(1) Let F be a countable subset of $\mathbb{R}^n$ then $m_*(F)$ equals

(A) 0
(B) ∞
(C) any positive number

Solutions: (A)
Reason: Outer measure of a singleton is 0 and any countable set is countable union of singletons.

(2) Which of the following are correct?

(A) Any countable set is measurable
(B) Any open set is measurable
(C) The set $\{x \in \mathbb{R} : \ |e^x \sin x| > 1\}$ is measurable

Solutions: (A), (B) and (C)
Reason: (A) and (B) is trivial.
For (C) the set is equal to $\{f^{-1}(1, \infty)\} \cup \{f^{-1}(-\infty, -1)\}$ where $f(x) = e^x \sin x$ is a continuous function.

(3) Which of the following sets are Borel sets?

(A) The subset of $[0, 1]$ whose decimal expansions starts with 2
(B) Subsets of $\mathbb{R}$ whose complements are countable

Solution: (A) and (B)
Reason:
For (A): The subset is of the form $[0.2, 0.3) = \cup_{n=1}^{\infty}(0.2 - \frac{1}{n}, 0.3)$, which is a Borel set.
For (B): Let the complement of set X be $\{r_n : n \in \mathbb{N}\}$ then $X = \cup_{n=1}^{\infty}(r_n, r_{n+1})$ which is a countable union of open intervals which are Borel sets.

(4) The outer measure of the set $\{0\} \times [-1, 1] \subset \mathbb{R}^2$ is

(A) 0
(B) 1
(C) 2

Solution: (A)
Reason: $\{0\} \times [-1, 1] = \cap_{n=1}^{\infty}[-\frac{1}{n}, \frac{1}{n}] \times [-1, 1]$ and
$m_*([-\frac{1}{n}, \frac{1}{n}] \times [-1, 1]) = \frac{1}{n} \to 0$ as $n \to \infty$.

(5) Let $E \subset \mathbb{R}^n$ be an unbounded set.

(A) Outer measure of E is infinity
(B) Outer measure of E is positive, but need not be infinity always
(C) There are unbounded sets whose outer measure is zero

Solution: (B)
Reason: Counter examples for (A) and (C) is the set of natural numbers in $\mathbb{R}$.

(6) Let $E \subset \mathbb{R}^n$ be such that $m_*(E) = 0$. Let O_k be the open set $O_k = \{y \in \mathbb{R}^n : \ d(y, E) < \frac{1}{k}\}$ where $d(y, E) = \inf_{x \in E} |x - y|$.

(A) $m_*(O_n) = \infty$ always
(B) $m_*(O_n)$ is finite always
(C) $m_*(O_n)$ is positive always

Solution: (C)
Reason:
Counter example for (A): Take a finite set $\{x_1, \ldots, x_k\}$. Then $O_n = \cup_{m=1}^{k} \{(x_m - \frac{1}{n}, x_m + \frac{1}{n})\}$ and hence, $m_*(O_n) = \frac{k}{2n}$ which is finite.
For (B): Similarly take an infinite set $\{x_n\}$ and arrive at a contradiction.

(7) Let $E \subset \mathbb{R}^n$ be such that $m_*(E) = 0$ and Let O_k be the open set $O_k = \{y \in \mathbb{R}^n : \ d(y, E) < \frac{1}{k}\}$ where $d(y, E) = \inf_{x \in E} |x - y|$. Then the statement $m_*(O_k) \to 0$ is

(A) True always
(B) True if E is closed
(C) True if E is bounded
(D) True if E is compact

Solution: (D)
Reason:
Counter-example for (A), (B) is the set of natural numbers.
Counter-example for (C) is the set of rationals in $[0, 1]$.
Reason for D: $\cap_{k=1}^{\infty} O_k = E$ and $m_*(O_1) < \infty$. Hence,
$m_*(O_k) \to m_*(E) = 0$.

(8) Let $E \subset \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Define $E + x = \{y + x : \ y \in E\}$. Suppose $m_*(E) = 0$. Which of the following are correct?

(A) $m_*(E + x) = 0$.
(B) $E + x$ is measurable
(C) $E + x$ need not be measurable

Solution: (A) and (B)
Reason: outer measure is translation invariant.

(9) Let $A_n = \{n\} \times [-n, n] \subset \mathbb{R}^2$. If $A = \cup_{n=1}^{\infty} A_n$, then

(A) $m_*(A) = 0$
(B) $m_*(A) = \infty$
(C) $0 < m_*(A) < \infty$

Solution: (A)
Reason: $m_*(A) \leq \sum_{n=1}^{\infty} m_*(A_n) = 0$ since, $m_*(A_n) = 0$ for $n \in \mathbb{N}$.

(10) Let $A = [0, 1]$. Which of the following are correct?

(A) $m_*(A) = \inf\{m_*(O) : \ A \subset O, O \quad \text{open} \quad \}$
(B) $m_*(A) = \sup\{m_*(K) : \ K \subset A, K \quad \text{compact} \quad \}$

Solution: (A) and (B)
Reason: Since $[0, 1]$ is a Lebesgue measurable set the outer and Lebesgue measure coincides. The Lebesgue measure is both inner and outer regular.