(1) Which of the following sets are of Lebesgue measure zero?

(A) Any countable subset of $\mathbb{R}$
(B) Cantor set
(C) The set of irrationals in the Cantor set
(D) $k-$dimensional subspace of $\mathbb{R}^n$, where $k < n$

Solution: All of the above.
Reason:
(A) Singletons have measure 0.
(B) The Cantor set $\mathcal{C} = \cap_{k=1}^{n} \mathcal{C}_k$ where $m(\mathcal{C}_k) < \infty$ and $m(\mathcal{C}_k) \to 0$ as $k \to \infty$.
(C) Subset of a measure 0 set.
(D) For $k < n$ a $k-$dimensional subspace $S \subset \mathbb{R}^n$ can be realized as image of a linear transformation A of rank k. That is, there exists $A : \mathbb{R}^n \to \mathbb{R}^n$ such that $A(\mathbb{R}^n) = S$ and A is singular. From the lectures, $m(S) = m(A(\mathbb{R}^n)) = |det(A)|m(\mathbb{R}^n) = 0$.

(2) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear map. Which of the following are correct?

(A) If E is a Lebesgue set then $T(E)$ is a Lebesgue set
(B) If E is a Borel set then $T(E)$ is a Borel set

Solution: (A),(B).
Reason: Follows from the relation $E = T^{-1}T(E)$.

(3) Which of the following are true?

(A) The outer measure m_* on $\mathbb{R}$ is translation invariant
(B) The Lebesgue measure m on $\mathbb{R}$ is translation invariant
(C) If μ is any Borel measure on $\mathbb{R}$ and $\mu(K) < \infty$ for all compact set K, then μ is a constant multiple of the Lebesgue measure

Solution: (A), (B)
Reason:
(A) and (B) are standard facts.
Counter-example for (C) is the Dirac measure δ_0 on $\mathbb{R}$.

(4) Let μ be a Borel measure on $\mathbb{R}$ such that $\mu(A+n) = \mu(A)$ for all Borel sets A and $n \in \mathbb{Z}$. Then which of the following are always correct?

(A) μ is the zero measure
(B) μ is a constant multiple of the Lebesgue measure
(C) Counting measure on $\mathbb{R}$ satisfies the property $\mu(A+n) = \mu(A)$ for all A and $n \in \mathbb{Z}$.

Solution:(C)
Reason:
Any Borel measure is translation invariant therefore, (A) and (B) does not hold.
(C) For any set $|A| = |A+n|$ for any $n \in \mathbb{Z}$.

(5) Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Which of the following are correct?

(A) If A is singular, then $A(E)$ is a Lebesgue set for all $E \subset \mathbb{R}^n$
(B) If A is invertible, then $A(E)$ is a Lebesgue set for all $E \subset \mathbb{R}^n$
(C) If A is invertible, then $A(E)$ is a Lebesgue set for all Lebesgue sets $E \subset \mathbb{R}^n$

Solution:(A) and (C)
Reason:
(A) if A is singular the $A(E)$ is a $k-$dimensional subspace of $\mathbb{R}^n$ where $k < n$. Therefore, $A(E)$ is a measure 0 set and hence, Lebesgue measurable.
Counter-example for (B) is the following: Let $\mathcal{N}$ be a non-measurable set in $\mathbb{R}^n$ then $A^{-1}(E)$ is never measurable. For otherwise, $E = A(A^{-1}(E))$ becomes measurable, which is a contradiction.
(C) follows from the relation $E = A^{-1}A(E)$.

(6) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a positive measurable function and let m denote the Lebesgue measure on $\mathbb{R}$. Which of the following are true?

(A) $\int_{E+x_0} f \; dm = \int_E f \; dm$ for all $E \in \mathcal{L}(R^n)$ and $x_0 \in \mathbb{R}^n$
(B) $\int_{\mathbb{R}^n} f(x+x_0) \; dx = \int_{\mathbb{R}^n} f \; dm$ for all $x_0 \in \mathbb{R}^n$
(C) $\int_{\mathbb{R}^n} f(Ax) \; dx = \int_{\mathbb{R}^n} f(x) \; dx$ for all linear maps $A : \mathbb{R}^n \to \mathbb{R}^n$

Solution: (B)
Reason:
Counter-example for (A): Let $E = (0,1)$ and $x_0 = 1$ then $E + x_0 = (1,2)$. Take $f = \chi_{(1,2)}$. Then, $\int_E f \; dm = 0$ but, $\int_{E+x_0} f \; dm = m((1,2)) = 1$.
(B) Let us first consider a simple function that is, $f = \sum_{k=1}^{m} a_k \chi_{A_k}$ wher A_k are measurable sets in R^n and $a_k \geq 0$. Then

$$\begin{aligned} \int_{\mathbb{R}^n} f(x + x_0) \; dx &= \sum_{k=1}^{m} a_k m(A_k + a) \\ &= \sum_{k=1}^{m} a_k m(A_k) = \int_{\mathbb{R}^n} f(x) \; dx \end{aligned}$$

Now, any positive measurable function can be pointwise approximated by monotone sequence of simple functions. Hence, by monotone convergence theorem we obtain the result.
Counter-example for (C): From the lectures it can be observed that $m(A(E)) = |det(A)|m(E)$ for $E \in \mathcal{L}(R^n)$. Therefore, if we take $f = \chi_E$ then, $\int_{\mathbb{R}^n} f(Ax) \; dx = m(A(E))$ and $\int_{\mathbb{R}^n} f(x) \; dx = m(E)$. So, if we take A to be singular then $\int_{\mathbb{R}^n} f(Ax) \; dx \neq \int_{\mathbb{R}^n} f(x) \; dx$.

(7) Which of the following are true?

(A) If $O \subset [0,1]$ is a dense open set then $m(O) = 1$
(B) If $O \subset [0,1]$ is open then $m(O) > 0$
(C) If $F \subset [0,1]$ is closed and has no interior then $m(F) = 0$
Solutions: (B)

Reason:
(A) Consider the set of rational numbers $\{r_n\}_n$ in $(0,1)$. Let us form the open set $U = \cup_{k=1}^{\infty}(r_k - \frac{1}{2^{k+1}}\frac{1}{m}, r_k + \frac{1}{2^{k+1}}\frac{1}{m})$ for some $m > 1$. Then U is an open dense set in $[0,1]$. But, $m(U) = \sum_{k=1}^{\infty} \frac{1}{2^k}\frac{1}{m} = \frac{1}{m} < 1$.
(B) Any open set is countable union of disjoint open intervals and since, any open interval has strictly positive measure.
(C) Consider the same open set U as in (A). Then complement of U in $[0,1]$ is a closed set with no interior. But, $m(U) = \frac{1}{m} < 1$ therefore, $m(U^{\complement}) \neq 0$.

(8) Which of the following sets are of Lebesgue measure zero?

(A) $C \times \mathbb{R} \subset \mathbb{R}^2$ where $C \subset [0,1]$ is the Cantor set
(B) $\mathbb{Q} \times \mathbb{R} \subset \mathbb{R}^2$
(C) Countable union of lines passing through the origin in $\mathbb{R}^2$
Solution: All of the above.

Reason:
(A) Cantor set has measure 0.
(B) $\mathbb{Q}$ has measure 0.
(C) Any line in $\mathbb{R}^2$ has measure 0.

(9) Which of the following sets have positive Lebesgue measure?

(A) Any unbounded set in $\mathbb{R}^2$
(B) Any unbounded closed set in $\mathbb{R}^2$
(C) $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R} \subset \mathbb{R}^2$

Solution: (C)
Reason:
Counter-examples of (A) and (B) is $\mathbb{R} \times \{0\}$.
(C) is true since, $\mathbb{R}^2 = \big((\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}\big) \sqcup \big(\mathbb{Q} \times \mathbb{R}\big)$ and measure of $(\mathbb{R} \setminus \mathbb{Q}) \times \mathbb{R}$ is 0 will imply measure of $\mathbb{R}^2$ to be 0.

(10) Which of the following are correct?

(A) The Cantor set is a closed set
(B) The Cantor set is a perfect set
(C) The complement of Cantor set in $[0,1]$ has positive Lebesgue measure
(D) The Cantor set is uncountable

Solution: All of the above.
Reason: (A) A Cantor set is countable intersection of closed sets.
(B) Every point of the Cantor set is a limit point.
(C) Since, Cantor set has Lebesgue measure 0.
(D) Every perfect set has to be uncountable.