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# Fair amenability for semigroups

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ABSTRACT

A new flavour of amenability for discrete semigroups is proposed that generalises group amenability and follows from a Følner-type condition. Some examples are explored, to argue that this new notion better captures some essential ideas of amenability. A semigroup $S$ is left fairly amenable if, and only if, it supports a mean $m \in \ell^\infty(S)^*$ satisfying $m(f) = m(s * f)$ whenever $s * f \in \ell^\infty(S)$, thus justifying the nomenclature "fairly amenable".



## 1. Introduction

Amenability begins in essence alongside modern analysis, as it is a central property lacking in a group used to show, for example, the Banach–Tarski paradox [22]. The first working definition for what is now called amenability was given by von Neumann [21], in terms of finitely-additive measures. A group $G$ is amenable if there is a finitely-additive measure $\mu$ such that $\mu(G) = 1$, and $\mu(gA) = \mu(A)$ for all $g \in G, A \subseteq G$ ($\mu$ is *left invariant*). This definition has the advantages of being easy to comprehend, hiding very little, and it is easy to show that the free group on two generators $\mathbb{F}_2$ does not support such a finitely-additive measure.

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The first modern definition of amenability, in its form as extended to semigroups, was given by Day [4], whose concept involved invariant means. A *mean* is a non-negative linear functional $m \in \ell^\infty(S)^*$ such that $m(\chi_S) = 1$. There is a bijective correspondence between means and finitely-additive measures: to obtain a mean from a finitely-additive measure, use the Lebesgue integral construction. An element $s \in S$ acts on a function $f \in \ell^\infty(S)$ (on the left), by setting $(s \cdot f)(t) := f(st)$ for all $t \in S$. Briefly, then, a semigroup $S$ is (classically) *left amenable* when there exists such an $m$ satisfying $m(s \cdot f) = m(f)$ for all $f \in \ell^\infty(S)$. For groups this coincides exactly with the von Neumann condition. However, for semigroups it does not: in fact, a mean satisfies the above condition if, and only if, the associated finitely-additive measure $\mu$ satisfies[1] $\mu\left(s^{-1}A\right) = \mu(A)$ for all $s \in S$ [17]. This might be called left *preimage* invariance of $\mu$. A simple but surprising consequence of this definition is that all semigroups with a zero element are both left and right amenable [4], yet they cannot have a (totally) invariant finitely-additive measure [20, p. 231]. On the other hand, all semigroups with more than one distinct left zero are not left amenable [17].

Numerous other alternative definitions for amenability from group theory disagree on semigroups in general. The Følner conditions, originally shown for groups by Følner [7] and of which there are now several flavours, have varying degrees of relation to left amenability of a semigroup. The Følner conditions are useful for showing when a group has amenability, and effectively describe the essential reason all Abelian groups are amenable. Følner's original conditions were first generalised to semigroups by Frey in 1960 and subsequently a simpler proof was given by Namioka [15]. Some of the Følner-type criteria that are sufficient for left amenability of a semigroup include the weak and strong Følner conditions [1] and the weak and strong Følner–Namioka conditions [23]. A *necessary* Følner-type condition for amenable semigroups is the one described by Namioka [15].

For some of these Følner conditions, and other related conditions, if the semigroup in question is *cancellative*, then there are improved results, since the inequality $2\,|F \backslash sF| \geq |sF \triangle F| \geq 2\,|sF \backslash F|$, true for any $s \in S$ and finite $F \subseteq S$, is then saturated. For example, Frey's thesis showed that if $S$ is a cancellative semigroup that contains no free subsemigroup on two generators, and is left amenable, then every subsemigroup of $S$ is left amenable. An improvement was made recently by Donnelly [5]: if $T$ is a subsemigroup of $S$, $S$ is cancellative, $T$ does not contain a free subsemigroup on two generators, and $S$ is left amenable, then $T$ is left amenable.

Another set of results concerns translating amenability between groups and algebras. A Banach algebra $\mathfrak{A}$ is called *amenable* if $\mathcal{H}^1(\mathfrak{A}, E^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule $E$ [19, p. 43]—this is equivalent to saying all derivations are inner derivations. It is the famous theorem of Johnson [9] that shows that the group $G$ is amenable if, and only if, $\ell^1(G)$ is amenable (as a convolution Banach algebra). However, for a semigroup $S$, the amenability of $\ell^1(S)$ does not relate well to the amenability of $S$.

[1] For any $s \in S$ and $A \subseteq S$, $s^{-1}A := \{t \in S : st \in A\}$.

One might hope that the situation would be less complicated when restricted to the class of inverse semigroups. Sticking to classical amenability, it is so much less complicated as to be almost trivial: Duncan and Namioka [6] showed that an inverse semigroup $S$ is amenable if and only if its maximal group homomorphic image (denoted $G(S)$) is amenable. As an example, if the inverse semigroup $S$ has a zero, then $G(S)$ is the trivial group, and therefore $S$ is amenable.

On the other hand, the convolution Banach algebra $\ell^1(S)$ is amenable if, and only if, the semilattice of idempotents (denoted $E(S)$) is finite and every subgroup of $S$ is amenable. This is regarded as too strong [13], since it eliminates many commutative inverse semigroups.

Paterson [18] suggested the following result points at one resolution: if the inverse semigroup $S$ has all maximal subgroups amenable, then $VN(S)$ (the von Neumann algebra of $S$) is amenable.

Milan [14] argued that the weak containment property—another generalisation of amenability for groups—is an appropriate notion of amenability for inverse semigroups, by showing the following. The free group on two generators with a zero adjoined, an example of a Clifford semigroup, does not have weak containment, but the commutative inverse semigroups all have weak containment. Therefore the weak containment property sits neatly between amenability of $S$ and amenability of $\ell^1(S)$. Milan [14] also showed that an $E$-unitary inverse semigroup has weak containment if, and only if, $G(S)$ is amenable, and that examples of inverse semigroups with weak containment include the graph inverse semigroups, which generalise and include the polycyclic monoids (see Jones and Lawson [10]).

Recall that for any given inverse semigroup $S$, the left regular representation $\pi_2$ of $s \in S$ on the Hilbert space $\mathcal{H} = \ell^2(S)$ is defined by

$$\pi_2(s) f := \sum_{tt^* \leq s^* s} f(t)\, st \quad \text{for all } f \in \ell^2(S)$$

[18]. This representation is central to the weak containment property. Due to the reliance the natural partial order to keep the above summation well-defined (consider $\pi_2(0) f$: the only idempotent bounded above by 0 is 0), this may not be generalisable to arbitrary semigroups.

In the remainder of this paper I describe a condition, similar to amenability, and given firstly in terms of finitely-additive measures, inspired by the results relating to cancellative semigroups and the regular representations of an inverse semigroup, that takes advantage of zeroes and other non-cancellative elements in a natural way. I then show an equivalent condition in terms of means. Table 1 lists some examples of semigroups and how amenability and *fair* amenability either agrees or differs. The Axiom of Choice shall be assumed throughout, though it will be mentioned where used.

## 2. Definitions

Let $S$ be a semigroup, and define the maps

$$\lambda_s(x) := sx; \quad \rho_s(x) := xs \quad \text{for all } s, x \in S.$$

$\lambda$ and $\rho$ are known as the *left regular* and *right regular* representations, respectively. (Note that these should not be confused with the regular representations on a Hilbert space described in the introduction.) For all $s \in S$, $\lambda_s$ and $\rho_s$ are elements of $\mathcal{T}_S$, the transformation semigroup of the set $S$. Note that $s^{-1}A$ (given in the footnote above as $s^{-1}A := \{t : st \in A\}$) corresponds to $\lambda_s^{-1}(A)$, and similarly, $As^{-1} = \rho_s^{-1}(A)$.

**Definition 2.1** *(Acting injectively).* If $\lambda_s|_A : A \to sA$ is an injection, then $s$ is said to *act injectively on the left of* $A$. If $\rho_s|_A : A \to As$ is an injection, then $s$ acts injectively on the *right* of $A$.

$s$ acts injectively on the left of $A$, if, and only if $sa = sb \Rightarrow a = b$ for any $a, b \in A$, i.e. $s$ is *left cancellative on* $A$; similarly on the right. A semigroup $S$ is left cancellative if, and only if, all $s \in S$ are left cancellative on $S$. Groups are totally cancellative on both sides, but there are non-group examples of left- and right-cancellative semigroups. The advantage with injective acts is that we may propose definitions and conditions on the "cancellative part" of any semigroup, rather than restricting our results to only cancellative semigroups.

**Lemma 2.2.** *For any $s \in S$ and $A \subseteq S$, the following are equivalent:*

(i) *$s$ acts injectively on the left of $A$;*
(ii) *For all two-element set $F \subseteq A$, $|sF| = |F|$;*
(iii) *For any finite set $F \subseteq S$, $|s(F \cap A)| = |F \cap A|$.*

**Definition 2.3** *(Subinvariant).* Let $S$ be a semigroup, and $\mu$ a finitely-additive measure on $S$ with finite total measure. If

$$\mu(sA) \leq \mu(A) \quad [\mu(As) \leq \mu(A)] \quad \text{for all } s \in S \text{ and } A \subseteq S,$$

then we say $\mu$ is *left [right] sub-invariant.*

Clearly $\mu(A) \geq \mu(sA)$ can be saturated: suppose $s$ is an identity. More generally, suppose for some $s, t \in S$ and $A \subseteq S$, we have $st$ acting as a permutation of $A$, in particular, $stA = A$. If $\mu$ is left sub-invariant, $\mu(A) = \mu(stA) \leq \mu(tA) \leq \mu(A)$, and thus $\mu(tA) = \mu(A)$. This suggests the next definition.

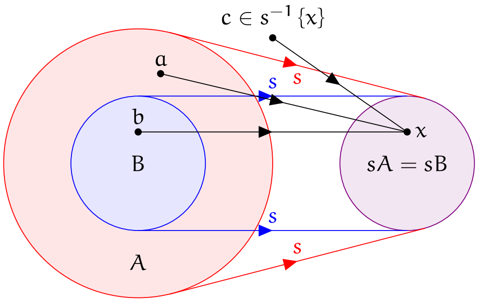


**Fig. 1.** For every element $s$ and set $A$ there is some subset $B$ such that $sA = sB$ and $s$ acts injectively on $B$.

**Definition 2.4** *(Fairly invariant, fairly amenable).* Let $S$ be any semigroup, let $\mu$ a finitely-additive measure on $S$ with $\mu(S) = 1$, and let $s \in S$ and $A \subseteq S$.

If whenever $s$ acts injectively on the left [right] of $A$,

$$\mu(sA) = \mu(A) \quad [\mu(As) = \mu(A)]$$

then $\mu$ is *fairly left [right] invariant*. If such a $\mu$ exists for a given semigroup $S$, then $S$ is *fairly left [right] amenable*.

In other words, for fairly amenable semigroups, invariance of $\mu$ is only required in the places where an element acts injectively on a set. As we shall see, this weakening of total invariance handles the issue discussed by van Douwen [20, p. 231].

**Lemma 2.5.** *For any semigroup $S$ and finitely-additive probability measure $\mu$, left [right] fair invariance of $\mu$ implies left [right] sub-invariance of $\mu$.*

**Proof.** See Fig. 1. First note that for any $A \subseteq S$ and $s \in S$ there exists a $B \subseteq A$ such that $sA = sB$ and $s$ is injective on $B$. To see this, use the Axiom of Choice to choose one $b \in s^{-1}\{x\} \cap A$ for each $x \in sA$. $B$ is simply the set of those choices. If $B \subseteq A \subseteq S$, and $sA = sB$, and $s$ acts injectively on $B$ (but not necessarily on $A$), then $\mu(A) \geq \mu(B) = \mu(sB) = \mu(sA)$, as required. □

**Remark 2.6.** What about selecting $\mu(sA) \geq \mu(A)$ as a condition ("super-invariance")? If $sA$ is a subset of $A$ then $\mu(sA) = \mu(A)$, and so disjoint subsets $sA$, $tA$ may lead to a contradiction.

## 3. Consequences

**Corollary 3.1.** *A group is amenable if, and only if, it is fairly amenable.*

**Proof.** This is trivial: for any group $G$, every $g \in G$ acts injectively on $G$ and all subsets of $G$, and so a finitely-additive measure on $G$ is invariant if, and only if, it is fairly invariant. □

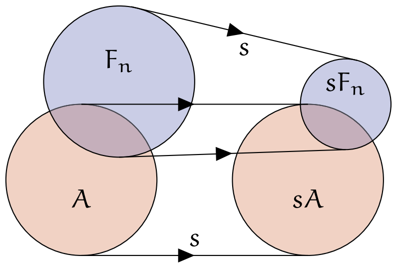


**Fig. 2.** If $s$ acts injectively on $A$, then it also acts injectively on the subset $A \cap F_n$ of $A$, and so $|A \cap F_n| = |s(A \cap F_n)|$. Note that $s(A \cap F_n) \subseteq sA \cap sF_n$ might not be saturated—consider disjoint $A$ and $F_n$.

Thus fair amenability for semigroups generalises amenability for groups. Similar to classical amenability, fair amenability is also a consequence of a Følner-type condition.

**Theorem 3.2.** *Let $S$ is a countable semigroup. If for each $s \in S$ there exists a sequence of non-empty finite sets $\{F_n\}_{n\in\mathbb{N}}$ eventually covering $S$ such that for all $A \subseteq S$,*

$$\lim_{n\to\infty} \frac{|s(A \cap F_n) \triangle (sA \cap F_n)|}{|F_n|} = 0,$$

*then $S$ is left fairly amenable. (Similarly on the right.)*

**Proof.** Fix a free ultrafilter $U$ over $\mathbb{N}$ and define $\mu$ through the ultralimit

$$\mu(A) := \lim_U \frac{|A \cap F_n|}{|F_n|} \quad \text{for all } A \subseteq S.$$

That $\mu$ is defined for all $A \subseteq S$, is finitely-additive, and $\mu(S) = 1$, follows as standard and easy applications of the Bolzano–Weierstraß Theorem and the Łos Theorem.

Suppose $s$ acts injectively on the left of $A$, and thus on the left of every subset of $A$, in particular $A \cap F_n$. Then $|A \cap F_n| = |s(A \cap F_n)|$ (see Fig. 2). Then,

$$\begin{aligned}\left|\frac{|A \cap F_n|}{|F_n|} - \frac{|sA \cap F_n|}{|F_n|}\right| &= \frac{||A \cap F_n| - |sA \cap F_n||}{|F_n|} \\ &= \frac{|s(A \cap F_n) \triangle (sA \cap F_n)|}{|F_n|} \to 0 \text{ as } n \to \infty\end{aligned}$$

by hypothesis, and hence $\mu(A) = \mu(sA)$, i.e. $\mu$ is left fairly invariant. □

**Remark 3.3.** While there are semigroups lacking strong Følner sequences that are also fairly amenable, this appears to be mitigated in the condition in Theorem 3.2 as the sets in the numerator are contained in the right ideal $sS$. Consider, for example, an infinite amenable group $G$ with zero adjoined ($G^0$), which is fairly amenable (see Corollary 3.14 below) and the zero element has no associated Følner sequence, however $0S = \{0\}$ and therefore any Følner sequence will do in Theorem 3.2.

**Corollary 3.4.** *All finite semigroups $S$ are fairly amenable (on both sides).*

**Proof.** $s$ is injective on the left of $A \subseteq S$ if, and only if, $|sA| = |A|$; similarly on the right. Therefore the counting measure suffices. Alternatively, use the constant Følner sequence $\{S\}_{n\in\mathbb{N}}$: for any $A \subseteq S$,

$$\frac{|s(A \cap S) \triangle (sA \cap S)|}{|S|} = \frac{|sA \triangle sA|}{|S|} = \frac{0}{|S|} = 0$$

as required by Theorem 3.2. □

**Remark 3.5.** Suppose that, given some set $A$ and measure $\mu$, $\mu(sA) = \mu(A)$ [$\mu(As) = \mu(A)$] for any $s$. We may describe $A$ as being a left [right] $\mu$-invariant set. In a fairly left [right] amenable semigroup $S$, every singleton set $\{x\}$ for $x \in S$ is guaranteed to be a left [right] invariant set.

**Lemma 3.6.** *Let $S$ be an infinite left [right] fairly amenable semigroup with measure $\mu$, having a left [right] zero $z \in S$. If $F$ is a finite subset of $S$, then $\mu(F) = 0$.*

**Proof.** For the singleton set case, we can "go via" $\{z\}$: for any $s, t \in S$,

$$\mu(\{s\}) = \mu(z\{s\}) = \mu(\{z\}) = \mu(z\{t\}) = \mu(\{t\}),$$

thus every singleton set has the same measure $k$. If $k > 0$ there exists some finite $N$ such that $Nk > 1$, i.e. the disjoint union of $N$ singletons would have measure greater than 1, contradicting $\mu(S) = 1$. Hence $k = 0$ and $\mu(F) = \sum_{f\in F} k = 0$. The right case holds similarly. □

**Corollary 3.7.** *Let $S$ be a non-trivial semigroup with zero. The finitely-additive measure $\delta_0$ given by*

$$\delta_0(A) = \delta_0\left(0^{-1}A\right) = \delta_0\left(A0^{-1}\right) = \begin{cases} 1 & \text{if } 0 \in A \\ 0 & \text{if } 0 \notin A \end{cases}$$

*(i.e. the measure obtained from the invariant mean $m \in \ell^\infty(S)^*$ given by $m(f) = f(0)$ for all $f \in \ell^\infty(S)$) cannot be fairly invariant.*

**Proof.** Let $a \in S$ where $a \neq 0$ and assume $\delta_0$ is fairly invariant. Since $0 \notin \{a\}$,

$$1 = \delta_0(\{0\}) = \delta_0(0\{a\}) = \delta_0(\{a\}) = 0$$

contradiction. □

**Question 3.8.** *Is there a left or right fairly amenable infinite semigroup with a finite subset having positive mass?*

Recall *Green's relations* $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$, $\mathcal{H}$, $\mathcal{J}$ on a semigroup. There are two easy lemmas.

**Lemma 3.9.** *If $S$ is left [right] fairly amenable with measure $\mu$, any finite subset $F$ of an infinite $\mathcal{L}$-class [$\mathcal{R}$-class] has $\mu(F) = 0$. It follows that in either case any finite subset $F$ of a $\mathcal{H}$-class has $\mu(F) = 0$, and that if $S$ is fairly amenable on both sides than any finite subset $F$ of a $\mathcal{D}$-class has $\mu(F) = 0$.*

**Proof.** By definition, for all $a, b \in S$ such that $a \mathcal{L} b$, there exists $s, s' \in S^1$ such that $sa = b, s'b = a$, and we only need one of these to establish that if $\mu$ is the left fairly invariant finitely-additive measure,

$$\mu(\{a\}) = \mu(s\{a\}) = \mu(\{sa\}) = \mu(\{b\}) \quad \text{for all } a, b \in S.$$

Thus every singleton subset of an $\mathcal{L}$-class has the same measure $k$. Using the final step of Lemma 3.6, we then have that every finite subset has measure 0. □

Green's Lemma [8, p. 43] states that for any $a, b \in S$ such that $a \mathcal{R} b$, the right regular representations restricted to $\mathcal{L}$-classes, $\rho_s|L_a$ and $\rho_{s'}|L_b$, are mutually inverse $\mathcal{R}$-class preserving bijections between the $\mathcal{L}$-classes $L_a$ and $L_b$. Put another way, there exists an $s \in S$ that acts injectively on the right of $L_a$ and an $s' \in S$ that acts injectively on the right of $L_b$.

**Lemma 3.10.** *Let $S$ be a semigroup. If $S$ is right fairly amenable then within each $\mathcal{D}$-class all $\mathcal{L}$ classes have the same measure. Similarly, if $S$ is left fairly amenable with $\mu$ then within each $\mathcal{D}$-class all $\mathcal{R}$ classes have equal measure. It follows that if $S$ is fairly amenable (on both sides) then all $\mathcal{D}$-related $\mathcal{H}$-classes have equal measure.*

**Proof.** Suppose $\mu$ is the finitely-additive right fairly invariant measure, and $L_a$, $L_b$ are $\mathcal{L}$-classes contained within the same $\mathcal{D}$-class. Using Green's Lemma, there exist $s, s' \in S^1$ such that $L_a = L_b s'$ and $L_b = L_a s$ are both examples of injective right acts. Thus $\mu(L_a) = \mu(L_a s) = \mu(L_b)$. □

What can we say about the value of a fairly invariant finitely-additive measure $\mu$ between distinct $\mathcal{D}$-classes in general? Probably not a lot (see Example 3.15 below).

A result for groups states that the direct product of finitely many amenable groups is also amenable. This is easily shown by noting that if $G = G_1 \times G_2$ then the subgroup $H = \{(g_1, 1_{G_2}) : g_1 \in G_1\} \cong G_1$, and $G/H \cong G_2$, so therefore the amenability of $G_1$ and $G_2$ imply the amenability of $H$ and $G/H$, and hence $G$. The fair amenability analogue of this result is shown in a more involved manner as follows.

**Theorem 3.11.** *Let $S$, $T$ be semigroups that are each left [right] fairly amenable. $S \times T$ is as well.*

**Proof.** Let $\mu_S$ and $\mu_T$ witness the left fair amenability of $S$ and $T$ respectively. Let $\pi_S$, $\pi_T$ denote the projections from $\mathcal{P}(S \times T)$ onto $\mathcal{P}(S)$ and $\mathcal{P}(T)$, respectively. Define $\mu$ firstly on the set of rectangles $R = A \times B$ where $A \subseteq S$ and $B \subseteq T$:

$$\mu(R) := \mu_S(\pi_S(R))\,\mu_T(\pi_T(R)) = \mu_S(A)\,\mu_T(B).$$

$\mu$ is clearly left fairly invariant and finitely-additive, and with $\mu(S \times T) = \mu_S(S)\,\mu_T(T) = 1$. Then

$$\mu\left(\bigcup_{i\in I} R_i\right) = \sum_{i\in I} \mu(R_i),$$

for each finite collection of disjoint rectangles[2] $\{R_i\}_{i\in I}$, and this is also left fairly invariant. If $(s,t)$ acts injectively on $\bigcup_{i\in I} R_i$, then $s$ acts on $\pi_S(R_i)$ injectively for each $i \in I$, likewise for $t \in T$ on $\pi_T(R_i)$. Furthermore, $(s,t)$ preserves the disjointness of $\{R_i\}_{i\in I}$. Let $C$ be an arbitrary subset of $S \times T$. $C$ is not necessarily a rectangle, so extend $\mu$ to all of $S \times T$ using

$$\mu(C) := \sup \mu\left(\bigcup_{i\in I} R_i\right),$$

for all $C \subseteq S \times T$, where the supremum is taken over all finite collections of subrectangles of $C$. If $(s,t) \in S \times T$ acts injectively on $C$ then it acts injectively on any finite collection of disjoint subrectangles of $C$. Each finite collection of disjoint subrectangles of $(s,t)C$ has the form $\{(s,t)R_i\}_{i\in I}$ for a finite collection of disjoint subrectangles $\{R_i\}_{i\in I}$ of $C$. Hence

$$\begin{aligned}\mu((s,t)C) &= \sup \mu\left(\bigcup_{i\in I}(s,t)R_i\right)\\ &= \sup \sum_{i\in I} \mu((s,t)R_i) = \sup \sum_{i\in I} \mu(R_i)\\ &= \sup \mu\left(\bigcup_{i\in I} R_i\right) = \mu(C),\end{aligned}$$

and thus $\mu$ is defined and is left invariant for all $C \subseteq S \times T$, as required. □

[2] Take care to avoid confusing *finite collections of rectangles* with *collections of finite rectangles*.

Another result for groups states that every left amenable group is also right amenable, and furthermore, a left invariant measure and right invariant measure can be combined to provide a bi-invariant measure [22, p. 148]. This result doesn't hold for all semigroups (either classically or fairly), but a similar proof technique can be applied to the fair amenability of semigroups with involution.

**Lemma 3.12.** *Let $S$ be a semigroup with involution $*$. If $S$ is left fairly amenable, then it is right fairly amenable (and vice-versa).*

**Proof.** $A^* := \{a^* : a \in A\}$, and so $(As)^* = s^* A^*$. Suppose that $S$ is left fairly amenable with $\mu$, and define $\nu$ by setting $\nu(A) := \mu(A^*)$ for all $A \subseteq S$. $\nu(S) = \nu(S^*) = \mu(S) = 1$. For all disjoint $A, B \subseteq S$,

$$\nu(A \cup B) = \mu((A \cup B)^*) = \mu(A^* \cup B^*) = \mu(A^*) + \mu(B^*) = \nu(A) + \nu(B).$$

Finally, if $s$ acts injectively on the *right* of $A$, then for $a, b \in A$,

$$s^* a^* = s^* b^* \Leftrightarrow (as)^* = (bs)^* \Leftrightarrow as = bs \Rightarrow a = b \Leftrightarrow a^* = b^*$$

and so $s^*$ acts injectively on the *left* of $A^*$. Then $\nu(As) = \mu(s^* A^*) = \mu(A^*) = \nu(A)$ wherever $s$ acts injectively on the right of $A$. Hence $\nu$ shows $S$ is right fairly amenable. □

Thus groups, inverse semigroups, semigroups of binary relations, and all other $*$-semigroups join the commutative semigroups as classes of semigroups where each example is either *fairly amenable (on both sides)*, or not at all. The question of semigroups with a finitely-additive measure that is simultaneously left and right fairly invariant is not here considered. In the next section, I give an example of a semigroup that is fairly amenable on one side but not the other.

Every subgroup of an amenable group is amenable, including those subgroups having measure zero in the supergroup. A quick summary of the proof goes as follows: let $G$ be an amenable group with measure $\mu$, $H$ a subgroup. Choose a set $M$ of representatives from each left coset of $H$, then define a measure $\nu$ on $H$ by setting $\nu(A) := \mu(MA)$ for all $A \subseteq H$ [22, p. 149]. It would be nice to emulate this in the semigroup case, but it seems there is no adequate analogue for semigroups of the coset structure of a group.

**Lemma 3.13.** *Let $S$ be a left [right] fairly amenable semigroup with measure $\mu$, and let $T$ be a subsemigroup of $S$ having $\mu(T) > 0$. $T$ is then left [right] fairly amenable.*

**Proof.** We may use $\nu$ as given by $\nu(A) = \mu(A) / \mu(T)$ for all $A \subseteq T$. □

This mirrors the classical case [4, p. 518]. In particular, any subgroup $G$ of a left or right fairly amenable semigroup is amenable *provided that* $\mu(G) > 0$.

**Corollary 3.14.** *Let $S$ be a semigroup without zero. $S^0$ is left [right] fairly amenable if and only if $S$ is. In particular, if $G$ is a group, $G^0$ is fairly amenable if and only if $G$ is amenable.*

**Proof.** Since the finite case is trivial, assume that $S$ is infinite. If $S^0$ is left fairly amenable with $\mu'$, since $S^0$ contains a zero, by Lemma 3.6 $\mu'(\{0\}) = 0$, which by finite additivity implies $\mu'(S) = 1$. By Lemma 3.13 $S$ is fairly amenable and, in the case of a group, amenable by Corollary 3.1.

Conversely, if $S$ is left fairly amenable with some $\mu$ then assigning $\mu'(A) = \mu(A \cap S)$ yields a fairly invariant measure $\mu'$ on $S^0$. The case on the right holds similarly. □

0-groups are examples of Clifford semigroups, which are characterised as being strong semilattices of groups [8, p. 94], and in turn are examples of inverse semigroups. What we can say about Clifford semigroups? The following furnishes us with an example of a fairly amenable Clifford semigroup that is not a 0-group, having a non-amenable subgroup.

**Example 3.15.** Let $S$ be the union of two free groups as follows: set $G \cong \mathbb{F}_2$ (not amenable), $H \cong \mathbb{F}_1$ (amenable), and let $\phi : G \to H$ be the homomorphism mapping $x \mapsto 1_H$ for all $x \in G$. Define the operation on $S$ as a strong semilattice $Y = (\{1, 0\}, \wedge)$ of the groups $G$, $H$, i.e. if one of $x$ or $y$ is in $H$ we map the other via $\phi$ into $H$ to compute $xy$. Despite the presence of $\mathbb{F}_2$, $S$ is fairly amenable.

**Proof.** Let $\mu_H$ witness the amenability of $H$. Define for $S$ the measure $\mu$ given by

$$\mu(A) := \mu_H(\phi(A \cap G) \cup (A \cap H)) \quad \text{for all } A \subseteq S,$$

which is invariant under action of $H$. Since $H$ is an infinite $\mathcal{H}$-class, $\mu_H(\{1_H\}) = 0$ by Lemma 3.9, and therefore $\mu(G) = 0$; it follows that $\mu(A) = \mu_H(A \cap H)$ for any $A \subseteq S$. If $A \subseteq H$ then $gA = A = Ag$ for all $g \in G$, so $\mu$ is trivially invariant under $G$, and thus $\mu$ suffices. □

The following shows an example of a fairly amenable Clifford semigroup that, as part of the semilattice, has no amenable subgroup whatsoever.

**Example 3.16.** Consider the semilattice on the integers $Y = (\mathbb{Z}, \wedge)$ where $a \wedge b = \min\{a, b\}$ for all $a, b \in Y$, together with a measure $\mu$ derived from the Følner sequence given by $F_n := [-n, n] \cap Y$ for each $n$.

Now $\mu(k \wedge Y) = \mu((-\infty, k] \cap Y) = \frac{1}{2}$ for all $k \in Y$, all finite sets have measure 0, and the semilattice is fairly amenable.

Suppose we take $S$ to be a strong semilattice of infinitely many non-amenable groups, as follows:

- Let the semilattice $Y$ be isomorphic to $(\mathbb{Z}, \wedge)$, as previously;
- For each $k \in \mathbb{Z}$ let $G_k$ be a non-amenable group;
- For each $k \in \mathbb{Z}$ let $\nu_k$ be any finitely-additive measure on $G_k$ with $\nu_k(G_k) = 1$ (which by the previous point is necessarily not invariant).

We can extend $\mu$ given on $Y$ to a fairly-invariant $\mu_S$ on $S$ by setting, for a fixed free ultrafilter $U$ over $\mathbb{N}$,

$$\mu_S(A) = \lim_U \frac{1}{2n+1} \sum_{k=-n}^{n} \nu_k (G_k \cap A).$$

While every $G_k$ is not amenable, $\mu_S$ witnesses the fair amenability of $S$. □

**Corollary 3.17.** *If the Clifford semigroup $S$ is a strong* finite *semilattice $Y$ of groups and $S$ is fairly amenable (on either side), at least one of the groups must be amenable.*

**Proof.** Suppose all the groups in $\{G_y : y \in Y\}$ are non-amenable, and $\mu$ witnesses the fair amenability of $S$, in particular, $\mu(S) = 1$. Using Lemma 3.13, $\mu(G_y) = 0$ for all $y \in Y$. $S = \bigcup_{y \in Y} G_y$, which is a disjoint union, and as there are only finitely many groups in the semilattice, $\mu(S) = 0$, contradiction. □

Finally, one theorem on groups that translates well to fairly amenable semigroups is the result that a directed union of amenable groups is also amenable.

**Theorem 3.18.** *If $S$ is the directed union of left [right] fairly amenable semigroups, then $S$ is left [right] fairly amenable.*

**Proof.** This proof uses essentially the same topological argument given in Wagon [22, p. 150]. Let $\{S_i : i \in I\}$ be the directed system of left fairly amenable semigroups whose union is $S$: i.e. for each $a, b \in I$ there exists a $c \in I$ such that $S_a$ and $S_b$ are subsemigroups of $S_c$, and, $S = \bigcup_{i \in I} S_i$. For each $i \in I$:

- let $\mu_i$ be the left fairly invariant finitely-additive measure corresponding to $S_i$;
- let $M_i$ be the set of finitely-additive measures $m : \mathcal{P}(S) \to [0, 1]$ such that $m(S) = 1$ and whenever $s \in S_i$ acts injectively on $A \subseteq S$, $m(sA) = m(A)$.

Define $m_i(A) := \mu_i(A \cap S_i)$ for all $A \subseteq S$. Clearly $m_i \in M_i$, i.e. $M_i$ is non-empty for all $i \in I$. Suppose $f \notin M_i$; either $f$ fails to be finitely additive, fails to be left fairly invariant for some $s \in S_i$, or $f(S) \neq 1$. It is possible to vary the "amount" by which each of the three conditions is violated (e.g. $1 - f(S) = \epsilon$), thus forming an open neighbourhood of $f$ consisting of points behaving similarly. This argument is similar to Wagon [22, p. 126]. Thus each $M_i$ is a closed subset of $[0, 1]^{\mathcal{P}(S)}$. If $S_a, S_b \subseteq S_c$ then $M_a \cap M_b \supseteq M_c$, since each member must be left fairly invariant for increasingly many elements. Thus the collection $\{M_i : i \in I\}$ has the finite intersection property. From Tychonoff's Theorem, the space $[0,1]^{\mathcal{P}(S)}$ is compact; equivalently, any collection of closed subsets with the finite intersection property is nonempty, and $\{M_i : i \in I\}$ is an example of such a collection. Therefore, there exists some $\mu \in \bigcap_{i\in I} M_i$ which is the required left fairly-invariant measure. The right case is handled analogously. □

## 4. Examples

**Proposition 4.1.** *Any finitely-generated free Abelian semigroup, such as* $(\mathbb{N}, +)$*, is fairly amenable.*

**Proof.** The free Abelian semigroup on $k$ generators is isomorphic to $(\mathbb{N} \cup \{0\})^k$ minus the origin, and again every action is injective. The Følner sequence given by

$$F_n = \{(a_1, a_2, \dots, a_k) : a_1, a_2, \dots a_k < n\}$$

suffices. □

**Proposition 4.2.** $(\mathbb{N}, \cdot)$ *(the natural numbers with multiplication) is also a cancellative Abelian semigroup. However, it is infinitely generated (by the primes). It is also fairly amenable.*

**Proof.** There exists a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$ where $F_n$ consists of the products of powers of the first $n$ primes, and each power lies in $[0, n]$, i.e.

$$F_n := \left\{p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n} : 0 \le i_j \le n, j = 1, \dots, n\right\},$$

as required. Bergelson [2] demonstrated a family of Følner sequences of this kind. □

**Example 4.3.** The free semigroup on two generators $FS_2 = \{a, b\}^+$ is neither left nor right fairly amenable.

**Proof.** See Fig. 3. Suppose $S = \{a, b\}^+$ is left fairly amenable and $\mu$ is the required measure. Note that $a$ and $b$ both act injectively on $S$ and so we require $\mu(aS) = \mu(S) = \mu(bS)$. But since $S = \{a, b\} \cup aS \cup bS$,

$$1 = \mu(S) = \mu(\{a, b\}) + \mu(aS) + \mu(bS) = \mu(\{a, b\}) + 1 + 1 \ge 2,$$

contradiction. By a similar argument, $FS_2$ is not right fairly amenable. (Alternatively, endow the semigroup with an involution $*$ where $a^* := b$ and vice-versa, and apply Lemma 3.12.) □

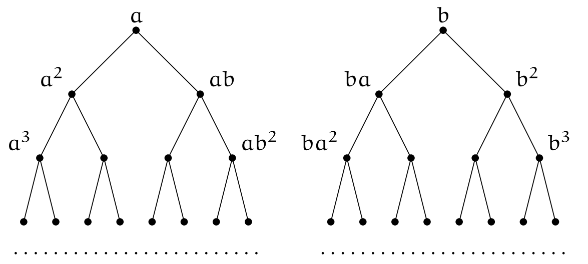


**Fig. 3.** The right Cayley graph for the free semigroup on two generators $\{a, b\}^+$.

**Remark 4.4.** Note that the previous argument can be adapted to any finite number of generators $n \geq 2$. Note also that $FS_2^0$ (the free semigroup on two generators with a zero adjoined) is now not fairly amenable either, in contrast to the classical case.

**Remark 4.5.** Another theorem on groups states that if a group $G$ is amenable and $N \triangleleft G$, then $G/N$ is also amenable; since every congruence on a group arises as the cosets of a normal subgroup this means that every quotient of an amenable group is amenable. Given $\mu$ on an amenable $G$ we may set $\nu$ on $G/N$ using

$$\nu(A) = \mu\left(\bigcup A\right).$$

The corresponding situation in fairly amenable semigroups encounters problems. Let $\sigma$ be a congruence on a fairly left amenable semigroup $S$ with measure $\mu$. Clearly $\nu$ over $S/\sigma$ (in place of $G/N$) has total measure 1 and is finitely-additive. However it is not always going to be left fairly invariant.

**Example 4.6.** As described in Proposition 4.1, the free Abelian semigroup on two generators $S$ is fairly amenable with the measure $\mu$. Let $\sigma$ be the congruence on $S$ with $(b, b^2), (b, ab) \in \sigma$, i.e.

$$S/\sigma \cong \operatorname{sgp}\left\langle a, b \middle| ab = ba = b^2 = b\right\rangle.$$

$S/\sigma$ is fairly amenable (it is a free commutative semigroup on one generator with a zero), however $\nu$ as in Remark 4.5 is not fairly invariant since $\nu(A) = \nu\left((b\sigma)^{-1}A\right)$ (the Dirac delta measure), via Lemma 3.9. □

Now we consider some bands. Recall that, in the classical sense, a right zero semigroup is left amenable but not right amenable. [17, Exercise 0.31].

**Example 4.7.** Let $S$ be a left (or right) zero semigroup. $S$ is fairly amenable (on both sides).

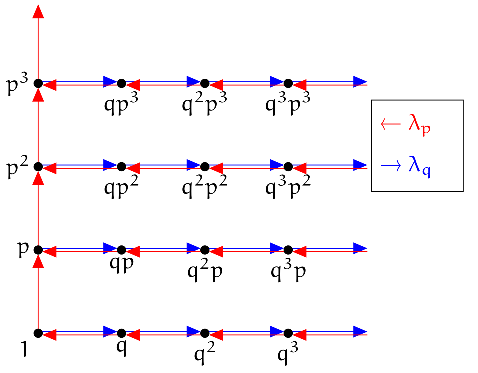


**Fig. 4.** Part of the left Cayley graph of the bicyclic monoid $B$.

**Proof.** The finite case is handled by Corollary 3.4, so assume $S$ is an infinite left zero semigroup.

For any $A \subseteq S$ and $s \in S$, $As = A$, so $\mu(As) = \mu(A)$ trivially. Thus any finitely-additive measure $\mu$ with $\mu(S) = 1$ is right fairly invariant. For any $A \subseteq S$ and $s \in S$, $sA = \{s\}$, and by Lemma 3.6 every $\mu(\{s\}) = 0$ if $\mu$ is fairly invariant, but since singletons are the only sets injectively acted on the left, the following suffices. Fix any free ultrafilter $U$, and define $\mu(A) = \chi_U(A)$. Thus there are infinitely many finitely-additive measures $\mu$ with $\mu(S) = 1$ that are left fairly invariant. The argument holds on the right analogously. □

**Example 4.8.** Every rectangular band is fairly amenable (on both sides).

**Proof.** We have just seen the specific examples of left and right zero semigroups (Example 4.7). Each rectangular band is isomorphic to the product of a left zero semigroup and a right zero semigroup, therefore by Theorem 3.11 all rectangular bands are fairly amenable. □

**Example 4.9.** The bicyclic monoid $B$ is fairly amenable (on both sides).

**Proof.** See Fig. 4. Recall that $B = \text{mon}\langle p, q | pq = 1\rangle = \{q^m p^n : m, n \in \mathbb{N} \cup \{0\}\}$.

Consider the increasing sequence of "squares" given by $\square_n = \left\{q^j p^k : j, k \leq n\right\}$ for all $n \in \mathbb{N}$. It will suffice to show this sequence is Følner for any element on the left.

The element $q$ acts injectively on the left of all $B$, so $|q\square_n| = |\square_n|$ and $|q\square_n \triangle \square_n| = 2n$. $p$ on the other hand does not act injectively on the left of $\square_n$, in which case $|p\square_n| \leq |\square_n|$. Since the minimal non-injective sets for each left multiplication by $p$ are $\left\{p^k, qp^{k+1}\right\}$ for each $k$, we can see exactly that $|p\square_n| = (n-1)n + 1$, and $|p\square_n \triangle \square_n| = n + 1$. For any arbitrary $x = q^j p^k$, then,

$$|x\square_n \triangle \square_n| = k + n(2j - k) \quad \text{for all } n > j$$

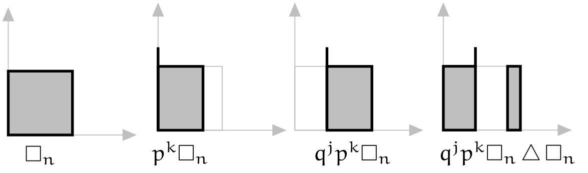


**Fig. 5.** Deriving $\left|q^j p^k \square_n \bigtriangleup \square_n\right|$ in the bicyclic monoid.

(depicted in Fig. 5) which is linear in $n$, and therefore the Følner sequence $\{\square_n\}_{n\in\mathbb{N}}$ suffices.

$B$ is inverse, so Lemma 3.12 applies and hence $B$ is fairly amenable on both sides. □

**Example 4.10.** The polycyclic monoid on two generators, $P_2$, is not fairly amenable. As described by Milan [14], $P_2$ has the weak containment property, so it follows that fair amenability is not equivalent to weak containment.

**Proof.** Recall that

$$P_2 = \mathrm{mon}^0 \left\langle p, q, p^{-1}, q^{-1} \middle| pp^{-1} = 1 = qq^{-1}, pq^{-1} = 0 = qp^{-1} \right\rangle,$$

and so every element other than 0 or 1 can be written canonically in the form $x^{-1}y$, where $x$, $y$ are (possibly empty) strings over the alphabet $\{p, q\}$ [12]. It follows that (at least) the free monoids $\left\{p^{-1}, q^{-1}\right\}^*$ and $\{p, q\}^*$ are embedded within $P_2$.

Assume $P_2$ is left fairly amenable with measure $\mu$, and for each $x \in P_2$ let $H_x \subseteq P_2$ consist of elements with their canonical form starting with the string $x$. $P_2$ can be decomposed like so:

$$P_2 = H_{p^{-1}} \uplus H_{q^{-1}} \uplus H_p \uplus H_q \uplus \{0, 1\}.$$

Consider the injective left actions $\lambda_{p^{-1}}$, $\lambda_{q^{-1}}$

$$p^{-1}P_2 = H_{p^{-1}} \uplus \{0\}, \quad q^{-1}P_2 = H_{q^{-1}} \uplus \{0\}.$$

Applying $\mu$ to these piece of $P_2$, we see that it is not left fairly invariant by contradiction:

$$\begin{aligned}
1 &= \mu(P_2) \\
&= \mu\left(H_{p^{-1}} \uplus H_{q^{-1}} \uplus H_p \uplus H_q \uplus \{0, 1\}\right) \\
&= \mu\left(H_{p^{-1}}\right) + \mu\left(H_{q^{-1}}\right) + \mu(H_p) + \mu(H_q) + \mu(\{0, 1\}) \\
&= \mu\left(H_{p^{-1}}\right) + \mu\left(H_{q^{-1}}\right) + \mu(H_p) + \mu(H_q) \quad \because \text{Lemma 3.6} \\
&= \mu\left(p^{-1}P_2\right) + \mu\left(q^{-1}P_2\right) + \mu(H_p) + \mu(H_q)
\end{aligned}$$

$$= 1 + 1 + \mu(H_p) + \mu(H_q) \quad \because \text{fair invariance}$$
$$\geq 2.$$

$P_2$ is also inverse, so by Lemma 3.12 it is not right fairly amenable either. □

**Remark 4.11.** As with $FS_2$ and greater, the previous argument can be adapted to any finite number of generators $n \geq 2$. $P_2$ is also an example of an inverse semigroup that is not fairly amenable, but is classically amenable because the maximal group homomorphic image (the trivial group) is amenable.

**Example 4.12.** For a Levi–Baer[3] semigroup $LB(p, q)$,

(i) $LB(p, q)$ is not left fairly amenable if $p = q$;
(ii) $LB(p, q)$ is not right fairly amenable for all $p$, $q$.

**Proof.** Recall that Levi–Baer semigroups are left cancellative, left simple, and have no idempotents. For succinctness let $S$ be shorthand for $LB(p, q)$.

On the left, let $a, b \in S$ be such that the right ideals $a \circ S$ and $b \circ S$ are disjoint. (There are two disjoint right ideals if, and only if, $p = q$.) For example, if $S$ is the Baer–Levi semigroup on $\mathbb{N}$, we may pick $a : n \mapsto 2n$ and $b : n \mapsto 2n+1$. Let $R = S \backslash ((a \circ S) \cup (b \circ S))$. Since $S$ is left cancellative, every left action is injective. Assume $S$ is left fairly amenable with measure $\mu$, then

$$\begin{aligned} 1 &= \mu(S) \\ &= \mu((a \circ S) \uplus (b \circ S) \uplus R) \quad \text{by definition} \\ &= \mu(a \circ S) + \mu(b \circ S) + \mu(R) \\ &= \mu(S) + \mu(S) + \mu(R) \quad \because \text{left fairly invariant} \\ &\geq 2, \end{aligned}$$

contradiction.

On the right, for each $s \in S$ let the equivalence relation $\theta_s$ be given by $a\,\theta_s\,b \Leftrightarrow a \circ s = b \circ s$ for all $a, b \in S$. Since $S$ consists of maps on the set $X$, $\theta_s$ depends only on $s(X)$, so $a\,\theta_s\,b \Leftrightarrow a|_{s(X)} = b|_{s(X)}$. For any $a, b, s \in S$,

$$\begin{aligned} a\,\theta_s\,b &\Leftrightarrow a \circ s = b \circ s \\ &\Leftrightarrow a(s(x)) = b(s(x)) \text{ for all } x \in X \end{aligned}$$

[3] A Baer–Levi semigroup $BL(p, q)$ is defined as being the set of injective maps $f$ on some infinite set $X$ having cardinality $p$, such that $|X \backslash f(X)|$ is some fixed infinite cardinal $q \leq |X| = p$ [3]. Conventionally, products in Baer–Levi semigroups are written in "algebraist" order—the composition of $f$ and $g$ is written $fg$—and hence the Baer–Levi semigroups are normally referred to as right cancellative and right simple. However, to remain consistent with notation of functions elsewhere in this paper, I shall use $\circ$ for composition and consider the equivalent "Levi–Baer" semigroup, which is left cancellative and left simple.

$$\begin{aligned}&\Leftrightarrow a(y) = b(y) \text{ for all } y \in s(X)\\&\Leftrightarrow a|_{s(X)} = b|_{s(X)}.\end{aligned}$$

By definition $|X \backslash s(X)|$ is some infinite cardinal, therefore a Baer–Levi semigroup on $X \backslash s(X)$ can be used to generate elements of each $\theta_s$-class. Thus for every $s \in S$, every $\theta_s$-equivalence class is nonempty and infinite. For each $s \in S$ define two disjoint sets $M_1$, $M_2$ by choosing two distinct elements from each $\theta_s$-class. $\theta_s$ partitions $S$ into sets that map to the same element under the right action of $s$, so $S \circ s = M_1 \circ s = M_2 \circ s$. For any $a, b \in M_1$, $a \circ s = b \circ s \Rightarrow a\,\theta_s\,b \Rightarrow a = b$, similarly for $M_2$. Thus $S \circ s = M_1 \circ s = M_2 \circ s$ and while the action $S \circ s$ is not injective, the actions on $M_1$ and $M_2$ are injective. Finally, assume that $S$ is right fairly amenable with measure $\nu$, and let $R = S \backslash (M_1 \cup M_2)$. Then,

$$\begin{aligned}1 &= \nu(S)\\&= \nu(M_1 \uplus M_2 \uplus R)\\&= \nu(M_1) + \nu(M_2) + \nu(R)\\&= \nu(M_1 \circ s) + \nu(M_2 \circ s) + \nu(R)\\&= \nu(S \circ s) + \nu(S \circ s) + \nu(R)\\&= \nu(S) + \nu(S) + \nu(R)\\&= 1 + 1 + \nu(R)\\&\geq 2,\end{aligned}$$

contradiction. □

**Example 4.13.** Left groups are left simple, right cancellative semigroups that are characterised as being direct products of groups and left zero semigroups. Let $Z$ be the left zero semigroup with elements from $\mathbb{N}$, and let $S$ be the left group $\mathbb{F}_{\{a,b\}} \times Z$. $S$ is left fairly amenable but is not right fairly amenable.

**Proof.** On the left: let $\xi$ be any finitely-additive measure on $\mathbb{F}_{\{a,b\}}$ with $\xi\left(\mathbb{F}_{\{a,b\}}\right) = 1$. $\xi$ is necessarily not invariant. Fix an ultrafilter $U$ over $\mathbb{N}$ and define the finitely-additive measure $\mu$ by setting

$$\mu(A) := \lim_{U} \frac{1}{n} \sum_{k=1}^{n} \xi\left(A \cap (\mathbb{F}_{\{a,b\}} \times \{k\}\right) \quad \text{for all } A \subseteq S.$$

$\mu$ exists, is finitely additive, and $\mu(S) = 1$, as usual. Suppose $(g, m) \in S$ acts injectively on the left of $A \subseteq S$: since $Z$ is left zero, this implies that $(x, m_1), (x, m_2) \in A \Rightarrow m_1 = m_2$ for all $x \in \mathbb{F}_2$ and $m_1, m_2 \in Z$, and thus $\mu(A) = 0$. Then,

$$\begin{aligned}\mu\left((g,n)\cdot A\right) &= \lim_U \frac{1}{n}\sum_{k=1}^{n} \xi\left((g,n)A \cap (\mathbb{F}_{\{a,b\}} \times \{k\}\right) \\ &\le \lim_U \frac{1}{n}\xi\left(\mathbb{F}_{\{a,b\}}\right) \\ &= 0,\end{aligned}$$

thus $\mu$ is left fairly invariant.

On the right: assume $S$ is right fairly invariant with measure $\nu$. The contradiction unfolds similarly the usual proof that the free group $\mathbb{F}_2$ is not amenable, as follows. Consider one set of words $F(a) \subset \mathbb{F}_{\{a,b\}}$, which end with the letter $a$. Then

$$\begin{aligned}(F(a)\times Z)\cdot(a^{-1},1) &= (F(a)a^{-1}\times Z) \\ &= S\backslash(F(a^{-1})\times Z),\end{aligned}$$

and similarly for $F(b)$; hence

$$\begin{aligned}1 &= \nu\left(S\right) \\ &= \nu\left((F(a)\cup F(a^{-1})\cup F(b)\cup F(b^{-1})\cup\{1\})\times Z\right) \\ &\ge \nu\left(F(a)\times Z\right)+\nu\left(F(a^{-1})\times Z\right)+\nu\left(F(b)\times Z\right)+\nu\left(F(b^{-1})\times Z\right) \\ &= \nu\left(F(a)a^{-1}\times Z\right)+\nu\left(F(a^{-1})\times Z\right)+\nu\left(F(b)b^{-1}\times Z\right)+\nu\left(F(b^{-1})\times Z\right) \\ &= \nu\left(S\right)+\nu\left(S\right) \\ &= 2,\end{aligned}$$

contradiction. □

**Example 4.14.** The free inverse semigroup on one generator $FIS_1$ is fairly amenable on both sides.

**Proof.** From Munn's Theorem on the structure of free inverse semigroups [11], elements of $FIS_1$ can be thought of as triples of integers

$$FIS_1 \cong \left\{(p,q,r)\in\mathbb{Z}^3 : p\ge 0, p+q\ge 0, q+r\ge 0, r\ge 0, p+q+r\ge 0\right\}$$

with the product defined by

$$(p,q,r)(p',q',r') := (\max\{p,p'-q\}, q+q', \max\{r', r-q'\})$$

for all $(p,q,r),(p',q',r')\in FIS_1$ [11, p. 193]. Consider the increasing sequence given by

$$F_n = \{(x,y,z)\in FIS_1 : x,y,z\le n\}.$$

**Table 1**
Amenability versus fair amenability on different semigroups.

| Kind of semigroup | Classically amenable | Fairly amenable |
|---|---|---|
| Finite | ⇔ Unique min. ideals | Yes (Corollary 3.4) |
| With zero | Yes | Sometimes (Corollary 3.14) |
| Monogenic | Yes | Yes (Proposition 4.1) |
| Free (≥2 gen.) | No | No (Example 4.3) |
| Abelian | Yes | ? |
| Clifford | Sometimes | Sometimes (Example 3.15) |
| Left/right zero sgp | Sided | Yes (Example 4.7) |
| Left/right group | ? | Sometimes (Sided; Example 4.13) |
| Baer–Levi | ? | No (Example 4.12) |
| Inverse | ⇔ Max grp hom. im. is | Sometimes |
| Bicyclic | Yes | Yes (Example 4.9) |
| Polycyclic | Yes (∵ zero) | No (Example 4.10) |
| Free monogenic inverse | Yes | Yes (Example 4.14) |

The sequence $\{|F_n|\}_{n\in\mathbb{N}}$ is the sequence of "house numbers" [16], given by

$$|F_n| = (n+1)^3 + \frac{1}{6}n(n+1)(2n+1)$$

and thus $(n \mapsto |F_n|) \in O(n^3)$. Let $(p,q,r) \in FIS_1$. By definition,

$$(p,q,r)F_n = \{(\max\{p, x-q\}, q+y, \max\{z, r-y\}) : (x,y,z) \in F_n\}.$$

For large $n$,

$$|(p,q,r)F_n| \approx |\{(x-q, q+y, z) : (x,y,z) \in F_n\}|$$

i.e. the left action of $(p,q,r)$ on $F_n$ is an almost-translation in $\mathbb{Z}^3$, and in particular

$$\begin{aligned} |F_n \triangle (p,q,r)F_n| &\approx |F_n \triangle \{(x-q, q+y, z) : (x,y,z) \in F_n\}| \\ &\approx 2qn^2. \end{aligned}$$

Thus $(n \mapsto |F_n \triangle (p,q,r)F_n|) \in O(n^2)$, and therefore the sequence $\{F_n\}_{n\in\mathbb{N}}$ is Følner. The right case holds similarly. □

Some of the examples and results from above are summarised in Table 1. The variety of interesting examples demonstrate that the "fair" modification of invariant finitely-additive measures interacts well with the structure of semigroups. Some important results from group amenability theory are preserved, and examples of fairly amenable semigroups, especially with zeroes, are more gratifying. The given examples of non-fairly amenable semigroups have a certain self-similarity which might be used to create Banach–Tarski-style paradoxes.

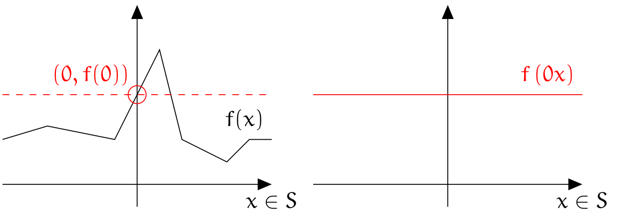


**Fig. 6.** The result of the dual left action of 0 on some $f \in \ell^{\infty}(S)$.

## 5. The convolution partial action

For real- or complex-valued functions $f : S \mapsto \mathbb{K}$ let the support of $f$ be denoted supp$(f)$, i.e.

$$\operatorname{supp}(f) := \{x \in S : f(x) \neq 0\}.$$

When two functions $f$ and $g$ have disjoint supports (i.e. $\operatorname{supp}(f) \cap \operatorname{supp}(g) = \emptyset$), we will simply say $f$ and $g$ are *disjoint*.

Recall that convolution of two functions $f, g \in \ell^1(S)$, denoted $f * g$, is defined by setting

$$\{f * g\}(x) := \sum_{st=x} f(s)\, g(t) \quad \text{for all } x \in S.$$

This extends to a left convolution "action" of $s \in S$ on $f \in \ell^{\infty}(S)$, denoted $s * f$, which may be defined by setting

$$\{s * f\}(x) := \sum_{st=x} f(t) \quad \text{for all } x \in S.$$

Equivalently, the summation can be taken over $t \in s^{-1}\{x\}$. For each $s \in S$, let the equivalence relation $\theta_s$ on $S$ be given by setting $x\,\theta_s\,y$ if and only if $sx = sy$, for all $x, y \in S$. Note that each $s^{-1}x$ is precisely a $\theta_s$-equivalence class.

Unsurprisingly, $*$ often fails to be an operation closed in $\ell^{\infty}(S)$, or even well-defined (excluding such things as Ramanujan summation). In contrast to the dual action which "flattens" along sections of the domain (see Fig. 6), the convolution "action" has the appearance of "bunching up" the values along the domain (Fig. 7). For an extreme example, suppose $S$ is an infinite semigroup with zero. Then

$$0 * \chi_S = \sum_{t\in S} \chi_{0\{t\}} = \sum_{t \in S} \chi_{\{0\}} = \delta_0,$$

which in one sense takes the "value" $|S| = \infty$ at 0. Less extreme cases can also fail to be defined along the entire domain $S$. Examples are depicted in Fig. 8. There are a few ways this situation might be treated.

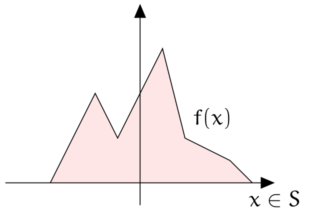


**Fig. 7.** The result of the left $*$-action of 0 on some $f \in \ell^1(S)$.

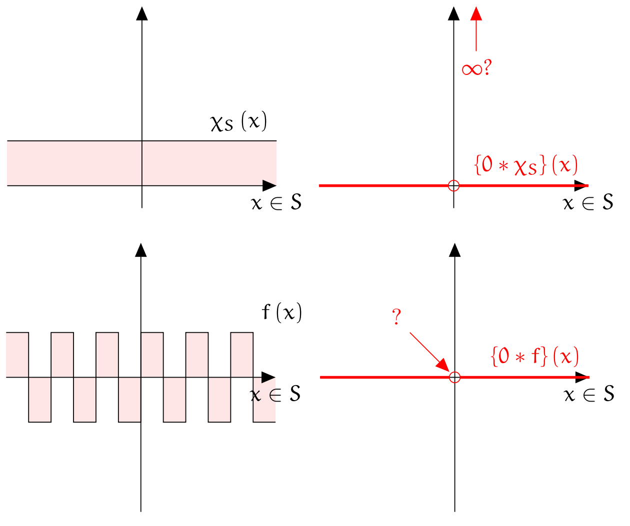


**Fig. 8.** Some example cases where the convolution partial action of 0 is not well-defined on $\ell^\infty(S)$.

(i) We could include, into the scope of discussion, unbounded functions and functions that possibly take the value $\infty$. This makes the $*$-"action" a mapping $S \times l^\infty(S) \to \mathbb{C}_\infty^S$. This approach is inclusive of degenerate cases such as $\delta_0$, but merely delays problems relating to singularities into a more complicated place. Additionally this approach does not address those $s * f$ which fail to be well-defined, but could still be argued to be bounded.

(ii) We could regard convolution as inducing a *partial* action, and include in our considerations only those combinations of $s \in S$ and $f \in \ell^\infty(S)$ where $s * f$ is well-defined everywhere and bounded, i.e. when $s * f \in \ell^\infty(S)$. Since $S$ is associative, with $\ell^\infty(S)$ as a set of objects, $S$ induces a set of arrows $A_S$, where for each $s \in S$ there is an arrow from each $f$ to $s * f$ wherever $s * f \in \ell^\infty(S)$, so $(\ell^\infty(S), A_S)$ defines a semi-category. If $S$ has an identity, then it is a category.

This last point seems interesting, not least because partial actions on C*-algebras are the subject of current research. Under what conditions is $s * f \in \ell^\infty(S)$, beyond $f \in \ell^1(S)$?

**Lemma 5.1.** *If $s$ acts injectively on the left on supp$(f)$, then $s * f \in \ell^\infty(S)$.*

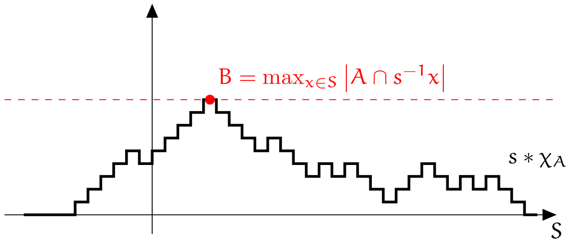


**Fig. 9.** Diagram accompanying Lemma 5.2.

**Proof.** By hypothesis, $\{s * f\}(t)$ is equal to $f(x)$ for some $x \in S$ $(sx = t)$ or zero (no such $x$). This is true for any $t \in \operatorname{supp}(f)$, and thus $s * f \in \ell^\infty(S)$. $\square$

In particular, $s * f$ exists and is bounded whenever $S$ is left cancellative (e.g. is a group). For a semigroup generally, however, the converse does not hold: there may be $f \in \ell^\infty(S)$ such that $s * f \in \ell^\infty(S)$ but $s$ is not injective on the support. For example, $f \in \ell^1(\mathbb{N}^0)$ given by $f(n) = 2^{-n}$, then $0 * f = \chi_{\{0\}}$.

**Lemma 5.2.** *For all $s \in S$ and $A \subseteq S$, the following conditions are equivalent.*

(i) $s * \chi_A \in \ell^\infty(S)$.
(ii) *There exists a finite partition $\{A_i\}_{i\in I}$ of $A$ such that $s$ acts injectively on the left of each $A_i$.*
(iii) *$s * \chi_A$ is simple. See Fig. 9.*

**Proof.**

(i) ⇒ (ii): Suppose $\|s * \chi_A\|_\infty = B < \infty$. $B$ is a non-negative integer which $s * \chi_A$ attains, since the value at each point is a sum of values in $\{0, 1\}$. For all $x \in S$ we have

$$\begin{aligned}\{s * \chi_A\}(x) &= \sum_{t\in s^{-1}x} \chi_A(t) \\ &= \left|A \cap s^{-1}x\right| \\ &\leq B \quad \text{by hypothesis.}\end{aligned}$$

For $i = 1, \ldots, B$ let $A_i$ consist of one choice element from each $(A \cap s^{-1}x)\setminus\bigcup_{j<i} A_j$ for $x \in S$ (where it is not empty).[4] Then $B$ choices are made, each $A \cap s^{-1}x$ is exhausted, and $I = \{1, \ldots, B\}$ is finite. The finite collection $\{A_i\}_{i\in I}$ is a partition of $A$, since the sets of the form $s^{-1}x$ for each

[4] The Axiom of Choice is not required because the set is finite.

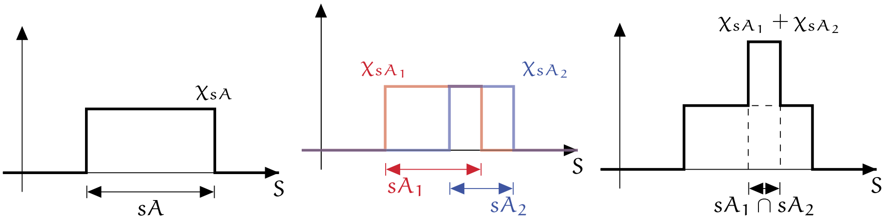


**Fig. 10.** An example of $s * \chi_A \geq \chi_{sA}$. $\chi_{sA} \leq \chi_{sA_1} + \chi_{sA_2}$, where $A = A_1 \cup A_2$ and $s$ acts injectively on $A_1$ and $A_2$ but not $A$ as a whole.

$x \in S$ are either empty, or distinct $\theta_s$-equivalence classes. $s$ acts injectively on the left of each $A_i$, as $A_i \cap s^{-1}x$ is either empty or a singleton set.

(ii) ⇒ (iii): Suppose there is a finite partition $\{A_i\}_{i\in I}$ of $A$ such that $s$ acts injectively on the left of $A_i$. Then $\chi_A = \sum_{i\in I} \chi_{A_i}$ and $s * \chi_{A_i} = \chi_{sA_i}$ for each $i \in I$, and thus

$$s * \chi_A = s * \sum_{i\in I} \chi_{A_i} = \sum_{i\in I} s * \chi_{A_i} = \sum_{i\in I} \chi_{sA_i},$$

which is a linear combination of finitely-many indicator functions, i.e. is simple.

(iii) ⇒ (i): If $s * \chi_A$ is simple then by definition it consists of a linear combination of finitely-many indicator functions, and thus attains some finite bound. □

The $*$ partial action, even on simple functions, can be difficult. Suppose $f \in \ell^\infty(S)$ is simple, i.e. there exists a finite collection of numbers $\{a_i \in \mathbb{C} : i \in I\}$ and sets $\{A_i \in \mathcal{P}(S) : i \in I\}$ such that $f = \sum_{i\in I} a_i\chi_{A_i}$. See Fig. 10. Where it exists, $*$ distributes over $+$, and clearly if $s*f$ is bounded then $s*\chi_{A_i}$ is also bounded for each $i \in I$. Therefore,

$$s * f = \sum_{i\in I} a_i \cdot (s * \chi_{A_i}).$$

However, if the action of $s$ is *not* injective on each $A_i$, $\sum_{i\in I} a_i\chi_{sA_i}$ could vary depending upon the selection of $\{A_i\}_{i\in I}$. Fortunately, if $s*f$ is bounded then each $s*\chi_{A_i}$ is bounded and therefore by Lemma 5.2 is simple, and also, there exists a finite partition $\{B_{ij}\}_{j\in J_i}$ of each $A_i$ such that $s$ acts injectively on the left of each $B_{ij}$. Thus

$$s * f = \sum_{i\in I} a_i \sum_{j\in J_i} \chi_{sB_{ij}},$$

and hence if $f$ is simple and $s * f \in \ell^\infty(S)$ then $s * f$ is also simple.

**Definition 5.3.** Let $m \in \ell^\infty(S)^*$. $m$ is *left $*$-invariant* if

$$m(f) = m(s * f)$$

for all $s \in S$ and $f \in \ell^\infty(S)$ wherever $s * f \in \ell^\infty(S)$.

**Theorem 5.4** *(Main Theorem). $S$ is left fairly amenable if, and only if, there exists a left $*$-invariant mean in $\ell^\infty(S)^*$.*

**Proof.** Assume $S$ is left fairly amenable with finitely-additive measure $\mu$. Then use $m : f \mapsto \int f \mathrm{d}\mu$ as the mean. Verifying left $*$-invariance is straightforward using Lemma 5.2, as follows.

Firstly, suppose $A \subseteq S$ and $s * \chi_A \in \ell^\infty(S)$. Lemma 5.2 gives us a finite partition $\{A_i\}_{i\in I}$ of $A$ such that $s$ acts injectively on the left of each $A_i$. Then,

$$\int (s * \chi_A)\,\mathrm{d}\mu = \sum_{i\in I} \mu(sA_i) = \sum_{i\in I} \mu(A_i) = \int \chi_A \mathrm{d}\mu.$$

Let $\ell^\infty_+(S)$ denote the subset of $\ell^\infty(S)$ consisting of bounded real-valued non-negative functions on $S$. Suppose $f \in \ell^\infty_+(S)$ is simple, i.e. there is a finite collection of sets $\{A_i\}_{i\in I}$ and values $a_i \in \mathbb{R}^+$ such that $f = \sum_{i\in I} a_i \chi_{A_i}$. If $s * f \in \ell^\infty(S)$ then $s * \chi_{A_i} \in \ell^\infty(S)$ for each $i \in I$, and using the above,

$$\int (s * f)\,\mathrm{d}\mu = \sum_{i\in I} a_i \int (s * \chi_{A_i})\,\mathrm{d}\mu = \sum_{i\in I} a_i \int \chi_{A_i} \mathrm{d}\mu = \int f \mathrm{d}\mu.$$

Thirdly, suppose $f \in \ell^\infty_+(S)$ is not simple, and $s * f \in \ell^\infty_+(S)$. Then,

$$\begin{aligned}\int (s * f)\,\mathrm{d}\mu &= \sup\left\{\int (s * g)\mathrm{d}\mu : (s * g) \le (s * f), g \text{ is simple}\right\}\\ &= \sup\left\{\int g\mathrm{d}\mu : g \le f, g \text{ is simple}\right\} = \int f \mathrm{d}\mu.\end{aligned}$$

From there it is both easy and standard to show that if $f \in \ell^\infty(S)$ is any real- or complex-valued function whatsoever and $s * f \in \ell^\infty(S)$, then $\int (s * f)\,\mathrm{d}\mu = \int f \mathrm{d}\mu$, and hence $m(f) = m(s * f)$.

Conversely, assume $m \in \ell^\infty(S)^*$ is a left $*$-invariant mean, and define $\mu \in [0, 1]^{\mathcal{P}(S)}$ by setting

$$\mu(A) := m(\chi_A) \quad \text{for all } A \in \mathcal{P}(S).$$

Then, if $s \in S$ acts injectively on the left of $A \in \mathcal{P}(S)$, $s * \chi_A = \chi_{sA}$, and then

$$\mu(sA) = m(\chi_{sA}) = m(s * \chi_A) = m(\chi_A) = \mu(A),$$

as required. □

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