(1) Let A be a subset of $[0,1]$ and m denote the Lebesgue measure on $\mathbb{R}$. Then which of the following are true?
(a) If A is closed then $m(A) > 0$
(b) If A is open then $m(A) = m(\bar{A})$, where $\bar{A}$ is the closure of A
(c) If $m(int(A)) = m(\bar{A})$ then A is (Lebesgue) measurable, where $int(A)$ is the interior of A.
(d) If $m(int(A)) = m(\bar{A})$ then A need not be measurable.

Solution : (c)
(a) is false because of singleton sets.
(b) is false because of we have a dense open set in $[0,1]$ with measure $1/2$. You can construct it by making small modification in cantor set and then take the complement.
(c) is true because Lebesgue measure is complete.

(2) Define an equivalence relation in $[1,2]$ by $x \sim y$ if $x - y$ is rational. Consider the set N consisting of precisely one element from each equivalence class. Then
(a) N is uncountable
(b) $[1,2] \setminus N$ is uncountable
(c) $m_*(N) = 0$
(d) $E \subset N$ measurable implies $m_*(E) = 0$

Solution : (a),(b),(d)
Here N is the non measurable set. So both N and $[1,2]\setminus N$ are uncountable and $m * (N) > 0$. (d) can be proved using the same arguments used to prove non measurability of N.

(3) Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then which of the following are necessarily true?

(a) If f is measurable, then $\phi \circ f$ is measurable, for any continuous real valued function ϕ

(b) If f^2 is measurable, then f is measurable

(c) If f is differentiable, then f' is measurable

(d) If $g : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $f = g$ $a.e$, then f is measurable

Solution : (a),(c),(d)
(a) and (d) are trivial.
(b) take $f = \chi_N - \chi_{[0,1]\setminus N}$ where N is a non measurable set. Then f is not measurable but f^2 is measurable.
(c) Since f' is a limit of measurable functions it is also measurable.

(4) Let $\{f_n\}$ be a sequence of real valued functions defined on $[0,1]$ which converges pointwise to a **continuous** real valued function f on $\mathbb{R}$. Then which of the following are necessarily true ?

(a) $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$

(b) If $0 \le f_n(x) \le f(x)\ \forall n \in \mathbb{N}$ *and* $x \in [0,1]$ then $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$

(c) If $|f_n(x)| \le \dfrac{1}{\sqrt{x}}\ \forall n \in \mathbb{N}$ *and* $x \in [0,1]$ then $\lim\limits_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$

(d) If $|f_n(x)| \le 1\ \forall n \in \mathbb{N}$ *and* $x \in [0,1]$ then $\lim\limits_{n\to\infty} \int_K f_n(x)dx = \int_K f(x)dx$ for all measurable $K \subset [0,1]$

Solution : (b),(c),(d)
By Dominated convergence theorem(DCT).

(5) Assume $\{f_n\}, \{g_n\}, f, g \in L^1(\mathbb{R}^n)$ be such that $f_n \longrightarrow f$ and $g_n \longrightarrow g$ pointwise a.e., then which of the following are true?

(a) $\int_{\mathbb{R}^n} (f_n + g_n)\ dm \longrightarrow \int_{\mathbb{R}^n} (f+g)\ dm$

(b) $|f_n| \le |f|$ a.e., $|g_n| \le |g|$ a.e. implies $\int_{\mathbb{R}^n} (f_n + g_n)\ dm \longrightarrow \int_{\mathbb{R}^n} (f+g)\ dm$

(c) $|f_n| \le |g|$ a.e. implies $\int_{\mathbb{R}^n} f_n\ dm \longrightarrow \int_{\mathbb{R}^n} f\ dm$

(d) $|f_n| \le |g_n|$ a.e. and $\int_{\mathbb{R}^n} g_n\ dm \longrightarrow \int_{\mathbb{R}^n} g\ dm$ implies $\int_{\mathbb{R}^n} f_n\ dm \longrightarrow \int_{\mathbb{R}^n} f\ dm$

Solution : (b),(c),(d)
(b) and (c) due to DCT
(d) from generalized DCT

(6) Consider the sequence of functions $f_n(x) = e^{-nx^2}$ on $[1, \infty)$. Which of the following are true?

(a) $\int_1^\infty f_n(x)dx \to 0$

(b) $\sup_n \|f_n\|_1 < \infty$

(c) f_n converges in $L^1[1, \infty)$

(d) f_n does not converge in $L^p[1, \infty)$ for any $1 \leq p \leq \infty$

Solution : (a),(b),(c)
Here f_n decreases to 0 and f_1 is integrable. Then use DCT.

(7) Let $\{E_n\}$ be a sequence of measurable sets in $\mathbb{R}$ such that $m(E_n) \to 0$ as $n \to \infty$ and $f \geq 0$ be measurable. Which of the following are true ?

(a) $\displaystyle\int_{E_n} f(x)dx \to 0$ as $n \to \infty$

(b) If $E_{n+1} \subset E_n$, $\forall n$ then $\displaystyle\int_{E_n} f(x)dx \to 0$ as $n \to \infty$

(c) If f is bounded, then $\displaystyle\int_{E_n} f(x)dx \to 0$ as $n \to \infty$

(d) If f is integrable and $E_{n+1} \subset E_n$, $\forall n$ then $\displaystyle\int_{E_n} f(x)dx \to 0$ as $n \to \infty$

Solution : (c),(d)
$E_n = (0, 1/n)$ and $f(x) = 1/x$ is a counter example for (a) and (b)
(c) is trivial
(d) follows from DCT.

(8) Let $f, f_n : (X, \mathcal{F}, \mu) \to \mathbb{R}$ be measurable functions. Then which of the following are true?

(a) If $0 \leq f_n$ converges to f uniformly, then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$

(b) If $\mu(X)$ is finite and $|f_n(x)| \leq 1$, $\forall x \in X$, and f_n converges to f *ae* then $\lim_{n\to\infty} \int_X g \circ f_n d\mu = \int_X g \circ f d\mu$, $\forall$ continuous function g on $\mathbb{R}$

(c) If $\mu(X) < \infty$ and if f_n are bounded by one, f_n converges to f a.e.(μ), then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$

(d) If $f_1 \leq f_2 \leq .... \leq f_n \leq f_{n+1} \leq ....$, and f_n converges to f *ae* then $\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$

Solutions : (b),(c)

(a) $f_n = \frac{1}{n}\chi_{[0,n]}$ converges to 0 uniformly but integrals converges to 1

(b) Since g is continuous on $\mathbb{R}$, g takes bounded sets to bounded sets. Hence we can apply DCT to the sequence $g \circ f_n$

(c) By DCT

(d) Following f_n gives a counter example

$$f_n(x) = \begin{cases} x + n & if\ x \leq -n \\ 0 & otherwise \end{cases}$$

(9) Let $\{f_n\}$ be a sequence of real valued measurable functions defined on $\mathbb{R}$ which converges uniformly to a real valued function f on $\mathbb{R}$. Then which of the following are necessarily true?

(a) $\lim_{n\to\infty} \int_{-\infty}^{\infty} f_n(x)dx = \int_{-\infty}^{\infty} f(x)dx$

(b) $\lim_{n\to\infty} \int_{1}^{\infty} f_n(x)dx = \int_{1}^{\infty} f(x)dx$

(c) $\lim_{n\to\infty} \int_{1}^{2} f_n(x)dx = \int_{1}^{2} f(x)dx$

(d) $\lim_{n\to\infty} \int_{K} f_n(x)dx = \int_{K} f(x)dx$ for any compact set $K \subset \mathbb{R}$

Solutions : (c),(d)

$f_n(x) = \frac{1}{n}\chi_{[0,n]}$ is a counter example for (a) and (b).

(c) and (d) follows trivially.

(10) Let $A \in \mathcal{L}(\mathbb{R}^n)$. Then which of the following are correct ?

(a) $\delta A \in \mathcal{L}(\mathbb{R}^n)$ for all $\delta > 0$

(b) $A + x \in \mathcal{L}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$

Solution : (a),(b)
Trivially follows from the properties of Lebesgue integration.