(1) Which of the following are true ?

(A)$\chi_{|x|\leq 1}(x)\ |x|^a\ (1-|x|)^b \in L^1(\mathbb{R}^n)$ iff $a > -n$ and $b > -1$
(B)$\chi_{|x|\leq 1}(x)\ |x|^a\ (1-|x|)^b \in L^1(\mathbb{R}^n)$ iff $a > -n$ and $b > 0$

Solutions: A
$\int_{\mathbb{R}^n} \chi_{|x|\leq 1}(x)\ |x|^a\ (1-|x|)^b = \int_0^1 r^a(1-r)^b r^{n-1}dr$
$= \int_0^1 r^{a+n-1}(1-r)^b dr = \int_0^{\frac{1}{2}} r^{a+n-1}(1-r)^b dr + \int_{\frac{1}{2}}^1 r^{a+n-1}(1-r)^b dr$
$= \int_0^{\frac{1}{2}} r^{a+n-1}(1-r)^b dr + \int_0^{\frac{1}{2}} r^b(1-r)^{a+n-1} dr$
$< \infty$ iff $a+n-1, b > -1$ iff $a > -n$ and $b > -1$

(2) Which of the following are correct?

(A) $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$
(B) $\mathcal{L}(\mathbb{R}) \times \mathcal{L}(\mathbb{R}) = \mathcal{L}(\mathbb{R}^2)$

Solutions: A
A is mentioned in Lectures.
$N \times \{0\}$ is a counter example for B where N is a non-measurable subset of $\mathbb{R}$

(3) Which of the following are correct?

(A) The sigma algebra generated by $\{[a_1, b_1] \times \cdots \times [a_n, b_n] : a_j, b_j \in \mathbb{Q}\}$ is $\mathcal{B}(\mathbb{R}^n)$
(B) The sigma algebra generated by $\{[a_1, b_1] \times \cdots \times [a_n, b_n] : a_j, b_j \in \mathbb{Q}\}$ is $\mathcal{L}(\mathbb{R}^n)$
(C) If $E \in \mathcal{L}(\mathbb{R}^2)$ then $E_x \in \mathcal{L}(\mathbb{R})$ for all $x \in \mathbb{R}$

Solutions: A A we have discussed already. Hence B is not true.
C is true for x *ae* only. Not for all x.

(4) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be Lebesgue measurable. Which of the following are correct?

(A) For each $x \in \mathbb{R}$, $f_x : \mathbb{R} \to \mathbb{R}$ (defined by $f_x(y) = f(x, y)$ is Lebesgue measurable
(B) If f is Borel measurable,then for each $x \in \mathbb{R} f_x$ is Borel measurable

Solutions: B
A)Consider $f = \chi_{\{0\}\times N}$ is Lebesgue Measurable, but $f_0 = \chi_N$

is not measurable where $N \subset \mathbb{R}$ is non-measurable. B)Since $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$, it follows by theorem 8.2 of Real and Complex analysis by Rudin.

(5) Consider the measure spaces $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ where m is the Lebesgue measure on $\mathbb{R}$ and $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$ where μ is the measure defined by $\mu(A) =$ number of rationals in $A \cap [0,1]$. Let $m \times \mu$ be the corresponding product measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$. Let $D = \{(x,x) : 0 \leq x \leq 1\} \subset \mathbb{R}^2$, which of the following is correct?

(A) $m \times \mu(D) = 0$
(B) $m \times \mu(D) = \infty$
(C) $m \times \mu(D) = 1$

Solution: A
Follows from the definition of product measures

(6) Consider the measure space $(\mathbb{R}, 2^{\mathbb{R}}, \mu)$ where μ is the measure defined by $\mu(A) =$ number of rationals in A. Let $\mu \times \mu$ be the corresponding product measure on $\mathbb{R}^2$. Let $D = \{(x,x) : 0 \leq x \leq 1\}$. Which of the following is correct?

(A) $\mu \times \mu(D) = 0$
(B) $\mu \times \mu(D) = \infty$
(C) $\mu \times \mu(D) = 1$

Solution: B
Follows from the definition of product measures

(7) Let $X = Y = [0,1]$, m Lebesgue measure on $[0,1]$, μ counting measure on Y. Put $f(x,y) = 1$ if $x = y$ and zero otherwise. Which of the following are correct?

(A) $\int_X f(x,y)\, dm(x) = 0$ for all $y \in Y$
(B) $\int_Y f(x,y)\, d\mu(y) = 1$ for all $x \in X$
(C) $\int_X \int_Y f(x,y)\, d\mu(y)\, dm(x) = \int_Y \int_X f(x,y)\, dm(x)\, d\mu(y)$
(D) μ is not $\sigma-$finite so the iterated integrals are not same

Solutions: A,B,D
A and B follows by direct computations.
C)LHS=1 and RHS=0 D)Counting measure is not $\sigma - finite$ and fubini theorem is valid for $\sigma - finite$ spaces only.

(8) Let $f : \mathbb{R} \to \mathbb{R}$ be a Borel measurable non-negative function. Let $A(f) = \{(x, y) :\ 0 < y < f(x)\}$. Which of the following are correct?

(A) $A(f) \in \mathcal{B}(\mathbb{R}^2)$
(B) Lebesgue measure of $A(f)$ equals $\int_{\mathbb{R}} f(x)\, dx$

Solutions: A,B
Proved Theorem in Real Analysis by Stein and Sackarchi(Refer applications of Fubini theoem)

(9) $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Which of the following are correct?

(A) The graph of f, $G(f) = \{(x, f(x)) :\ x \in \mathbb{R}\}$ is a Borel set in $\mathbb{R}^2$
(B) Lebesgue measure of $G(f)$ is zero
(C) Lebesgue measure of $G(f)$ is infinity

A,B
A)It is the zero set of the continuous function $F(x, y) = y - f(x)$
B)By Fubini's theorem $m(G(f)) = \int_{\mathbb{R}} m(G(f)_x) dx = 0$

(10) Let $(X, \mathcal{F}, \mu)$ be a $\sigma - finite$ measure space and f a positive measurable function on X. For $t \geq 0$ define $F_f(t) = \mu\{x \in X :\ f(x) > t\}$. Which of the following are correct?

(A) F_f is non-increasing and hence Borel measurable
(B) $\int_X f\, d\mu = \int_0^\infty F_f(t)\, dt$

Solutions: A,B
Refer Distribution functions form chapter 8 of Real and Complex analysis by Rudin