Inversions of time-domain spectral induced polarization data using stretched exponential

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SUMMARY
We provide a two-stage approach to extract spectral induced polarization (SIP) information from time-domain IP data. In the first stage we invert dc data to recover the background conductivity. In the second, we solve a linear inverse problem and invert all time channels simultaneously to recover the IP parameters. The IP decay curves are represented by a stretched exponential (SE) rather than the traditional Cole–Cole model, and we find that defining the parameters in terms of their logarithmic values is advantageous. To demonstrate the capability of our simultaneous SIP inversion we use synthetic data simulating a porphyry mineral deposit. The challenge is to image a mineral body that is hosted within an alteration halo having the same chargeability but a different time constant. For a 2-D problem, we were able to distinguish the body using our simultaneous inversion but we were not successful in using a sequential (or conventional) SIP inversion approach. For the 3-D problem we recovered 3-D distributions of the SIP parameters and used those to construct a 3-D rock model having four rock units. Three chargeable units were distinguished. The compact mineralization zone, having a large time constant, was distinguished from the circular alteration halo that had a small time constant. Finally, to promote the use of the SIP technique, and to have further development of SIP inversion, all examples presented in this paper are available in our open source resources (https://github.com/simpeg-research/kang-2018-spectral-inducedpolarization).

Key words: Electrical properties; Electromagnetic theory; Inverse theory.

1 INTRODUCTION
Earth materials can accumulate electrical charges when an electric field is applied and, when this occurs, the material is said to be ‘chargeable’. There are different physical models associated with this effect: electrode polarization, membrane polarization, electrical double layer and Maxwell–Wagner polarization (Marshall & Madden 1959; Wong 1979; Revil et al. 2015), but they all produce a polarization of electrical charges. Macroscopically, we see this effect as a frequency dependence of the electrical conductivity such as shown in Fig. 1(a). The shape of the conductivity spectrum, differences between high and low limits of the real part, the location of the peak in the imaginary part and width of the transition will differ between rocks. Thus, if we are able to evaluate the complex conductivity, we can discriminate between different materials such as clays, sulfides and graphites. (Pelton et al. 1978).

The potential for achieving this has long been realized and it has spawned spectral induced polarization (SIP) surveys, which record earth responses over a broad frequency range (e.g. 0.01–1000 Hz). Surveys can also be carried out in the time-domain, and the response function, obtained by taking the Fourier transform of the frequency-domain response, yields time-domain decays like that shown in Fig. 1(b).

In practice, the challenge is to carry out electromagnetic surveys, and invert the acquired data, to find $\sigma(x, y, z; \omega)$, or its time-domain counterpart, within the earth. These are 4-D functions and hence the inversion is much more challenging than just trying to find a 3-D frequency-independent conductivity. Usually, the number of unknowns is reduced by introducing a parametrization such as Cole–Cole (Cole & Cole 1941; Tarasov & Titov 2013):

$$\sigma_{cc}(\omega) = \sigma_{\infty} - \frac{\eta_{cc}\sigma_{\infty}}{1+(i\omega\tau_{cc})^{\gamma_{cc}}}, \quad (1)$$

where $\sigma_{\infty}$ is the conductivity at infinite frequency, $\eta_{cc} = \sigma_{\infty} - \sigma_{0}$ is the chargeability, $\tau_{cc}$ is the time constant (s), and $\gamma_{cc}$ is the frequency exponent; the subscript $cc$ indicates Cole–Cole and $\sigma_{0}$ is conductivity at zero frequency. Depending upon the Cole–Cole parameters, complex conductivity spectra will vary. For instance, in the frequency-domain, as $\tau_{cc}$ decreases, the peak of the imaginary part moves to higher frequency (Fig. 1a). Similarly, in the time-domain, time conductivity curve decays slowly with increasing $\tau_{cc}$ as shown in Fig. 1(b).

With the aid of parametrization, SIP inversion approaches in both frequency-domain (Kemna et al. 2004) and time-domain (Yuval & Oldenburg 1997; Höning & Tezkan 2007; Fendaca et al. 2013) have been developed. These have ignored EM induction effects but some
more recent studies have attempted to deal with the complications introduced by EM coupling. (Commer et al. 2011; Xu & Zhdanov 2015; Kang & Oldenburg 2017; Belliveau & Haber 2018).

In this paper, we focus on the time-domain surveys and assume that there are no EM induction effects, or that they have been removed in a pre-processing step (Kang & Oldenburg 2016). Two main approaches are then commonly used to extract SIP information. In the first, all four Cole–Cole parameters (Fiandaca et al. 2013) are sought at once. In principle, this is appealing but solving the large multiparameter system can be challenging because of the non-uniqueness of the problem and getting trapped in various local minima. A second approach implements a two-stage strategy where the dc data are first inverted to provide the conductivity. The sensitivities from the final stage of the dc inversion provide a mapping that linearly relates IP data to a pseudo-chargeability (Oldenburg & Li 1994). This approach is reasonably valid so long as the intrinsic chargeability is small (say \( \eta_0 < 0.2 \)). Pseudo-chargeability is often interpreted directly in field applications but, because it is a nonlinear function of the Cole–Cole parameters, it may be further analysed to reveal the SIP parameters (Yuval & Oldenburg 1997; Höning & Tezkan 2007) by inverting time channels individually and then inverting the recovered pseudo-chargeabilities to find Cole–Cole parameters. That has drawbacks because there is no connection between the individual inversions; each has its own trade-off parameter and convergence criterion. We anticipate that higher quality information can be obtained by inverting all IP data at once.

Although the Cole–Cole representation works well for frequency-domain and time-domain data, it is more computationally intensive to implement for the time-domain because of the need to carry out the required Fourier transforms. This becomes particularly evident when working in 3-D and using many cells. To reduce the computation cost, we use a stretched exponential (SE) function (Kohlrausch 1854; Hilfer 2002) to represent the IP decays. This has an analytic form for the impulse conductivity and hence it is computationally efficient. Recently, Belliveau & Haber (2018) used the SE function to simulate IP effects in time-domain EM data. They use the SE function to convert Ohm’s law into an ordinary differential equation, whereas we use the SE function to derive an impulse conductivity function in time-domain.

The main goal of this paper is to develop a computationally efficient algorithm that uses a two-stage procedure to invert all of the IP data at once to recover the SE parameters that define IP characteristics for earth materials. The most interesting structures are 3-D (e.g. isolated targets and rough terrains) and hence applying this SIP inversion in 3-D is an important task. For time-domain problems it is important to incorporate the repetitive input current waveform (Fiandaca et al. 2012) and so we address this issue as well.

Our paper proceeds as follows. First, we introduce the SE conductivity and compare it to the Cole–Cole conductivity (Section 2.1). With the SE conductivity, we then show how time-domain IP data are simulated (Section 2.2) and develop our SIP inversion methodology (Section 2.3). Finally the technique is applied to a synthetic example emulating a porphyry deposit where we attempt to discover a mineralized zone lying in an alteration halo with equal chargeability. We compare the results obtained from individual inversions with those from our all-at-once approach. With the latter approach the mineralized zone can be distinguished from the halo because of the different time constant values (Section 3).

2 METHODOLOGY

2.1 SE and Cole–Cole functions

Using the Cole–Cole parametrization for an IP response in the time-domain is computationally expensive because of the need to carry out all of the required Fourier transforms. It would be advantageous to have a parametrization in the time-domain. In this section we use the SE and show how its parameters relate to those of the Cole–Cole model.

The time domain SE function (Kohlrausch 1854) can be defined as

\[
f(t) = \exp \left( -\left( \frac{t}{\tau_{se}} \right)^{\nu_{se}} \right).\]  

(2)

This can be used to define step-off conductivity response:

\[
s_{se}^{\text{step-off}} = \sigma_{se} \otimes (1 - \delta_{se}^{\text{step-on}})
\]

\[
= \begin{cases} 
  \sigma_{se}(1 - \eta_{se}), & t \leq 0; \\
  -\sigma_{se} \eta_{se} \exp \left( -\left( \frac{t}{\tau_{se}} \right)^{\nu_{se}} \right), & t > 0.
\end{cases}
\]  

(3)

where \( \delta_{se}^{\text{step-on}}(t) \) is the Heaviside step function, and subscript \( se \) stands for SE. Here \( \otimes \) indicates a convolution operator. For instance \( a \otimes h \) can be defined as

\[
a \otimes h = \int_{-\infty}^{\infty} a(u)h(t-u)du.
\]  

(4)

As in the Cole–Cole representation there are three parameters \( (\eta, \tau \) and \( c) \). Putting the SE values into the CC model does not yield the
same response as evaluating the Fourier transform of eq. (1) with these same values. However, the correspondence is close enough so that intuition previously developed by using CC parameterization is still useful.

When \( c_{cc} = 1 \) (often called Debye model), the two response functions are equivalent. For different sets of parameters the functions differ and finding explicit a relationship between the SE and the Cole–Cole parameters is not possible (Hilfer 2002). However, for a given time range (e.g. time domain dc-IP: 0.1 ms–4 s) and with an error threshold (2 per cent standard deviation), there can be good agreement between the two. To show this, we fix \( \sigma_\infty = 1 \, \text{S m}^{-1} \) and \( \eta_{cc} = 100 \, \text{mV V}^{-1} \), and compute the step-off conductivity for the Cole–Cole model for variable \( c_{cc} \) and \( \tau_{cc} \); these are shown in Fig. 2 (solid lines). As \( c_{cc} \) increases (green to red curves), the time decay gets broader, and as \( \tau_{cc} \) decreases, the response decays faster (Figs 2a and b). We carry out a small parametric inversion to fit these Cole–Cole decays with the SE function to estimate the equivalent SE parameters. The results are presented in Fig. 2 and Table 1. Overall, the SE function closely matches the Cole–Cole response for a given time range. As shown in Table 1, \( \eta_{cc} \) is coincident with \( \eta_{cc} \), but \( \tau_{cc} \) overestimates \( \tau_{cc} \), and \( c_{cc} \) underestimates \( c_{cc} \); this discrepancy gets smaller as \( c_{cc} \) increases. Therefore, when interpreting the SE parameters, the readers can use their understanding based upon Cole–Cole parameters qualitatively, but for quantitative interpretation extra care might be necessary.

We now want to use this SE function and find an expression for the SE conductivity, \( \sigma_{se} \), which can be considered as an impulse response. This can be obtained by taking the derivative of \( \sigma_{step-off} \) with respect to time and multiplying by \(-1\):

\[
\sigma_{se}(t) = \sigma_\infty \delta(t) - \sigma_\infty \eta_{se} \exp\left(-\frac{t}{\tau_{se}}\right) \exp\left(-\frac{t}{\tau_{se}}\right).
\]

(5)

By using 5, in the following section, we derive the linear IP equation, which will allow us to compute time-domain IP responses.

Note that the impulse SE function derived in Belliveau & Haber (2018) is slightly different from eq. (5). This difference arises due to two reasons. First, we used a Cole–Cole model from Tarasov & Titov (2013), but Belliveau & Haber (2018) used one from Pelton et al. (1978). Secondly, we used a SE function shown in eq. (2), whereas they used:

\[
f(t) = \exp\left(-\frac{t}{\tau}\right).
\]

Our choice of the SE function results in a closer connection between Cole–Cole and SE parameters.

### 2.2 Simulating time-domain IP data with SE function

The linear IP function (Seigel 1959; Oldenburg & Li 1994), which has been broadly used to invert IP data, can be written as

\[
d^{IP}(t) = J[\sigma_\infty][\tilde{\eta}(t)],
\]

(6)

where \( d^{IP}(t) \) is an IP datum at time \( t \), \( J \) is a sensitivity function related to the background conductivity \( (\sigma_\infty(x, y, z)) \) and \( \tilde{\eta}(x, y, z; t) \) is a pseudo-chargeability, which includes information about IP effects. The sensitivity function is:

\[
J = \frac{\partial F_{dc}[\sigma_\infty]}{\partial \log(\sigma_\infty)},
\]

(7)

where \( F_{dc} \cdot [\cdot] \) is the Maxwell dc operator that computes the dc response for a given conductivity model. To use the SE function with this linear IP equation, it is necessary to define pseudo-chargeability and derive an explicit form.

We rewrite the impulse SE conductivity, \( \sigma_{se} \) as

\[
\sigma_{se}(t) = \sigma_\infty \delta(t) - \sigma_\infty \eta_{se} \tilde{\eta}(t),
\]

(8)

where the impulse pseudo-chargeability, \( \tilde{\eta}(t) \), is

\[
\tilde{\eta}_{se}(t) = \eta_{se} \exp\left(-\frac{t}{\tau_{se}}\right) \exp\left(-\frac{t}{\tau_{se}}\right).
\]

(9)

From Kang & Oldenburg (2016), the pseudo-chargeability, \( \tilde{\eta}(t) \), can be defined as

\[
\tilde{\eta}(t) = \int_{-\infty}^{t} \tilde{\eta}(t - b) w(b) db,
\]

(10)

where \( w(t) \) is the normalized input current waveform. When a step-off current is injected, \( w(t) \) is equal to \( 1 - u^{step-off}(t) \) and therefore the pseudo-chargeability can be written as

\[
\tilde{\eta}(t) = \left\{ \begin{array}{ll}
\eta_{se} \exp\left(-\frac{t}{\tau_{se}}\right), & t > 0 \\
0, & t \leq 0
\end{array} \right.
\]

(11)

With eqs (11) and (6), the observed time-domain dc-IP data can be written as

\[
d(t) = \left\{ \begin{array}{ll}
F_{dc}[\sigma_\infty] + J[\sigma_\infty][\tilde{\eta}(t), & t \leq 0 \\
F_{dc}[\sigma_\infty] + J[\sigma_\infty][\tilde{\eta}(t), & t > 0
\end{array} \right.
\]

(12)

Here, \( t \leq 0 \) and \( t > 0 \) indicates when the input current is on and off, respectively. Note that when the current is on, there is a dc signal, but it does not exist in the off-time. Hence, any off-time data can be considered as IP data (assuming of course that there is no EM coupling). Considering that an IP datum is a small perturbation of dc datum (Oldenburg & Li 1994), we ignore \( \delta^0 \) when \( t \leq 0 \), and hence the time-domain dc-IP data can be written as

\[
d(t) = \left\{ \begin{array}{ll}
F_{dc}[\sigma_\infty], & t \leq 0 \\
F_{dc}^{} \delta^0(t), & t > 0
\end{array} \right.
\]

(13)

The assumption made here is equivalent to assuming \( \eta \ll 1 \) resulting in

\[
\sigma_{se} \approx \sigma_\infty \approx (1 - \eta),
\]

(14)

where \( \sigma_0 \) is conductivity at zero frequency (or dc conductivity).

The explicit form of the pseudo-chargeability (eq. 11) clearly shows the numerical advantage of the SE conductivity function compared to the Cole–Cole function for computing time-domain IP data. If the Cole–Cole conductivity is chosen then to evaluate eq. (10) we first need to compute \( \tilde{\eta}(\omega) \) and \( w(\omega) \) in the frequency domain, multiply them, and convert into time-domain for each 3-D voxel. Now, for a given distribution of \( \sigma_\infty, \eta_{se}, \tau_{se} \) and \( c_{se} \), time-domain IP data during the off-time can be computed directly using eq. (6).

For general situations where repetitive rectangular pulses are injected, a linear combination of the step-off response (eq. 11) is required to compute IP data. The details for this are described in Appendix A.

### 2.3 Spectral IP inversion methodology

To obtain a 3-D distribution of the conductivity and SE parameters from time-domain dc-IP data, we develop a gradient-based SIP inversion algorithm using the linear IP function. Our dc-SIP inversion is similar to the conventional dc-IP inversion approach (Oldenburg...
The sensitivity function (eq. 7) is evaluated and hence IP signals as

\[ \phi = \beta \phi_m(m) \]

where \( \phi \) is a measure of data misfit, \( \phi_m \) is a model regularization (or model norm) and \( \beta \) is trade-off (or Tikhonov) parameter. We use the sum of the squares to measure data misfit:

\[ \phi_d = \| W_d(d^{\text{pred}}(m) - d) \|_2^2 = \sum_{j=1}^{N} \left( \frac{d^{\text{pred}}_j - d_{\text{obs}}_j}{\epsilon_j} \right)^2, \]

where \( N \) is the number of the observed data and \( W_d \) is a diagonal data weighting matrix which contains the reciprocal of the estimated uncertainty of each datum (\( \epsilon_j \)) on the main diagonal, \( d \) is the observed data. The uncertainty for the \( j \)th datum, \( \epsilon_j \), is defined as

\[ \epsilon_j = 10^{-2} \cdot \text{percent} |d| + \text{floor}, \]

where percent, and floor, are percentage error (per cent) and noise floor, respectively. Percentage error handles error proportional to the amplitude of the observed data, whereas noise floor captures constant noise level (e.g. sensor sensitivity). For instance, in Section 3, we will use 5 per cent percent error and 1 mV V\(^{-1}\) noise floor for SIP inversions. Because the gradient-based inversion is used, sensitivity of the predicted data with regard to the inversion model, \( \frac{\partial d^{\text{pred}}}{\partial m} \), must be obtained, and its derivation is shown in Appendix B.

Note that this sensitivity function is different from that arising from the petrophysical work (Pelton et al. 1978) is \( \eta_{se} \), \( \log(\tau_{se}) \), and \( c_{se} \), but other combinations of logarithms or scaled representations might have practical advantages. Rather than making those decisions now we generically write our inversion model as \( m \) and postpone a final decision until we perform numerical analyses in Section 3.3.

Our inversion model, \( m \), is written as

\[ m = \begin{bmatrix} m^\eta \\ m^\tau \\ m^c \end{bmatrix}, \]

where \( m \) with the superscript \( \eta, \tau \) or \( c \), indicates the model related to each of the SE parameters. Note that, if \( m^\phi \) is a \( (M \times 1) \) vector, then the inversion model is \( (3M \times 1) \).

To solve our inverse problem a gradient-based approach is used to minimize an objective function
smooth model which is close to a reference model, \( \mathbf{m}_{\text{ref}} \). The inversion model includes information about three different SE models in 3-D space and \( \phi_{a} \) can be written as

\[
\phi_{a} = \sum_{i=m,x,y,z} \phi_{a,i},
\]

where the regularization for each SE parameter is

\[
\phi_{a,i} = \alpha_{i} \| W_{i}(\mathbf{m} - \mathbf{m}_{\text{ref}}) \|_{2}^{2} + \sum_{j=m,x,y,z} \alpha_{j} \| W_{j}(\mathbf{m} - \mathbf{m}_{\text{ref}}) \|_{2}^{2},
\]

where \( W_{i} \) is a diagonal matrix containing volumetric information of the cells, and \( W_{i}, W_{j}, \) and \( W_{l} \) are discrete approximations of the first derivative operator in \( x, y \) and \( z \) directions, respectively. The \( \alpha_{i}'s \) are weighting parameters that balance the relative importance of producing small or smooth models (Tikhonov & Arsenin 1977; Oldenburg & Li 2005).

Bound constraints, based upon prior knowledge or physical principles (e.g. \( 0 \leq \eta_{se} < 1 \) and \( 0 \leq c_{se} \leq 1 \)) can be included in the optimization

\[
\text{minimize } \phi_{d}(\mathbf{m}) + \beta \phi_{a}(\mathbf{m})
\]

subject to \( \mathbf{m}_{l} \leq \mathbf{m} \leq \mathbf{m}_{u} \)

\[
\phi_{d} \leq \phi_{d}^{*},
\]

where \( \mathbf{m}_{l} \) and \( \mathbf{m}_{u} \) are lower and upper bounds. We solve this problem using a projected Gauss–Newton (GN) method (Kelley 1999). The trade-off parameter \( \beta \) is determined using a cooling technique where \( \beta \) is progressively reduced from some high value (Oldenburg & Li 2005; Kang et al. 2014). The inversion is started with an initial model, \( \mathbf{m}_{0} \), and the non-linear iterations are stopped when the target misfit, \( \phi_{d}^{*} \), is reached (Oldenburg & Li 2005). A target misfit of \( \phi_{d}^{*} = n_{d} \), is equivalent to a root-mean-square misfit (rms) of 1. Details of how we are setting up inversion parameters will be discussed in Section 3.3. Both 2-D and 3-D SIP inversion algorithms are developed, although the above description is based upon 3-D case. SIMPEG’s inversion framework (Cockett et al. 2015; Heagy et al. 2017) is used for implementation of the SIP inversion algorithms and packaged up as a SimPEG-SIP module. This SimPEG-SIP module is available through main SIMPEG package (https://github.com/simpeg/simpeg).

### 3 SYNTHETIC EXAMPLES

#### 3.1 Setup and physical property

To demonstrate the capability of our dc-SIP inversion procedure, we consider a 3-D synthetic model of a mineralized porphyry deposit shown in Fig. 3. The central stock is surrounded by an alteration unit that forms a halo structure (donut-shape in Fig. 3). The top part of the porphyry has been eroded and is covered by overburden in which patches of clay are embedded. A rectangular block-like mineralized zone exists within the halo structure.

We suppose that the halo structure includes fine-grained mineralization (e.g. disseminated sulphides), whereas the mineralization zone has a much greater grain size; this results in the two units having distinctive IP characteristics. Table 1 shows the values of conductivity and IP parameters for each geological unit and Fig. 4 shows the vertical section of conductivity and IP parameters at \( y = 0 \) m. The halo and clay units have lower resistivities (300 and 500 m\( \Omega \)m, respectively) than the background (\( \sim 1000 \) m\( \Omega \)m). The porphyry and the stock have greater resistivity (5000 m\( \Omega \)m). Hence, the conductive halo and clay can be distinguished from the resistive porphyry and the stock based upon their resistivity values. The three chargeable units: clay, halo and mineralization have the same chargeability of 100 m\( V -1 \), and hence the mineralization is not distinguishable from the clay and halo units based upon chargeability alone. However, the mineralization has a greater time constant (5s) than the other two chargeable units (\( \leq 0.5 \) s). These results because of its greater grain size (Pelton et al. 1978; Revil et al. 2015). The time constant in the SIP decays is thus a potentially diagnostic property that can be used to distinguish the mineralization from the halo structure. The frequency exponent of the halo and the mineralization are equal to 0.5, whereas that of the clay is 0.8 and hence it might also be possible to distinguish between the alteration halo and clay. Thus, if we can recover 3-D distributions of spectral IP information (e.g. \( \tau_{se}, c_{se} \)), the three chargeable units: mineralization, halo and clay, should be distinguishable.

In the following numerical experiments, we generate synthetic dc and IP data for the porphyry model, and investigate SIP information in the data by fitting them using the SE function (Section 3.2). To extract more rigorous information, we first invert the synthetic dc and IP data in 2-D, then in 3-D. The 2-D inversions, in Section 3.3, are used to answer the question about whether we want to invert for the spectral parameters or their logarithms. It also allows us to compare the results from our simultaneous inversion with those from a conventional approach (Yuval & Oldenburg 1997), which separately inverts multiple time channels of the IP data and then extracts spectral IP information by fitting resultant pseudo-chargeabilities to a Cole–Cole model. Based upon these results we proceed with our 3-D dc and SIP inversions and ultimately recover a 3-D rock model in Section 3.4.

#### 3.2 DC and IP data

Five lines of dc-IP data are acquired with a dipole–dipole array, with 50 m electrode spacing and n-spacing = 10; each sounding has 10 stations resulting in maximum spacing of 500 m between source and receiver pairs. There are 135 dc data for each line and 675 in total. The IP data consist of 21 time channels, logarithmically sampled from 1 ms to 4 s in the off-time. The input current is a half-duty cycle waveform having 4 s on- and off-time (\( \tau_{se, \text{on}}=2 \); see Appendix A). There are 2835 IP data for each line and hence 14 175 in total. For the discretization of the 3-D tensor mesh, we use 12.5 \( \times \) 25 \( \times \) 10 m cell for the core region; the total number of cells is 100 \( \times \) 50 \( \times \) 25 = 130 000. To generate dc data, the SIMPEG-de package is used. The IP data are generated using the linear IP function (eq. 6). Five percent Gaussian noise is added to both dc and IP data. Fig. 5(a) shows the simulated dc data (apparent resistivity), and Figs 6(b) and (c) show the IP data (apparent chargeability) at 1 ms and 4 s, respectively. The IP data show different characteristics because of the spectral polarization effects.

To investigate the SIP information content in the IP data, we fit each IP decay with the SE parameters. The predicted IP decay with the step-off input current can be written as

\[
d^{IP} = \frac{V_{i}(t)}{R_{0}} \times 10^{\eta_{se}} \times \eta_{se} \exp \left( -\frac{t}{\tau_{se}} \right) \times 10^{c_{se}},
\]

By fitting all of the IP decays, we obtain three SE parameters: \( \eta_{se}, \tau_{se} \) and \( c_{se} \) for each point in the pseudo-section (Fig. 5). Note that each point in the pseudo-section has an IP decay consisting of 21 time channels from 1 ms to 4 s.
Figure 3. 3-D porphyry model. Left-hand panel: plan map at \( z = -95 \) m. Right-hand panel: vertical section at \( y = 0 \) m. Seven geological units are marked as different colours. White crosses on the plan map indicates electrode locations used for the dc-IP survey.

Figure 4. Stretched exponential parameters in section view at \( y = 0 \) m. (a) conductivity at infinite frequency \( (\sigma_\infty) \), (b) chargeability \( (\eta_{se}) \), (c) time constant \( (\tau_{se}) \) and (d) frequency exponent \( (c_{se}) \).
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3.3 2-D inversion

Before inverting in 3-D, there are two questions to be answered: (1) How do we define the model parameters $m_1$ and $m_2$? and (2) How much resolution can be added by simultaneously inverting time-domain IP data compared to the conventional SIP inversion approach where each time channel is inverted separately? These questions will be addressed by working in 2-D.

For the 2-D model, we take a cross-section of Fig. 4, along the centreline ($y = 0$ m) and assume this a 2-D structure. We discretize this using $25 \text{ m} \times 15 \text{ m}$ cells. To answer the questions, we first invert dc data to recover a conductivity model. Then we apply the conventional IP inversion approach to recover 2-D distributions of pseudo-chargeability at individual time channels; estimates of SE parameters are obtained from analysing these. Next, we use our SIP inversion approach to estimate SE parameters. Finally, the SE parameters from the two approaches are then compared.

We start with the conventional IP inversion approach. dc data are first inverted to recover the background conductivity. For the dc inversion, a 1000 $\Omega \cdot \text{m}$ half-space is used for an initial model. Fig. 7(a) shows the recovered conductivity model, $\sigma_{est}$. The resistive stock and surrounding conductive halo structures are successfully imaged. Then by using the recovered conductivity, we compute the sensitivity function ($\mathbf{J}[\sigma_{est}]$), and proceed with the IP inversion using the linear IP function (eq. 6). We invert IP data at 1 ms (Fig. 5b) to recover pseudo-chargeability. Fig. 7(a) shows the recovered pseudo-chargeability, and all chargeable zones corresponding to clay, halo,
mineralization are imaged well. However, distinguishing these three different chargeable units are not possible just with the recovered pseudo-chargeability model since all of them are imaged with high pseudo-chargeability values.

We would like to extract more polarization information entangled in the multiple time channels of the IP data such as those shown in Fig. 6. Following Yuval & Oldenburg (1997), we successively invert other time channels of the IP data (1 ms–4 s), and obtain pseudo-chargeabilities at multiple time channels. Fig. 7(b) shows the pseudo-chargeability at the last time channel (4 s). Interestingly, near surface chargeable materials have disappeared due to their fast-decaying nature (small $\tau_{se}$ and large $c_{se}$). More quantitative information can be extracted by interpreting the recovered pseudo-chargeability at multiple times.

To extract SE parameters from pseudo-chargeability at each 2-D prism, we set up a small inverse problem that fits the pseudo-chargeability at multiple times to obtain three SE parameters. Rather than doing this on a pixel by pixel basis we solve an inverse problem that fits all prisms of the pseudo-chargeability together and recover a distribution of the SE parameters. Here the observed data are the
recovered pseudo-chargeability, \( \tilde{\eta}_{cse} \), and the predicted data can be written as

\[
d^\text{pred} = \tilde{y}(\tilde{\eta}_{cse}, \tau_{cse}, c_{cse})
\]  

(26)

This fitting procedure is similar to how we fit the IP decays with the SE function (eq. 25).

Our inversion methodology is the same as that provided in Section 2.3 but we have not yet addressed the parametrization for the inversion. As stated in Section 2.3, for each of the SE parameters one can choose either a linear or log model. As a first trial, we choose \( m = [\eta_{cse}, \log(\tau_{cse}), c_{cse}] \) and carry out the inversion. The initial model, \( m_0 \), is a homogeneous earth having the uniform SE parameters: \( \eta_{cse} = 0.01, \tau_{cse} = 0.1, c_{cse} = 1 \); 5 per cent Gaussian errors are assigned as uncertainties. To ignore any impact from regularization, we set the trade-off parameter \( \beta = 0 \). Unfortunately, this inversion does not converge and is trapped in a local minimum. Considering that we are using a gradient-based inversion, the reason for this is likely associated with the scale of the gradients. For instance, if the chargeability has much greater sensitivity compared to the others, our gradient-based inversion will try to update chargeability, but not the other two parameters. To investigate this, we compute the gradient of the data misfit term, \( \nabla_m \phi_d \):

\[
\nabla_m \phi_d = \left( \frac{\partial d^\text{pred}}{\partial m} \right)^T r
\]  

(27)

where the residual, \( r \), is

\[
r = W_d(d^\text{pred} - d^{\text{obs}})
\]  

(28)

The gradient term, \( \nabla_m \phi_d \), can be separated into gradients for each of three model parameters: \( m^\eta, m^\tau, m^c \). Fig. 8(a) shows the histogram for \( \log(\eta_{cse}) \). The gradient of \( \eta_{cse} \) generally dominates and thus the inversion will preferentially update chargeability, but not the other two parameters. We believe this is the main reason why the inversion did not converge. This can be rectified by using the logarithm of each parameter and hence we try \( m = [\log(\eta_{cse}), \log(\tau_{cse}), \log(c_{cse})] \). With this choice, the inversion converges after five iterations. Fig. 8(b) shows the gradients for each SE parameter. Character of the gradient distributions (e.g. mean and variance) for each of three SE parameters shown in Fig. 8(b) are more similar compared to those shown in Fig. 8(a). This results in more uniform weighting for each SE parameter when computing the gradient of \( \phi_d \). Based upon this experiment, we chose our inversion model as

\[
m = \begin{bmatrix} m^\eta \\ m^\tau \\ m^c \end{bmatrix} = \begin{bmatrix} \log(\eta_{cse}) \\ \log(\tau_{cse}) \\ \log(c_{cse}) \end{bmatrix}
\]  

(29)

This choice of the inversion parameter is same as Fiandaca et al. (2012), although the reasons for that choice were not provided in their paper.

The recovered \( \eta_{cse}, \tau_{cse}, c_{cse} \) are presented in Figs 9(a)–(c), respectively. The recovered \( \eta_{cse} \) is similar to the recovered pseudo-chargeability at 1 ms (Fig. 7b). In the chargeable regions, \( \tau_{cse} \) has a greater value than the background, whereas both chargeable anomalies at the left- and right-hand sides show similar \( \tau_{cse} \) values. Chargeable regions at the near surface (clay) show large \( c_{cse} \) values (~1) compared to the other chargeable regions, and hence the chargeable clay can be distinguished from two other chargeable units. The cross-plot of the recovered \( \tau_{cse} \) and \( c_{cse} \) shown in Fig. 11(a) clearly shows two distinct clusters of points due to different \( c_{cse} \) values, whereas the recovered \( \tau_{cse} \) shows similar values (~0.5). Therefore, with this sequential SIP inversion approach we can obtain the location of chargeable regions, and distinguish the clay from the halo and mineralization. However, the distinction between the chargeable halo and the mineralization is not evident.

We now use our approach, which simultaneously inverts 21 time channels of the IP data, to recover the distribution of \( \eta_{cse}, \tau_{cse} \) and \( c_{cse} \). Although not shown here, a similar analysis for the gradients was performed, and based upon that, we chose the same parametrization of the inversion model by taking the logarithm of all three SE parameters. The same homogeneous halfspace model (\( \eta_{cse} = 0.01, \tau_{cse} = 0.1, c_{cse} = 1 \)) was used as the initial and reference model; the other parameters of the inversion are summarized in Table 3.

This SIP inversion converges after nine iterations. Fig. 10 shows the recovered SE parameters. The recovered \( \eta_{cse} \) is similar to that from the previous inversion (Fig. 9a). However, the recovered \( \tau_{cse} \) shows significant differences, particularly for the chargeable region at the right hand side compared to Fig. 9(b); a much greater \( \tau_{cse} \) value is recovered (~2 s) as shown in Fig. 10(b). The cross-plot of the recovered \( \tau_{cse} \) and \( c_{cse} \) in Fig. 11(b) differs from the sequential inversion results (Fig. 11a); another cluster having large \( \tau_{cse} \) and small \( c_{cse} \) is observed. Hence, with the simultaneous SIP inversion, a chargeable region having large \( \tau_{cse} \) can be distinguished from two other chargeable units. In addition, similarly with the recovered \( c_{cse} \), clay can be distinguished from the halo and mineralized zone (Fig. 11b). In summary, our 2-D analysis has shown that, with a proper setup of the SIP inversion, the simultaneous SIP inversion can obtain important spectral information about the mineralization (large time constant). We now address the 3-D problem.

### 3.4 3-D inversion

We now apply our 3-D dc-SIP inversion procedure. It includes two steps: (a) dc inversion to recover conductivity model and (b) simultaneous SIP inversion to recover SE parameters: \( \eta_{cse}, \tau_{cse}, c_{cse} \). For both dc and SIP inversions we use the same inversion parameters used in the 2-D inversions; both inversions reached the target misfit.

The 3-D dc inversion recovers a conductivity model, \( \sigma(x, y, z) \). Fig. 12(a) shows the plan map (\( z = -95 \, m \)) and vertical section (\( y = 0 \, m \)) of the recovered model. In the plan map, the resistive centre (the stock) and the conductive circular structure (the halo) are successfully imaged. In the vertical section, the conductive near surface is also imaged. In the second step we recover three SE parameters by simultaneously inverting 21 time channels of IP data. Fig. 12(b) shows the recovered chargeability model. The chargeable volumes are colocated with conductive anomalies seen in the dc inversion. The time constants for the halo region are generally high but the region on the right-hand side, associated with the mineralized zone, has a significantly higher \( \tau_{cse} \) value. The cross-plot of the recovered \( \tau_{cse} \) and \( c_{cse} \) with colored resistivity is shown in Fig. 13(a); here \( \eta_{cse} \) values greater than 80 mV V\(^{-1}\) are plotted. Using the recovered SE distributions and their cross-plot, we analyse the clustering of SE parameters and distinguish different rock units.

All chargeable volumes show low resistivity, and spatially there are two main features: circular features due to halo, and isolated patches at near surface. The overall circular volume shows large \( \tau_{cse} \) and small \( c_{cse} \), but the right side shows greater \( \tau_{cse} \) than the left-hand side (right-hand panel of Fig. 12c). This distinction is clearly shown on the x-axis of Fig. 13(a). The near surface patches show higher \( c_{cse} \) compared to the rest of the chargeable volumes (right panel of Fig. 12d), and this trend is presented in the y-axis of Fig. 13(a). Therefore, three main rock units: R1–R3 can be distinguished in the chargeable volumes:

(i) R1: small \( \tau_{cse} \) and small \( c_{cse} \).
Figure 8. Histograms of the gradient for the data misfit, $\nabla_m \phi_d$. Blue, orange and green colours indicate histogram correspond to each of the inversion models: $m^\eta$, $m^\tau$ and $m^c$. Two sets of the inversion models are considered: (a) $\eta_{se}$, $\log(\tau_{se})$, $c_{se}$ and (b) $\log(\eta_{se})$, $\log(\tau_{se})$, $\log(c_{se})$.

Figure 9. Recovered distributions of SE parameters extracted from the pseudo-chargeability at 21 time channels (e.g. Figs 7b and c). (a) Chargeability ($\eta_{se}$), (b) time constant ($\tau_{se}$) and (c) frequency exponent ($c_{se}$).

(ii) R2: small $\tau_{se}$ and large $c_{se}$
(iii) R3: large $\tau_{se}$ and small $c_{se}$

This rock classification is also shown in Fig. 13(b) with the cross-plot. The final 3-D rock model is shown in Fig. 14. Considering this is a porphyry deposit, we interpret four rock units. The non-chargeable rock unit, R0, can be considered as host rock. R1, which shows up as a chargeable circular feature in the plan map, is interpreted as the alteration halo; R3, is interpreted to be a mineralized zone (the large grain size associated with mineralization should generate a large $\tau$); R2 having large $c_{se}$ is interpreted as clay. Threshold values of the SE parameters to interpret three rock units are summarized in Table 4. This classification of three different rock types: halo, clay and mineralization would not have been possible if a single time channel of IP data, or integrated IP data (e.g. Newmont standard time windows: 0.8 s–1.4 s), was inverted.

3.5 Summary for the synthetic example

By using the synthetic porphyry model as a motivating example we have investigated three crucial items. The first concerned the choice of parameters for the SIP inversion. Choosing logarithmic parameters, $m = \{\log(\eta_{se}, \tau_{se}, c_{se})\}$, balanced the gradient ($\nabla_m \phi$) for each SE parameters (Fig. 8). This resulted in a more robust convergence and the algorithm was not so easily trapped in a local
Inversions of time-domain SIP data

Figure 10. Recovered distributions of SE parameters extracted from the SIP inversion, which simultaneously inverts 21 time channels of the IP data. (a) chargeability ($\eta_{se}$), (b) time constant ($\tau_{se}$) and (c) frequency exponent ($c_{se}$).

Figure 11. Cross-plots of the recovered time constant ($\tau_{se}$) and frequency component ($c_{se}$) from (a) sequential and (b) simultaneous SIP inversions with colored chargeability ($\eta_{se}$).

Table 3. Inversion parameters of dc and SIP inversions for synthetic examples.

<table>
<thead>
<tr>
<th></th>
<th>dc</th>
<th>SIP</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncertainty (per cent, floor)</td>
<td>(5 per cent, 1 mV V$^{-1}$)</td>
<td>(5 per cent, 1 mV V$^{-1}$)</td>
</tr>
<tr>
<td>$m_0$</td>
<td>1000 $\Omega$m (10 mV V$^{-1}$)</td>
<td>0.1 s, 1</td>
</tr>
<tr>
<td>$m_{sef}$</td>
<td>1000 $\Omega$m (10 mV V$^{-1}$)</td>
<td>0.1 s, 1</td>
</tr>
<tr>
<td>$m_i$</td>
<td>$-\infty$</td>
<td>(0.01 mV V$^{-1}$, 10$^{-6}$ s, 0.01)</td>
</tr>
<tr>
<td>$m_u$</td>
<td>$+\infty$</td>
<td>(1000 mV V$^{-1}$, 10 s, 1)</td>
</tr>
<tr>
<td>$\alpha_s$</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>$\alpha_z$</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 4. Interpretation of the SE parameters to construct a 3D rock model.

<table>
<thead>
<tr>
<th>Rock unit</th>
<th>$\sigma_\infty$ (S m$^{-1}$)</th>
<th>$\eta_{se}$ (mV V$^{-1}$)</th>
<th>$\tau_{se}$ (s)</th>
<th>$c_{se}$</th>
<th>Interpretation</th>
</tr>
</thead>
<tbody>
<tr>
<td>R0</td>
<td>Low</td>
<td>$\eta_{se} &lt; 80$</td>
<td>N/A</td>
<td>N/A</td>
<td>Host</td>
</tr>
<tr>
<td>R1</td>
<td>High</td>
<td>$\eta_{se} &gt; 80$</td>
<td>$\tau_{se} &lt; 1.1$</td>
<td>$c_{se} &lt; 0.6$</td>
<td>Halo</td>
</tr>
<tr>
<td>R2</td>
<td>High</td>
<td>$\eta_{se} &gt; 80$</td>
<td>$\tau_{se} &lt; 1.1$</td>
<td>$c_{se} &gt; 0.6$</td>
<td>Clay</td>
</tr>
<tr>
<td>R3</td>
<td>High</td>
<td>$\eta_{se} &gt; 80$</td>
<td>$\tau_{se} &gt; 1.1$</td>
<td>$c_{se} &lt; 0.6$</td>
<td>Mineralization</td>
</tr>
</tbody>
</table>

The second item pertained to the SIP parameters obtained from sequential inversion versus simultaneous inversions. Using the 2-D porphyry model, we showed that better resolution can be obtained with the simultaneous inversion. In particular, we were able to distinguish a chargeable mineralization zone that had large $\tau_{se}$,
A simultaneous inversion of data at all time channels is also more robust in the presence of problematic data. For instance, low quality data at a particular time channel can greatly affect the chargeability recovered from that time channel and therefore cause problems in a subsequent parametric inversion that recovers IP parameters. However, the effects are ameliorated if those data are incorporated into the ensemble of data inverted in a simultaneous approach. The third item showed that, by carrying out the SIP inversion in 3-D, we can identify volumetric regions with different SIP parameters and construct a rock model. We applied our simultaneous SIP inversion to the porphyry model and recovered 3-D distributions of $\eta_{Se}$, $\tau_{Se}$, $c_{Se}$. This allowed us to distinguish three different chargeable volumes identified as: halo, clay and mineralization in a non-chargeable background earth. This information would have not been obtained if the usual IP inversion approach, which inverts a single channel of IP data, or a time-integral of IP data, was used.

4 CONCLUSIONS

Earth materials (e.g. clays, minerals and ice) have distinct conductivity spectra and hence their polarization characteristics can be diagnostic in geoscience applications such as mining, groundwater, and environment. In some of these problems it may be sufficient to recognize that there is a difference in conductivity between low and high frequencies (generally referred to as ‘chargeability’) and all that is necessary is to find regions that are chargeable. For time-domain IP surveys, this could be accomplished by inverting data at a single time channel, or inverting the integral of the decay curve over some time gates. Despite the success of that procedure it has been
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A long standing goal to extract more information about the rocks and to discriminate between rocks that have same chargeability. The full complex conductivity spectrum is therefore needed and this requires high quality data measured over a broad time range; modern systems now accomplish this. The next stage, and that considered here, is to invert these multi-channel data to recover the spectral information about the conductivity.

We tackle this problem in the following manner:

(i) We assume that our data are not contaminated with EM effects and that the Maxwell dc equations are valid. If EM effects are prevalent, then they must be handled in a pre-processing step (e.g. Kang & Oldenburg 2016).

(ii) We parametrize the conductivity response function as a SE. This representation has four parameters ($\sigma_\infty, \eta, \tau, c$). These are the same parameters as in the frequency Cole–Cole parametrization, but the analytic representation of the SE response function greatly speeds up computation.

(iii) Rather than solving for all parameters at once, we adopt a two-stage process where we first invert the dc data to estimate $\sigma_\infty$, and use the sensitivity from that solution to generate a linear mapping between IP data and pseudo-chargeability. This requires that the chargeability is small ($\eta < \sim 0.3$).

(iv) The IP data at multiple time channels are inverted simultaneously to recover the parameters ($\eta, \tau, c$). We use a gradient-based Gauss–Newton algorithm to solve the inverse problem and define the logarithms of $\eta, \tau, c$ as parameters in the inversion. This equalizes the distribution of the gradients and results in better convergence.

To test our algorithm we used simulated data from a 3-D porphyry model. Application of the developed SIP inversion showed that we can distinguish a chargeable mineralized zone that has a large time constant, from other chargeable volumes (clay and an alteration halo). We would have not been able to accomplish this by inverting a single time channel of IP data.

In conclusion, the inversion procedure we proposed in this paper provides a methodology for extracting not only chargeability, $\eta$, from the IP decays but also other spectral parameters: $\tau$ and $c$. As shown by the test example, knowing these parameters may play an essential role in answering geologic questions. In addition to the important next step of applying this to a field data set, there are still further items to be worked on at the algorithmic level. The first item is non-uniqueness issue due to increased number of parameters ($\tau$ and $c$), and how it can be addressed. The second item is what to do when $\eta$ is large (say $\eta > 0.5$) and there is potential breakdown of our assumptions. Careful analysis of these items, and applying the 3-D SIP inversion to field data sets, are required to demonstrate the practicality of our SIP technique. In spite of these missing ingredients, our work in this paper constructs a cornerstone to make the time-domain SIP inversion technique useful in practical applications. To promote useability of the methodology, and to invite others to contribute to this goal, the codes used for the examples presented in this

Figure 13. Cross-plot of the recovered time constant ($\tau_{se}$) and frequency exponent ($c_{se}$) from 3-D SIP inversion. (a) Cross-plot of $\tau_{se}$ and $c_{se}$ with colored resistivity. (b) Rock classification result in the cross-plot.

Figure 14. 3-D rock model constructed by using four different physical properties: $\sigma_{est}, \eta_{se}, \tau_{se}, c_{se}$ obtained from 3-D dc and SIP inversions. The rock model includes four different units, and their features are summarized in Table 3.


REFERENCES


Wong, J., 1979. An electrochemical model of the induced-polarization phenomenon in disseminated sulfide ores, Geophysics, 44(7), 1245.


APPENDIX A: HANDLING REPETITIVE WAVEFORMS

In Section 2.2, we have derived an explicit form of pseudo-chargeability assuming the input current is a step-off \((1 - u^{\text{step-on}})\) as shown in eq. (10). However, in practice we use periodic half-duty cycle waveforms having on- and off-time, then stack them to obtain both dc (on-time) and IP data (off-time). Fiandaca et al. (2012) showed the importance of considering this repetitive waveform when simulating and inverting time-domain dc-IP data. This is important when \(\tau_{se}\) is large and IP data do not reach steady-state in the prescribed on-time. This consideration of waveform is important when quantitative analyses are required (e.g. actual values of IP parameters), whereas if one’s goal is obtaining anomalous chargeable volumes and their distinction with other SE parameter (\(\tau_{se}\) and \(c_{se}\)), this consideration of actual waveform is less important. Our SIP inversion code is designed so that the user can input the number of stacks \(n_{\text{stack}}\). In this section, we illustrate how this functionality is implemented.

With a normalized current waveform, \(w(t) = l(t)/\max(l)\), pseudo-chargeability, \(\bar{\eta}(t)\), can be written as

\[
\bar{\eta}(t) = \hat{I}(t) \otimes u(t).
\]

Here \(l(t)\) is the input current. Then IP data is

\[
d^{IP}(t) = J\bar{\eta}(t).
\]

So, if we can obtain stacked pseudo-chargeability, we can compute stacked IP data. Now, consider half-duty cycle waveform having on- and off-time with period of \(T\) as shown in Fig. A1. In a simple manner, by convolving this with \(\hat{I}\) and stacking off-times \(T/2\) and \(3T/4-T\), we can obtain stacked pseudo-chargeability, \(\bar{\eta}_{\text{stack}}\). Here the number of stacks \(n_{\text{stack}}\) is 2, but if we have more cycles then \(n_{\text{stack}}\) will be increased. Pseudo-chargeability for the first pulse, \(\bar{\eta}_{1}(t)\), can be written as

\[
\bar{\eta}_{1}(t) = \bar{\eta}_{\text{stack}}(t - T/4) - \bar{\eta}_{\text{off}}(t).
\]

Figure A1. Half-duty cycle waveform having a period of \(T\).
where $\eta_{\text{off}}$ is a step-off pseudo-chargeability (eq. 11). Similarly, for the second pulse, $\tilde{\eta}_2(t)$.

$$\tilde{\eta}_2(t) = -(\tilde{\eta}_{\text{off}}(t - T/4) - \tilde{\eta}_{\text{off}}(t)) + (\tilde{\eta}_{\text{off}}(t - T/2) - \tilde{\eta}_{\text{off}}(t - T/4)).$$

(A4)

Hence, $\tilde{\eta}_{\text{stack}}(t)$ can be defined as

$$\tilde{\eta}_{\text{stack}}(t) = \frac{1}{2}(\tilde{\eta}_1(t) - \tilde{\eta}_2(t)).$$

(A5)

This can be extended to $n_{\text{stack}}$ case with a series form:

$$\tilde{\eta}_{\text{stack}}(t) = \frac{1}{n_{\text{stack}}} \left( \sum_{i=0}^{n_{\text{stack}}-1} \tilde{\eta}_{\text{off}}(t - T/2i) \cdot (-1)^i \cdot (n_{\text{stack}} - i) \right).$$

(A6)

Similarly, for on-time, stacked pseudo-chargeability, $\tilde{\eta}_{\text{stack}}$, can be written as

$$\tilde{\eta}_{\text{stack}}(t) = \frac{1}{n_{\text{stack}}} \left( \sum_{i=1}^{n_{\text{stack}}-1} \tilde{\eta}_{\text{off}}(t + T/4 + T/2(i - 1) \cdot (-1)^i \cdot (n_{\text{stack}} - i) \right).$$

(A7)

Therefore, to obtain stacked pseudo-chargeability we just need a capability to evaluate $\tilde{\eta}_{\text{off}}$. With the stacked pseudo-chargeability, IP data can be written as

$$d^{IP}(t) = J\tilde{\eta}_{\text{stack}}(t).$$

(A8)

Suppose $t_0$ is the moment current is turned off. With the assumption (eq. 14), IP data for the on-time is ignored resulting in observed dc-IP data:

$$d(t) = \begin{cases} F_{dc} \sigma_{\infty}, & t \leq t_0 \\ d^{IP}(t) = J\tilde{\eta}_{\text{stack}}(t), & t > t_0. \end{cases}$$

(A9)

Although not shown here, we have found that assuming $n_{\text{stack}} = 2$ is often enough, even when the actual number of stack is much greater than 2 (e.g. 16). Hence we have used $n_{\text{stack}} = 2$ for both synthetic and field examples in Section 3.

**APPENDIX B: SENSITIVITY**

Because we are using gradient-based inversion, obtaining a sensitivity function for each model parameter is necessary. The predicted IP data can be written as

$$d^{IP}(t) = J\tilde{\eta}(t).$$

(B1)

Required sensitivity functions are the derivative of $d^{IP}$ with respect to each SE parameter: $\eta_{se}$, $\tau_{se}$, $c_{se}$. Here, we show the derivation for these sensitivity functions when input current is a step-off.

$$\frac{\partial d^{IP}(t)}{\partial \eta_{se}} = J \frac{\partial \tilde{\eta}(t)}{\partial \eta_{se}} = \exp\left(-\left(\frac{t}{\tau_{se}}\right)^{\epsilon_{se}}\right)$$

(B2)

$$\frac{\partial d^{IP}(t)}{\partial \tau_{se}} = J \frac{\partial \tilde{\eta}(t)}{\partial \tau_{se}} = \frac{c_{se} \epsilon_{se}}{\tau_{se}} \left(\frac{t}{\tau_{se}}\right)^{\epsilon_{se}} \exp\left(-\left(\frac{t}{\tau_{se}}\right)^{\epsilon_{se}}\right)$$

(B3)

$$\frac{\partial d^{IP}(t)}{\partial c_{se}} = J \frac{\partial \tilde{\eta}(t)}{\partial c_{se}} = \eta_{se} \left(\frac{t}{\tau_{se}}\right)^{\epsilon_{se}} \exp\left(-\left(\frac{t}{\tau_{se}}\right)^{\epsilon_{se}}\right) \log\left(\frac{t}{\tau_{se}}\right)$$

(B4)

By using these sensitivity functions for the step-off case with the stacked functions shown in eq. (A7), we can obtain the needed sensitivity functions. If logarithmic variables are used the corresponding chain rule is applied (e.g. $\frac{\partial \log(\eta_{se})}{\partial \eta_{se}} = \frac{1}{\eta_{se}}$).