(1) Let $\{f_n\}$ be a sequence in $L^1(\mathbb{R})$ which converges to f in $L^1(\mathbb{R})$. Which of the following are correct?

(A) $\int_{\mathbb{R}} |f_n(x)|\, dx$ converges to $\int_{\mathbb{R}} |f(x)|\, dx$
(B) $\{f_n\}$ converges to f almost everywhere on $\mathbb{R}$
(C) There exists a subsequence of $\{f_n\}$ which converges to f almost everywhere on $\mathbb{R}$
(D) $\int_{\mathbb{R}} f_n(x)\; g(x)\; dx$ converges to $\int_{\mathbb{R}} \; f(x)\; g(x)\; dx$ for any $g \in L^\infty(\mathbb{R})$

Solutions: A,C,D
A) $\left| \int |f_n| - \int |f| \right| \leq \int \left| \; |f_n| - |f| \; \right| \leq \int |f_n - f| \to 0$
B) Consider the sequence $f_1 = \chi_{[0,\frac{1}{2}]}, f_2 = \chi_{[\frac{1}{2},1]}, f_3 = \chi_{[0,\frac{1}{4}]}, f_4 = \chi_{[\frac{1}{4},\frac{1}{2}]}, f_5 = \chi_{[\frac{1}{2},\frac{3}{4}]}, f_6 = \chi_{[\frac{3}{4},1]}, f_7 = \chi_{[0,\frac{1}{8}]} \cdots$ Then f_n converges to 0 in $L^1(\mathbb{R})$ but does not converge pointwise at any point in $[0,1]$.
C) Refer Theorem 3.12 of Rudin-Real and Complex
D)$| \int f_n g - \int fg| \leq \int |(f_n - f)g| \leq \|g\|_\infty \int |f_n - f| \to 0$

(2) Let $\nu(A) = \int_A \frac{1}{1+x^2}\, dx$ for $A \in \mathcal{B}(\mathbb{R})$. Which of the following are correct?

(A) ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$
(B) Lebesgue measure on $\mathbb{R}$ is absolutely continuous with respect to ν

Solutions: A,B
Since $\frac{1}{1+x^2}$ is positive everywhere.

(3) Which of the following are correct statements?

(A) $T(f) = \int_0^1 f(x)dx$ is a continuous linear functional on $L^1[0,1]$
(B) $T(f) = \int_0^1 f(x)dx$ is a continuous linear functional on $L^2(\mathbb{R})$
(C) $T(f) = \int_0^1 f(x)dx$ is a continuous linear functional on $L^p[0,1]$ for all $1 \leq p \leq \infty$

Solution: A,B,C
A and B follows directly
C)It is a finite measure space and hence, $L^p \subset L^1$ for all $p > 1$.

(4) Let T be an $n \times n$ invertible real matrix. Let μ be the measure defined by $\mu(A) = m(TA)$ where m is the Lebesgue measure on $\mathbb{R}^n$. Which of the following are correct?

(A) μ is absolutely continuous with respect to m
(B) m is absolutely continuous with respect to μ

Solution: A,B
$\mu(A) = m(TA) = det(T)m(A)$ and $det(T) \neq 0$. Hence $\mu(A) = 0 \iff m(A) = 0$

(5) For $A \in \mathcal{B}(\mathbb{R}^2)$ define $A_{\mathbb{R}} = \{(x, 0) \in A : \ x \in \mathbb{R}\} = A \cap (\mathbb{R} \times \{0\})$. Define a measure ν on $\mathcal{B}(\mathbb{R}^2)$ by $\nu(A) = \int_{A_{\mathbb{R}}} e^{-x^2} \, dx$. Which of the following statements are correct?

(A) ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$
(B) μ is mutually singular with respect to the Lebesgue measure on $\mathbb{R}^2$

Solutions: B
A) $[0, 1] \times \{0\}$ gives counter example.
B)μ concentrates on $\mathbb{R} \times \{0\}$ while Lebesgue Measure concentrates in its complement.

(6) Let μ be a non-zero complex measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ which is absolutely continuous with respect to the Lebesgue measure m on $\mathbb{R}$. Let h denote the Radon-Nikodym derivative $\frac{d\mu}{dm}$. Which of the following are possible?

(A) h is zero outside $[0, 1]$
(B) h is zero on irrationals
(C) h is one on the set $\{\frac{1}{n} : \ n \in \mathbb{N}\}$ and zero otherwise

Solutions: A
B and C implies μ is zero.

(7) Let m be the Lebesgue measure on $\mathbb{R}$ and μ be the measure defined by $\mu(A) = m(A) + 1$ if $0 \in A$, $\mu(A) = m(A)$ otherwise. Which of the following are correct?

(A) m is not absolutely continuous with respect to μ
(B) m is absolutely continuous with respect to μ

(C) μ is absolutely continuous with respect to m

Solutions: B
Follow directly from definitions.

(8) Let δ_0 be the measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ defined by $\delta_0(A) = 1$ if $0 \in A$ and zero otherwise. Which of the following is correct?

(A) $m - \delta_0$ is absolutely continuous with respect to m
(B) $m - \delta_0$ is not absolutely continuous with respect to m

Solutions: B
Follows directly from definitions.

(9) Let $f \in L^p[0,1]$ for some $1 < p < \infty$. Define $T(g) = \int_0^1 f(x)g(x)dx$. Which of the following are correct?

(A) T defines a continuous linear functional on $L^\infty[0,1]$
(B) T defines a continuous linear functional on $L^2[0,1]$
(C) T defines a continuous linear functional on $L^q[0,1]$ for all $q \geq p^*$ where $\frac{1}{p} + \frac{1}{p^*} = 1$

Solutions: A,C
A)$|Tg| \leq \int_0^1 |fg| \leq \|f\|_p \; \|g\|_{p^*} \leq \|f\|_p \|g\|_\infty$ where $\frac{1}{p} + \frac{1}{p^*} = 1$
B)If $p < 2$ then it is not even well defined.
C)$|Tg| \leq \int_0^1 |fg| \leq \|f\|_p \; \|g\|_{p^*} \leq \|f\|_p \|g\|_q$

(10) Let $f_1, f_2 \in L^2(\mathbb{R})$ and let $T(g) = \int_{\mathbb{R}} f_1(x)f_2(x)g(x)dx$. Which of the following is correct?

(A) T defines a continuous linear functional on $L^2(\mathbb{R})$
(B) T defines a continuous linear functional on $L^1(\mathbb{R})$
(C) T defines a continuous linear functional on $L^\infty(\mathbb{R})$

Solutions: C
A) Product of 3 L^2 functions need not be integrable and hence T is not well defined.
B) Same reason as A
C)Since $f_1, f_2 \in L^2(\mathbb{R})$, $f_1 f_2 \in L^1(\mathbb{R})$ and hence T is a continuous linear functional.