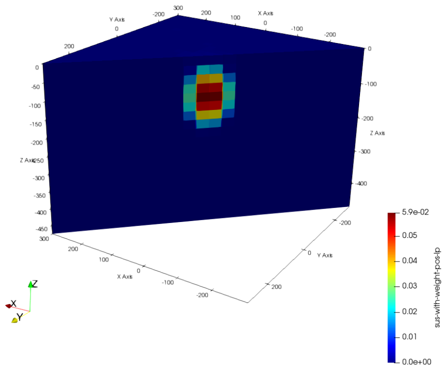
$$\phi_m = \alpha_s \int_v w_s \left|\kappa - \kappa_{\text{ref}}\right|^{p_s} dv + \alpha_x \int_v w_x \left|\frac{d\kappa}{dx}\right|^{p_x} dx + \alpha_y \int_v w_y \left|\frac{d\kappa}{dy}\right|^{p_y} dy + \alpha_z \int_v w_z \left|\frac{d\kappa}{dz}\right|^{p_z} dz$$

- Non-linear inversion
- Balancing each term is not trivial (Fournier and Oldenburg, 2019)
- Provides a capability to explore model space
  - Altering each parameter → generates a different model

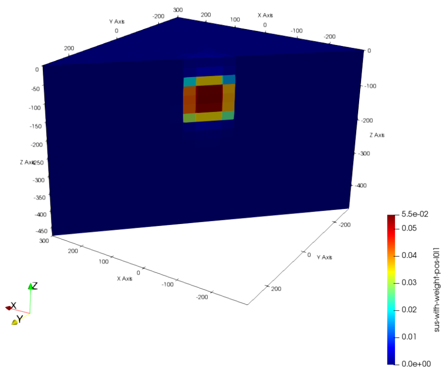
$p_s=p_x=p_y=p_z=2$



$p_s=0,\ p_x=p_y=p_z=2$



$p_s=0,\ p_x=p_y=p_z=1$

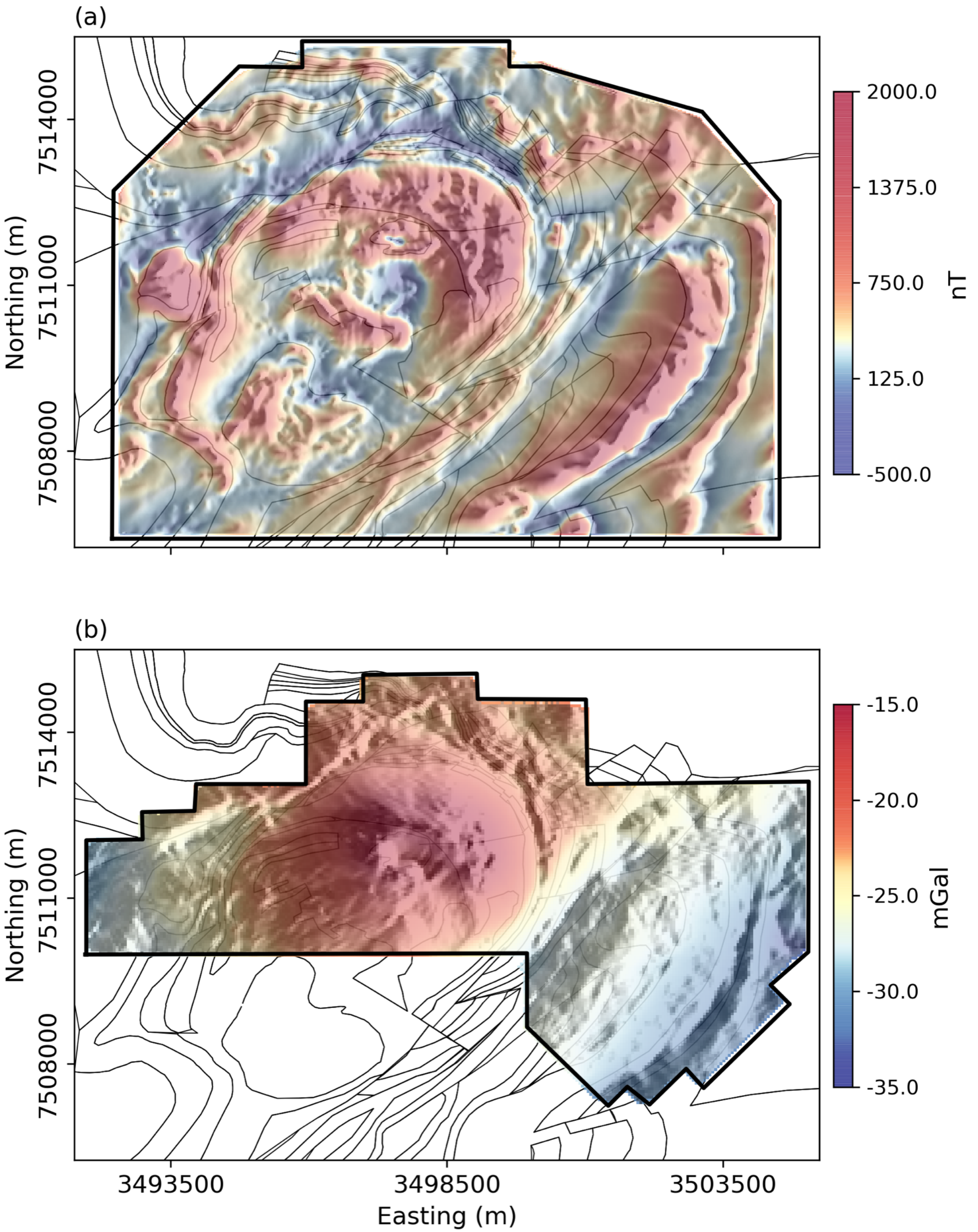


Field example: Kevitsa Ni-Cu-PGE, Finland

- Discovered in early 1980's
- New Boledan
- 160 M tons of Nickel

Kevitsa

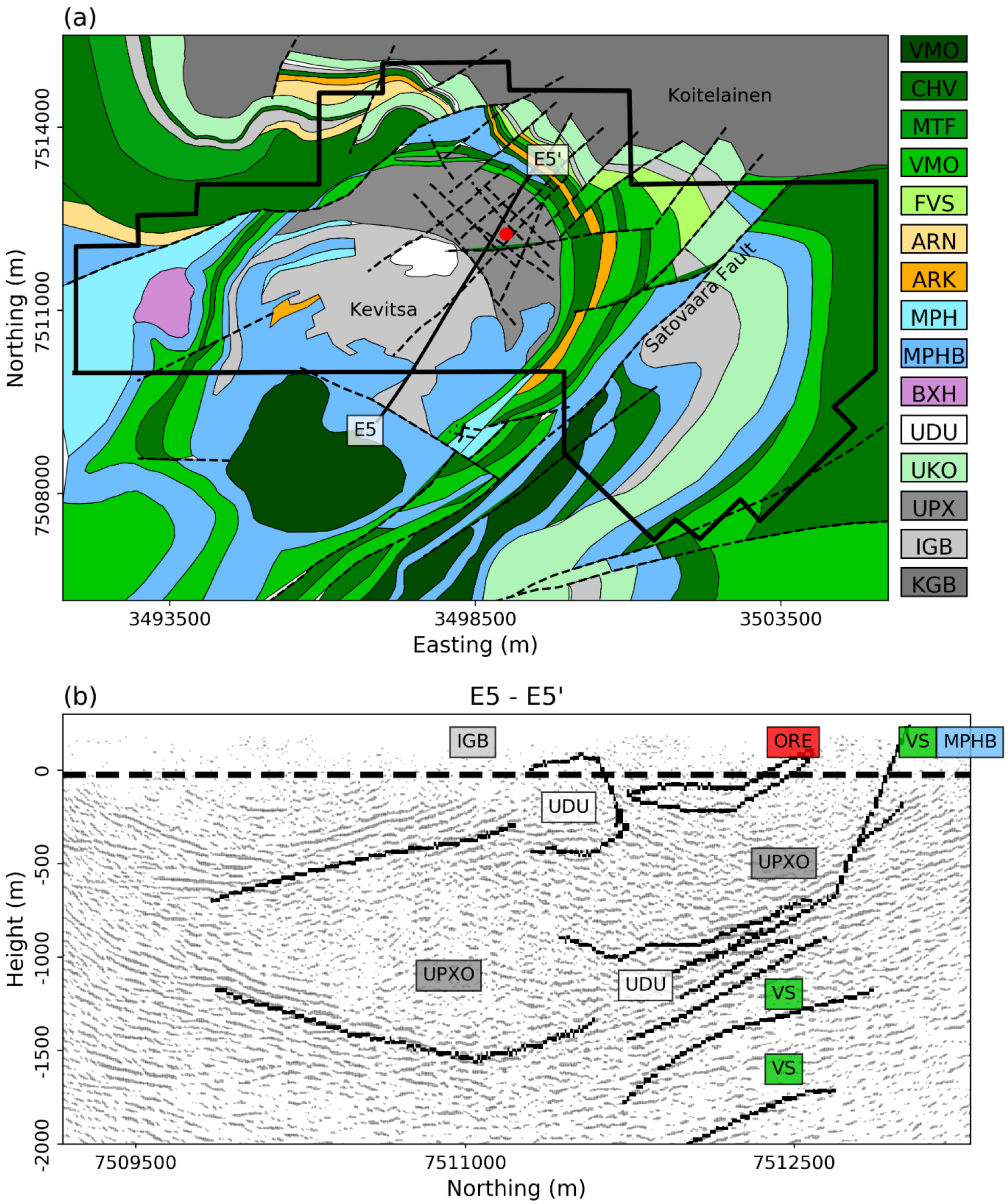
- Large number of acquired datasets
  - DC-Resistivity and IP
  - Magnetotelluric
  - 2D-3D Seismic refraction
  - Airborne TEM + FEM
  - Magnetics
  - Gravity



Kevitsa

- Complex geology
  - >2.04 Ga
  - Ultra-mafic intrusion
  - Volcanic and sedimentary host
  - Important folding and faulting

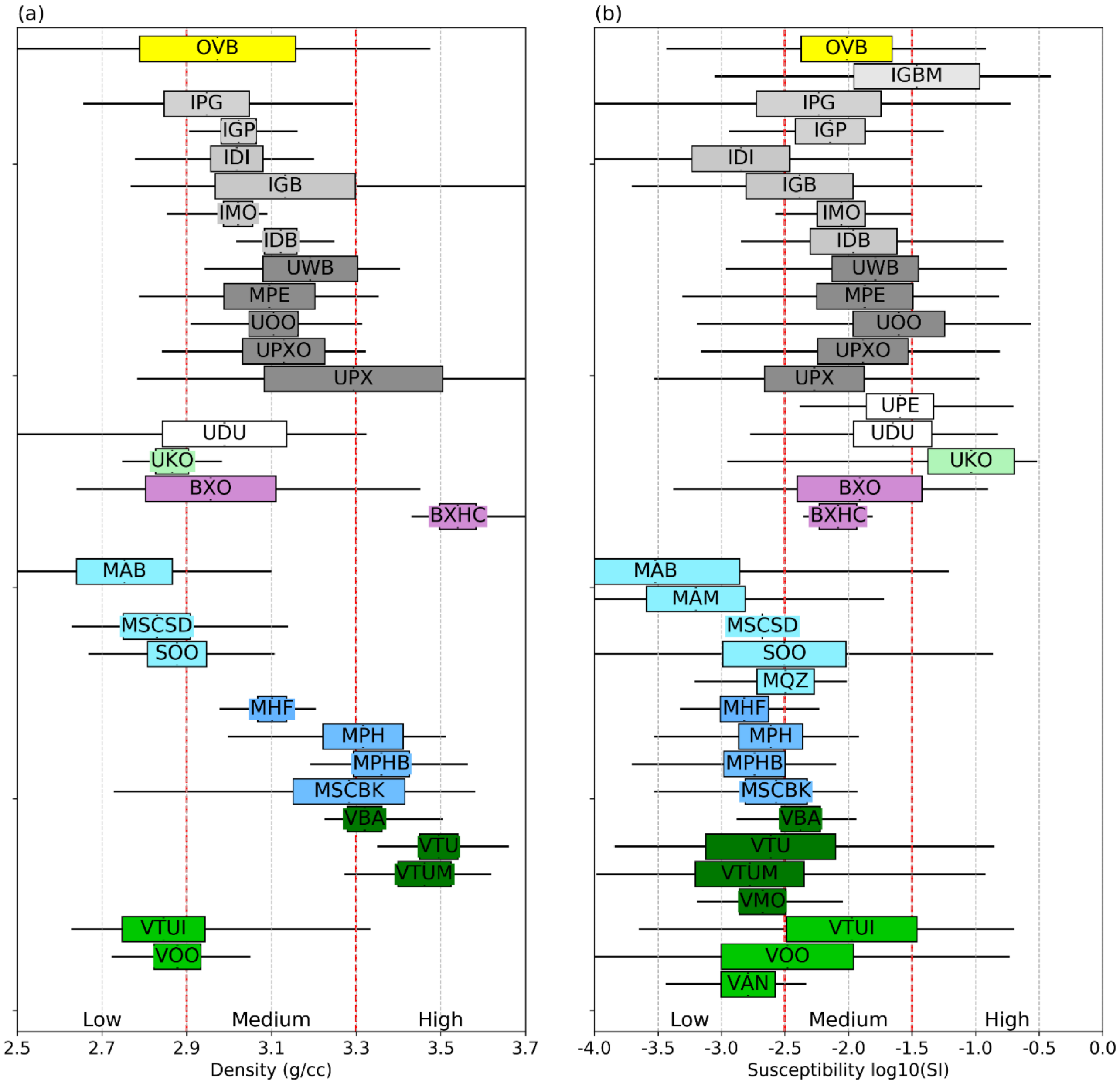
- Seismic refraction profile



Kevitsa

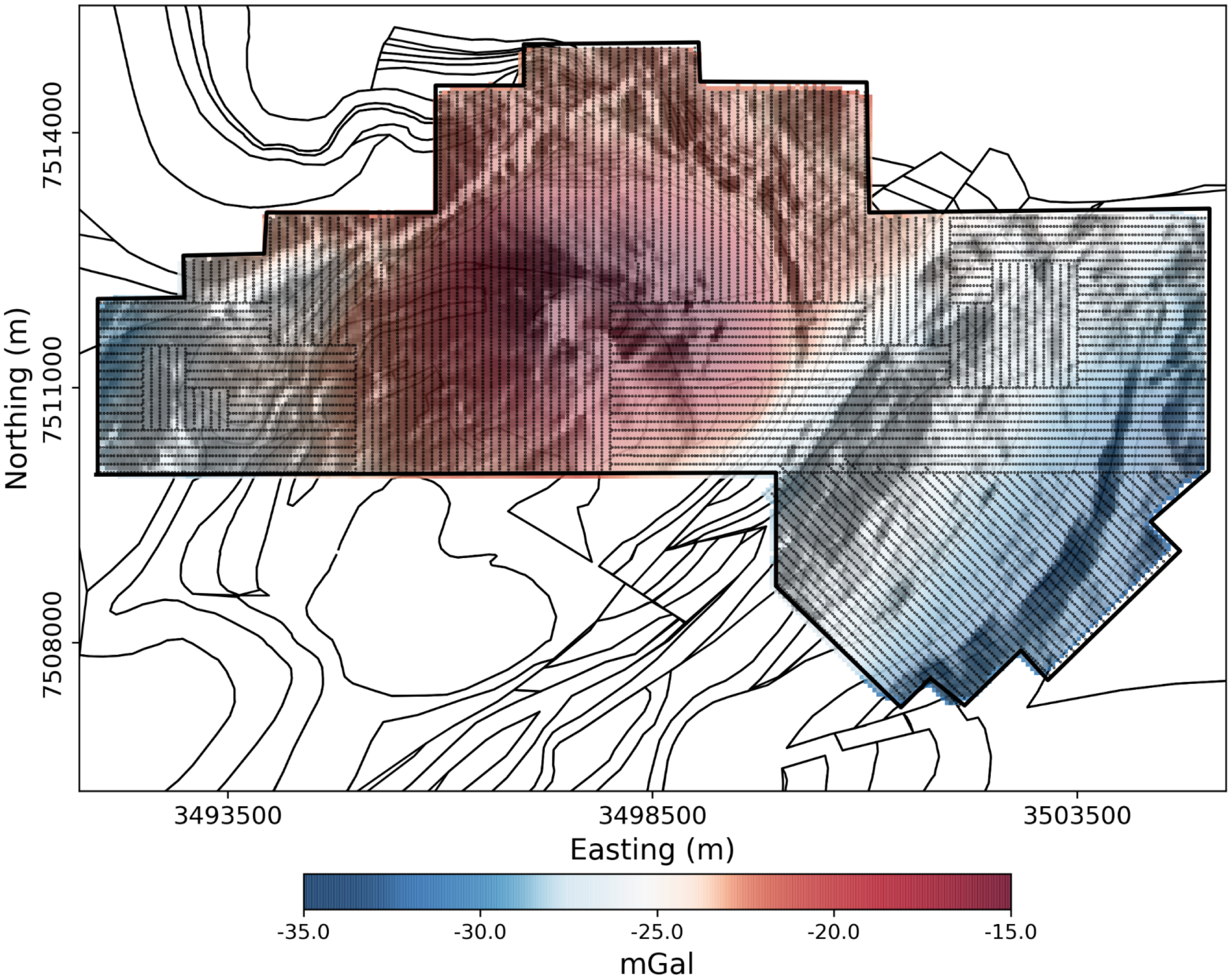
- Physical property logs
  - 279 bore holes
  - > 500k samples

	Code	Density
	OVB: Overburden	Low
	VBA: Basalt	Medium
	IGB: Gabbro	Medium
	UPX: Pyroxenite	High
	UDU: Dunite	Low
	UKO: Komatiite	Low
	BXH: Breccia	Medium
	MPH: Phyllite	Low
	MPHB: Carboneceous Phyllite MSCBK: Black Schist	Medium High
	VMO: Volcanic - Mafic	Medium
	VIO: Volcanic	Low



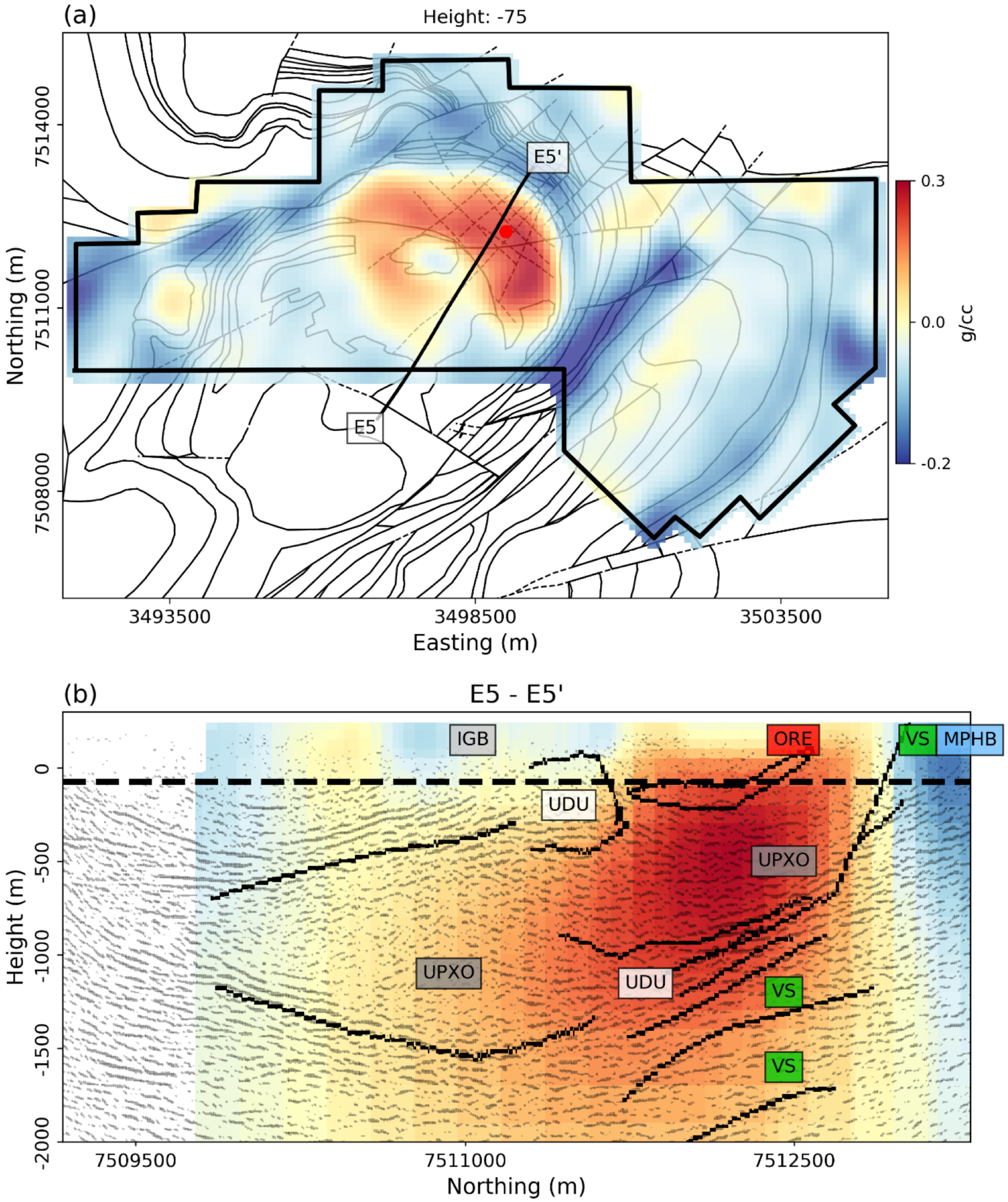
Kevitsa

- Kevitsa (Gravity)



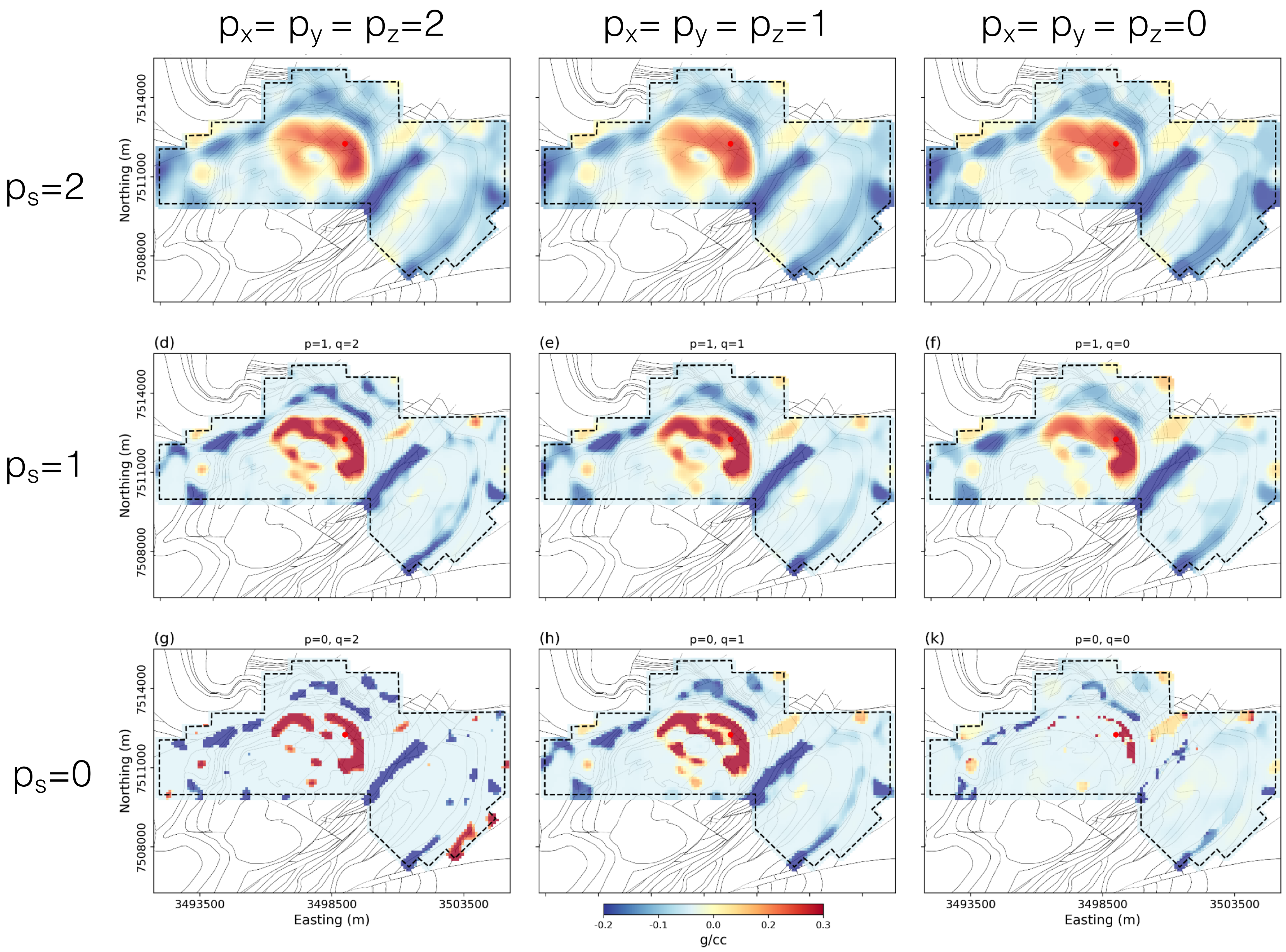
Kevitsa

- Smooth model (conventional L_2-norm)



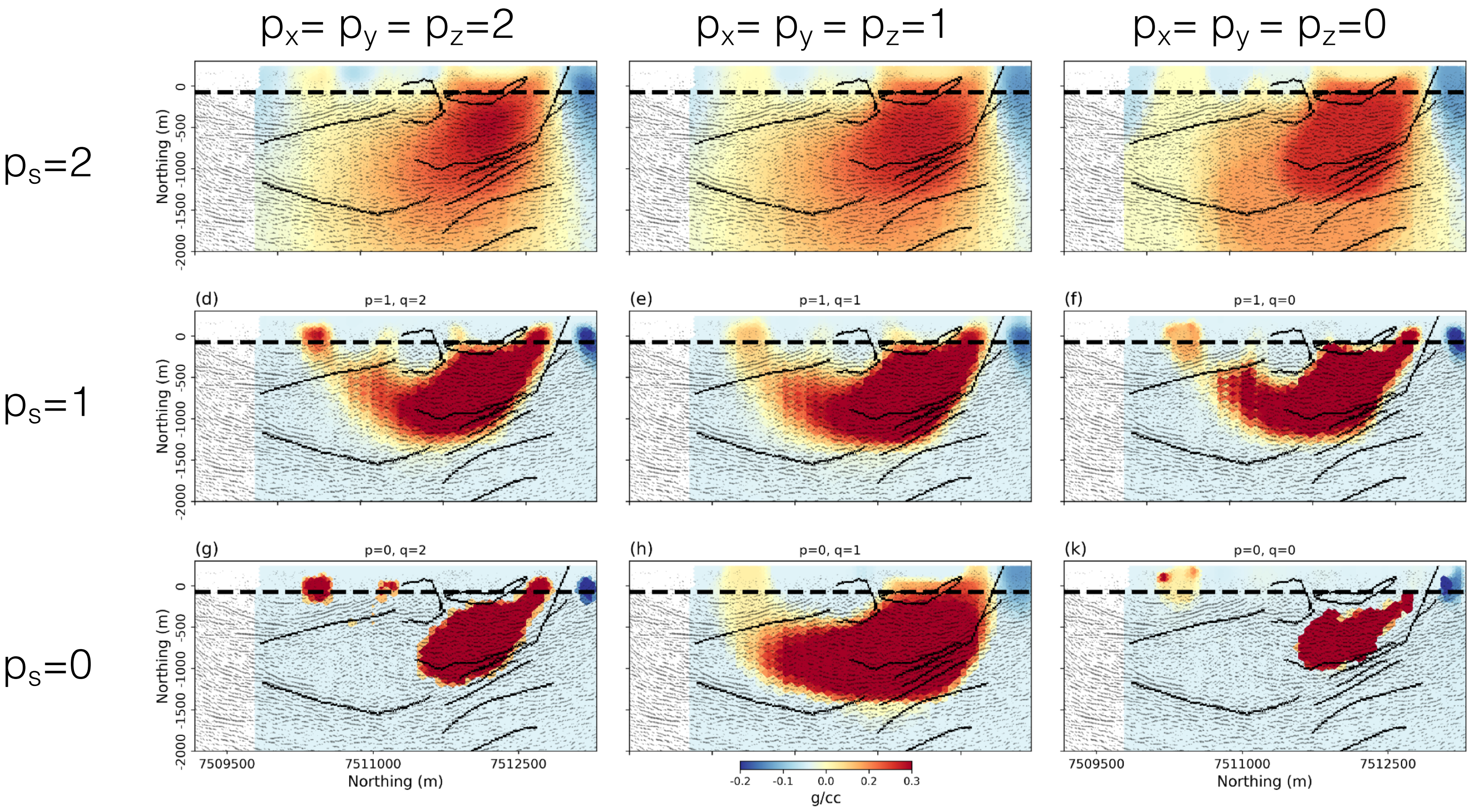
Kevitsa

Exploring the model space



Kevitsa

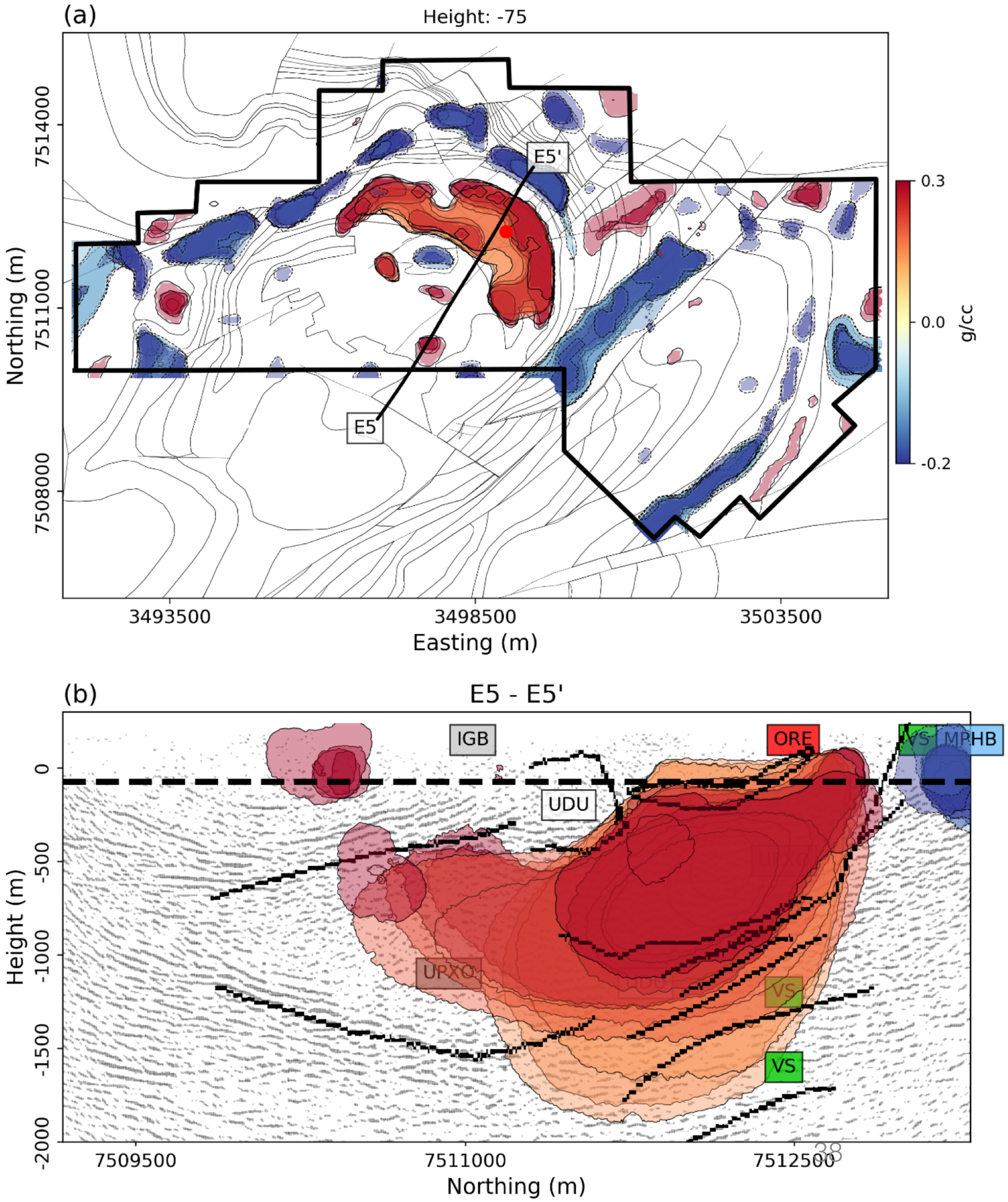
Exploring the model space



Kevitsa

- Plot iso-contours of density anomalies from 9 models

- Interpreting with variability



Outline

- Why beyond L_2?
- L_p norms
- 2D cross-well tomography
- 3D magnetics
- Case history: 3D gravity at Kevitsa

Next up …

Overview Tikhonov SVD Linear Nonlinear Lp-norms Field scale The future