A binary operation on $\ell^\infty(S)$ that is suspiciously similar to (but not the same as) convolution, and an application

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July 16, 2013

About Me

What is not in this talk

- Topology

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- Topology
- Graphs
- Differential operators
- Physics
- Cat pictures

Semigroups

Recall that a semigroup S is a set S together with an associative binary operation $\cdot$, i.e.

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- The transformations of a set, with composition:

$$\mathcal{T}_X = \{f : X \mapsto X\}$$

Semigroup elements as transformations

Consider the maps in $\mathcal{T}_S$:

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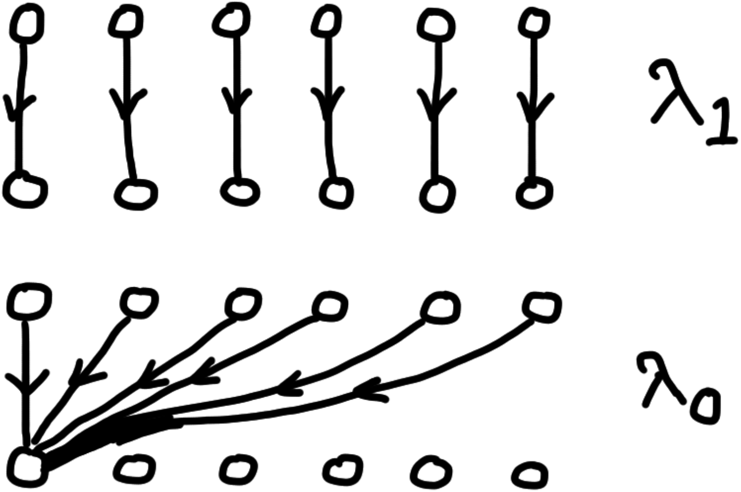
- If S is a group, λ_a, ρ_a are bijections (hence Sym_S - Cayley's theorem)

Identity and zero



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- $\ell^\infty(S)$ is the space of bounded functions:

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- All are Banach spaces – complete as normed spaces.
- $\ell^2(S)$ is a Hilbert space.
- S is faithfully embedded using δ:

$$\delta_a(t) = \begin{cases} 1 & \text{if } a = t \\ 0 & \text{otherwise} \end{cases} = \chi_{\{a\}}(t)$$

Use these as a "basis", i.e.

$$f = \sum_{s \in S} f(s)s$$

(where s is identified with δ_s.)

Convolution

Convolution of two functions $f, g \in \ell^1(S)$:

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For any semigroup S, convolution is defined on all $\ell^1(S)$, and makes it a Banach algebra – the *semigroup algebra* of S.

Convolution failure

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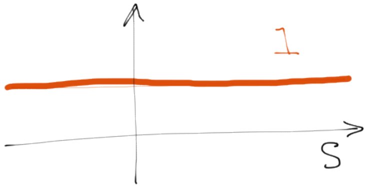
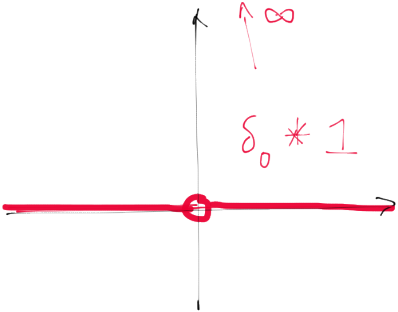
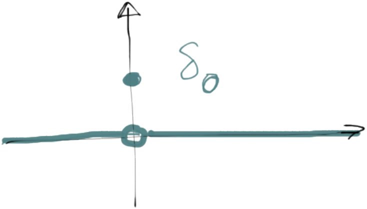
Suppose S is infinite and has a zero. $\delta_0 \in \ell^1(S)$ and $1 \in \ell^\infty(S)$. Attempted convolution:

$$\begin{aligned}(\delta_0 * 1)(u) &= \sum_{st=u} \delta_0(s)1(t) \\ &= \sum_{0t=u} 1 \\ &= \begin{cases} |S| & \text{if } u = 0 \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Clearly $\delta_0 * 1 \notin \ell^\infty(S)$...


1
S


∞
δ0 * 1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$g \in G$ permutes the elements of G so $\delta_g * 1 = 1$ for any $g \in G$. In general

$$\delta_g * f \in \ell^p(S) \quad \text{because} \quad \|\delta_g * f\|_p = \|f\|_p$$

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However, $1 * 1 = |G|$.

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“$\ell^\infty(S)$ generalises subsets, and $\ell^1(S)$ generalises finite subsets.”

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What about semigroups?

A solution

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Basic case: restrict attention to the non-negative real-valued functions in $\ell^\infty(S)$. How about:

$$(f \circledast g)(u) := \sup_{st=u} f(s)g(t)$$

for non-negative $f, g \in \ell^\infty(S)$?

Properties of $\circledast$

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(Recall $AB = \{ab : a \in A, b \in B\}$.)

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- $\chi_A \circledast \chi_B = \chi_{AB}$ *for sets* A, B;
- *If* $S = S^1$ *then* δ_1 *is an identity.*

(Recall $AB = \{ab : a \in A, b \in B\}$.)

Example

Try δ_0 and 1 again...

$$
\begin{aligned}
(\delta_0 \circledast 1)(u) &= \sup_{st=u} \delta_0(s)1(t) \\
&= \sup_{0t=u} 1 \\
&= \begin{cases} 1 & \text{if } u = 0 \\ 0 & \text{otherwise} \end{cases} \\
&= \delta_0(u).
\end{aligned}
$$

So $\delta_0 \circledast 1 = \delta_0$. (Alternatively, $\chi_{\{0\}} \circledast \chi_S = \chi_{\{0\}S} = \chi_{\{0\}}$.)

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3. Define
$$\delta_s \circledast (\mathfrak{R}(f) + i\mathfrak{I}(f)) := (\delta_s \circledast \mathfrak{R}(f)) + i(\delta_s \circledast \mathfrak{I}(f)).$$

Applications

- A measure μ is (left) *fairly invariant* if for all $s \in S$ and all subsets $A \subseteq S$ such that $\lambda_s|_A$ is a bijection, $\mu(sA) = \mu(A)$. (D 2012)

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Theorem (D)

If μ is a left fairly invariant finitely-additive probability measure on S, then there is a corresponding mean $m \in \ell^\infty(S)^*$ such that

$$m(\delta_s \circledast f) = m(f)$$

for all $s \in S, f \in \ell^\infty(S)$ such that λ_s restricted to $support(f) = \{s : f(s) \neq 0\}$ is a bijection, and vice-versa.

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Question / Riddle

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If a *Banach algebra* is a complete normed vector space that is also a ring, ...

what do you call a complete normed vector space without additive inverses that is also a semiring without distributivity?

References

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Acknolwedgements

AustMS Student Support Scheme

AMSSC Committee

Questions

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