(1) Let μ be a complex measure on $(X, \mathcal{F})$. For $E \in \mathcal{F}$ define $\nu(E)$ to be the supremum of $\{\sum |\mu(E_j)|\}$ where the supremum is taken over *finite* measurable partitions $\{E_j\}$ of E. Which of the following are correct?

(A) $\nu = |\mu|$
(B) There exists $E \in \mathcal{F}$ such that $\nu(E) < |\mu|(E)$
(C) ν is not a measure

Solution: A
From the definition we have $\nu(E) \leq |\mu|(E)$ for $E \in \mathcal{F}$.
Now let $\{E_i\}$ be a countable partition of E and $\epsilon > 0$. The series $\sum |\mu(E_i)|$ is convergent, hence there is an N such that $|\mu(E_N)| + |\mu(E_{N+1}| + \cdots < \epsilon$. Define the finite partition $\{F_i\}$ of E by $F_i = E_i$ for $i = 1, 2, \cdots N-1,\ F_n = E_n \cup E_{N+1} \cup \cdots$. It follows that $\sum |\mu(E_i)| < \sum_{i=1}^{N} |\mu(F_i)| + \epsilon$. Hence $|\mu|(E) \leq \nu(E) + \epsilon \implies |\mu|(E) \leq \nu(E)$ for $E \in \mathbb{F}$. Hence $\nu = \mu$

(2) Let μ be a real measure defined on $(\mathbb{N}, 2^{\mathbb{N}})$, $\mu(\{j\}) = a_j$ where $a_j \in \mathbb{R}$ and $\sum |a_j| < \infty$. Which of the following are correct?

(A) $\mu^+(A) = \sum_{\{j \in A; a_j \geq 0\}} a_j$
(B) $\mu^-(A) = -\sum_{\{j \in A; a_j < 0\}} a_j$
(C) $|\mu|(A) = \sum_{j \in A} |a_j|$

Solutions: A,B,C
Follows directly from definitions.

(3) Which of the following are correct?

(A) For a complex measure λ, λ is concentrated on A then $|\lambda|$ is concentrated on A
(B) For a complex measure λ, if $|\lambda|$ is concentrated on A then so is λ

Solutions: A,B
Refer Theorem 6.8 of Rudin-Real and Complex analysis

(4) Let λ be the Borel measure defined by $\lambda(A) = \sum_{n \in \mathbb{Z} \cap A} \frac{(i)^n}{n^2}$, $A \in \mathcal{B}(\mathbb{R})$. Which of the following are correct?

(A) λ is concentrated on the set $\{\frac{1}{n^2} :\ n \in \mathbb{Z}\}$
(B) λ is concentrated on $\mathbb{Z}$

(C) $|\lambda|$ is concentrated on $\mathbb{Z}$

Solutions: B,C

Follows directly from definition.

(5) Let m be the Lebesgue measure on $\mathbb{R}$ and let μ be the measure defined by $\mu(A) =$ number of rationals in A, for $A \in \mathcal{B}(\mathbb{R})$. Which of the following is correct?

(A) μ is mutually singular to m
(B) μ is absolutely continuous with respect to m

Solutions: A

A)μ is concentrated on rationals and m is concentrated on Irrationals.
B)Singleton set $\{1\}$ is a counter example

(6) Let $(X, \mathcal{F}, \mu)$ be a positive measure space. Which of the following sets are convex?

(A) $\{f \in L^2(\mu) : \int_X |f|^2 \, d\mu \leq 1\}$
(B) $\{f \in L^2(\mu) : \int_X |f|^2 \, d\mu = 1\}$
(C) $\{f \in L^2(\mu) : 1 \leq \int_X |f|^2 \, d\mu \leq 2\}$

Solutions: A

A is the unit ball in the Hilbert space and hence convex.
For B and C we can find counter examples from $\mathbb{R}^2$ which also can be considered as L^2 space.

(7) Let $(X, \mathcal{F}, \mu)$ be a positive measure space and let $f \in L^2(\mu)$. Which of the following are convex sets?

(A) $\{g \in L^2(\mu) : \int_X fg \, d\mu = 2\}$
(B) $\{g \in L^2(\mu) : \left|\int_X fg\right| \leq 1\}$
(C) $\{g \in L^2(\mu) : \left|\int_X fg\right| \geq 2\}$

Solutions: A,B

A and B follows from direct computation.
For C we can get counter example from $\mathbb{R}$ which is also a L^2 space.

(8) Let $(X, \mathcal{F}, \mu)$ be a positive measure space and $f \in L^2(\mu)$ be a non-zero function. Let $M = \{g \in L^2(\mu) : \int_X fgd\mu = 0\}$. Which of the following is correct?

(A) M is closed
(B M is closed and convex
(C) The vector with minimal norm in M is f

Solution: A,B
Here M is the null space of a continuous linear functional hence it is closed and convex.
Vector with minimal norm in M is 0, not f.

(9) Which of the following sets have a vector with minimal norm?

(A)$\{\frac{n+1}{n} f_n : f_n \in L^2[0,1], \|f_n\|_2 = 1, \langle f_n, f_m \rangle = 0$ for $n \neq m\}$
(B) $\{g \in L^2(\mathbb{R}) : \left|\int_0^1 g(t)\ dt\right| > 1\}$
(C) $\{g \in L^2[0,1] : T(g) = 2\}$ where $T : L^2[0,1] \to \mathbb{C}$ is a continuous linear functional.

Solutions: C
A)Minimum norm is 1, but all the elements in set have norm greater than 1.
B)Consider the sequence $g_n = (1+\frac{1}{n})\chi_{[0,1]}$ in $L^2(\mathbb{R})$ with $\|g_n\| = 1 + \frac{1}{n}$ and hence belongs to the given set. But $1 < |\int_0^1 g| \leq \|g\|_2$ for all g in the given set. Hence there is no element with minimum norm.
C)Closed and convex set in a Hilbert space has an element with minimum norm by projection theorem.

(10) Let $(X, \mathcal{F}, \mu)$ be a positive measure space and $X = \cup_{n=1}^{\infty} A_n$ where $A_n \in \mathcal{F}$ and $A_k \cap A_j = \phi$ if $k \neq j$. Let $\{a_n\}$ be a sequence of complex numbers and consider the map $T : L^2(\mu) \to \mathbb{C}$ defined by $T(f) = \sum_n a_n \int_{A_n} f\ d\mu$. Which of the following are correct?
(A) T is a continuous linear functional if and only if $\sum |a_n| < \infty$
(B) T is continuous linear functional if and only if $\sum |a_n|^2 < \infty$
(C) T is a continuous linear functional if and only if $\sum |a_n|^2 \mu(A_n) < \infty$

Solutions: C

$T(f) = \sum a_n \int_{A_n} f \; d\mu = \sum \int_X \chi_{A_n} f \; d\mu = \sum a_n \langle f, \chi_{A_n} \rangle = \sum \langle f, \bar{a_n} \chi_{A_n} \rangle \leq \sum \|f\|_2^2 \; \|a_n \chi_{A_n}\|_2^2 = \|f\|_2^2 \; \sum |a_n|^2 \mu(A_n)$