(1) Let $(X, \mathcal{F}, \mu)$ be a measure space. Then,

(A) $f, g \in L^1(\mu)$ implies $fg \in L^1(\mu)$
(B) $f, g \in L^2(\mu)$ implies $fg \in L^2(\mu)$
(C) $f, g \in L^1(\mu)$ implies $fg \in L^1(\mu)$
(D) $f \in L^1(\mu)$ and $f \in L^\infty(\mu)$ implies $f \in L^2(\mu)$

Solutions: C,D.
Reason:
(A) $x^{-\frac{1}{2}} \in L^2(0,1)$ but $x^{-1} \notin L^1(0,1)$
(B) $x^{-\frac{1}{4}} \in L^2(0,1)$ but $x^{-\frac{1}{2}} \notin L^1(0,1)$
(C) By Holder's inequality.
(D) $\int |f|^2 d\mu \leq \| f \|_\infty \int |f|^2 d\mu < \infty$

(2) Which of the following are true?

(A) $L^1[0,1] \subset L^2[0,1]$
(B) $L^1[0,\infty) \subset L^2[0,\infty)$
(C) $L^2[0,1] \subset L^1[0,1]$
(D) $L^2[0,\infty) \subset L^1[0,\infty)$

Solutions: C
Reason:
(A) Counter example: $\frac{1}{\sqrt{x}}$
(B) Counter example: $\frac{1}{\sqrt{x}}\chi_{(0,1)}$
(C) $\int_0^1 |f| d\mu = (\int_0^1 |f|^2 d\mu)^{\frac{1}{2}} (\int_0^1 1^2 d\mu)^{\frac{1}{2}} < \infty$
(D) Counter example: $\frac{1}{x}\chi_{(1,\infty)}$

(3) Let $f = \chi_{[0,\frac{1}{2}]}$. Then,

(A) f is continuous almost every where with respect to the Lebesgue measure on $\mathbb{R}$
(B) f can be approximated by continuous functions in the L^∞ norm
(C) There exists a continuous function g such that $f = g$ almost every where

Solutions: C
Reason:
(A) f is continuous except at 0 and $\frac{1}{2}$.

(B) Limit of continuous functions in sup norm is also continuous.
(C) Since there is a jump at 2 points we can't find such a function.

(4) Which of the following are correct?

(A) $\chi_{|x|\leq 1}(x)\ |x|^a \in L^1(\mathbb{R}^n)$ iff $a > -n$
(B) $\chi_{|x|\leq 1}(x)\ |x|^a \in L^1(\mathbb{R}^n)$ iff $a < -n$
(C)$\chi_{|x|\geq 1}(x)\ |x|^a \in L^1(\mathbb{R}^n)$ iff $a > -n$
(D) $\chi_{|x|\geq 1}(x)\ |x|^a \in L^1(\mathbb{R}^n)$ iff $a < -n$

Solutions: A,D
Reason:
It is a well known result.

(5) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Then,

(A)$f \in L^1(\mathbb{R})$ implies f is bounded.
(B)$f \in L^1(\mathbb{R})$ and f is continuous implies f is bounded.
(C)$f \in L^1(\mathbb{R})$ and f is continuous implies $\lim_{|x|\to\infty} |f| = 0$.
(D)$f \in L^1(\mathbb{R})$ and f is uniformly continuous implies f is bounded.

Solutions: D
Reason:
Counter example for (A), (B) and (C) is the following function: $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = 0, x \in [-\infty, 1]$, $f(n) = n$ for $n \geq 2$, $f(n - \frac{1}{n^3}) = 0 = f(n + \frac{1}{n^3})$, f is affine (tent-like) in the interval $[n - \frac{1}{n^3}, n + \frac{1}{n^3}]$ and $f = 0$ elsewhere.
(D) Show that $\lim_{|x|\to\infty} |f| = 0$ and hence, the conclusion follows.

(6) Let $f_n : [0, 1] \to \mathbb{R}$ be defined by $f_n(x) = x^n, x \in [0, 1]$ for $n = 1, 2, 3, \cdots$. Which of the following are correct?

(A) f_n converges to zero uniformly in $[0, 1]$
(B) f_n converges to zero in $L^1[0, 1]$
(C) f_n converges to zero in $L^p[0, 1]$ for all $1 \leq p < \infty$

Solutions: B,C
Reason:
Follows from direct arguments.

(7) Let $f_n : [1, \infty) \to \mathbb{R}$ be defined by $f_n(x) = x^{-n}, x \in [1, \infty)$ for $n = 1, 2, 3, \cdots$. Which of the following are correct?

(A) f_n converges to zero uniformly
(B) f_n converges to zero in $L^1[1, \infty)$
(C) f_n converges to zero in $L^p[1, \infty)$ for all $1 \leq p < \infty$

Solutions: B,C
Reason:
Follows from direct arguments.

(8) Let $f_n : [2, \infty) \to \mathbb{R}$ be defined by $f_n(x) = x^{-n}, x \in [2, \infty)$ for $n = 1, 2, 3, \cdots$. Which of the following are correct?

(A) f_n converges to zero uniformly
(B) f_n converges to zero in $L^1[2, \infty)$
(C) f_n converges to zero in $L^p[2, \infty)$ for all $1 \leq p \leq \infty$

Solutions: A,B,C
Reason:
Follows from direct arguments.

(9) Let $(X, \mathcal{F}, \mu)$ be a measure space and $1 \leq p, r, s \leq \infty$. Which of the following are correct?

(A) If $p < r < s$, then $L^p \cap L^s(\mu) \subset L^r(\mu)$
(B) If $\mu(X) < \infty$, then $L^p(\mu) \subset L^r(\mu)$ if $r < p$
(C) If $\mu(X) < \infty$ and $f \in L^\infty(\mu)$ then $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$

Solutions: A,B,C
Reason:
(A) Let $0 < p < q < \infty$ and $r = \lambda p + (1 - \lambda q)$. Take $f \in L^p \cap L^q$ then

$$\int |f|^r = \int |f|^{p\lambda} |f|^{q(1-\lambda)} d\mu$$

Then by applying Holder's inequality

$$\int |f|^r \leq (\int |f|^p)^\lambda (\int |f|^q)^{1-\lambda} = \| f \|_p{}^{p\lambda} \| f \|_q{}^{q(1-\lambda)}$$

In the case $q = \infty$,

$$\| f \|_r = \int_X |f(x)|^{r-p}|f(x)|^p = (\text{esssup}_{x\in X}|f(x)|^{r-p})^{\frac{1}{r}}(\int_X |f(x)|^p dx)^{\frac{1}{r}}$$
$$= \| f \|_p{}^{\frac{p}{r}} \| f \|_\infty{}^{1-\frac{p}{r}}$$

(B) $\int |f|^r d\mu = \int (|f|^p)^{\frac{r}{p}} d\mu \leq \int (|f|^r)^{\frac{p}{r}} \mu(X)^{1-\frac{r}{p}} < \infty$
(C) Let $\delta > 0$ and let $X_\delta := \{x \in X : |f(x)| > \| f \|_\infty - \delta\}$ then,

$$\| f \|_p \geq (\int_{X_\delta} (\| f \|_\infty - \delta)^p d\mu)^{\frac{1}{p}} = (\| f \|_\infty - \delta)\mu(X_\delta)^{\frac{1}{p}}$$

hence, $\liminf_{p\to\infty} \| f \|_p \geq \| f \|_\infty$.
But, again we have for $p > q$

$$\| f \|_p \leq (\int_X |f(x)|^{p-q}|f(x)|^q d\mu)^{\frac{1}{p}} \leq \| f \|_\infty^{\frac{p-q}{p}} \| f \|_q^{\frac{q}{p}} .$$

Hence, the conclusion follows.

(10) Let $(X, \mathcal{F}, \mu)$ be a measure space and let f and g be positive measurable functions such that $fg \geq a$ for some $a > 0$. Then,

(A) If $\mu(X) = 1$, $\left(\int_X \ f d\mu\right)\left(\int_X \ g \ d\mu\right) \geq a$
(B) If $\mu(X) < 1$, $\left(\int_X \ f d\mu\right)\left(\int_X \ g \ d\mu\right) \geq a$

Solutions: A
Reason:
(A) Apply Holder's inequality to $\sqrt{(}fg)$
(B) Counter example: $X = [0, \frac{1}{2}]$, $f = g = \frac{1}{2}$, $a = \frac{1}{4}$